IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Transforming Differential Changes between Coordinate Frames

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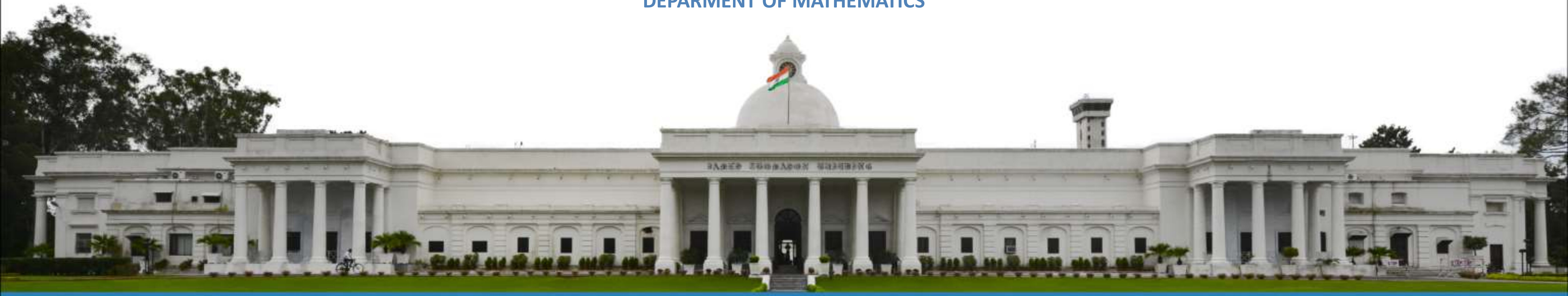

Transforming Differential Changes between Coordinate Frames

- We have been discussing differential changes made with respect to a given coordinate frame or with respect to base coordinates. In this section we will evaluate the transformation of differential changes between coordinate frames. That is, given $\mathrm{D}, what\ is\ {}^{T}\mathrm{D}$?

$$\mathrm{D}\,T = T\ {}^{T}\mathrm{D}$$

So we can have:

$${}^{\mathrm{T}}\mathrm{D} = \mathrm{T}^{-1}\mathrm{D}\ \mathrm{T}$$

- The above equation is important as it relates differential changes between coordinate frames. Before we use this result we will first expand the matrix product on the right hand side of the above equation.

Transforming Differential Changes between Coordinate Frames

- If we represent the elements of differential coordinate transformations T in terms of the vectors n, o, a and p as follows with δ and d as differential rotation and translation respectively :

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{D}\,T = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & \mathrm{d_y} \\ -\delta_y & \delta_x & 0 & \mathrm{d_z} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- We can express product of right hand two transformations as follows:

$$\mathrm{D}\,T = \begin{bmatrix} (\delta \times n)_x & (\delta \times o)_x & (\delta \times a)_x & ((\delta \times p) + d)_x \\ (\delta \times n)_y & (\delta \times o)_y & (\delta \times a)_y & ((\delta \times p) + d)_y \\ (\delta \times n)_z & (\delta \times o)_z & (\delta \times a)_z & ((\delta \times p) + d)_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- So we can obtain:

$$\mathrm{T}^{-1}\mathrm{D}\,T = \begin{bmatrix} n.(\delta\times n) & n.(\delta\times o) & n.(\delta\times a) & n.((\delta\times p)+d) \\ o.(\delta\times n) & o.(\delta\times o) & o.(\delta\times a) & o.((\delta\times p)+d) \\ a.(\delta\times n) & a.(\delta\times o) & a.(\delta\times a) & a.((\delta\times p)+d) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- By using identities:

$$f.(g\times h) = -g.(f\times h) = g.(h\times f)$$

and

$$f.(f\times h) = 0$$

we have:

$$^{\mathrm{T}}\mathrm{D} = \begin{bmatrix} 0 & -\delta.(n\times o) & \delta.(a\times n) & \delta.(p\times n)+d.n \\ \delta.(n\times o) & 0 & -\delta.(o\times a) & \delta.(p\times o)+d.o \\ -\delta.(a\times n) & \delta.(o\times a) & 0 & \delta.(p\times a)+d.a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Further we have

$$n \times o = a;$$
$$a \times n = o;$$
$$o \times a = n;$$

- Finally we can have following equation:

$$^{T}D = \begin{bmatrix} 0 & -\delta.a & \delta.o & \delta.(p \times n) + d.n \\ \delta.a & 0 & -\delta.n & \delta.(p \times o) + d.o \\ -\delta.o & \delta.n & 0 & \delta.(p \times a) + d.a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We also have

$$^{T}D = \begin{bmatrix} 0 & -{}^{T}\delta_z & {}^{T}\delta_y & {}^{T}d_x \\ {}^{T}\delta_z & 0 & -{}^{T}\delta_x & {}^{T}d_y \\ {}^{T}\delta_y & {}^{T}\delta_x & 0 & {}^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Comparing respective entries in equations,we can obtain the differential translation and rotation vectors described with respect to coordinate frame T(${}^{T}\delta$ and ${}^{T}d$)in terms of the differential translation and rotation vectors described with respect to base coordinates (δ and d):

$$
\begin{aligned}
{}^{T}d_x &= \delta.(p \times n) + d.n \\
{}^{T}d_y &= \delta.(p \times o) + d.o \\
{}^{T}d_z &= \delta.(p \times a) + d.a
\end{aligned}
$$

$$
\begin{aligned}
{}^{T}\delta_x &= \delta.n \\
{}^{T}\delta_y &= \delta.o \\
{}^{T}\delta_z &= \delta.a
\end{aligned}
$$

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- Finally we can have the following 6×6 Matrix:

$$\begin{bmatrix} {}^T d_x \\ {}^T d_y \\ {}^T d_z \\ {}^T \delta_x \\ {}^T \delta_y \\ {}^T \delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_y & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

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- Computationally we can also have following important results:

$$
\begin{aligned}
{}^{T}d_x &= n.((\delta \times p) + d) \\
{}^{T}d_y &= o.((\delta \times p) + d) \\
{}^{T}d_z &= a.((\delta \times p) + d)
\end{aligned}
$$

$$
\begin{aligned}
{}^{T}\delta_x &= n.\delta \\
{}^{T}\delta_y &= o.\delta \\
{}^{T}\delta_z &= a.\delta
\end{aligned}
$$

Example

Let

$$T = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and differential translation and rotation w.r.t. F be:

$$d = i + 0.5k$$

$$\delta = 0.1j$$

What is the equivalent differential translation and rotation in coordinate frame M ?

Solution

$$n = j$$
$$o = k$$
$$a = i$$
$$p = 10i + 5j$$

We first form

$$\delta \times p = -k$$

Then add d to it

$$\delta \times p + d = i - 0.5k$$

Now we will evaluate ${}^{\mathrm{M}}d \ and \ {}^{M}\delta$

$$^{\mathrm{M}}d = -0.5j + k$$
$$^{\mathrm{M}}\delta = 0.1i$$

Solution

We can check this result by evaluating dT

$$dT = T\ {}^{T}\Delta$$

Now we can have

$$^{T}\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathrm{dT} = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thanks!