Chapter 2

Energy and Dynamics of SHM 1.15.2 1.15,3

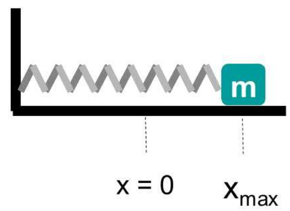
Back in PH121 we started with position, velocity, and acceleration to describe motion. We have done that again for a simple harmonic oscillator. In PH121, after basic motion, we found that we could use the idea of a force and Newton's second law to find the acceleration for our description of motion. Then we changed view points and used the idea of energy to find motion. The energy picture was easier in some ways. Let's try to develop and energy picture for simple harmonic oscillators. Of course the goal in physics is to describe the motion of the object, so we will say that we are looking for the *dynamics* of simple harmonic oscillators when we look for the position, velocity, and acceleration as a function of time.

Fundamental Concepts

- Initial Conditions
- Energy and SHM
- Equation of motion
- Vertical oscillations

2.1 Initial Conditions

Usually, we need to know how we start our oscillator to solve a problem. Let's see how this works.



Suppose we start the motion of a mass attached to a spring (a harmonic oscillator) by pulling the mass to $x = x_{\max}$ and releasing it at $t = 0$. Let's see if we can find the phase. Our initial conditions require

$$\begin{aligned} x(0) &= x_{\max} \\ v(0) &= 0 \end{aligned}$$

Using our formula for $x(t)$ and $v(t)$ we have

$$\begin{aligned} x(0) &= x_{\max} = x_{\max}\cos(0+\phi_o) \\ v(0) &= 0 = -v_{\max}\sin(0+\phi_o) \end{aligned}$$

If we choose $\phi_o = 0$, these conditions are met.

Notice that we needed to know the starting time and the position and the velocity at that time. These are what we call *initial conditions*. It is still true that $x(t)$ and $v(t)$ are out of phase. But we found $\phi_o = 0$. There is another phase term hiding in our expression for $v(t)$ and to find it we need a small trig identity.

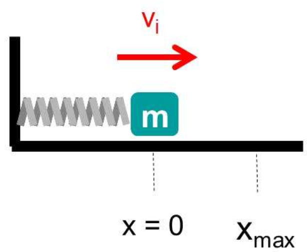
$$\begin{aligned} \sin(\alpha+\phi_o) &= \cos\left(\alpha - \left(\frac{\pi}{2} - \phi_o\right)\right) \\ &= \cos\left(\alpha - \frac{\pi}{2} + \phi_o\right) \end{aligned}$$

so we could write

$$\begin{aligned} v(t) &= -\omega x_{\max}\sin(\omega t+\phi_o) \\ &= -\omega x_{\max}\cos\left(\alpha - \frac{\pi}{2} + \phi_o\right) \end{aligned}$$

and we can see that there was a phase shift of $-\pi/2$ hiding in our sine function. So $v(t)$ really must be out of phase with $x(t)$.

2.1.1 A second example



Using the same equipment, let's start with

$$\begin{aligned} x(0) &= 0 \\ v(0) &= v_i \end{aligned}$$

that is, the mover object is already moving when we start our experiment, and we start watching just as it passes the equilibrium point.

$$\begin{aligned} x(0) &= 0 = x_{\max}\cos(0+\phi_o) \\ v(0) &= v_i = -v_{\max}\sin(0+\phi_o) \end{aligned}$$

from the first equation we have

$$0 = x_{\max}\cos(\phi_o)$$

and that gives us

$$\phi_o = \cos^{-1}(0)$$

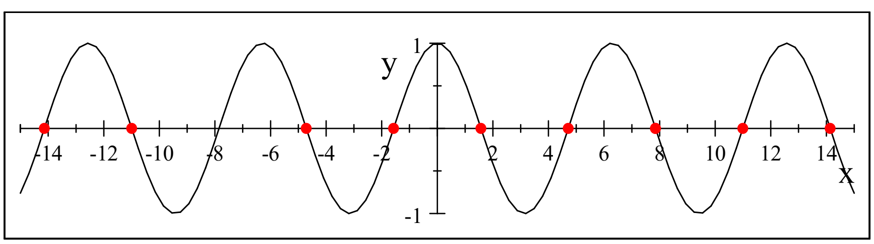
so

$$\phi_o = \pm\frac{\pi}{2}$$

really this is

$$\phi_o = \pm\frac{\pi}{2} \pm n\pi \qquad n = 0, 1, 2, \cdots$$

This gives the red dots in the next plot.



