Chapter 4

Forces Oscillations and Waves 1.15.6

We studied oscillation with mass-spring systems and pendula. In today's lecture we will add in forces to make the oscillators start to move. Oscillation is a motion of a mass, but with another object (a spring or string) participating in the motion. The air in our classroom is an example of this type of motion in a way, but instead of one object on a spring, there are millions of objects, the air molecules. The molecules move randomly, but with a specific distribution of speeds. But what would happen if all the molecules moved together in the same direction (more or less) and at the same speed (more or less)? This is what we call wind! Of course we can have bulk motion of millions of objects, like wind. We are going to study an even more specific motion of millions of object that is not random like thermal motion. This specific motion of the objects we will call a *wave.*

Fundamental Concepts

1. Forced Oscillations
2. A wave requires a disturbance, and a medium that can transfer energy
3. Waves are categorized as longitudinal or transverse (or a combination of the two).

4.1 Forced Oscillations

We found in the last section that if we added a force like

$$\mathbf{F}_d = -b\mathbf{v}$$

our oscillation died out. An example would be a small child on a swing. You give them a push, but eventually they stop swinging.

Suppose we want to keep the child going? You know what to do, you push! But you have to push at just the right time.

Suppose we make a machine that can push the child on the swing. We could apply a periodic force like

$$F(t) = F_o \sin(\omega_f t)$$

where ω_f is the angular frequency of this new driving force and where F_o is a constant.

$$\Sigma F = F_o \sin(\omega_f t) - kx - bv_x = ma$$

1. When this system starts out, the solution is very messy. It is so messy that we will not give it in this class! (So maybe this isn't really a good way to drive a child on a swing). But after a while, a steady-state is reached. In this state, the energy added by our driving force $F_o \sin(\omega_f t)$ is equal to the energy lost by the drag force, and we have

   $$x(t) = A\cos(\omega_f t + \phi)$$

   our old friend! BUT NOW the amplitude is given by

   $$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2 - \omega_o^2\right)^2 + \left(\frac{b\omega_f}{m}\right)^2}}$$

   where

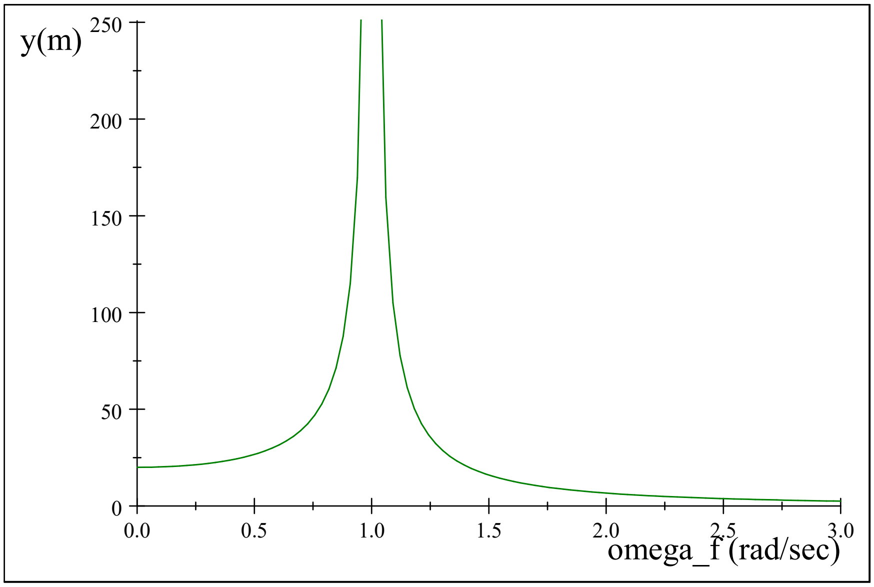
   $$\omega_o = \sqrt{\frac{k}{m}} \tag{4.1}$$

   is the natural frequency. This is the frequency we had before when we weren't pushing the oscillator. And recall that ω_f is the frequency of our force. So now our solution looks more like our original SHM solution except for the wild formula for A. In fact, it operates very like our SHM solution. But what does the new version of A mean?

