IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Differential Transformation

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Differential Transformation

- Let T denote the position and orientation of a frame M with respect to fixed frame F. Let T + ΔT denote the new position and orientation of M after undergoing a rotation about a vector k by an angle $\Delta\theta$and a translation (Δx,Δy,Δz) w.r.t F. Then

$$T + \Delta T = Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta).T$$

From above we can get the following:

$$\Delta\, T = (Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta)\ - I).T$$

Differential Transformation

- Now let the T + ΔT be obtained by translation and a rotation about by an angle w.r.t M itself (i.e. current frame) Then

$$T + \Delta T = T.Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta)$$
$$\Delta T = T.(Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta)\ - I)$$

Differential Transformation

The 4 ×4 matrix $(Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta) - I)$is called differential transformation w.r.t. frame F denoted by

$$\mathrm{D} = Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta) - \mathrm{I}$$

And thus, change in position and orientation of T is given by

$$\Delta T = \mathrm{D\ T}$$

Similarly

$$\Delta T = T\ {}^{T}\mathrm{D}$$

Where ^{T}D is given by $Trans(\Delta x^{T}, \Delta y^{T}, \Delta z^{T})\ Rot(k^{T}, \Delta\theta^{T}) - \mathrm{I}$

Differential Transformation

Let

$$Trans(d) = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

denotes small translation w.r.t. F and

$$Rot(k, \Delta\theta) = \begin{bmatrix} k_x k_x v(\Delta\theta) + \cos\Delta\theta & k_y k_x v(\Delta\theta) - k_z \sin\Delta\theta & k_z k_x v(\Delta\theta) - k_y \sin\Delta\theta & 0 \\ k_y k_x v(\Delta\theta) - k_z \sin\Delta\theta & k_y k_y v(\Delta\theta) + \cos\Delta\theta & k_z k_y v(\Delta\theta) - k_x \sin\Delta\theta & 0 \\ k_z k_x v(\Delta\theta) - k_y \sin\Delta\theta & k_z k_y v(\Delta\theta) - k_x \sin\Delta\theta & k_z k_z v(\Delta\theta) + \cos\Delta\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

denotes rotation about a vector k by an angle Δθ where v(Δθ)=1-cos(Δθ)

Differential Transformation

If $\Delta\theta$ is very small

$$\sin\Delta\theta \to \Delta\theta$$

$$\cos\Delta\theta \to 1$$

$$\operatorname{vers}\Delta\theta \to 0$$

Hence for small values of angle we get

$$Rot(k,\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta & 0 \\ k_z\Delta\theta & 1 & -k_x\Delta\theta & 0 \\ -k_y\Delta\theta & k_x\Delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Transformation

$$D = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta & 0 \\ k_z\Delta\theta & 1 & -k_x\Delta\theta & 0 \\ -k_y\Delta\theta & k_x\Delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So

$$D = \begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta & \Delta x \\ k_z\Delta\theta & 0 & -k_x\Delta\theta & \Delta y \\ -k_y\Delta\theta & k_x\Delta\theta & 0 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} \ldots\ldots\ldots\ldots (A)$$

Differential Motion

- The differential motions of a frame can be divided into the following: 1.Differential Translations 2.Differential Rotations 3.Differential Transformations
- The Differential Operator is a way to account for "small Motions" (DT)
- It can be used to study movement of the End Frame over a short time intervals (Δt)

Fundamental Rotation Approximation

- $$Rot(X, \Delta\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\Delta\theta_x) & -\sin(\Delta\theta_x) & 0 \\ 0 & \sin(\Delta\theta_x) & \cos(\Delta\theta_x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If$\Delta\theta$ is very small

$$Rot(X, \Delta\theta_x) \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\Delta\theta_x & 0 \\ 0 & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Rot\left(Y, \Delta\theta_y\right) \approx \begin{bmatrix} 1 & 0 & \Delta\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\Delta\theta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Rot(Z, \Delta\theta_z) \approx \begin{bmatrix} 1 & -\Delta\theta_z & 0 & 0 \\ \Delta\theta_z & 1 & 0 & 0 \\ 0 & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying the Rotation Matrices, we get a general rotation matrix as

$$Rot(X,\Delta\theta_x)\ Rot(Y,\Delta\theta_y)\ Rot(Z,\Delta\theta_z)$$

$$Gen\_Rot \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & 0 \\ \Delta\theta_z & 1 & -\Delta\theta_x & 0 \\ -\Delta\theta_y & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: we have neglected higher order product terms

$$Gen\_Movement \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting the matrices, we get:

$$T + \Delta T \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . T$$

Solving for the differential motion (ΔT)

$$\Delta T \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . T - T$$

Further Simplifying:

$$\Delta T \approx \begin{bmatrix} 0 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 0 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 0 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} . \mathrm{T} = \mathrm{D\,T} \ldots\ldots\ldots\ldots (\mathrm{B})$$

We will call this matrix the del operator:

Comparing D from equations (A) and (B), we get:

$$k_x\Delta\theta = \Delta\theta_x$$
$$k_y\Delta\theta = \Delta\theta_y$$
$$k_z\Delta\theta = \Delta\theta_z$$