Notice that at each of these locations $\cos(\phi)$ is zero. but let's make an agreement that we will choose the smallest value for ϕ_o that makes $\cos(\phi_o) = 0$. That is our

$$\phi_o = \pm\frac{\pi}{2}$$

But positive and negative $\pi/2$ are the same "smallness." We don't know the sign. Using our initial velocity condition will let us determine the sign. Let's try it

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\frac{\pi}{2}\right) \\ v_i &= \pm v_{\max} \\ v_i &= \pm\omega x_{\max} \end{aligned}$$

From this we can see

$$x_{\max} = \pm\frac{v_i}{\omega}$$

We defined the initial velocity as positive, and we insist on having positive amplitudes, so then we choose

$$\phi_o = -\frac{\pi}{2}$$

so

$$\sin\left(-\frac{\pi}{2}\right) = -1$$

and then this minus sign will cancel the one in our initial velocity equation that to make our initial velocity positive.

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\frac{\pi}{2}\right) \\ &= -v_{\max}(-1) \\ &= v_{\max} \end{aligned}$$

Our solutions are

$$\begin{aligned} x(t) &= \frac{v_i}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right) \\ v(t) &= v_i\sin\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$

Note that our solution is a set of equations! It isn't a set of numbers. We saw this sometimes in PH121. But we will see it more in our study of oscillations and waves. Sometimes the solutions are equations. But we did fill in our equations with the number for the phase constant. it would be even better if we had numbers for ω and v_i.

Generally to have a complete solution, you must find all the constants based on the initial conditions. This would mean we need $x_{\max}$, ω, and ϕ_o to have a complete solution.

Let's try a complete example problem.

2.1.2 Example

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is 2.00 cm, and the frequency is 1.50 Hz.

a) show that the position of the particle is given by

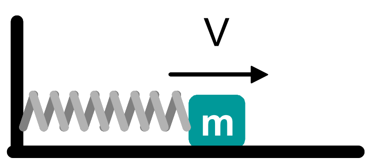
$$x = (2.00\,\text{cm})\sin(3.00\pi t)$$

determine

b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed,

c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration, and

d) the total distance traveled between $t = 0$ and $t = 1.00\,\text{s}$



Basic Equations

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

Variables

t	time, initial time =0	$t_i = 0$
x	Position, Initial position =0	$x(0) = 0$
v		
a		
$x_{\max}$	x amplitude	$x_{\max} = 2.00\,\mathrm{cm}$
v_m	v amplitude	
a_m	a amplitude	
ω	angular frequency	
ϕ_o	phase constant	
f	frequency	$f = 1.50\,\mathrm{Hz}$

Symbolic Solution
Part (a)
We can start by recognizing that we know ω because we know the frequency.

$$\begin{aligned}\omega &= 2\pi f \\ &= 2\pi\,(1.50\,\mathrm{Hz}) \\ &= 9.4248\,\mathrm{rad\,Hz}\end{aligned}$$

We also know the amplitude $A = x_{\max}$ which is given.

$$A = 2.00\,\mathrm{cm}$$

Knowing that at $t = 0$

$$x(0) = 0 = x_{\max}\cos(0 + \phi_o)$$

which we have seen before! We can guess that

$$\phi_o = \pm\frac{\pi}{2}$$

Using

$$v(0) = -\omega x_{\max}\sin\left(0 \pm \frac{\pi}{2}\right)$$

and demanding that amplitudes be positive values, and noting that at $t = 0$ the velocity is positive from the initial conditions:

$$\phi = -\frac{\pi}{2}$$

We also note from our trig identity that we used above

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

that we have

$$\begin{aligned}x(t) &= x_{\max}\cos\left(2\pi f t - \frac{\pi}{2}\right) \\ &= x_{\max}\sin(2\pi f t)\end{aligned}$$

which is in the form we want. All we have to do is put in numbers.

$$\begin{aligned} x(t) &= x_{\max}\sin(2\pi f t) \\ &= (2.00\,\mathrm{cm})\sin(3.00\pi t) \end{aligned}$$

And once again our solution for a) is an equation.

