Chapter 27

27 Conservation of Energy for Fluid Flow 1.14.5, 1.14.6

So far, we have only dealt with fluids in equilibrium. The topic of fluid dynamics is a complicated mathematical field that requires differential equations to do with any exactness. There are whole upper division classes that study this called *Fluid Dynamics*. But with some simplifying assumptions, we can get a feel for how fluids flow.

Fundamental Concepts

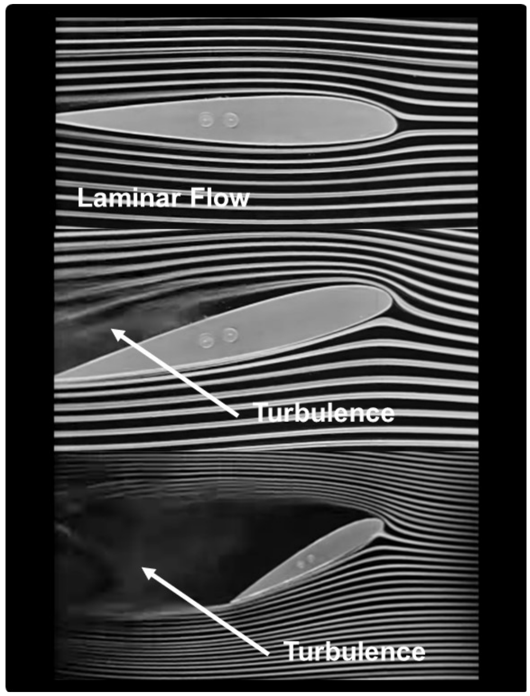
1. Continuity

2. Bernoulli's Equation

27.1 Moving Fluids

To start with, let's make some definitions.

New Definitions:



1. Laminar: Steady flow, if each particle of the fluid follows a smooth path, such that the paths of the different particles never cross each other.

2. Turbulent: non-laminar flow characterized by whirlpool or eddy regions

3. Viscosity: term describing the internal friction of a fluid (Honey is more viscus than water)

4. Irrotational: flow having no angular momentum

5. Streamline: a line indicating the path taken by a particle under steady flow

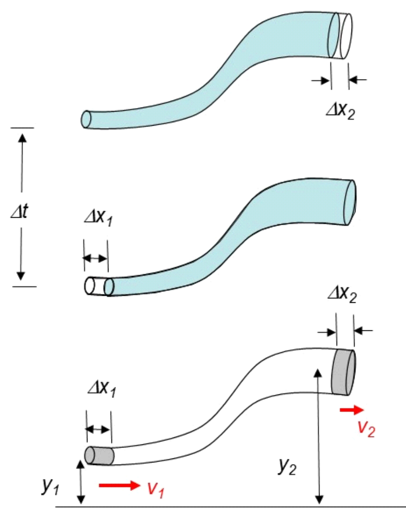
In the figure above we can see streamlines going around airfoils. The reason we can see them is that small particles are being deposited by evenly spaced sources in an air tunnel. This makes is so we can see the streamlines! Here is the same type of thing for a soccor (futbol) being tested by NASA.

NASA Ames measuring turbulence from a soccer ball. There is a cool video at this URL! https://www.nasa.gov/centers-and-facilities/ames/nasa-turns-world-cup-into-lesson-in-aerodynamics/

Sadly, we will limit our study to laminar, non-viscus, irrotational, incompressible fluids. This is a little bit like saying we will only study frictionless surfaces in PH121! But to go beyond non-viscus laminar flow we need the math of partial differential equations, which is not a prereq for this class. So we will have to content ourselves with this restriction. For those of you who are interested in learning more (or are ME majors) you can take our junior level fluid dynamic course taught by the Mechanical Engineering Department.

27.2 Equation of continuity

Let's start our discussion of fluid flow with a pipe that changes diameter along it's length.



