Fundamentals of Physics II
Wave Motion, Thermal Physics, and Optics

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The Date

Contents

[1] Comparing physical theory to LDS thought has it's dangers, I am aware. But as scientists, we can't escape thinking about this. Just as a caution, John A. Widsoe wrote a book called *Joseph Smith as Scientist* in which Widsoe tried very hard to show that LDS though is in harmony with Universal Ether Theory. But we now know Universal Ether Theory is not correct, so being in harmony with it is meaningless. So we should be cautious!

Preface

This set of notes is intended to be an aid to the student. It is what I intend to present in class. There are likely errors and mistakes, so use these notes, but don't expect perfection. If there are things that are confusing, please talk to me or ask questions in class.

I consulted many authors and colleagues in creating these notes. Principle among the authors was Serway and Jewett and Haliday and Resnic. Colleagues of special note are Kevin Kelly, Brian Pyper, Steve Tercotte, and Ryan Nielson. I especially appreciate the PH123 students of the Fall of 2006. They were the first to test run this sets of notes, and I appreciate their feedback (and patience).

Part I

Oscillation and Waves

Chapter 1

Introduction to the Course, Simple Harmonic Motion 1.15.1

Fundamental Concepts

- Simple Harmonic Motion
- Frequency and Period
- Mathematical description of Simple Harmonic Motion

1.1 Oscillation and Simple Harmonic Motion

Last semester, in PH121 (or Dynamics) you studied how things move. We identified a moving object (I often refer to this object as the *mover object*) and other objects that exerted forces on the mover (I often refer to these other objects as the *environmental objects*). You learned about forces and torques which get mover objects moving. You should remember Newton's Second Law and Newton's Second Law for rotation.

You also learned and practiced a lot of math. We will continue to use the math you learned in PH121 this semester.

But we will go beyond what we learned in PH121 to study new types of motion, and new objects that move.

This semester, we will start with a very special type of motion. It is the motion that results from oscillation. We call this very special type of motion, *simple harmonic motion*.

Simple harmonic motion (SHM) means a motion that repeats in the most special, simple way. Some characteristics of this type of motion are as follows:

1. The motion repeats in a regular way, like a grandfather clock pendulum swings back and forth in a set amount of time. This set amount of time is called a period.

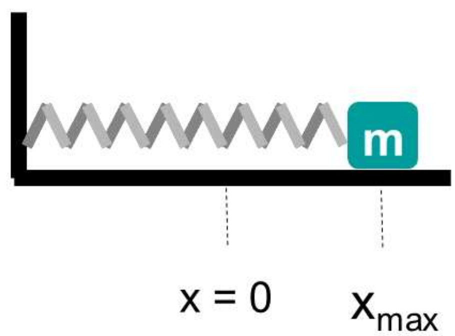
2. The mover object moves about a center position and is symmetric about that position. This center position is the bottom of the swing for our grandfather clock. The idea of symmetry just means that the pendulum reaches the same height on both sides as it swings

3. The mover object's motion can be traced out in a position vs. time graph. For SHM, the shape of the position vs. time graph is a sinusoid.

4. The velocity of the mover object constantly changes. It is zero at the extreme points (the points farthest from the center, or the largest positive and the largest negative displacements). That is where it turns around and goes back the other way. It stops there, but just for a split second. The velocity is largest at the equilibrium position (the center position). If we plotted a velocity vs. time graph, the velocity would also be a sinusoid or snake-like shape.

5. The acceleration also constantly changes. And it is also a sinusoid.

That is a lot of criteria for a motion that is supposed to be simple! Let's take an example system to see how simple harmonic motion works. I have drawn a mass (marked m) attached to a spring. And lets assume that the mass is on a frictionless surface and there is no air drag force. If we pull the mass so the string stretches, we would get simple harmonic motion.



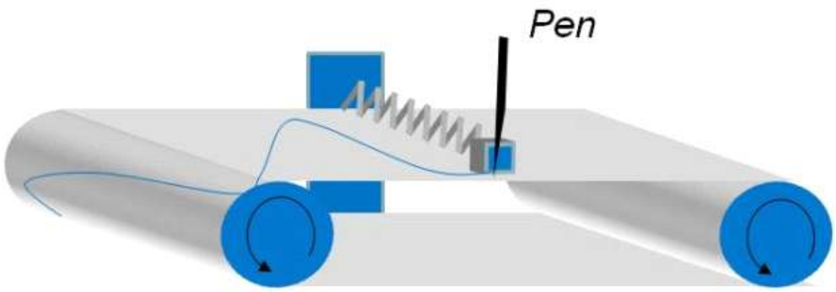
The center position for our mass is called the Equilibrium position, and for convenience we often define it as the origin of our coordinate system.

Equilibrium Position: The position of the mover mass (not the spring, which is an environmental object acting on the mover mass) when the spring is neither stretched nor compressed.

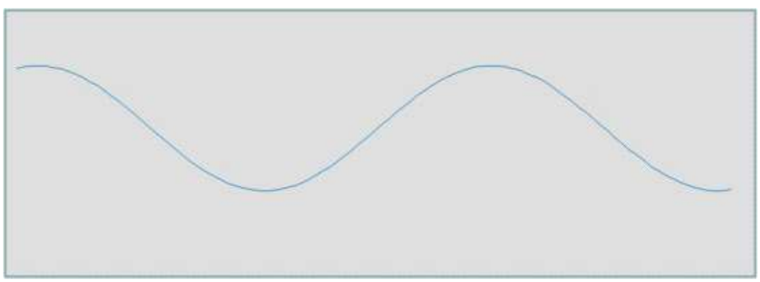
Let's draw a picture of simple harmonic motion. First we need a device to do this. Suppose you go to the supermarket[1]. But instead of putting your ramen

[1] Like Albertsons, where they still have checkers.

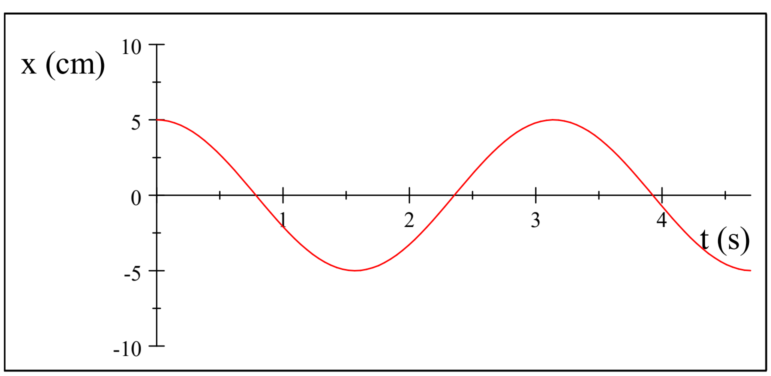
and granola bars on the belt at the checkout counter, you strap on a device like this[2]



You can see the mover object on the spring. But we have placed a pen on the object and that pen is tracing out a pattern as the belt moves. You will recognize this pattern as a trig function. The result might look something like this if you removed the belt.



Of course, what we have made is a position vs. time graph. We remember these from PH121! This gives us a record of the past motion of the object.



where in this graph, $x_{\max} = 5\,\mathrm{cm}$. Having the power of mathematics, we know we can write an equation that would describe this curve. From your Trigonometry

[2] Please don't really try this at the local supermarkets. They don't seem to have any sense scientife inquarey or even a sense of humor about such things at all.

experience, we can guess that an equation for our mover object motion might look something like

$$x(t) = x_{\max}\cos(\theta)$$

Notice that the instantaneous position, $x(t)$ has to be a function of time. Back in trigonometry we would have said a cosine function was a function of an angle, θ. But we know this is a position vs. time graph, so our angle must be different for different times. Let's write

$$\theta(t) = \omega t$$

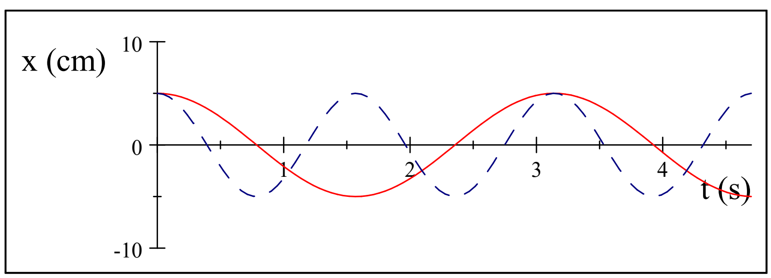
as strange sort of an angle. This "angle" is a function of time, and let's say how fast our "angle" is changing is given by

$$\frac{d\theta}{dt} = \omega$$

This is a sort of speed for how fast our angle is changing. So our equation must be

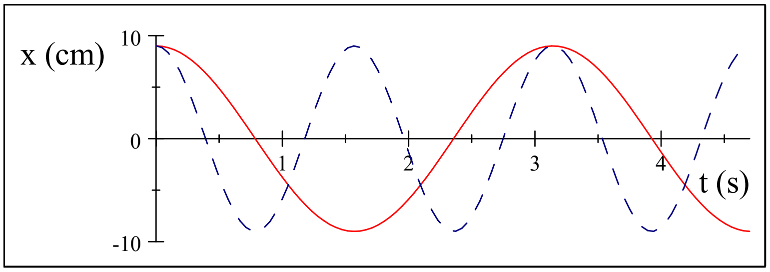
$$x(t) = x_{\max}\cos(\omega t)$$

In the next figure $x(t) = x_{\max}\cos(\omega t)$ is plotted with two different values for ω.



We can see what ω does for us. It stretches out or compresses our curve. In the blue (dashed) curve our angle changes quickly. In the red (solid) curve the angle changes slowly. Note that we are not plotting position vs. angle. Both plots would be the same if we did! We are plotting position vs. time. And for the blue curve our mass is oscillating much more quickly than it is for the red curve. The quantity ω tells us something about how fast our mover object is oscillating.

The quantity $x_{\max}$ is the maximum displace of our mover object from the equilibrium position. This maximum displacement from equilibrium has a name. It is called the *amplitude*. In our equation I have used $x_{\max}$ where most books use the letter A for amplitude to emphasize that the amplitude is the maximum displacement of the mover mass from the mover mass equilibrium position. In our coordinate system, the mass is going back and forth in the x direction. Since the amplitude means the maximum displacement, $x_{\max}$ is a good way to write amplitude. Here is the same graph but with $x_{\max} = 9\,\mathrm{cm}$



The two graphs for the two different ω values are not more stretched out in time, but now they are taller along the position axis. This means that the mass is moving farther from the center as it oscillates.

You will remember from your trigonometry class that the *period*, T, tells us the time it takes for the oscillation to go through a complete cycle. A complete cycle is when the object, say, goes from $x_{\max}$ to $-x_{\max}$ and then back to $x_{\max}$. You can probably guess that how long it takes to oscillate and how often it oscillates would be related. How often the oscillator completes a cycle is called the frequency. The longer the period, the lower the frequency.

$$f = \frac{1}{T}$$

Think of cars passing you on your way to class. Period is like how long you wait in between cars. Frequency is like how often cars pass. If you wait less time between cars, the cars pass more frequently. And that is just what our equation says! We can see from our graphs that our stretching quantity ω, must be related to the frequency of our oscillation. If the frequency is high, then ω must be large so that we reach different "angles" faster. But for the cosine function to work we need angle units. *We will choose radians* for our units and we will write our stretching quantity as

$$\omega = 2\pi f$$

This works! If ω is bigger then our oscillation happens more frequently. The 2π has units of radians. So ω has units of $\mathrm{rad\,Hz}$ or more commonly $\mathrm{rad/s}$. That matches our derivative above. Let's give a name to the quantity ω. Since it has radians in it we might guess that it has something to do with circular motion (more on this later) and it has frequency in it. So we will call ω the *angular frequency*.

1.2 Velocity and Acceleration

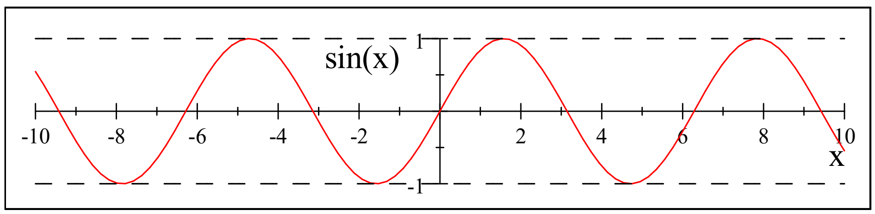
So far we have guessed the descriptive equation for SHM.

$$x(t) = x_{\max}\cos(\omega t) \tag{1.1}$$

This gives us the position of the particle at any given time. Since knowing the position as a function of time is a good description of the motion of our mover mass, we can call this equation the *equation of motion* for our mass. We will see soon that this equation is correct. But let's pretend that we are earlier researchers and we just know the equation makes the right shape. Still, we can learn much from knowing this equation. Since we know how to take derivatives, and we know the derivative of position with respect to time is the velocity, we can see that the velocity of the mass at any given time is given by

$$v(t) = \frac{dx(t)}{dt} = -\omega x_{\max} \sin(\omega t) \tag{1.2}$$

From our fond memories of our trigonometry class we know the maximum of a sine function is always 1.



Notice that the ω and the $x_{\max}$ aren't changing for our oscillation mover mass. They are constant. Then if we want the maximum speed we can simply set the $\sin(\omega t) = 1$. Then the maximum speed will be

$$v_{\max} = \omega x_{\max}(1) \tag{1.3}$$

Notice that this does not tell us when the speed is maximum. Just what the maximum speed is. We then have an equation for the speed of our object as a function of time

$$v(t) = -v_{\max} \sin(\omega t)$$

We will often use this trick of knowing the maximum of sine is one.

We can also find the acceleration. We just take another derivative.

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d^2 x(t)}{dt^2} \\ &= -\omega^2 x_{\max} \cos(\omega t) \end{aligned}$$

This is the acceleration of the object attached to the spring. It's also true that the maximum for a cosine function is 1, so the maximum acceleration would be

$$a_{\max} = \omega^2 x_{\max}(1) \tag{1.4}$$

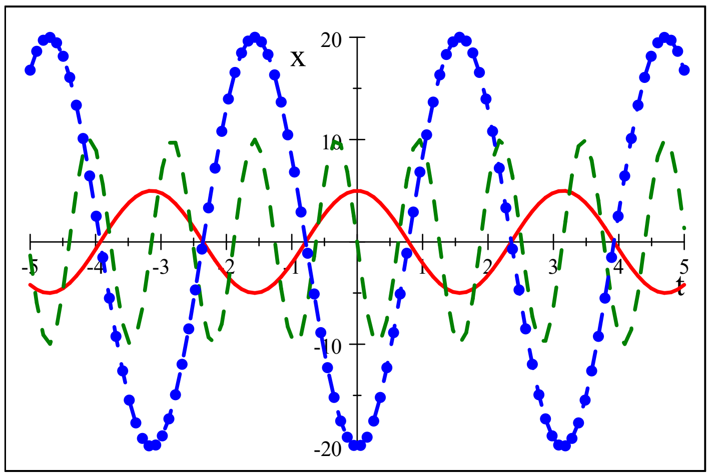
so we could write our instantaneous acceleration as

$$a(t) = -a_{\max} \cos(\omega t)$$

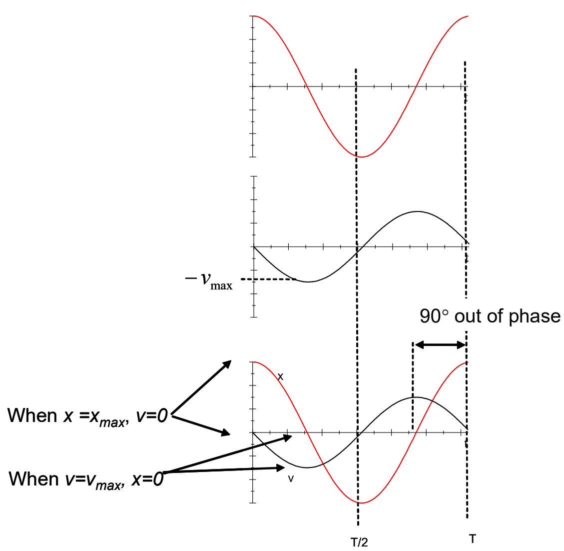
These are significant results, so let's summarize. For a simple harmonic oscillator, the instantaneous position, speed, and acceleration are given by

$$
\begin{aligned}
x(t) &= x_{\max}\cos(\omega t) \qquad (1.5)\\
v(t) &= -\omega x_{\max}\sin(\omega t)\\
a(t) &= -\omega^2 x_{\max}\cos(\omega t)
\end{aligned}
$$

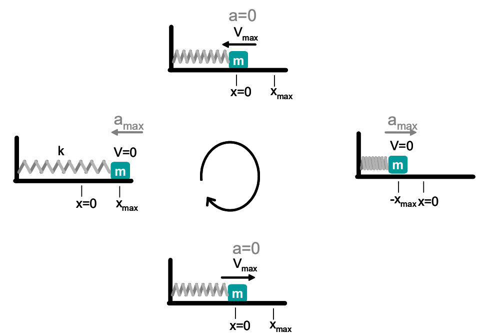
Let's plot $x(t)$, $v(t)$, and $a(t)$ for a specific case



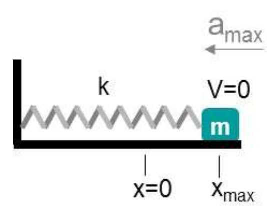
Red (solid) is the displacement, green (dashed) is the velocity, and blue (dot-dashed) is the acceleration. Note that each as a different maximum amplitude. Also note that they don't rise and fall at the same time. We will describe this as being *not in phase*.



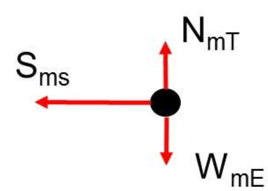
The acceleration is 90° *out of phase* from the velocity. Let's think about why this would be. Suppose we attach a mass to a spring and allow the mass to slide on a frictionless surface.



Let's start by stretching the spring by pulling the mass to the right and releasing. This is the situation in the left hand part of the last figure.