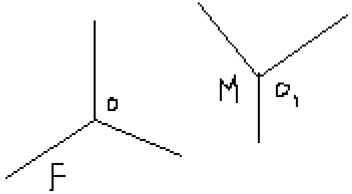
$$\frac{\Delta T}{\Delta t} \approx \begin{bmatrix} 0 & -\frac{\Delta\theta_z}{\Delta t} & \frac{\Delta\theta_y}{\Delta t} & \frac{\Delta x}{\Delta t} \\ \frac{\Delta\theta_z}{\Delta t} & 0 & -\frac{\Delta\theta_x}{\Delta t} & \frac{\Delta y}{\Delta t} \\ -\frac{\Delta\theta_y}{\Delta t} & \frac{\Delta\theta_x}{\Delta t} & 0 & \frac{\Delta z}{\Delta t} \\ 0 & 0 & 0 & 0 \end{bmatrix} . \mathrm{T} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} . T$$

The above equation represents the velocity of frame T.

Velocity of a Frame

- Let F be a fixed frame and M be a moving frame. Let ${}^{F}T_{M} = T$. If the frame M has translational velocities d_x, d_y, d_z and rotational velocities $\delta_x, \delta_y, \delta_z$ about x,y,z axes of F, then:

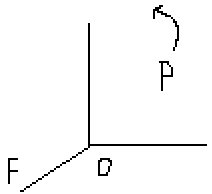
$$\frac{dT}{dt} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} . T$$



Velocity of a Point

- Let F be a coordinate frame and P, a point in F. If P has translational velocities d_x, d_y, d_z along x, y and z directions and rotational velocities $\delta_x, \delta_y, \delta_z$ about the x, y and z axes of F, then the velocity of P in F is given by:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ 1\end{bmatrix} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Differential Transformation

- Previous equation gives the derivative of T w.r.t time t when the entries in the matrix representing small translation and rotation are replaced by translational and rotational velocities w.r.t the frame F.
- Let ${}^{F}X$ and ${}^{M}X$ denote a point X in F and M frames respectively. Then we know that

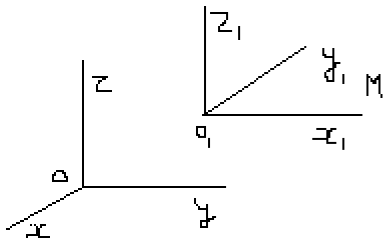
${}^{F}X = T\,{}^{M}X$ where T is the homogeneous transformation matrix relating M w.r.t. F.

Then

$$\frac{d\,{}^{F}X}{dt} = T\frac{d\,{}^{M}X}{dt} + \frac{dT}{dt}{}^{M}X = T\frac{d\,{}^{M}X}{dt} + \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} . T\,{}^{M}X$$

Example

- Let the coordinate of O_1 w.r.t F be (2, 1, 3) and x_1 axis is parallel to y-axis, y_1 is parallel to negative x-axis and z_1 is parallel to z- axis of F. Let the M-frame moves with a rotational velocity 0.004 rad/sec about y-axis of F and a translational velocity 0.1 cm/sec along the z-axis of F. Let P be a point in M with coordinates (1, 2, 1) w.r.t M. Let P moves with a translation velocity of 0.2 cm/sec along z_1-direction of M and a rotational velocity of 0.05 rad/sec about x_1-axis of M. Then find the velocity of P as viewed from F.



Solution

$$^{F}[P] = ^{F}T_M {}^{M}[P] = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{M}[P] \; therefore \; \frac{d\,^{F}P}{dt} = T\frac{d\,^{M}P}{dt} + \frac{dT}{dt}{}^{M}P$$

$$\frac{d^{M}P}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0.05 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{M}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$, \frac{dT}{dt} = \begin{bmatrix} 0 & 0 & 0.004 & 0 \\ 0 & 0 & 0 & 0 \\ -0.004 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} T$$

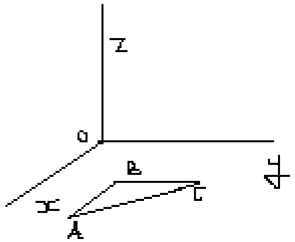
Solution

Therefore, the velocity of the point P at (1,2,1) $\frac{d^F P}{dt}$ is:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0.05 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{M}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0.004 & 0 \\ 0 & 0 & 0 & 0 \\ -0.004 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{M}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Example

The three vertices of a triangle ABC rotated about the y-axis with angular velocity 0.03rad/sec. Find the coordinates of the vertices as functions of t. Assume that x at time t=0 the vertices are A(2,1,0), B(1.2.0) and C(1,2,0).



Solution

For any point (x, y, z) ∈ F, $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.03 & 0 \\ 0 & 0 & 0 & 0 \\ -0.03 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$,

=> $\dot{x} = 0.03z$, $\dot{y} = 0$, $\dot{z} = -0.03x$. Applying it for the vertex A , $\dot{x}_1 = 0.03z_1$, $\dot{y}_1 = 0$, $\dot{z}_1 = -0.03x_1$

with initial condition $x_1(0) = 2, y_1(0) = 1, z_1(0) = 0$

=> $\ddot{x}_1 = 0.03\dot{z}_1 = -(0.03)^2.x_1$: $x_1(0) = 2$, $\dot{x}_1(0) = 0$ $\Rightarrow x_1(t) = 2\cos(0.03)t$. Similarly others are formed.

Thanks!