   Lets look at A for some values of ω_f. I will pick some nice numbers for the other values.

$F_o = 2\,\mathrm{N}$
$b = 0.5\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.5\frac{\mathrm{N}}{\mathrm{m}}$
$m = 0.5\,\mathrm{kg}$
$\phi_o = 0$

   The graph looks like this:



Now let's calculate ω_o

$$\begin{aligned}\omega_o &= \sqrt{\frac{0.5\,\frac{\mathrm{N}}{\mathrm{m}}}{0.5\,\mathrm{kg}}} \\ &= \frac{1.0}{\mathrm{s}}\end{aligned}$$

Notice that right at $\omega_f = \omega_o$ our solution gets very big. This is called *resonance*. To see why this happens, think of the velocity for a simple harmonic oscillator

$$\frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o)$$

Our driving force is still

$$F(t) = F_o \sin(\omega t)$$

And remember that work is given by

$$w = \int \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}}$$

or just

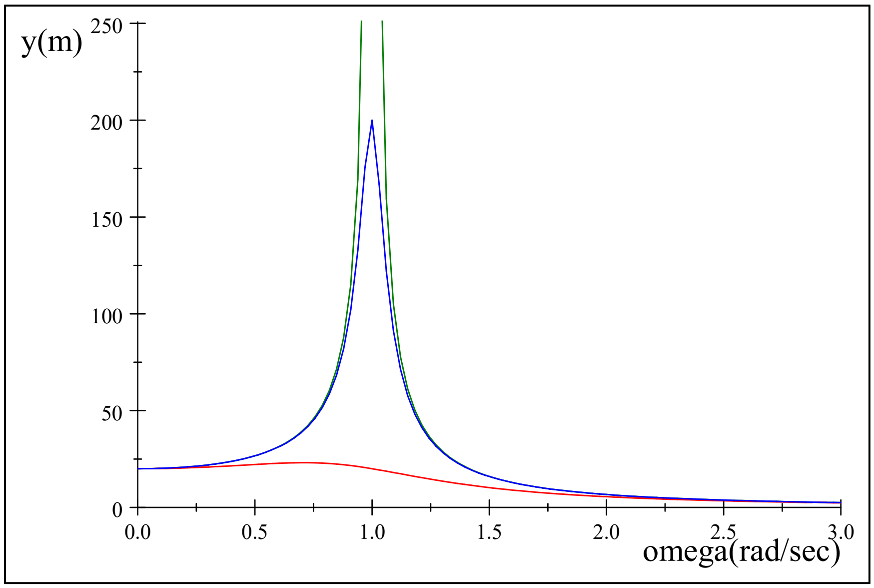
$$w = \overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}$$

if our force is constant. The rate at which work is done (power) is

$$\begin{aligned}\mathcal{P} &= \frac{\overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}}{\Delta t} = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{v}} &(4.2)\\ &= -F_o \omega A \sin(\omega t)\sin(\omega t + \phi_o) &(4.3)\end{aligned}$$

if F and v are in phase ($\phi_o = 0$), the power will be at a maximum! Think of pushing the child on the swing. You push in phase with the oscillation of the child. When you do this the amplitude of the swing get's bigger.

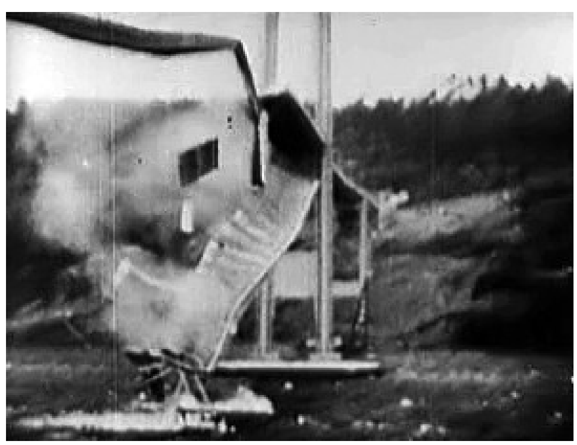
We can plot A for several values of b, our damping (friction) coefficient.



