Chapter 3

More Oscillators, Forces and Friction 1.15.4 1.15.5 1.15.6

You have probably wondered if anyone actually uses mass-spring systems. And we do. They were more common in the past where springs were used to store energy (U_s) to run clocks and toys and machines. But simple harmonic motion can describe other systems as well. You have probably seen an old fashioned clock with a pendulum. A pendulum almost experiences simple harmonic motion. Let's see how this works.

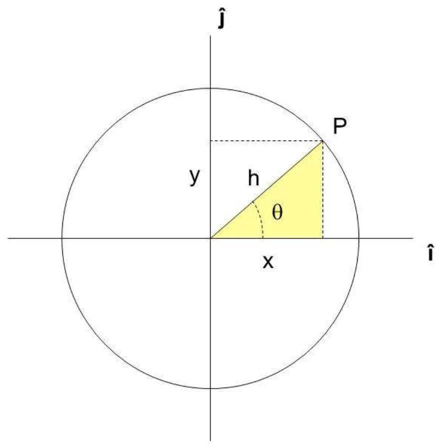
Fundamental Concepts

- Relationship between Simple Harmonic Motion and circular motion.
- Pendula
- Damping
- Driving
- Resonance

3.1 Comparing Simple Harmonic Motion with Uniform Circular Motion

You might have objected to our use of ω as angular frequency. Didn't ω mean angular speed in PH121?

That circular motion and SHM are related should not be a surprise once we found the solutions to the equations of motion were trig functions. Recall that the trig functions are defined on a unit circle

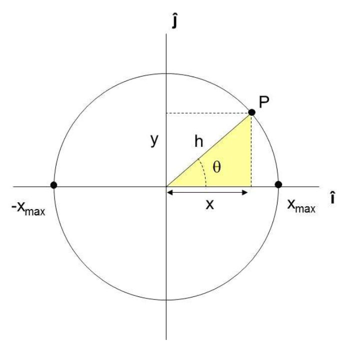


$$\tan\theta = \frac{x}{y} \tag{3.1}$$

$$\cos\theta = \frac{x}{h} \tag{3.2}$$

$$\sin\theta = \frac{y}{h} \tag{3.3}$$

Let's relate this to our SHM equations of motion.



Look at the projection x of the point P on the x axis. This is just the x-component of the position! Lets follow this projection as P travels around the circle. We find the projection ranges from $-x_{\max}$ to $x_{\max}$. If we watch closely we find the projection's velocity is zero at the extreme points and is a maximum in the middle. This projection is given as the cosine of the vector from the origin to P. It is just taking the x-component! This projection, indeed fits our SHO solution.

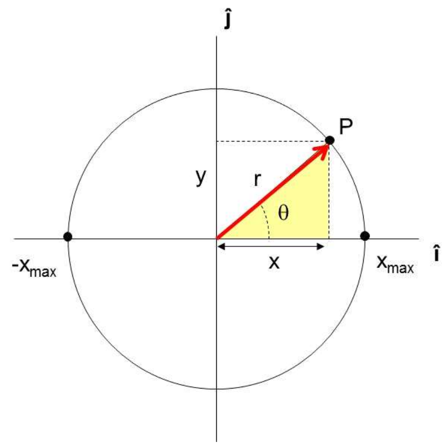
Now lets define a projection of P onto the y axis. Again we have SHM, but this time the projection is a sine function. Because

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta) \tag{3.4}$$

we can see that this is just a SHO that is $90\,^\circ = \frac{\pi}{2}$ rad out of phase. It is probably worth recalling that a projection of one vector on another can be expressed by a dot product. We could express our length x as

$$x = \overrightarrow{\mathbf{r}} \cdot \hat{\imath} = r\cos\theta$$

where r is the radius of the circle.



We can see that when $\theta = 0$ we have $x = r$ and this will be the largest x value, so $r = x_{\max}$.

