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NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Stability of a Dynamical System

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Linear and Non Linear Systems

- Consider the dynamical system of the form:

$$\dot{x} = f(t, x)$$

where $x(t) \in R^n$ for each $t \geq 0$ and $f: [0, \infty) \times R^n \to R^n$ is a nonlinear vector function. i.e.,

$$f(t, x) = f(t, x_1, x_2 \dots \dots \dots x_n)$$

$$= \begin{Bmatrix} f_1(t, x_1, x_2 \dots \dots \dots x_n) \\ f_2(t, x_1, x_2 \dots \dots \dots x_n) \\ . \\ . \\ f_n(t, x_1, x_2 \dots \dots \dots x_n) \end{Bmatrix}$$

Linear and Non Linear Systems

- Example:

(i)

$$\dot{x}_1 = 2x_1 + 3x_2$$
$$\dot{x}_2 = x_1 - x_2$$

is a linear system

(ii)

$$\dot{x}_1 = x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2 + 2x_1 x_2$$

is a nonlinear system

Equilibrium Point

A point $x_e \in R^n$ is said to be an equilibrium point of the system

$$\dot{x} = f(t, x)$$

If $f(t, x) = 0$ for all t.

In previous example:

For system (i), $(0,0)$ is an equilibrium point.

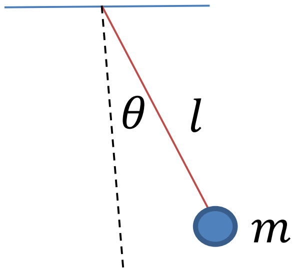
For system (ii), $(0,0)$ and $\left(\frac{1}{2}, 1\right)$ are equilibrium points.

Equilibrium Point

- Example:

Consider the zero control pendulum equation:

$$ml\ddot{\theta} + mgl\sin\theta = 0$$



Put $x_1 = \theta$ and $x_2 = \dot{\theta}$

We get the dynamical system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

So

$$(n\pi, 0): n = 0, \pm 1, \pm 2, \dots\dots\dots \text{ are equilibrium points.}$$

Stability of an equilibrium point x_e

Consider the equilibrium point x_e of the system

$$\dot{x} = f(x)$$

i.e.

$$f(x_e) = 0$$

Stability: x_e is said to be stable if for any $\epsilon > 0$, there exists a $\delta > 0$ such that:
If $\|x(t_0) - x_e\| < \delta$
Then $\|x(t) - x_e\| < \epsilon$
Asymptotic Stability: x_e is said to be asymptotically stable if it is stable and
Let $x(t) = x_e$
$t \to \infty$
Unstable: x_e is said to be unstable if for any x_0 in neighborhood of x_e and for any $\epsilon > 0$, there exists $t_1 > 0$ *such that*:
$\|x(t_1) - x_e\| > \epsilon$ *or* $\|x(t) - x_e\| \to \infty$

Lyapunov Method

Consider the time-invariant dynamical system

$$\dot{x} = f(x)$$

where f is such that $f(0) = 0$.
Then $x \equiv 0$ is an equilibrium point of the system

We define a Lyapunov function V (x) as follows:

(i) $V(x)$ and all its partial derivatives $\frac{\partial V}{\partial x_i}$ are continuous.

(ii) $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighbourhood $\|x\| \leq k$ of the origin.

(iii) The derivative of V with respect to (1),namely

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots \frac{\partial V}{\partial x_n}\dot{x}_n \\ &= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + \cdots \frac{\partial V}{\partial x_n}f_n\end{aligned}$$

is negative semidefinite (i.e. $\dot{V}(0) = 0$, and for all x in $\|x\| \leq k, \quad \dot{V}(x) \leq 0$).

If a Lyapunov function exists then the system is stable at the equilibrium point $x = 0$. In other words the trivial solution $x \equiv 0$ is stable.

If the condition (iii) is replaced by negative definiteness then the trivial solution is asymptotically stable.

But in general a lyapunov function need not represent the energy of a given system.

