Chapter 21

21 Thin Films and Light Interference 3.3.4, 3.3.5

Last lecture we reminded ourselves that light is a wave. In this lecture let's look at some examples of how the wave nature of light can be used. Applications for the using light waves stretch from radar to digital communications to atmospheric chemical analysis. But let's take on the ideas of interference from thin films and from different path lengths.

Fundamental Concepts

- Thin films can cause reflections that create interference
- There may be a phase shift when a wave reflects (the wave may invert-this is a review)
- If two waves are out of phase by half a wavelength we have total destructive interference (review)
- If two waves are out of phase by a full wavelength we have total constructive interference (review)
- Other phases provide partial constructive or partial destructive interference

21.1 Mathematical treatment of single frequency interference

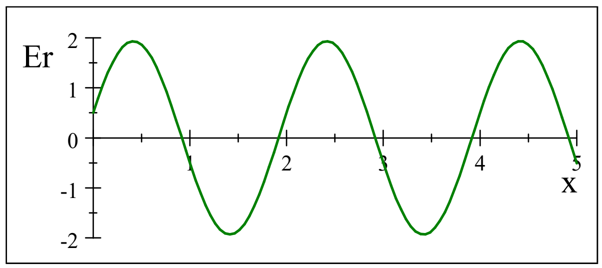
Let's review our equations for interference of two wave. We start with two waves in the same medium. Only now our waves are light waves. So the medium is the electromagnetic field. So we can say our amplitude is $E_{\max}$

$$\begin{aligned} E_1 &= E_{\max} \sin(kx_1 - \omega t + \phi_1) \\ E_2 &= E_{\max} \sin(kx_2 - \omega t + \phi_2) \end{aligned}$$

Each wave has its own phase constant. Each wave starts from a different position (one at x_1 and the other at x_2), The superposition yields.

$$E_r = E_{\max} \sin(kx_1 - \omega t + \phi_1) + E_{\max} \sin(kx_2 - \omega t + \phi_2)$$

which is graphed in the next figure.



Notice that the wave form is taller (larger amplitude). Noticed it is shifted along the x axis. This graph is not surprising to us now, because we have done a case like this before. We can find the shift in general rewriting y_r. We need a trig identity

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

then let $a = kx_2 - \omega t + \phi_2$ and $b = kx_1 - \omega t + \phi_1$

$$\begin{aligned} E_r &= E_{\max} \sin(kx_2 - \omega t + \phi_2) + E_{\max} \sin(kx_1 - \omega t + \phi_1) \\ &= 2E_{\max} \cos\left(\frac{(kx_2 - \omega t + \phi_2) - (kx_1 - \omega t + \phi_1)}{2}\right) \sin\left(\frac{(kx_2 - \omega t + \phi_2) + (kx_1 - \omega t + \phi_1)}{2}\right) \\ &= 2E_{\max} \cos\left(\frac{kx_2 - kx_1}{2} + \frac{\phi_2 - \phi_1}{2}\right) \sin\left(\frac{kx_2 + kx_1 - 2\omega t + \phi_2 + \phi_1}{2}\right) \\ &= 2E_{\max} \cos\left(k\frac{x_2 - x_1}{2} + \frac{\phi_2 - \phi_1}{2}\right) \sin\left(k\frac{x_2 + x_1}{2} - \omega t + \frac{\phi_2 + \phi_1}{2}\right) \\ &= 2E_{\max} \cos\left(k\frac{\Delta x}{2} + \frac{\Delta\phi_o}{2}\right) \sin\left(k\frac{x_2 + x_1}{2} - \omega t + \frac{\phi_2 + \phi_1}{2}\right) \\ &= 2E_{\max} \cos\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right)\right) \sin\left(k\frac{x_2 + x_1}{2} - \omega t + \frac{\phi_2 + \phi_1}{2}\right) \\ &= 2E_{\max} \cos\left(\frac{1}{2}(\Delta\phi)\right) \sin\left(k\frac{x_2 + x_1}{2} - \omega t + \frac{\phi_2 + \phi_1}{2}\right) \end{aligned}$$

where the last line is just a rearrangement to match the form we got last time. As usual, let's look at the sine part first. It is still a wave. We can see that more clearly if we define a variable