So by projecting circular motion onto the x-axis

$$x(t) = x_{\max}\cos(\theta)$$

But θ changes in time. We can recall from our PH121 or Dynamics experience that the angular speed

$$\omega = \frac{\Delta\theta}{\Delta t}$$

or, if we agree to start from $\theta_i = 0$ and $t_i = 0$,

$$\omega = \frac{\theta}{t}$$

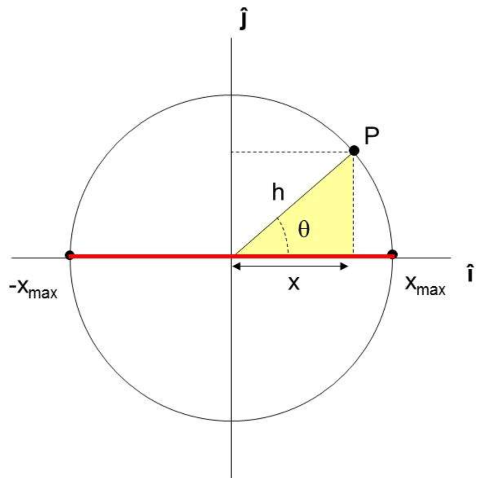
so

$$\theta = \omega t$$

then we have

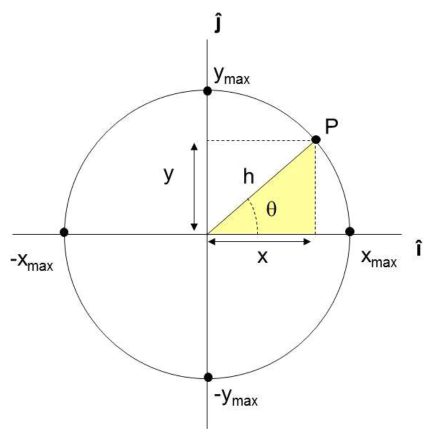
$$x(t) = x_{\max}\cos(\omega t)$$

Which is just our equation for SHM. Now we can see why we used "ω" for both angular speed and angular frequency. Really they are very related. Both tell us something about how fast a cyclic event happens. For motion around a circle, one is how fast the point P goes around the circle, and the other is how often the projection goes back and forth. It makes sense that these have to be the same rate.



The projection of circular motion onto the x-axis gives simple harmonic motion.

Let's go back to our projection on the y-axis.



We found that we can describe this projection as

$$y(t) = y_{\max} \sin(\omega t)$$

We will choose the cosine function, but from our trig experience it should be clear that these projections are equivalent, just 90° out of phase

$$\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp \sin\theta$$

So

$$x(t) = x_{\max} \cos(\omega t + \phi_o)$$

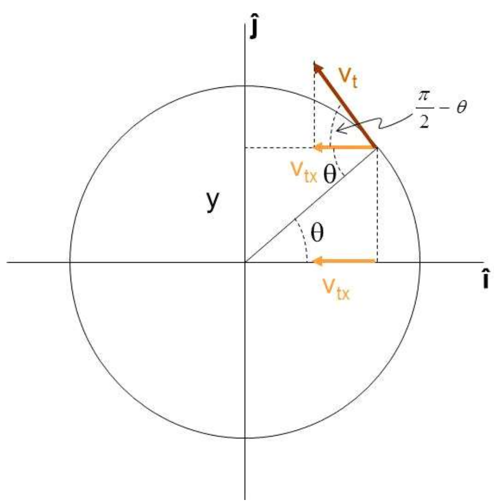
could be a sine function if $\phi_o = \pm\pi/2$. In this way we have incorporated both possibilities into one function.

Note that this is just the function we guessed from our observation!

We see that uniform circular motion can be thought of as the combination of two SHOs, with a phase difference of $\pi/2$ rad.

