Chapter 5

5 Sinusoidal Waves in one and Two Dimensions 1.16.2 1.16.3

Fundamental Concepts

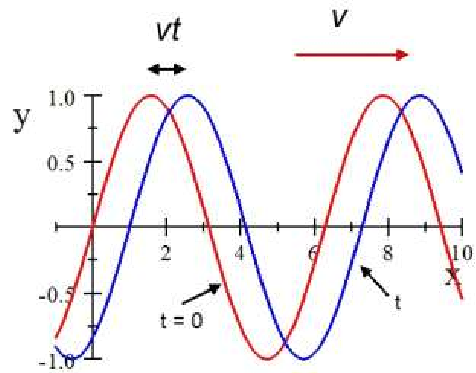
- It takes two variables to describe a wave, position and time

- Wave graphs have many named parts, amplitude, period, wavelength crest, trough, etc.

- There is a spatial analog to temporal frequency called spacial frequency and it is represented by the factor k in our equations

- Waves from point sources are spherical

5.1 Sinusoidal Waves



A sinusoidal graph should be familiar from our study of oscillation. Simple harmonic oscillators are described by the function

$$y\left(t\right)=y_{\max}\cos\left(\omega t+\phi\right)\qquad \text{SHM} \tag{5.1}$$

but this only gave us a vertical displacement. Now our sinusoidal function must also be a function of position along the wave.

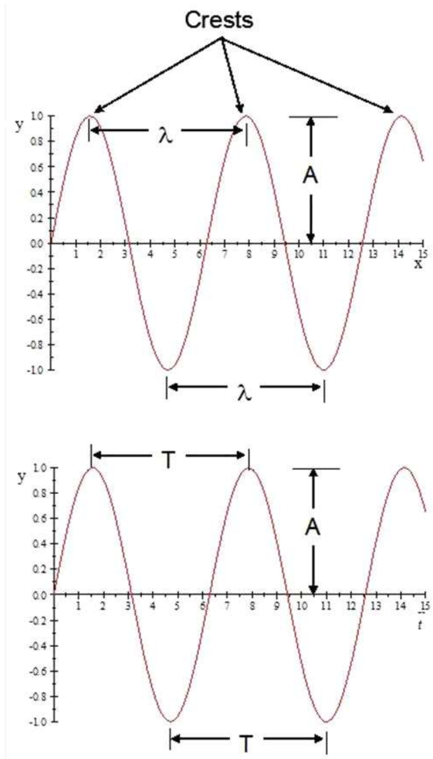
$$y\left(x,t\right)=y_{\max}\sin\left(ax-\omega t+\phi\right)\qquad \text{waves} \tag{5.2}$$

but before we study the nature of this function, lets see what we can learn from the graph of a sinusoidal wave. We will need both our two views, the camera snapshot and the video of a point. Look at figure **??**. This is two camera snap shots superimposed. The red curve shows the wave (y position for each value of x) at $t=0$. At some some later time t, the wave pattern has moved to the right as shown by the blue curve. The shift is by an amount $\Delta x=v\Delta t$.

5.2 Parts of a wave

The peak of a wave is called the crest. For a sine wave we have a series of crests. We define the wavelength as the distance between any to identical points (e.g. crests) on adjacent waves in a snapshot view.

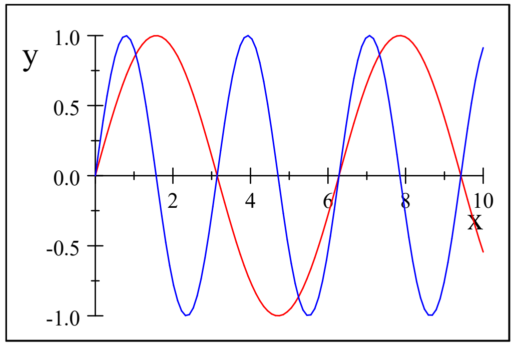
Notice that this is very similar to the definition of the period, T, when we graphed SHM on a y,t set of axis. In fact, this similarity is even more apparent if we plot a sinusoidal wave using our two wave pictures. In the next figure, the snap-shot comes first. We can see that there will be crests. The distance between the crests is given the name *wavelength*. This is not the entire length of the whole wave. But is is a characteristic length of part of the wave that is easy to identify. The next figure shows all this using our snapshot and history graph for a sine wave.



