Chapter 22

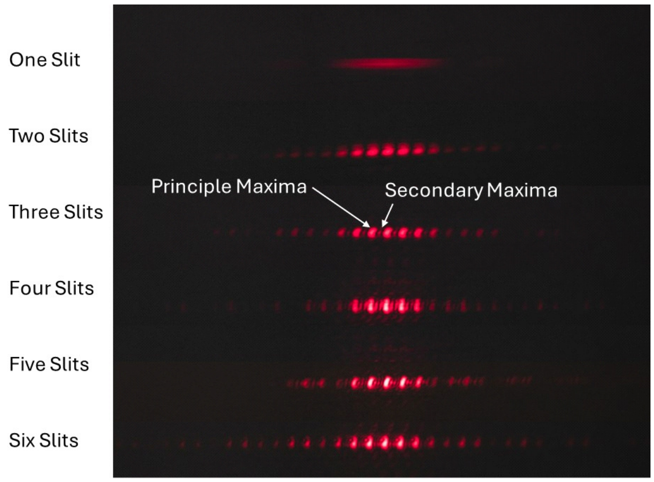
22 Slit Intensity 3.4.1, 3.4.2

Fundamental Concepts

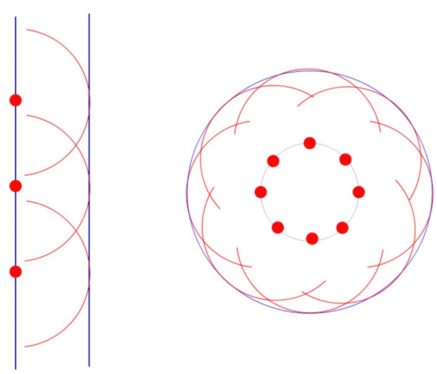
- Single slit minima
- Phasors
- Single slit intensity pattern
- The sinc function

22.1 Single Slits

We have looked at interference from two slits, and for many slits. Here is the figure again that shows how the constructive and destructive interference pattern changes as we change the number of slits or openings. Earlier we gave a name for these patterns of constructive and destructive interference. We called them diffraction patterns.



The two slits acted like two sources. We might expect that a single slit will give only a single bright spot. But if you look carefully at the last figure, you can see that there is something odd about the one slit pattern. The spot of light is stretched out. And if you look very carefully it almost looks like there are faint splotches of red light to the sides of the brighter fringe of light. let's consider a single slit very closely. To do this, let's return to the work of Huygens.[1] His idea for the nature of light was simple. He suggested that every point on the wave front of a light wave was the source (the disturbance) for a new set of small spherical waves. In optics, the crests of the waves are often called wavefronts. The next wavefront would be formed by the superposition of the little "wavelets." Here is an example for a plane wave and a spherical wave.



[1] Huygens method is technically not a correct representation of what happens. The actual wave leaving the single opening is a superposition of the original wave, and the wave scattered from the sides of the opening. You can see this scattering by tearing a small hole in a piece of paper and looking through the hole at a light source. You will see the bright ring around the hole where the edges of the paper are scattering the light. But the mathematical result we will get using Huygens method gives a mathematically identical result for the resulting wave leaving the slit with much less high power math. So we will stick with Huygens in this class.

In each case we have drawn spots on the wave front and drawn spherical waves around those spots. where the wavefronts of the little wavelets combine, we have new wave front of our wave. This is sort of what happens in bulk matter. Remember that light is absorbed and re-emitted by the atoms of the material. This is why light slows down in a material. Because of the time it is absorbed, it effectively goes slower. But the light is not necessarily re-emitted in the same direction. Sometimes it is, but sometimes it is not. This creates a small, spherical wave (called a wavelet) that is emitted by that atom. So Huygens idea is not too bad.

We can use this idea for a single slit and look at what happens as the light goes through. Here is such a slit.