The angular velocity is given by

$$\omega = \frac{v}{r} \tag{3.5}$$



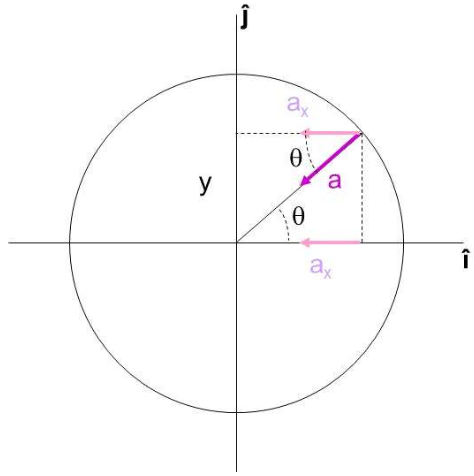
A particle traveling on the x-axis in SHM will travel from $x_{\max}$ to $-x_{\max}$ and from $-x_{\max}$ to $x_{\max}$ (one complete period, T) while the particle traveling with P makes one complete revolution. Thus, the angular frequency ω of the SHO and the angular velocity of the particle at P are the same. The magnitude of the tangential velocity is then

$$v_t = \omega r = \omega x_{\max} \tag{3.6}$$

and the projection of this velocity onto the x-axis is

$$v_{tx} = -\omega x_{\max} \sin(\omega t + \phi_o) \tag{3.7}$$

Which is just what we expected from our earlier observation!



The centripetal acceleration of a particle at P is given by

$$a_c = \frac{v_t^2}{r} = \frac{v_t^2}{x_{\max}} = \frac{\omega^2 x_{\max}^2}{x_{\max}} = \omega^2 x_{\max} \tag{3.8}$$

The direction of the acceleration is inward toward the origin. Of course, we just want the x-component of this, so again we make a projection. If we project this onto the x-axis we have

$$a_x = -\omega^2 x_{\max} \cos(\omega t + \phi) \tag{3.9}$$

Again this is just want we expected from our observation.

So now we have shown that our set of equations

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t) \\ v(t) &= -\omega x_{\max} \sin(\omega t) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t) \end{aligned} \tag{3.10}$$

is correct for our harmonic oscillator.

3.2 Our first problem type: Simple Harmonic Motion

You have probably thought by now that we have a new problem type. We can call it the *simple harmonic motion* problem type or SHM. The equations we have so far for this problem type are

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max} \sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t + \phi_o) \end{aligned} \tag{3.11}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

$$U_s = \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi)$$

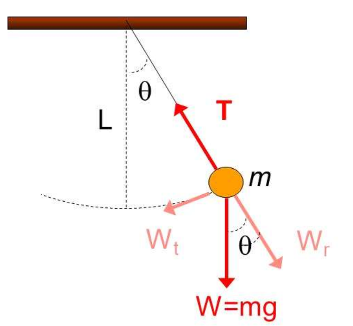
$$K = \frac{1}{2} m \omega^2 x_{\max}^2 \sin^2(\omega t + \phi)$$

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{3.12}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3.13}$$

We have used energy to describe simple harmonic motion, and we have found our equation of motion using a differential equation for mass-spring systems. And we have simple harmonic motion in the y-direction including the weight force due to gravity for mass-spring systems. In our next lecture, we will study simple harmonic motion for different systems, pendula, and other things. It turns out that SHM is a good model for many different systems.

3.3 The Simple Pendulum



A simple pendulum is a mass on a string. The mass is called a "bob." Usually we study the motion of the pendulum bob, so let's consider the pendulum the mover object. A simple pendulum bob exhibits periodic motion, but not exactly simple harmonic motion. The forces on the bob are $\overrightarrow{\mathbf{W}}$, and $\overrightarrow{\mathbf{T}}$ the tension on the string. The tangential component of W is always directed toward $\theta = 0$. This is a restoring force!