And just for good measure we can have the pipe go up in position. We allow an ideal (non-compressable) fluid to flow through the pipe in laminar flow. The segment of fluid is the shaded part in the top part of figure **??**. In a time Δt the fluid at the left hand side moves a distance

$$\Delta x_1 = v_1 \Delta t$$

If A_1 is the area of the pipe at the left hand side (LHS) then the mass contained in the shaded region (bottom left part of figure **??**) is

$$m_1 = \rho \cancel{V}_1 = \rho A_1 \Delta x_1 = \rho A_1 v_1 \Delta t$$

where ρ is the density of the fluid. Likewise, for the RHS region in the bottom of figure **??**

$$m_2 = \rho \cancel{V}_2 = \rho A_2 \Delta x_2 = \rho A_2 v_2 \Delta t$$

Unless we create, destroy, or pool mass, the mass that crosses A_2 must equal the mass that crosses A_1

$$\begin{aligned} m_1 &= m_2 \\ \rho A_1 v_1 \Delta t &= \rho A_2 v_2 \Delta t \end{aligned}$$

or

$$A_1 v_1 = A_2 v_2 \tag{27.1}$$

This is the *equation of continuity for fluids* (remember that we assumed ideal fluids).

Since we assumed that no mass was created, destroyed, or pooled as the fluid flowed, this is what we might call "conservation of mass." And as usual, when something is conserved, we can use it to solve problems. With this equation, knowing the diameter of a pipe and how it changes tells us about the speed of the fluid in the pipe and how it changes. Of course, there could be situations where mass is not conserved (say, a leaky pipe). So conservation of mass is an assumption that we have to check. But for now let's assume that the equation of continuity holds for our problems.

The quantity Av has a name. We call it the flow rate and for some reason I don't know we give it the symbol $\mathbb{Q}$

$$\mathbb{Q} = Av$$

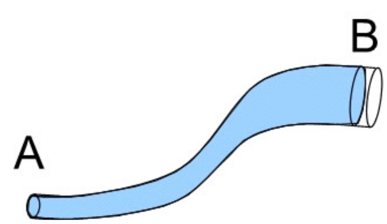
27.3 Bernoulli's Equation

It would be great if we could find the pressure as well as the speed of the fluid inside our non-uniform pipe. We would need an equation that included both pressure and speed. Let's try to find such and equation!

In PH121 we developed to ways of looking at motion problems. One was the force picture where we had to deal with the vector nature of forces and motion. The other was the energy picture where we could solve problems without considering directions (but we gave up being able to find directions). Usually the energy picture made solving problems easier. And except for the flow direction, we don't really want to know the direction of the motion of each molecule in our fluid, so giving up direction doesn't seem too bad. So, the work energy theorem is something we know from PH121. Let's start there,

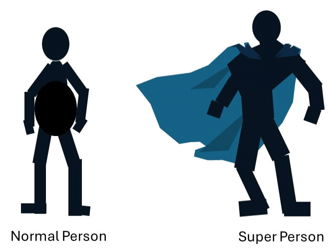
$$W = \Delta K + \Delta U$$

We still need to use forces to find the work done on the parcel of water, but if we choose our axes carefully, we can keep the work equation in one dimension.

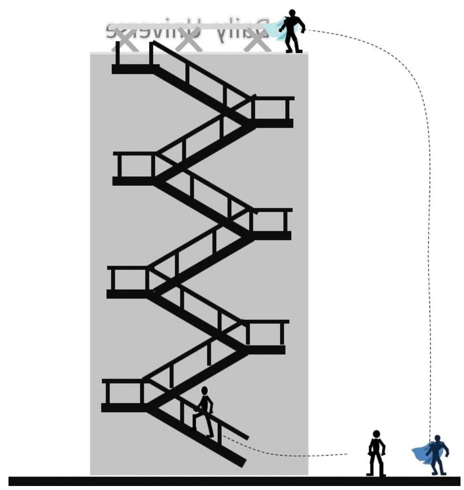


27.3.1 Work done in Δt

Let's review a little of what we learned in PH121. Suppose we have two characters, Normal Person and Super Person.