Example 1: Consider the dynamics of the damped pendulum

$$mL^2\ddot{\theta} + mgL\sin\theta + b\dot{\theta} = 0$$

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\frac{g}{L}\sin x_1 - \frac{b}{mL^2}x_2 \end{array}\right\}$$

If we take $m = L = 1$, then

$$\ddot{\theta} + g\sin\theta + b\dot{\theta} = 0$$

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -g\sin x_1 - bx_2 \end{array}\right\}$$

Let

$$
\begin{aligned}
V &= \alpha(1-\cos x_1)+\beta x_2^2, \quad \alpha,\beta>0 \\
\dot{V} &= \alpha\sin x_1(x_2)+2\beta x_2(-g\sin x_1-bx_2) \\
&= \alpha x_2\sin x_1-2\beta x_2\sin x_1-2\beta b x_2^2 \\
&= x_2\sin x_1(\alpha-2\beta)-2\beta b x_2^2
\end{aligned}
$$

If $x_2=0$, then

$$
\begin{aligned}
\dot{x}_2 &= 0, \quad \dot{x}_1=0, \quad \dot{x}_2=-g\sin x_1 \\
\Rightarrow \sin x_1 &= 0 \\
\Rightarrow x_1 &= k\pi \\
\therefore \quad \dot{V} &= 0 \quad \text{if} \quad x_1,x_2=0 \\
\dot{V} &< 0 \quad \text{if} \quad \alpha=2\beta, \quad \beta>0, \quad x_2\neq 0
\end{aligned}
$$

Example 2: Now consider the dynamics of the controlled pendulum

$$mL^2\ddot{\theta} + mgL\sin\theta + b\dot{\theta} = \tau$$

which can be written in the following control system form

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{\tau}{mL^2} - \frac{g}{L}\sin x_1 - \frac{b}{mL^2}x_2 \end{array}\right\}$$

If we take $m = L = 1$, then

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \tau - g\sin x_1 - bx_2 \end{array}\right\}$$

Problem

Find a control torque τ such that the pendulum stabilizes at $x_1 = \theta_d = \pi/4$.

Let

$$x_d = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then the tracking error between x and x_d is

$$\begin{aligned} e(t) &= x - x_d \\ \Rightarrow \dot{e} &= \dot{x} \end{aligned}$$

Let

$$\begin{aligned} e_1 &= x_1 - x_{1d} \\ \Rightarrow e_2 &= x_2 - x_{2d} = x_2 \end{aligned}$$

$$
\begin{aligned}
\therefore \quad \dot{e}_1 &= e_2 \\
\dot{e}_2 &= \tau - g\sin(e_1 + \pi/4) - be_2
\end{aligned}
$$

we want to find a control τ so that the equilibrium point $(0,0)$ is asymptotically stable.
Let

$$
\begin{aligned}
V &= \alpha e_1^2 + e_2^2 \\
\text{Then} \quad \dot{V} &= 2\alpha e_1 e_2 + 2e_2(\tau - g\sin(e_1 + \pi/4) - be_2)
\end{aligned}
$$

Chose the control as

$$\begin{aligned} \tau &= -\alpha e_1 + g\sin(e_1 + \pi/4) \\ \Rightarrow \quad \dot{V} &= 2\alpha e_1 e_2 - 2\alpha e_1 e_2 - 2be_2^2 < 0 \\ (\text{If} \quad e_2 &= 0, \quad \dot{e}_2 = 0 \quad \text{implies} \quad \alpha e_1 = 0) \end{aligned}$$

Hence the system is asymptotically stable.

Example 3: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \beta x_2 + u \end{aligned}$$

Let $x_{1d} = L/2$, $x_{2d} = 0$.
Let

$$\begin{aligned} e_1 &= x_1 - L/2 \\ e_2 &= x_2 \end{aligned}$$

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= u - \alpha(e_1 + L/2) - \beta e_2.
\end{aligned}
$$

Let $u = -k_1 e_1 - Le_2 + \alpha(L/2)$
Then

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= k_1 e_1 + Le_2 + \alpha L/2 - \alpha(e_1 + L/2) - \beta e_2.
\end{aligned}
$$

Consider

$$
\begin{aligned}
V(e_1, e_2) &= ke_1^2 + e_2^2 \\
\Rightarrow \dot{V} &= 2ke_1 e_2 + 2e_2(-ke_1 - Le_2 + \alpha L/2) - \alpha(e_1 + L/2) - \beta e_2 \\
\Rightarrow \dot{V} &= -2Le_2^2 - 2\alpha e_1 e_2 + e_1 e_2(2k - 2k_1 - 2\alpha) - 2\beta e_2^2
\end{aligned}
$$

By suitable choice of k and k_1 the above system can be made asymptotically stable.

Thanks!