Chapter 7

7 Standing Waves and Sound Waves 1.16.6, 1.17.1

Fundamental Concepts

- Waves traveling different directions can create static patterns of constructive and destructive interfearance.
- These patterns are called "standing waves"
- The frequencies that make the standing wave patterns are quantized
- Sound is a wave
- We can express sound waves in terms displacement of air molecules or as pressure changes

7.1 Mathematical Description of Superposition

We know what superposition is, but we don't really want to add values for millions of points in a medium to find out what a combination of waves will look like. At the very least, we want to make a computer do that (and programs like OpenFoam do something very akin to this!). But where we can, we would like to combine wave functions algebraically. Let's see how this can work.

Lets define two wave functions

$$y_1 = y_{\max} \sin(kx - \omega t)$$

and

$$y_2 = y_{\max} \sin(kx - \omega t + \phi_o)$$

These are two waves with the same frequency and wave number traveling the same direction in the medium, but they at $t = 0$ the y values are not the same because of the phase shift. The graph of y_2 is shifted by an amount ϕ_o.

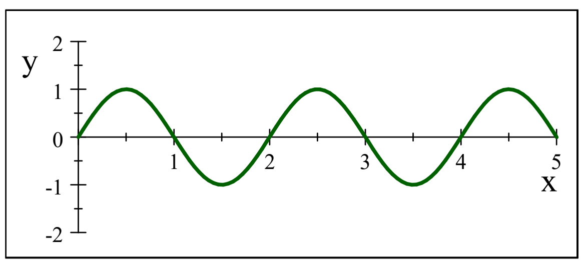
I will pick some values for the constants

$$\begin{array}{l} \lambda = 2 \\ k = \frac{2\pi}{\lambda} \\ \omega = 1 \\ \phi_o = \frac{\pi}{6} \\ t = 0 \\ A = 1 \end{array}$$

then for y_1 we have

$$\begin{aligned} y_1 &= (1)\sin\left(\frac{2\pi}{\lambda}x - (1)\,t\right) \\ &= \sin\left(\frac{2\pi}{2}x - (1)\,t\right) \\ &= \sin(\pi x - t) \end{aligned}$$

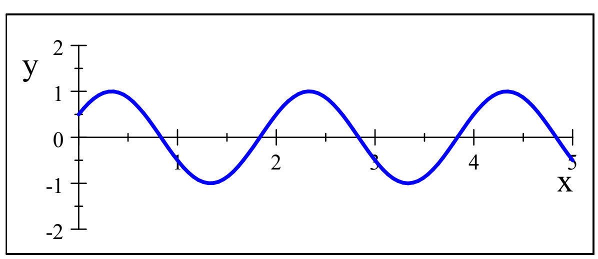
here is a plot of the wave function, y_1



Now let's consider y_2. Using the values we chose, y_2 can be written as

$$\begin{aligned} y_2 &= y_{\max}\sin(kx - \omega t + \phi_o) \\ &= \sin\left(\pi x - t + \frac{\pi}{6}\right) \end{aligned}$$

which looks like this



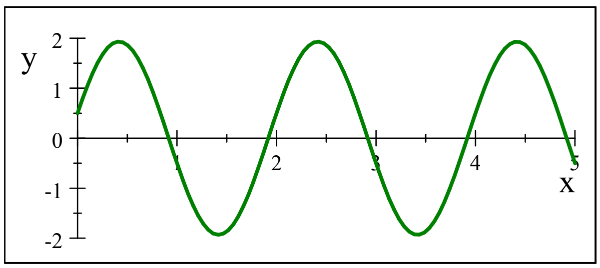
What does it look like if we add these waves using superposition? Symbolically we have

$$y_r = y_{\max} \sin(kx - \omega t) + y_{\max} \sin(kx - \omega t + \phi_o) \tag{7.1}$$

and putting in the numbers gives

$$y_r = \sin(\pi x - t) + \sin\left(\pi x - t + \frac{\pi}{6}\right)$$

which is shown in the next graph.



Notice that the wave form is taller (larger amplitude). Noticed it is shifted along the x axis.