Part (b)

We have a formula from our previous work for $v_{\max}$

$$\begin{aligned} v_{\max} &= \omega x_{\max} \\ &= 2\pi f x_{\max} \end{aligned}$$

so let's use it. To find when $v_{\max}$ happens, take a derivative of $x(t)$

$$v(t) = v_{\max} = -\omega x_{\max}\sin\left(2\pi f t - \frac{\pi}{2}\right)$$

and recognize that $\sin(\theta) = 1$ is at a maximum and this happens when $\theta = \pi/2$. So we take the stuff that is in the parenthesis for the sine function and set it equal to our angle that makes the sine function equal to 1.

$$\frac{\pi}{2} = 2\pi f t - \frac{\pi}{2}$$

We can simplify this and solve for t

$$\pi = 2\pi f t$$

$$\frac{1}{2f} = t$$

$$t = \frac{1}{2\,(1.50\,\mathrm{Hz})} = 0.333\,33\,\mathrm{s}$$

Part (c) Like with the velocity we must the formula. But we know it is just taking the derivative of $v(t)$

$$a(t) = -\omega^2 x_{\max}\cos(\omega t + \phi_o)$$

but recognize that the maximum is achieved when $\cos(\omega t + \phi_o) = 1$ or when $\omega t + \phi_o = 0$. We can solve this for t.

$$\begin{aligned} t &= \frac{\phi_o}{\omega} \\ &= \frac{-\frac{\pi}{2}}{2\pi f} \\ &= \frac{-1}{4f} \\ &= -0.166\,67\,\mathrm{s} \end{aligned}$$

The formula for $a_{\max}$ is also already in our set of equations

$$\begin{aligned} a_{\max} &= -\omega^2 x_{\max} \\ &= -(2\pi f)^2 x_m \end{aligned}$$

Putting in numbers gives

$$\begin{aligned} a_{\max} &= (2\pi 1.5\,\mathrm{Hz})^2\,(2.00\,\mathrm{cm}) \\ &= 1.776\,5\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Part (d)

We know the period is

$$T = \frac{1}{f}$$

We should find the number of periods in $t = 1.00\,\mathrm{s}$ and find the distance traveled in one period,

$$N = \frac{t}{T}$$

and multiply them together. In one period the distance traveled is

$$d = 4x_m$$

$$d_{tot} = d * N = d * \frac{t}{T} = 4 f x_m t$$

Putting in numbers gives

$$\begin{aligned} d_{tot} &= 4 f x_m t \\ &= 8.00\,\mathrm{cm} * 1.50\,\mathrm{Hz} * 1.00\,\mathrm{s} \\ &= 0.12\,\mathrm{m} \end{aligned}$$

We have come far in only one lecture! We have a set of new equations and a new problem type. In what we did in this lecture we used Newton's Second Law. But in PH121 we learned that often using the idea of energy made problems easier. Can we use energy with simple harmonic motion problems? That is what we will talk about in our next lecture.

2.2 Oscillators and Energy

You might have noticed that we are calling mover objects that experience simple harmonic motion (SHM) by the name *simple harmonic oscillators (SHO).* Let's consider such a SHO. Because our SHO is moving, we know there must be energy associated with it. To understand the energy involved, let's start with kinetic energy. Recall from PH121 that

$$K = \frac{1}{2} m v^2 \tag{2.1}$$

and we recall that for a spring, we have the spring potential energy given by

$$U_s = \frac{1}{2}kx^2 \tag{2.2}$$

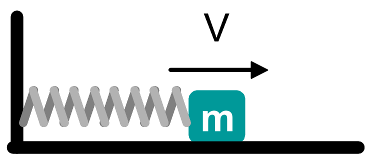
Let's try a problem.

Using energy techniques, find the maximum velocity of the mass in terms of the Amplitude and the angular frequency.

