Chapter 17

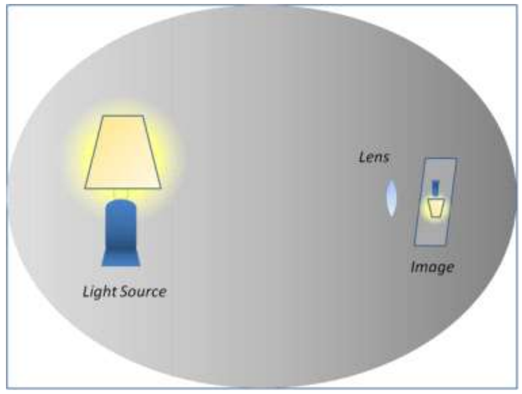
17 Thin Lenses 3.2.4, 3.2.5

Fundamental Concepts

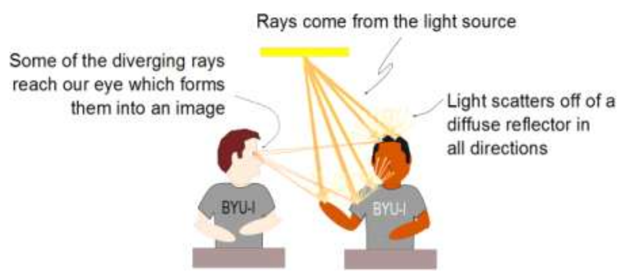
- For a thin lens, we can describe where the light goes using $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$ where $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$
- There is a sign convention for all of these equations.
- The eye is like a semi-nfinate double lens with the cornea and the crystalline lens as the two lenses
- The crystalline lens does focusing but in biology it is called accommodation
- We have three color sensors for red green and blue light
- Myopia and hyperopia are problems with the shape of the eye and are corrected with additional lenses
- Optical lenses are measured in optical power which is $1/f\,(m)$ with the unit if a diopter.
- Medical people measure the power of a lens in diopters. A diopter is one over the focal length with the focal length measured in meters.

17.1 Images from lenses

Let's think once again about what an image is. If you haven't done this before, take a piece of paper and a lens, and hold up the lens in a darkened room that has some bright object in it. Move the lens or the paper back and forth, and at just the right distance, a miniature picture of the bright object will appear. We should think about what the word "picture" means in this sense.



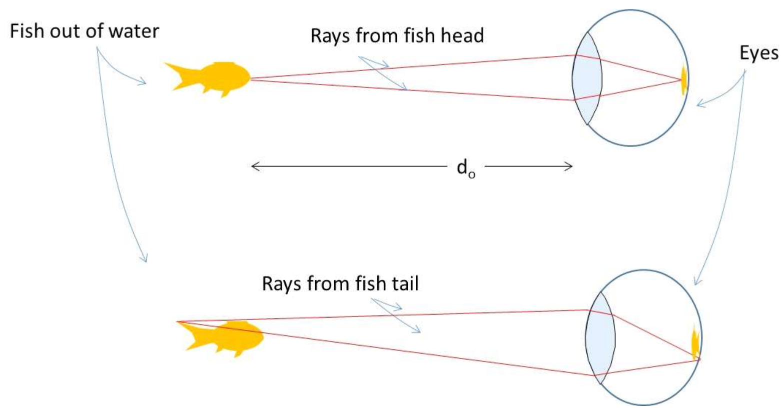
We have talked about how we see objects. Remember the BYU-I guys from last time.



Our eyes gather rays that are diverging from the object because light has bounced off of the object. Our eyes intersect a diverging set of rays that form a definite pattern. That diverging set of rays forming a pattern is the picture of the object.

