IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Dynamics of Manipulator (cont.)

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DEPARMENT OF MATHEMATICS

Lagrangian Formulation for Manipulator Dynamics

- Example 4: Two degrees of freedom (uniformly distributed mass(density ρ)(Method 1)

$OA = L_1 \; AB = L_2$

Let P be a point on OA with

Distance l from O.

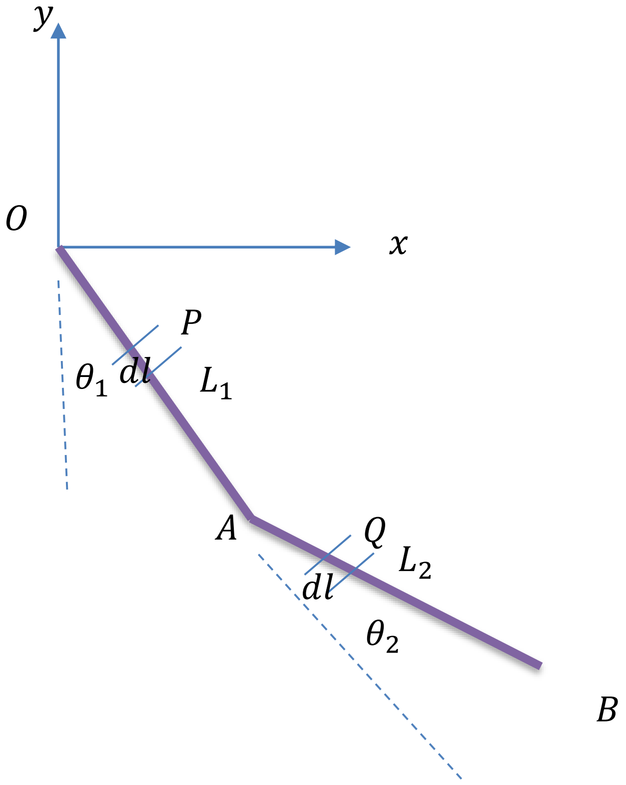
Coordinates of $P = (l \sin\theta_1, -l\cos\theta_1)$

Velocity of $P = (l\cos\theta_1 . \ \dot{\theta}_1, \ l\sin\theta_1 . \dot{\theta}_1)$

$|V_P|^2 = l^2\dot{\theta}_1^2$

$$dK_1 = \frac{1}{2} . \rho . dl . l^2\dot{\theta}_1{}^2$$

$$Then \ K_1 = \int_0^{L_1} dk = \frac{1}{6} . M_1 . L_1^2 . \ \dot{\theta}^2$$



Lagrangian Formulation for Manipulator Dynamics

- Example 4: Two degrees of freedom (uniformly distributed mass(density ρ)(Method 1)

$OA = L_1 \; AB = L_2$

Let Q be a point on AB with

Distance l from A.

Coordinates of Q $= (L_1 \sin\theta_1 + l\sin(\theta_1+\theta_2), -L_1\cos\theta_1 \;\; - l\cos(\theta_1+\theta_2))$

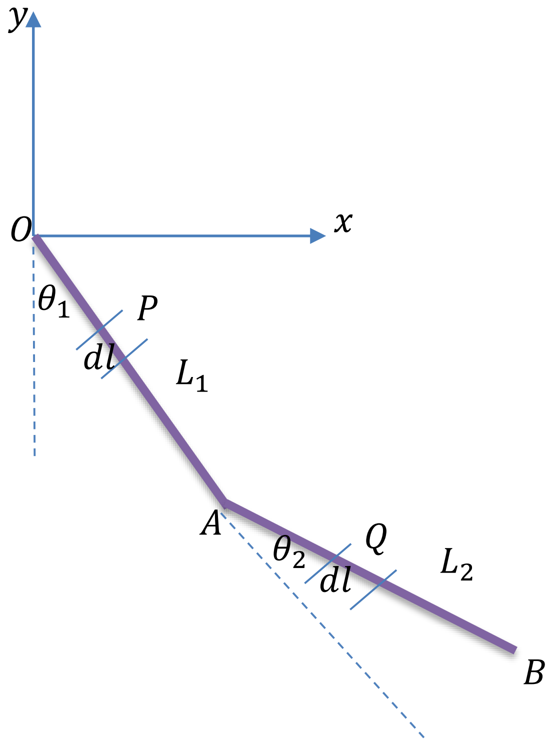
Velocity of $P = (L_1\cos\theta_1.\ \dot{\theta}_1 + l\cos(\theta_1+\theta_2).\left(\dot{\theta}_1+\dot{\theta}_2\right), L_1\sin\theta_1.\ \dot{\theta}_1 + l\sin(\theta_1+\theta_2).(\dot{\theta}_1+\dot{\theta}_2)$