The spring is pulling strongly on the mass. We could draw a free body diagram for this situation. We will need the spring force



Back in PH121 we said that Hooke's Law is not something that is always true, but by "law" we mean a mathematical representation (and equation) that comes from our mental model of how the universe works. In this case, Hooke's law is an equation that comes from Hooke's model of how springs work. It is a good model for most springs as long as we don't stretch them too far. You remember Hooke's law from PH 121. It tells us that the spring force is proportional to how far we stretch or compress the spring (Δx_e) and how stiff the spring is (k).

$$\begin{aligned} F_s &= S = -k\Delta x_e \qquad (1.6) \\ &= -k\left(x - x_e\right) \end{aligned}$$

where x_e is the *equilibrium position of the mass.* If we assume the equilibrium position is at $x = 0$ then we can write our spring force as

$$S = -kx \tag{1.7}$$

And if we write out Newton's second law we get

$$\begin{aligned} F_{net_x} &= -ma_x = -S_{ms} \\ F_{net_y} &= ma_y = N_{mT} - W_{mE} \end{aligned}$$

We can see that a_y should be zero because the mass won't lift off the table in the y-direction. but in the x-direction

$$-a_x = -\frac{S_{ms}}{m}$$

and knowing that

$$S_{ms} = -kx$$

from Hooke's law, we can see that

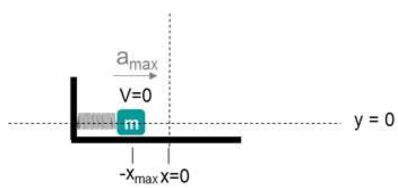
$$-a_x = +\frac{kx}{m}$$

or

$$a_x = -\frac{kx}{m}$$

Since at our starting point x is big our acceleration is big. But there is a minus sign. The minus sign tells us that a_x must be to the left.

But suppose the mass was to the left of $x = 0$



Now $x < 0$ so it is negative. That makes

$$a_x = -\frac{kx}{m}$$

positive. The acceleration always seems to point toward $x = 0$, the equilibrium position for the mass. This is an important part of the definition of simple harmonic motion, having an acceleration that always points toward the equilibrium position. We call a force that makes the acceleration point toward the equilibrium position *restoring force.*

Restoring force: A force that is always directed toward the equilibrium position

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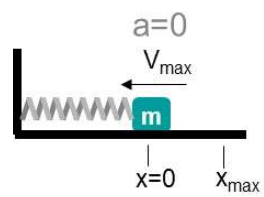
Now lets look at our object a short time later



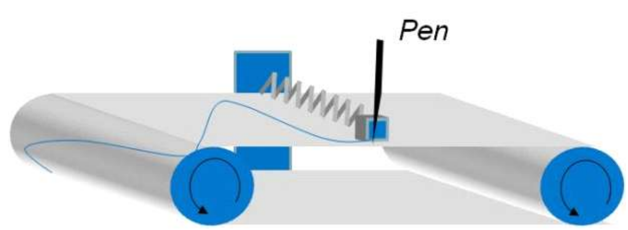
We can see that in this position, $x = 0$ so

$$a_x = \frac{k\,(0)}{m} = 0$$

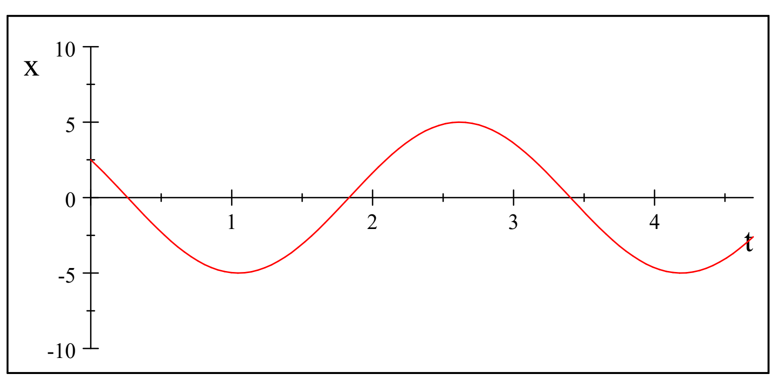
so at this position the acceleration is zero. That is because right at $x = 0$ the spring is neither pushing nor pulling. There is no net force so no acceleration when $x = 0$.

1.3 The Idea of Phase

We said before that $x\,(t)\,,\ v\,(t)\,,$ and $a\,(t)$ are "out of phase." Let's look at the idea of "phase" more carefully. Suppose you return to the grocery store and start your SHM device.



But this time you work with a lab partner, and the lab partner tries to start a stopwatch when you let go of the mass. But, due to having a slow reaction time, your partner starts the clock too late. The resulting graph looks like this

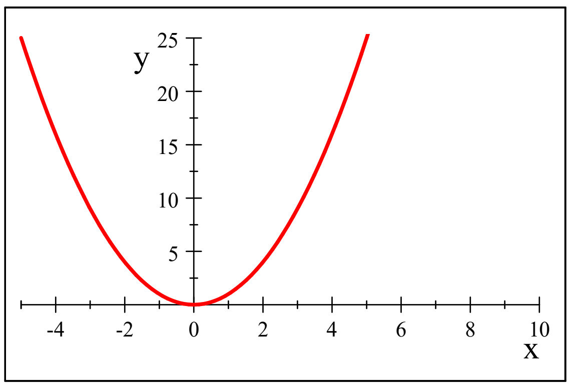


But we know that the only difference between this graph and the one we had before is that the lab partner was slow, so the graph is shifted on the time axes. We expect we can use the same mathematical model for SHM, but we must need to change something.

We know from our algebra classes what a shift looks like. Take the expression

$$y = x^2$$

We can plot this to get a parabola



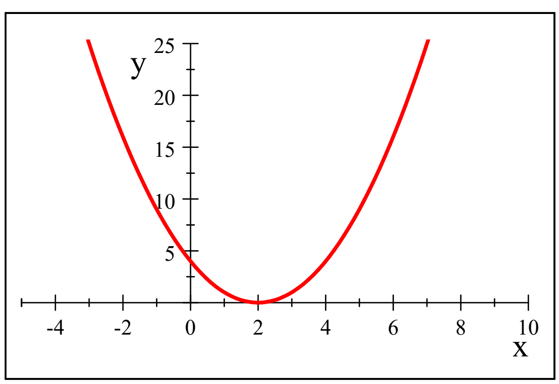
A shifted parabola would be expressed as

$$y = (x - \phi_o)^2$$

where ϕ_o is the amount of the shift. Suppose $\phi_o = 2$, then

$$y = (x - 2)^2$$

and our shifted parabola looks like this



Recall that a shift like

$$y = (x - \phi_o)^2$$

will move the parabola to the right while

$$y = (x + \phi_o)^2$$

will move the parabola to the left.

We can use this idea to write our SHM expression for our situation with the slow lab partner.

$$x(t) = x_{\max} \cos(\omega (t \pm \tau_o))$$

In our case, the lab partner was late by $\tau_o = 4.7124\,\mathrm{s}$. This is not usually how we express the shift, however. We usually distribute the ω so our equation looks like

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t \pm \omega \tau_o) \\ &= x_{\max} \cos(\omega t \pm \omega \tau_o) \end{aligned}$$

We usually use the symbol $\phi_o = \omega \tau_o$ so we will write our SHM expression for position as a function of time as

$$x(t) = x_{\max} \cos(\omega t \pm \phi_o)$$

We could call ϕ_o the *slow lab partner constant,* but that is long and not very kind. So let's call ϕ_o by the name *phase constant.* It is also customary to call the entire expression in parenthesis, $(\omega t \pm \phi_o)$, the phase of the cosine function. This is especially used in the fields of Optics and Electrodynamics.

From what we did before, we know that $x(t)$, $v(t)$, and $a(t)$ are out of phase, so there must be a phase constant involved somehow. Let's look for it in what follows.

Chapter 2

Energy and Dynamics of SHM 1.15.2 1.15,3

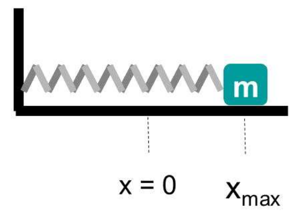
Back in PH121 we started with position, velocity, and acceleration to describe motion. We have done that again for a simple harmonic oscillator. In PH121, after basic motion, we found that we could use the idea of a force and Newton's second law to find the acceleration for our description of motion. Then we changed view points and used the idea of energy to find motion. The energy picture was easier in some ways. Let's try to develop and energy picture for simple harmonic oscillators. Of course the goal in physics is to describe the motion of the object, so we will say that we are looking for the *dynamics* of simple harmonic oscillators when we look for the position, velocity, and acceleration as a function of time.

Fundamental Concepts

- Initial Conditions
- Energy and SHM
- Equation of motion
- Vertical oscillations

2.1 Initial Conditions

Usually, we need to know how we start our oscillator to solve a problem. Let's see how this works.



Suppose we start the motion of a mass attached to a spring (a harmonic oscillator) by pulling the mass to $x = x_{\max}$ and releasing it at $t = 0$. Let's see if we can find the phase. Our initial conditions require

$$\begin{aligned} x(0) &= x_{\max} \\ v(0) &= 0 \end{aligned}$$

Using our formula for $x(t)$ and $v(t)$ we have

$$\begin{aligned} x(0) &= x_{\max} = x_{\max}\cos(0+\phi_o) \\ v(0) &= 0 = -v_{\max}\sin(0+\phi_o) \end{aligned}$$

If we choose $\phi_o = 0$, these conditions are met.

Notice that we needed to know the starting time and the position and the velocity at that time. These are what we call *initial conditions.* It is still true that $x(t)$ and $v(t)$ are out of phase. But we found $\phi_o = 0$. There is another phase term hiding in our expression for $v(t)$ and to find it we need a small trig identity.

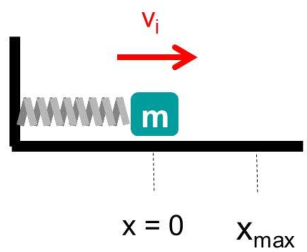
$$\begin{aligned} \sin(\alpha+\phi_o) &= \cos\left(\alpha - \left(\frac{\pi}{2} - \phi_o\right)\right) \\ &= \cos\left(\alpha - \frac{\pi}{2} + \phi_o\right) \end{aligned}$$

so we could write

$$\begin{aligned} v(t) &= -\omega x_{\max}\sin(\omega t+\phi_o) \\ &= -\omega x_{\max}\cos\left(\alpha - \frac{\pi}{2} + \phi_o\right) \end{aligned}$$

and we can see that there was a phase shift of $-\pi/2$ hiding in our sine function. So $v(t)$ really must be out of phase with $x(t)$.

2.1.1 A second example



Using the same equipment, let's start with

$$\begin{aligned} x(0) &= 0 \\ v(0) &= v_i \end{aligned}$$

that is, the mover object is already moving when we start our experiment, and we start watching just as it passes the equilibrium point.

$$\begin{aligned} x(0) &= 0 = x_{\max} \cos(0 + \phi_o) \\ v(0) &= v_i = -v_{\max} \sin(0 + \phi_o) \end{aligned}$$

from the first equation we have

$$0 = x_{\max} \cos(\phi_o)$$

and that gives us

$$\phi_o = \cos^{-1}(0)$$

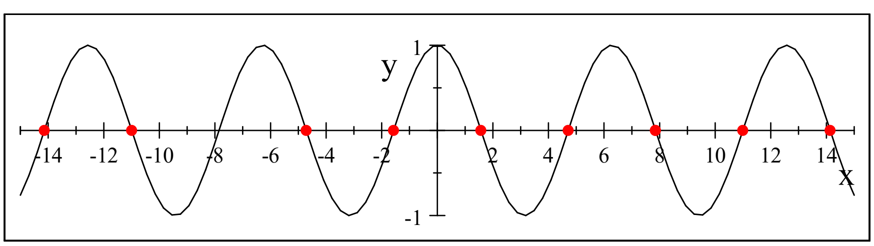
so

$$\phi_o = \pm\frac{\pi}{2}$$

really this is

$$\phi_o = \pm\frac{\pi}{2} \pm n\pi \qquad n = 0, 1, 2, \cdots$$

This gives the red dots in the next plot.



Notice that at each of these locations $\cos(\phi)$ is zero. but let's make an agreement that we will choose the smallest value for ϕ_o that makes $\cos(\phi_o) = 0$. That is our

$$\phi_o = \pm\frac{\pi}{2}$$

But positive and negative $\pi/2$ are the same "smallness." We don't know the sign. Using our initial velocity condition will let us determine the sign. Let's try it

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\frac{\pi}{2}\right) \\ v_i &= \pm v_{\max} \\ v_i &= \pm\omega x_{\max} \end{aligned}$$

From this we can see

$$x_{\max} = \pm\frac{v_i}{\omega}$$

We defined the initial velocity as positive, and we insist on having positive amplitudes, so then we choose

$$\phi_o = -\frac{\pi}{2}$$

so

$$\sin\left(-\frac{\pi}{2}\right) = -1$$

and then this minus sign will cancel the one in our initial velocity equation that to make our initial velocity positive.

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\frac{\pi}{2}\right) \\ &= -v_{\max}(-1) \\ &= v_{\max} \end{aligned}$$

Our solutions are

$$\begin{aligned} x(t) &= \frac{v_i}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right) \\ v(t) &= v_i\sin\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$

Note that our solution is a set of equations! It isn't a set of numbers. We saw this sometimes in PH121. But we will see it more in our study of oscillations and waves. Sometimes the solutions are equations. But we did fill in our equations with the number for the phase constant. it would be even better if we had numbers for ω and v_i.

Generally to have a complete solution, you must find all the constants based on the initial conditions. This would mean we need $x_{\max}$, ω, and ϕ_o to have a complete solution.

Let's try a complete example problem.

2.1.2 Example

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is 2.00 cm, and the frequency is 1.50 Hz.

a) show that the position of the particle is given by

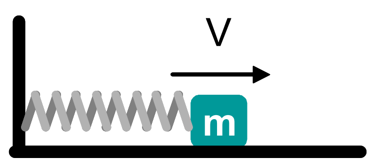
$$x = (2.00\,\mathrm{cm})\sin(3.00\pi t)$$

determine

b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed,

c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration, and

d) the total distance traveled between $t = 0$ and $t = 1.00\,\mathrm{s}$



Basic Equations

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

Variables

t	time, initial time =0	$t_i = 0$
x	Position, Initial position =0	$x(0) = 0$
v		
a		
$x_{\max}$	x amplitude	$x_{\max} = 2.00\,\text{cm}$
v_m	v amplitude	
a_m	a amplitude	
ω	angular frequency	
ϕ_o	phase constant	
f	frequency	$f = 1.50\,\text{Hz}$

Symbolic Solution
Part (a)
We can start by recognizing that we know ω because we know the frequency.

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi\,(1.50\,\text{Hz}) \\ &= 9.424\,8\,\text{rad Hz} \end{aligned}$$

We also know the amplitude $A = x_{\max}$ which is given.

$$A = 2.00\,\text{cm}$$

Knowing that at $t = 0$

$$x(0) = 0 = x_{\max}\cos(0 + \phi_o)$$

which we have seen before! We can guess that

$$\phi_o = \pm\frac{\pi}{2}$$

Using

$$v(0) = -\omega x_{\max}\sin\left(0 \pm \frac{\pi}{2}\right)$$

and demanding that amplitudes be positive values, and noting that at $t = 0$ the velocity is positive from the initial conditions:

$$\phi = -\frac{\pi}{2}$$

We also note from our trig identity that we used above

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

that we have

$$\begin{aligned} x(t) &= x_{\max}\cos\left(2\pi f t - \frac{\pi}{2}\right) \\ &= x_{\max}\sin(2\pi f t) \end{aligned}$$

which is in the form we want. All we have to do is put in numbers.

$$\begin{aligned} x(t) &= x_{\max}\sin(2\pi f t) \\ &= (2.00\,\mathrm{cm})\sin(3.00\pi t) \end{aligned}$$

And once again our solution for a) is an equation.

Part (b)

We have a formula from our previous work for $v_{\max}$

$$\begin{aligned} v_{\max} &= \omega x_{\max} \\ &= 2\pi f x_{\max} \end{aligned}$$

so let's use it. To find when $v_{\max}$ happens, take a derivative of $x(t)$

$$v(t) = v_{\max} = -\omega x_{\max}\sin\left(2\pi f t - \frac{\pi}{2}\right)$$

and recognize that $\sin(\theta) = 1$ is at a maximum and this happens when $\theta = \pi/2$. So we take the stuff that is in the parenthesis for the sine function and set it equal to our angle that makes the sine function equal to 1.

$$\frac{\pi}{2} = 2\pi f t - \frac{\pi}{2}$$

We can simplify this and solve for t

$$\pi = 2\pi f t$$

$$\frac{1}{2f} = t$$

$$t = \frac{1}{2\,(1.50\,\mathrm{Hz})} = 0.333\,33\,\mathrm{s}$$

Part (c) Like with the velocity we must the formula. But we know it is just taking the derivative of $v(t)$

$$a(t) = -\omega^2 x_{\max}\cos(\omega t + \phi_o)$$

but recognize that the maximum is achieved when $\cos(\omega t + \phi_o) = 1$ or when $\omega t + \phi_o = 0$. We can solve this for t.

$$\begin{aligned} t &= \frac{\phi_o}{\omega} \\ &= \frac{-\frac{\pi}{2}}{2\pi f} \\ &= \frac{-1}{4f} \\ &= -0.166\,67\,\mathrm{s} \end{aligned}$$

The formula for $a_{\max}$ is also already in our set of equations

$$\begin{aligned} a_{\max} &= -\omega^2 x_{\max} \\ &= -(2\pi f)^2 x_m \end{aligned}$$

Putting in numbers gives

$$\begin{aligned} a_{\max} &= (2\pi 1.5\,\mathrm{Hz})^2\,(2.00\,\mathrm{cm}) \\ &= 1.776\,5\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Part (d)
We know the period is

$$T = \frac{1}{f}$$

We should find the number of periods in $t = 1.00\,\mathrm{s}$ and find the distance traveled in one period,

$$N = \frac{t}{T}$$

and multiply them together. In one period the distance traveled is

$$d = 4x_m$$

$$d_{tot} = d * N = d * \frac{t}{T} = 4fx_m t$$

Putting in numbers gives

$$\begin{aligned} d_{tot} &= 4fx_m t \\ &= 8.00\,\mathrm{cm} * 1.50\,\mathrm{Hz} * 1.00\,\mathrm{s} \\ &= 0.12\,\mathrm{m} \end{aligned}$$

We have come far in only one lecture! We have a set of new equations and a new problem type. In what we did in this lecture we used Newton's Second Law. But in PH121 we learned that often using the idea of energy made problems easier. Can we use energy with simple harmonic motion problems? That is what we will talk about in our next lecture.