We want to use our problem solving steps:

Type of problem: This is an energy and a SHO problem:

Drawing:



Variables:

E	energy
K	Kinetic energy
x	Position
v	velocity or speed
m	mass of the object
k	spring stiffness constant

Basic Equations:

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ U_s &= \frac{1}{2}kx^2 \end{aligned}$$

Symbolic Answer

You might say, this is an easy problem, we know from last time that

$$v_{\max} = x_{\max}\omega$$

but let's find this again using the ideas of energy. In PH121 we often found that using energy made problems easier, so this might be worth a little more work now. The total mechanical energy is

$$E = K + U_s + U_g$$

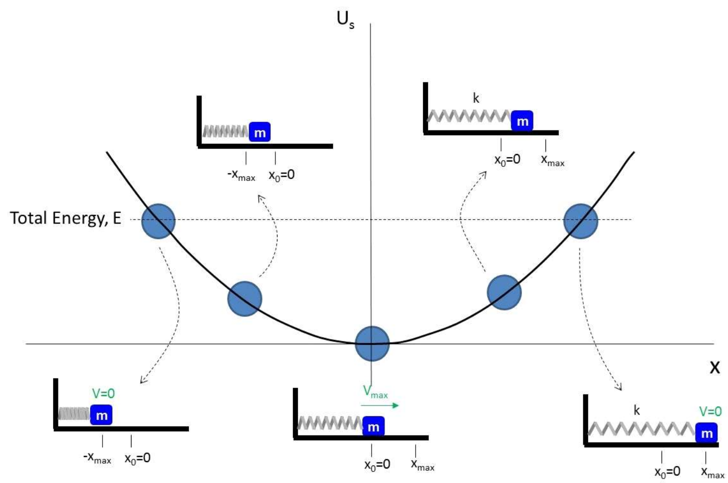
Let's say that our oscillator is moving horizontally just like last time, and that we define the center of mass of our object so that it's y-position is right at $y = 0$ so our object has zero gravitational potential energy.

$$E = K + U_s + 0$$

And let's say we have no friction, so that we can say that energy is conserved. We set the initial and final energies equal to each other.

$$\begin{aligned} E_i &= E_f \\ K_i + U_{si} &= K_f + U_{sf} \\ \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 \end{aligned}$$

And let's start with our initial condition being where the spring is stretched to $x_{\max}$. Then at that moment, $v_i = 0$. And let's take our final position when the mass is moving as fast as possible (because that is what we are looking for, $v_{\max}$).



We know from our PH121 experience that this will be right when $U_{sf} = 0$ (so all the energy is kinetic). Then

$$0 + \frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv_{\max}^2 + 0$$

or

$$\frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx_{\max}^2$$

We can solve for $v_{\max}$

$$v_{\max}^2 = \frac{kx_{\max}^2}{m}$$

$$v_{\max} = x_{\max}\sqrt{\frac{k}{m}}$$

But is this what we wanted? We expected that

$$v_{\max} = x_{\max}\omega$$

And this is what we got so long as

$$\omega = \sqrt{\frac{k}{m}}$$

And this is always true for a mass on a spring. We don't need a numeric answer, This is reasonable (just what we expect) and the units check.

Let's look at kinetic and potential energy as a function of time. For our Simple Harmonic Oscillator (SHO) we know the velocity as a function of time,

$$v(t) = -\omega x_{\max} \sin(\omega t + \phi)$$

so the kinetic energy as a function of time must be

$$\begin{aligned} K &= \frac{1}{2} m \left(-\omega x_{\max} \sin(\omega t + \phi)\right)^2 \\ &= \frac{1}{2} m \omega^2 x_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

and now we know that $\omega = \sqrt{k/m}$, so we can write the kinetic energy as

$$\begin{aligned} K &= \\ &= \frac{1}{2} m \left(\sqrt{\frac{k}{m}}\right)^2 x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} m \frac{k}{m} x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} k x_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

As the spring is stretched or compressed we store energy as spring potential energy. The potential energy due to a spring acting on our SHO (mover mass) is given by (from your PH121 class)

