Analysis of Bialinycki-Birula Unitary Cellular Automata

Original unitary matrices as defined in Bialinycki-Birula: Phys. Rev. D 49, 6920 (1994)

```
> restart;with(LinearAlgebra):
> p1:=<<1|0>,<1|0>>;p2:=<<0|1>,<0|1>>;p3:=<<1|0>,<-1|0>>;p4:=<<0|-1>,<0|1>>;
```

$$p1 := \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$p2 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$p3 := \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$p4 := \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

```
> phi1:=<psi1,xi1>:phi2:=<psi2,xi2>:phi3:=<psi3,xi3>:phi4:=<psi4, xi4>:phi5:=<psi5, xi5>:phi6:=<psi6, xi6>:phi7:=<psi7, xi7>:phi8:=<psi8, xi8>:
```

Neighborhood Summand from original Birula unitary matrices:

```
>
N:=a*MatrixVectorMultiply(p1,phi8)+a*MatrixVectorMultiply(p2,phi2)+a*MatrixVectorMultiply(p3,phi5)+b*MatrixVectorMultiply(p2,phi4)+b*MatrixVectorMultiply(p3,phi7)+b*MatrixVectorMultiply(p4,phi1)+b*MatrixVectorMultiply(p1,phi6)+a*MatrixVectorMultiply(p4,phi3);
```

$$N := \begin{bmatrix} a\,\psi 8 + a\,\xi 2 + a\,\psi 5 + b\,\xi 4 + b\,\psi 7 - b\,\xi 1 + b\,\psi 6 - a\,\xi 3 \\ a\,\psi 8 + a\,\xi 2 - a\,\psi 5 + b\,\xi 4 - b\,\psi 7 + b\,\xi 1 + b\,\psi 6 + a\,\xi 3 \end{bmatrix}$$

Build a symmetric Scattering Matrix (1/4)*K (Upper & lower two rows exchange Psi for Xi conjugates).

```
> K0:=<<a|b|b|a>,<-a|b|-b|a>,<-b|a|-a|b>,<b|a|a|b>>;
```

$$K0 := \begin{bmatrix} a & b & b & a \\ -a & b & -b & a \\ -b & a & -a & b \\ b & a & a & b \end{bmatrix}$$

Properties of K matrix:

```
> K:=subs(a=1+I,subs(b=1-I,K0)):simplify(eval(Determinant(K)));F,Q=FrobeniusForm(K,output=['F','Q']);U,V=SmithForm(K,output=['U','V']);
```

$$F, Q = \left(\begin{bmatrix} 0 & 0 & 0 & 64 \\ 1 & 0 & 0 & -16-16I \\ 0 & 1 & 0 & 8I \\ 0 & 0 & 1 & 2-2I \end{bmatrix}, \begin{bmatrix} 1 & 1+I & 4I & -8-8I \\ 0 & -1-I & -4I & 8-8I \\ 0 & -1+I & -4I & -8-8I \\ 0 & 1-I & -4I & 8-8I \end{bmatrix} \right)$$

$$U, V = \left(\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}+\frac{1}{2}I \\ 0 & 0 & \frac{1}{4}-\frac{1}{4}I & \frac{1}{4}-\frac{1}{4}I \\ 0 & \frac{-1}{4}-\frac{1}{4}I & 0 & \frac{1}{4}-\frac{1}{4}I \\ \frac{1}{8}-\frac{1}{8}I & \frac{1}{8}-\frac{1}{8}I & \frac{1}{8}+\frac{1}{8}I & \frac{1}{8}+\frac{1}{8}I \end{bmatrix}, \begin{bmatrix} 1 & -I & -I & 1 \\ 0 & 1 & 0 & I \\ 0 & 0 & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

K has an anti-Hadamard property and a permutation property:

```
> MatrixMatrixMultiply(K, Transpose(K))/8;
```

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

```
> P:=MatrixMatrixMultiply(Transpose(K), K)/8;
```

$$P := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Original summand becomes (K.Psi) + ((P.K).Xi) -> Out:

```
> psi:=<psi5, psi6, psi7, psi8>:xi:=<xi1, xi2, xi3, 
xi4>:k1:=MatrixVectorMultiply(K,psi):k2:=
MatrixVectorMultiply(K,xi):kk:=k1+MatrixVectorMultiply(P,k2
):New1:=simplify(eval(kk));
```

