IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Manipulator Dynamics
Multiple Degree of Freedom

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Manipulator Dynamics
Multiple Degree of Freedom

Let

$$T = {}^{0}T_n = {}^{0}T_1 . {}^{1}T_2 \ldots\ldots\ldots\ldots {}^{n-1}T_n$$

be the arm matrix of the manipulator. Then ${}^{i-1}T_i$ is a function of q_i, where q_i is the joint variable (joint angle or joint distance) where

$${}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{D} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{\mathrm{T}}\mathrm{D} = \begin{bmatrix} 0 & -{}^{T}\delta_z & {}^{T}\delta_y & {}^{T}d_x \\ {}^{T}\delta_z & 0 & -{}^{T}\delta_x & {}^{T}d_y \\ -{}^{T}\delta_y & {}^{T}\delta_x & 0 & {}^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let

$$T_i = {}^{i-1}T_i$$

$$\frac{dT_i}{dq_i} = D_i . T_i$$

where

$$\mathrm{D_i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots\ldots \textit{if joint i is revolute}$$

$$\mathrm{D_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots\ldots \textit{if joint i is prismatic}$$

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Thus

$$\frac{dT}{dt} = \frac{\partial T}{\partial q_1}.\dot{q_1} + \frac{\partial T}{\partial q_2}.\dot{q_2} + \ \dots\dots\dots\dots + \frac{\partial T}{\partial q_n}.\dot{q_n}$$

Now consider

$$^{0}T_j = {}^{0}T_1.{}^{1}T_2 \dots\dots\dots {}^{j\,-\,1}T_j$$

and

$$u_{jk} = \frac{\partial}{\partial q_k}\left({}^{0}T_j\right) = T_1.T_2 \dots\dots\dots T_{j-1}.D_k.T_k \dots\dots\dots T_j \qquad (if\ k \leq j)$$

Let

$$u_{jkl} = \frac{\partial}{\partial q_l}\left(u_{jk}\right) \qquad ; l \leq j$$

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Thus

$$u_{jkl} = T_1 . T_2 \ldots \ldots D_l . T_l \ldots \ldots D_k . T_k \ldots \ldots T_j \qquad (if\,(l < k < j)$$

Let P be a point on j^{th} link with an element of length dl. Let r_j be the vector representing the point P in the j^{th} coordinate frame, then the vector representing P in the base coordinate frame is:

$$P_j = {}^0T_j . r_j$$

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Its velocity is:

$$V_j = \frac{dP_j}{dt} = \frac{d}{dt}\left({}^0T_j . r_j\right)$$

$$= \left[\sum_{k=1}^{j} \frac{\partial}{\partial q_k}\left({}^0T_j\right)\dot{q}_k\right] r_j$$

$$= \left[\sum_{k=1}^{j} u_{jk}\dot{q}_k\right] r_j$$

If ρ is the uniform density then the kinetic energy of the element dl is given by:

$$dK_j = \frac{1}{2}\left|V_j\right|^2 . \rho . dl$$

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Let

$$P_j = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} \quad w.r.t.\,base$$

and

$$V_j = \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{bmatrix} \quad the\ velocity\ vector$$

Then

$$V_j.V_j^T = \begin{bmatrix} \dot{x}_j^2 & \dot{x}_j.\dot{y}_j & \dot{x}_j.\dot{z}_j \\ \dot{y}_j.\dot{x}_j & \dot{y}_j^2 & \dot{y}_j.\dot{z}_j \\ \dot{z}_j.\dot{x}_j & \dot{z}_j.\dot{y}_j & \dot{z}_j^2 \end{bmatrix}$$

$$Trace\left(V_j.V_j^T\right) = \dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2$$

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So

$$dK_j = \frac{1}{2} Trace \left[\sum_{k=1}^{j} u_{jk} \dot{q}_k \right] r_j . \left[\left[\sum_{k=1}^{j} u_{jk} \dot{q}_k \right] r_j \right]^T dm$$

$$= \frac{1}{2} Trace \sum_{k=1}^{j} \sum_{l=1}^{j} u_{jk} . r_j . r_j^T . u_{jl}^T . \dot{q}_k . \dot{q}_l . dm$$

$$K_j = \int dK_j = \frac{1}{2} Trace \sum_{k=1}^{j} \sum_{l=1}^{j} u_{jk} . \int r_j . r_j^T dm . u_{jl}^T . \dot{q}_k . \dot{q}_l$$

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$$r_j . r_j^T = \begin{bmatrix} x_j^2 & x_j . y_j & x_j . z_j & x_j \\ x_j . y_j & y_j^2 & y_j . z_j & y_j \\ x_j . z_j & y_j . z_j & z_j^2 & z_j \\ x_j & y_j & z_j & 1 \end{bmatrix}$$

$$\int_{Every\ link\ j} r_j . r_j^T dm = \begin{bmatrix} \frac{1}{2}(-I_{xx} + I_{yy} + I_{zz}) & I_{xy} & I_{xz} & m.\bar{x} \\ I_{xy} & \frac{1}{2}(I_{xx} - I_{yy} + I_{zz}) & I_{yz} & m.\bar{y} \\ I_{xz} & I_{yz} & \frac{1}{2}(I_{xx} + I_{yy} - I_{zz}) & m.\bar{z} \\ m.\bar{x} & m.\bar{y} & m.\bar{z} & 1 \end{bmatrix}$$

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$$I_{xx} = \rho \int (y^2 + z^2) dl$$

$$I_{yy} = \rho \int (x^2 + z^2) dl$$

$$I_{zz} = \rho \int (x^2 + y^2) dl$$

$$I_{xy} = \int x.y\, dm$$

$$I_{yz} = \int y.z\, dm$$

$$I_{zx} = \int x.z\, dm$$

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Total Kinetic Energy

$$K = \sum_{j=1}^{n} K_j$$

$$K = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{l=1}^{j} u_{jk}.I_j.u_{jl}^T.\dot{q}_k.\dot{q}_l + \frac{1}{2}\sum_{j=1}^{n} I_j.\dot{q}_j^2$$

where I_j is the inertia of the j^{th} actuator.

Total Potential Energy

$$P.E. = \sum_{j=1}^{n} P_j$$

$$P_j = \sum_{j=1}^{n} m_j.g_j^T\left({}^0T_j.\bar{r}_j\right)$$

$$where\ \bar{r}_j\ is\ the\ center\ of\ mass\ of\ j^{th}\ link$$

So,

$$L = K - P = \sum_{j=1}^{n} K_j - \sum_{j=1}^{n} P_j = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{l=1}^{j} u_{jk}.I_j.u_{jl}^T.\dot{q}_k.\dot{q}_l + \frac{1}{2}\sum_{j=1}^{n} I_j.\dot{q}_j^2 - \sum_{j=1}^{n} m_j.g_j^T\left({}^0T_j.\bar{r}_j\right)$$

Thanks!