Let's call the path the bob takes "s." Then from Jr. High geometry we recall[1]

$$s = L\theta \tag{3.14}$$

We will use a the cylindrical or rtz coordinate system. The radial axis is directed along the string. The tangential direction is along the circular path the bob takes and is always tangent to the path. In this coordinate system, we can solve for the part of the force directed along the path. This is the restoring part of the net force. Remember from Newton's second law the tangential and radial components of the force are

$$\begin{aligned} F_t &= ma_t \\ F_r &= ma_r \end{aligned}$$

and

$$a_t = \frac{d^2 s}{dt^2}$$

Let's write out Newton's second with the sum of the forces part.

$$\begin{aligned} ma_t &= -W\cos(90 - \theta) \\ -ma_r &= -T + W\sin(90 - \theta) \end{aligned}$$

then, using a trig identity (but only a small one)

$$\begin{aligned} a_t &= -\frac{W}{m}\sin(\theta) \\ &= -g\sin\theta \end{aligned}$$

We have two expressions for $a_{t.}$ We can set them equal

$$\frac{d^2 s}{dt^2} = -g\sin\theta \tag{3.15}$$

Remember that $s = L\theta$ so we could write the left hand side as

$$\frac{d^2 s}{dt^2} = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2}{dt^2}(\theta)$$

then equation (3.15) becomes

$$L\frac{d^2}{dt^2}(\theta) = -g\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

[1] Really I did not recall this, I have to look it up every time, but s is called the *arclength*.

This is a differential equation much like our differential equation for a harmonic oscillator,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

except it has a sine function in it. But, if we take θ as a very small angle, then

$$\sin(\theta) \approx \theta \tag{3.16}$$

This approximation has a name, it is called the "small angle approximation."

In this approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

and we have a differential equation we recognize! If we compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

we see that it is a match if

$$\omega^2 = \frac{g}{L} \tag{3.17}$$

we have all the same solutions for θ that we found last time for x. Since ω changed, the frequency and period will now be in terms of g and L.

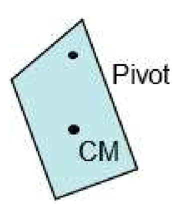
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.18}$$

For a pendulum that oscillates only over small angles, the period and frequency depend only on L and g!

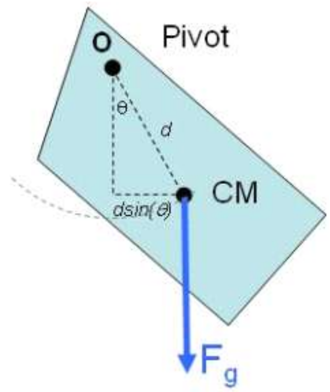
The analysis we just did for the pendulum we can do for other simple harmonic oscillators (or near simple harmonic oscillators). Let's try a few.

3.3.1 Physical Pendulum

Usually when we build a pendulum, we assume the string is so small compared to the bob, that we can ignore it's mass. What if this is not true? Suppose we build a pendulum by making a large solid object swing from one point. Can we describe it's motion?



Let's pull it to the right



then we consider that because of F_g we will have a torque about an axis through O. Last pendulum we used Newton's Second Law. This time let's use Newton's second law for rotation.

$$\tau = \mathbf{r} \times \mathbf{F}$$

In our case this is

$$\begin{aligned} \tau &= \mathbf{d} \times \mathbf{F}_g \\ &= -F_g d \sin\theta \\ &= -mgd \sin\theta \end{aligned}$$

Remember that an extended body has a moment of inertia, $\mathbb{I}$. Remember also that angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

From the rotational form of Newton's Second Law for rotation.

$$\Sigma\tau = \mathbb{I}\alpha$$

we can write

$$-mgd \sin\theta = \mathbb{I}\frac{d^2\theta}{dt^2}$$

Getting all the constants together gives

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}} \sin(\theta)$$

and again if we let θ be small so that $\sin(\theta) \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}}\theta$$

which we can compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

and we can see that we have the same differential equation if

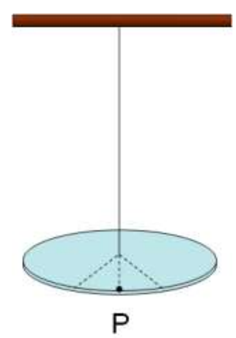
$$\omega^2 = \frac{mgd}{\mathbb{I}} \tag{3.19}$$