We can find out by how much by rewriting y_r. We need a trig identity

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

Then let $a = kx - \omega t$ and $b = kx - \omega t + \phi$

$$\begin{aligned}
y_r &= y_{\max} \sin(kx - \omega t) + A \sin(kx - \omega t + \phi_o) \\
&= 2y_{\max} \cos\left(\frac{(kx - \omega t) - (kx - \omega t + \phi_o)}{2}\right) \sin\left(\frac{(kx - \omega t) + (kx - \omega t + \phi_o)}{2}\right) \\
&= 2y_{\max} \cos\left(\frac{-\phi_o}{2}\right) \sin\left(\frac{2kx - 2\omega t + \phi_o}{2}\right) \\
&= 2y_{\max} \cos\left(\frac{-\phi_o}{2}\right) \sin\left(kx - \omega t + \frac{\phi_o}{2}\right) \\
&= 2y_{\max} \cos\left(\frac{\phi_o}{2}\right) \sin\left(kx - \omega t + \frac{\phi_o}{2}\right)
\end{aligned}$$

where we use the fact that $\cos(-\theta) = \cos(\theta)$.

Let's look at the parts of this expression. First take the sine part.

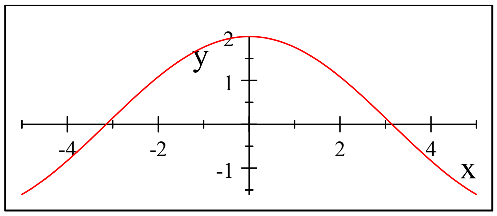
$$\sin\left(kx - \omega t + \frac{\phi_o}{2}\right) \tag{7.2}$$

This part is a traveling wave with the same k and ω as our original waves, but it has a phase of $\phi_o/2$. So our combined wave is shifted by $\phi_o/2$ or half the phase shift of y_2.

Now let's look at other factor

$$2y_{\max}\cos\left(\frac{\phi_o}{2}\right) \tag{7.3}$$

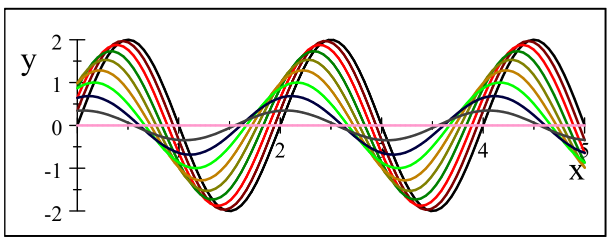
The sine part of our wave equation is multiplied by all of this factor. So all of this part is the new amplitude. It has a maximum value when $\phi_o = 0$



When $\phi_o = \pi$, then

$$2y_{\max}\cos\left(\frac{\pi}{2}\right) = 0$$

so when $\phi_o = 0$ we have a new maximum amplitude of $2y_{\max}$ and when $\phi_o = \pi$ we have a zero amplitude. Here is our wave for several choices of ϕ_o.



7.1.1 Mathematical description of standing waves

Standing Wave Demo

Now that we have a way to make two waves to superimpose, we can study the special case of standing waves.

A standing wave pattern is the result of the superposition of two traveling waves with the same frequency going in opposite directions

$$\begin{aligned} y_1 &= y_{\max}\sin(kx - \omega t) \\ y_2 &= y_{\max}\sin(kx + \omega t) \end{aligned}$$

The sum is

$$y_r = y_1 + y_2 = y_{\max}\sin(kx - \omega t) + A\sin(kx + \omega t)$$

To gain insight into what these two waves produce, we use another of our favorite trig identities

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

to get

$$\begin{aligned}
y_r &= y_{\max}\sin(kx-\omega t)+y_{\max}\sin(kx+\omega t)\\
&= y_{\max}\sin(kx)\cos(\omega t)-y_{\max}\cos(kx)\sin(\omega t)+y_{\max}\sin(kx)\cos(\omega t)+y_{\max}\cos(kx)\sin(\omega t)\\
&= 2y_{\max}\sin(kx)\cos(\omega t)\\
&= (2y_{\max}\sin(kx))\cos(\omega t)
\end{aligned}$$

