Chapter 20

20 Double Slits 3.3.2, 3.3.3

Fundamental Concepts

- Two wave mixing works for light
- Visible light wavelengths are small, we can use the small angle approximation
- Multiple slits give areas of constructive and destructive interference.

20.0.1 Constructive Interference

We can find the condition for getting a bright or a dark band if we think about it a bit. The two waves coming from the two slits are just two different waves. We can use our two wave mixing analysis with our constructive and destructive interference criteria. For constructive interference, the difference in phase must be a multiple of 2π. Let's review this, but instead of $y_{\max}$ being the displacement of the waves, let's write them as $E_{\max}$ because it is the electric field that is carrying the wave and it doesn't really go up or down, it just gets a higher value or lower value. Our waves will be

$$\begin{aligned} E_1 &= E_{\max} \sin\left(k_2 r_2 - \omega_2 t_2 + \phi_2\right) \\ E_2 &= E_{\max} \sin\left(k_1 r_1 - \omega_1 t_1 + \phi_1\right) \end{aligned}$$

and the resulting wave will be

$$\begin{aligned} E_r &= E_{\max}\sin(k_2r_2-\omega_2t_2+\phi_2)+A\sin(k_1r_1-\omega_1t_1+\phi_1) \\ &= 2E_{\max}\cos\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \\ &\quad \times\sin\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)+(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\left[(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)\right]\right) \\ &\quad \times\sin\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)+(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \end{aligned}$$

As we are now well aware, the sine part is a combined wave, and the cosine part is part of the amplitude. The amplitude can be written as

$$A=2E_{\max}\cos\left(\frac{1}{2}\left[(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)\right]\right)$$

We should pause and think about which of our values will change. The frequency won't change, $\omega_2=\omega_1=\omega$. There is no slower material, so the wavelength won't change, $k_2=k_1=k$. We want the waves to mix at the same time so $t_2=t_1=t$. And the new waves are created from the old wave hitting the slits. As long as the original wave hits both slits at once, then $\phi_2=\phi_1=\phi_o$. We are left with

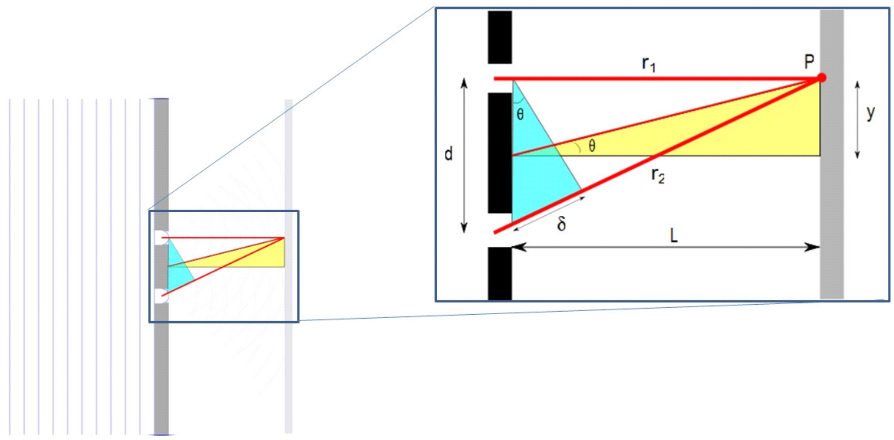
$$\begin{aligned} A &= 2E_{\max}\cos\left(\frac{1}{2}\left[(kr_2-\omega t+\phi_o)-(kr_1-\omega t+\phi_0)\right]\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\left[(kr_2)-(kr_1)\right]\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}k\left(r_2-r_1\right)\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}k\Delta r\right) \end{aligned}$$

That means that to have constructive interference the path difference between the two slit-sources must be an even number of wavelengths. We have been calling the path difference in the total phase Δx, or for spherical waves Δr, but in optics it is customary to call this path difference δ or even $\Delta\ell$. So

$$\delta=\Delta r$$

But let's write the amplitude function as

$$A=2E_{\max}\cos\left(\frac{1}{2}k\Delta r\right)$$



so our total phase equation becomes

$$\Delta\phi = k\Delta r$$

We want to use this in our criteria for constructive or destructive interference. But first, let's express $\delta = \Delta r$ in terms of geometry that is easy to measure in our experiment. In this set up, the screen is much farther away than d, the slit distance, we can say that the blue triangle is almost a right triangle, and then δ is

$$\Delta r = r_2 - r_1 \approx d\sin\theta$$

Our wave repeats every 2π radians or every wavelength, λ, then we have constructive interference (a bright spot) when

$$\Delta\phi = k\Delta r \approx kd\sin\theta = 2\pi m \quad (m = 0, \pm1, \pm2\ldots)$$

We can solve this for Δr

$$\Delta r = d\sin\theta = m\lambda \quad (m = 0, \pm1, \pm2\ldots)$$

and we can give m a name. It is called the *order number*. That is, if we are off by any number of whole wavelengths then our total phase due to path difference will be a multiple of 2π.