2.2 Oscillators and Energy

You might have noticed that we are calling mover objects that experience simple harmonic motion (SHM) by the name *simple harmonic oscillators (SHO).* Let's consider such a SHO. Because our SHO is moving, we know there must be energy associated with it. To understand the energy involved, let's start with kinetic energy. Recall from PH121 that

$$K = \frac{1}{2}mv^2 \tag{2.1}$$

and we recall that for a spring, we have the spring potential energy given by

$$U_s = \frac{1}{2}kx^2 \tag{2.2}$$

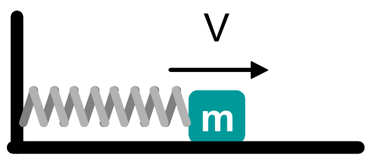
Let's try a problem.

Using energy techniques, find the maximum velocity of the mass in terms of the Amplitude and the angular frequency.

We want to use our problem solving steps:

Type of problem: This is an energy and a SHO problem:

Drawing:



Variables:

E	energy
K	Kinetic energy
x	Position
v	velocity or speed
m	mass of the object
k	spring stiffness constant

Basic Equations:

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ U_s &= \frac{1}{2}kx^2 \end{aligned}$$

Symbolic Answer

You might say, this is an easy problem, we know from last time that

$$v_{\max} = x_{\max}\omega$$

but let's find this again using the ideas of energy. In PH121 we often found that using energy made problems easier, so this might be worth a little more work now. The total mechanical energy is

$$E = K + U_s + U_g$$

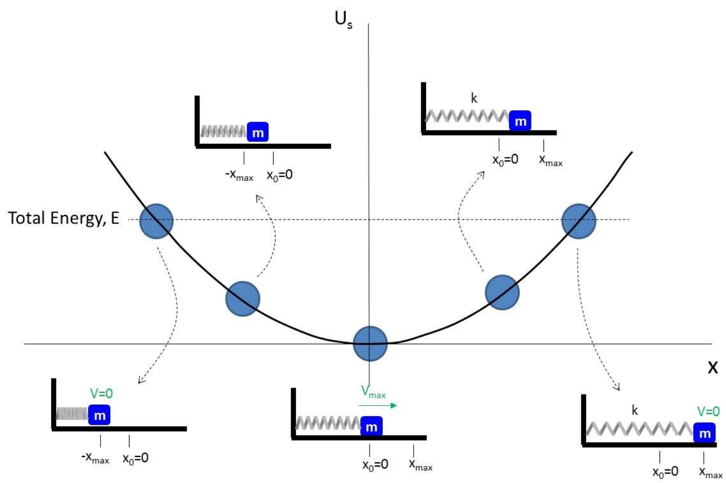
Let's say that our oscillator is moving horizontally just like last time, and that we define the center of mass of our object so that it's y-position is right at $y = 0$ so our object has zero gravitational potential energy.

$$E = K + U_s + 0$$

And let's say we have no friction, so that we can say that energy is conserved. We set the initial and final energies equal to each other.

$$\begin{aligned} E_i &= E_f \\ K_i + U_{si} &= K_f + U_{sf} \\ \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 \end{aligned}$$

And let's start with our initial condition being where the spring is stretched to $x_{\max}$. Then at that moment, $v_i = 0$. And let's take our final position when the mass is moving as fast as possible (because that is what we are looking for, $v_{\max}$).



We know from our PH121 experience that this will be right when $U_{sf} = 0$ (so all the energy is kinetic). Then

$$0 + \frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv_{\max}^2 + 0$$

or

$$\frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx_{\max}^2$$

We can solve for $v_{\max}$

$$v_{\max}^2 = \frac{kx_{\max}^2}{m}$$

$$v_{\max} = x_{\max}\sqrt{\frac{k}{m}}$$

But is this what we wanted? We expected that

$$v_{\max} = x_{\max}\omega$$

And this is what we got so long as

$$\omega = \sqrt{\frac{k}{m}}$$

And this is always true for a mass on a spring. We don't need a numeric answer, This is reasonable (just what we expect) and the units check.

Let's look at kinetic and potential energy as a function of time. For our Simple Harmonic Oscillator (SHO) we know the velocity as a function of time,

$$v(t) = -\omega x_{\max} \sin(\omega t + \phi)$$

so the kinetic energy as a function of time must be

$$\begin{aligned} K &= \frac{1}{2} m \left(-\omega x_{\max} \sin(\omega t + \phi)\right)^2 \\ &= \frac{1}{2} m \omega^2 x_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

and now we know that $\omega = \sqrt{k/m}$, so we can write the kinetic energy as

$$\begin{aligned} K &= \\ &= \frac{1}{2} m \left(\sqrt{\frac{k}{m}}\right)^2 x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} m \frac{k}{m} x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} k x_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

As the spring is stretched or compressed we store energy as spring potential energy. The potential energy due to a spring acting on our SHO (mover mass) is given by (from your PH121 class)

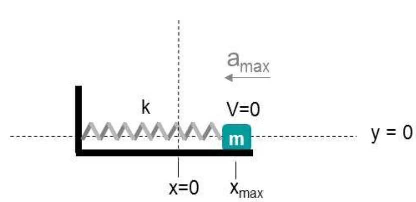
$$U_s = \frac{1}{2} k x^2 \tag{2.3}$$

For our SHO we also know the position as a function of time

$$x(t) = x_{\max} \cos(\omega t + \phi_o)$$

So, the potential energy as a function of time must be

$$U_s = \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi)$$



Let's say, again, that our oscillator is moving horizontally, and that we define the center of mass of our object so that it's y-position is right at $y = 0$ so our object has zero gravitational potential energy

$$U_g = mgy = 0$$

so the total mechanical energy is given by

$$E = K + U_s$$

which we can write out as

$$\begin{aligned} E &= K + U_s \\ &= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi) + \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi) \\ &= \frac{1}{2}kx_{\max}^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) \\ &= \frac{1}{2}kx_{\max}^2 \end{aligned}$$

Where we have used the trig identity that $\sin^2(\theta) + \cos^2(\theta) = 1$.

It tells us that if there are no loss mechanisms (e.g. no friction) then the energy in a harmonic oscillator never changes. And we remember that we would say that energy is conserved for such a situation, which is not surprising because we have already used conservation of energy for springs in PH121 and in the previous example. But let's give a new name to a quantity that does not change. Let's call it a *constant of motion*. So we can make a statement about the total energy for our SHO.

The total mechanical energy of an ideal SHO is a constant of motion

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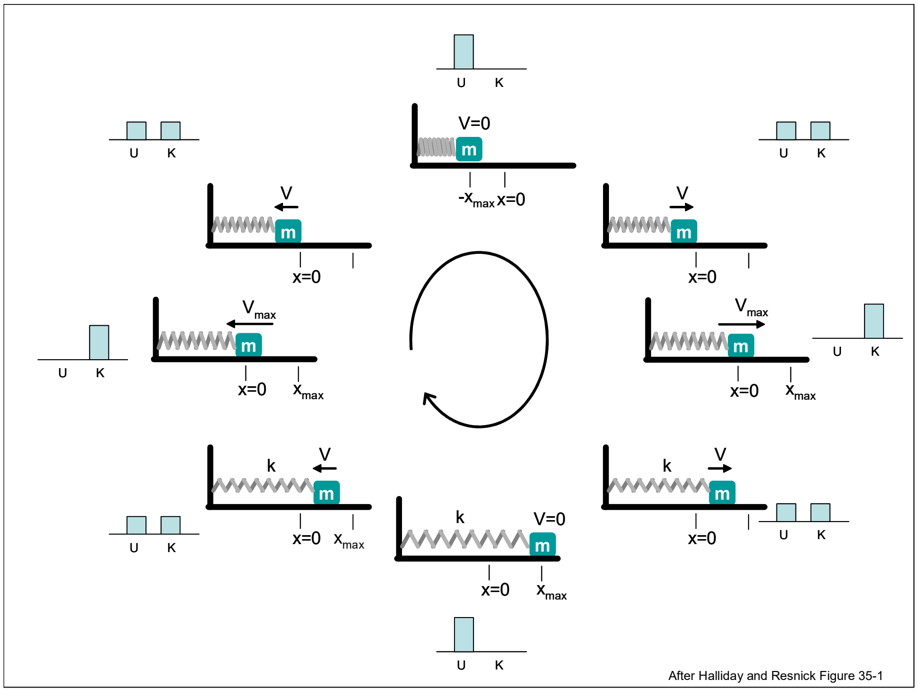
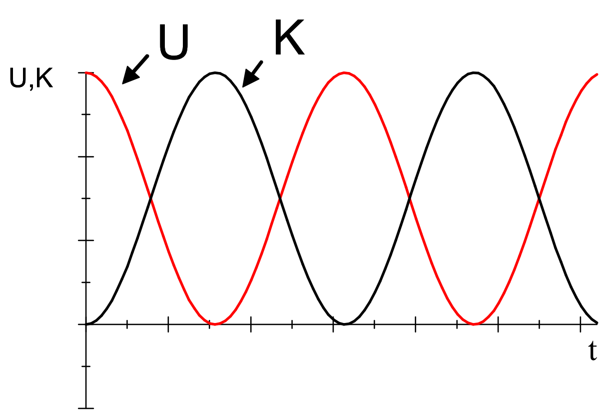
If we plot the amount of kinetic and potential energy for an oscillator we might find something like this:



After Halliday and Resnick Figure 35-1

Note that the kinetic and potential energy are out of phase with each other. If we plot them on the same scale (for the case $\phi = 0$) we have



Let's try another problem using energy of a SHO.

2.3 Mathematical Representation of Simple Harmonic Motion

We have a mathematical representation of simple harmonic motion from looking at our graph of position vs. time. But as a problem, let's use math and what we know from PH121 to show that our equation

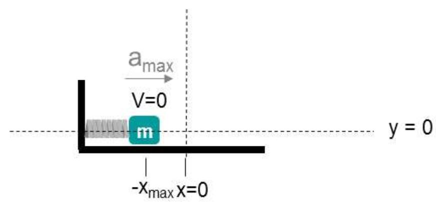
$$x(t) = x_{\max} \cos(\omega t)$$

must be right.

Recall from PH 121

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.4}$$

and let's assume we have SHO moving only in the x-direction.



Further assume the surface the object rests on is frictionless. Also let's say that the equilibrium position $x_{em} = 0$. Then we can write Newton's second law as

$$\begin{aligned} F_{net_x} &= ma_x = -kx \qquad (2.5) \\ m\frac{d^2x}{dt^2} &= -kx \end{aligned}$$

We have a new kind of equation. If you are taking this freshman class as a... well... freshman, you may not have seen this kind of equation before. It is called a differential equation. The solution of this equation is a function or functions that will describe the motion of our mass-spring system as a function of time. It says that the way the object moves is an equation where the second derivative is almost the same as the original function. The only difference is some constants that are multiplied.

It is this function that we want, so let's see how we can find it.

Start by getting all the constants on the same side of the equation.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

It would be tidier if we defined a quantity ω as

$$\omega^2 = \frac{k}{m} \tag{2.6}$$

why define ω^2? Think of our previous example. We found that $\omega = \sqrt{k/m}$. This is just the square of what we found in our example. Then we can write our differential equation as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{2.7}$$

We need a function who's second derivative is the negative of itself with just a constant out front. From Math 112 we know a few of these

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ x(t) &= A\sin(\omega t + \phi_o) \end{aligned}$$

where A, ω, and ϕ_o are constants that we must find. Let's choose the cosine function and explicitly take it's derivatives to see if this function does solve our differential equation

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \qquad (2.8) \\ \frac{dx(t)}{dt} &= -\omega A\sin(\omega t + \phi_o) \\ \frac{d^2x(t)}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

Let's substitute these expressions into our differential equation for the motion

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ -\omega^2 A\cos(\omega t + \phi_o) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

As long as the constant ω^2 is our $\omega^2 = k/m$ we have a solution. We could have found that

$$\omega^2 = \frac{k}{m}$$

by solving this differential equation, but it might not have been as meaningful that way. We can identify ω as the angular frequency.

$$\omega = 2\pi f$$

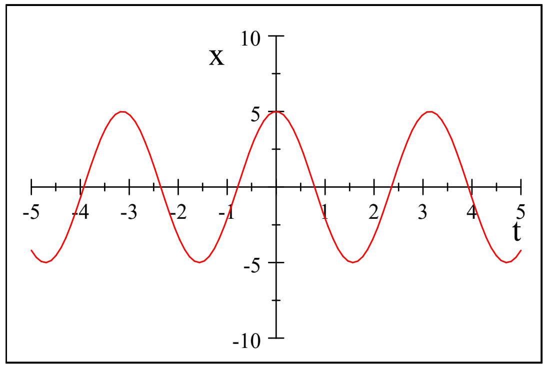
Thus

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f \tag{2.9}$$

This says that if the spring is stiffer, we get a higher frequency, or if the mass is larger, we get a lower frequency.

We still don't have a complete solution, because we don't know A and ϕ_o. We recognize ϕ_o as the initial phase angle. We will have to find this by knowing the initial conditions of the motion. A is the amplitude. That must be the maximum displacement $x_{\max}$. Let's look at a specific case

$$\boxed{\begin{array}{|l|} \hline x_{\max} = 5 \\ \hline \phi_o = 0 \\ \hline \omega = 2 \\ \hline \end{array}} \tag{2.10}$$



We can easily see that the amplitude A corresponds to the maximum displacement $x_{\max}$. (how would you prove this?). We know from trigonometry that a cosine function has a period T.

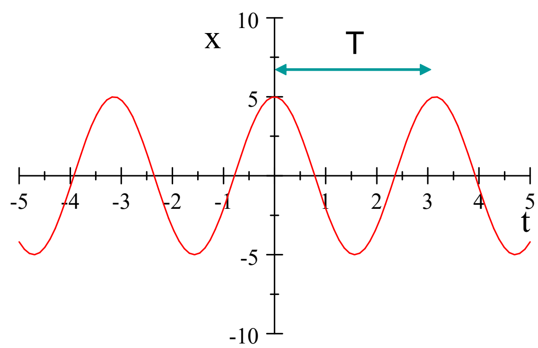


Figure 2.1:

The period is related to the frequency

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{2.11}$$

We can write the period and frequency in terms of our mass and spring constant

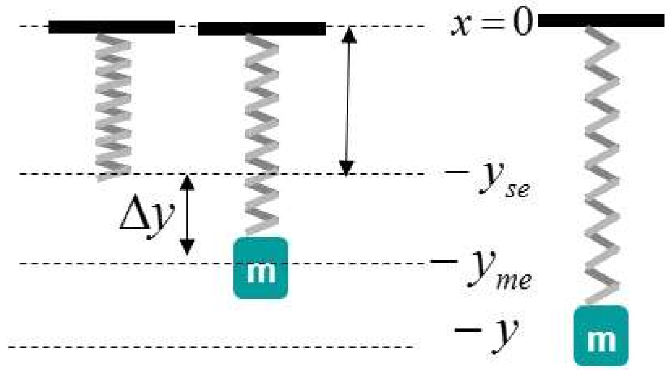
$$T = 2\pi\sqrt{\frac{m}{k}} \tag{2.12}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.13}$$

From our graph we were able to complete our problem. But more importantly we have two new equations for T and f that will help us solve more problems.

2.3.1 Hanging Springs

Let's do a problem. We now know we have forces involved with our SHOs. So let's do a force problem. In class so far, I have always hung our masses and springs, but we have used horizontal systems for calculations. Let's find the equation of motion for a mass hanging from a spring.



From observation, we would guess that we can just choose the new equilibrium point to be $y = 0$ and use the same equation

$$y(t) = y_{\max} \cos(\omega t - \phi_o)$$

let's see if that is true. Start with a free body diagram for the mass (the hanging mass is our mover, the spring is part of the environment). There are two forces acting on the mass. The force due to gravity, and the force due to the spring.

$$\Sigma F_y = ma_y = S_{ms} - W_{mE}$$

We know the form of these forces

$$\begin{aligned} S_{ms} &= -k\Delta y \\ W_{mE} &= -mg \end{aligned}$$

but we need to carefully choose our origin. Let's try the top of the spring where it attaches to the stand. If the mass just hangs there we would expect the spring to stretch to an equilibrium length y_e and any other motion would either shorten or lengthen the spring. We can write our force equation as

$$\begin{aligned} ma_y &= S_{ms} - W_{mE} \\ &= -k(y - y_e) - mg \end{aligned}$$

where y can be positive or negative. If $y = 0$ and the mass is just sitting there, not oscillating, there is no acceleration. Then the stretched length is just y_e. In this case

$$\begin{aligned} m(0) &= k(0 - y_e) - mg \\ ky_{em} &= mg \end{aligned}$$

But now let's let our mass move again. We can substitute our stationary mass answer this into the previous equation for a moving mass

$$\begin{aligned} ma &= -k(y - y_e) - mg \\ &= ky_e - ky - mg \\ &= mg - ky - mg \\ &= -ky \end{aligned}$$

This gives a net force for the system of

$$F_{net} = -ky$$

It is as though the system were horizontal with no gravitational force and only a spring force. We can see that we are justified in claiming that we could simply choose the origin at the distance y_{em} from the top of the spring, and we can use the equation

$$y(t) = y_{\max} \cos(\omega t - \phi_o)$$

as our equation of motion.

Chapter 3

More Oscillators, Forces and Friction 1.15.4 1.15.5 1.15.6

You have probably wondered if anyone actually uses mass-spring systems. And we do. They were more common in the past where springs were used to store energy (U_s) to run clocks and toys and machines. But simple harmonic motion can describe other systems as well. You have probably seen an old fashioned clock with a pendulum. A pendulum almost experiences simple harmonic motion. Let's see how this works.

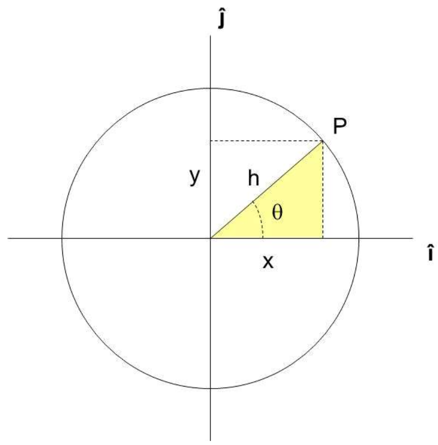
Fundamental Concepts

- Relationship between Simple Harmonic Motion and circular motion.
- Pendula
- Damping
- Driving
- Resonance

3.1 Comparing Simple Harmonic Motion with Uniform Circular Motion

You might have objected to our use of ω as angular frequency. Didn't ω mean angular speed in PH121?