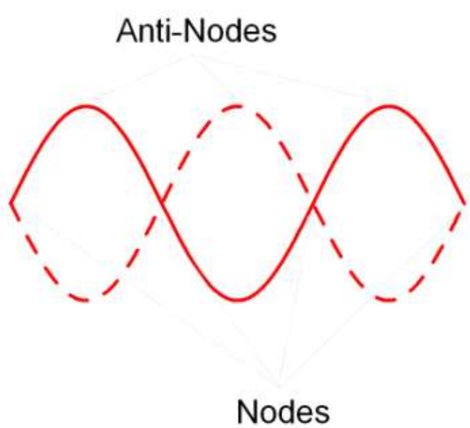
This looks like the harmonic oscillator equation

$$y = y_{\max}\cos(\omega t+\phi_o)$$

with $\phi_o = 0$ and with another complicated amplitude

$$2y_{\max}\sin(kx)$$

That is, we have a set of harmonic oscillators who's amplitude is different for each value of x.



We can identify spots along the x axis where the amplitude is always zero! we will call these spots *nodes.* These happen when $\sin(kx) = 0$ or when

$$kx = n\pi$$

By using

$$k = \frac{2\pi}{\lambda}$$

we have

$$\begin{aligned}
\frac{2\pi}{\lambda}x &= n\pi\\
\frac{2}{\lambda}x &= n\\
x &= n\frac{\lambda}{2}
\end{aligned}$$

We can also find the places along x where the amplitude will be largest. this occurs when $\sin(kx) = 1$ or when

$$kx = n\frac{\pi}{2}$$

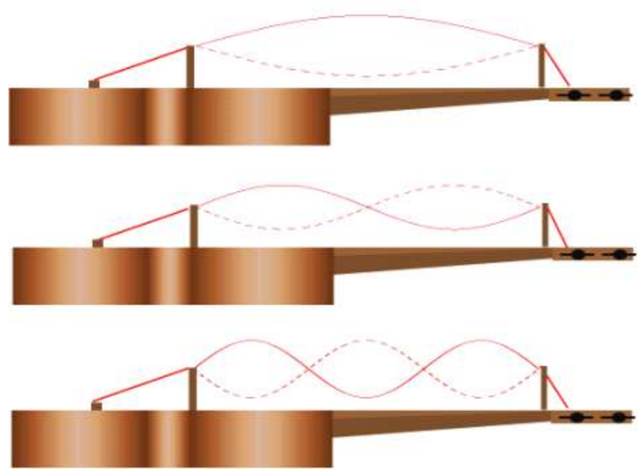
or

$$\begin{aligned} \frac{2\pi}{\lambda}x &= n\frac{\pi}{2} \\ x &= n\frac{\lambda}{4} \end{aligned}$$

these are called *antinodes*.

We can also create standing waves with sound or even light waves!

7.1.2 Standing Waves in a String Fixed at Both Ends



If we attach a string to something on both ends, we find something interesting in the standing wave pattern. Not all possible standing waves can be realized. Some frequencies are preferred, and some never show up. The standing wave pattern is *quantized*. The patterns that are allowed are called *normal modes*. We will see this any time a wave confined by boundary conditions (light in a resonant cavity, radio waves in a wave guide, electrons in an atom, etc.).

The figure shows some normal modes for a string.

We find which modes are allowed by first imposing the boundary condition that each end must be a node. We start with

$$y = 2y_{\max}\sin(kx)\cos(\omega t)$$

and recognize that we have one condition met because

$$y = 0$$

when
$$x = 0$$
we need $y = 0$ when $x = L$. That happens when
$$kL = n\pi$$
I will write this as
$$k_n L = n\pi$$
to indicate there are many values of k that can work. Solving this for λ_n gives
$$\begin{aligned} \frac{2\pi}{\lambda_n} L &= n\pi \\ \frac{2L}{n} &= \lambda_n \end{aligned}$$
Which says there are many wavelengths that will work. Let's see how this works, the first mode will have
$$\lambda_1 = 2L$$
where L is the length of the string. Looking at the figure, this works.

