Chapter 1

Introduction to the Course, Simple Harmonic Motion 1.15.1

Fundamental Concepts

- Simple Harmonic Motion
- Frequency and Period
- Mathematical description of Simple Harmonic Motion

1.1 Oscillation and Simple Harmonic Motion

Last semester, in PH121 (or Dynamics) you studied how things move. We identified a moving object (I often refer to this object as the *mover object*) and other objects that exerted forces on the mover (I often refer to these other objects as the *environmental objects*). You learned about forces and torques which get mover objects moving. You should remember Newton's Second Law and Newton's Second Law for rotation.

You also learned and practiced a lot of math. We will continue to use the math you learned in PH121 this semester.

But we will go beyond what we learned in PH121 to study new types of motion, and new objects that move.

This semester, we will start with a very special type of motion. It is the motion that results from oscillation. We call this very special type of motion, *simple harmonic motion*.

Simple harmonic motion (SHM) means a motion that repeats in the most special, simple way. Some characteristics of this type of motion are as follows:

1. The motion repeats in a regular way, like a grandfather clock pendulum swings back and forth in a set amount of time. This set amount of time is called a period.

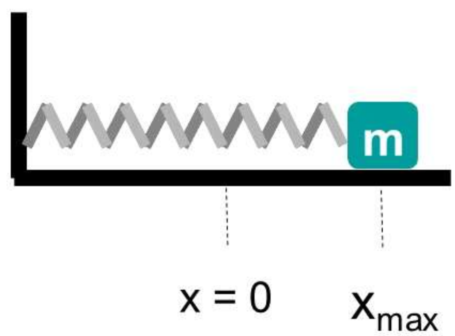
2. The mover object moves about a center position and is symmetric about that position. This center position is the bottom of the swing for our grandfather clock. The idea of symmetry just means that the pendulum reaches the same height on both sides as it swings

3. The mover object's motion can be traced out in a position vs. time graph. For SHM, the shape of the position vs. time graph is a sinusoid.

4. The velocity of the mover object constantly changes. It is zero at the extreme points (the points farthest from the center, or the largest positive and the largest negative displacements). That is where it turns around and goes back the other way. It stops there, but just for a split second. The velocity is largest at the equilibrium position (the center position). If we plotted a velocity vs. time graph, the velocity would also be a sinusoid or snake-like shape.

5. The acceleration also constantly changes. And it is also a sinusoid.

That is a lot of criteria for a motion that is supposed to be simple! Let's take an example system to see how simple harmonic motion works. I have drawn a mass (marked m) attached to a spring. And lets assume that the mass is on a frictionless surface and there is no air drag force. If we pull the mass so the string stretches, we would get simple harmonic motion.



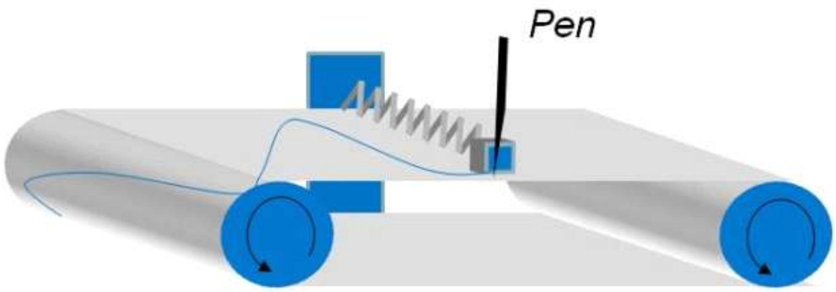
The center position for our mass is called the Equilibrium position, and for convenience we often define it as the origin of our coordinate system.

Equilibrium Position: The position of the mover mass (not the spring, which is an environmental object acting on the mover mass) when the spring is neither stretched nor compressed.

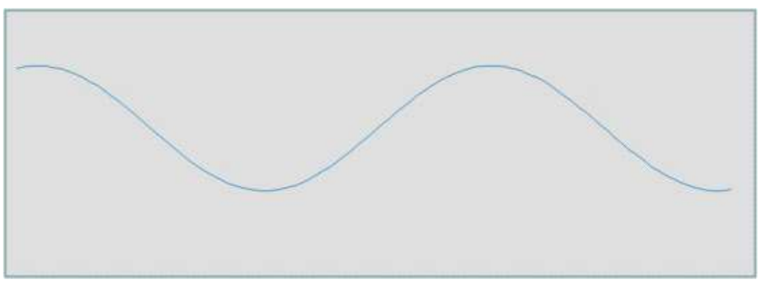
Let's draw a picture of simple harmonic motion. First we need a device to do this. Suppose you go to the supermarket[1]. But instead of putting your ramen

[1] Like Albertsons, where they still have checkers.

