IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Trajectory Planning

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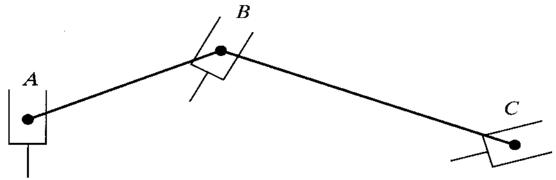
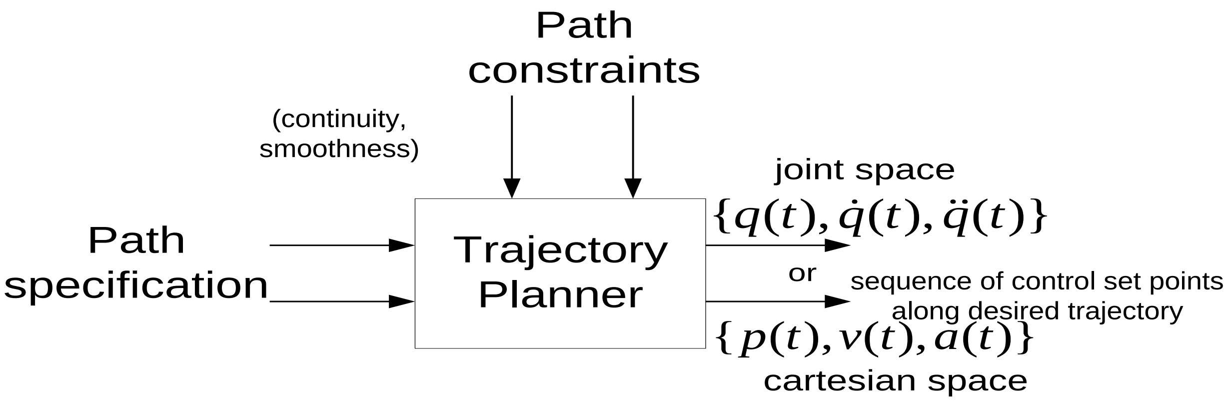
Path v/s Trajectory

- Path: A sequence of robot configurations in a particular order without regard to the timing of these configurations.
- Trajectory: It is concerned about when each part of the path must be attained, thus specifying timing.

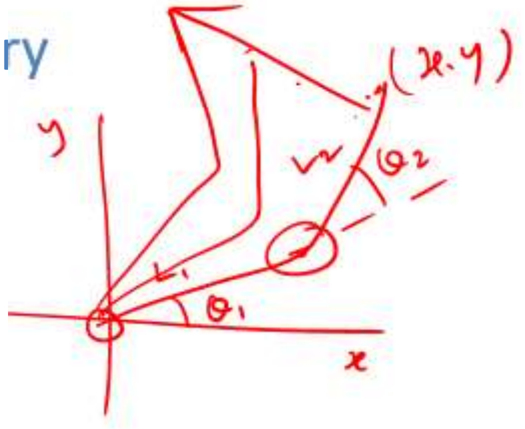


movements in a path
Sequential robot

Trajectory Planning



Cartesian and Joint Trajectory



- Consider a two arm manipulator equation:

$$x = L_1 C\theta_1 + L_2 C(\theta_1 + \theta_2)$$
$$y = L_1 S\theta_1 + L_2 S(\theta_1 + \theta_2)$$

The Inverse Kinematics solution is:

$$\theta_2 = \arccos\left(\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1 L_2}\right)$$

$$\theta_1 = \tan^{-1}\left(\frac{ay - bx}{ax + by}\right)$$

where

$$\frac{a}{b} = \frac{L_1 + L_2 C(\theta_2)}{L_2 S(\theta_2)}$$

Cartesian and Joint Trajectory

- If $\left(x(t), y(t)\right)$ is given as a vector function of time t, then we can obtain $\left(\theta_1(t), \theta_2(t)\right)$as functions of t.
- Here $\left(x(t), y(t)\right)$ is called Cartesian Trajectory and $\left(\theta_1(t), \theta_2(t)\right)$is called Joint Trajectory.
- For a n-arm manipulator, if the arm-matrix ${}^0T_n$ is given as a function of t, then it is called the Cartesian Trajectory.
- The corresponding inverse kinematics solution $q(t) = (q_1(t), q_2(t) \ldots \ldots \ldots q_n(t))$ is called the Joint Trajectory.

