Chapter 16

16 Images formed by Curved Mirrors and by Refraction 3.2.2, 3.2.3

Back in Newton's day, there were no electronic computers, or calculators, or mechanical adding machines. Early optics researchers did math with a pen and paper. This is one reason they liked small angle approximations. The approximation allowed them to do harder problems using easy math. And so long as the things they built worked, the approximations were good enough. We are going to use another approximation in this lecture. It is called the thin mirror approximation. It will make the math that describes mirrors, for say, telescopes, much easier. We will also introduce a way to draw curved mirrors and light that will tell you how the mirror works. We will use this drawing approach to do homework and examination problems. Professional optical designers use computer codes that do this drawing very accurately to help the optical engineer understand the system they are designing and to look for mistakes. So this drawing scheme is very useful. Because it makes optical systems so much easier to understand, let's start with the drawings and then work toward a mathematical description of thin mirrors.

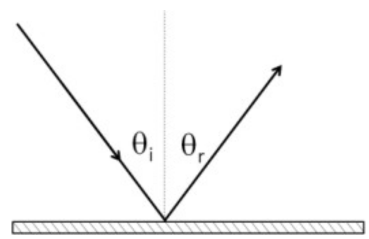
Fundamental Concepts

- Curved Mirrors can form images
- We can write the magnification in terms of the object and image distances $m = -\frac{d_i}{d_o}$
- For "thin" mirrors $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$
- The focal length of a "thin" mirror is. $f = \frac{R}{2}$

- We can describe how a mirror operates with just three easy-to-draw rays.
- A semi-infinite bump (single sided lens) can be described by the equation $\frac{1}{d_o} + \frac{1}{d_i} = \frac{(n_2 - n_1)}{R}$

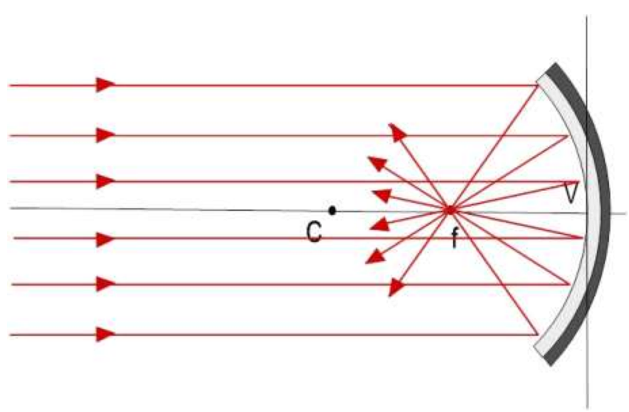
16.0.3 Concave Mirrors

Concave mirrors can form images. I'm sure you know that many telescopes are made with mirrors. We should see how this works. We recall the law of reflection



$$\theta_i = \theta_r$$

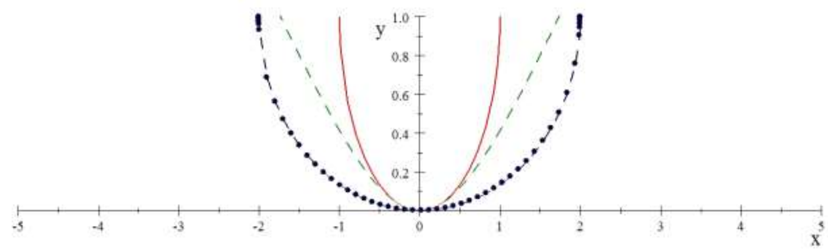
Armed with this, we can see what would happen if we curved the mirror surface. Each ray has a different normal due to the curvature of the mirror. The result is that parallel rays all meet at a spot on the axis.