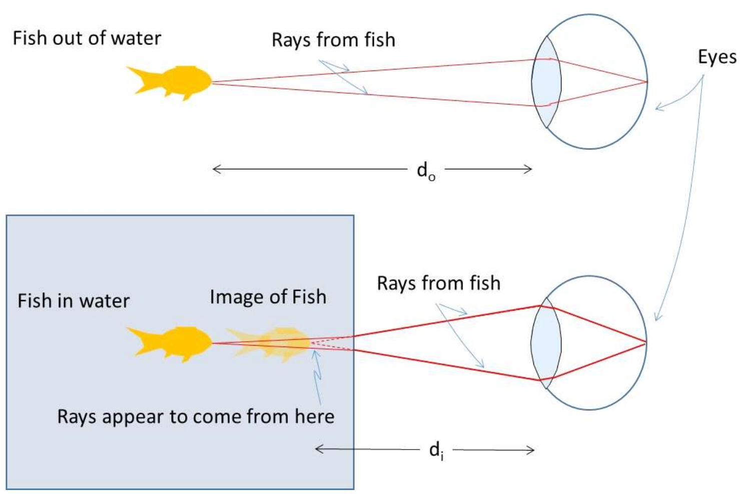
So when we say that the lens has formed a miniature picture of our object, we mean that the lens has somehow formed a diverging set of rays that form a pattern that looks like the pattern formed by the diverging set of rays coming from the object, itself. In other words, the object forms a diverging set of rays. And our lens forms a duplicate set of rays in the same pattern, so we see the same thing. The lens' version is smaller, upside down, and backwards, but it is still essentially the same pattern. We saw this with our semi-infinite lens, but it must be true for less infinite lenses as well.

As a first step to see how this works, consider our fish tank again. It would be bad on the fish, but think about looking at a fish in air. The room light would bounce off of the fish, and we would have a diverging set of rays from every point on the fish (see next figure).



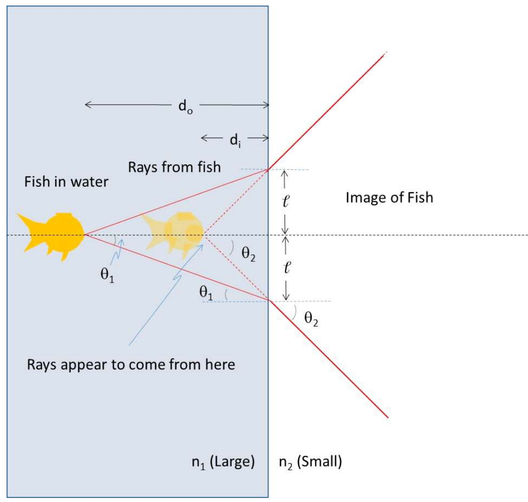
We can see that the picture is made from every point on the fish being "imaged" to a point on our retina. We collect the rays leaving every point on the fish, and bring them to corresponding points on the retina to make the picture.

It will take us a few lectures to see exactly how this is done, but as a first step, let's consider the fish tank, itself. Put the fish back in the tank and look at it (next figure).



Rays still come from the fish. But we now know that the change from a slow material to a fast material will bend the light. These bent rays are collected by our eyes, and the picture of the fish is formed on the retina just as before. But our eyes interpret the light as though it went in straight lines with no bends (dotted lines in the last figure). our visual processing center in our brain is designed to believe light travels in straight lines, so our mind tells us there is a fish, but that the fish head (and every other part of the fish) is closer than it really is. We call this apparent fish at the closer location an image of the fish, because this is where we think the diverging set of rays come from that form the fish pattern.

The next figure shows the details of the rays leaving a dot on the fish head (with the angles exaggerated so it's easier to see them).



A spot on the fish head is our object for this set of rays. The distance from the fish-head dot and the edge of the water/air boundary is our *object distance,* d_o.

The distance from the image of the fish-head dot to the edge of the water/air boundary is our *image distance* and is given the symbol d_i, because it appears to be where the rays come from.

We can find where the image is (d_i) knowing d_o. We can see from the figure that

$$\begin{aligned} \ell &= d_o \tan\theta_1 \\ &= d_i \tan\theta_2 \end{aligned}$$

so

$$d_o \tan\theta_1 = d_i \tan\theta_2$$

or

$$d_o \frac{\tan\theta_1}{\tan\theta_2} = d_i$$

from Snell's law, we know that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$

Usually we can take the small angle approximation. This would limit our analysis to rays that are near the central axes. We have done this before with mirrors. We call this central axis the *optics axis* and the rays that are not too far away from this axis *paraxial rays.* Then for our small angles we can write

$$\tan\theta_i \approx \sin\theta_i$$

so

$$d_o \frac{\tan\theta_1}{\tan\theta_2} \approx d_o \frac{\sin\theta_1}{\sin\theta_2} = d_o \frac{n_2}{n_1} = d_i$$