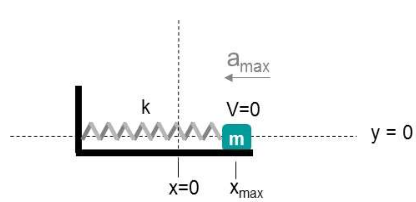
$$U_s = \frac{1}{2} k x^2 \tag{2.3}$$

For our SHO we also know the position as a function of time

$$x(t) = x_{\max} \cos(\omega t + \phi_o)$$

So, the potential energy as a function of time must be

$$U_s = \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi)$$



Let's say, again, that our oscillator is moving horizontally, and that we define the center of mass of our object so that it's y-position is right at $y = 0$ so our object has zero gravitational potential energy

$$U_g = mgy = 0$$

so the total mechanical energy is given by

$$E = K + U_s$$

which we can write out as

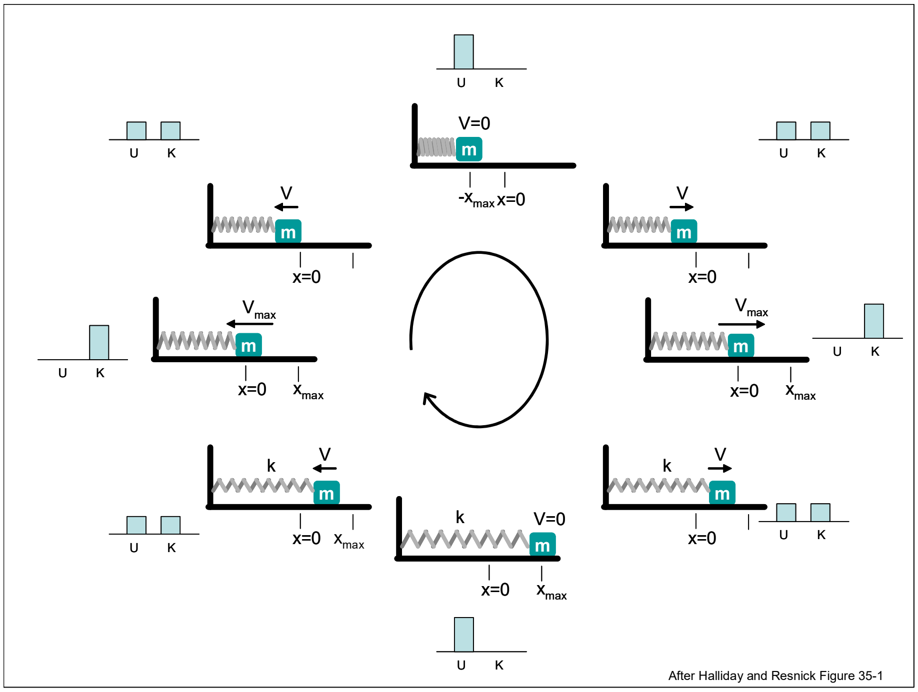
$$\begin{aligned} E &= K + U_s \\ &= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi) + \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi) \\ &= \frac{1}{2}kx_{\max}^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) \\ &= \frac{1}{2}kx_{\max}^2 \end{aligned}$$

Where we have used the trig identity that $\sin^2(\theta) + \cos^2(\theta) = 1$.

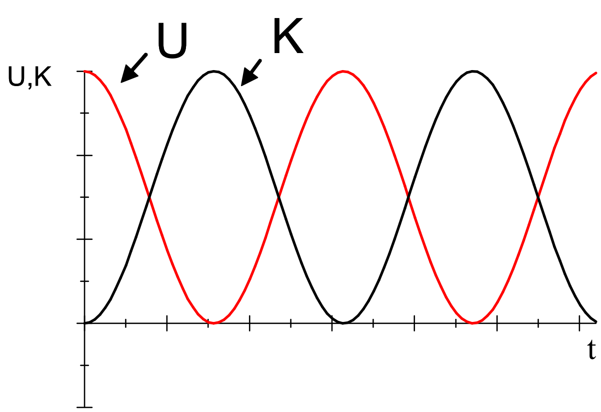
It tells us that if there are no loss mechanisms (e.g. no friction) then the energy in a harmonic oscillator never changes. And we remember that we would say that energy is conserved for such a situation, which is not surprising because we have already used conservation of energy for springs in PH121 and in the previous example. But let's give a new name to a quantity that does not change. Let's call it a *constant of motion*. So we can make a statement about the total energy for our SHO.