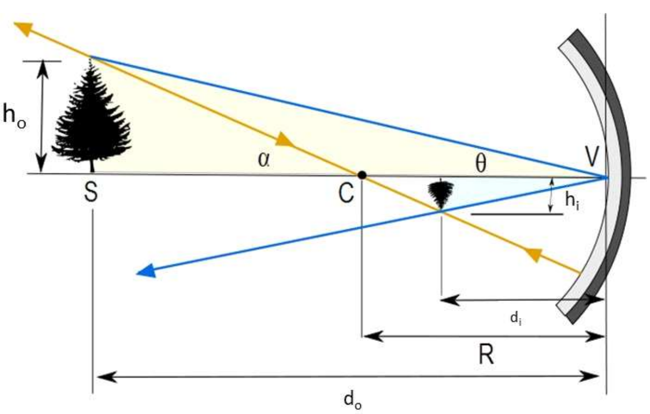
If we place something at this location, we could start a fire! We have just re-invented the solar cooker. Look at the point where all the rays meet. Let's give this special place a name. We call it the focal point. It is the place where the rays come together if they came into the mirror parallel.

16.0.4 Paraxial Approximation for Mirrors

The correct shape of a mirror to make all the parallel rays meet at the focal point is more like a parabola, but parabolas are hard to machine or build. Spherical shapes are relatively easy. So we often see spherical mirrors used where we should have parabolas. This will work so long as we allow only rays that strike the mirror not too far from the principal axis. We can see why this works if we plot a sphere and a parabola (and a hyperbola). For small deviations from the center, the shape of the functions all look alike.



We would expect the reflections to be similar under these circumstances. Using only the part of a mirror where the spherical and parabolic shapes are essentially the same is called the *paraxial approximation*. So, if we meet the criteria for the paraxial approximation, our spherical mirrors should work. Note that when you need the entire mirror, say, in a communications antenna, you must do better than a spherical approximation to the correct shape for your mirror. Your satellite dish is likely not a spherical section.



Like with our flat mirror, we will measure distances from the mirror surface (from point V in the figure). We can find the image location, d_i, by taking two rays. We could use any of billions of rays. But let's try to pick rays that are easy to draw. One convenient ray is the ray that passes through the center of curvature, C. This ray will strike the mirror surface at right angles and bounce back along the same path. The incidence angle will be zero, so the reflected angle must be zero by the law of reflection. That is the yellow ray in the figure.

Another convenient ray is the ray from the tip of the object to point V. This ray will bounce back with angle θ. Right at point V the mirror surface is perpendicular to the optic axis. This makes the bounce of the blue ray just like the a bounce from a flat mirror! That is easy to draw. Where these two reflected rays cross, we will find the image of the tip of our arrow. Knowing the shape of the arrow and that the bottom is on the axis, we can fill in the rest of the image.

We can calculate the magnification for this case. We use the gold triangle to determine that

$$\tan\theta = \frac{h_o}{d_o}$$

and the light blue triangle to determine that

$$\tan\theta = \frac{-h_i}{d_i}$$

And you might object to the negative sigh. But we want to indicate that the image is inverted by making it's sign negative. So because the image is upside down h_i is defined to be negative. To make our tangent equation work, we need to insert another minus sign. This gives us $\tan\theta$ instead of $-\tan\theta$ when h' is inverted (when h', itself, is negative). When we decide to make the sine of a quantity negative or positive just because we want it to be that way we cay it is negative (or positive) *by convention.* And we have done this here. It is just a convention. But we will use it. So our magnification would be

$$m = \frac{h_i}{h_o}$$

But we can write

$$\begin{aligned} h_o &= d_o \tan\theta \\ h_i &= -d_i \tan\theta \end{aligned}$$

so the magnification could be written as

$$m = \frac{h_i}{h_o} = \frac{-d_i \tan\theta}{d_o \tan\theta} = -\frac{d_i}{d_o}$$

which gives us the magnification in terms of how far away the object and image are.