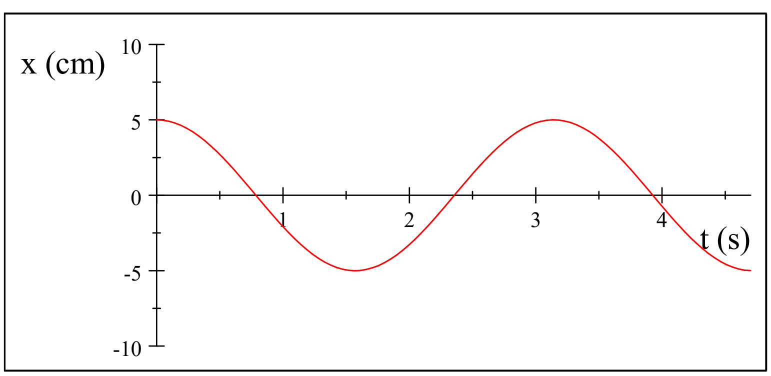
and granola bars on the belt at the checkout counter, you strap on a device like this[2]



You can see the mover object on the spring. But we have placed a pen on the object and that pen is tracing out a pattern as the belt moves. You will recognize this pattern as a trig function. The result might look something like this if you removed the belt.



Of course, what we have made is a position vs. time graph. We remember these from PH121! This gives us a record of the past motion of the object.



where in this graph, $x_{\max} = 5\,\mathrm{cm}$. Having the power of mathematics, we know we can write an equation that would describe this curve. From your Trigonometry

[2] Please don't really try this at the local supermarkets. They don't seem to have any sense scientife inquarey or even a sense of humor about such things at all.

experience, we can guess that an equation for our mover object motion might look something like

$$x(t) = x_{\max} \cos(\theta)$$

Notice that the instantaneous position, $x(t)$ has to be a function of time. Back in trigonometry we would have said a cosine function was a function of an angle, θ. But we know this is a position vs. time graph, so our angle must be different for different times. Let's write

$$\theta(t) = \omega t$$

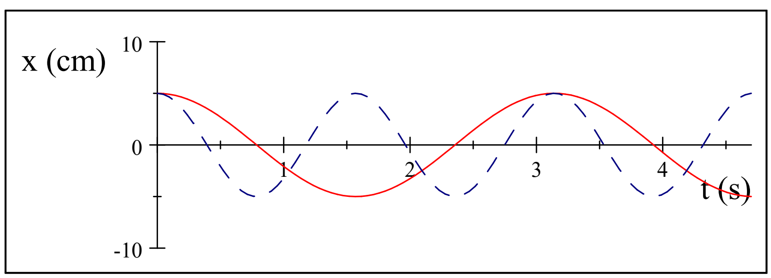
as strange sort of an angle. This "angle" is a function of time, and let's say how fast our "angle" is changing is given by

$$\frac{d\theta}{dt} = \omega$$

This is a sort of speed for how fast our angle is changing. So our equation must be

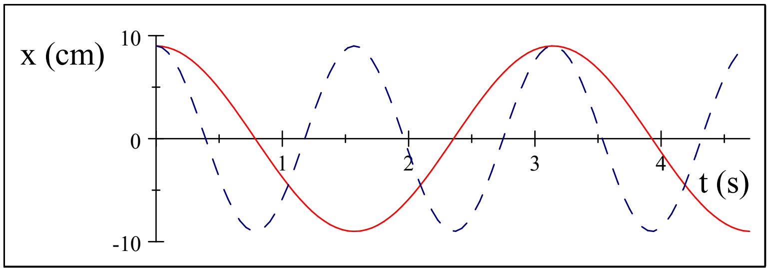
$$x(t) = x_{\max} \cos(\omega t)$$

In the next figure $x(t) = x_{\max} \cos(\omega t)$ is plotted with two different values for ω.