$$\mathsf{x} = \frac{x_2 + x_1}{2}$$

and another

$$\overline{\phi} = \frac{\phi_2 + \phi_1}{2}$$

Then the sine part is just

$$\sin\left(k\mathsf{x} - \omega t + \overline{\phi}\right)$$

and we can see it is just our basic wave equation form. The rest of the equation must be the amplitude and we can clearly see that the amplitude depends on what we call the phase difference,

$$\Delta\phi = \frac{2\pi}{\lambda}(\Delta x) + \Delta\phi_o$$

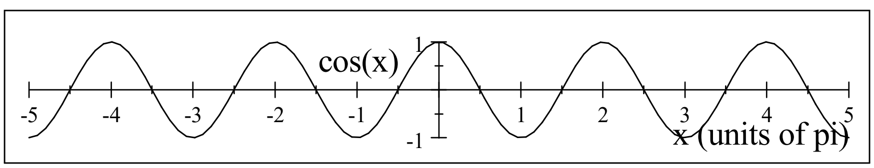
And from this we can see that for a single frequency situation there can be two sources two sources of phase difference. One can be from the two waves traveling different paths and then combining (Δx) and the other is from the two waves starting with a different phase to begin with, $\Delta\phi_o$. If the total phase difference between the two waves is a multiple of 2π, then the two waves will experience constructive interference

$$\Delta\phi = m2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Let's see that this works. Our amplitude is

$$A = 2E_{\max}\cos\left(\frac{1}{2}(\Delta\phi)\right)$$

and if we look at a cosine function we see that $\cos(\theta)$ is either 1 or -1 at $\theta = n\pi$.



So if $\Delta\phi = m2\pi$ then the amplitude is

$$\begin{aligned} A &= 2E_{\max}\left(\frac{1}{2}(m2\pi)\right) \\ &= 2E_{\max}\cos(m\pi) \\ &= \pm 2E_{\max} \end{aligned}$$

We don't really care if the amplitude function is big positively or negatively. So we get constructive interference for $\cos(m\pi)$ being either 1 or -1. Then, this our case for constructive interference.

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right) = m2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

How about for destructive interference? We start again with our amplitude function

$$A = 2E_{\max}\cos\left(\frac{1}{2}(\Delta\phi)\right)$$

but now we want when the cosine part needs to be zero.

$$\cos(\theta) = 0$$

Looking at our cosine graph again, that happens for cosine when $\theta = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\cdots$. We could write this as $\theta = \left(m + \frac{1}{2}\right)\pi$ for $m = 0, \pm1, \pm2, \pm3, \cdots$. But remember that in our amplitude function, we already have the 1/2 in the function, so we want $\Delta\phi$ to have just the odd integer multiple of π. We could write this as

$$\Delta\phi = (2m+1)\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots$$

So our condition for destructive interference is

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right) = (2m+1)\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots$$

We have developed a useful matched set of equations that will tell us if we mix two waves when we will have constructive and destructive interference:

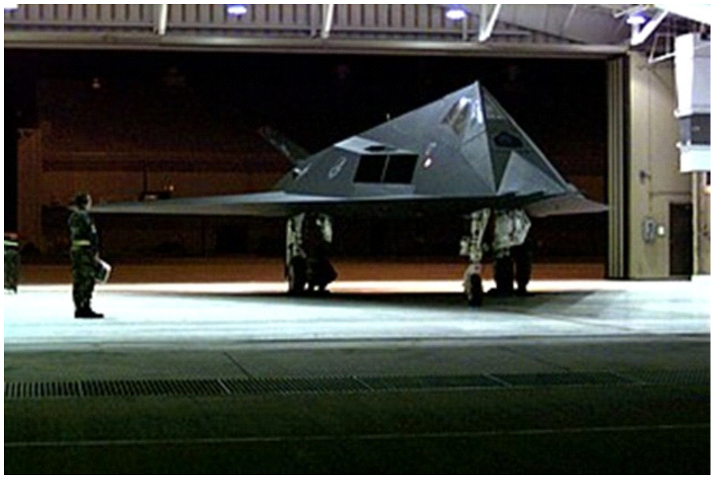
$$\begin{aligned}
\Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = m2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive} \\
\Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = (2m+1)\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}
\end{aligned}$$

Let's take an example to see how this can be used.