Note that there are crests in the history graph view as well. That is because one marked part of the medium is being displaced as a function of time. Think of a marked piece of the rope going up and down, or think of floating in the ocean at one point, you travel up and down (and a little bit back and forth) as the waves go by. But now the horizontal axis is time. There will be a characteristic time between crests. That time is called the *period*. This is just like oscillatory motion–because it is oscillatory motion! Like the wavelength is not the length of the whole wave, the period is not the time the whole wave exists. It is just the time it takes the part of the medium we are watching to go through one complete cycle. Notice that this video picture is exactly the same as a plot of the motion of a simple harmonic oscillator! For a sinusoidal wave, each part of the medium experiences simple harmonic motion.

We remember frequency from simple harmonic motion. But now we have a wave, and the wave is moving. We can extend our view of frequency by defining it as follows:

The frequency of a periodic wave is the number of crests (or any other point of the wave) that pass a given point in a unit time interval.



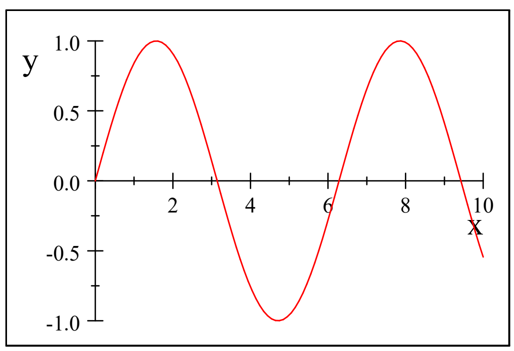
In the previous figure, the blue curve has twice the frequency as the red curve. Notice how it has two crests for every red crest. The maximum displacement of the wave is called the *amplitude* just as it was for simple harmonic oscillators.

5.3 Wavenumber and wave speed

Consider again a sinusoidal wave.

$$y\left(x,t\right)=y_{\max}\sin\left(kx-\omega t+\phi_{o}\right) \tag{5.3}$$

We have drawn the wave in the snapshot picture mode



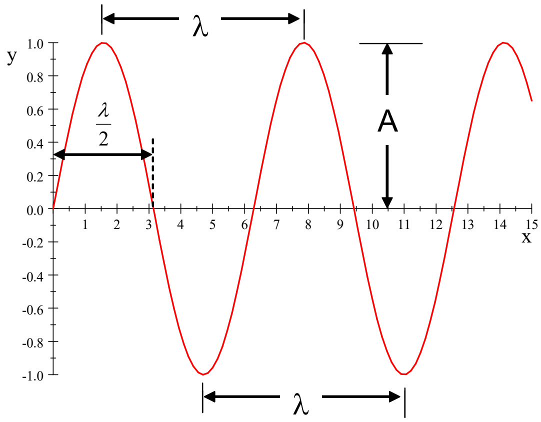
To make this graph, we set $t=0$ and plotted the resulting function

$$y\left(x,0\right)=y_{\max}\sin\left(kx+0\right) \tag{5.4}$$

$y_{\max}$ is the amplitude. I want to investigate the meaning of the constant k. Lets find k like we did for SHM when we found ω. Consider the point $x=0$. At this point

$$y\left(0,0\right)=y_{\max}\sin\left(k\left(0\right)\right)=0 \tag{5.5}$$

The next time $y=0$ is when $x=\frac{\lambda}{2}$



then

$$y\left(\frac{\lambda}{2},0\right)=y_{\max}\sin\left(k\frac{\lambda}{2}\right)=0 \tag{5.6}$$

From our trigonometry experience, we know that this is true when

$$k\frac{\lambda}{2}=\pi \tag{5.7}$$

solving for k gives

$$k=\frac{2\pi}{\lambda} \tag{5.8}$$

Then we now have a feeling for what k means. It is 2π over the spacing between the crests. The 2π must have units of radians attached. Then

$$y\left(x,0\right)=A\sin\left(\frac{2\pi}{\lambda}x+0\right) \tag{5.9}$$

We have a special name for the quantity k. It is called the *wave number*.

$$k\equiv\frac{2\pi}{\lambda} \tag{5.10}$$

Both the name and the symbol are somewhat unfortunate. Neither gives much insight into the meaning of this quantity. But from what we have done in studying oscillation, we can understand this new quantity. For a harmonic oscillator, we know that

$$y\left(t\right)=y_{\max}\sin\left(\omega t\right)$$

where

$$\omega=2\pi f=\frac{2\pi}{T}$$

T is how far, *in time*, the crests are apart, and the inverse of this, $\frac{1}{T}$ is the frequency. The frequency tells us how often we encounter a crest as we march along in time. So $\frac{1}{T}$ must be how many crests we have in a unit amount of time.