We can see what ω does for us. It stretches out or compresses our curve. In the blue (dashed) curve our angle changes quickly. In the red (solid) curve the angle changes slowly. Note that we are not plotting position vs. angle. Both plots would be the same if we did! We are plotting position vs. time. And for the blue curve our mass is oscillating much more quickly than it is for the red curve. The quantity ω tells us something about how fast our mover object is oscillating.

The quantity $x_{\max}$ is the maximum displace of our mover object from the equilibrium position. This maximum displacement from equilibrium has a name. It is called the *amplitude*. In our equation I have used $x_{\max}$ where most books use the letter A for amplitude to emphasize that the amplitude is the maximum displacement of the mover mass from the mover mass equilibrium position. In our coordinate system, the mass is going back and forth in the x direction. Since the amplitude means the maximum displacement, $x_{\max}$ is a good way to write amplitude. Here is the same graph but with $x_{\max} = 9\,\text{cm}$



The two graphs for the two different ω values are not more stretched out in time, but now they are taller along the position axis. This means that the mass is moving farther from the center as it oscillates.

You will remember from your trigonometry class that the *period*, T, tells us the time it takes for the oscillation to go through a complete cycle. A complete cycle is when the object, say, goes from $x_{\max}$ to $-x_{\max}$ and then back to $x_{\max}$. You can probably guess that how long it takes to oscillate and how often it oscillates would be related. How often the oscillator completes a cycle is called the frequency. The longer the period, the lower the frequency.

$$f = \frac{1}{T}$$

Think of cars passing you on your way to class. Period is like how long you wait in between cars. Frequency is like how often cars pass. If you wait less time between cars, the cars pass more frequently. And that is just what our equation says! We can see from our graphs that our stretching quantity ω, must be related to the frequency of our oscillation. If the frequency is high, then ω must be large so that we reach different "angles" faster. But for the cosine function to work we need angle units. *We will choose radians* for our units and we will write our stretching quantity as

$$\omega = 2\pi f$$

This works! If ω is bigger then our oscillation happens more frequently. The 2π has units of radians. So ω has units of $\mathrm{rad\,Hz}$ or more commonly $\mathrm{rad/s}$. That matches our derivative above. Let's give a name to the quantity ω. Since it has radians in it we might guess that it has something to do with circular motion (more on this later) and it has frequency in it. So we will call ω the *angular frequency.*

1.2 Velocity and Acceleration

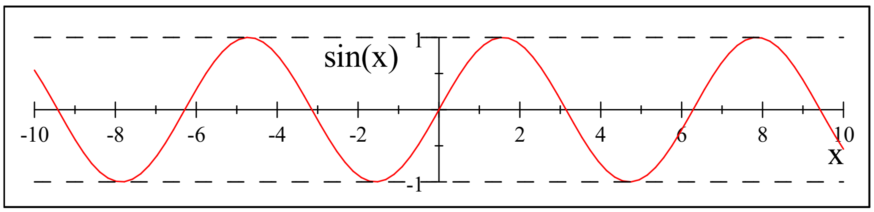
So far we have guessed the descriptive equation for SHM.

$$x(t) = x_{\max}\cos(\omega t) \tag{1.1}$$

This gives us the position of the particle at any given time. Since knowing the position as a function of time is a good description of the motion of our mover mass, we can call this equation the *equation of motion* for our mass. We will see soon that this equation is correct. But let's pretend that we are earlier researchers and we just know the equation makes the right shape. Still, we can learn much from knowing this equation. Since we know how to take derivatives, and we know the derivative of position with respect to time is the velocity, we can see that the velocity of the mass at any given time is given by

$$v\left(t\right)=\frac{dx\left(t\right)}{dt}=-\omega x_{\max}\sin\left(\omega t\right) \tag{1.2}$$

From our fond memories of our trigonometry class we know the maximum of a sine function is always 1.



Notice that the ω and the $x_{\max}$ aren't changing for our oscillation mover mass. They are constant. Then if we want the maximum speed we can simply set the $\sin\left(\omega t\right)=1$. Then the maximum speed will be

$$v_{\max}=\omega x_{\max}\left(1\right) \tag{1.3}$$

Notice that this does not tell us when the speed is maximum. Just what the maximum speed is. We then have an equation for the speed of our object as a function of time

$$v\left(t\right)=-v_{\max}\sin\left(\omega t\right)$$

We will often use this trick of knowing the maximum of sine is one.