That circular motion and SHM are related should not be a surprise once we found the solutions to the equations of motion were trig functions. Recall that the trig functions are defined on a unit circle

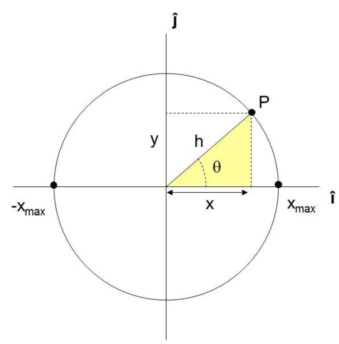


$$\tan\theta = \frac{x}{y} \tag{3.1}$$

$$\cos\theta = \frac{x}{h} \tag{3.2}$$

$$\sin\theta = \frac{y}{h} \tag{3.3}$$

Let's relate this to our SHM equations of motion.



Look at the projection x of the point P on the x axis. This is just the x-component of the position! Lets follow this projection as P travels around the circle. We find the projection ranges from $-x_{\text{max}}$ to x_{max}. If we watch closely we find the projection's velocity is zero at the extreme points and is a maximum in the middle. This projection is given as the cosine of the vector from the origin to P. It is just taking the x-component! This projection, indeed fits our SHO solution.

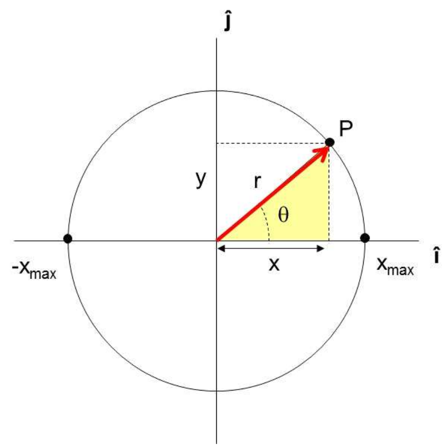
Now lets define a projection of P onto the y axis. Again we have SHM, but this time the projection is a sine function. Because

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\left(\theta\right) \tag{3.4}$$

we can see that this is just a SHO that is $90\,^{\circ} = \frac{\pi}{2}$ rad out of phase. It is probably worth recalling that a projection of one vector on another can be expressed by a dot product. We could express our length x as

$$x = \overrightarrow{\mathbf{r}} \cdot \hat{\imath} = r\cos\theta$$

where r is the radius of the circle.



We can see that when $\theta = 0$ we have $x = r$ and this will be the largest x value, so $r = x_{\mathrm{max}}$.

So by projecting circular motion onto the x-axis

$$x\left(t\right) = x_{\mathrm{max}}\cos\left(\theta\right)$$

But θ changes in time. We can recall from our PH121 or Dynamics experience that the angular speed

$$\omega = \frac{\Delta\theta}{\Delta t}$$

or, if we agree to start from $\theta_i = 0$ and $t_i = 0$,

$$\omega = \frac{\theta}{t}$$

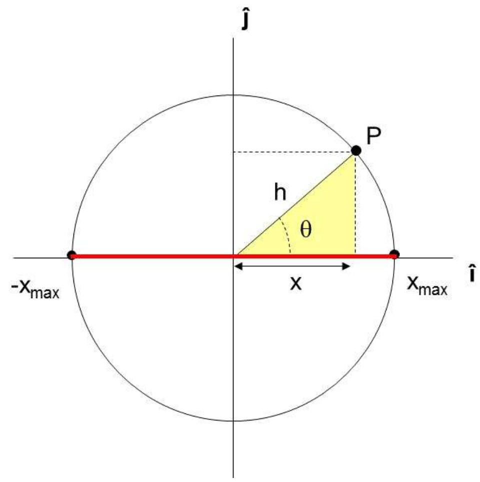
so

$$\theta = \omega t$$

then we have

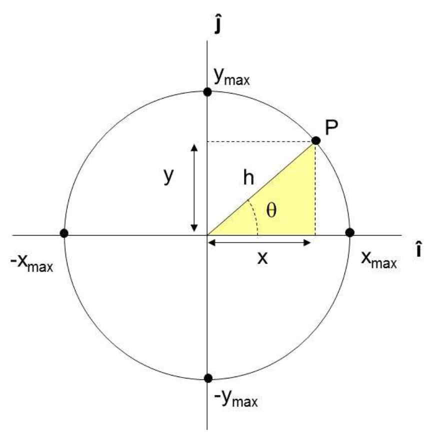
$$x\left(t\right) = x_{\mathrm{max}}\cos\left(\omega t\right)$$

Which is just our equation for SHM. Now we can see why we used "ω" for both angular speed and angular frequency. Really they are very related. Both tell us something about how fast a cyclic event happens. For motion around a circle, one is how fast the point P goes around the circle, and the other is how often the projection goes back and forth. It makes sense that these have to be the same rate.



The projection of circular motion onto the x-axis gives simple harmonic motion.

Let's go back to our projection on the y-axis.



We found that we can describe this projection as

$$y\left(t\right)=y_{\max}\sin\left(\omega t\right)$$

We will choose the cosine function, but from our trig experience it should be clear that these projections are equivalent, just 90° out of phase

$$\cos\left(\theta\pm\frac{\pi}{2}\right)=\mp\sin\theta$$

So

$$x\left(t\right)=x_{\max}\cos\left(\omega t+\phi_{o}\right)$$

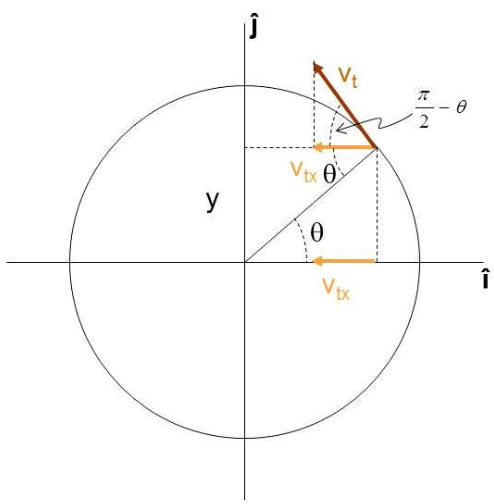
could be a sine function if $\phi_o = \pm\pi/2$. In this way we have incorporated both possibilities into one function.

Note that this is just the function we guessed from our observation!

We see that uniform circular motion can be thought of as the combination of two SHOs, with a phase difference of $\pi/2$ rad.

The angular velocity is given by

$$\omega = \frac{v}{r} \tag{3.5}$$



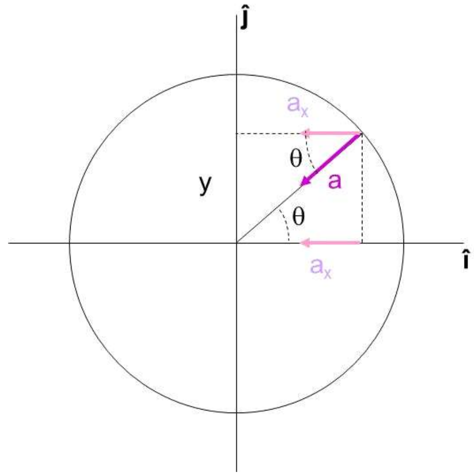
A particle traveling on the x-axis in SHM will travel from $x_{\max}$ to $-x_{\max}$ and from $-x_{\max}$ to $x_{\max}$ (one complete period, T) while the particle traveling with P makes one complete revolution. Thus, the angular frequency ω of the SHO and the angular velocity of the particle at P are the same. The magnitude of the tangential velocity is then

$$v_t = \omega r = \omega x_{\max} \tag{3.6}$$

and the projection of this velocity onto the x-axis is

$$v_{tx} = -\omega x_{\max} \sin(\omega t + \phi_o) \tag{3.7}$$

Which is just what we expected from our earlier observation!



The centripetal acceleration of a particle at P is given by

$$a_c = \frac{v_t^2}{r} = \frac{v_t^2}{x_{\max}} = \frac{\omega^2 x_{\max}^2}{x_{\max}} = \omega^2 x_{\max} \tag{3.8}$$

The direction of the acceleration is inward toward the origin. Of course, we just want the x-component of this, so again we make a projection. If we project this onto the x-axis we have

$$a_x = -\omega^2 x_{\max} \cos(\omega t + \phi) \tag{3.9}$$

Again this is just want we expected from our observation.

So now we have shown that our set of equations

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t) \qquad (3.10) \\ v(t) &= -\omega x_{\max} \sin(\omega t) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t) \end{aligned}$$

is correct for our harmonic oscillator.

3.2 Our first problem type: Simple Harmonic Motion

You have probably thought by now that we have a new problem type. We can call it the *simple harmonic motion* problem type or SHM. The equations we have so far for this problem type are

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t + \phi_o) \qquad (3.11) \\ v(t) &= -\omega x_{\max} \sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

$$U_s = \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi)$$

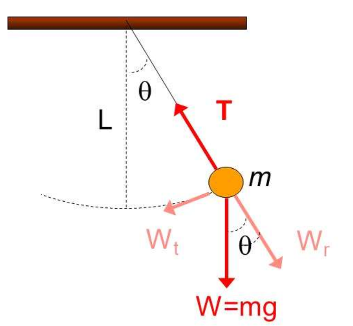
$$K = \frac{1}{2} m\omega^2 x_{\max}^2 \sin^2(\omega t + \phi)$$

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{3.12}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3.13}$$

We have used energy to describe simple harmonic motion, and we have found our equation of motion using a differential equation for mass-spring systems. And we have simple harmonic motion in the y-direction including the weight force due to gravity for mass-spring systems. In our next lecture, we will study simple harmonic motion for different systems, pendula, and other things. It turns out that SHM is a good model for many different systems.

3.3 The Simple Pendulum



A simple pendulum is a mass on a string. The mass is called a "bob." Usually we study the motion of the pendulum bob, so let's consider the pendulum the mover object. A simple pendulum bob exhibits periodic motion, but not exactly simple harmonic motion. The forces on the bob are $\overrightarrow{\mathbf{W}}$, and $\overrightarrow{\mathbf{T}}$ the tension on the string. The tangential component of W is always directed toward $\theta = 0$. This is a restoring force!

Let's call the path the bob takes "s." Then from Jr. High geometry we recall[1]

$$s = L\theta \tag{3.14}$$

We will use a the cylindrical or rtz coordinate system. The radial axis is directed along the string. The tangential direction is along the circular path the bob takes and is always tangent to the path. In this coordinate system, we can solve for the part of the force directed along the path. This is the restoring part of the net force. Remember from Newton's second law the tangential and radial components of the force are

$$\begin{aligned} F_t &= ma_t \\ F_r &= ma_r \end{aligned}$$

and

$$a_t = \frac{d^2 s}{dt^2}$$

Let's write out Newton's second with the sum of the forces part.

$$\begin{aligned} ma_t &= -W\cos(90-\theta) \\ -ma_r &= -T + W\sin(90-\theta) \end{aligned}$$

then, using a trig identity (but only a small one)

$$\begin{aligned} a_t &= -\frac{W}{m}\sin(\theta) \\ &= -g\sin\theta \end{aligned}$$

We have two expressions for a_t. We can set them equal

$$\frac{d^2 s}{dt^2} = -g\sin\theta \tag{3.15}$$

Remember that $s = L\theta$ so we could write the left hand side as

$$\frac{d^2 s}{dt^2} = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2}{dt^2}(\theta)$$

then equation (3.15) becomes

$$L\frac{d^2}{dt^2}(\theta) = -g\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

[1] Really I did not recall this, I have to look it up every time, but s is called the *arclength*.

This is a differential equation much like our differential equation for a harmonic oscillator,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

except it has a sine function in it. But, if we take θ as a very small angle, then

$$\sin(\theta) \approx \theta \tag{3.16}$$

This approximation has a name, it is called the "small angle approximation."

In this approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

and we have a differential equation we recognize! If we compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

we see that it is a match if

$$\omega^2 = \frac{g}{L} \tag{3.17}$$

we have all the same solutions for θ that we found last time for x. Since ω changed, the frequency and period will now be in terms of g and L.

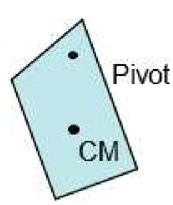
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.18}$$

For a pendulum that oscillates only over small angles, the period and frequency depend only on L and g!

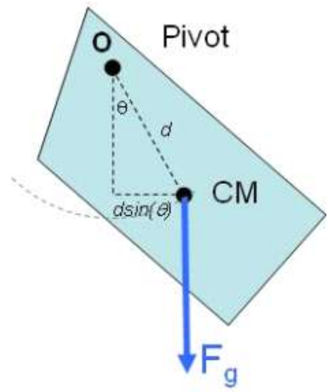
The analysis we just did for the pendulum we can do for other simple harmonic oscillators (or near simple harmonic oscillators). Let's try a few.

3.3.1 Physical Pendulum

Usually when we build a pendulum, we assume the string is so small compared to the bob, that we can ignore it's mass. What if this is not true? Suppose we build a pendulum by making a large solid object swing from one point. Can we describe it's motion?



Let's pull it to the right



then we consider that because of F_g we will have a torque about an axis through O. Last pendulum we used Newton's Second Law. This time let's use Newton's second law for rotation.

$$\tau = \mathbf{r} \times \mathbf{F}$$

In our case this is

$$\begin{aligned} \tau &= \mathbf{d} \times \mathbf{F}_g \\ &= -F_g d \sin\theta \\ &= -mgd \sin\theta \end{aligned}$$

Remember that an extended body has a moment of inertia, $\mathbb{I}$. Remember also that angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

From the rotational form of Newton's Second Law for rotation.

$$\Sigma\tau = \mathbb{I}\alpha$$

we can write

$$-mgd \sin\theta = \mathbb{I}\frac{d^2\theta}{dt^2}$$

Getting all the constants together gives

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}} \sin(\theta)$$

and again if we let θ be small so that $\sin(\theta) \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}}\theta$$

which we can compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

and we can see that we have the same differential equation if

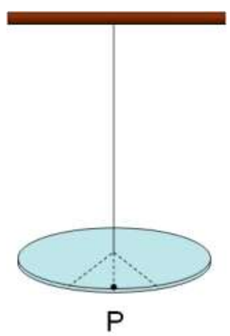
$$\omega^2 = \frac{mgd}{\mathbb{I}} \tag{3.19}$$

In this case

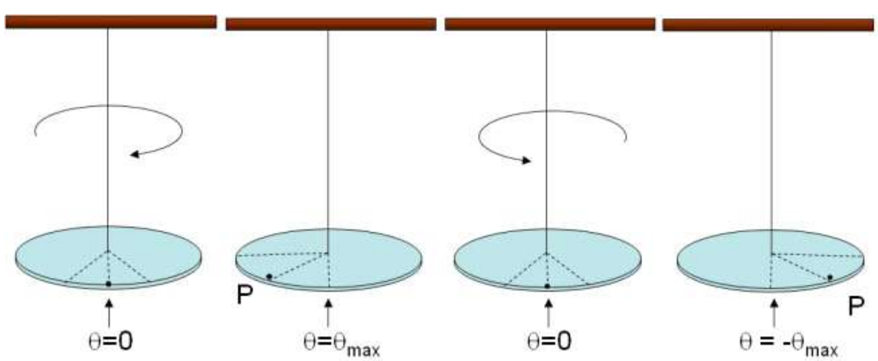
$$T = 2\pi\sqrt{\frac{\mathbb{I}}{mgd}} \tag{3.20}$$

3.3.2 Torsional Pendulum

Physics majors and those taking PH220 in the future, you will see a torsional pendulum. Mechanical engineering majors might use a torsional pendulum. A torsional pendulum is made by suspending a rigid object from a wire.



In this case, a rigid disk. The object can rotate. But wires don't like to twist. The spring-like molecular bonds resist the twisting.



The twisted wire exerts a restoring torque on the body that is proportional to the angular position (sound familiar).

$$\boldsymbol{\tau} = -\kappa\theta$$

This looks like a greek version of $F = -kx$! Again, let's use Newton's Second Law for rotation.

$$\begin{aligned}\Sigma\tau &= \mathbb{I}\alpha \\ &= \mathbb{I}\frac{d^2\theta}{dt^2}\end{aligned}$$

so

$$\begin{aligned}\mathbb{I}\frac{d^2\theta}{dt^2} &= -\kappa\theta \\ \frac{d^2\theta}{dt^2} &= -\frac{\kappa}{\mathbb{I}}\theta\end{aligned}$$

Once again we have our favorite differential equation so long as

$$\omega^2 = \frac{\kappa}{\mathbb{I}} \tag{3.21}$$

which makes the period of the oscillation

$$T = 2\pi\sqrt{\frac{\mathbb{I}}{\kappa}} \tag{3.22}$$

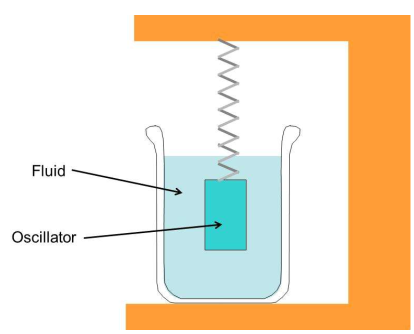
So far we have used only idea springs or wires or frictionless pendula with no air drag. This made the equations easier. But what if there is friction of some sort? Let's deal with this next.

3.4 Damped Oscillations

You remember friction from PH121. So far we have only allowed frictionless oscillators to make the math easy. But what if we do have friction? To investigate this,suppose we add in another force

$$\mathbf{D} = -b\mathbf{v} \tag{3.23}$$

This force is proportional to the velocity. This a dissipative (friction-like) force typical of what we find when we moves objects through viscus fluids. This is a drag force, but a more extreme drag force than we used in PH121 where we only had air to make the drag force.



We call b the damping coefficient and it depends on how much friction the fluid can supply. Now, from Newton's second law,

$$\Sigma F = -kx - bv_x = ma$$

We can write the acceleration and velocity as derivatives of the position just like we have done before

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is another differential equation! But it is harder to guess its solution, and finding that solution is a subject for a differential equations class like M316 or PH332, so we won't learn how to find the solution here, but we can use the results from our trusted colleagues in the math department[2]. Here is the solution:

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos(\omega t + \phi_o) \tag{3.24}$$

which looks simple enough, but now we have the added complication that ω is more complex

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{3.25}$$

so we get quite a mess if we write equation (3.24) with this new ω.

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) t + \phi_o\right) \tag{3.26}$$

To see what this solution means, we should study three cases:

1. the damping force is small: $(bv_{\max} < kx_{\max})$ The system oscillates, but the amplitude is smaller as as time goes on. We call this "underdamped."