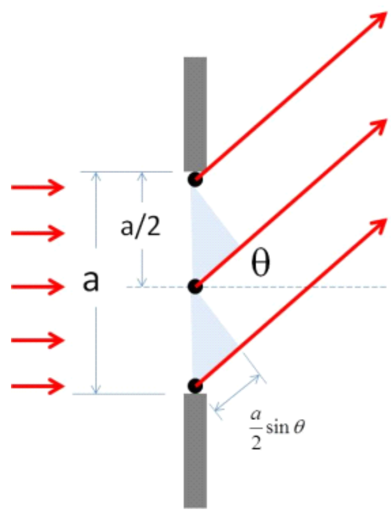
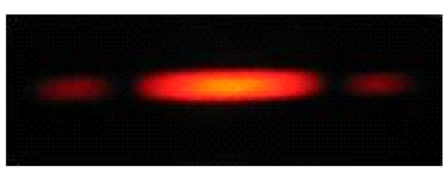


Figure 22.1:

In the figure above, we have divided a single slit of width a into two parts, each of size $a/2$. According to Huygens' principle, each position of the slit acts as a source of light rays. So we can treat half a slit as two coherent sources. These two sources should interfere. So what do we see when we perform such an experiment? Here is a brighter version of our light pattern from a single slit.



The figure shows a diffraction pattern for a thin slit. There are several terms that are in common use to describe the pattern

1. Central Maximum: The broad intense central band.

2. Secondary Maxima:The fainter bright bands to both sides of the central maxima

3. Minima: The dark bands between the maxima

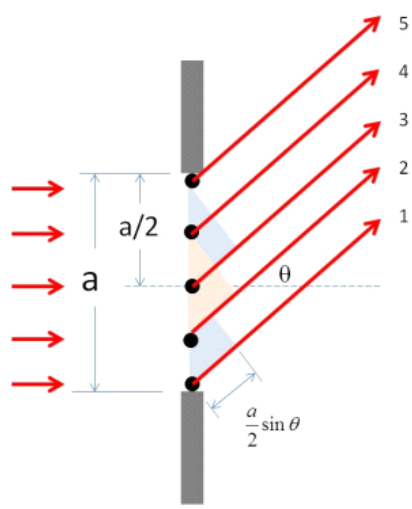
Notice, if we take two of the rays that hit one of our constructive interference spots, we get a path difference Δr that is

$$\Delta r = \frac{a}{2}\sin\theta$$

This would give a bright spot.

22.2 Narrow Slit Minima

Let's use figure 22.1 to find the *dark* minima of the single slit pattern. First we should notice that figure 22.1 could have another set of rays that contribute to the *bright* spot because they will also have a path difference of $(a/2)\sin\theta$. Let's fill these in. They are rays 2 and 4 of the next figure.



Before we started with what we are now calling rays 1 and 3. Ray 1 travels a distance

$$\Delta r = \frac{a}{2}\sin(\theta) \tag{22.1}$$

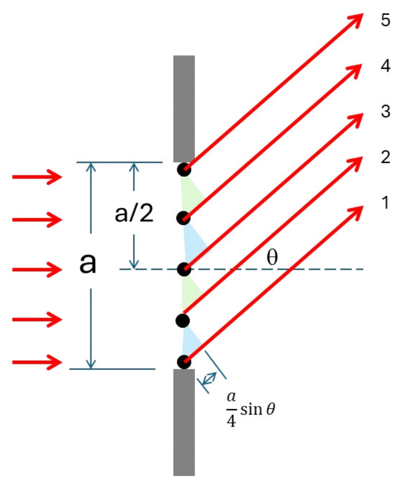
farther than ray 3. As we just argued, rays 2 and 4 also have the same path difference, and so do rays 3 and 5. If this path difference is a multiple of λ we get constructive interference, but If this path difference is $\lambda/2$ then we will have destructive interference. The condition for a minima is then

$$\frac{a}{2}\sin(\theta) = \pm\frac{\lambda}{2} \tag{22.2}$$

or

$$\sin(\theta) = \pm\frac{\lambda}{a} \tag{22.3}$$

But look at our figure. We could just as well divide the slit into four equal parts



and imagine light traveling from each of those parts to our viewing screen. Then we have a path difference of

$$\Delta r = \frac{a}{4}\sin(\theta) \tag{22.4}$$

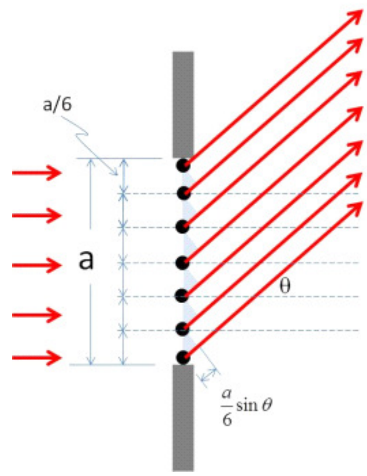
And to have destructive interference we need this path difference to be $\lambda/2$

$$\frac{a}{4}\sin(\theta) = \pm\frac{\lambda}{2} \tag{22.5}$$

or

$$\sin(\theta) = \pm\frac{2\lambda}{a} \tag{22.6}$$

But there is no reason to stop at four equal parts. Suppose we we divide our opening into six equal parts.