The total mechanical energy of an ideal SHO is a constant of motion

If we plot the amount of kinetic and potential energy for an oscillator we might find something like this:



Note that the kinetic and potential energy are out of phase with each other. If we plot them on the same scale (for the case $\phi = 0$) we have



Let's try another problem using energy of a SHO.

2.3 Mathematical Representation of Simple Harmonic Motion

We have a mathematical representation of simple harmonic motion from looking at our graph of position vs. time. But as a problem, let's use math and what we know from PH121 to show that our equation

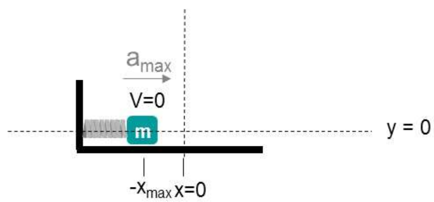
$$x(t) = x_{\max} \cos(\omega t)$$

must be right.

Recall from PH 121

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.4}$$

and let's assume we have SHO moving only in the x-direction.



Further assume the surface the object rests on is frictionless. Also let's say that the equilibrium position $x_{em} = 0$. Then we can write Newton's second law as

$$\begin{aligned} F_{net_x} &= ma_x = -kx \\ m\frac{d^2x}{dt^2} &= -kx \end{aligned} \tag{2.5}$$

We have a new kind of equation. If you are taking this freshman class as a... well... freshman, you may not have seen this kind of equation before. It is called a differential equation. The solution of this equation is a function or functions that will describe the motion of our mass-spring system as a function of time. It says that the way the object moves is an equation where the second derivative is almost the same as the original function. The only difference is some constants that are multiplied.

It is this function that we want, so let's see how we can find it.

Start by getting all the constants on the same side of the equation.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

It would be tidier if we defined a quantity ω as

$$\omega^2 = \frac{k}{m} \tag{2.6}$$

why define ω^2? Think of our previous example. We found that $\omega = \sqrt{k/m}$. This is just the square of what we found in our example. Then we can write our differential equation as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{2.7}$$

We need a function who's second derivative is the negative of itself with just a constant out front. From Math 112 we know a few of these

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ x(t) &= A\sin(\omega t + \phi_o) \end{aligned}$$

where A, ω, and ϕ_o are constants that we must find. Let's choose the cosine function and explicitly take it's derivatives to see if this function does solve our differential equation

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \qquad (2.8) \\ \frac{dx(t)}{dt} &= -\omega A\sin(\omega t + \phi_o) \\ \frac{d^2x(t)}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

Let's substitute these expressions into our differential equation for the motion

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ -\omega^2 A\cos(\omega t + \phi_o) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

As long as the constant ω^2 is our $\omega^2 = k/m$ we have a solution. We could have found that

$$\omega^2 = \frac{k}{m}$$

by solving this differential equation, but it might not have been as meaningful that way. We can identify ω as the angular frequency.

$$\omega = 2\pi f$$

Thus

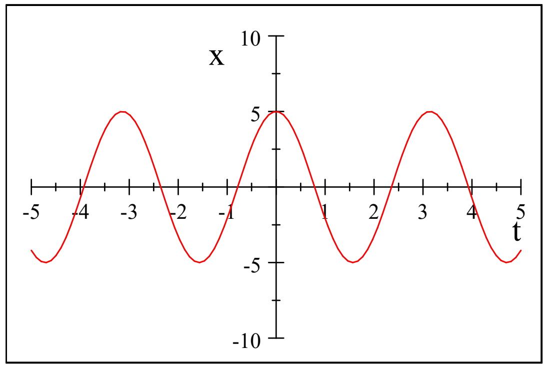
$$\omega = \sqrt{\frac{k}{m}} = 2\pi f \tag{2.9}$$

This says that if the spring is stiffer, we get a higher frequency, or if the mass is larger, we get a lower frequency.

We still don't have a complete solution, because we don't know A and ϕ_o. We recognize ϕ_o as the initial phase angle. We will have to find this by knowing the initial conditions of the motion. A is the amplitude. That must be the maximum displacement $x_{\max}$. Let's look at a specific case

$x_{\max} = 5$
$\phi_o = 0$
$\omega = 2$

(2.10)



We can easily see that the amplitude A corresponds to the maximum displacement x_{max}. (how would you prove this?). We know from trigonometry that a cosine function has a period T.