2. the damping force is large: $(bv_{\max} > kx_{\max})$The system does not oscillate. we call this "overdamped." We can also say that $\frac{b}{2m} > \omega_o$ (after we define ω_o below)

3. The system is "critically damped" (see below).

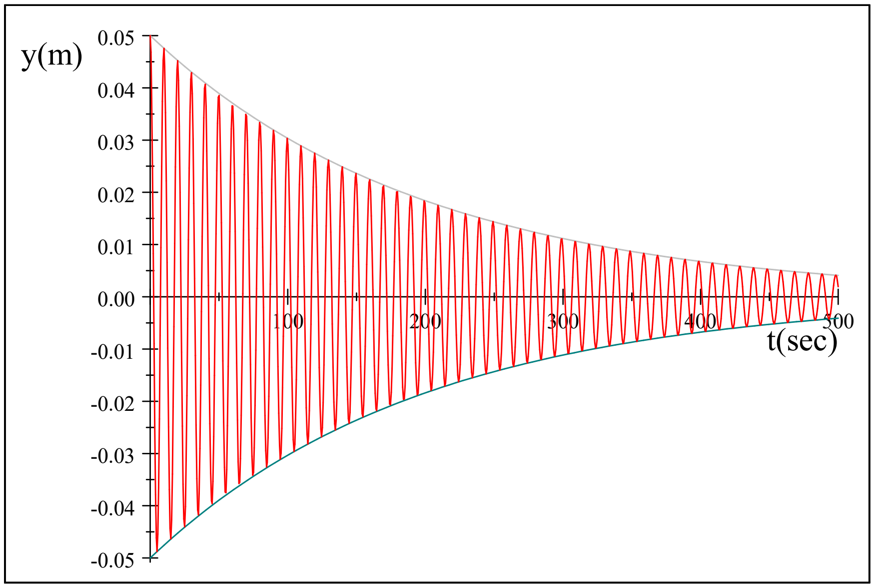
Let's look at an example. Suppose we have an oscillator with the following characteristics:

1.

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.005\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$

[2] That is, until you finish M316, then you will know how to do this kind of problem yourself!

Graphing the equation of motion $x(t)$, we we get a graph that looks like this



The gray lines are given by

$$\pm x_{\max} e^{-\frac{b}{2m}t}$$

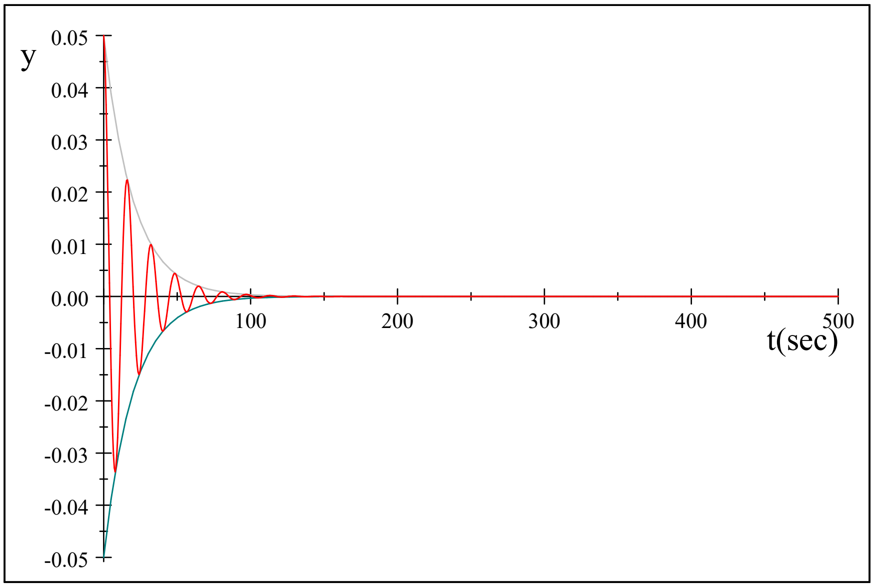
Notice that this quantity is as a collection of terms multiplies the cosine part. It is marked in curly braces blow

$$x(t) = \left\{ x_{\max} e^{-\frac{b}{2m}t} \right\} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \right) t + \phi_o \right)$$

We know that the part of the equation that multiplies the cosine part is the amplitude. But now the amplitude is more than just $x_{\max}$. And notice that the amplitude $\left\{ x_{\max} e^{-\frac{b}{2m}t} \right\}$ changes with time. The gray lines in the figure show how the amplitude changes. We call this the *envelope* of the curve. The oscillation fits within the gray lines (like an old fashioned letter fits inside a paper envelope).

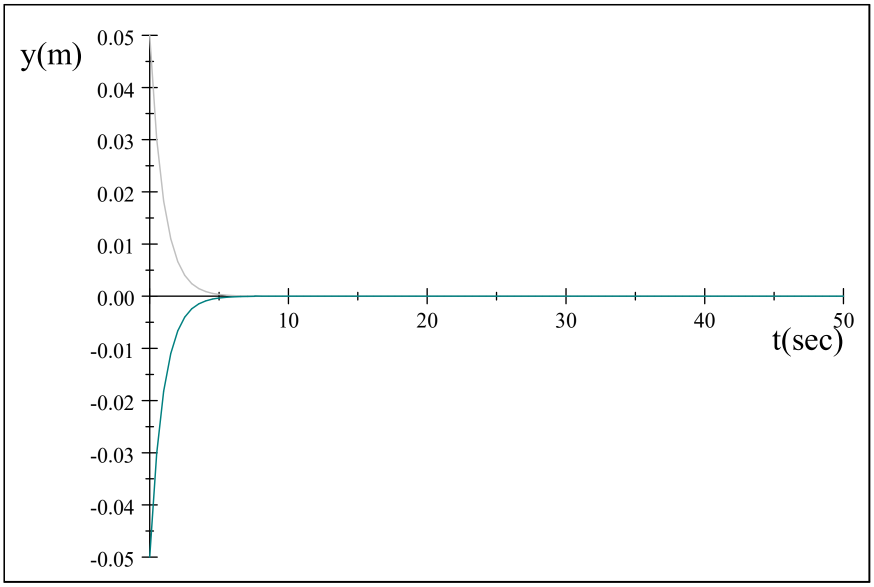
Now let's change b to a larger value

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.05\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$



we see we have less oscillation. The envelope has become more restrictive, making the oscillation die out more quickly. This is a little bit like going over a bump in your car. The car may go up and down a few times, but not many. Now let's increase b even more.

$$\begin{array}{|l|}\hline x_{\max} = 5\,\mathrm{cm} \\ \hline b = 0.5\frac{\mathrm{kg}}{\mathrm{s}} \\ \hline k = 0.2\frac{\mathrm{N}}{\mathrm{m}} \\ \hline m = .5\,\mathrm{kg} \\ \hline \phi_o = 0 \\ \hline \end{array} \tag{3.27}$$



What happened?

When the damping force gets bigger, the oscillation eventually stops. Only the exponential decay is observed. This happens when

$$\frac{b}{2m} = \sqrt{\frac{k}{m}}$$

When that is true,

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0$$

We call this situation *critically damped*. We are just on the edge of oscillation. We define

$$\omega_o = \sqrt{\frac{k}{m}}$$

as the *natural frequency* of the system. Then the value of b that gives us critically damped behavior is

$$b_c = 2m\omega_o$$

When $\frac{b}{2\pi} \geq \omega_o$ the solution in equation (3.24) is not valid! If you are a physicist or a mechanical engineer you will find out more about this situation in your advanced mechanics classes.

Chapter 4

Forces Oscillations and Waves 1.15.6

We studied oscillation with mass-spring systems and pendula. In today's lecture we will add in forces to make the oscillators start to move. Oscillation is a motion of a mass, but with another object (a spring or string) participating in the motion. The air in our classroom is an example of this type of motion in a way, but instead of one object on a spring, there are millions of objects, the air molecules. The molecules move randomly, but with a specific distribution of speeds. But what would happen if all the molecules moved together in the same direction (more or less) and at the same speed (more or less)? This is what we call wind! Of course we can have bulk motion of millions of objects, like wind. We are going to study an even more specific motion of millions of object that is not random like thermal motion. This specific motion of the objects we will call a *wave.*

Fundamental Concepts

1. Forced Oscillations
2. A wave requires a disturbance, and a medium that can transfer energy
3. Waves are categorized as longitudinal or transverse (or a combination of the two).

4.1 Forced Oscillations

We found in the last section that if we added a force like

$$\mathbf{F}_d = -b\mathbf{v}$$

our oscillation died out. An example would be a small child on a swing. You give them a push, but eventually they stop swinging.

Suppose we want to keep the child going? You know what to do, you push! But you have to push at just the right time.

Suppose we make a machine that can push the child on the swing. We could apply a periodic force like

$$F(t) = F_o \sin(\omega_f t)$$

where ω_f is the angular frequency of this new driving force and where F_o is a constant.

$$\Sigma F = F_o \sin(\omega_f t) - kx - bv_x = ma$$

1. When this system starts out, the solution is very messy. It is so messy that we will not give it in this class! (So maybe this isn't really a good way to drive a child on a swing). But after a while, a steady-state is reached. In this state, the energy added by our driving force $F_o \sin(\omega_f t)$ is equal to the energy lost by the drag force, and we have

   $$x(t) = A\cos(\omega_f t + \phi)$$

   our old friend! BUT NOW the amplitude is given by

   $$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2 - \omega_o^2\right)^2 + \left(\frac{b\omega_f}{m}\right)^2}}$$

   where

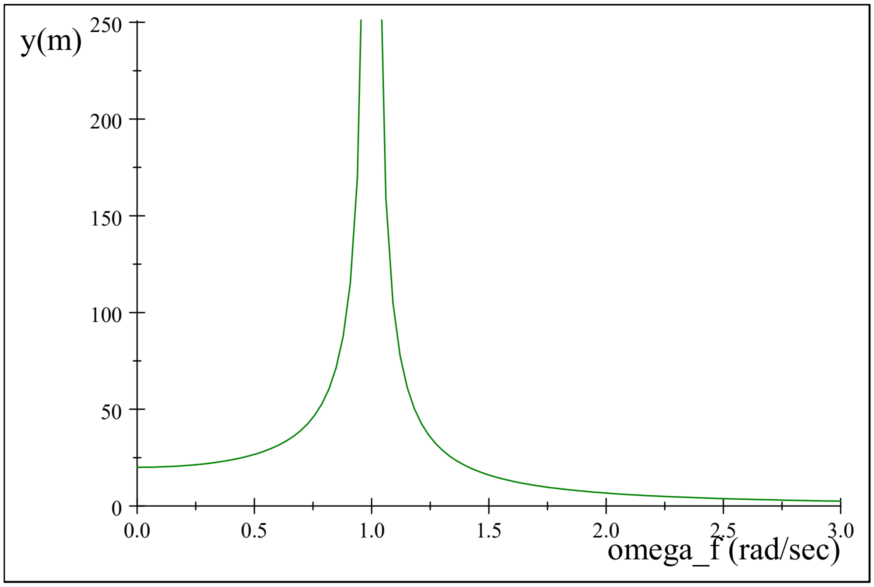
   $$\omega_o = \sqrt{\frac{k}{m}} \tag{4.1}$$

   is the natural frequency. This is the frequency we had before when we weren't pushing the oscillator. And recall that ω_f is the frequency of our force. So now our solution looks more like our original SHM solution except for the wild formula for A. In fact, it operates very like our SHM solution. But what does the new version of A mean?

   Lets look at A for some values of ω_f. I will pick some nice numbers for the other values.

$F_o = 2\,\mathrm{N}$
$b = 0.5\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.5\frac{\mathrm{N}}{\mathrm{m}}$
$m = 0.5\,\mathrm{kg}$
$\phi_o = 0$

   The graph looks like this:



Now let's calculate ω_o

$$\begin{aligned}\omega_o &= \sqrt{\frac{0.5\frac{\mathrm{N}}{\mathrm{m}}}{0.5\,\mathrm{kg}}} \\ &= \frac{1.0}{\mathrm{s}}\end{aligned}$$

Notice that right at $\omega_f = \omega_o$ our solution gets very big. This is called *resonance*. To see why this happens, think of the velocity for a simple harmonic oscillator

$$\frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o)$$

Our driving force is still

$$F(t) = F_o \sin(\omega t)$$

And remember that work is given by

$$w = \int \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}}$$

or just

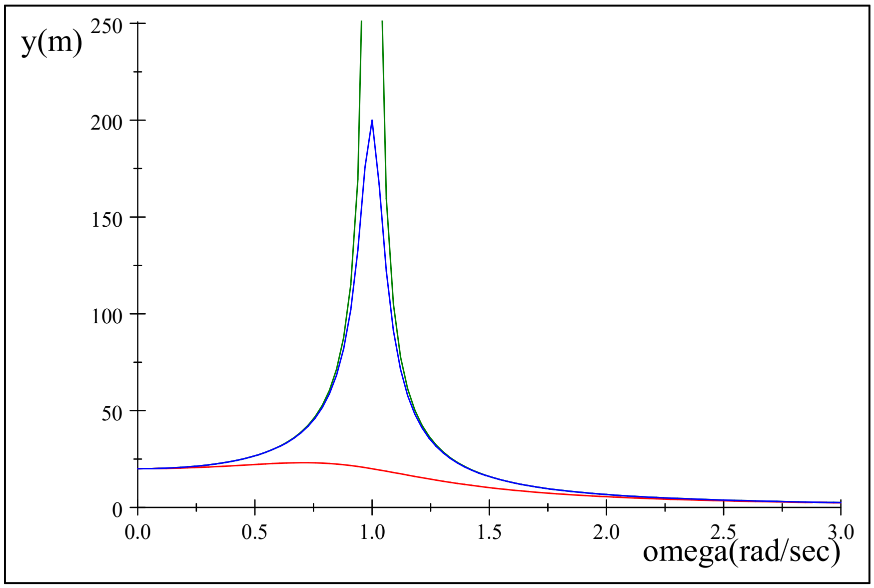
$$w = \overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}$$

if our force is constant. The rate at which work is done (power) is

$$\begin{aligned}\mathcal{P} &= \frac{\overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}}{\Delta t} = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{v}} & (4.2) \\ &= -F_o \omega A \sin(\omega t) \sin(\omega t + \phi_o) & (4.3)\end{aligned}$$

if F and v are in phase ($\phi_o = 0$), the power will be at a maximum! Think of pushing the child on the swing. You push in phase with the oscillation of the child. When you do this the amplitude of the swing get's bigger.

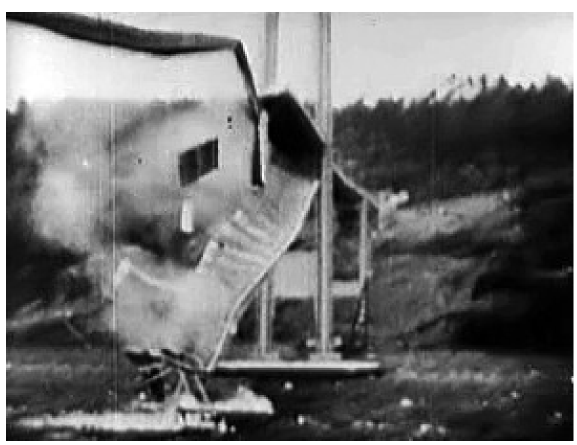
We can plot A for several values of b, our damping (friction) coefficient.



Green: b=0.005kg/s; Blue: b=0.05kg/s; Red b=0.01 kg/s

As $b \to 0$ we see that our resonance peak gets larger. In real systems b can never be zero, but sometimes it can get small. As $b \to$large, the resonance dies down and our A gets small.

Resonance can be great, it can make a musical instrument sound louder (more on this later). But it can also be bad. Here is a picture of the Tacoma Narrows Bridge.



Fall of the Tacoma Narrows Bridge (Image in the Public Domain)

The wind gusts formed a periodic driving force that allowed a driving harmonic oscillation to form. Since the bridge was resonant with the gust frequency, the amplitude grew until the bridge materials broke. Resonance can be a bad thing for structures.

4.2 What is a Wave?

Waves are organized motions in a group of objects. We will give a name to the group of objects, we will call a *medium*.

Spring Demo

4.2.1 Criteria for being a wave

Waves involve energy transfer, but in the case of waves the energy is transferred through space without transfer of matter. Winds transfers both energy and matter. So waves are different than wind. In a wave the objects don't move far from where they start. Think of an oscillation. The mass does move, but never gets too far away from its equilibrium position. Waves are very like this. This is a very specific kind of motion. To be a wave, the motion must have the following characteristics"

Spring Demo-marked part

1. some source of disturbance
2. a medium (group of objects) that can be disturbed
3. some physical mechanism by which the objects of the medium can influence each other

A wave made by a rock thrown into a pond will go out in all directions away from the place where the rock started the wave. This is a normal way that waves are formed. A disturbance starts the wave (the rock disturbs the water) and the energy from the disturbance moves away from the disturbance as a wave. But if we have a wave in a rope or string, the wave can't go in all directions because the string does not go in all directions.

Let's take on this one-dimensional case of a wave on a rope or string first. In the limit that the string mass is negligible we represent a one-dimensional wave mathematically as a function of two variables, position and time, $y(x,t)$.

4.2.2 Longitudinal vs. transverse

We divide the various kinds of waves that occur into two basic types:

Transverse wave: a traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation

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Longitudinal wave: a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation

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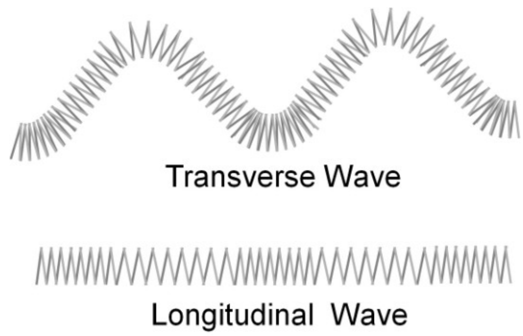
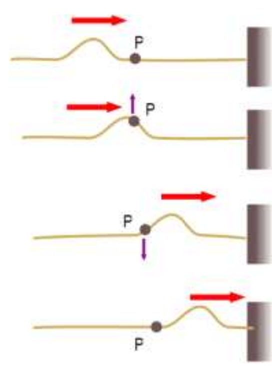


4.2.3 Examples of waves:

Let's look at some specific cases of wave motion.