16.1 Mirror Equation

We can further exploit this geometry to get a relationship between d_o, d_i, and R. Notice that

$$\tan\alpha = \frac{h_o}{d_o - R}$$

and that

$$\tan\alpha = \frac{-h_i}{R - d_i}$$

Then

$$\frac{h_o}{d_o - R} = \frac{-h_i}{R - d_i}$$

or

$$\frac{R - d_i}{d_o - R} = -\frac{h_i}{h_o}$$

We can use our magnification definition to replace h_i/h_o

$$\frac{R - d_i}{d_o - R} = \frac{d_i}{d_o}$$

And we have a useful equation, but not in the form you usually see it. Let's perform some algebra

$$\begin{aligned}
(R - d_i)\, d_o &= d_i\,(d_o - R) \\
-d_i d_o + R d_o &= p d_i - R d_i \\
R d_o + R d_i &= d_o d_i + d_i d_o \\
\frac{R d_i}{R d_o d_i} + \frac{R d_o}{R d_o d_i} &= \frac{2 d_o d_i}{R d_o d_i} \\
\frac{1}{d_o} + \frac{1}{d_i} &= \frac{2}{R}
\end{aligned}$$

We are close. But let's make one more change to this equation. And it will involve the focal point.

Focal Point

Remember that special place where if the light comes into the mirror parallel, the reflected rays all meet? We called that the focal point, f.

Now that we know the mirror equation, let's let d_o be very large (for example, let d_o be the distance to the Sun). Then the rays would come in parallel. And all the rays would come together at the focal point. We would get an image of the Sun right at the focal point! Mathematically this would be

$$\begin{aligned}
\frac{1}{\infty} + \frac{1}{d_i} &\approx \frac{2}{R} \\
0 + \frac{1}{d_i} &\approx \frac{2}{R} \\
\frac{1}{d_i} &\approx \frac{2}{R}
\end{aligned}$$

or

$$d_i \approx \frac{R}{2}$$

This is a special image point. But we really already know what to call this special place where parallel rays come together. We call it the *focal point* and the distance from the mirror to the point f is called the *focal length*. We see that, indeed

$$f = \frac{R}{2} \tag{16.1}$$

so we can write the mirror equation as

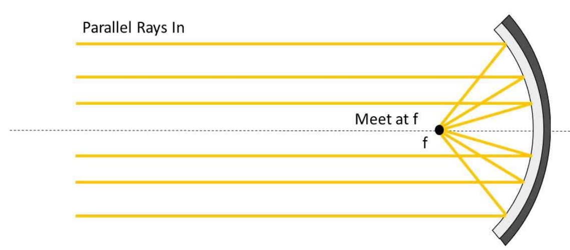
$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \tag{16.2}$$

This equation is very useful. Even when you use powerful optical design software it can form the beginning of your optical system design.

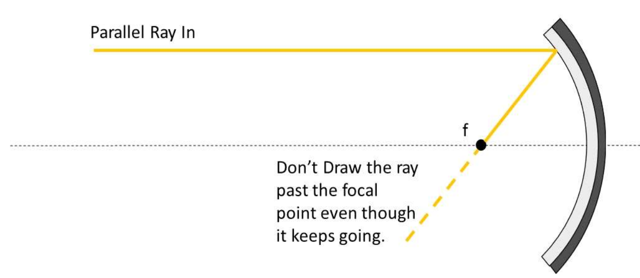
16.1.1 Ray Diagrams for Mirrors

For our mirror problems we will need to draw a diagram of the situation. And often the secret to solving the problem is in making the diagram! So, let's learn how to do this. We could dray an infinite number of rays to find where the image will be. And, indeed, optical design software does draw many rays to check our designs. But for us we need to draw just a few easy rays. But what rays are easy to draw?

Let's think about what we already know.



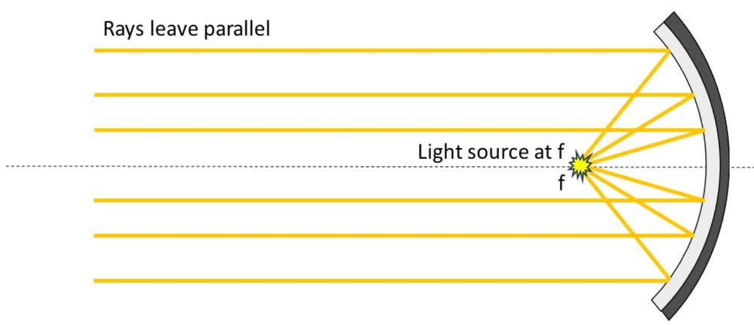
If parallel rays come into the mirror, they meet at the focal point. They don't really stop there unless we put something there.



