IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Coordinate Frames and Homogeneous Transformations-II

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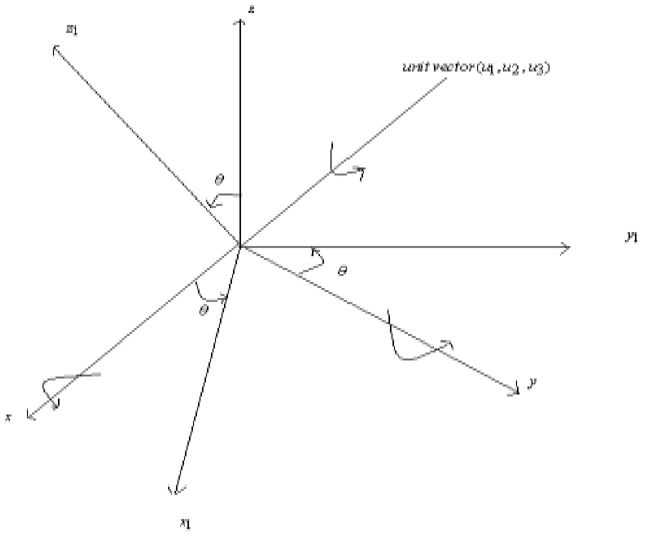
ROTATION ABOUT A UNIT VECTOR BY AN ANGLE θ

- If M is obtained from F by rotation about the unit vector $\vec{r}(u_1, u_2, u_3)$by an angle θ ,then

$$
{}^{F}T_M = \begin{bmatrix} u_1^2(1-\cos\theta)+\cos\theta & u_1u_2\,(1-\cos\theta)-u_3\sin\theta & u_1u_3\,(1-\cos\theta)+u_2\sin\theta \\ u_1u_2\,(1-\cos\theta)+u_3\sin\theta & u_2^2(1-\cos\theta)+\cos\theta & u_2u_3\,(1-\cos\theta)-u_1\sin\theta \\ u_1u_3\,(1-\cos\theta)+u_2\sin\theta & u_1u_3\,(1-\cos\theta)+u_2\sin\theta & u_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}
$$

ROTATION ABOUT A UNIT VECTOR BY AN ANGLE θ

- Let R be the Rotation matrix then



$$\text{Trace of } R = 1 + 2\cos\theta$$

$$\theta = \pm\cos^{-1}\left(\frac{\text{Trace of } (R) - 1}{2}\right)$$

$$r_{32} - r_{23} = 2u_1\sin\theta$$

$$u_1 = \frac{r_{32} - r_{23}}{2\sin\theta};$$

$$u_2 = \frac{r_{13} - r_{31}}{2\sin\theta};$$

$$u_3 = \frac{r_{21} - r_{12}}{2\sin\theta}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Homogeneous Transformation

- A homogenous transformation matrix represent both a rotation and a translation
- Special cases
  - 1. Translation

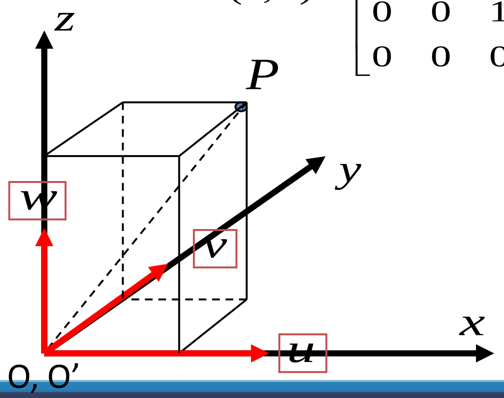
$$ {}^{A}T_{B} = \begin{bmatrix} I_{3\times3} & {}^{A}r^{o'} \\ 0_{1\times3} & 1 \end{bmatrix} $$

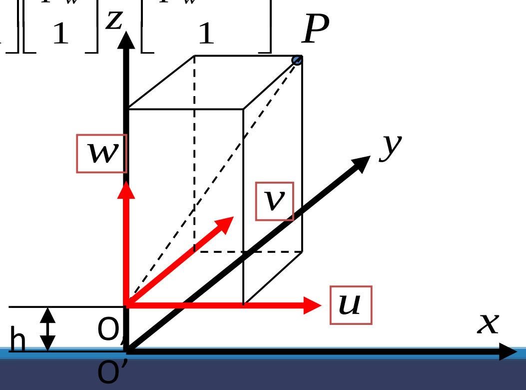
  2. Rotation

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} $$

- Example : Translation along Z-axis with h:

$$ Trans(z,h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix} = \begin{bmatrix} p_u \\ p_v \\ p_w + h \\ 1 \end{bmatrix} $$





Composite Homogeneous Transformation Matrix

Rules:

– Transformation (rotation/translation) w.r.t (X,Y,Z) (OLD FRAME), using pre-multiplication

– Transformation (rotation/translation) w.r.t (U,V,W) (NEW FRAME), using post-multiplication

Example:

➢ **Find the homogeneous transformation matrix (T) for the following operations:**

Rotation α about OX axis

Translation of a along OX axis

Translation of d along OZ axis

Rotation of θ about OZ axis

$$T = T_{z,\theta} T_{z,d} T_{x,a} T_{x,\alpha} I_{4\times 4}$$

$$Answer: = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thanks!