Then we would find

$$\Delta r = \frac{a}{6}\sin(\theta) \tag{22.7}$$

and we still want a dark spot so

$$\frac{a}{6}\sin(\theta) = \pm\frac{\lambda}{2} \tag{22.8}$$

or

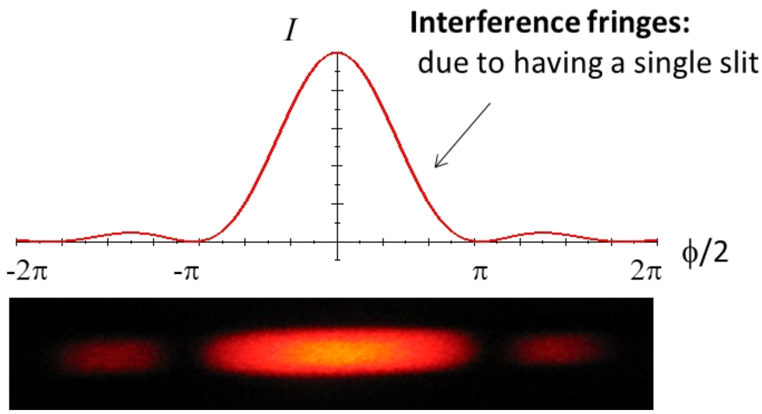
$$\sin(\theta) = \pm\frac{3\lambda}{a} \tag{22.9}$$

and in general at the angle θ that makes the sine of that angle equal to $m\lambda/a$ we get a dark spot.

$$\sin(\theta) = m\frac{\lambda}{a} \quad m = \pm 1, \pm 2, \pm 3 \ldots \tag{22.10}$$

You might object that we did not find the bright spots, only the dark spots. That's fine because the bright spots have to be in between the dark spots. But we can do better. Let's look at the full intensity pattern for a single slit.

22.3 Intensity of the single-slit pattern



We have found the dark minima, now we would like to find the shape of the bright bands. We will use phasors to do this. If you have already taken Principles of Physics III (PH220) you know what a phasor is, but for the rest of us we need a quick introduction.

22.3.1 Phasors

Let's take two waves

$$E_1 = E_{o1} \sin(\omega t + \alpha_1) \tag{22.11}$$

and

$$E_2 = E_{o2} \sin(\omega t + \alpha_2) \tag{22.12}$$

where I have hidden the position dependence in the α's

$$\begin{aligned} \alpha_1 &= kx_1 + \phi_1 \\ \alpha_2 &= kx_2 + \phi_2 \end{aligned}$$

We wish to find the resultant wave

$$E = E_{o1} + E_{o2} \tag{22.13}$$

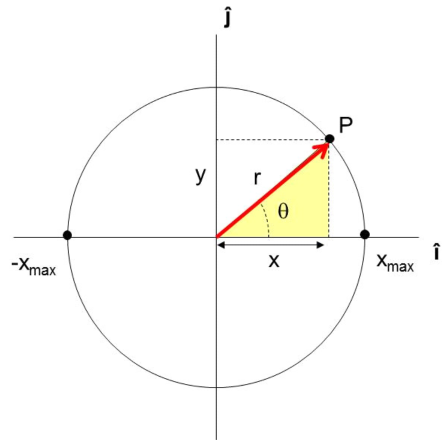
We actually know how to do the algebra to combine these waves (we have done it enough times!), but let's find another way to do the math that can extend to many many waves being added up.

Think of our equation for a wave.

$$E_1 = E_{o1} \sin(\omega t + \alpha_1) \tag{22.14}$$

We have made our wave equation look like the equation for a harmonic oscillator. And so long as we don't change x_1 or x_2 this is fine. The electric field at our position where we are mixing the waves will oscillate like a harmonic oscillator.

Way back when we studied oscillation we said the equation for a harmonic oscillator position was like the projection of circular motion onto an axis.