In this case

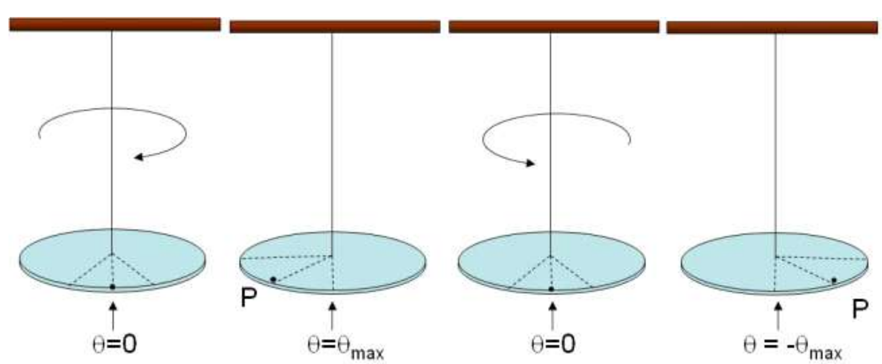
$$T = 2\pi\sqrt{\frac{\mathbb{I}}{mgd}} \tag{3.20}$$

3.3.2 Torsional Pendulum

Physics majors and those taking PH220 in the future, you will see a torsional pendulum. Mechanical engineering majors might use a torsional pendulum. A torsional pendulum is made by suspending a rigid object from a wire.



In this case, a rigid disk. The object can rotate. But wires don't like to twist. The spring-like molecular bonds resist the twisting.



The twisted wire exerts a restoring torque on the body that is proportional to the angular position (sound familiar).

$$\boldsymbol{\tau} = -\kappa\theta$$

This looks like a greek version of $F = -kx$! Again, let's use Newton's Second Law for rotation.

$$\begin{aligned}\Sigma\tau &= \mathbb{I}\alpha \\ &= \mathbb{I}\frac{d^2\theta}{dt^2}\end{aligned}$$

so

$$\begin{aligned}\mathbb{I}\frac{d^2\theta}{dt^2} &= -\kappa\theta \\ \frac{d^2\theta}{dt^2} &= -\frac{\kappa}{\mathbb{I}}\theta\end{aligned}$$

Once again we have our favorite differential equation so long as

$$\omega^2 = \frac{\kappa}{\mathbb{I}} \tag{3.21}$$

which makes the period of the oscillation

$$T = 2\pi\sqrt{\frac{\mathbb{I}}{\kappa}} \tag{3.22}$$

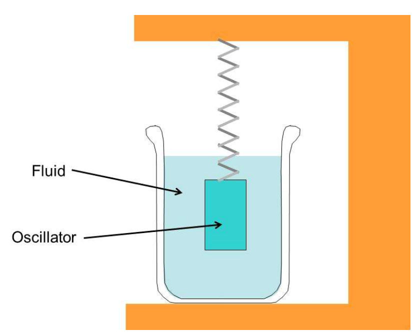
So far we have used only idea springs or wires or frictionless pendula with no air drag. This made the equations easier. But what if there is friction of some sort? Let's deal with this next.

3.4 Damped Oscillations

You remember friction from PH121. So far we have only allowed frictionless oscillators to make the math easy. But what if we do have friction? To investigate this,suppose we add in another force

$$\mathbf{D} = -b\mathbf{v} \tag{3.23}$$

This force is proportional to the velocity. This a dissipative (friction-like) force typical of what we find when we moves objects through viscus fluids. This is a drag force, but a more extreme drag force than we used in PH121 where we only had air to make the drag force.



We call b the damping coefficient and it depends on how much friction the fluid can supply. Now, from Newton's second law,

$$\Sigma F = -kx - bv_x = ma$$

We can write the acceleration and velocity as derivatives of the position just like we have done before

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is another differential equation! But it is harder to guess its solution, and finding that solution is a subject for a differential equations class like M316 or PH332, so we won't learn how to find the solution here, but we can use the results from our trusted colleagues in the math department[2]. Here is the solution:

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos(\omega t + \phi_o) \tag{3.24}$$

which looks simple enough, but now we have the added complication that ω is more complex

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{3.25}$$

so we get quite a mess if we write equation (3.24) with this new ω.