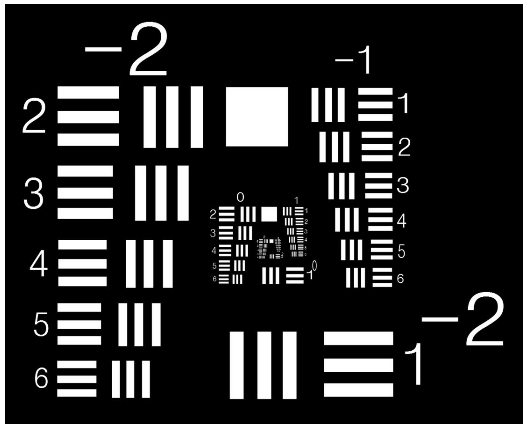
Now think of the relationship between the snapshot and the video representation for a sinusoidal wave. We have a new quantity

$$k = \frac{2\pi}{\lambda}$$

where λ is how far, *in distance*, the crests are apart. This implies that $\frac{1}{\lambda}$ plays the same role in the snap shot graph that f plays in the video graph. It must tell us how many crests we have, but this time it is how many crests in a given amount of distance. We found above that k told us something about how often the zeros (well, every other zero) will occur. But the crests must occur at the same rate. So k tells us how often we encounter a crest in our snapshot graph. This is not too strange. It would be meaningful, for example, to say that a farmer had plowed his or her field to have two furrows per meter.

The frequency in the video graph is how often we encounter a crest, $\frac{1}{\lambda}$ is how often we encounter a crest in the snap shot graph. Thus $\frac{1}{\lambda}$ is playing the same role for a snap shot graph as frequency plays for a history graph. We could call $\frac{1}{\lambda}$a *spatial frequency.* It is how often we encounter a crest as we march along in position, or how many crests we have in a unit amount of distance. And λ could be called a *spatial period*. Both $1/T$ and $1/\lambda$ answer the question "how often something happens in a unit of something" but one asks the question in time and the other in position along the wave.

My mental image for this is the set of groves on the edge of a highway. There is a distance between them, like a wavelength, and how often I encounter one as I move a distance along the road is the spatial frequency. You could say that there are, maybe, $3/\,\mathrm{m}$. That is a spatial frequency! It is how many of something happens in a unit distance. We use this concept in optics to test how well an optical system resolves details in a photograph. The next figure is a test image. A good camera will resolve all spatial frequencies equally well. Notice the test image has sets of bars with different spatial frequencies. By forming an image of this pattern, you can see which spacial frequencies are faithfully represented by the optical system.



Resolution test target based on the USAF 1951 Resolution Test Pattern (not drawn to exact specifications).

In class you will see that our projector does not represent all spatial frequencies equally well! You can also see this now in the copy you are reading. If you are reading on-line or an electronic copy, your screen resolution will limit the representation of some spatial frequencies. Look for the smallest set of three bars where you can still tell for sure that there are three bars without zooming. A printed version that has been printed on a laser printer will usually allow you to see even smaller sets of three bars clearly.

Let's place k in the full equation for the sine wave for any time, t.

$$y(x,t) = y_{\max} \sin(kx - \omega t + \phi_o) \tag{5.11}$$

Sometimes you will see this equation written in a slightly different way. We could change our variable from x to $x - vt$ so that

$$y(x,t) = y(x - vt, 0)$$

With a little algebra we can do this

$$\begin{aligned} y(t) &= y_{\max} \sin(kx - \omega t + \phi_o) \\ &= y_{\max} \sin\left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t + \phi_o\right) \\ &= y_{\max} \sin\left(\frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T} t\right) + \phi_o\right) \end{aligned}$$

This is in the form of a wave function so long as

$$v = \frac{\lambda}{T} \tag{5.12}$$

which is just how far the wave travels from crest to crest divided by how long it takes to travel that distance. The wave travels one wavelength in one period.

Indeed, this is the wave speed! Then

$$y(x,t) = y_{\max} \sin\left(\frac{2\pi}{\lambda}(x - vt) + \phi_o\right) \tag{5.13}$$

or, using k

$$y(x,t) = y_{\max} \sin\left(k(x - vt) + \phi_o\right) \tag{5.14}$$

Wave speed forms

We just found

$$v = \frac{\lambda}{T} \tag{5.15}$$

but it is easy to see that

$$v = \frac{2\pi\lambda}{2\pi T} = \frac{\omega}{k} \tag{5.16}$$

and

$$v = \lambda f \tag{5.17}$$

This last formula is, perhaps, the most common form encountered in our study of light.