That gave us

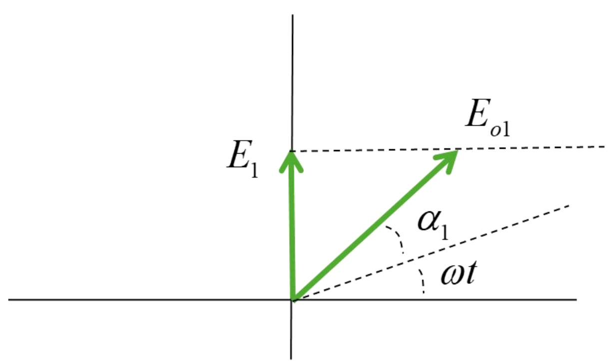
$$x(t) = x_{\max} \cos(\omega t + \phi_o)$$

or

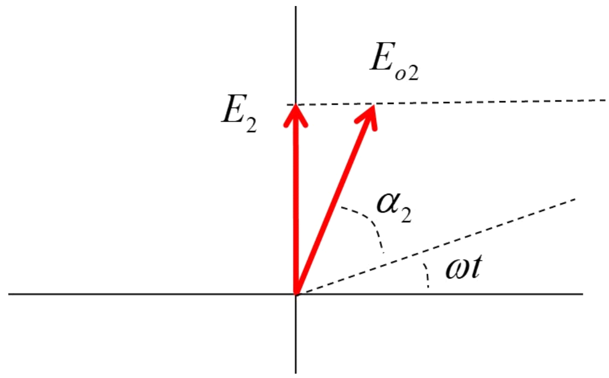
$$y(t) = y_{\max} \sin(\omega t + \phi_o)$$

as our oscillator equations. And see how much our modified wave equation looks like the second one of these oscillator equations! This gives us an idea of how to use vector addition to add up waves!

Let's represent E_{o1} as a rotating vector with angular speed ω. We get projections of E_{o1} on the x and y axis. In this discussion, we will consider the vertical (y) axis but we could choose either axis (but if we use the x-axis we would then have cosine waves).



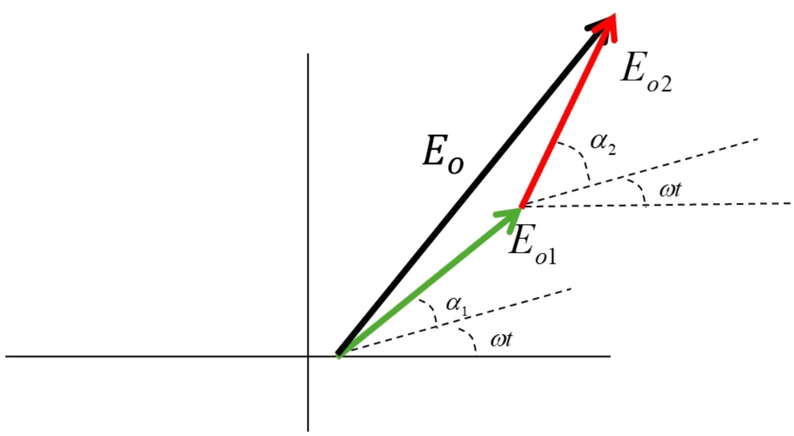
At any given time, E_1 will be the projection $E_{o1} \sin(\omega t + \alpha_1)$. We can also take a second rotating vector for E_2



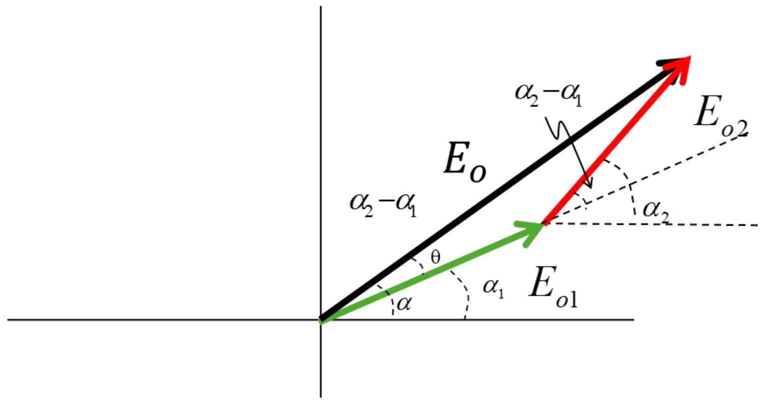
Note that E_2 is projected onto the vertical axis through an angle $\omega t + \alpha_2$. So the projection E_2 is $E_{o2} \sin(\omega t + \alpha_2)$.