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) t + \phi_o\right) \tag{3.26}$$

To see what this solution means, we should study three cases:

1. the damping force is small: ($bv_{\max} < kx_{\max}$) The system oscillates, but the amplitude is smaller as as time goes on. We call this "underdamped."

2. the damping force is large: ($bv_{\max} > kx_{\max}$)The system does not oscillate. we call this "overdamped." We can also say that $\frac{b}{2m} > \omega_o$ (after we define ω_o below)

3. The system is "critically damped" (see below).

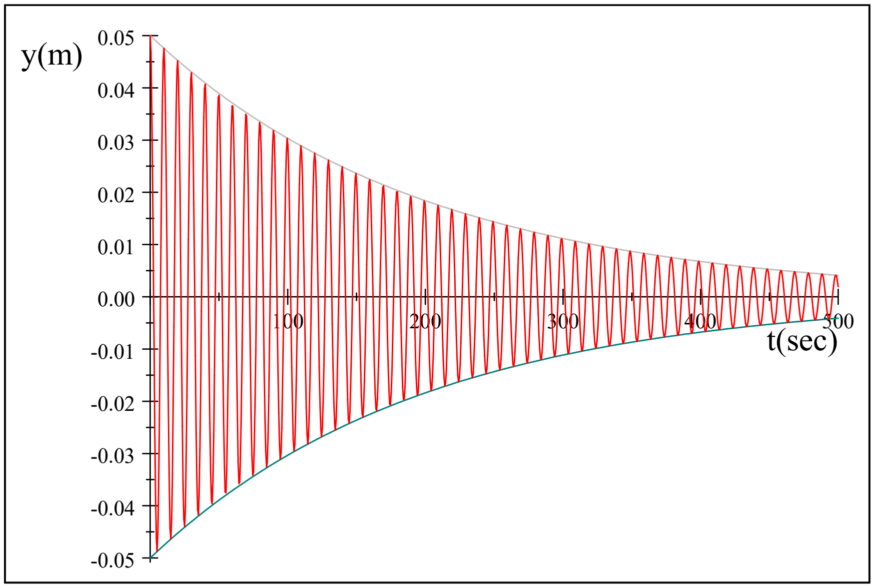
Let's look at an example. Suppose we have an oscillator with the following characteristics:

1.

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.005\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$

[2] That is, until you finish M316, then you will know how to do this kind of problem yourself!

Graphing the equation of motion $x(t)$, we we get a graph that looks like this



The gray lines are given by

$$\pm x_{\max} e^{-\frac{b}{2m}t}$$

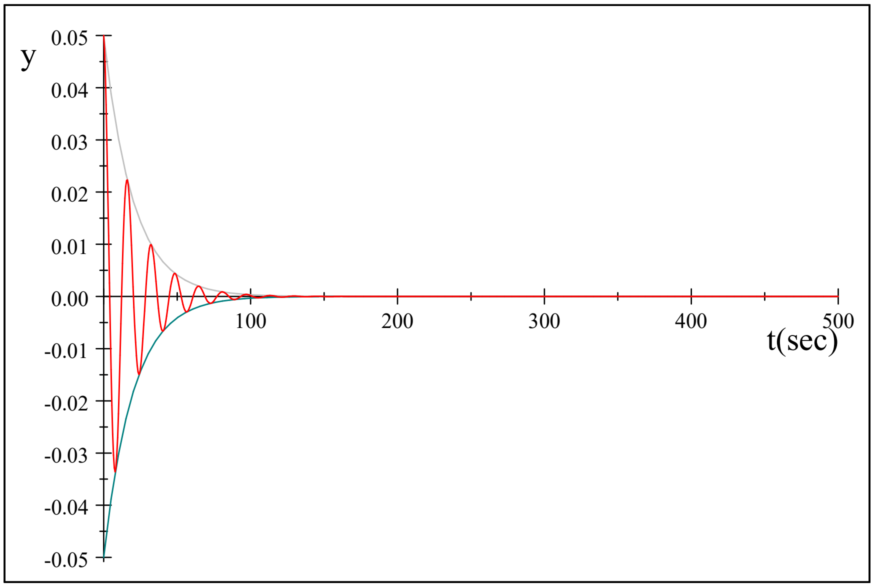
Notice that this quantity is as a collection of terms multiplies the cosine part. It is marked in curly braces blow

$$x(t) = \left\{ x_{\max} e^{-\frac{b}{2m}t} \right\} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \right) t + \phi_o \right)$$