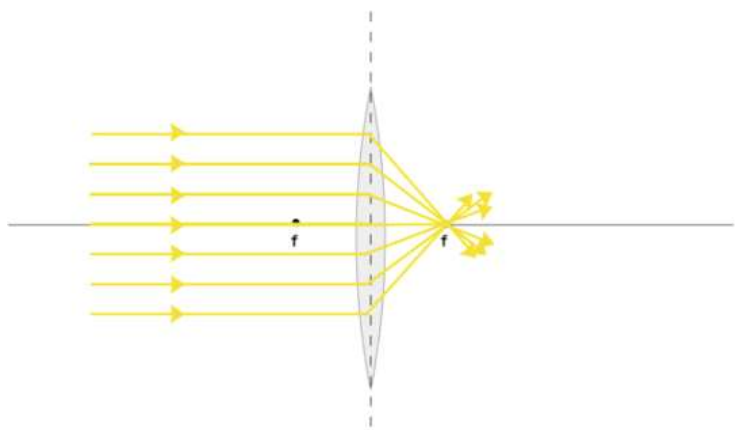
and we have the image distance

$$d_i = d_o \frac{n_2}{n_1}$$

Note that once again we have a virutual image. The place where we see the fish does not have rays of light coming from it! But our brain processing center believes the rays of light didn't bend, so it exends the rays we see bacward into the fish tank and it interprets what we see as a fish being where those extended rays meet.

This is not so useful unless you have some burning need to know where fish are in a tank. But we now have the vocabulary to discuss the larger problem of how a lens works, which we will take up next time.

Before we do a lot of math to describe how lenses work, lets think about our early childhood experiences. You may have burned things with a magnifying glass[1]. Using the idea of a ray diagram, here is what happens.



We know that the rays from the Sun come from so far away that they are essentially parallel. We know that these rays come together to a fine point that can start a fire. The point where these rays converge is important to us. We found such a point from our mirrors. We called this point the *focal point*.

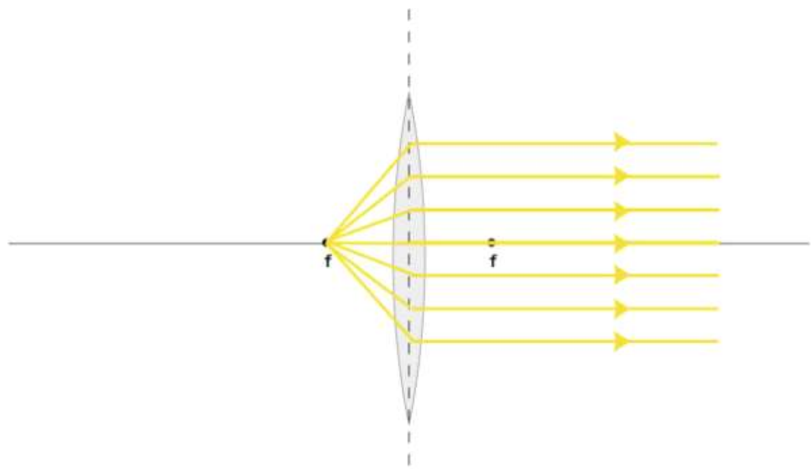
Now think about Snell's law. We have

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

The light will bend at a surface, but snell's law doesn't care if $n_1 > n_2$ of if $n_2 > n_1$. It works both ways. The light will follow the same path either direction. We would expect that if we put a light source at the focal distance,

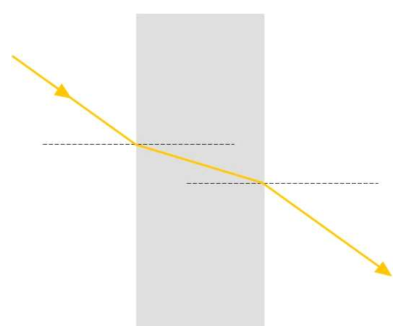
[1]If you didn't do this as a child, consider trying it now. Be responsible and safe, but it is valuable to see how this works.

the rays should come out parallel.