These rotating vectors have a name. We call them *phasors.*

To add the phasors, we draw phasor two on the end of phasor 1. It is just like adding vectors back in Principles of Physics I (PH121)!



The sum is the resultant phasor like in vector addition. Only, remember now that the phasors are rotating. But because they have the same angular frequency, ω, phasor 1 and phasor 2 rotate together as a unit. The projection of this rotating group of phasors gives the sum we are looking for. But notice the sum's value would change in time as the phasors rotate. We can choose any time to find the resulting vector, so we might as well take $t = 0$, then $\omega t = 0$ and the vectors will be rotated as shown.



Using the redrawn phasors in the diagram directly above, we can see from the law of cosines

$$E_0^2 = E_{o1}^2 + E_{o2}^2 - 2E_{o1}E_{o2}\cos\left(\pi - (\alpha_2 - \alpha_1)\right) \tag{22.15}$$

or

$$E_0^2 = E_{o1}^2 + E_{o2}^2 + 2E_{o1}E_{o2}\cos\left(\alpha_2 - \alpha_1\right) \tag{22.16}$$

using the trig identity that $\cos(\pi - \gamma) = -\cos\gamma$. This is the magnitude of our resulting vector. For the case of a double slit we know that $E_{o1} = E_{o2}$ so our resulting vector would be

$$E_0^2 = E_{o1}^2 + E_{o1}^2 + 2E_{o1}E_{o1}\cos\left(\alpha_2 - \alpha_1\right) \tag{22.17}$$

$$= 2E_{o1}^2 + 2E_{o1}^2\cos\left(\alpha_2 - \alpha_1\right) \tag{22.18}$$

We can see that when $\cos(\alpha_2 - \alpha_1) = -1$ we get zero. This is destructive interference!

We should go on to find the angle. In the figure it is called just α and it is α_1 plus a little bit more. It's not thrilling, but since we have the sides of the triangle we could just use the law of cosines again to find the little bit more angle

$$E_{o1}^2 = E_0^2 + E_{o1}^2 + 2E_0E_{o2}\cos(\theta)$$

so

$$\theta = \cos^{-1}\frac{E_{o2}^2 - E_0^2 + E_{o1}^2}{2E_0E_{o2}}$$

or when $E_{o1} = E_{o2}$

$$\theta = \cos^{-1}\frac{2E_{o1}^2 - E_0^2}{2E_0E_{o1}}$$

and

$$\alpha = \alpha_1 + \theta$$

Now we ask what the question, "what is the projection of this resultant phasor on the vertical axis?" Since we chose $t = 0$ for our picture, it is easy to pick off the fact that the resultant vector has a phase angle of α. thus

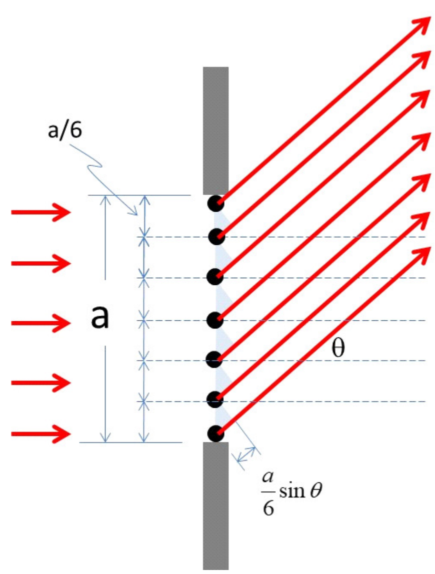
$$E_R = E_0 \sin(\omega t + \alpha) \tag{22.19}$$

This seems like a lot of work for adding just two waves together, but we will use it to add many many waves together.

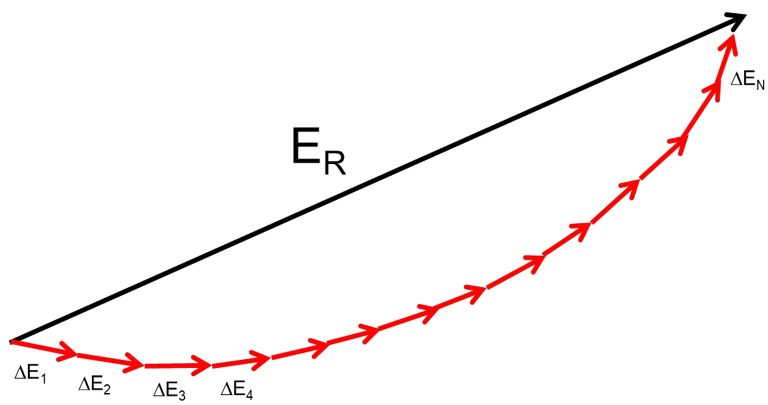
22.3.2 Using phasors to find the intensity pattern

Now that we know what a phasor is, let's try to use them to find the single slit intensity pattern.

