IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Dynamics of Manipulator

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Dynamics of Manipulator

- Kinematics of a robot manipulator deals with the relation between position and orientation of the end effector and the joint variable of the manipulator without considering the mass of the links or the forces/torques acting on the manipulator.
- Dynamics of a robot deals with the relation between position, velocity and acceleration of the robot's joints along with the force/torque exerted at each joint to cause the motion.

Dynamics of Manipulator

- Robot arm dynamics deals with the mathematical formulations of the equations of robot arm motion.
- It provides an insight into the structure of the robot system.
- Dynamics is a basis for model based control systems.
- A platform for computer simulations.

Different Approaches

- There can be two approaches to solve dynamics:

i. Lagrange-Euler approach: Energy based approach.

ii. Newton-Euler approach: Based on Force/Momentum

- We will focus on Lagrangian Dynamics as it is simple and systematic.

Newton's Formulation

- The relation between the joint positions, velocities and accelerations of the manipulator and the torques/forces applied by the actuators and external forces which caused the motion.
- As we know, if we want a mass to accelerate linearly, we need to exert a force and we have following equation Newton's 2nd law:

$$F = m.a$$

- Similarly for angular acceleration, torque need to be applied and we have following equation:

$$\tau = I.\alpha$$

Lagrangian Formulation

- Based on differentiation of Energy terms w.r.t. system's variables and time.
- If $(q_1, q_2 \dots \dots \dots q_n)$ and $(\dot{q}_1, \dot{q}_2, \dots \dots \dots \dot{q}_n)$ are the joint position and velocity variables (generalized coordinates) of a robot manipulator respectively, then its kinetic energy K is a function of $\dot{q}_i$ and potential energy P is a function of q_i.
- Lagrangian is defined as:

$$L = K - P$$

where L is lagrangian, K and P are kinetic and potential energies of the system respectively.

- Then we have following generalized second order ordinary differential equations for both linear and rotational motions:

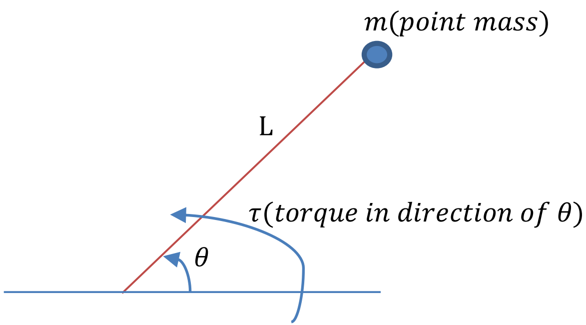
$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \dots \dots \dots (1)$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} \dots \dots \dots (2)$$

where F is summation of all external forces for linear motion, τ is summation of all torques in rotational motion, q is θ in case of revolute joint and x in case of prismatic joint denotes system variables.

Lagrangian Formulation for Manipulator Dynamics

- Example 1: One degree of freedom:



$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(L\dot{\theta}\right)^2 = \frac{1}{2}mL^2\dot{\theta}^2$$

$$P = mgLsin\theta$$

$$L = K - P = \frac{1}{2}mL^2\dot{\theta}^2 - mgLsin\theta$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mL^2\left(2\dot{\theta}\right) = mL^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mglcos\theta$$

$$\tau = mL^2\ddot{\theta} + mglcos\theta$$

Lagrangian Formulation for Manipulator Dynamics

- Example 2: One degree of freedom (Uniformly Distributed mass)
- Let $OA = L_1$ and ρ be the

Uniform density of the rod.

Then mass of the rod is:

$M = \rho . L_1$

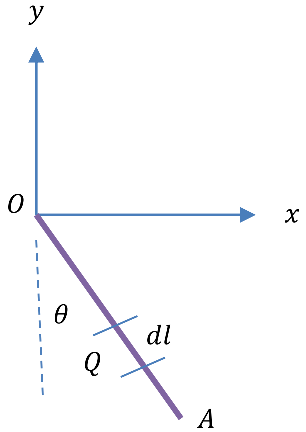
Let Q be a point on the rod at

A distance l $from$ O.

Then the coordinated of Q are:

$(l \sin\theta, -l\cos\theta)$.

The velocity is $(l\cos\theta\ \dot{\theta}, l\sin\theta\ \dot{\theta})$



Lagrangian Formulation for Manipulator Dynamics

- Example 2: One degree of freedom (Uniformly Distributed mass)

So:

$$v^2 = l^2 \dot{\theta}^2$$

Let *dl* be a small element then the kinetic energy of the element is:

$$dK = \frac{1}{2}.\rho.dl.l^2\dot{\theta}^2$$

$$Then\ K = \int_0^{L_1} dk = \frac{1}{6}.M.L_1^2.\ \dot{\theta}^2$$

Potential Energy of the element dl is:

$$dP = (\rho.dl)g\ (L_1 - l\cos\theta) \Rightarrow P = \int_0^{L_1} dp = \rho.g.L_1^2 - \frac{\rho g L_1^2}{2}$$

Lagrangian Formulation for Manipulator Dynamics

- Example 2: One degree of freedom (Uniformly Distributed mass)