This is easy to draw! And we know $f = R/2$ so it is even easy to find f. This is one way to make a solar cooker.

so we could call any ray that comes in parallel and goes through the focal point a “solar cooker ray.”

But it turns out we can use this geometry twice. Think, the law of reflection tells us that every ray that bounced off of the mirror did so with equal angles $\theta_i = \theta_r$. So what would happen if we put a light source at the focal point? The rays would come out parallel!



This would be easy to draw! And this is also useful. Old fashioned flashlights used this to make the light that came out in all directions go in just one direction. Big versions of this were used to send light out parallel to find enemy aircraft in World Ward II. These devices were called *spot lights* so we can call rays that act like this “spot light” rays.

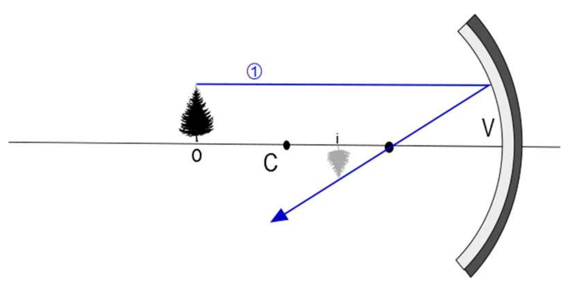
Now days, you might only see such a device used in movies about your favorite super hero (murciélago boy) used to call on the super hero for help.

There is one more ray that is easy to dray. We have already encountered it. It is the ray that goes through the center of curvature for the spherical mirror so it hits with $\theta_i = 0$ and bounces off with $\theta_r = 0$.

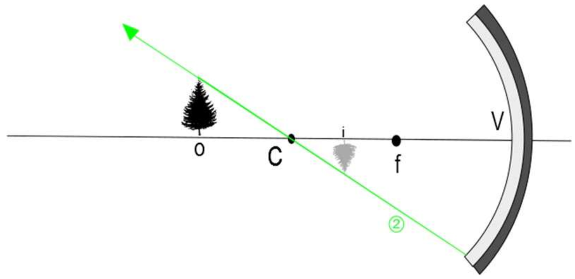
Let's put all of these together to dray ray diagrams for a spherical mirror.

Principal rays for a concave mirror:

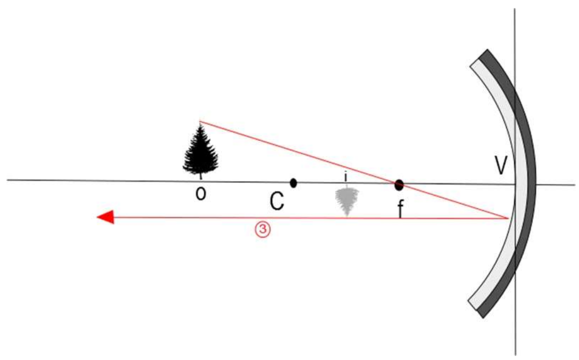
1. Ray 1 is drawn from the top of the object such that its reflected ray must pass through f.



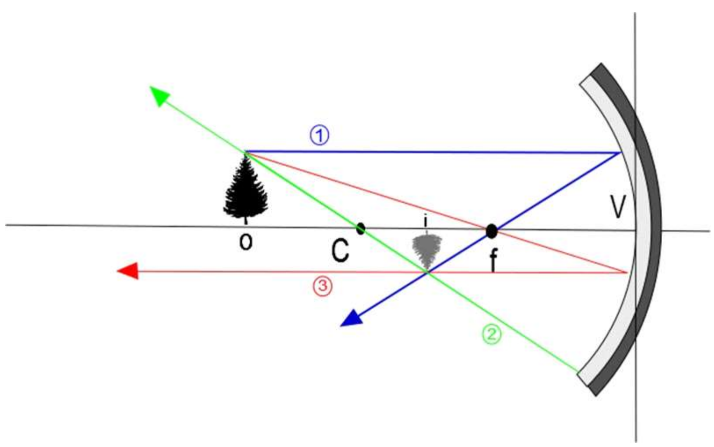
2. Ray 2 is drawn from the top of the object through the center of curvature. This ray will be incident on the mirror surface at a right angle and will be reflected back on itself.