We can also find the acceleration. We just take another derivative.

$$\begin{aligned}a\left(t\right) &= \frac{dv}{dt}\\ &= \frac{d^{2}x\left(t\right)}{dt^{2}}\\ &= -\omega^{2}x_{\max}\cos\left(\omega t\right)\end{aligned}$$

This is the acceleration of the object attached to the spring. It's also true that the maximum for a cosine function is 1, so the maximum acceleration would be

$$a_{\max}=\omega^{2}x_{\max}\left(1\right) \tag{1.4}$$

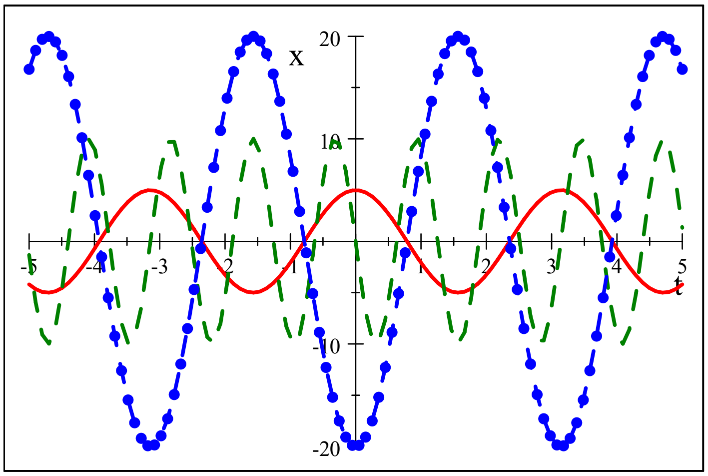
so we could write our instantaneous acceleration as

$$a\left(t\right)=-a_{\max}\cos\left(\omega t\right)$$

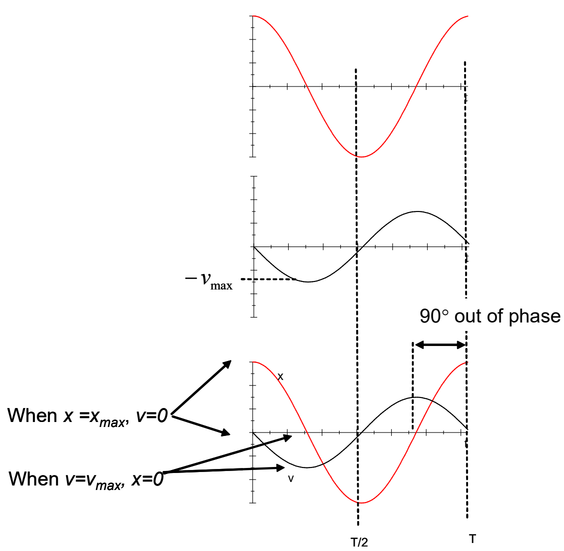
These are significant results, so let's summarize. For a simple harmonic oscillator, the instantaneous position, speed, and acceleration are given by

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t) \\ v(t) &= -\omega x_{\max}\sin(\omega t) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t) \end{aligned} \tag{1.5}$$

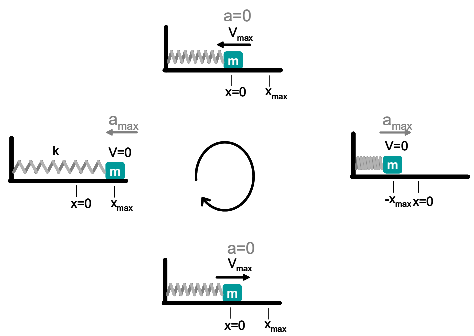
Let's plot $x(t)$, $v(t)$, and $a(t)$ for a specific case



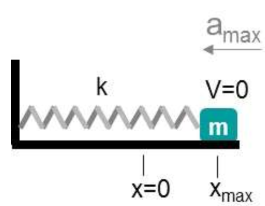
Red (solid) is the displacement, green (dashed) is the velocity, and blue (dot-dashed) is the acceleration. Note that each as a different maximum amplitude. Also note that they don't rise and fall at the same time. We will describe this as being *not in phase*.