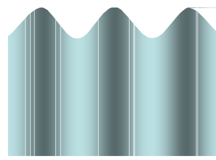
A pulse on a rope:



In the picture above, you see wave that has just one peak traveling to the right. We call such a wave a *pulse*. Notice how the piece of the rope marked P moves up and down, but the wave is moving to the right. This pulse is a transverse wave because the parts of the medium (observe point P) move perpendicular to the direction the wave is moving.

An ocean wave:

Of course, some waves are a combination of these two basic types. You may have noticed that in Physics we tend to define basic types of things, and then use these basic types to define more complex objects. Water waves, for example, are transverse at the surface of the water, but are longitudinal throughout the water.



Earthquake waves:

Earthquakes produce both transverse and longitudinal waves. The two types of waves even travel at different speeds! P waves are longitudinal and travel faster, S waves are transverse and slower.

4.3 Wave speed

We can perform an experiment with a rope or a long spring. Make a wave on the rope or spring. Then pull the rope or spring tighter and make another wave. We see that the wave on the tighter spring travels faster.

It is harder to do, but we can also experiment with two different ropes, one light and one heavy. We would find that the heaver the rope, the slower the wave. We can express this as

$$v = \sqrt{\frac{T_s}{\mu}}$$

where T_s is the tension in the rope, and μ is the linear mass density

$$\mu = \frac{m}{L}$$

where m is the mass of the rope, and L is the length.

The term μ might need an analogy to make it seem helpful. So suppose I have an iron bar that has a mass of 200 kg and is 2 m long. Further suppose I want to know how much mass there would be in a 20 cm section cut of the end of the rod. How would I find out?

This is not very hard, We could say that there are 200 kg spread out over 2 m, so each meter of rod has 100 kg of mass, that is, there is 100 kg/ m of mass per unit length. Then to find how much mass there is in a 0.20 m section of the rod I take

$$m = 100\frac{\mathrm{kg}}{\mathrm{m}} \times 0.20\,\mathrm{m} = 20.0\,\mathrm{kg}$$

The 100 kg/ m is μ. It is how much mass there is in a unit length segment of something In this example, it is a unit length of iron bar, but for waves on string, we want the mass per unit length of string.

If you are buying stock steel bar, you might be able to buy it with a mass per unit length. If the mass per unit length is higher then the bar is more massive. The same is true with string. The larger μ, the more massive equal string segments will be.

We should note that in forming this relationship, we have used our standard introductory physics assumption that the mass of the rope can be neglected. Let's consider what would happen if this were not true. Say we make a wave in a heavy cable that is suspended. The mass at the lower end of the cable pulls down on the upper part of the cable. The tension will actually change along the length of the cable, and so will the wave speed. Such a situation can't be represented by a single wave speed. But for our class, we will assume that any such changes are small enough to be ignored.

4.4 One dimensional waves

To mathematically describe a wave we will define a function of both time and position.

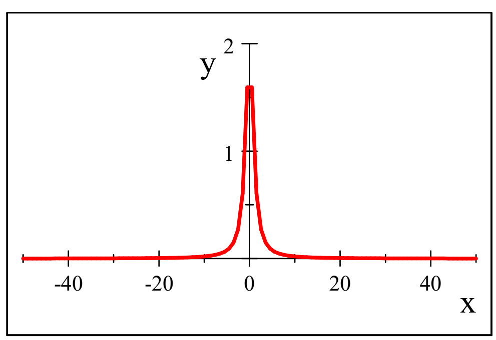
$$y(x,t) \tag{4.4}$$

Note, that this is new in our physics experience. Before we usually were concerned about only one variable at a time. For oscillation we had just $y(t)$ for example. But now we will be concerned about two variables, x, and t.

let's take a specific example[1]

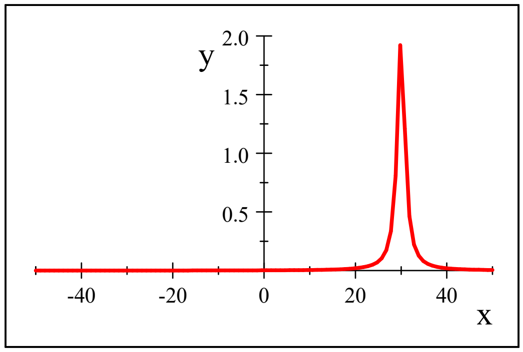
$$y = \frac{2}{(x-3.0t)^2+1} \tag{4.5}$$

Let's plot this for $t = 0$



what will this look like for $t = 10$?

[1] This is not an important wave function, just one I picked because it makes a nice graphic example.



The pulse travels along the x-axis as a function of time. Note that there is a value for y for every x position and that these y values change for different times. That is what we mean by saying we have a function of both x and t.

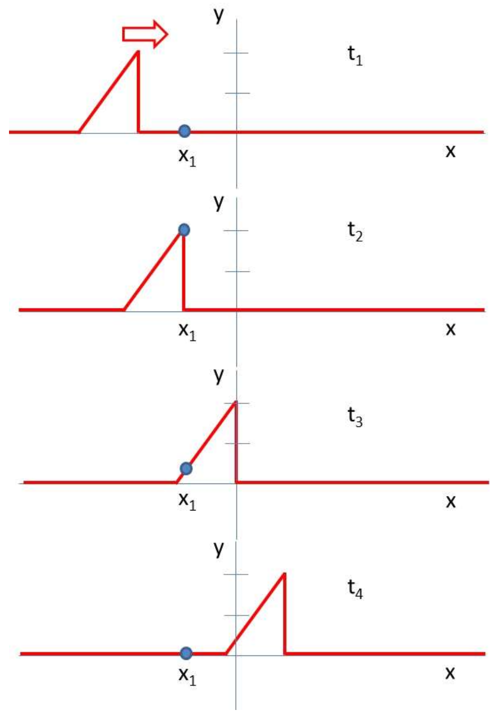
We denote the speed of the pulse as v, then we can define a function

$$y(x,t) = y(x - vt, 0) \tag{4.6}$$

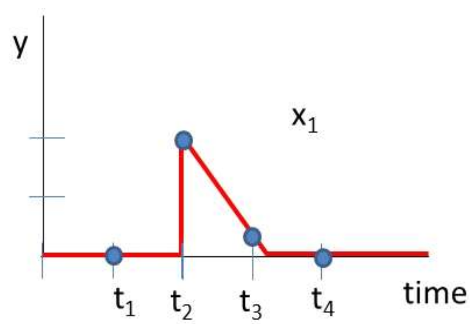
that describes a pulse as it travels. An element of the medium (rope, string, etc.) at position x at some time t, will have the displacement that an element had earlier at $x - vt$ when $t = 0$.

We will give $y(x - vt, 0)$ a special name, the *wave function*. It represents the y position, the transverse position in our example, of any element located at a position x at any time t.

Notice that wave functions depend on two variables, x, and t. It is hard to draw a wave so that this dual dependence is clear. Often we draw two different graphs of the same wave so we can see independently the position and time dependence. So far we have used one of these graphs. A graph of our wave at a specific time, t_o. This gives $y(x, t_o)$. This representation of a wave is very like a photograph of the wave taken with a digital camera. It gives a picture of the entire wave, but only for one time, the time at which the photograph was taken. Of course we could take a series of photographs, but still each would be a picture of the wave at just one time. Here is such a series of graphs at times t_1 through t_4.



The second representation is to observe the wave at just one point in the medium, but for many times. This is very like taking a video camera and using it to record the displacement of just one part of the medium for many times. You could envision marking just one part of a rope, and then using the video recorder to make a movie of the motion of that single part of the rope. We could then go frame by frame through the video, and plot the displacement of our marked part of the rope as a function of time. Such a graph is sometimes called a history graph of the wave. Here is such a graph for the position x_1.



To go from the time graphs to the history graph you observe what happens at the location x_1 for each of the times. Then plot those y positions on the y vs.

t graph. Then you must connect the points. This takes some thinking to make sure you connect them right (or a whole lot of points). But with practice, this is not hard and both view points are valuable ways to look at a wave.

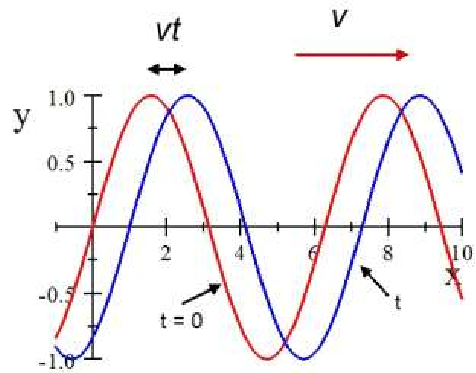
Chapter 5

5 Sinusoidal Waves in one and Two Dimensions 1.16.2 1.16.3

Fundamental Concepts

- It takes two variables to describe a wave, position and time
- Wave graphs have many named parts, amplitude, period, wavelength crest, trough, etc.
- There is a spatial analog to temporal frequency called spacial frequency and it is represented by the factor k in our equations
- We can also express sound waves in terms of pressure changes
- We don't hear all frequencies equally well
- Waves from point sources are spherical
- Light waves are waves in the electromagnetic field

5.1 Sinusoidal Waves



A sinusoidal graph should be familiar from our study of oscillation. Simple harmonic oscillators are described by the function

$$y\left(t\right)=y_{\max}\cos\left(\omega t+\phi\right)\qquad\text{SHM}\tag{5.1}$$

but this only gave us a vertical displacement. Now our sinusoidal function must also be a function of position along the wave.

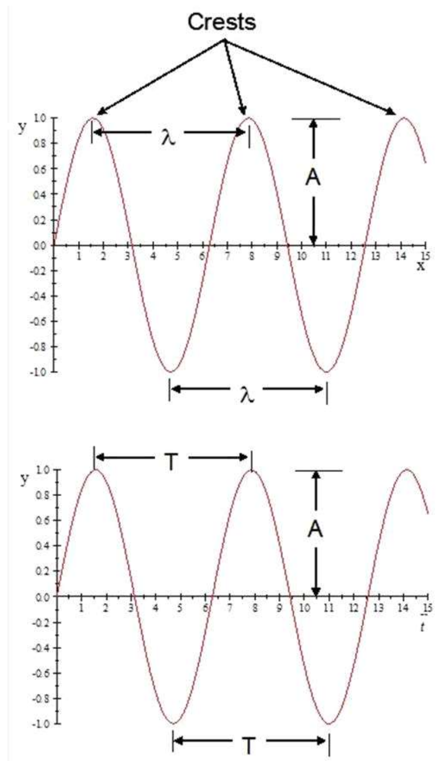
$$y\left(x,t\right)=y_{\max}\sin\left(ax-\omega t+\phi\right)\qquad\text{waves}\tag{5.2}$$

but before we study the nature of this function, lets see what we can learn from the graph of a sinusoidal wave. We will need both our two views, the camera snapshot and the video of a point. Look at figure **??**. This is two camera snap shots superimposed. The red curve shows the wave (y position for each value of x) at $t=0$. At some some later time t, the wave pattern has moved to the right as shown by the blue curve. The shift is by an amount $\Delta x=v\Delta t$.

5.2 Parts of a wave

The peak of a wave is called the crest. For a sine wave we have a series of crests. We define the wavelength as the distance between any to identical points (e.g. crests) on adjacent waves in a snapshot view.

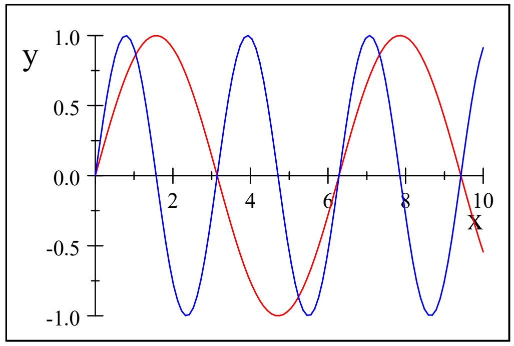
Notice that this is very similar to the definition of the period, T, when we graphed SHM on a y,t set of axis. In fact, this similarity is even more apparent if we plot a sinusoidal wave using our two wave pictures. In the next figure, the snap-shot comes first. We can see that there will be crests. The distance between the crests is given the name *wavelength*. This is not the entire length of the whole wave. But is is a characteristic length of part of the wave that is easy to identify. The next figure shows all this using our snapshot and history graph for a sine wave.



Note that there are crests in the history graph view as well. That is because one marked part of the medium is being displaced as a function of time. Think of a marked piece of the rope going up and down, or think of floating in the ocean at one point, you travel up and down (and a little bit back and forth) as the waves go by. But now the horizontal axis is time. There will be a characteristic time between crests. That time is called the *period*. This is just like oscillatory motion–because it is oscillatory motion! Like the wavelength is not the length of the whole wave, the period is not the time the whole wave exists. It is just the time it takes the part of the medium we are watching to go through one complete cycle. Notice that this video picture is exactly the same as a plot of the motion of a simple harmonic oscillator! For a sinusoidal wave, each part of the medium experiences simple harmonic motion.

We remember frequency from simple harmonic motion. But now we have a wave, and the wave is moving. We can extend our view of frequency by defining it as follows:

The frequency of a periodic wave is the number of crests (or any other point of the wave) that pass a given point in a unit time interval.



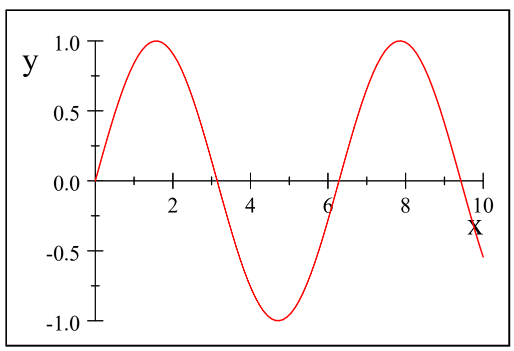
In the previous figure, the blue curve has twice the frequency as the red curve. Notice how it has two crests for every red crest. The maximum displacement of the wave is called the *amplitude* just as it was for simple harmonic oscillators.

5.3 Wavenumber and wave speed

Consider again a sinusoidal wave.

$$y\,(x,t) = y_{\max} \sin\left(kx - \omega t + \phi_o\right) \tag{5.3}$$

We have drawn the wave in the snapshot picture mode



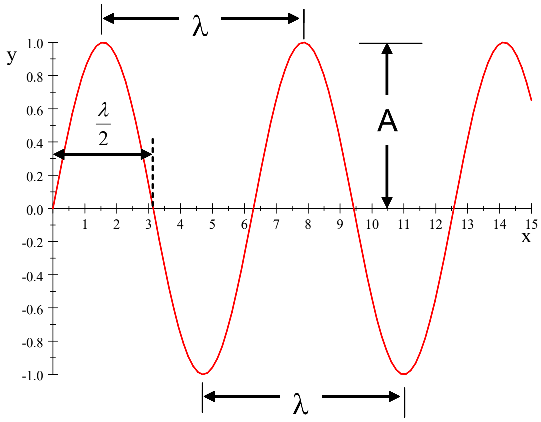
To make this graph, we set $t = 0$ and plotted the resulting function

$$y\,(x,0) = y_{\max} \sin\left(kx + 0\right) \tag{5.4}$$

$y_{\max}$ is the amplitude. I want to investigate the meaning of the constant k. Lets find k like we did for SHM when we found ω. Consider the point $x = 0$. At this point

$$y\,(0,0) = y_{\max} \sin\left(k\,(0)\right) = 0 \tag{5.5}$$

The next time $y = 0$ is when $x = \frac{\lambda}{2}$



then

$$y\left(\frac{\lambda}{2},0\right)=y_{\max}\sin\left(k\frac{\lambda}{2}\right)=0 \tag{5.6}$$

From our trigonometry experience, we know that this is true when

$$k\frac{\lambda}{2}=\pi \tag{5.7}$$

solving for k gives

$$k=\frac{2\pi}{\lambda} \tag{5.8}$$

Then we now have a feeling for what k means. It is 2π over the spacing between the crests. The 2π must have units of radians attached. Then

$$y\left(x,0\right)=A\sin\left(\frac{2\pi}{\lambda}x+0\right) \tag{5.9}$$

We have a special name for the quantity k. It is called the *wave number*.

$$k\equiv\frac{2\pi}{\lambda} \tag{5.10}$$

Both the name and the symbol are somewhat unfortunate. Neither gives much insight into the meaning of this quantity. But from what we have done in studying oscillation, we can understand this new quantity. For a harmonic oscillator, we know that

$$y\left(t\right)=y_{\max}\sin\left(\omega t\right)$$

where

$$\omega=2\pi f=\frac{2\pi}{T}$$

T is how far, *in time*, the crests are apart, and the inverse of this, $\frac{1}{T}$ is the frequency. The frequency tells us how often we encounter a crest as we march along in time. So $\frac{1}{T}$ must be how many crests we have in a unit amount of time.

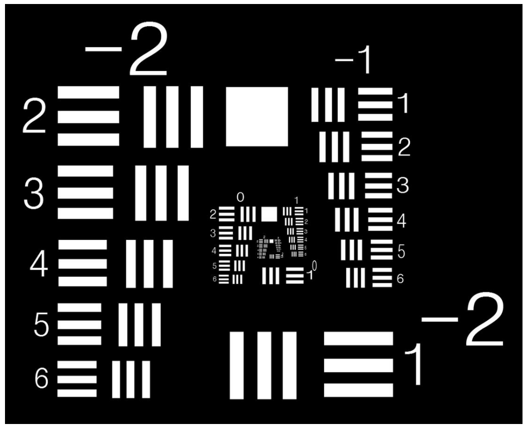
Now think of the relationship between the snapshot and the video representation for a sinusoidal wave. We have a new quantity

$$k = \frac{2\pi}{\lambda}$$

where λ is how far, *in distance*, the crests are apart. This implies that $\frac{1}{\lambda}$ plays the same role in the snap shot graph that f plays in the video graph. It must tell us how many crests we have, but this time it is how many crests in a given amount of distance. We found above that k told us something about how often the zeros (well, every other zero) will occur. But the crests must occur at the same rate. So k tells us how often we encounter a crest in our snapshot graph. This is not too strange. It would be meaningful, for example, to say that a farmer had plowed his or her field to have two furrows per meter.

The frequency in the video graph is how often we encounter a crest, $\frac{1}{\lambda}$ is how often we encounter a crest in the snap shot graph. Thus $\frac{1}{\lambda}$ is playing the same role for a snap shot graph as frequency plays for a history graph. We could call $\frac{1}{\lambda}$a *spatial frequency.* It is how often we encounter a crest as we march along in position, or how many crests we have in a unit amount of distance. And λ could be called a *spatial period*. Both $1/T$ and $1/\lambda$ answer the question "how often something happens in a unit of something" but one asks the question in time and the other in position along the wave.