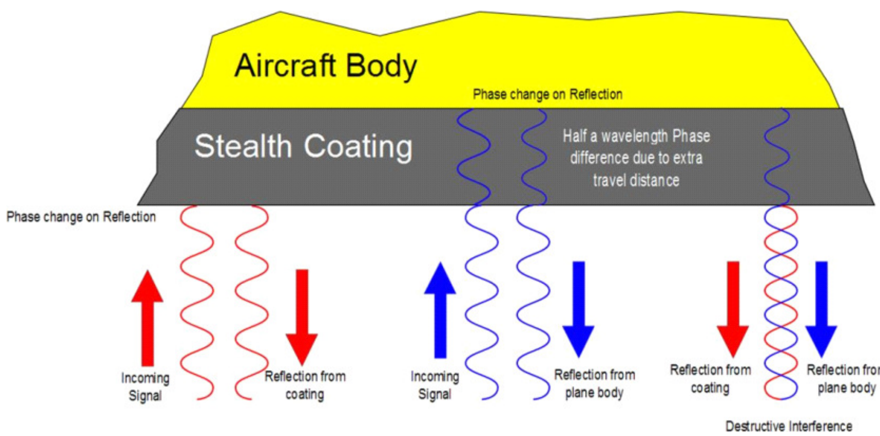
21.1.1 Example of two wave interference: Stealth Fighter

The U.S. Air Force F-117 Nighthawk Stealth Fighter is an aircraft that is designed to be mostly invisible to radar. How could this work?

The answer is that I don't know[1] . But we can make some guesses. The Stealth Fighter is kept in temperature and humidity controlled hangers.



So maybe there is a coating on the aircraft that needs protecting. Let's see if we could make an airplane "stealthy" by applying a coating.



So suppose the stealth fighter is coated with an anti-reflective polymer that is part of it's mechanism for making the plane invisible to radar. Suppose we have a radar system with a wavelength of 3.00 cm. Further suppose that the index of refraction of the anti-reflective polymer is $n = 1.50$, and that the aircraft index of refraction is very large, how thick would you make the coating?

We want destructive interference, so let's start with our destructive interfer-

[1] I don't really know that this is true. And if I did, I wouldn't be able to use this as an example! So we are reverse engineering the stealth fighter technology. But I think it is a good guess.

ence condition

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = (2m+1)\,\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}$$

The radar waves all hit the plane in phase. From the figure, we see that the radar wave will reflect off of the coating. Because the index of refraction of the coating is large, this is like a fixed end of a rope. There will be an inversion.

But some of the wave will penetrate the polymer. This will reflect off of the plane body. The plane body has a very large index of refraction, so once again the wave will experience an inversion. The outgoing waves would then both be in phase as they leave and create constructive interference (if there were not a path difference) because

$$\Delta\phi_o = \pi - \pi = 0$$

at this point. Thus

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r\right) = (2m+1)\,\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}$$

But $\Delta\phi_o$ is not the only term in our equation, we also have to remember the path difference, Δr. The part of the wave that entered the polymer travels farther. If that path difference, Δr, is just right we can get destructive interference. For the $m = 0$ case

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r\right) = (2\,(0)+1)\,\pi = \pi$$

then the amplitude function would be

$$\begin{aligned} A &= 2E_{\max}\cos\left(\frac{2\pi}{\lambda}\Delta r\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\,(\pi)\right) \\ &= 0 \end{aligned}$$

and we have destructive interference. Note that these are electromagnetic waves, so instead of $y_{\max}$ we have used $E_{\max}$ as the individual wave amplitude. But the important thing is that the plane cannot be seen by the radar! Of course, this works for $m = 1$ and $m = 2$, etc. as well. Any odd multiple of π will work.

$$\frac{2\pi}{\lambda}\Delta r = (2m+1)\,\pi \qquad m = 0, 1, 2, \cdots$$

so that we are guaranteed an odd multiple of π. This is our condition for destructive interference.