The second mode has three nodes (one on each end and one in the middle). This gives
$$\lambda_2 = L$$
We can keep going, the third mode will have five nodes
$$\lambda_3 = \frac{3L}{2}$$
and so forth to give
$$\lambda_n = \frac{2L}{n}$$
We use our old friend
$$v = f\lambda$$
to find the frequencies of the modes
$$f = \frac{v}{\lambda}$$
$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$
or, in general
$$\begin{aligned} f_n &= \frac{v}{\lambda_n} = n\frac{v}{2L} \\ &= \frac{n}{2L} v \\ &= \frac{n}{2L}\sqrt{\frac{T}{\mu}} \end{aligned}$$

The lowest frequency has a special name, the *fundamental frequency.* The higher frequencies are integer multiples of the fundamental. When this happens we say that the frequencies form a *harmonic series,* and the modes are called *harmonics.* Since only certain frequencies work, we say that the frequencies of waves on the string that make standing waves are *quantized*! Quantization means that some values are allowed. This idea is the basis behind Quantum mechanics (which views light and even matter as waves).

7.1.3 Musical Strings

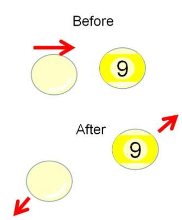
So how do we get different notes on a guitar or Piano?

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{7.4}$$

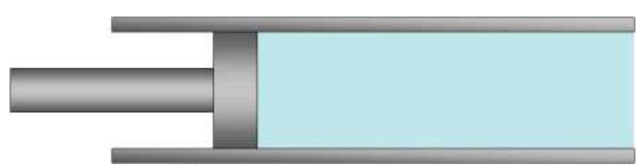
A guitar uses tension to change the frequency or pitch (tuning) and length of string (your fingers pressing on the strings) to change notes. A Piano uses both tension and length of string (and mass per unit length as well!). What do you expect and organ will do?

7.2 Sound waves

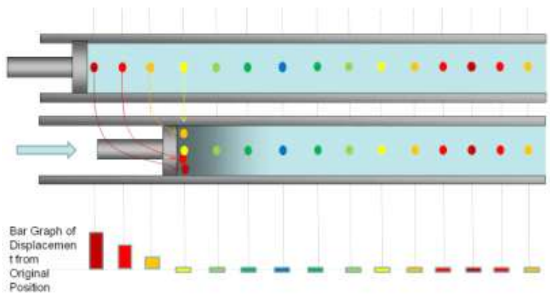
Sound is a wave. The medium is air particles. The transfer of energy is done by collision.



The wave will be a longitudinal wave. Let's see how it forms. We can take a tube with a piston in it.

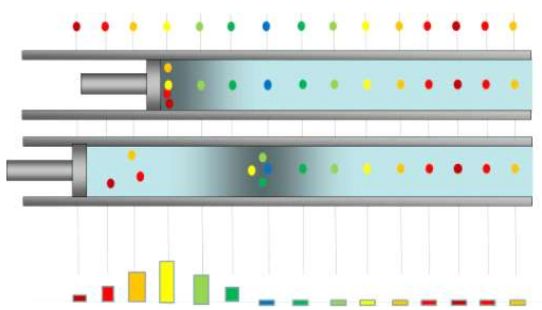


As we exert a force on the piston, the air molecules are compressed into a group. In the next figure, each dot represents a group of air molecules. In the top picture, the air molecules are not displaced. But when the piston moves, the air molecules receive energy by collision. They bunch up. We see this in the second picture.

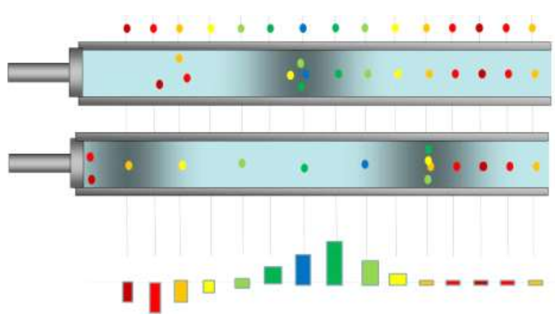


The graph below the two pictures shows how much displacement each molecule group experiences.