My mental image for this is the set of groves on the edge of a highway. There is a distance between them, like a wavelength, and how often I encounter one as I move a distance along the road is the spatial frequency. You could say that there are, maybe, $3/\mathrm{m}$. That is a spatial frequency! It is how many of something happens in a unit distance. We use this concept in optics to test how well an optical system resolves details in a photograph. The next figure is a test image. A good camera will resolve all spatial frequencies equally well. Notice the test image has sets of bars with different spatial frequencies. By forming an image of this pattern, you can see which spacial frequencies are faithfully represented by the optical system.



Resolution test target based on the USAF 1951 Resolution Test Pattern (not drawn to exact specifications).

In class you will see that our projector does not represent all spatial frequencies equally well! You can also see this now in the copy you are reading. If you are reading on-line or an electronic copy, your screen resolution will limit the representation of some spatial frequencies. Look for the smallest set of three bars where you can still tell for sure that there are three bars without zooming. A printed version that has been printed on a laser printer will usually allow you to see even smaller sets of three bars clearly.

Let's place k in the full equation for the sine wave for any time, t.

$$y\left(x,t\right)=y_{\max}\cos\left(kx-\omega t+\phi_o\right) \tag{5.11}$$

Sometimes you will see this equation written in a slightly different way. We could change our variable from x to $x-vt$ so that

$$y\left(x,t\right)=y\left(x-vt,0\right)$$

With a little algebra we can do this

$$\begin{aligned}
y\left(t\right) &= y_{\max}\cos\left(kx-\omega t+\phi_o\right)\\
&= y_{\max}\cos\left(\frac{2\pi}{\lambda}x-\frac{2\pi}{T}t+\phi_o\right)\\
&= y_{\max}\cos\left(\frac{2\pi}{\lambda}\left(x-\frac{\lambda}{T}t\right)+\phi_o\right)
\end{aligned}$$

This is in the form of a wave function so long as

$$v=\frac{\lambda}{T} \tag{5.12}$$

which is just how far the wave travels from crest to crest divided by how long it takes to travel that distance. The wave travels one wavelength in one period. Indeed, this is the wave speed! Then

$$y\left(x,t\right)=y_{\max}\sin\left(\frac{2\pi}{\lambda}\left(x-vt\right)\right) \tag{5.13}$$

Wave speed forms

We just found

$$v = \frac{\lambda}{T} \tag{5.14}$$

but it is easy to see that

$$v = \frac{2\pi\lambda}{2\pi T} = \frac{\omega}{k} \tag{5.15}$$

and

$$v = \lambda f \tag{5.16}$$

This last formula is, perhaps, the most common form encountered in our study of light.

Phase

You may be wondering about the phase constant we learned about in our study of SHM. We have ignored it up to now. But of course we can shift our sine just like we did for our plots of position vs. time for oscillation. Only now with a wave we have two graphs, a history and snapshot graphs, so we could shift along the x in a snapshot graph or along the t axes in a history graph. So the sine wave has the form.

$$y(x,t) = A\sin(kx - \omega t + \phi_o) \tag{5.17}$$

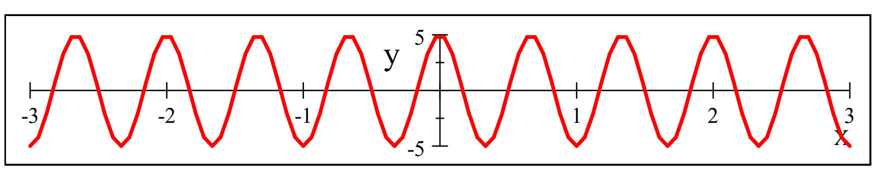
were ϕ_o will need to be determined by initial conditions just like in SHM problems and those initial conditions will include initial positions as well as initial times.

Let's consider that we have two views of a wave, the snapshot and history view. Each of these looks like sinusoids for a sinusoidal wave. Let's consider a specific wave,

$$y(x,t) = 5\sin\left(3\pi x - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$

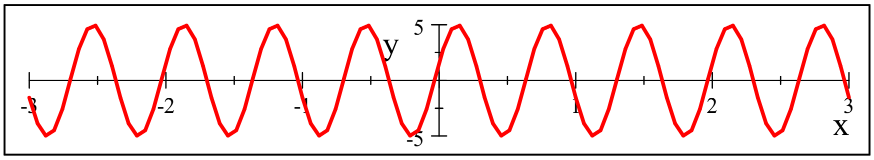
we look at a snapshot graph at $t = 0$

$$y(x,0) = 5\sin\left(3\pi x - \frac{\pi}{5}(0) + \frac{\pi}{2}\right)$$



and another at $t = 2\,\mathrm{s}$

$$y(x, 2\,\mathrm{s}) = 5\sin\left(3\pi x - \frac{\pi}{5}(2\,\mathrm{s}) + \frac{\pi}{2}\right)$$



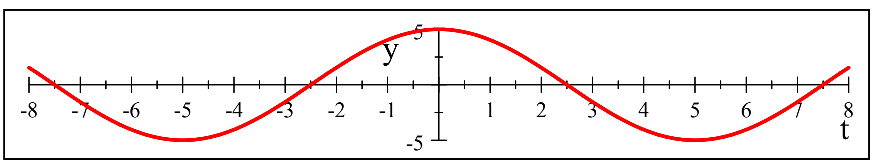
Comparing the two, we could view the latter as having a different phase constant that is the sum of what we have called the phase constant, ϕ_o plus $\omega\Delta t$, which tells us how different the times are between the two graphs. This time difference is what is shifting the graph

$$\phi_{total} = \omega\Delta t + \phi_o = -\frac{\pi}{5}\left(2\,\mathrm{s}\right) + \frac{\pi}{2}$$

that is, within the snapshot view, the time dependent part of the argument of the sign acts like an additional phase constant.

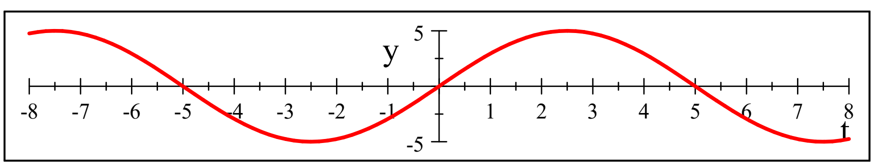
Likewise, in the history view, we can plot our wave at $x = 0$

$$y\left(0, t\right) = 5\sin\left(3\pi\left(0\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



and at $x = 1.5\,\mathrm{m}$

$$y\left(1.5\,\mathrm{m}, t\right) = 5\sin\left(3\pi\left(1.5\,\mathrm{m}\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



Within the history view, the kx part of the argument acts like a phase constant.

$$\phi_{total} = k\Delta x + \phi_o = 3\pi\left(1.5\,\mathrm{m}\right) + \frac{\pi}{2}$$

Of course neither kx nor ωt are constant, But within individual views of the wave we have set them constant to form our snapshot and history representations. We can see that any part of the argument of the sine, $kx - \omega t + \phi_o$ could contribute to a phase shift, depending on the view we are taking.

Because of this, it is customary to call the entire argument of the sine function, $\phi = kx - \omega t + \phi_o$ the *phase of the wave*. Where ϕ_o is the phase constant, ϕ is the phase. Of course then, ϕ must be a function of x and t, so we have a different value for $\phi\left(x, t\right)$ for every point on the wave for every time. This can be a little confusing, ϕ and ϕ_o look a lot the same, but they are different.

5.4 The Speed of Waves

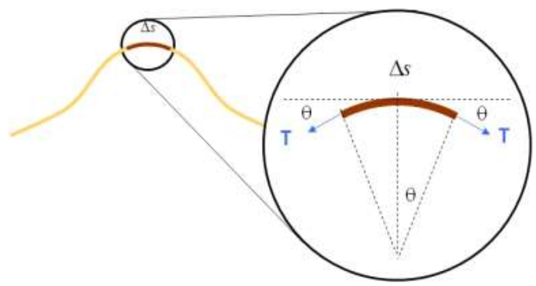
We have already considered the speed of waves on strings

$$v = \sqrt{\frac{T}{\mu}}$$

we should see where this comes from so we know it's limitations. We would expect that there may be things that change the speed of sound waves in air as well (like a tension or a massiveness of the medium). Let's start with a formal derivation of our string speed formula, then take on sound waves in air.

5.4.1 Derivation of the string wave speed formula

Let's work a problem together. Let's find an expression for the speed of the wave as it travels along a string.



What are the forces acting on an element of string?

1. Tension on the RHS of the element from the rest of the string on the right, T_r
2. Tension on the LHS of the element from the rest of the string on the left, T_l
3. F_g

Lets assume that the element of string, Δs, at the crest is approximately an arc of a circle with radius R.

There is a force pulling left on the left end of the element that is tangent to the arc, there is a force pulling right at the right end of the element which is also tangent to the arc. These forces produce centripetal accelerations

$$a = \frac{v^2}{R} \tag{5.18}$$

The horizontal components of the forces cancel $(T\cos\theta)$. The vertical component, $(T\sin(\theta))$ is directed toward the center of the arc. If the rope is not moving in the x direction, then $T_l = T_r$.

Then, the radial force F_r will have matching components from each side of the element that together are $2T\sin(\theta)$. Notice that to make sure the components are the same we must assume that the string is uniform and that it is not too massive (almost massless string approximation!). Since the element is small,

$$F_r = 2T\sin(\theta) \approx 2T\theta \tag{5.19}$$

The element has a mass m.

$$m = \mu\Delta s \tag{5.20}$$

where μ is the mas per unit length. Using the arc length formula

$$\Delta s = R(2\theta) \tag{5.21}$$

so

$$m = \mu\Delta s = 2\mu R\theta \tag{5.22}$$

and finally we use the formula for the radial acceleration

$$F_r = ma = (2\mu R\theta)\frac{v^2}{R} \tag{5.23}$$

Combining these two expressions for F_r

$$2T\theta = (2\mu R\theta)\frac{v^2}{R} \tag{5.24}$$

$$T = (\mu R)\frac{v^2}{R} \tag{5.25}$$

$$\frac{T}{\mu} = v^2 \tag{5.26}$$

and we find that

$$v = \sqrt{\frac{T}{\mu}} \tag{5.27}$$

which is just what we stated before, only now we see just how much approximation there was in building this formula. We might guess that it would not work so well for large cables supporting bridges or with cables that change size along their length.

5.4.2 Speed of the medium

Let's review for a moment. Suppose we take a jump rope, and shake one end up and down while a lab partner keeps his or her end stationary. You can make a sine wave in the rope. But being good physics students, let's make our "shake" very symmetric. Let's start our wave using simple harmonic motion.

Let's call an element of the rope Δx. Here the "Δ" is being used to mean "a small amount of." We are taking a small amount of the rope and calling it's length Δx. And it is the motion of this small amount of rope that I want to think about now.

Each element of the rope (Δx_i) will also oscillate with SHM (think of a driven SHO). Note that the elements of the rope oscillate in the y direction, but the wave travels in x. This is a transverse wave.

Let's describe the motion of an element of the string at point P.

At $t = 0$,

$$y = A \sin (kx - \omega t + \phi_o) \tag{5.28}$$

(where I have chosen $\phi_o = 0$ for this example). The element does not move in the x direction. So we define the *transverse speed*, v_y, and the *transverse acceleration,* a_y, as the velocity and acceleration of the element of rope in the y direction. These are not the velocity and acceleration of the wave, just the velocity and acceleration of the element Δx at a point P.

Because we are doing this at one specific x location we need partial derivatives to find the velocity

$$v_y = \left.\frac{dy}{dt}\right]_{x=\text{constant}} = \frac{\partial y}{\partial t} \tag{5.29}$$

That is, we take the derivative of y with respect to t, but we pretend that x is not a variable because we just want one x position. Then

$$v_y = \frac{\partial y}{\partial t} = -\omega A \cos (kx - \omega t + \phi_o) \tag{5.30}$$

and

$$a_y = \frac{\partial v_y}{\partial t} = -\omega^2 A \sin (kx - \omega t + \phi_o) \tag{5.31}$$

These solutions should look very familiar! We expect them to be the same as a harmonic oscillator except that we now have to specify which oscillator–which part of the rope–we are looking at. That is what the kx part is doing.

5.5 The Linear Wave Equation

We added in some new math in our last two lectures. Differential equations! We identified simple harmonic motion using a differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Perhaps we could find an equation like this for waves and then identify waves using that equation. We can start the same way we did for oscillation. Let's write the acceleration for a part of the medium.

$$a_y = \frac{d^2y}{dt^2} = -\omega^2 A \sin (kx - \omega t + \phi_o) \tag{5.32}$$

But for waves we have two variables. Let's guess that we might need to take derivatives with respect to x as well as for t.

$$\frac{dy}{dx} = kA\cos(kx - \omega t + \phi_o)$$

This isn't all that useful on it's own, but it does tell us how the slope of the curve changes in a snapshot graph. Let's take another derivative

$$\frac{d^2y}{dx^2} = -k^2 A\sin(kx - \omega t + \phi_o)$$

We are back to a sine function. That is not a surprise. But it looks a lot like our time second derivative. We could take the ratio of the two equations to see just how different they are

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = \frac{-\omega^2 A\sin\left(kx - \omega t + \phi_o\right)}{-k^2 A\sin(kx - \omega t + \phi_o)}$$

Of course the sine parts cancel, and so do the A's

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = \frac{-\omega^2}{-k^2}$$

and we know that

$$\frac{\omega}{k} = v$$

the wave speed (not the speed of a piece of the medium, but the speed of the wave)

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2y}{dx^2}} = v^2$$

If a function satisfies this equation, we can identify it as a wave. But this form is awkward, let's rewrite it as

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$$

This is the *linear wave equation.* There are waves that don't follow this equation. They are called non-linear waves. But that is a subject for an upper division course.

5.6 Waves in two and three dimensions

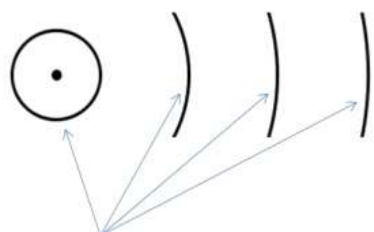
So far we have written expressions for waves, but our experience tells us that waves don't usually come as one dimensional phenomena. In the next figure, we see the disturbance (a drop) creating a water wave.

Picture of a water drop (Jon Paul Johnson, used by permision)

You can see the wave forming, but the wave is clearly not one dimensional. It appears nearly circular. In fact, it is closer to hemispherical, and this limit is only true because the disturbance is at the air-water boundary. Most waves in a uniform medium will be roughly spherical. As such a wave travels away from the source, the energy traveling gets more spread out. This causes the amplitude to decrease. Think of a sound wave, it gets quieter the farther you are from the source. We change our equation to account for this by making the amplitude a function of the distance, r, from the source

$$y(r,t) = A(r)\sin(kr - \omega t + \phi_o) \tag{5.33}$$

Of course, if we look at a very large wave, but we only look at part of the wave, we see that our part looks flatter as the wave expands.



Portion of a Spherical Wave: Wave becomes more flat as it expands

Very far from the source, our wave is flat enough that we can ignore the curvature across it's wave fronts. We call such a wave a *plane wave.* There are no true plane waves in nature, but this idealization makes our mathematical solutions simpler and many waves come close to this approximation.

We have said that sound is a longitudinal wave with a medium of air. Really any solid, liquid, or gas will work as a medium for sound. For our study, we will take sound to be a longitudinal wave and treat liquids and gasses. Solids have additional forces involved due to the tight bonding of the atoms, and therefore are more complicated. Technically in a solid sound can be a transverse wave as well a longitudinal wave, but we usually call transverse waves of this nature *shear waves.*

Chapter 6

Power and Superposition 6 1.16.4 1.16.5

Fundamental Concepts

- Because waves are three dimensional, describing the power or energy delivered per time of the wave is not enough, We describe how spatially spread out that power is. We call this spread power *intensity*
- If we make more than one wave in a medium, the waves "add up" or *superimpose.*

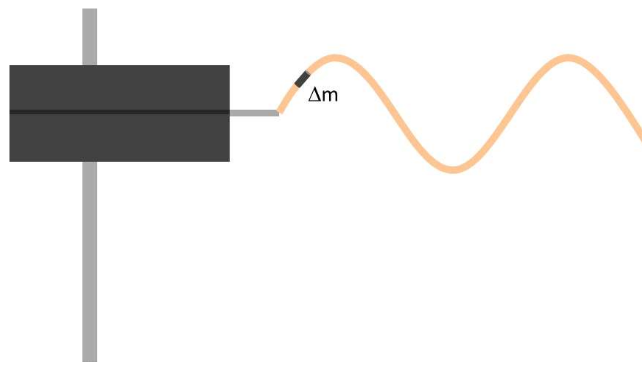
6.1 Energy and Power in waves

We have said that waves don't transport mass, they transport energy. We used the example of sound waves. If the air molecules move to another location , we call it wind. But if the air molecules just vibrate around an equilibrium position, we called that sound waves. The molecules don't end up somewhere else. But something does move.

Maybe a better example is a water wave in a swimming pool. Suppose someone jumps into a calm pool. The person is a disturbance and the water is a medium. We will get waves. But the water doesn't end up bunched together on the other side of the pool. The water is still everywhere in the pool, even around the person that jumped in. So what did move away from the person? It is energy.

For sinusoidal waves, the amount of energy in the wave is related to both it's amplitude and it's frequency. And the specifics of that relationship depend on the type of wave. We will study both sound and light waves later in our course and we will find that relationship between energy, amplitude, and frequency are different for the two different kinds of waves.

Let's look at a specific case of a rope attached to a mechanical oscillator. You can see the situation in the next figure. The oscillator (black thing) has a piece that goes up and down (silver thing sticking out to the right). The rope is attached to this silver oscillating piece. The oscillatory acts as a disturbance. A wave is formed in the rope.