Green: b=0.005kg/s; Blue: b=0.05kg/s; Red b=0.01 kg/s

As $b \to 0$ we see that our resonance peak gets larger. In real systems b can never be zero, but sometimes it can get small. As $b \to$large, the resonance dies down and our A gets small.

Resonance can be great, it can make a musical instrument sound louder (more on this later). But it can also be bad. Here is a picture of the Tacoma Narrows Bridge.



Fall of the Tacoma Narrows Bridge (Image in the Public Domain)

The wind gusts formed a periodic driving force that allowed a driving harmonic oscillation to form. Since the bridge was resonant with the gust frequency, the amplitude grew until the bridge materials broke. Resonance can be a bad thing for structures.

4.2 What is a Wave?

Waves are organized motions in a group of objects. We will give a name to the group of objects, we will call a *medium*.

Spring Demo

4.2.1 Criteria for being a wave

Waves involve energy transfer, but in the case of waves the energy is transferred through space without transfer of matter. Winds transfers both energy and matter. So waves are different than wind. In a wave the objects don't move far from where they start. Think of an oscillation. The mass does move, but never gets too far away from its equilibrium position. Waves are very like this. This is a very specific kind of motion. To be a wave, the motion must have the following characteristics"

Spring Demo-marked part

1. some source of disturbance
2. a medium (group of objects) that can be disturbed
3. some physical mechanism by which the objects of the medium can influence each other

A wave made by a rock thrown into a pond will go out in all directions away from the place where the rock started the wave. This is a normal way that waves are formed. A disturbance starts the wave (the rock disturbs the water) and the energy from the disturbance moves away from the disturbance as a wave. But if we have a wave in a rope or string, the wave can't go in all directions because the string does not go in all directions.

Let's take on this one-dimensional case of a wave on a rope or string first. In the limit that the string mass is negligible we represent a one-dimensional wave mathematically as a function of two variables, position and time, $y(x,t)$.

4.2.2 Longitudinal vs. transverse

We divide the various kinds of waves that occur into two basic types:

Transverse wave: a traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation

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Longitudinal wave: a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation

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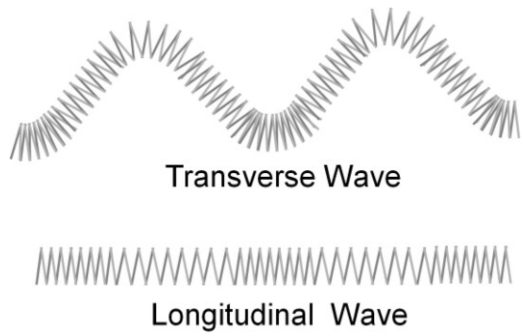
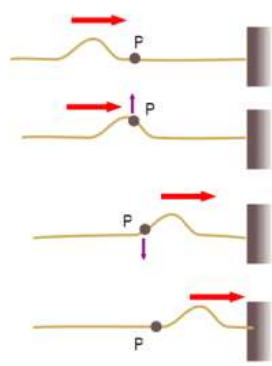


4.2.3 Examples of waves:

Let's look at some specific cases of wave motion.

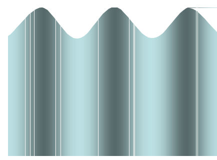
A pulse on a rope:



In the picture above, you see wave that has just one peak traveling to the right. We call such a wave a *pulse*. Notice how the piece of the rope marked P moves up and down, but the wave is moving to the right. This pulse is a transverse wave because the parts of the medium (observe point P) move perpendicular to the direction the wave is moving.