If we assume that $\lambda \ll d$ we can find the distance from the axis for each fringe more easily. This guarantees that θ will be small. Using the yellow triangle we see

$$\tan\theta = \frac{y}{L}$$

but if θ is small this is just about the same as

$$\sin\theta = \frac{y}{L}$$

because for small angles $\tan\theta \approx \sin\theta \approx \theta$. So if θ is small then

$$\begin{aligned}\Delta r &= d\sin\theta \\ &= d\frac{y}{L}\end{aligned}$$

and for a bright spot or *"fringe"* we find

$$d\frac{y}{L} = m\lambda$$

Solving for the position of the bright spots gives

$$y_{bright} \approx \frac{\lambda L}{d} m \quad (m = 0, \pm 1, \pm 2 \ldots) \tag{20.1}$$

We can measure up from the central spot and predict where each successive bright spot will be.

20.0.2 Destructive Interference

We can also find a condition for destructive interference. We know that a path difference of an odd multiple of a half wavelength will give distractive interference. so

$$\Delta\phi = k\Delta r \approx kd\sin\theta = \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

or using $k = 2\pi/\lambda$

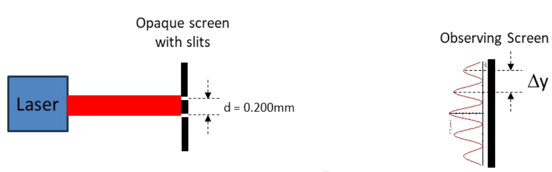
$$\Delta r = d\sin\theta = \left(m + \frac{1}{2}\right) \lambda \quad (m = 0, \pm 1, \pm 2 \ldots)$$

will give a dark fringe. The location of the dark fringes will be

$$y_{dark} \approx \frac{\lambda L}{d} \left(m + \frac{1}{2}\right) \quad (m = 0, \pm 1, \pm 2 \ldots) \tag{20.2}$$

Let's try an example problem.

Suppose we have a laser beam that produces pretty red light with $\lambda = 632.8\,\text{nm}$. And suppose we shine this laser on an opaque screen with two slits that are $0.020\,\text{mm}$ apart. And suppose we let the light hit a screen that is $5.00\,\text{m}$ away from the slits. How far apart will the bright interference fringes be on the screen?



PT: This is a two slit problem

BE

$$y_{bright} = \frac{\lambda L}{d} m \qquad m = 0, \pm 1, \pm 2, \cdots$$

$$\begin{aligned}
\lambda &= 632.8\,\mathrm{nm} \\
d &= 0.020\,\mathrm{mm} \\
L &= 5.00\,\mathrm{m}
\end{aligned}$$

Solution

We want the space between two bright fringes. Δy, this is

$$\begin{aligned}
\Delta y &= \frac{\lambda L}{d}(m+1) - \frac{\lambda L}{d}(m) \\
&= \frac{\lambda L}{d}((m+1)-(m)) \\
&= \frac{\lambda L}{d}
\end{aligned}$$

or

$$\Delta y = \frac{\lambda L}{d}$$

$$\begin{aligned}
\Delta y &= \frac{(632.8\,\mathrm{nm})\,(5.00\,\mathrm{m})}{0.020\,\mathrm{mm}} \\
& 0.158\,2\,\mathrm{m} \\
&= 15.82\,\mathrm{cm}
\end{aligned}$$