Phase

You may be wondering about the phase constant we learned about in our study of SHM. We have ignored it up to now. But of course we can shift our sine just like we did for our plots of position vs. time for oscillation. Only now with a wave we have two graphs, a history and snapshot graphs, so we could shift along the x in a snapshot graph or along the t axes in a history graph. So the sine wave has the form.

$$y(x,t) = A\sin\left(kx - \omega t + \phi_o\right) \tag{5.18}$$

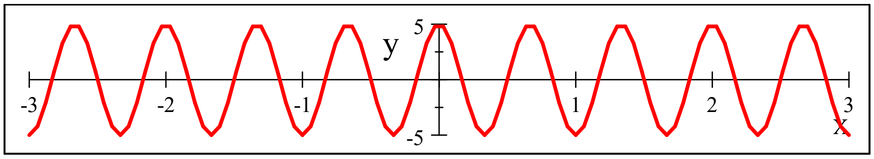
were ϕ_o will need to be determined by initial conditions just like in SHM problems and those initial conditions will include initial positions as well as initial times.

Let's consider that we have two views of a wave, the snapshot and history view. Each of these looks like sinusoids for a sinusoidal wave. Let's consider a specific wave,

$$y(x,t) = 5\sin\left(3\pi x - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$

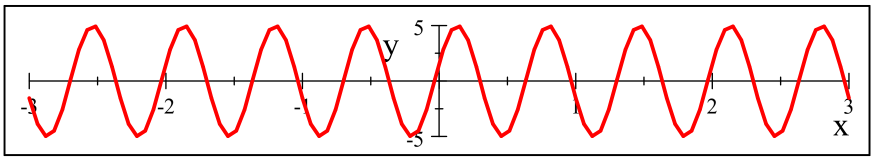
we look at a snapshot graph at $t = 0$

$$y(x,0) = 5\sin\left(3\pi x - \frac{\pi}{5}(0) + \frac{\pi}{2}\right)$$



and another at $t = 2\,\mathrm{s}$

$$y(x, 2\,\mathrm{s}) = 5 \sin\left(3\pi x - \frac{\pi}{5}(2\,\mathrm{s}) + \frac{\pi}{2}\right)$$



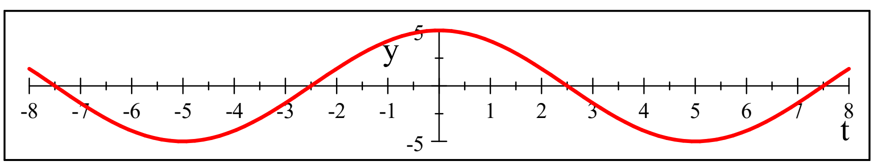
Comparing the two, we could view the latter as having a different phase constant that is the sum of what we have called the phase constant, ϕ_o plus $\omega\Delta t$, which tells us how different the times are between the two graphs. This time difference is what is shifting the graph

$$\phi_{total} = \omega\Delta t + \phi_o = -\frac{\pi}{5}(2\,\mathrm{s}) + \frac{\pi}{2}$$

that is, within the snapshot view, the time dependent part of the argument of the sign acts like an additional phase constant.

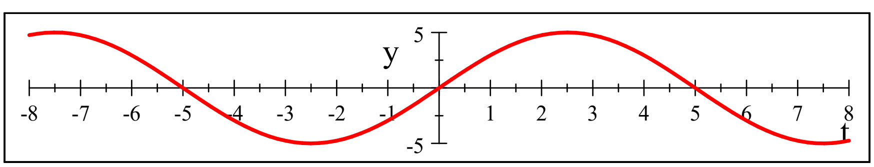
Likewise, in the history view, we can plot our wave at $x = 0$

$$y(0, t) = 5 \sin\left(3\pi(0) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



and at $x = 1.5\,\mathrm{m}$

$$y(1.5\,\mathrm{m}, t) = 5 \sin\left(3\pi(1.5\,\mathrm{m}) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



Within the history view, the kx part of the argument acts like a phase constant.

$$\phi_{total} = k\Delta x + \phi_o = 3\pi\,(1.5\,\mathrm{m}) + \frac{\pi}{2}$$

Of course neither kx nor ωt are constant, But within individual views of the wave we have set them constant to form our snapshot and history representations. We can see that any part of the argument of the sine, $kx - \omega t + \phi_o$ could contribute to a phase shift, depending on the view we are taking.