And suppose they want to get to the top of a tall building. Normal Person has to take the stairs, but Super Person can jump right up to the top.

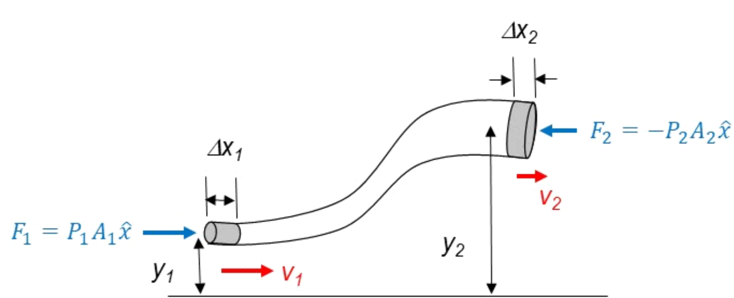


Who has done more work?

We know the change in potential energy is the same for both. And we know that

$$W = -\Delta U$$

so the amount of work is the same for both cases! Now let's return to our fluid flow.



On the left hand side (LHS) of our pipe we have a force due to pressure from the fluid in the rest of the LHS of the pipe. That fluid has molecules that will bang in to our small marked parcel of fluid. The force from the LHS fluid in the rest of the pipe will be

$$F_1 = P_1 A_1 \widehat{\mathbf{x}} \tag{27.2}$$

On the right hand side (RHS) there is also more fluid in the rest of the RHS of the pipe. Our fluid will bang into that fluid, and since the fluid is non-compressible, the rest of the fluid in the RHS will push back. The force will be

$$F_2 = -P_2 A_2 \widehat{\mathbf{x}} \tag{27.3}$$

From PH121 we remember that works is

$$w = \int \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{dx}}$$

and since our force is not changing this would be simply

$$w = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{\Delta x}} \tag{27.4}$$

We need to find the work done in moving our particular parcel of fluid from the left hand side to the right hand side. As the parcel moves in the $\widehat{\mathbf{x}}$ direction the amount of work will slowly change (like Normal Person going up the stairs) but really the amount of work only depends on the beginning and ending conditions. So we could ignore all the details of moving the parcel and envision the parcel jumping from the left up to the right side of the pipe (like in the Super Person case).

Then for the LHS

$$W_1 = P_1 A_1 \Delta x_1$$

and for the RHS

$$W_2 = -P_2 A_2 \Delta x_2$$

The total work will be the sum of these two works. But before we add them together, let's make some substitutions that will make our final formula for the total work more meaningful.

Lets define the volume of our fluid segment

$$\cancel{V} = A_1 \Delta x_1 \tag{27.5}$$

Note that this volume is NOT the volume of the entire segment. It is just the part that is Δx_1 long. Then from the same arguments that lead to the equation of continuity, we see that the volume of the marked part of the fluid must not change as it flows

$$\cancel{V} = A_1 \Delta x_1 = A_2 \Delta x_2$$

The we can write the work equations as

$$W_1 = P_1 V$$

and

$$W_2 = -P_2 \not{V}$$

So finally, the total work done is

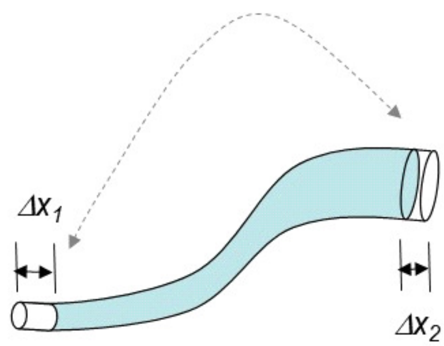
$$\begin{aligned} W &= W_1 + W_2 \\ &= P_1 \not{V} - P_2 \not{V} \\ &= (P_1 - P_2) \not{V} \end{aligned} \quad (27.6)$$

so, from the work-energy theorem

$$\begin{aligned} W &= \Delta K + \Delta U \\ (P_1 - P_2) \not{V} &= \Delta K + \Delta U \end{aligned}$$