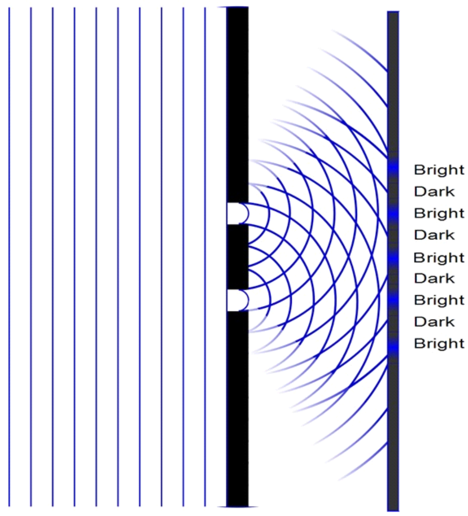
Units check

Reasonableness check

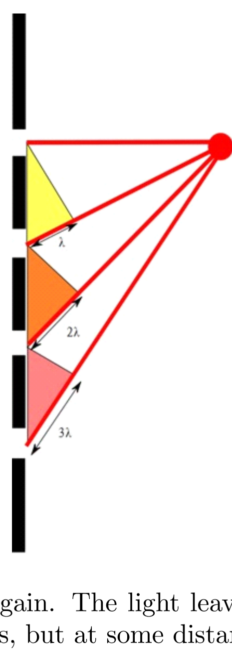
This may seem like a mostly useless thing to measure, but let's think of this problem backwards. If we have two slits and measure a pattern of 15.82 cm peak separation we can know that the slits were 0.200 mm. Light interference is used in industry for measuring very small distances on the order of tens of microns.

20.1 Multiple Slits

What happens if we have more than two slits?



At some point, two of the slits will have a path difference that is a whole wavelength, and we would expect a bright spot. But what about the other slits? If we have a slit spacing such that each of the succeeding slits has a path difference that is just an additional wavelength, then each of the slits will contribute to the constructive interference at our point, and the point will become a bright spot.



Let's look at just two slits again. The light leaves each slit in phase with the light from the rest of the slits, but at some distance L away and at some angle

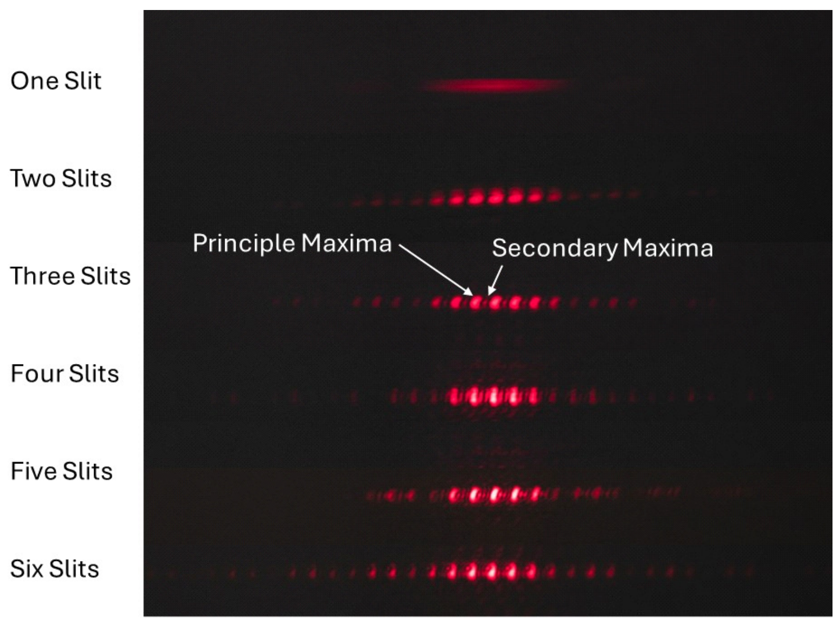
θ we will have a path difference

$$\delta = d \sin\left(\theta_{bright}\right) = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots \tag{20.3}$$

because the path lengths are not all the same. But we can see that if we have more than two slits, say, four slits, then this same equation would work to find a bright spot for the top two slits. And it would also work for the bottom two slits. And it would work for the middle two slits! Then for every set of two slits we get constructive interference at the same angle. That means at that spot on that angle all the constructive interference from sets of two slits add up to make a super constructive interference. This will be a very bright spot. We call this a principle maxima.

Of course we could find a spot where two of the slits were $\lambda/2$ off so there will be destructive interference from these two slits. But then a third slit would still send light at that angle. So there would be some brightness. So this isn't total destructive interference. There will be a spot less bright than the principle maxima, but not totally dark. We call these secondary maxima.

In the next figure, you can see laser light that passes through increasing numbers of slits.



Notice that as the number of slits grows, the principle maxima get brighter until they turn white because they are oversaturating the camera sensor.