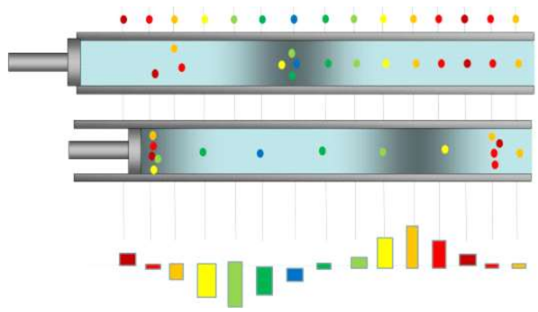
Suppose we now pull the piston back. This would allow the molecules to bounce back to the left, but the molecules that they have collided with will receive some energy and go to the right. This is shown in the next figure. Color coded dots are displayed above the before and after picture so you can see where the molecule groups started.



If we pull the piston back further, the molecules can pass their original positions.



Then we can push inward again and compress the gas.



This may seem like a senseless thing to do, but it is really what a speaker does to produce sound. In particular, a speaker is a harmonic oscillator. The simple harmonic motion of the speaker is the disturbance that makes the sound wave.

7.2.1 Periodic Sound Waves, Pressure

Vollyball Demo

Suppose I have a ball and I ask six people from the class to come up and press on the ball from all directions. This is a new force situation, unlike most we dealt with in Dynamics or PH121. Each person exerts a force on the ball. The person uses the area of their hand to exert the force. The motion of the ball, and even it's shape depend on both the force (magnitude and direction) and the area involved in each push.

From our demo, it seems that the force and area of the ball could be related to better describe the situation. Let's look at the ratio

$$\frac{F}{A}$$

What does it represent? This ratio tells us how spread out an applied force may be. The area is important. Think of the sides of a the eraser of a pencil.

It is coinvent to give this concept a name. We will call it *pressure.*

Pressure is a scalar value that describes how a force acts over an area

$$P = \frac{F}{A}$$

But in a fluid, what is the force? Lets consider a ball hitting a wall. Is there a force? During the collision, there is a force of the ball on the wall, and a Newton's third law force of the wall molecules pushing back on the ball. The ball will bounce back.

Ping Pong Ball Demo

This force is small and only lasts during the collision. But now suppose we have many balls, and all the balls impact the wall. Further suppose that every time a ball bounces back is ends up headed back to the wall and bounces again. If the balls keep coming, there will be a force on the wall quite a bit of the time. At least, on average there is a force, anyway. This is the force that causes air pressure. The air molecules in this room are like the small balls. we will find that they have an amount of kinetic energy. They impact the walls (and us) all over our surface area. The result is air pressure.

The water pressure in a swimming pool is caused by moving water particles. You should convince yourself that the reason the water stays in the pool is partly because the air molecules bounce against the water surface exerting a pressure on the waver!

Let's go back to making sounds. Suppose we push our piston as we did before in figure **??**. When we push in the piston, it creates a region of higher pressure next to it.

When we pull back the piston the fluid expands to fill the void. We create a rarefaction next to the piston.

Suppose we drive the piston sinusoidally. Can we describe the motion of the particles and of the wave?