Because of this, it is customary to call the entire argument of the sine function, $\phi = kx - \omega t + \phi_o$ the *phase of the wave*. Where ϕ_o is the phase constant, ϕ is the phase. Of course then, ϕ must be a function of x and t, so we have a different value for $\phi(x,t)$ for every point on the wave for every time. This can be a little confusing, ϕ and ϕ_o look a lot the same, but they are different.

5.4 The Speed of Waves

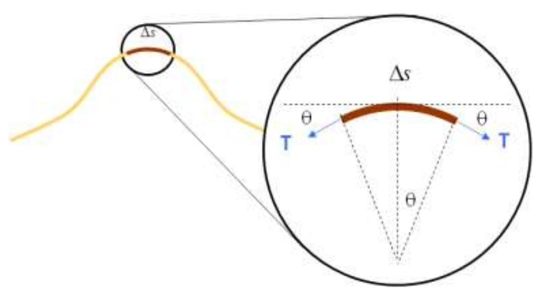
We have already considered the speed of waves on strings

$$v = \sqrt{\frac{T}{\mu}}$$

we should see where this comes from so we know it's limitations. We would expect that there may be things that change the speed of sound waves in air as well (like a tension or a massiveness of the medium). Let's start with a formal derivation of our string speed formula, then take on sound waves in air.

5.4.1 Derivation of the string wave speed formula

Let's work a problem together. Let's find an expression for the speed of the wave as it travels along a string.



What are the forces acting on an element of string?

1. Tension on the RHS of the element from the rest of the string on the right, T_r

2. Tension on the LHS of the element from the rest of the string on the left, T_l

3. F_g

Lets assume that the element of string, Δs, at the crest is approximately an arc of a circle with radius R.

There is a force pulling left on the left end of the element that is tangent to the arc, there is a force pulling right at the right end of the element which is also tangent to the arc. These forces produce centripetal accelerations

$$a = \frac{v^2}{R} \tag{5.19}$$

The horizontal components of the forces cancel $(T\cos\theta)$. The vertical component, $(T\sin(\theta))$ is directed toward the center of the arc. If the rope is not moving in the x direction, then $T_l = T_r$.

Then, the radial force F_r will have matching components from each side of the element that together are $2T\sin(\theta)$. Notice that to make sure the components are the same we must assume that the string is uniform and that it is not too massive (almost massless string approximation!). Since the element is small,

$$F_r = 2T\sin(\theta) \approx 2T\theta \tag{5.20}$$

The element has a mass m.

$$m = \mu\Delta s \tag{5.21}$$

where μ is the mas per unit length. Using the arc length formula

$$\Delta s = R(2\theta) \tag{5.22}$$

so

$$m = \mu\Delta s = 2\mu R\theta \tag{5.23}$$

and finally we use the formula for the radial acceleration

$$F_r = ma = (2\mu R\theta)\frac{v^2}{R} \tag{5.24}$$

Combining these two expressions for F_r

$$2T\theta = (2\mu R\theta)\frac{v^2}{R} \tag{5.25}$$

$$T = (\mu R)\frac{v^2}{R} \tag{5.26}$$

$$\frac{T}{\mu} = v^2 \tag{5.27}$$

and we find that

$$v = \sqrt{\frac{T}{\mu}} \tag{5.28}$$

which is just what we stated before, only now we see just how much approximation there was in building this formula. We might guess that it would not work so well for large cables supporting bridges or with cables that change size along their length.

5.4.2 Speed of the medium

Let's review for a moment. Suppose we take a jump rope, and shake one end up and down while a lab partner keeps his or her end stationary. You can make a sine wave in the rope. But being good physics students, let's make our "shake" very symmetric. Let's start our wave using simple harmonic motion.

Let's call an element of the rope Δx. Here the "Δ" is being used to mean "a small amount of." We are taking a small amount of the rope and calling it's length Δx. And it is the motion of this small amount of rope that I want to think about now.

Each element of the rope (Δx_i) will also oscillate with SHM (think of a driven SHO). Note that the elements of the rope oscillate in the y direction, but the wave travels in x. This is a transverse wave.

Let's describe the motion of an element of the string at point P.

At $t = 0$,

$$y = A \sin (kx - \omega t + \phi_o) \tag{5.29}$$

(where I have chosen $\phi_o = 0$ for this example). The element does not move in the x direction. So we define the *transverse speed*, v_y, and the *transverse acceleration,* a_y, as the velocity and acceleration of the element of rope in the y direction. These are not the velocity and acceleration of the wave, just the velocity and acceleration of the element Δx at a point P.