But we are interested in the thickness. We realize that Δr is about twice the thickness, since the wave travels though the coating and back out. So let's let $\Delta r \approx 2t$

$$\frac{2\pi}{\lambda}2t \approx (2m+1)\,\pi$$

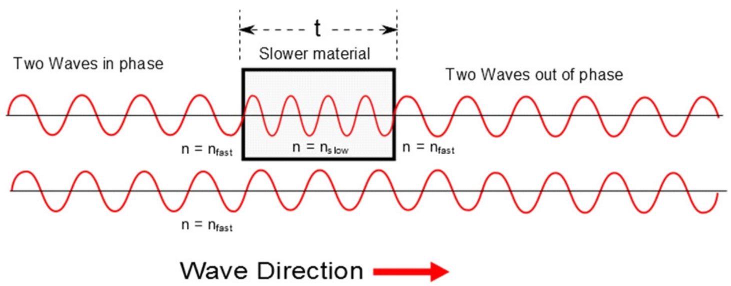
$$2t \approx (2m+1)\frac{\lambda}{2}$$
$$2t \approx \left(m+\frac{1}{2}\right)\lambda$$
$$t \approx \left(m+\frac{1}{2}\right)\frac{\lambda}{2}$$

But there is a further complication. We should write our thickness equation as

$$t \approx \left(m+\frac{1}{2}\right)\frac{\lambda_{in}}{2}$$

because Δr has to provide an odd integer times the wavelength *inside the coating* for the phase to be right. After all, the wave is traveling inside the coating. We know that the wavelength will change as we enter the slower material.

To see this, consider two waves traveling to the right in the figure below. One passes through a slower medium. We expect the wavelength to shorten. We can see that, depending on the thickness t, the wave may be in phase or out of phase as it leaves. In the figure, the thickness is just right so that we have destructive interference.



We have such a wavelength shift in the coating. But we don't know the wavelength inside the coating. All we know is the radar wavelength, λ_{out}. We can fix it by writing the wavelength inside in terms of the wavelength outside. Earlier in our studies we found that the new wavelength will be given by equation (**??**)

$$\lambda_f = \frac{v_f}{v_i}\lambda_i$$

Let's rewrite this for our case

$$\lambda_{in} = \frac{v_{in}}{v_{out}}\lambda_{out}$$

We can express this in terms of the index of refraction

$$n = \frac{c}{v}$$

by multiplying the left hand side by c/c then

$$\lambda_{in} = \frac{cv_{in}}{cv_{out}}\lambda_{out}$$

or

$$\begin{aligned}\lambda_{in} &= \frac{\frac{c}{v_{out}}}{\frac{c}{v_{in}}}\lambda_{out} \\ &= \frac{n_{out}}{n_{in}}\lambda_{out}\end{aligned}$$

in the case of our aircraft coating the outside medium is air so $n_{out} \approx 1$

$$\lambda_{in} = \frac{1}{n_{in}}\lambda_{out}$$

This is this wavelength we need to match as the radar signal enters the medium.

Using this expression for λ_{in} in

$$t \approx \left(m + \frac{1}{2}\right)\frac{\lambda_{in}}{2} \qquad m = 0, 1, 2, \cdots$$

will give us the condition for destructive interference. Let's rewrite our λ_{in} equation for our case of a coating and air

$$\lambda_{in} = \lambda_{coating} = \frac{1}{n_{in}}\lambda_{out} = \frac{1}{n_{coating}}\lambda_{air}$$

thus

$$t \approx \left(m + \frac{1}{2}\right)\frac{1}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right) \qquad m = 0, 1, 2, \cdots$$

is our condition for being stealthy.

Let's assume we want the thinnest coating possible, so we set $m = 0$. Then

$$t \approx \left(\frac{1}{4}\right)\left(\frac{\lambda_{air}}{n_{coating}}\right)$$

and our thickness would be

$$t \approx \left(\frac{1}{4}\right)\left(\frac{3.00\,\mathrm{cm}}{1.50}\right) = 0.5\,\mathrm{cm}$$

This seems doable for an aircraft coating!

Of course we could also make a plane that would be more visible to radar by choosing the constructive interference case. Suppose we are building a search and rescue plane. We want to enhance it's ability to be seen by radar in fog. We start with the condition for constructive interference

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = m2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive}$$

It will still be true that $\Delta\phi_o = 0$.