We have an expression for the left hand side of our work-energy theorem! Now for the right hand side $\Delta K + \Delta U$

27.3.2 Kinetic Energy



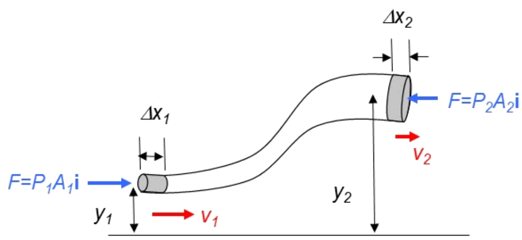
If we think about the energy of the segment, we will realize that after time Δt, it is as though most of the fluid did not move. We can't tell one part of the fluid from another. The shaded region in figure **??** is occupied before and after Δt. We can treat this problem as if we moved fluid from region 1 to region 2 (Super Person vs. Normal Person again!) and left the rest of the segment alone!

So, ignoring the rest of the fluid, we can see that if the velocity of our parcel of water changed, then the kinetic energy of the parcel must change as well

$$\Delta K = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad (27.7)$$

(the mass is the same, as we found before).

27.3.3 Potential Energy



Let's look at the change in potential energy for the parcel of fluid. Once again we can consider just the beginning and ending case (think of Super Person and Normal Person once again). We work as though we moved a mass $m = m_1 = m_2$ from y_1 to y_2.

$$\Delta U = mgy_2 - mgy_1 \tag{27.8}$$

Since our pipe went upward as the parcel of water flows, we gained potential energy. But if the pipe went downward we would loose potential energy.

27.3.4 Total Work done on the system

Let's now assemble our work-energy theorem. We had found the work so far,

$$(P_1 - P_2)\,\cancel{V} = \Delta K + \Delta U$$

but now we know ΔK and ΔU We have calculated W, ΔK, and ΔU for our parcel of water, so let's substitute in what we have found.

$$((P_1 - P_2)\,\cancel{V}) = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgy_2 - mgy_1) \tag{27.9}$$

This equation describes our fluid flow much like the equation of continuity, but now it has pressure involved! We have achieved our goal. But we can make this equation a little easier to understand if we play a trick by dividing by $\cancel{V}$

$$((P_1 - P_2)) = \left(\frac{1}{2}\frac{m}{\cancel{V}}v_2^2 - \frac{1}{2}\frac{m}{\cancel{V}}v_1^2\right) + \left(\frac{m}{\cancel{V}}gy_2 - \frac{m}{\cancel{V}}gy_1\right) \tag{27.10}$$

and recalling that

$$\rho = \frac{m}{\cancel{V}}$$

then

$$(P_1 - P_2) = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + (\rho gy_2 - \rho gy_1)$$

Notice that we have subscripts 1 and 2. Let's put all the 1's on one side of our equation and all the 2's on the other side.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \tag{27.11}$$

which completes our goal. We have an equation that contains the velocity *and* the pressure of the fluid.

But notice! The left hand side looks just the same as the right hand side except for the subscripts! We have a before and after picture like we did in energy and momentum problems back in Principles of Physics I. Something is being conserved. Sometimes you see this written as

$$P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant} \tag{27.12}$$

This is another conservation equation! The thing that is being conserved is $P + \frac{1}{2}\rho v^2 + \rho g y$. We have not named this quantity, but whatever it is, it is not changing as the fluid flows. Our experience with conservation equations tells us that this new conservation law will help us solve problems. This equation is so useful we give it a name. It is called *Bernoulli's equation* (after the scientist that came up with it). What is really being conserved is energy. We found Bernoulli's equation from conservation of energy. But this form of conservation of energy is useful for solving pressure and fluid flow problems.