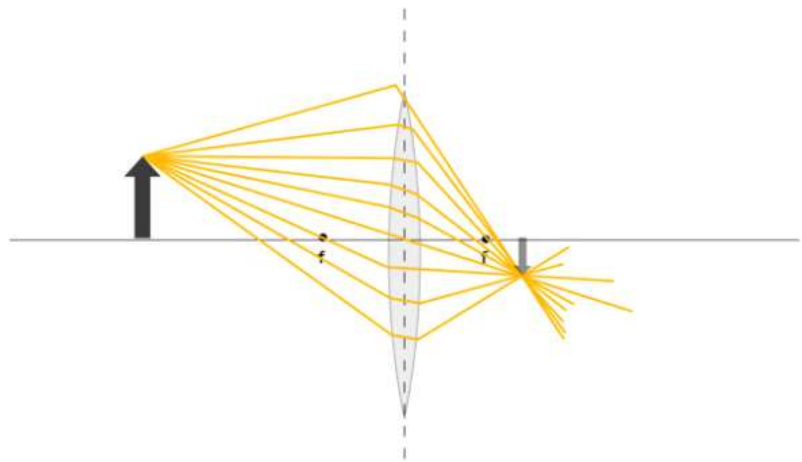
This is one way manufacturers make LED flashlights.

We need one other bit of information that we already have seen from basic refraction.

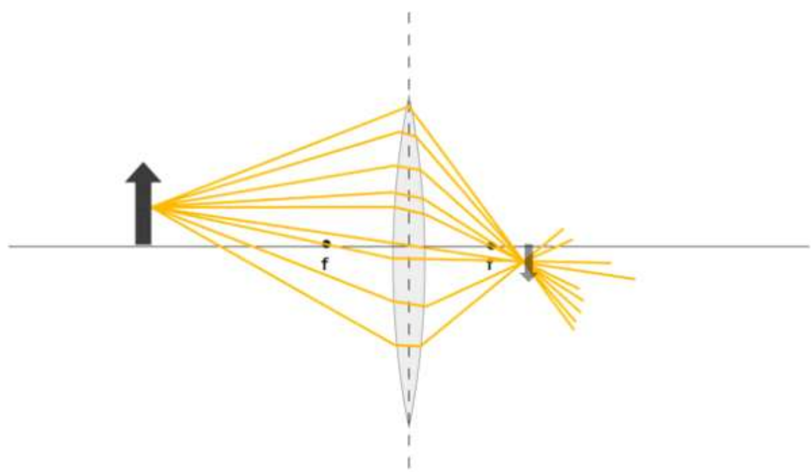


A flat block does refract the light, but when the light leaves the block it is only displaced, it retains the original direction. With these three ray scenarios, we can describe how a lens works.

We know that for every point on the object, we get millions of reflected rays that diverge. In the next figure, rays of light are reflecting off of the large, black arrow. The lens must collect these rays together to form the corresponding point on the image.



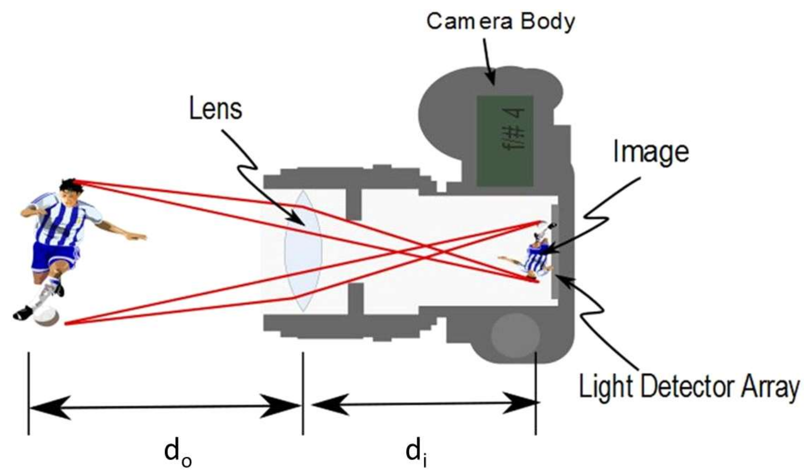
Of course, this happens for every point on the image.