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r\right) = m2\pi$$

and it is still true that $\Delta r \approx 2t$.

$$\left(\frac{2\pi}{\lambda}2t\right) \approx m2\pi$$

then

$$\left(\frac{1}{\lambda}2t\right) \approx m$$

$$t \approx \frac{1}{2}m\lambda$$

and we still have to adjust for the coating index of refraction

$$t \approx \frac{m}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right)$$

And once again we have several choices for m

$$t \approx \frac{m}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right) \qquad m = 0, 1, 2, \cdots$$

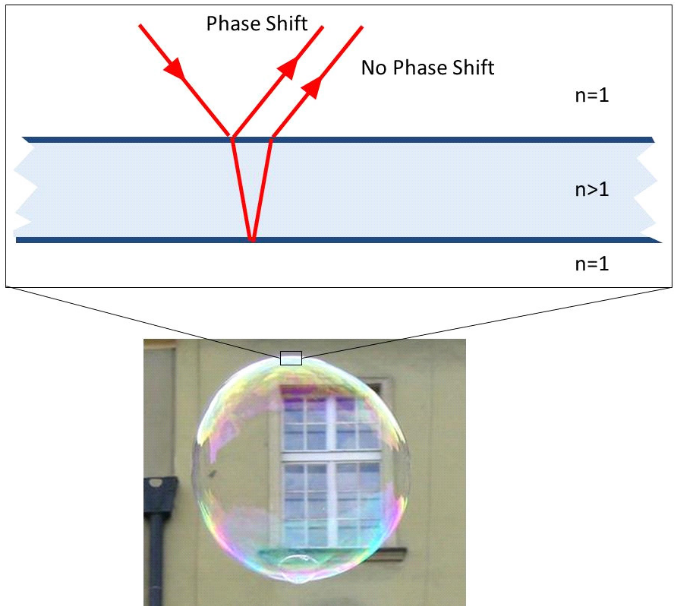
But now the coating will provide constructive interference, making it easier to track on radar from the command center. For the thinnest possibility, set $m = 1$ because the $m = 0$ case doesn't give us any thickness.

$$\begin{aligned} t &\approx m\frac{1}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right) \\ &= \frac{1}{2}\left(\frac{3.00\,\mathrm{cm}}{1.50}\right) \\ &= 1\,\mathrm{cm} \end{aligned}$$

Note that we reasoned out these equations for the boundary conditions that we have in our problem (inversion on reflection from both the coating and the plane body). If the boundary conditions change, so do the equations.

21.1.2 Example of two wave interference: soap bubble

Take a soap bubble for example.



Interference from a soap bubble. (Bubble image in the Public Domain, courtesy Marcin Deręgowski)

Now we have a phase shift on the first reflection, but not one on the reflection from the inside surface of the bubble because the bubble is full of air. The index of refraction of air is less than that for the bubble material. So as we leave the bubble material it is more like having a free end of a rope. As the waves leave the surface, they are half a wavelength out of phase due to $\Delta\phi_o$ because of the single inversion from the bubble outer surface. We would have destructive interference due to just this, but we also have to account for the bubble thickness. If this thickness is a multiple of a wavelength, then we are still have half a wavelength out of phase and we have destructive interference.

Here are our basic equations

$$\begin{aligned} \Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = m2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \text{ Constructive} \\ \Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = (2m+1)\,\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \text{ Destructive} \end{aligned}$$

Suppose we want want constructive interference to get our colors, so we take the first

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta r + \Delta\phi_o\right) = m2\pi$$

and this time we have

$$\begin{aligned}\Delta\phi_o &= \phi_{transmitted} - \phi_{reflected} \\ &= 0 - \pi \\ &= -\pi\end{aligned}$$

It is still true that $\Delta r \approx 2t$ so from our constructive interference equation

$$\frac{2\pi}{\lambda} 2t - \pi = m2\pi$$

$$\frac{2}{\lambda} t - \frac{1}{2} = m$$

$$\frac{2}{\lambda} t = m + \frac{1}{2}$$

$$t = \frac{\lambda}{2}\left(m + \frac{1}{2}\right)$$

We again have the problem that this wavelength must be the wavelength inside the bubble material $\lambda = \lambda_{in}$. But we see the outside wavelength λ_{out}. We can reuse our conversion from outside to inside wavelength from our last problem because we are once again in air and $n_{air} \approx 1$.

$$\lambda_{in} = \frac{1}{n_{in}} \lambda_{out}$$

then

$$t = \frac{\lambda_{out}}{2n_{in}}\left(m + \frac{1}{2}\right) \qquad m = 0, 1, 2, \cdots$$

Or writing this with $n_{in} = n_{\text{bubble}}$ to make it clear that the inside material is the bubble solution,

$$t = \left(m + \frac{1}{2}\right)\frac{1}{2}\left(\frac{\lambda_{air}}{n_{\text{bubble}}}\right) \qquad m = 0, 1, 2, \cdots$$

but this was the equation for destructive interference for the plane! We can see that memorizing the thickness equations won't work. We need to start with our conditions on $\Delta\phi$ for constructive and destructive interference to be safe!