Trajectory Planning

- **Point to point motion**
  - Teach initial and final points; intermediate path is not critical and is computed by the controller
  - Applications: Moving of parts, spot welding, automated loading and unloading of machines; pick-and-place motion
- **Continuous path motion**
  - Used when there is a need to follow a complex path through 3-D space, possibly at high speeds (spray painting, welding, polishing)
  - Points generally taught by manual lead through with high speed automatic sampling

Cubic Polynomial Trajectories

- Single joint (1 DOF):
  - We Know $\theta(0), \theta(t_f)$
  - We also Know $\dot{\theta}(0), \dot{\theta}(t_f)$
  - Want to find $\theta(t), \dot{\theta}(t), \ddot{\theta}(t)$
  - So we have four Constraints: Lets try a Cubic polynomial

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Cubic Polynomial Trajectories

- To determine the coefficients we need to look at our boundary conditions as follows:

  - ✓ Position at t=0
  - ✓ Velocity at t=0
  - ✓ Position at t=final
  - ✓ Velocity at t=final

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\theta(t) = a_1 + 2a_2 t + 3a_3 t^2$$

Cubic Polynomial Trajectories

So we can evaluate the coefficients as follows:

$$\theta(0) = a_0$$
$$\dot{\theta}(0) = a_1$$

$$\theta(t_f) = \theta(0) + \dot{\theta}(0).t_f + a_2.t_f^2 + a_3.t_f^3$$
$$\dot{\theta}(t_f) = \dot{\theta}(0) + 2.a_2.t_f + 3.a_3.t_f^2$$

$$a_2 = \frac{3}{t_f^2}.\left(\theta(t_f) - \theta(0)\right) - \frac{2}{t_f}\dot{\theta}(0) - \frac{1}{t_f}\dot{\theta}(t_f)$$
$$a_3 = -\frac{2}{t_f^3}.\left(\theta(t_f) - \theta(0)\right) - \frac{1}{t_f^2}.(\dot{\theta}(t_f) - \dot{\theta}(0)$$

Cubic Polynomial Trajectories

If the initial and final velocity are zero then,

$$a_0 = \theta(0)$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}.\left(\theta(t_f) - \theta(0)\right)$$

$$a_3 = -\frac{2}{t_f^3}.\left(\theta(t_f) - \theta(0)\right)$$

So finally, the acceleration is given by:

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

$$\ddot{\theta}(t) = \frac{6}{t_f^2}.\left(\theta(t_f) - \theta(0)\right) - \frac{12}{t_f^3}\left(\theta(t_f) - \theta(0)\right).\mathrm{t}$$

Quintic Polynomial Trajectory

- Cubic allow us to define the position and velocity at each location in the trajectory but not the acceleration.
- If we also want to specify the acceleration we would need a Quintic or order 5 polynomial.

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

- Use the initial and final positions, velocities and accelerations as our boundary conditions to solve for the coefficients.

Quintic Polynomial Trajectory

$$\begin{bmatrix} \theta_0 \\ \omega_0 \\ \alpha_0 \\ \theta_f \\ \omega_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix}^{-1} \begin{bmatrix} \theta_0 \\ \omega_0 \\ \alpha_0 \\ \theta_f \\ \omega_f \\ \alpha_f \end{bmatrix}$$

Example

Consider a SCARA Robot manipulator:

The arm matrix

$$
{}^{0}T_4 = \begin{bmatrix} C_{1-2-4} & S_{1-2-4} & 0 & a_1C_1 + a_2C_{1-2} \\ S_{1-2-4} & -C_{1-2-4} & 0 & a_1S_1 + a_2S_{1-2} \\ 0 & 0 & -1 & d_1 - d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$OA = d_1 = 10$

$AB = a_1 = 5$

$BC = d_2 = 3$

$CE = d_3(variable)$

$EF = d_4 = 4$

Example

If

$$T(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 3 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{DT}{dt}(0) = 0_{4\times4}$$

$$T(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{DT}{dt}(10) = 0_{4\times4}$$

Example

Let

$$T(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3$$

$$\Rightarrow T(0) = A_0, \frac{dT}{dt}(0) = A_1 = 0$$

$$T(10) = A_0 + 10.A_1 + 100.A_2 + 1000.A_3$$

$$\dot{T}(10) = A_1 + 20.A_2 + 300.A_3$$

$$T(0) = T_0 = A_0 \quad \frac{dT}{dt} = T_1 = A_1$$

Thanks!