$New1 :=$

$$\begin{bmatrix} \psi5 - \psi6\,I + \psi6 - \psi7\,I + \psi7 + \xi2\,I + \psi8 + \xi1\,I - \xi1 + \psi8\,I + \xi2 - \xi4\,I - \xi3 - \xi3\,I + \xi4 + \psi5\,I \end{bmatrix}$$
$$\begin{bmatrix} -\psi5 - \psi6\,I + \psi6 + \psi7\,I - \psi7 + \xi2\,I + \psi8 - \xi1\,I + \xi1 + \psi8\,I + \xi2 - \xi4\,I + \xi3 + \xi3\,I + \xi4 - \psi5\,I \end{bmatrix}$$
$$\begin{bmatrix} -\psi5 + \psi6\,I + \psi6 - \psi7\,I - \psi7 - \xi2\,I + \psi8 + \xi1\,I + \xi1 - \psi8\,I + \xi2 + \xi4\,I + \xi3 - \xi3\,I + \xi4 + \psi5\,I \end{bmatrix}$$
$$\begin{bmatrix} \psi5 + \psi6\,I + \psi6 + \psi7\,I + \psi7 - \xi2\,I + \psi8 - \xi1\,I - \xi1 - \psi8\,I + \xi2 + \xi4\,I - \xi3 + \xi3\,I + \xi4 - \psi5\,I \end{bmatrix}$$

Verify against original summand:

```
> New2:=simplify(eval(subs(a=1+I,subs(b=1-I,N)))):New1[1] -
New2[1];New1[2] - New2[2];evalc(conjugate(New1[1]) -
New1[4]);evalc(conjugate(New1[2]) -New1[3]);
```

$$0$$

$$0$$

$$0$$

$$0$$

Alternative: K.Psi + K1.Xi, K1 = P.K:

```
> K1:=MatrixMatrixMultiply(P,
K):simplify(eval(Determinant(K1)));MatrixMatrixMultiply(K1,
Transpose(K1))/8;MatrixMatrixMultiply(Transpose(K1),K1)/8;
```

$$-64$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Alternative separation of original summand separating real and imaginary coefficients:

```
> evalc(New1[1] + New1[4])/2;evalc(New1[2] + New1[3])/2;
```

$$\psi5 + \psi6 + \psi7 + \psi8 - \xi1 + \xi2 - \xi3 + \xi4$$

$$-\psi5 + \psi6 - \psi7 + \psi8 + \xi1 + \xi2 + \xi3 + \xi4$$

Introduce two new matrices

```
> H1:=<<1|1|1|1>,<1|-1|-1|1>,<-1|1|-1|1>,<-1|-
1|1|1>>;H2:=<<1|1|1|1>,<-1|1|1|-1>,<-1|1|-1|1>,<1|1|-1|-
1>>;
```

$$H1 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$H2 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

Hadamard property holds for both:

```
> MatrixMatrixMultiply(H1,Transpose(H1))/4;MatrixMatrixMultiply(H2,Transpose(H2))/4;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We also need two Connectivity matrices:

```
> P1:=<<1|I|0|0>,<0|0|1|I>,<0|0|1|I>,<1|I|0|0>>;P2:=<<0|0|1|I>,<1|I|0|0>,<1|I|0|0>,<0|0|1|I>>;
```

$$P1 := \begin{bmatrix} 1 & I & 0 & 0 \\ 0 & 0 & 1 & I \\ 0 & 0 & 1 & I \\ 1 & I & 0 & 0 \end{bmatrix}$$

$$P2 := \begin{bmatrix} 0 & 0 & 1 & I \\ 1 & I & 0 & 0 \\ 1 & I & 0 & 0 \\ 0 & 0 & 1 & I \end{bmatrix}$$

Introduce new pair of scattering matrices as S1 = P1.H1, S2 = P2:H2 to make original summand in the form (S1.Psi) + (S2.Xi) -> Out:

```
> S1:=MatrixMatrixMultiply(P1,H1):S2:=MatrixMatrixMultiply(P2,H2):pp:=MatrixVectorMultiply(S1,psi)+MatrixVectorMultiply(S2,xi):New3:=simplify(eval(pp));
```

$New3 :=$

$$\begin{bmatrix} \psi5 - \psi6\, I + \psi6 - \psi7\, I + \psi7 + \xi2\, I + \psi8 + \xi1\, I - \xi1 + \psi8\, I + \xi2 - \xi4\, I - \xi3 - \xi3\, I + \xi4 + \psi5\, I \\ -\psi5 - \psi6\, I + \psi6 + \psi7\, I - \psi7 + \xi2\, I + \psi8 - \xi1\, I + \xi1 + \psi8\, I + \xi2 - \xi4\, I + \xi3 + \xi3\, I + \xi4 - \psi5\, I \\ -\psi5 - \psi6\, I + \psi6 + \psi7\, I - \psi7 + \xi2\, I + \psi8 - \xi1\, I + \xi1 + \psi8\, I + \xi2 - \xi4\, I + \xi3 + \xi3\, I + \xi4 - \psi5\, I \\ \psi5 - \psi6\, I + \psi6 - \psi7\, I + \psi7 + \xi2\, I + \psi8 + \xi1\, I - \xi1 + \psi8\, I + \xi2 - \xi4\, I - \xi3 - \xi3\, I + \xi4 + \psi5\, I \end{bmatrix}$$

Verify against original:

```
> New3[1]-New2[1];New3[2]-New2[2];New3[3]-New2[2];New3[4]-New2[1];
```

$$0$$

$$0$$

$$0$$

$$0$$

```
>
```