IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Coordinate Frames and Homogeneous Transformations

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Three Dimensional Space

- Three dimensional space R^3 is defined as

$$R^3 = \{(x, y, z)\} \mid x, y, z \in R \qquad (1)$$

Any point P in R^3 is denoted by (x, y, z) using three real numbers x, y, and z, where (x, y, z) are called the co-ordinate of point P. The point $(0, 0, 0)$is called the origin and denoted O. The directed line segment OP is called a vector $\vec{r} = \overrightarrow{OP}$. The directed line segments from the point O to the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are denoted by $\vec{i}\,, \vec{j}\ and\ \vec{k}$ respectively, which are called standard basis vector. It can be easily seen that the vector $\vec{r}$ can be represented as equation (2)

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad (2)$$

- The magnitude of $\vec{r}$ is length of the vector is given by equation

$$\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2} \qquad (3)$$

- $|\vec{r}| = 1$, then $\vec{r}$ is called unit vector.

Dot and Cross Product

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points and $\overrightarrow{a} = \overrightarrow{OA}$ and $\overrightarrow{b} = \overrightarrow{OB}$ are the position vectors then the dot product and the cross product of the vectors is defined in equation (4) and (5) respectively.
- $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}| \cdot Cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \qquad (4)$
- $\overrightarrow{a} \times \overrightarrow{b} = \begin{bmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = (y_1 z_2 - y_2 z_1)i + (x_1 z_2 - x_2 z_1)j + (x_1 y_2 -$

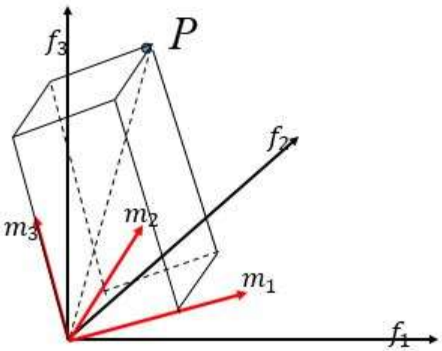
Dot and Cross Product

- Two vectors are called orthonormal (perpendicular) if $\overrightarrow{a} \cdot \overrightarrow{b} = 0.$
- The standard basic vectors $\vec{i}, \vec{j}$ and $\overrightarrow{k}$ form the right handed system because if the fingers of the right hand curl from first vector to second then the third vector points in the direction of the thumb. Instead of $\vec{i}, \vec{j}$ and $\overrightarrow{k}$ we can consider any three mutually $\perp$ unit vectors $\overrightarrow{f_1}, \overrightarrow{f_2}$ and $\overrightarrow{f_3}$ to form a right handed system F, then

$\overrightarrow{f_\alpha} \cdot \overrightarrow{f_n} = 0$ if $\alpha \neq n$

= 1, if $\alpha = n$

Coordinate Transformations



- Let $F = (\vec{f}_1, \vec{f}_2, \vec{f}_3)$ and $M = (\vec{m}_1, \vec{m}_2, \vec{m}_3)$ be orthonormal coordinate frames in R^3 having the same origin and let R be any 3x3 matrix defined by $a_{kj} = \vec{f}_k . \vec{m}_j$ for $1 \le k, j \le 3$ Then for each point (vector) r in R^3:

$$^{F}[r] = R\ ^{M}[r]$$

Coordinate Transformations

- As the matrix R gives coordinates of a point r with respect to F if r is given with respect to M, we denote the matrix R by ${}^{F}R_{M}$
- The Equation can be written as:

$$
{}^{F}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{m_1} \cdot \overrightarrow{f_1} & \overrightarrow{m_2} \cdot \overrightarrow{f_1} & \overrightarrow{m_3} \cdot \overrightarrow{f_1} \\ \overrightarrow{m_1} \cdot \overrightarrow{f_2} & \overrightarrow{m_2} \cdot \overrightarrow{f_2} & \overrightarrow{m_3} \cdot \overrightarrow{f_2} \\ \overrightarrow{m_1} \cdot \overrightarrow{f_3} & \overrightarrow{m_2} \cdot \overrightarrow{f_3} & \overrightarrow{m_3} \cdot \overrightarrow{f_3} \end{bmatrix} {}^{M}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}
$$

Proof

- Any point P(x, y, z) can be written in vector form as:

$$\vec{r} = x_1\vec{f}_1 + y_1\vec{f}_2 + z_1\vec{f}_3 \qquad (1)$$

Where $\vec{f}_1, \vec{f}_2, \vec{f}_3$denote orthonormal coordinate axes of the fixed frame F. (x,y,z) are the coordinates of the point P w.r.to F. Let M be the moving frame . Let $\overrightarrow{m}_1, \overrightarrow{m}_2, \overrightarrow{m}_3$be orthonormal vectors denoting the coordinates axis of frame M. Then the vector $\vec{r}$can be written as :

$$\vec{r} = x_2\overrightarrow{m}_1 + y_2\overrightarrow{m}_2 + z_2\overrightarrow{m}_3 \qquad (2)$$

Where (x_1, y_1, z_1)denotes the co-ordinate of point P in M.