Therefore

$$L = K - P$$

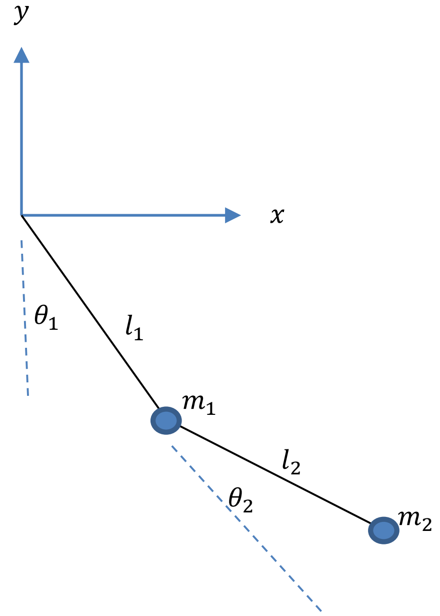
$$= \frac{1}{6} M . L_1^2 . \dot{\theta}^2 - \rho . g . L_1^2 + m . \frac{g}{2} . L_1 \cos\theta$$

The dynamic equation is:

$$\frac{1}{3} M L_1^2 \ddot{\theta} + \frac{mg}{2} L_1 \sin\theta = \tau$$

Lagrangian Formulation for Manipulator Dynamics

- Example 3: Two degrees of freedom (point mass)



Two degrees of freedom (point mass)

$$K = K_1 + K_2$$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^{\ 2}$$

To calculate K_2, position equations for mass m_2 needed, which are given by:

$$x_2 = l_1 sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) = l_1 S_1 + l_2 S_{12}$$
$$y_2 = -l_1 C_1 - l_2 C_{12}$$

$$V_2 = \dot{x}^2 + \dot{y}^2$$

$$\dot{x_2} = l_1 C_1 \dot{\theta}_1 + l_2 C_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y_2} = l_1 S_1 \dot{\theta}_1 + l_2 S_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

So

$$V_2^2 = l_1^2 \dot{\theta}_1^{\ 2} + l_2^2 \left(\dot{\theta}_1^{\ 2} + \dot{\theta}_2^{\ 2} + 2\dot{\theta}_1 \dot{\theta}_2 \right) + 2 l_1 l_2 C_2 (\dot{\theta}_1^{\ 2} + \dot{\theta}_1 \dot{\theta}_2)$$

Two degrees of freedom (point mass)

$$K_2 = \frac{1}{2}m_2 V_2^2 = \frac{1}{2}m_2 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\left(\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta}_1\dot{\theta}_2\right) + \frac{1}{2}m_2 2 l_1 l_2 C_2(\dot{\theta_1}^2 + \dot{\theta}_1\dot{\theta}_2)$$

So

$$K = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\left(\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta}_1\dot{\theta}_2\right) + \frac{1}{2}m_2 2 l_1 l_2 C_2(\dot{\theta_1}^2 + \dot{\theta}_1\dot{\theta}_2)$$

Similarly we can find potential energy as

$$P_1 = -m_1 g l_1 C_1$$

$$P_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12}$$

$$P = P_1 + P_2 = -(m_1 + m_2) g l_1 C_1 - m_2 g l_2 C_{12}$$

$$L = K - P$$

$$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\left(\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta}_1\dot{\theta}_2\right) + \frac{1}{2}m_2 2 l_1 l_2 C_2\left(\dot{\theta_1}^2 + \dot{\theta}_1\dot{\theta}_2\right) + (m_1 + m_2) g l_1 C_1$$

$$+ m_2 g l_2 C_{12}$$

Two degrees of freedom (point mass)

Derivatives of Lagrangian are:

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2m_2 l_1 l_2 C_2\dot{\theta}_1 + m_2 l_1 l_2 C_2 \dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2 - 2m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^{\,2}$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g l_1 S_1 - m_2 g l_2 S_{12}$$

So from Eq. (2)

$$\tau_1 = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2 - 2m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$
$$- m_2 l_1 l_2 S_2 \dot{\theta}_2^{\,2} + (m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12}$$

Two degrees of freedom (point mass)

Similarly,

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 C_2 \dot{\theta}_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 S_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2\right) - m_2 g l_2 S_{12}$$

So,

$$\tau_2 = (m_2 l_2^2 + m_2 l_1 l_2 C_2)\ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_1{}^2 + m_2 g l_2 S_{12}$$

Two degrees of freedom (point mass)

Writing equations in matrix form:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2 & [m_2 l_2^2 + m_2 l_1 l_2 C_2 \\ (m_2 l_2^2 + m_2 l_1 l_2 C_2) & m_2 l_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}0 & -m_2 l_1 l_2 S_2\\ m_2 l_1 l_2 S_2 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1{}^2\\ \dot{\theta}_2{}^2\end{bmatrix}$$

$$+ \begin{bmatrix}-m_2 l_1 l_2 S_2 & -m_2 l_1 l_2 S_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2\dot{\theta}_1\end{bmatrix} + \begin{bmatrix}(m_1+m_2) g l_1 S_1 + m_2 g l_2 S_{12}\\ m_2 g l_2 S_{12}\end{bmatrix}$$

So we can write in generalized form as:

$$M(q,\dot{q})\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

$$where\ M(q,\dot{q})\ \ is\ Inertia\ matrix, C(q,\dot{q})\ denotes\ coriolis\ and\ centrifugal\ forces\ and\ G(q)\ signifies\ conservative\ forces\ (gravity)$$

Thanks!