But we usually pick the top of the object. If we place the bottom of the object on the *optic axis*[2], the bottom of the image will also be on the optic axis. So knowing where the bottom of the image is, and finding the top of the image gives a pretty good idea of where the rest of the image must be. So we will draw diagrams for the top of the object to find the top of the image.

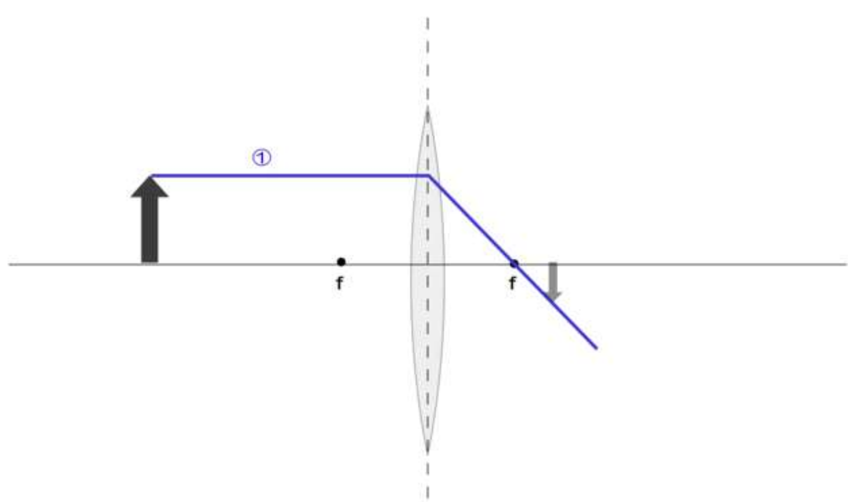
But suppose this is not true? For example, when we use a camera, we do not align the optical system on an axis before we shoot.

[2] The line drawn on the figure thorugh the middle of the lens.

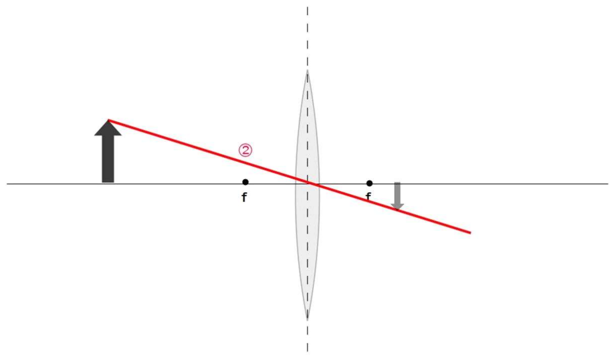


We can, of course, trace two rays for the bottom of the image in this case and find the location of the bottom of the image as well.

We know that light is a supperposition of millions of waves and that when light reflects off of something like your neighbor or a fish or a soccor playe there will be millions of rays. Drawing millions of rays is impractical, and fortunately, not needed. We can choose three simple rays that leave the top of the object , and see where these rays converge to form the top of the image. Let's start with a ray that travels from the top of the object and travels parallel to the optic axis. We recognize this ray as being like one of the rays from the Sun. It comes in parallel, so it will leave the lens and travel through the focal point.

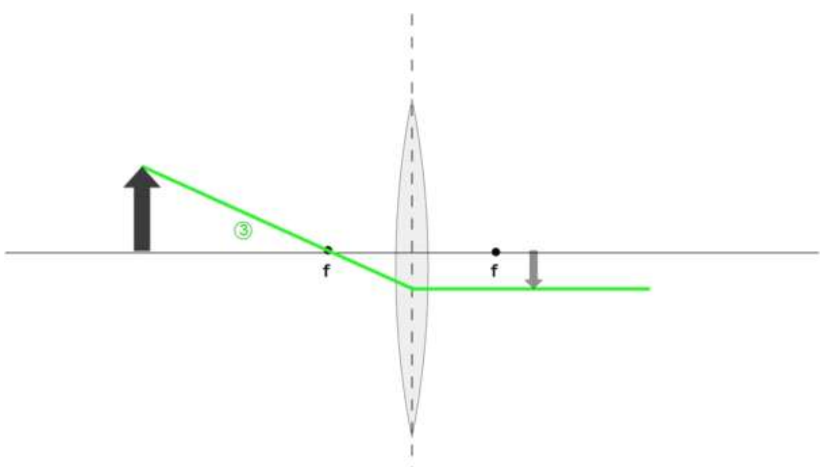


For a second easy ray, lets take the case that is like our flat block. Near the center of the lens, the sides are nearly flat. So we expect that the ray will leave in about the same direction as it was going before it struck the lens.