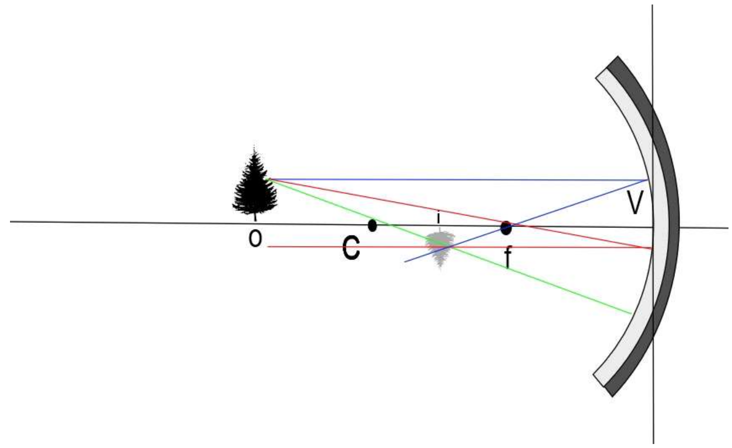
3. Ray 3 is drawn from the top of the object through the focal point to reflect parallel to the principal axis.



We can put these together, and where they meet will be where the image will form.

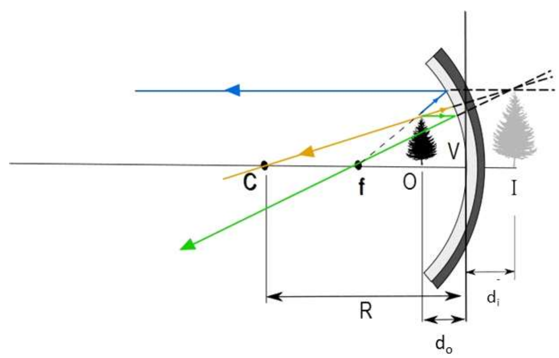


Of course, we did this for the top of the tree. We could do the same for the middle of the tree and we would get



and we can do this for every point on the tree and find the corresponding image point. But it turns out that if we do the top of the object we pretty much know where the rest of the image will be. So with just three rays we can find the image of our object!

We can do the same for an object closer than a focal length



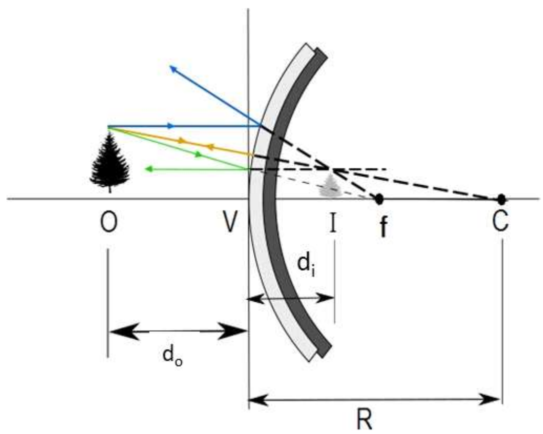
16.1.2 Principal rays for a convex mirror:

We also may have a mirror that curves, but curves the other way. We can use three rays, but this time we need to be careful.

1. Ray 1 is drawn from the top of the object such that its reflected ray appears to have come from f.

2. Ray 2 is drawn from the top of the object so that it appears to have come from the center of curvature. This ray will be incident on the mirror surface at a right angle and will be reflected back on itself.

3. Ray 3 is drawn from the top of the object to reflect parallel to the principal axis by aiming it at f.



We should tabulate our sign convention for mirrors. So far we know that if h_i is upside down it will be negative. But we can assign signs for all the other distances in our ray diagrams.