- Now in F (Equation 1)

$$\vec{r} = (\vec{r}.\vec{f}_1)\vec{f}_1 + (\vec{r}.\vec{f}_2)\vec{f}_2 + (\vec{r}.\vec{f}_3)\vec{f}_3$$

And in M (Equation 2)

$$\vec{r} = (\vec{r}.m_1)\vec{m}_1 + (\vec{r}.m_2)m_2 + (\vec{r}.m_3)m_3$$

Since $\vec{m}_1, \vec{m}_2, \vec{m}_3$ are vectors in F we can write (Equation 3):

$$\vec{m}_i = (\vec{m}_i.\vec{f}_1)\vec{f}_1 + (\vec{m}_i.\vec{f}_2)\vec{f}_2 + (\vec{m}_i.\vec{f}_3)\vec{f}_3 i = 1,2,3$$

- From Equation 2 and 3 we can get

$$\vec{r} = \sum_{i=1}^{3}(\vec{r}.\vec{m}_i)[\,(\vec{m}_i.\vec{f}_1)\vec{f}_1+(\vec{m}_i.\vec{f}_2)\vec{f}_2 + (\vec{m}_i.\vec{f}_3)\vec{f}_3]$$

Hence we get (Equation 4)

$$\vec{r} = \sum_{j=1}^{3}[\sum_{i=1}^{3}[(\vec{r}.\vec{m}_i)(\vec{m}_i.\vec{f}_j)]\vec{f}_j]$$

Hence

$$^{F}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{m_1}.\overrightarrow{f_1} & \overrightarrow{m_2}.\overrightarrow{f_1} & \overrightarrow{m_3}.\overrightarrow{f_1} \\ \overrightarrow{m_1}.\overrightarrow{f_2} & \overrightarrow{m_2}.\overrightarrow{f_2} & \overrightarrow{m_3}.\overrightarrow{f_2} \\ \overrightarrow{m_1}.\overrightarrow{f_3} & \overrightarrow{m_2}.\overrightarrow{f_3} & \overrightarrow{m_3}.\overrightarrow{f_3} \end{bmatrix} {}^{M}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$^{F}[r] = {}^{F}\, {R_M}^{M}[r]$$

Example

- Let F be the coordinate frame with $\vec{i}, \vec{j}, \vec{k}$ as the coordinate axes and M be the coordinate frame with $\vec{j}, -\vec{i}, \vec{k}$ as its coordinate axes. Suppose the coordinates of a point p with respect to the frame M are measured and found to be ${}^{M}[P] = [0.6, 0.5, 1.4]^T$. What are the coordinates of p with respect to the fixed coordinates frame F.

Solution

$$^{F}[p] = {}^{F}R_{M}{}^{M}[r]$$

$$= \begin{bmatrix} (\vec{m}_1.\vec{f}_1) & (\vec{m}_2.\vec{f}_1) & (\vec{m}_3.\vec{f}_1) \\ (\vec{m}_1.\vec{f}_2) & (\vec{m}_2.\vec{f}_2) & (\vec{m}_3.\vec{f}_2) \\ (\vec{m}_1.\vec{f}_3) & (\vec{m}_2.\vec{f}_3) & (\vec{m}_3.\vec{f}_3) \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.5 \\ 1.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.5 \\ 1.4 \end{bmatrix}$$

$$= [-0.5, 0.6, 1.4]^{T}$$

Fundamental Rotation Matrices

- Rotation about x-axis by an angle θ

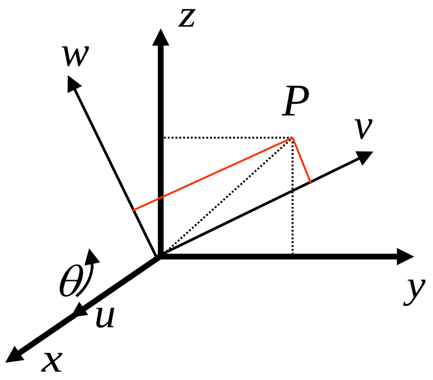
$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & C\theta & -S\theta\\ 0 & S\theta & C\theta\end{bmatrix}$$

- Rotation about y-axis by an angle θ

$$Rot(y,\theta)=\begin{bmatrix}C\theta & 0 & S\theta\\ 0 & 1 & 0\\ -S\theta & 0 & C\theta\end{bmatrix}$$

- Rotation about z-axis by an angle θ

$$Rot(z,\theta)=\begin{bmatrix}C\theta & -S\theta & 0\\ S\theta & C\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$



$$P_{xyz}=RP_{uvw}$$

$$P_{uvw}=QP_{xyz}$$

$$Q=R^{-1}=R^{T}$$

Example

- A point $a_{uvw} = (4,3,2)$ is attached to a rotating frame, which was obtained by rotating 60 degree about the OZ axis of the reference frame (x-y-z –frame). Find the coordinates of the point relative to the reference frame.

$$a_{xyz} = Rot(z,60)a_{uvw}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

Composite Rotation Matrix

Example

- Find the rotation matrix for the following operations:

1. Rotation ϕ about OY axis 2. Rotation θ about OW axis 3. Rotation α about OU axis

$$R = Rot(y,\phi) I_3 Rot(w,\theta) Rot(u,\alpha)$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

Post-multiply if rotation is with respect to current frame

Pre-multiply if rotation is with respect to base frame

Thanks!