How about the dark parts of the bubble with no color (the parts we can see through). These would be destructive interference

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right) = (2m + 1)\pi$$

We can fill in the pieces to obtain

$$\begin{aligned}
\left(\frac{2\pi}{\lambda}2t-\pi\right) &= (2m+1)\,\pi \\
\frac{2}{\lambda}2t-1 &= (2m+1) \\
\frac{2}{\lambda}2t &= (2m+1)+1 \\
\frac{4}{\lambda}t &= (2m+1)+1 \\
t &= \frac{\lambda}{4}\,(2m+2) \\
t &= \frac{\lambda}{2}\,(m+1) \\
t &= \frac{m+1}{2}\left(\frac{\lambda_{out}}{n_{\text{bubble}}}\right)
\end{aligned}$$

This is our condition for destructive interference for the bubble. We don't have to, but we could write $m+1=p$ where p is an integer that starts at 1 instead of zero.

$$t=\frac{p}{2}\left(\frac{\lambda_{out}}{n_{\text{bubble}}}\right) \qquad p=1,2,\cdots$$

But this is very like the condition for constructive interference for the plane.

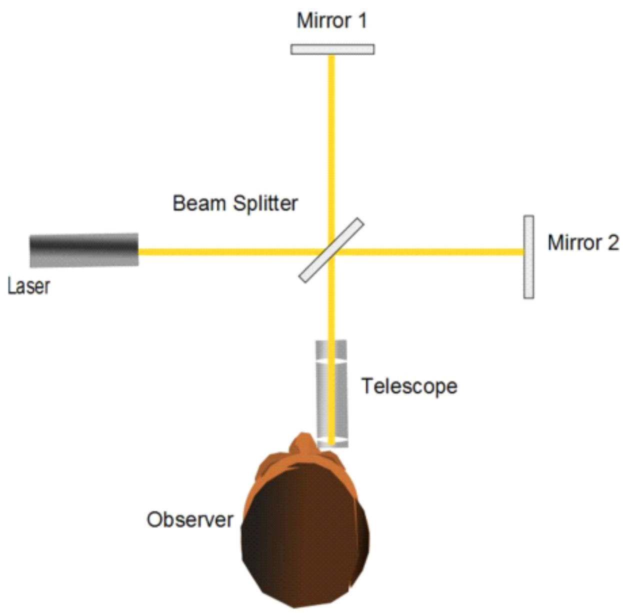
Hopefully, it is apparent that we have to start with our basic equations

$$\begin{aligned}
\Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta x+\Delta\phi_o\right)=m2\pi \qquad m=0,\pm1,\pm2,\pm3,\cdots \text{ Constructive} \\
\Delta\phi &= \left(\frac{2\pi}{\lambda}\Delta x+\Delta\phi_o\right)=(2m+1)\,\pi \qquad m=0,\pm1,\pm2,\pm3,\cdots \text{ Destructive}
\end{aligned}$$

each time we attempt an interference problem because the outcome depends on both Δx and $\Delta\phi_o$. We have to construct the equation each time for the interference condition we want (constructive or destructive) finding Δx and $\Delta\phi_o$ for the boundary conditions we have.

21.2 The Michelson Interferometer

The Michelson interferometer is another device that uses path differences to create interference fringes.



The device is shown in the figure. A coherent light source is used. The light beam is split into two beams that are usually at 90° apart. The beams are reflected off of two mirrors back along the same path and are mixed at the telescope. The result (with perfect alignment) is a target fringe pattern like the first two shown below.

Bright Center Dark Center Small Off-Center Fringes

If the alignment is off, you get smaller fringes, but the system can still work. This is shown in the last image in the previous figure.

In the figure, we have constructive interference in the center, but if we move one of the mirrors half a wavelength, we would have destructive interference and would see a dark spot in the center. This device gives us the ability to measure distances on the order of the wavelength of the light. When the distance is continuously changed, the pattern seems to grow from the center (or collapse into the center).

Notice that if the mirror is moved $\frac{\lambda}{2}$, the path distance changes by λ because the light travels the distance to the mirror and then back from the mirror (it travels the path twice!).