Quantity	**Positive if**	**Negative if**
Object location (d_o)	Object is in front of surface (real object)	Object is in back of surface (virtual object)
Image location (d_i)	Image is in front of surface (real image)	Image is in back of surface (virtual image)
Image height (h_i)	Image is upright	Image is inverted
Radius (R_1 and R_2)	Center of curvature is in front of surface	Center of curvature is in back of surface
Focal length (f)	In front of surface	In back of surface

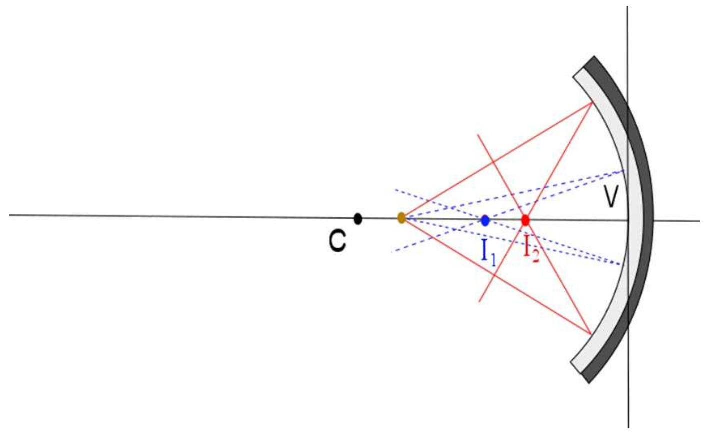
16.2 Aberrations

It's time to ask what happens when we don't use paraxial rays (when the paraxial ray approximation is not valid) but our equations are based on using only small angles or paraxial rays. You might guess that we will have problems in our imagery. That sounds bad, so let's use a special word for "problems" in imagery. That word is *aberration.*

Our first aberration comes from making the mirror the wrong shape. It should be a parabola, but instead it is a section of a sphere. We will call the problem this creates, *spherical aberration.*

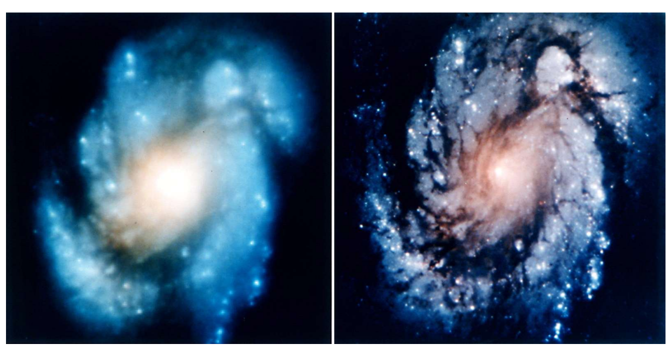
Remember that reason the we use spherical shapes is that spherical shapes are easier to make than parabolas or hyperbole, or other shapes. So optics manufacturers have been using spherical optics for centuries. Later, when we do lenses we will find there are ways to correct for spherical aberration.

But what does spherical aberration do? We end up with different images of our object at different image locations. The center rays (blue dashed lines in the next figure) make an image at I_1 but the outside (non-paraxial) rays focus closer at I_2. If we put up a screen to see the image, we will have a blurry image because some of the light is out of focus at I_1 but a different bit of light is out of fucus at I_2. There is no one place where all the light is in focus.



Most of the time for this class, we will point our optics so the object is near the principal axis, so we can make the paraxial approximation that fixes this problem. But in real optical systems we can't do this.

This optical problem was the problem with the Hubble Space Telescope!