Because we are doing this at one specific x location we need partial derivatives to find the velocity

$$v_y = \left.\frac{dy}{dt}\right]_{x=\text{constant}} = \frac{\partial y}{\partial t} \tag{5.30}$$

That is, we take the derivative of y with respect to t, but we pretend that x is not a variable because we just want one x position. Then

$$v_y = \frac{\partial y}{\partial t} = -\omega A \cos (kx - \omega t + \phi_o) \tag{5.31}$$

and

$$a_y = \frac{\partial v_y}{\partial t} = -\omega^2 A \sin (kx - \omega t + \phi_o) \tag{5.32}$$

These solutions should look very familiar! We expect them to be the same as a harmonic oscillator except that we now have to specify which oscillator–which part of the rope–we are looking at. That is what the kx part is doing.

5.5 The Linear Wave Equation

We added in some new math in our last two lectures. Differential equations! We identified simple harmonic motion using a differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Perhaps we could find an equation like this for waves and then identify waves using that equation. We can start the same way we did for oscillation. Let's write the acceleration for a part of the medium.

$$a_y = \frac{d^2y}{dt^2} = -\omega^2 A \sin\left(kx - \omega t + \phi_o\right) \tag{5.33}$$

But for waves we have two variables. Let's guess that we might need to take derivatives with respect to x as well as for t.

$$\frac{dy}{dx} = kA\cos(kx - \omega t + \phi_o)$$

This isn't all that useful on it's own, but it does tell us how the slope of the curve changes in a snapshot graph. Let's take another derivative

$$\frac{d^2y}{dx^2} = -k^2 A \sin(kx - \omega t + \phi_o)$$

We are back to a sine function. That is not a surprise. But it looks a lot like our time second derivative. We could take the ratio of the two equations to see just how different they are

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = \frac{-\omega^2 A \sin\left(kx - \omega t + \phi_o\right)}{-k^2 A \sin(kx - \omega t + \phi_o)}$$

Of course the sine parts cancel, and so do the A's

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = \frac{-\omega^2}{-k^2}$$

and we know that

$$\frac{\omega}{k} = v$$

the wave speed (not the speed of a piece of the medium, but the speed of the wave)

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = v^2$$

If a function satisfies this equation, we can identify it as a wave. But this form is awkward, let's rewrite it as

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2}$$

This is the *linear wave equation.* There are waves that don't follow this equation. They are called non-linear waves. But that is a subject for an upper division course.

5.6 Waves in two and three dimensions

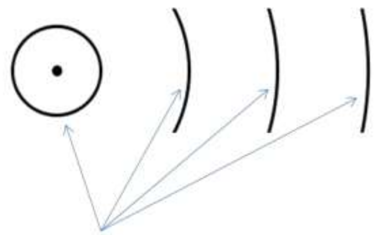
So far we have written expressions for waves, but our experience tells us that waves don't usually come as one dimensional phenomena. In the next figure, we see the disturbance (a drop) creating a water wave.

Picture of a water drop (Jon Paul Johnson, used by permision)

You can see the wave forming, but the wave is clearly not one dimensional. It appears nearly circular. In fact, it is closer to hemispherical, and this limit is only true because the disturbance is at the air-water boundary. Most waves in a uniform medium will be roughly spherical. As such a wave travels away from the source, the energy traveling gets more spread out. This causes the amplitude to decrease. Think of a sound wave, it gets quieter the farther you are from the source. We change our equation to account for this by making the amplitude a function of the distance, r, from the source

$$y(r,t) = A(r)\sin(kr - \omega t + \phi_o) \tag{5.34}$$

Of course, if we look at a very large wave, but we only look at part of the wave, we see that our part looks flatter as the wave expands.



Portion of a Spherical Wave: Wave becomes more flat as it expands

Very far from the source, our wave is flat enough that we can ignore the curvature across it's wave fronts. We call such a wave a *plane wave*. There are no true plane waves in nature, but this idealization makes our mathematical solutions simpler and many waves come close to this approximation.

We have said that sound is a longitudinal wave with a medium of air. Really any solid, liquid, or gas will work as a medium for sound. For our study, we will take sound to be a longitudinal wave and treat liquids and gasses. Solids have additional forces involved due to the tight bonding of the atoms, and therefore are more complicated. Technically in a solid sound can be a transverse wave as well a longitudinal wave, but we usually call transverse waves of this nature *shear waves.*