We can consider dividing the slit into small regions Δy in length.



Each piece of the slit contributes ΔE to the electric field. We can use superposition to find the resultant wave for a given point P. The phase diagram would look something like this.



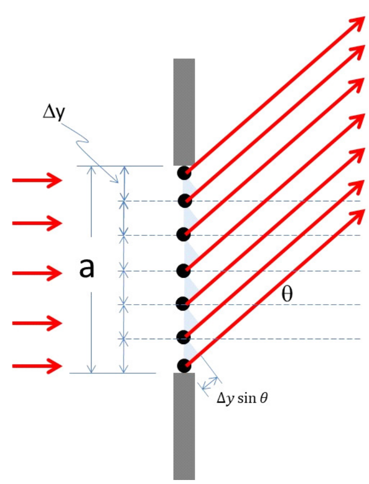
We add up all the little ΔE parts, but each has a slightly different distance, so it has a slightly different phase because kr is part of the phase. We can put the phase difference due to the different paths into the phase constant.

Our set of little waves might be like

$$\begin{aligned} \Delta E_1 &= E_o \sin(\omega t) \\ \Delta E_2 &= E_o \sin(\omega t + \phi_2) \\ \Delta E_3 &= E_o \sin(\omega t + \phi_3) \\ &\vdots \end{aligned} \tag{22.20}$$

Note that these are still waves. We have just put the kr_i part into the phase ϕ_i like we did in the α's in the last section.

Note that each wave has an additional phase shift, and each has the amplitude $E_{o.}$ But we need to find these phase shifts. We will call the phase shift $\Delta\phi$ to remind us that it is small. Look at the next figure.



By thinking about this figure we can see that

$$\Delta r = \Delta y \sin\theta \tag{22.21}$$

for each pair of adjacent rays. and that gives a phase difference of

$$\Delta\phi = k\Delta r = \frac{2\pi}{\lambda}\Delta y \sin\theta \tag{22.22}$$

Let's suppose that as we cross the slit we have N waves each of magnitude ΔE. Then the total phase difference between waves from the top to the bottom of the slit is

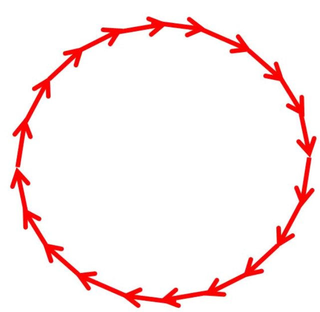
$$\phi = N\Delta\phi = N\frac{2\pi}{\lambda}\Delta y \sin\theta \tag{22.23}$$

$$= \frac{2\pi}{\lambda} a \sin\theta \tag{22.24}$$

where a is our slit width and

$$a = N\Delta y \tag{22.25}$$

We are finding dark spots in our diffraction pattern, and the first dark spot will be when the projection of our phasor sum, $E_R = 0$. And think about this, when you add vectors you put the vectors tip-to-tail, and when you are done you start at the tail of the first and draw a new vector to the tip of the last vector you added. This new vector is the sum. And to make that new vector have a length $E_R = 0$ all we have to do is to have the vectors we add make a circle. Then the distance from the tail of the first vector to the tip of the last vector is exactly zero! We are doing this with phasors, so we place our phasors tip-to-tail and find the first minimum or first dark band when $\phi = 2\pi$ or when the phasor diagram circles back on itself.