An ocean wave:

Of course, some waves are a combination of these two basic types. You may have noticed that in Physics we tend to define basic types of things, and then use these basic types to define more complex objects. Water waves, for example, are transverse at the surface of the water, but are longitudinal throughout the water.



Earthquake waves:

Earthquakes produce both transverse and longitudinal waves. The two types of waves even travel at different speeds! P waves are longitudinal and travel faster, S waves are transverse and slower.

4.3 Wave speed

We can perform an experiment with a rope or a long spring. Make a wave on the rope or spring. Then pull the rope or spring tighter and make another wave. We see that the wave on the tighter spring travels faster.

It is harder to do, but we can also experiment with two different ropes, one light and one heavy. We would find that the heaver the rope, the slower the wave. We can express this as

$$v = \sqrt{\frac{T_s}{\mu}}$$

where T_s is the tension in the rope, and μ is the linear mass density

$$\mu = \frac{m}{L}$$

where m is the mass of the rope, and L is the length.

The term μ might need an analogy to make it seem helpful. So suppose I have an iron bar that has a mass of 200 kg and is 2 m long. Further suppose I want to know how much mass there would be in a 20 cm section cut of the end of the rod. How would I find out?

This is not very hard, We could say that there are 200 kg spread out over 2 m, so each meter of rod has 100 kg of mass, that is, there is 100 kg/ m of mass per unit length. Then to find how much mass there is in a 0.20 m section of the rod I take

$$m = 100\frac{\mathrm{kg}}{\mathrm{m}} \times 0.20\,\mathrm{m} = 20.0\,\mathrm{kg}$$

The 100 kg/ m is μ. It is how much mass there is in a unit length segment of something In this example, it is a unit length of iron bar, but for waves on string, we want the mass per unit length of string.

If you are buying stock steel bar, you might be able to buy it with a mass per unit length. If the mass per unit length is higher then the bar is more massive. The same is true with string. The larger μ, the more massive equal string segments will be.

We should note that in forming this relationship, we have used our standard introductory physics assumption that the mass of the rope can be neglected. Let's consider what would happen if this were not true. Say we make a wave in a heavy cable that is suspended. The mass at the lower end of the cable pulls down on the upper part of the cable. The tension will actually change along the length of the cable, and so will the wave speed. Such a situation can't be represented by a single wave speed. But for our class, we will assume that any such changes are small enough to be ignored.

4.4 One dimensional waves

To mathematically describe a wave we will define a function of both time and position.

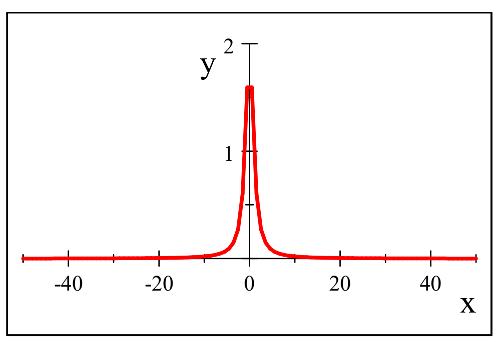
$$y\left(x,t\right) \tag{4.4}$$

Note, that this is new in our physics experience. Before we usually were concerned about only one variable at a time. For oscillation we had just $y\left(t\right)$ for example. But now we will be concerned about two variables, x, and t.

let's take a specific example[1]

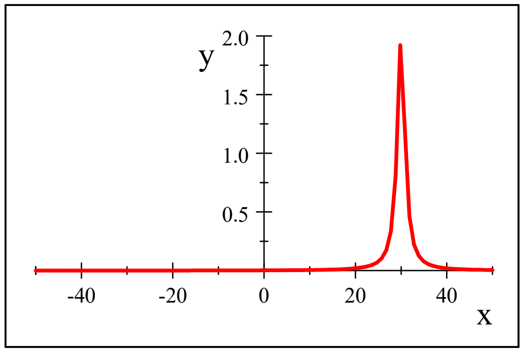
$$y = \frac{2}{\left(x - 3.0t\right)^2 + 1} \tag{4.5}$$

Let's plot this for $t = 0$



what will this look like for $t = 10$?