$$\left|V_Q\right|^2 = L_1^2\dot{\theta}_1^2 + l^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + 2L_1.l.\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\cos\theta_2$$

$$dK_2 = \frac{1}{2}.\rho.dl.\left(L_1^2\dot{\theta}_1^2 + l^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + 2L_1.l\left(\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2\right)\cos\theta_2\right)$$

$$Then\ K_2 = \int_0^{L_2} dk = \frac{1}{2}M_2L_1^2\dot{\theta}_1^2 + \frac{1}{2}M_2\frac{L_2^2}{3}.\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \frac{1}{2}M_2L_1L_2\left(\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2\right)\cos\theta_2$$

$$K = K_1 + K_2 = \frac{1}{6}.M_1.L_1^2.\ \dot{\theta}^2 + \frac{1}{2}M_2L_1^2\dot{\theta}_1^2 + \frac{1}{2}M_2\frac{L_2^2}{3}.\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \frac{1}{2}M_2L_1L_2\left(\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2\right)\cos\theta_2$$



Lagrangian Formulation for Manipulator Dynamics

- Example 4: Two degrees of freedom (uniformly distributed mass(density ρ)(Method 1)

Similarly we can calculate potential energies:

$$dP_1 = \rho . dl . g(L_1 + L_2 - l\cos\theta_1)$$

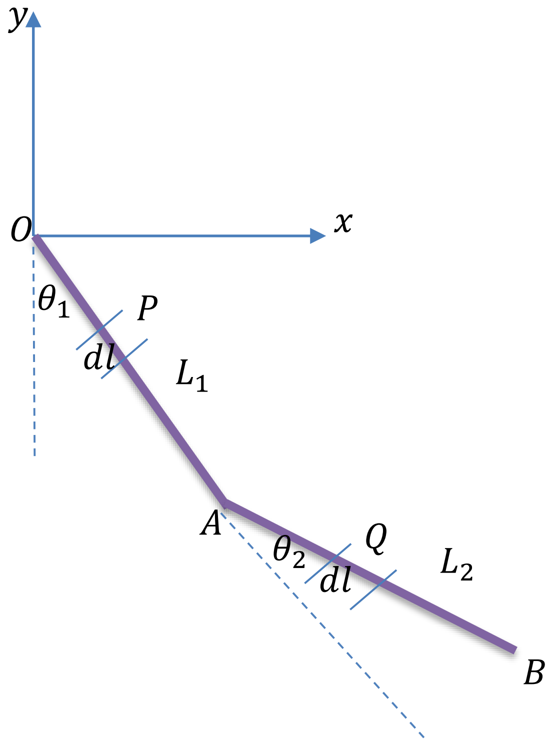
$$P_1 = M_1 . g(L_1 + L_2 - \frac{L_1}{2}\cos\theta_1)$$

$$dP_2 = \rho . dl . g(L_1 + L_2 - (L_1\cos\theta_1 + l\cos(\theta_1+\theta_2))$$

$$P_1 = M_2 . g(L_1 + L_2 - L_1\cos\theta_1 - \frac{L_2}{2}\cos(\theta_1+\theta_2))$$

$$P = P_1 + P_2 = M_1 . g(L_1 + L_2 - \frac{L_1}{2}\cos\theta_1) + M_2 . g(L_1 + L_2 - L_1\cos\theta_1 - \frac{L_2}{2}\cos(\theta_1+\theta_2))$$

$$L = K - P$$



Lagrangian Formulation for Manipulator Dynamics

- Example 4: Two degrees of freedom (uniformly distributed mass(density ρ)(Method 1)

Similarly we can calculate potential energies:

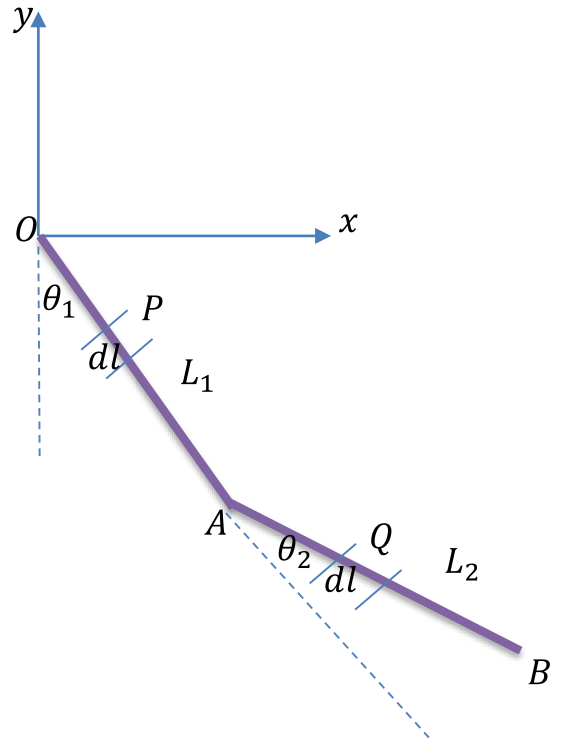
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{M_1 L_1^2}{3}\ddot{\theta}_1 + M_2 L_1^2 \ddot{\theta}_1 + \frac{M_2}{3} L_2^2 \ddot{\theta}_1 + \frac{M_2}{3} L_2^2 \ddot{\theta}_2$$

$$+M_2 L_1 L_2 \ddot{\theta}_1 \cos\theta_2 - M_2 L_1 L_2 \dot{\theta}_1 \sin\theta_2 \dot{\theta}_2 + \frac{1}{2} M_2 L_1 L_2 \ddot{\theta}_2 \cos\theta_2 - \frac{M_2}{2} L_1 L_2 \dot{\theta}_2 \sin\theta_2 \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} M_2 L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) \sin\theta_2 + M_2 g.\frac{L_2}{2} \sin(\theta_1 + \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = M_1 g.\frac{L_1}{2}.\sin\theta_1 + M_2 g L_1 \sin\theta_1 + M_2 g.\frac{L_2}{2}.\sin(\theta_1 + \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3} M_2 L_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) + \frac{1}{2} M_2 L_1 L_2 \dot{\theta}_1 \cos\theta_2$$



Lagrangian Formulation for Manipulator Dynamics

- Example 4: Two degrees of freedom (uniformly distributed mass(density ρ)(Method 1)

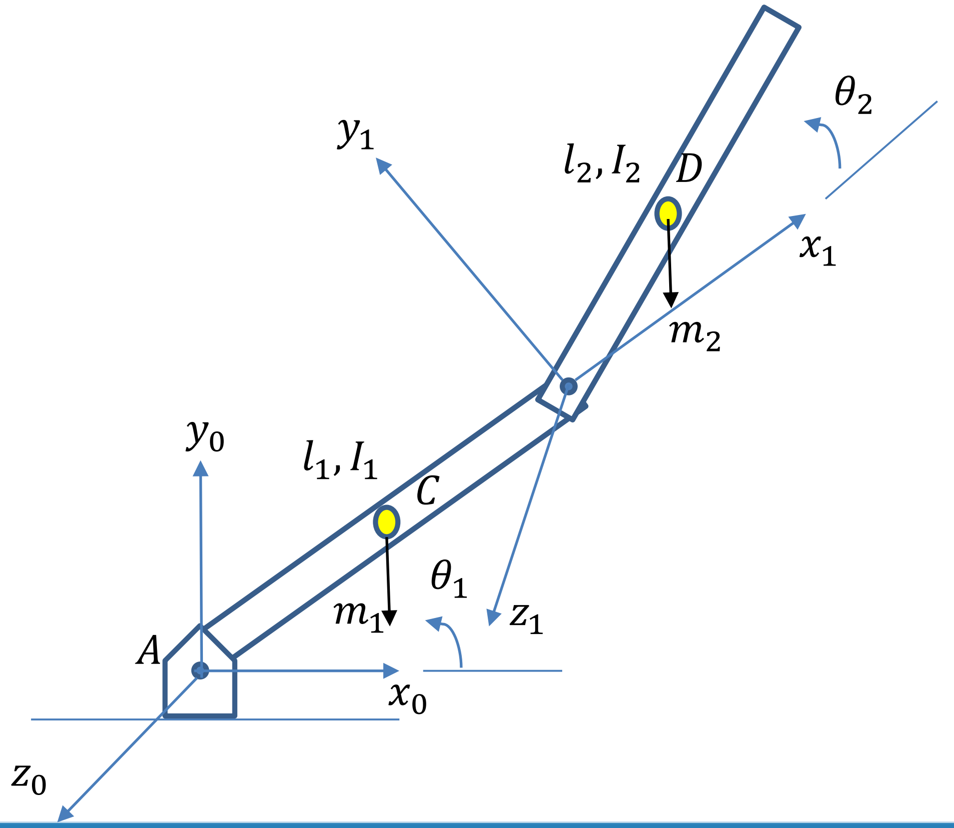
Using lagrangian derivatives and substituting in Eq. 2, we have following torque equations:

$$\tau_1 = \left(\frac{1}{3}M_1L_1^2 + M_2L_1^2 + \frac{1}{3}M_2L_2^2 + M_2L_1L_2C_2\right)\ddot{\theta_1} + \left(\frac{1}{3}M_2L_2^2 + \frac{1}{2}M_2L_1L_2C_2\right)\ddot{\theta_2}$$
$$-(M_2L_1L_2S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}M_2L_1L_2S_2\right)\dot{\theta}_2{}^2 + \left(\frac{1}{2}M_1 + M_2\right)gL_1S_1 + \frac{1}{2}M_2gL_2S_{12}$$

$$\tau_2 = \left(\frac{1}{3}M_2L_2^2 + \frac{1}{2}M_2L_1L_2C_2\right)\ddot{\theta_1} + \left(\frac{1}{3}M_2L_2^2\right)\ddot{\theta_2} + \left(\frac{1}{2}M_2L_1L_2S_2\right)(\dot{\theta}_1{}^2 + \dot{\theta}_1\dot{\theta}_2) + \frac{1}{2}M_2gL_2S_{12} + \frac{1}{2}M_2L_1L_2\dot{\theta}_1 \sin\theta_2\dot{\theta}_2$$

Lagrangian Formulation for Manipulator Dynamics

- Example 5: Two degrees of freedom (uniformly distributed mass)(Method 2)



I_1 *and* I_2 denotes moment of inertia

Two degrees of freedom (uniformly distributed mass)(Method 2)

$$x_D = l_1 C_1 + 0.5 l_2 C_{12} \qquad \dot{x_D} = -l_1 S_1 \dot{\theta}_1 - 0.5 l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$y_D = l_1 S_1 + 0.5 l_2 S_{12} \qquad \dot{x_D} = l_1 C_1 \dot{\theta}_1 - 0.5 l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

So

$$V_D^2 = \dot{x_D}^2 + \dot{y_D}^2 = \dot{\theta}_1^{\,2}(l_1^2 + 0.25 l_2^2 + l_1 l_2 C_2) + \dot{\theta}_2^{\,2}(0.25 l_2^2) + \dot{\theta}_1 \dot{\theta}_2 (0.5 l_2^2 + l_1 l_2 C_2)$$

$$K = K_1 + K_2 = \left[\frac{1}{2} I_A \dot{\theta}_1^{\,2}\right] + \left[\frac{1}{2} I_D \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} m_2 V_D^2\right]$$

$$I_A = \frac{1}{3} m_1 l_1^2 \quad I_D = \frac{1}{12} m_2 l_2^2$$

$$K = \dot{\theta}_1^{\,2}\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^{\,2}\left(\frac{1}{6} m_2 l_2^2\right) + \dot{\theta}_1 \, \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)$$

Two degrees of freedom (uniformly distributed mass)(Method 2)

Potential Energy of system is given by:

$$P = \frac{m_1 g l_1}{2} S_1 + m_2 g (l_1 S_1 + \frac{l_2}{2} S_{12})$$

$$L = K - P$$

$$= \dot{\theta}_1{}^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) + \dot{\theta}_2{}^2 \left(\frac{1}{6} m_2 l_2^2 \right) + \dot{\theta}_1\, \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) - \frac{m_1 g l_1}{2} S_1$$
$$- m_2 g (l_1 S_1 + \frac{l_2}{2} S_{12})$$

Two degrees of freedom (uniformly distributed mass)(Method 2)

Using lagrangian derivatives and substituting in Eq. 2, we have following torque equations:

$$\tau_1 = \left(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_2 - (m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_2^{\,2} + \left(\frac{1}{2}m_1 + m_2\right)g l_1 C_1 + \frac{1}{2}m_2 g l_2 C_{12}$$

$$\tau_2 = \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_1^{\,2} + \frac{1}{2}m_2 g l_2 C_{12}$$

Thanks!