IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Manipulator Jacobian

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Manipulator Jacobian

For a robot manipulator with n Degrees of Freedom (DoF), the manipulator Jacobian matrix is a $6 \times n$ matrix which relates the end effector velocity and joints velocity in the following way:

$$\begin{bmatrix} {}^{T}d_x \\ {}^{T}d_y \\ {}^{T}d_z \\ {}^{T}\delta_x \\ {}^{T}\delta_y \\ {}^{T}\delta_z \end{bmatrix} = J_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ . \\ . \\ . \\ \dot{q}_n \end{bmatrix}$$

where the LHS denotes the linear and angular velocity of the end –effector w.r.t. end-effector coordinate frame and $\dot{q}_i$ denotes the angular velocity (in case of revolute joint) or linear velocity (in case of prismatic joint) of the actuator at the i^{th} joint.

Manipulator Jacobian

Let

$$T = {}^{0}T_{n} = {}^{0}T_{1}.\,{}^{1}T_{2} \ldots\ldots\ldots\ldots {}^{n-1}T_{n}$$

be the arm matrix of the manipulator.

where

$$ {}^{i-1}T_{i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Manipulator Jacobian

We know that, the velocity

$$\frac{dT}{dt} = D.T = T.{}^{T}D$$

where

$$^{\mathrm{T}}\mathrm{D} = \begin{bmatrix} 0 & -^{T}\delta_z & ^{T}\delta_y & ^{T}d_x \\ ^{T}\delta_z & 0 & -^{T}\delta_x & ^{T}d_y \\ -^{T}\delta_y & ^{T}\delta_x & 0 & ^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

represents the Differential operator containing velocity components w.r.t. the current frame (end-effector)

Let

$$T = {}^{0}T_n = {}^{0}T_1(q_1).{}^{1}T_2(q_2) \ldots\ldots\ldots\ldots {}^{n-1}T_n(q_n)$$

$$\frac{dT}{dt} = (D^{0}T_1)({}^{1}T_n) + {}^{0}T_1(D^{1}T_2)({}^{2}T_n) + \ldots\ldots\ldots\ldots\ldots + {}^{0}T_{n-1}(D^{n-1}T_n)$$

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$$\frac{dT}{dt} = \frac{\partial T}{\partial q_1}.\dot{q_1} + \frac{\partial T}{\partial q_2}.\dot{q_2} + \ \dots\dots\dots\dots + \frac{\partial T}{\partial q_n}.\dot{q_n}$$

$$\frac{\partial T}{\partial q_i} = {}^0T_1.\,{}^1T_2 \dots\dots\, {}^{i-1}T_{i-1}.\frac{d}{dq_i}({}^{i-1}T_i)\,{}^iT_n$$

$$= {}^0T_{i-1}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}({}^{i-1}T_i)\,{}^iT_n \dots (in\ case\ of\ revolute\ q_i)$$

$$= {}^0T_{i-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}({}^{i-1}T_i)\,{}^iT_n \dots (in\ case\ q_i\ is\ prismatic)$$

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$$= ({}^{0}T_{i-1}).D_i\,.({}^{i\,-\,1}T_n)$$

$$= {}^{0}T_{i-1}.{}^{i\,-\,1}T_n.({}^{i\,-\,1}T_n)^{-1}.D_i\,.{}^{i\,-\,1}T_n$$

$$= T.({}^{i\,-\,1}T_n)^{-1}.D_i\,.{}^{i\,-\,1}T_n$$

$$= T.{}^{T}D_i$$

For revolute joint:

$$\vec{\delta} = \vec{k} \;\; and \;\; \vec{d} = 0$$

For prismatic joint:

$\vec{\delta} = 0$and $\vec{d} = \vec{k}$

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Therefore

$$^{T}D = \begin{bmatrix} 0 & -{}^{i}\delta_z & {}^{i}\delta_y & {}^{i}d_x \\ {}^{i}\delta_z & 0 & -{}^{i}\delta_x & {}^{i}d_y \\ -{}^{i}\delta_y & {}^{i}\delta_x & 0 & {}^{i}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is obtained by:

$$^{i}d_x = {}^{i}n.((\delta \times {}^{i}p) + d)$$
$$= {}^{i}n.\left(-\mathrm{p_y}\hat{\imath} + \mathrm{p_x}\hat{\jmath}\right)$$
$$^{i}d_x = -n_x p_y + n_y p_x$$

Similarly:

$$^{i}d_y = -o_x p_y + o_y p_x$$
$$^{i}d_z = -a_x p_y + a_y p_x$$
$$^{i}\delta_x = \delta.n = n_z$$
$$^{i}\delta_y = \delta.o = o_z$$
$$^{i}\delta_z = \delta.a = a_z$$

$(Here\ \ {}^{i}n, {}^{i}o, {}^{i}a\ and\ {}^{i}p\ are\ columns\ of\ {}^{i\,-\,1}T_n)$

Manipulator Jacobian

$$\frac{dT}{dt} = T[{}^{T}D_1.\dot{q_1} + {}^{T}D_2.\dot{q_2} + \ \dots\dots\dots\dots + {}^{T}D_n.\dot{q_n}]$$

$$= T[{}^{T}D]$$

$$\Rightarrow \ {}^{T}D = {}^{T}D_1.\dot{q_1} + {}^{T}D_2.q_2 + \ \dots\dots\dot{\dots} + {}^{T}D_n.\dot{q_n}$$

Summing the RHS and writing in the linear and angular velocity components in the matrix ${}^{T}D$ in vector form, we get:

$$\begin{bmatrix} {}^{T}d_x \\ {}^{T}d_y \\ {}^{T}d_z \\ {}^{T}\delta_x \\ {}^{T}\delta_y \\ {}^{T}\delta_z \end{bmatrix} = \begin{bmatrix} {}^{1}d_x & {}^{2}d_x & .. & .. & {}^{n}d_x \\ {}^{1}d_y & {}^{2}d_y & .. & .. & {}^{n}d_y \\ {}^{1}d_z & {}^{2}d_z & .. & .. & {}^{n}d_z \\ {}^{1}\delta_x & {}^{2}\delta_x & .. & .. & {}^{n}\delta_x \\ {}^{1}\delta_y & {}^{2}\delta_y & .. & .. & {}^{n}\delta_y \\ {}^{1}\delta_z & {}^{2}\delta_z & .. & .. & {}^{n}\delta_x \end{bmatrix} . \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ . \\ . \\ . \\ \dot{q_n} \end{bmatrix}$$

Thanks!