This before (left) WFPC and after (right) WFPC2 image of the core of the galaxy M100 shows the dramatic improvement in Hubble's view of the universe after the first servicing mission in December 1993. (Images Courtesy NASA)

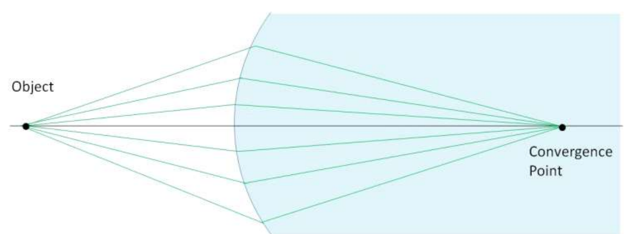
And it took some work to correct this problem. We will learn more about this as we study lenses next.

16.3 Semi-Infinite Lenses and the Semi-Infinite Lens Image Equation

We learned how to find an image location graphically for mirrors. But many optical systems including contacts and glasses, use lenses. Let's look at lenses next.

Let's start by thinking of a special case for refraction. A curved surface on a very large piece of glass. We will assume that the piece of glass is semi-infinate, but all it has to be is very large.

We can call this a semi-infinate bump of glass. And it may seem like a far fetched situation, but it turns out that our eyes are essentially a system like this because our eyes are filled with a jelly like substance, and the light in our eyes stays in this substance until it is detected in our retina. So let's look at this situation.

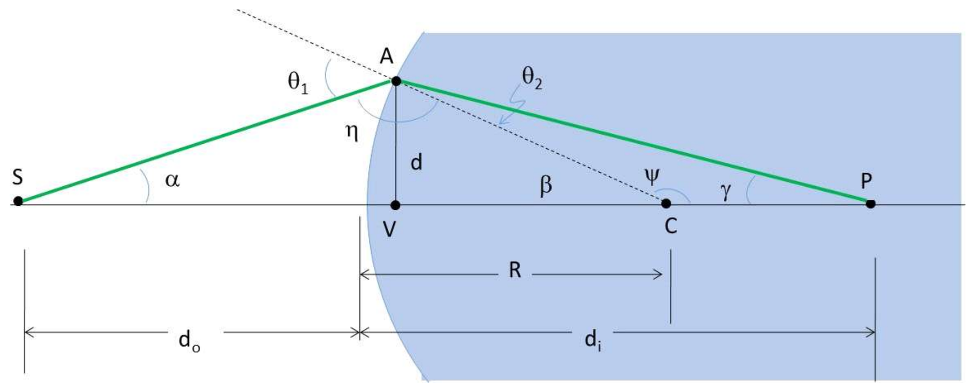


Take a point object that either glows, or has rays of light reflecting from it. The rays leave the object and reach the surface of the glass. The rays will refract at the surface. Each bends toward the normal, but because of the curvature of the glass, the rays all converge toward the center. We can identify this convergence point as the image of the point object. For mirrors we used the law of reflection to know where the light would go. But for lenses, the light is transmitted. So at the surface we can find the refracted angles using Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

We will again use the small angle approximation. Then θ_1 and θ_2 are small and none of the rays are very far away from the axis. We know this as the paraxial approximation. Snell's law becomes

$$n_1 \theta_1 = n_2 \theta_2$$



Using the more detailed figure above with it's exaggerated angles (not so paraxial but easier to see), we observe triangles SAC and PAC. We can see that for triangle SAC the top angle labeled η, plus θ_1 must be 180.

$$\eta+\theta_1=180\,^{\circ}$$

We also know that the sum of interior angles of a triangle must equal 180. So

$$180\,^{\circ}=\alpha+\beta+\eta$$

then

$$\begin{aligned}\eta+\theta_1 &= \alpha+\beta+\eta \\ \theta_1 &= \alpha+\beta\end{aligned}$$

Likewise, from triangle PAC,

$$\beta+\psi=180\,^{\circ}$$

$$180\,^{\circ}=\psi+\gamma+\theta_2$$

so then

$$\beta=\theta_2+\gamma$$

then,

$$\theta_2=\beta-\gamma$$

and we can write our paraxial Snell's law as

$$\begin{aligned}n_1\theta_1 &= n_2\theta_2 \\ n_1\left(\alpha+\beta\right) &= n_2\left(\beta-\gamma\right) \\ n_1\alpha+n_1\beta &= n_2\beta-n_2\gamma \\ n_1\alpha+n_2\gamma &= n_2\beta-n_1\beta \\ n_1\alpha+n_2\gamma &= \beta\left(n_2-n_1\right)\end{aligned}$$