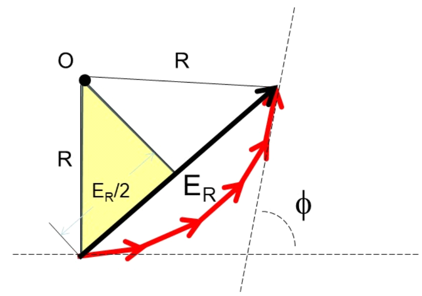
$$2\pi = \frac{2\pi}{\lambda} a \sin\theta \tag{22.26}$$

We can cancel and rearrange this to give

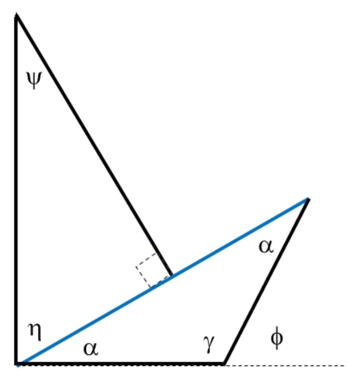
$$\frac{\lambda}{a} = \sin\theta \tag{22.27}$$

which is what we expected for the first minimum! This seems to be working. Let's now find the intensity pattern.

To find the intensity pattern on any point of the screen (not just the minima), we let Δy get very small. We add up $N \to \infty$ terms. The curve becomes smooth (not shown as smooth in the next figure, though).



The quantity E_R is the resultant from adding together all of our phasors. We can see the phasor addition in the figure. The angle at the top of the yellow triangle is $\phi/2$. One way to see this is to note that the lower triangle has angles α, γ, and α



We see that

$$\gamma = 180 - \phi \tag{22.28}$$

and

$$\phi = 2\alpha \tag{22.29}$$

Note also that

$$\eta = 90 - \alpha \tag{22.30}$$

For the upper triangle,

$$\begin{aligned} 180 &= \psi + \eta + 90 && (22.31) \\ 90 &= \psi + \eta && (22.32) \end{aligned}$$

we know η, so

$$90 = \psi + 90 - \alpha \tag{22.33}$$

or

$$\psi = \alpha \tag{22.34}$$

but

$$\alpha = \frac{\phi}{2} \tag{22.35}$$

so

$$\psi = \frac{\phi}{2} \tag{22.36}$$

Then

$$\sin\left(\frac{\phi}{2}\right) = \frac{\frac{E_R}{2}}{R} = \frac{E_R}{2R} \tag{22.37}$$

where R is the radius of curvature, and from the archlength formula

$$s = R\phi \tag{22.38}$$

and since we are finding the arc length, we know it will be nearly the sum of each little phasor vector length $s = NE_o$ so

$$s = R\phi = NE_o$$

and

$$R = \frac{NE_o}{\phi}$$

then our resultant wave phasor has length

$$E_R = 2R\sin\left(\frac{\phi}{2}\right) \tag{22.39}$$

$$= 2\frac{NE}{\phi}\sin\left(\frac{\phi}{2}\right) \tag{22.40}$$

$$= NE_o\frac{\sin\left(\frac{\phi}{2}\right)}{\frac{\phi}{2}} \tag{22.41}$$

It is silly, but we usually define $\beta = \phi/2$ so

$$E_R = NE_o\frac{\sin(\beta)}{\beta}$$

Earlier we learned that the intensity will go as the field squared

$$I = I_{\max}\left(\frac{\sin(\beta)}{\beta}\right)^2 \tag{22.42}$$

And here is a tricky part. We get the angle ϕ from adding up the angles from each of our little ΔE pieces. Each of them has an angle

$$\Delta\phi = k\Delta r = \frac{2\pi}{\lambda}\Delta y \sin\theta \tag{22.43}$$

and N of these will be

$$\phi = N\frac{2\pi}{\lambda}\Delta y \sin\theta$$

So we can see that the angle ϕ is the same as our total phase. If we have an aperture width of a then $a = N\Delta y$ and

$$\phi = N\frac{2\pi}{\lambda}\Delta y \sin\theta = \frac{2\pi}{\lambda}a\sin\theta$$