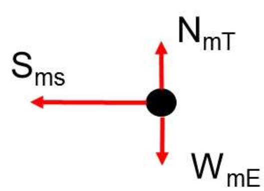
The acceleration is 90° *out of phase* from the velocity. Let's think about why this would be. Suppose we attach a mass to a spring and allow the mass to slide on a frictionless surface.



Let's start by stretching the spring by pulling the mass to the right and releasing. This is the situation in the left hand part of the last figure.



The spring is pulling strongly on the mass. We could draw a free body diagram for this situation. We will need the spring force



Back in PH121 we said that Hooke's Law is not something that is always true, but by "law" we mean a mathematical representation (and equation) that comes from our mental model of how the universe works. In this case, Hooke's law is an equation that comes from Hooke's model of how springs work. It is a good model for most springs as long as we don't stretch them too far. You remember Hooke's law from PH 121. It tells us that the spring force is proportional to how far we stretch or compress the spring (Δx_e) and how stiff the spring is (k).

$$\begin{aligned} F_s &= S = -k\Delta x_e \\ &= -k\left(x - x_e\right) \end{aligned} \tag{1.6}$$

where x_e is the *equilibrium position of the mass.* If we assume the equilibrium position is at $x = 0$ then we can write our spring force as

$$S = -kx \tag{1.7}$$

And if we write out Newton's second law we get

$$\begin{aligned} F_{net_x} &= -ma_x = -S_{ms} \\ F_{net_y} &= ma_y = N_{mT} - W_{mE} \end{aligned}$$

We can see that a_y should be zero because the mass won't lift off the table in the y-direction. but in the x-direction

$$-a_x = -\frac{S_{ms}}{m}$$

and knowing that

$$S_{ms} = -kx$$

from Hooke's law, we can see that

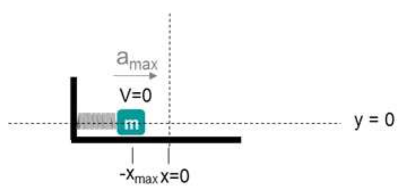
$$-a_x = +\frac{kx}{m}$$

or

$$a_x = -\frac{kx}{m}$$

Since at our starting point x is big our acceleration is big. But there is a minus sign. The minus sign tells us that a_x must be to the left.

But suppose the mass was to the left of $x = 0$



Now $x < 0$ so it is negative. That makes

$$a_x = -\frac{kx}{m}$$

positive. The acceleration always seems to point toward $x = 0$, the equilibrium position for the mass. This is an important part of the definition of simple harmonic motion, having an acceleration that always points toward the equilibrium position. We call a force that makes the acceleration point toward the equilibrium position *restoring force.*

Restoring force: A force that is always directed toward the equilibrium position

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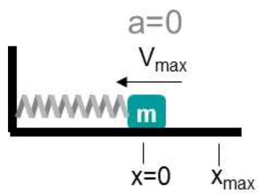
Now lets look at our object a short time later



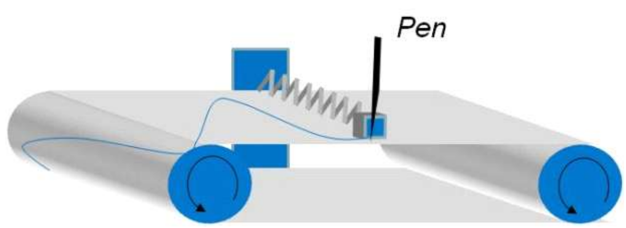
We can see that in this position, $x = 0$ so

$$a_x = \frac{k(0)}{m} = 0$$

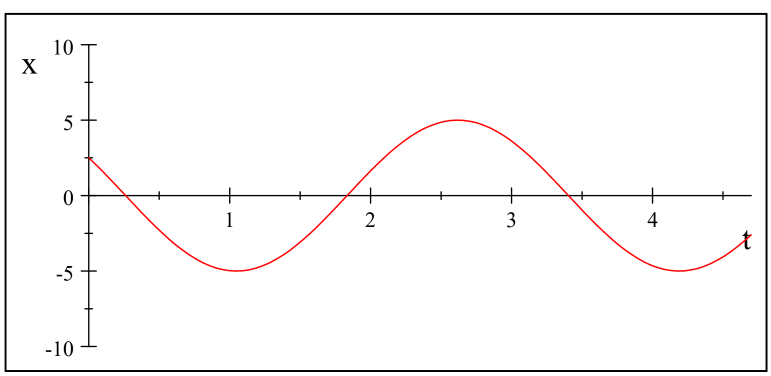
so at this position the acceleration is zero. That is because right at $x = 0$ the spring is neither pushing nor pulling. There is no net force so no acceleration when $x = 0$.