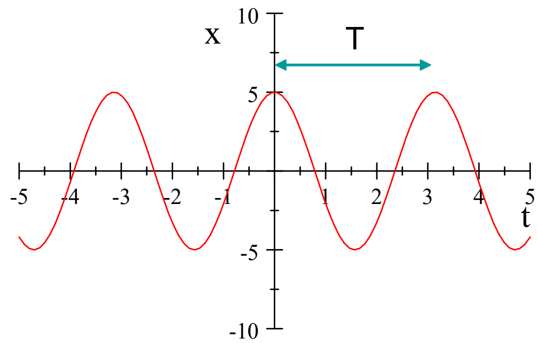


Figure 2.1:

The period is related to the frequency

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{2.11}$$

We can write the period and frequency in terms of our mass and spring constant

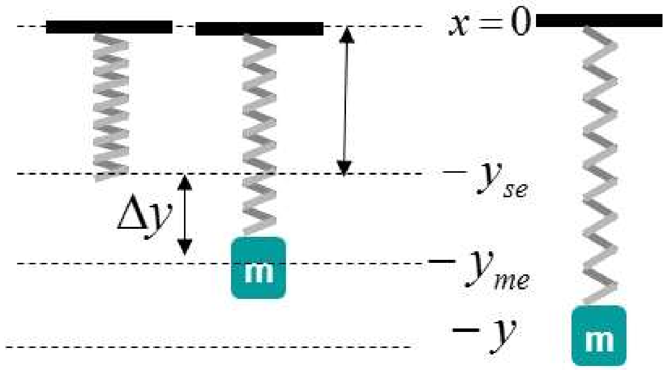
$$T = 2\pi\sqrt{\frac{m}{k}} \tag{2.12}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.13}$$

From our graph we were able to complete our problem. But more importantly we have two new equations for T and f that will help us solve more problems.

2.3.1 Hanging Springs

Let's do a problem. We now know we have forces involved with our SHOs. So let's do a force problem. In class so far, I have always hung our masses and springs, but we have used horizontal systems for calculations. Let's find the equation of motion for a mass hanging from a spring.



From observation, we would guess that we can just choose the new equilibrium point to be $y = 0$ and use the same equation

$$y(t) = y_{\max} \cos(\omega t - \phi_o)$$

let's see if that is true. Start with a free body diagram for the mass (the hanging mass is our mover, the spring is part of the environment). There are two forces acting on the mass. The force due to gravity, and the force due to the spring.

$$\Sigma F_y = ma_y = S_{ms} - W_{mE}$$

We know the form of these forces

$$\begin{aligned} S_{ms} &= -k\Delta y \\ W_{mE} &= -mg \end{aligned}$$

but we need to carefully choose our origin. Let's try the top of the spring where it attaches to the stand. If the mass just hangs there we would expect the spring to stretch to an equilibrium length y_e and any other motion would either shorten or lengthen the spring. We can write our force equation as

$$\begin{aligned} ma_y &= S_{ms} - W_{mE} \\ &= -k(y - y_e) - mg \end{aligned}$$

where y can be positive or negative. If $y = 0$ and the mass is just sitting there, not oscillating, there is no acceleration. Then the stretched length is just y_e. In this case

$$\begin{aligned} m\left(0\right) &= k\left(0-y_{e}\right)-mg \\ ky_{em} &= mg \end{aligned}$$

But now let's let our mass move again. We can substitute our stationary mass answer this into the previous equation for a moving mass

$$\begin{aligned} ma &= -k\left(y-y_{e}\right)-mg \\ &= ky_{e}-ky-mg \\ &= mg-ky-mg \\ &= -ky \end{aligned}$$

This gives a net force for the system of

$$F_{net}=-ky$$

It is as though the system were horizontal with no gravitational force and only a spring force. We can see that we are justified in claiming that we could simply choose the origin at the distance y_{em} from the top of the spring, and we can use the equation

$$y\left(t\right)=y_{\max}\cos(\omega t-\phi_{o})$$

as our equation of motion.