We know that the part of the equation that multiplies the cosine part is the amplitude. But now the amplitude is more than just $x_{\max}$. And notice that the amplitude $\left\{ x_{\max} e^{-\frac{b}{2m}t} \right\}$ changes with time. The gray lines in the figure show how the amplitude changes. We call this the *envelope* of the curve. The oscillation fits within the gray lines (like an old fashioned letter fits inside a paper envelope).

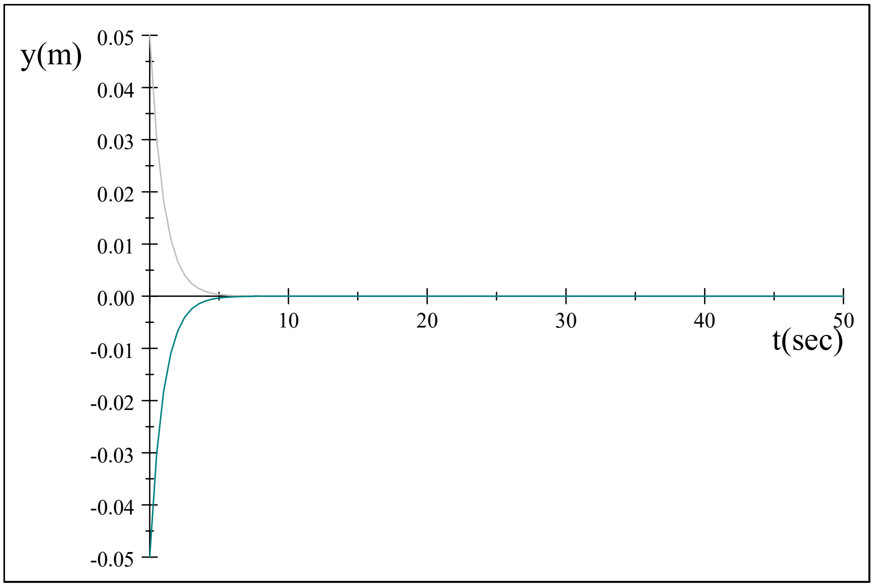
Now let's change b to a larger value

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.05\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$



we see we have less oscillation. The envelope has become more restrictive, making the oscillation die out more quickly. This is a little bit like going over a bump in your car. The car may go up and down a few times, but not many. Now let's increase b even more.

$$\begin{array}{|l|}\hline x_{\max} = 5\,\mathrm{cm} \\ \hline b = 0.5\frac{\mathrm{kg}}{\mathrm{s}} \\ \hline k = 0.2\frac{\mathrm{N}}{\mathrm{m}} \\ \hline m = .5\,\mathrm{kg} \\ \hline \phi_o = 0 \\ \hline \end{array} \tag{3.27}$$



What happened?

When the damping force gets bigger, the oscillation eventually stops. Only the exponential decay is observed. This happens when

$$\frac{b}{2m} = \sqrt{\frac{k}{m}}$$

When that is true,

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0$$

We call this situation *critically damped*. We are just on the edge of oscillation. We define

$$\omega_o = \sqrt{\frac{k}{m}}$$

as the *natural frequency* of the system. Then the value of b that gives us critically damped behavior is

$$b_c = 2m\omega_o$$

When $\frac{b}{2\pi} \geq \omega_o$ the solution in equation (3.24) is not valid! If you are a physicist or a mechanical engineer you will find out more about this situation in your advanced mechanics classes.