and that makes

$$\beta = \frac{\frac{2\pi}{\lambda}a\sin\theta}{2}$$

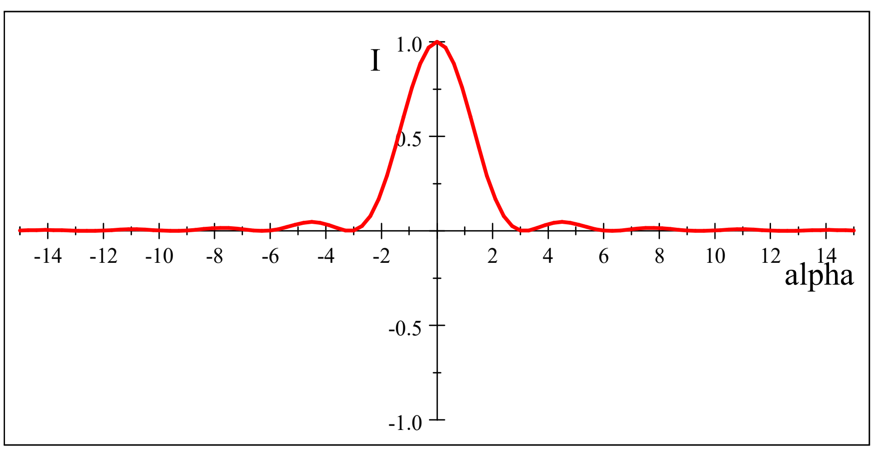
We can write our irradiance function as

$$I = I_{\max}\left(\frac{\sin\left(\frac{\frac{2\pi}{\lambda}a\sin\theta}{2}\right)}{\frac{\frac{2\pi}{\lambda}a\sin\theta}{2}}\right)^2 \tag{22.44}$$

or more simply

$$I = I_{\max}\left(\frac{\sin\left(\frac{\pi}{\lambda}a\sin\theta\right)}{\frac{\pi}{\lambda}a\sin\theta}\right)^2 \tag{22.45}$$

The function is plotted in the next figure

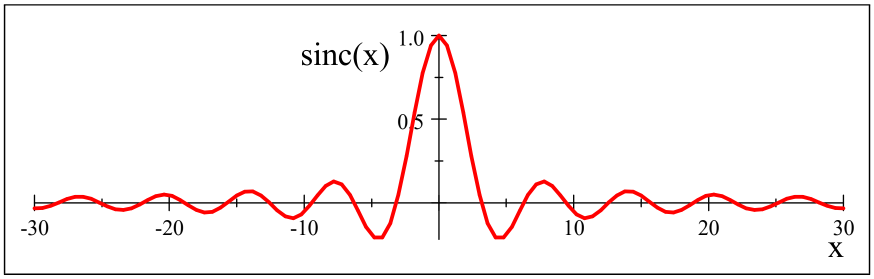


From this we can find our peaks where the pattern is bright.

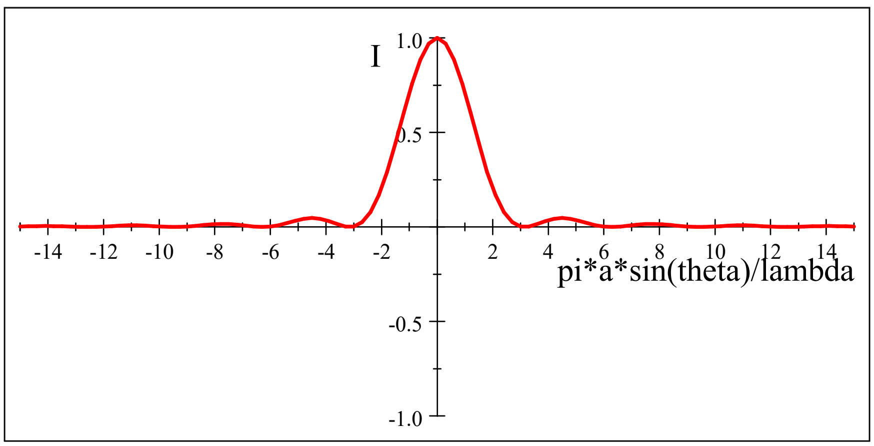
Notice that our field equation this has the form

$$\frac{\sin x}{x}$$

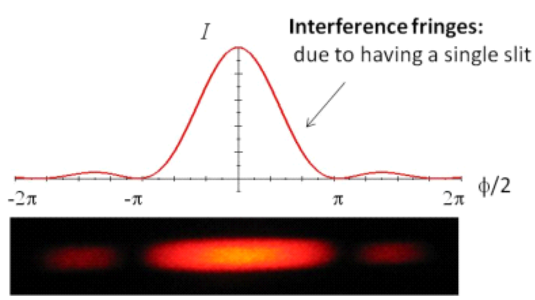
which has a distinctive shape.



this is known as a sinc function (pronounced like "sink"). It has a central maximum as we would expect. Of course our pattern has a sinc squared



You can see the central maximum and the much weaker minima produced by this function. Indeed, it seems to match what we saw very well. Putting it all together, our pattern looks like this.



You might ask, can we use our phasor approach to find the intensity pattern for double slit diffraction? Of course, the answer is yes, but I have had enough phasor fun for this class. You will see phasors again in Principles of Physics III.