Once again let's take a small part of our medium. Before we took a pice that was Δx long, but this time let's describe our piece of rope as a small bit of mass Δm. We already know about linear mass densities.

$$\mu = \frac{M}{L}$$

and since our rope is uniform, the linear mass density is the same for each part of the rope. So for our marked part

$$\mu = \frac{\Delta m}{\Delta x}$$

Then for this part of the rope medium there will be a kinetic energy

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}\Delta m v_y^2 \end{aligned}$$

using our linear mass density we can write

$$\Delta m = \mu \Delta x$$

so the kinetic energy is

$$K = \frac{1}{2}\left(\mu \Delta x\right) v_y^2$$

and from before we know that $v_y = -\omega A \cos\left(kx - \omega t + \phi_o\right)$ so our kinetic energy is

$$K = \frac{1}{2}\left(\mu \Delta x\right)\left(-\omega A \cos\left(kx - \omega t + \phi_o\right)\right)^2$$

we can clean this up a bit

$$K=\frac{1}{2}\mu\Delta x\omega^2A^2\cos^2\left(kx-\omega t+\phi_o\right)$$

Now let's simplify our situation by letting $\phi_o=0$ and let's take a snap shot view with $t=0$.

$$K=\frac{1}{2}\mu\Delta x\omega^2A^2\cos^2\left(kx\right)$$

Then, if we take only part of the energy associated with one very tiny part of our rope medium, our Δx becomes just dx. We don't have all the energy of the wave, just a small part of it. So we can write this as

$$dK=\frac{1}{2}\mu\omega^2A^2\cos^2\left(kx\right)dx$$

and we can integrate this up to find the energy of just one wavelength of the wave.

$$\int_0^{K_\lambda}dK=\int_0^\lambda\frac{1}{2}\mu\omega^2A^2\cos^2\left(kx\right)dx$$

A lot of the parts of our equation are constants, let's take them out front.

$$\int_0^{K_\lambda}dK=\frac{1}{2}\mu\omega^2A^2\int_0^\lambda\cos^2\left(kx\right)dx$$

The left hand side of this equation is just K_λ

$$K_\lambda=\frac{1}{2}\mu\omega^2A^2\int_0^\lambda\cos^2\left(kx\right)dx$$

but if you are like me, the right hand side is an integral that you have conveniently forgotten. But we can look up this integral in an integral table. My table gave the form

$$\int\left(\cos^2\left(ax\right)\right)dx=\frac{1}{2}x+\frac{1}{4a}\sin\left(2ax\right)$$

We have to match this to our equation. We see $a=k$ so

$$\begin{aligned}
K_\lambda &= \frac{1}{2}\mu\omega^2A^2\left[\frac{1}{2}x+\frac{1}{4k}\sin\left(2kx\right)\right|_0^\lambda \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{1}{4k}\sin\left(2k\lambda\right)\right\}-\left\{\frac{1}{2}\left(0\right)+\frac{1}{4k}\sin\left(2k\left(0\right)\right)\right\}\right] \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{1}{4\frac{2\pi}{\lambda}}\sin\left(2\frac{2\pi}{\lambda}\lambda\right)\right\}-\frac{1}{4k}\sin\left(2k\left(0\right)\right)\right] \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{\lambda}{8\pi}\sin\left(4\pi\right)\right\}\right] \\
&= \frac{1}{4}\mu\omega^2A^2\lambda
\end{aligned}$$

This wavelength sized piece of tope will also have some potential energy because the fibers that make the rope are stretched to make the wave. They aren't really quite like little springs, but it's not to far wrong to model them as though they were. So let's take the spring potential energy

$$U = \frac{1}{2}k_s\left(y - y_e\right)^2$$

and let $y_e = 0$ for our piece of rope. We know that our piece of rope will go up and down in SHM. The natural frequency of our rope piece is

$$\omega = \sqrt{\frac{k_s}{m}}$$

We can solve this for the spring constant

$$k_s = m\omega^2$$

or for our little piece of the rope

$$k_s = \Delta m\omega^2$$

Let's put this into our potential energy equation

$$U = \frac{1}{2}\Delta m\omega^2\left(y\right)^2$$

and substitute $\Delta m = \mu\Delta x$ again

$$U = \frac{1}{2}\mu\Delta x\omega^2\left(y\right)^2$$

and of course, let our Δx get small so we have a small bit of potential energy from a small bit of rope.

$$dU = \frac{1}{2}\mu\omega^2\left(y\right)^2 dx$$

and integrate this to find the potential energy in the stretchy bonds of one wavelength worth of rope.

$$\int_0^{U_\lambda} dU = \int_0^{\lambda}\frac{1}{2}\mu\omega^2\left(y\right)^2 dx$$

We need to put in our equation for the position for our wave.

$$y\left(x,t\right) = A\sin\left(kx - \omega t + \phi_o\right)$$

but once again let's choose $\phi_o = 0$ and $t = 0$ so that

$$y\left(x,0\right) = A\sin\left(kx\right)$$

then

$$\int_0^{U_\lambda} dU = \int_0^{\lambda}\frac{1}{2}\mu\omega^2\left(A\sin\left(kx\right)\right)^2 dx$$

The left hand side is easy. And there are, once again, constants

$$U_\lambda = \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda (\sin(kx))^2 \, dx$$

and we have another integral I don't remember how to do. But I still have my table of integrals and my table tells me

$$\int \left(\sin^2(ax)\right) dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Which feels very familiar. Lets let $a = k$ again. Then

$$\begin{aligned} U_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \left[\frac{1}{2}x - \frac{1}{4k}\sin(2kx)\right|_0^\lambda \\ & \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda - \frac{1}{4k}\sin(2k\lambda)\right\} - \left\{\frac{1}{2}0 - \frac{1}{4k}\sin(2k(0))\right\}\right] \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda \end{aligned}$$

The total energy in the λ-sized rope piece would be

$$\begin{aligned} E_\lambda &= K_\lambda + U_\lambda \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda + \frac{1}{4}\mu\omega^2 A^2 \lambda \\ &= \frac{1}{2}\mu\omega^2 A^2 \lambda \end{aligned}$$

We know that energy is being transferred by the wave, whether it is a light or sound wave or any other mechanical wave. We should wonder, how fast is energy transferring. This can mean the difference between a warm ray of sin on a cool spring day and being burned by a laser beam. We will start by considering the rate of energy transfer, *power*. The concept of power should be familiar to us from PH121. We can find the power as the rate at which energy is transferred.

$$\mathcal{P} = \frac{\Delta E}{\Delta t}$$

Since we picked the amount of energy in one wavelength, and we know the time it takes for one wavelength of the wave to pass by is T. Then the power is

$$P = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T}$$

but remember

$$v = \frac{\lambda}{T}$$

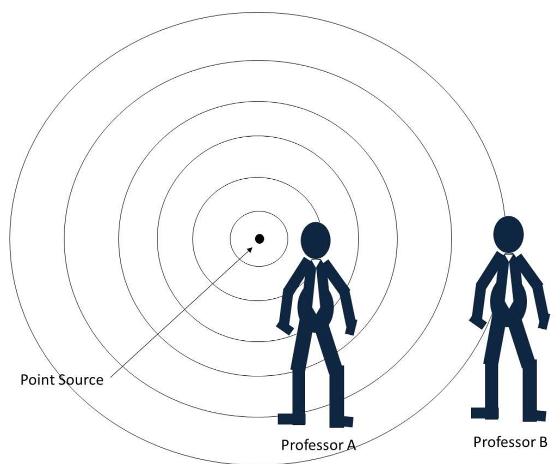
so we can write our power as

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

Power is important. Most detectors (like our ears and eyes) really detect the power delivered by a wave. And we can see that the power is proportional to the amplitude squared and the frequency squared for the wave. More on this as we study light and sound.

6.2 Power and Intensity

What we have done works fine for linear waves. But if we consider our spherical wave from a point source, we can see that this description isn't good enough. In the next figure we have two professors.



Thinking from our experience we would say that the sound will seem louder for professor A. This is because the energy in the sound wave is being spread over the surface of the wave, and that surface is getting bigger as the wave moves outward. The energy is more spread out by the time it gets to professor B.

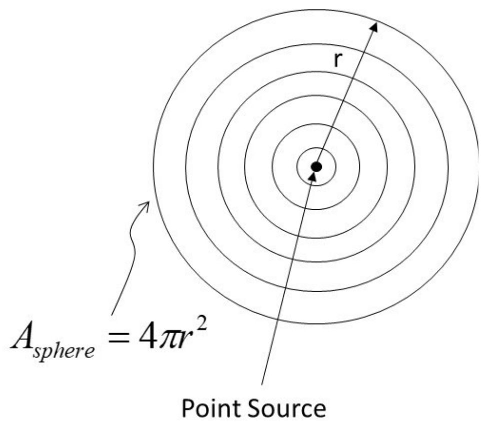
Tuning Fork Demo

To be able to describe how much energy we get from our wave we need define something new.

$$\mathcal{I} \equiv \frac{\mathcal{P}}{A} \tag{6.1}$$

that is, the power divided by an area. But what does it mean?

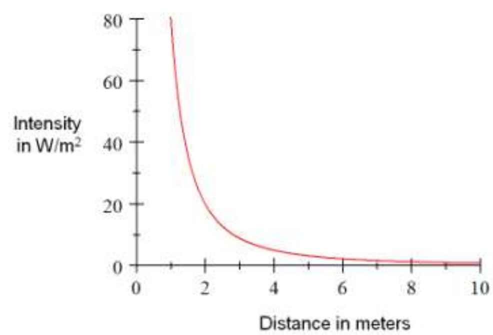
Consider a point source. This could be a loud speaker, or a buzzer, or a baby crying, etc.



it sends out waves in all directions. The wave crests will define a sphere around the points source (the figure shows a cross section but remember it is a wave from a point source, so we are really drawing concentric spheres like balloons inside of balloons.). Then form our point source

$$\mathcal{I} = \frac{\mathcal{P}}{4\pi r^2} \tag{6.2}$$

As the wave travels, its the power per unit area decreases with the square of the distance (think gravity) because the area is getting larger.



This quantity that tells us how spread out our power has become is called the *intensity* of the wave. Professor A would agree with us that the wave he heard was more intense than the wave heard by Professor B. That is because the wave was less spread out for Professor A.

Suppose we cup our hand to our ear. What are we doing? We are increasing the area of our ear. Our ears work by transferring the energy of the sound wave to a mechanical-electro-chemical device that creates a nerve signal.[1] The more

[1] The inner hair cells in the organ of Corti in the cochlea.

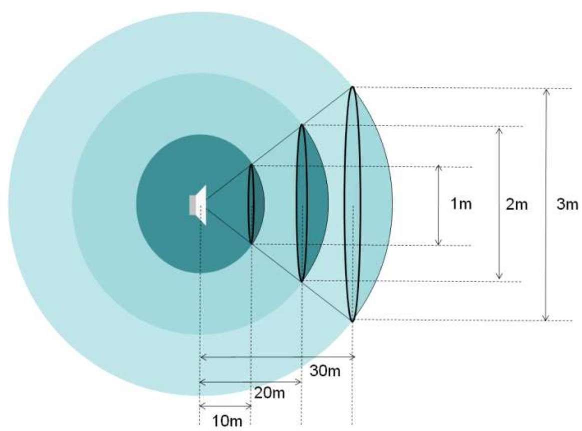
energy, the stronger the signal. If we are a distance r away from the source of the sound then the intensity is

$$\mathcal{I} = \frac{\mathcal{P}_{source}}{A_{wave}}$$

But we are collecting the sound wave with another area, the area of our hand. The power received is

$$\begin{aligned} \mathcal{P}_{received} &= \mathcal{I} A_{hand} \\ &= \frac{A_{hand}}{A_{wave}} \mathcal{P}_{source} \end{aligned}$$

and we can see that, indeed, the larger the hand, the more power, and therefore more energy we collect. This is the idea behind a dish antenna for communications and the idea behind the acoustic dish microphones we see at sporting events. In next figure, we can see that it would take an increasingly larger dish to maintain the same power gathering capability as we get farther from the source.



6.3 Superposition Principle

Wave Machine Demo

What happens if we have more than one wave propagating in a medium? If you remember being a little child in a bath tub, you will probably remember making waves in the water. If you made a wave with each hand, the two waves seemed to "pile up" in the middle and make a big splash. We should expect something like this for any kind of wave. We call the "piling up" of waves *superposition.* The word literally means putting one wave on top of another.

Superposition: If two or more traveling waves are moving through a medium, the resultant wave formed at any point is the algebraic sum of the values of the individual wave forms.

So if we have

$$y_1(x,t) \tag{6.3}$$

and

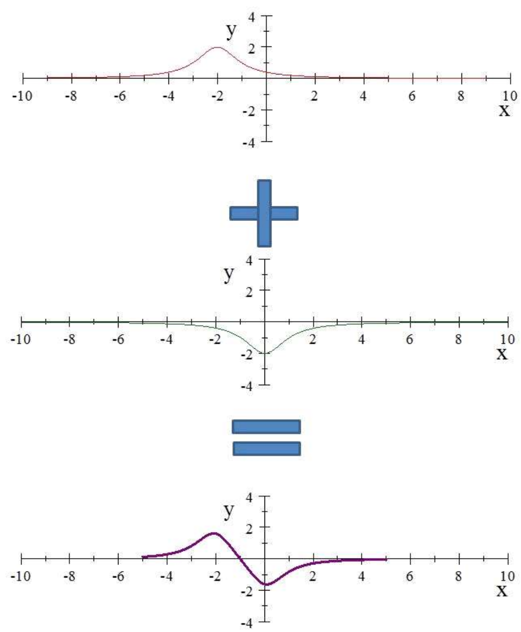
$$y_2(x,t) \tag{6.4}$$

both propagating on a string, then we would see

$$y_r(x,t) = y_1(x,t) + y_2(x,t) \tag{6.5}$$

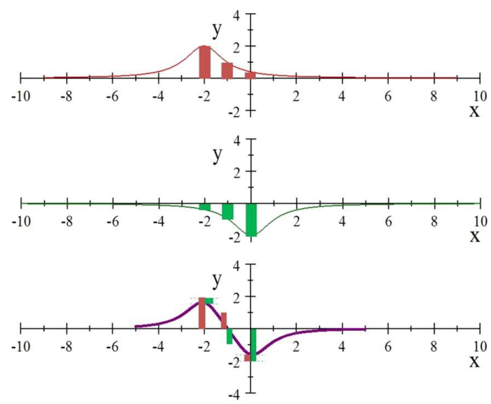
This is a fantastically simple way for the universe to act!

Let's look at an example. let's add the top wave (red) to the middle wave(green). We get the bottom wave (purple)



Of course we are adding these in the snapshot view. So this is all done for just one instant of time.

Let's see how to do this.



Start at $x = -2$. In the figure, I drew a red bar to show the y value at $x = -2$ for the red curve. Likewise, I have a green bar sowing the value of y at $x = -2$ for the green wave. Note that this is negative. On the bottom graph, the bars have been repeated, and we can see that the red bar minus the green bar brings us to the value for the resulting wave at the point $x = -2$. We need to do this at every point along all the waves for this instant of time.

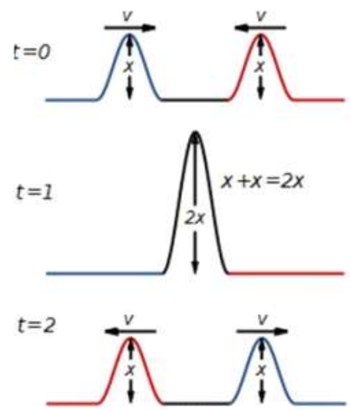
This is tedious by hand, so we won't generally do this calculation by hand. But a computer can do it easily.

Note that this is really only true for *linear* systems. Let's take the example of a slinky. If we form two waves in the slinky, they behave according to the superposition principle most of the time. But suppose we make the amplitude of the individual waves large. They may travel individually OK, but when the amplitudes add we may overstretch the slinky. Then it would never return to it's original shape. The wave form would have to change. Such a wave is not linear. There is a good rule of thumb for when waves are linear.

A wave is generally linear when its amplitude is much smaller than its wavelength.

6.4 Consequences of superposition

Linear waves traveling in media can pass through each other without being destroyed or altered!

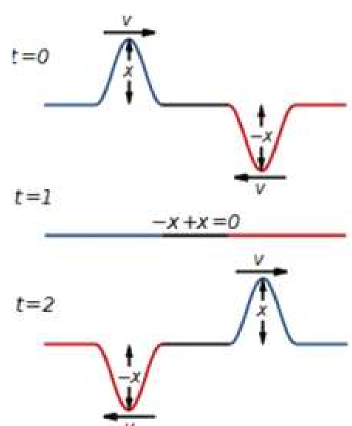


Constructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Constructive_interference.svg)

Our wave on the string makes the string segments move in the y direction. Both waves do this. So putting the two waves together just makes the string segments move more! There is a special name for what we observe

1. *interference*: The combination of separate waves in the same region of space to produce a resultant wave.

What happens if one of the pulses is inverted?



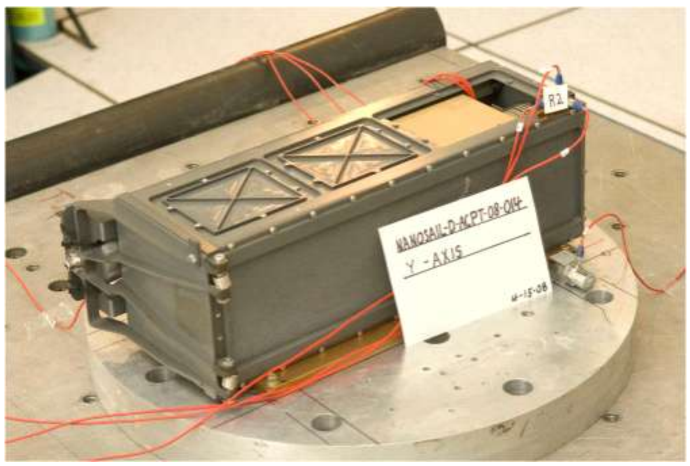
Destructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Destructive_interference1.svg)

When the two pulses meet, they "cancel each other out." But do they go away? No! the energy is still there, the string segment motions have just summed vectorially to zero, the energy carried by each wave is still there. We have a few more definitions. The type of interference we have just seen is the first

1. *Destructive Interference*: Interference between waves when the displacements caused by the two waves are opposite in direction

2. *Constructive Interference*: interference between waves when the displacements caused by the two waves are in the same direction

The combination of waves is important for both scientists and engineers. In engineering this is the hart of vibrometry.



Marshall and Cal Poly technicians wired the NanoSail-D spacecraft to accelerometers, instruments which measure vibration response during simulated launch conditions. Image couracy NASA, image in the Public Domain.

Mechanical systems have moving parts. These moving parts can be the disturbance that creates a wave. If more than one wave crest arrives at a location in the device, the amplitude at that location could become large. The oscillation of this part of the device could rattle apart welds or bolts, destroying the device. Later, as we study spectroscopy, we will see how to diagnose such a problem and hint at how to correct it.