1.3 The Idea of Phase

We said before that $x(t)$, $v(t)$, and $a(t)$ are "out of phase." Let's look at the idea of "phase" more carefully. Suppose you return to the grocery store and start your SHM device.



But this time you work with a lab partner, and the lab partner tries to start a stopwatch when you let go of the mass. But, due to having a slow reaction time, your partner starts the clock too late. The resulting graph looks like this

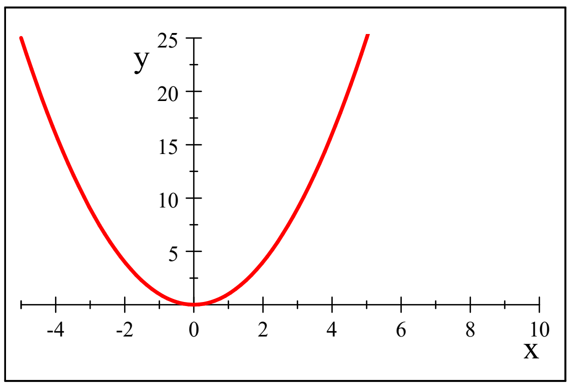


But we know that the only difference between this graph and the one we had before is that the lab partner was slow, so the graph is shifted on the time axes. We expect we can use the same mathematical model for SHM, but we must need to change something.

We know from our algebra classes what a shift looks like. Take the expression

$$y = x^2$$

We can plot this to get a parabola



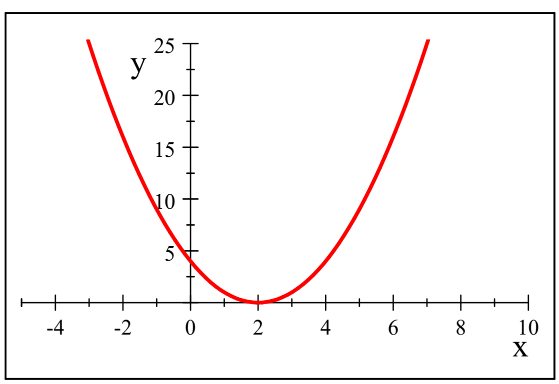
A shifted parabola would be expressed as

$$y = (x - \phi_o)^2$$

where ϕ_o is the amount of the shift. Suppose $\phi_o = 2$, then

$$y = (x - 2)^2$$

and our shifted parabola looks like this



Recall that a shift like

$$y = (x - \phi_o)^2$$

will move the parabola to the right while

$$y = (x + \phi_o)^2$$

will move the parabola to the left.

We can use this idea to write our SHM expression for our situation with the slow lab partner.

$$x(t) = x_{\max} \cos(\omega(t \pm \tau_o))$$

In our case, the lab partner was late by $\tau_o = 4.7124\,\mathrm{s}$. This is not usually how we express the shift, however. We usually distribute the ω so our equation looks like

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t \pm \omega\tau_o) \\ &= x_{\max} \cos(\omega t \pm \omega\tau_o) \end{aligned}$$

We usually use the symbol $\phi_o = \omega\tau_o$ so we will write our SHM expression for position as a function of time as

$$x(t) = x_{\max} \cos(\omega t \pm \phi_o)$$

We could call ϕ_o the *slow lab partner constant,* but that is long and not very kind. So let's call ϕ_o by the name *phase constant*. It is also customary to call the entire expression in parenthesis, $(\omega t \pm \phi_o)$, the phase of the cosine function. This is especially used in the fields of Optics and Electrodynamics.

From what we did before, we know that $x(t)$, $v(t)$, and $a(t)$ are out of phase, so there must be a phase constant involved somehow. Let's look for it in what follows.