Looking at the figure. We see that d is a leg of three different right triangles (SAV, ACV, and PAV). The ray in the figure is clearly not a paraxial ray. If we use a paraxial ray, then the point V will approach the air-glass boundary. When this happens, then $SV = d_o$, $VC = R$, and $VP = d_i$. So we can write (using the small angle approximation)

$$\begin{aligned}
\tan\alpha &\approx \alpha \approx \frac{d}{d_o} \\
\tan\beta &\approx \beta \approx \frac{d}{R} \\
\tan\gamma &\approx \gamma \approx \frac{d}{d_i}
\end{aligned}$$

So, our Snell's law becomes

$$\begin{aligned}
n_1\alpha + n_2\gamma &= \beta\left(n_2 - n_1\right) \\
n_1\frac{d}{d_o} + n_2\frac{d}{d_i} &= \frac{d}{R}\left(n_2 - n_1\right)
\end{aligned}$$

We can divide out the $d's$

$$\frac{n_1}{d_o} + \frac{n_2}{d_i} = \frac{(n_2 - n_1)}{R} \tag{16.3}$$

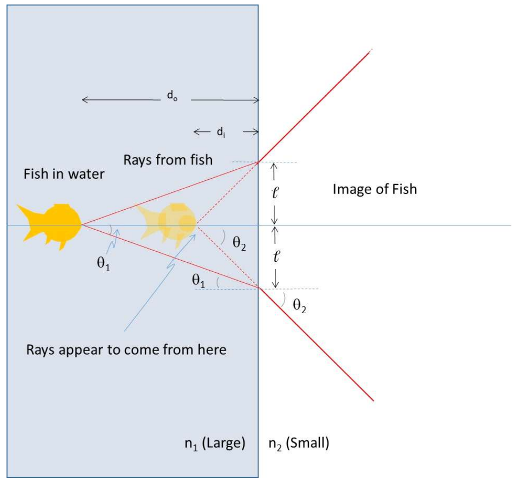
We can use this formula to convince ourselves that no matter what the angle is (providing it is small), the rays will form an image at P.

Real images will be in the glass for this special case (we will fix this with non-infinite lenses soon) so we need to change sign conventions.

Quantity	**Positive if**	**Negative if**
Object location (d_o)	Object is in front of interface surface	Object is in back of interface surface (virtual object)
Image location (d_i)	Image is in back of interface surface (real image)	Image is in front of interface surface (virtual image)
Image height (h_i)	Image is upright	Image is inverted
Radius (R)	Center of curvature is in back of interface surface	Center of curvature is in front of interface surface

We could go through the entire derivation and switch the indices of refraction. It turns out we get the same equation. The results will be different, but the equation is the same.

16.3.1 Flat Refracting surfaces



Let's consider a fish tank. The fish tank has an interface, but it is flat. Can we use our equation (16.3) to describe this?

The answer is yes, if we let $R = \infty$. This makes sense for a flat surface. If we have an infinitely large sphere, then our small part of that spherical surface that makes up the fish tank wall will be very flat.

Then

$$\frac{n_1}{d_o} + \frac{n_2}{d_i} = \frac{(n_2 - n_1)}{\infty}$$

or

$$\frac{n_1}{d_o} + \frac{n_2}{d_i} = 0$$

we see that

$$d_i = -d_o \frac{n_2}{n_1}$$

We can see from this that, since $n_2 < n_1$ for this case d_i is less than d_o. The image is in the water! It is in *front* of the water/air boundary of the tank. But light is not actually coming from that location. What has happened is that the light path has bent, but our brain image processing center thinks light travels only in straight lines. So it tells us there is a fish there, but the fish appears closer than it is. The light doesn't come from where we think the fish is. And as we know, this means the image is virtual, just like for flat mirrors!