1. Compression: A local region of higher pressure in a fluid
2. Rarefaction: A local region of lower pressure in a fluid

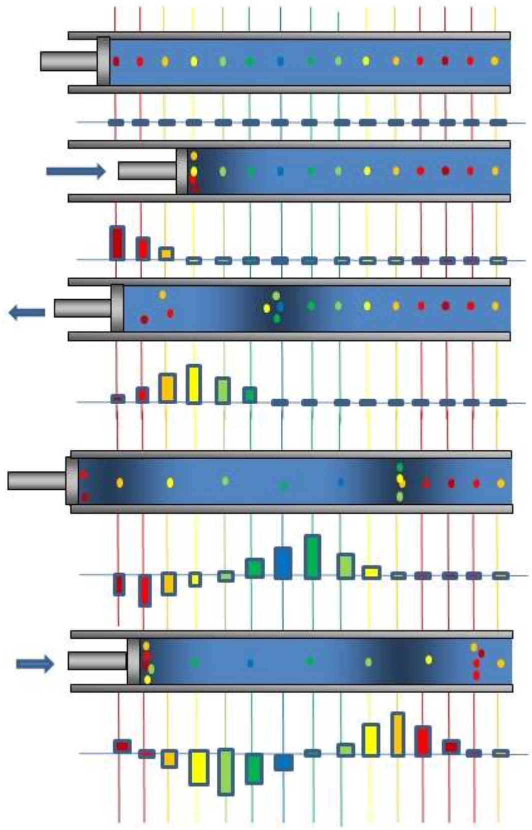
We can identify the distance between two compressions as λ.

We define $s(x,t)$ (like we defined a wave function, $y(x,t)$) as the displacement a particle of fluid relative to its equilibrium position.

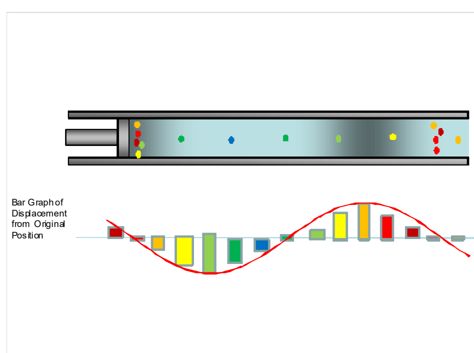
$$s(x,t) = s_{\max}\cos(kx - \omega t) \tag{7.5}$$

but what is $s_{\max}$?

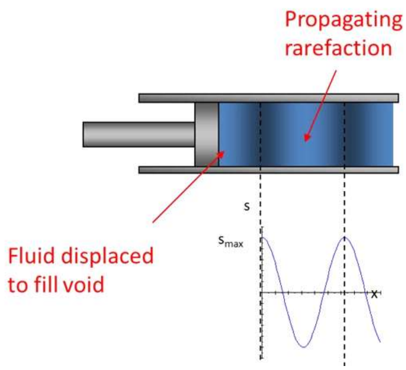
We remember that $s_{\max}$ is the maximum displacement of a particle of fluid from its equilibrium position. We plotted this using a bar graph to show displacement from the equilibrium position for our molecules. As we push the piston in and out we will get something like this.



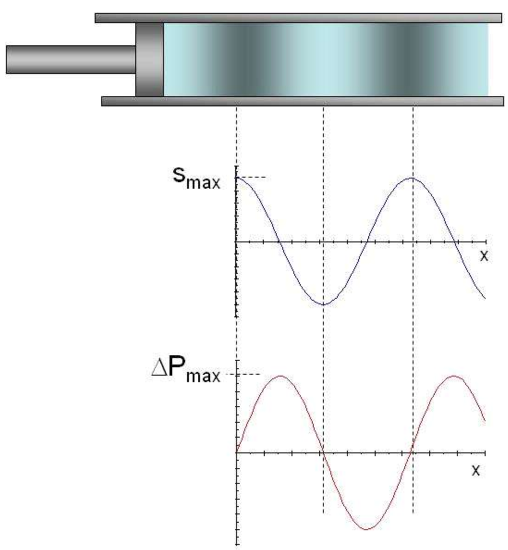
found before that we get something that looks like a sine wave, but remember what the bars represent. They represent the displacement from original position.



We don't usually draw bar graphs, we usually just draw the sine wave.



The variation of the gas pressure ΔP measured from its equilibrium is also periodic



which is why we often refer to a sound wave as a pressure wave. Think of when the wave gets to your ear. the wave consists of a group of particles all headed for your ear drum. When they hit, they exert a force. Pressure is a force spread over an area,

$$P = \frac{F}{A}$$

so in a sense, we hear changes in air pressure!