[1] This is not an important wave function, just one I picked because it makes a nice graphic example.



The pulse travels along the x-axis as a function of time. Note that there is a value for y for every x position and that these y values change for different times. That is what we mean by saying we have a function of both x and t.

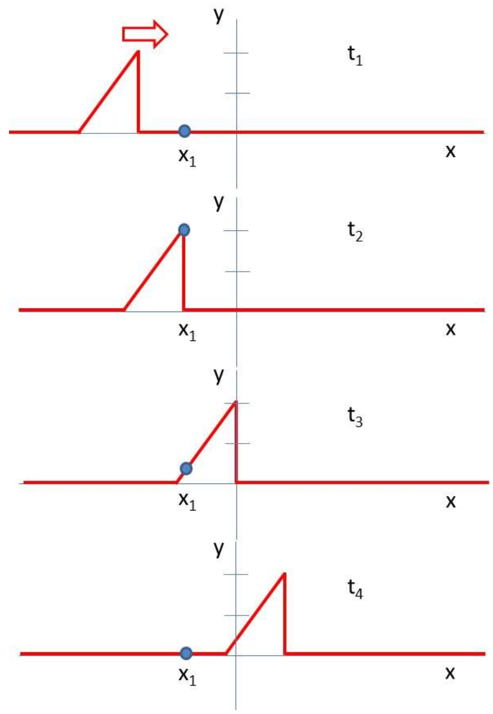
We denote the speed of the pulse as v, then we can define a function

$$y(x,t) = y(x - vt, 0) \tag{4.6}$$

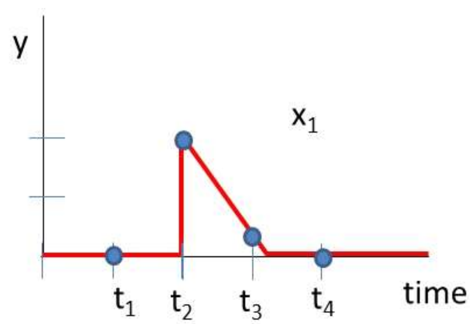
that describes a pulse as it travels. An element of the medium (rope, string, etc.) at position x at some time t, will have the displacement that an element had earlier at $x - vt$ when $t = 0$.

We will give $y(x - vt, 0)$ a special name, the *wave function*. It represents the y position, the transverse position in our example, of any element located at a position x at any time t.

Notice that wave functions depend on two variables, x, and t. It is hard to draw a wave so that this dual dependence is clear. Often we draw two different graphs of the same wave so we can see independently the position and time dependence. So far we have used one of these graphs. A graph of our wave at a specific time, t_o. This gives $y(x, t_o)$. This representation of a wave is very like a photograph of the wave taken with a digital camera. It gives a picture of the entire wave, but only for one time, the time at which the photograph was taken. Of course we could take a series of photographs, but still each would be a picture of the wave at just one time. Here is such a series of graphs at times t_1 through t_4.



The second representation is to observe the wave at just one point in the medium, but for many times. This is very like taking a video camera and using it to record the displacement of just one part of the medium for many times. You could envision marking just one part of a rope, and then using the video recorder to make a movie of the motion of that single part of the rope. We could then go frame by frame through the video, and plot the displacement of our marked part of the rope as a function of time. Such a graph is sometimes called a history graph of the wave. Here is such a graph for the position x_1.



To go from the time graphs to the history graph you observe what happens at the location x_1 for each of the times. Then plot those y positions on the y vs.

t graph. Then you must connect the points. This takes some thinking to make sure you connect them right (or a whole lot of points). But with practice, this is not hard and both view points are valuable ways to look at a wave.