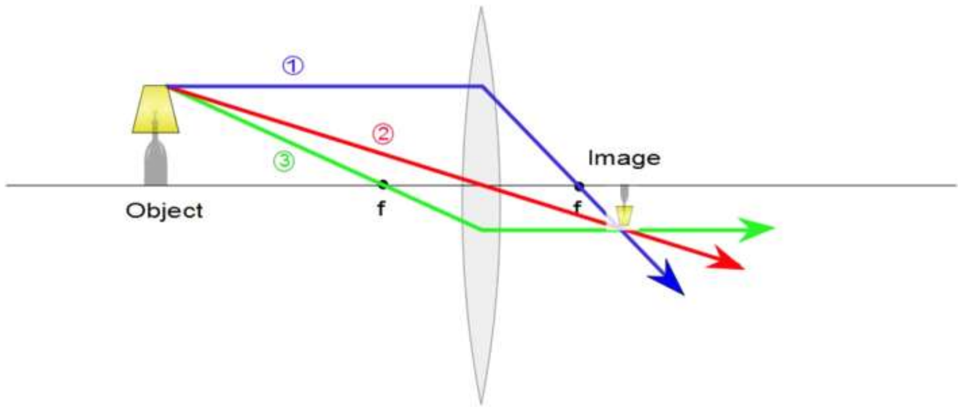


Technically this ray should be shifted. This is where our thin lens approximation comes in. If the lens is thin, then the ray going through the center of the lens won't be shifted much, and we can ignore the shift. We just dray ray 2 as a straight line.

Two rays are really enough to determine where the top of the image will be, but there is a third ray that is easy to draw, so let's draw it to give us more confidence in our answer. That ray is one that leaves the top of the object and passes through the focal point on the object side of the lens. This situation we also recognize. Since the ray goes through the focal point, it is as though the light came from that point. This is like our LED flashlight case. This ray will leave the lens parallel to the optic axis.



Where all three rays intersect, we will have the top of the image.



Notice that in this case, the image is upside down. That is normal. Also notice that it is smaller than the object. We say that the image is magnified, which may still seem a little bit strange. But in optics, a magnification of greater than one means that the image is bigger than the object. This is like a movie projector that makes a large image of a small film segment. The magnification can be equal to one, meaning the object and image are the same size. And finally the magnification can be less than one. This means that the image is smaller than the object. This is a convenient definition, because then we can use the same equation to describe all three situations.

$$m \equiv \frac{\text{Image height}}{\text{Object height}} = \frac{h_i}{h_o}$$

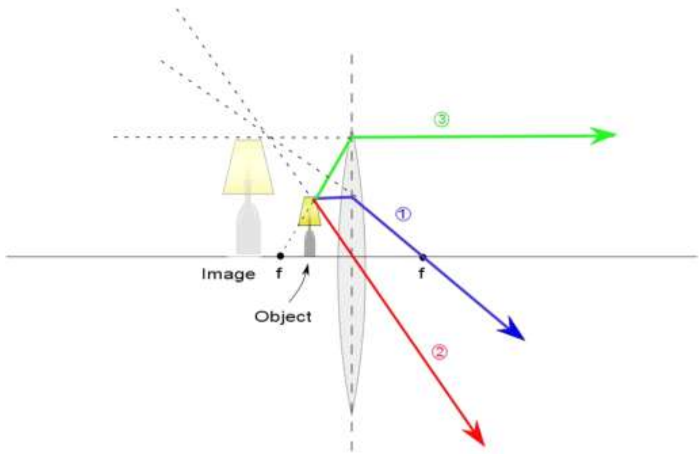
where h_o is the object height, and h_i is the image height. Notice the negative sign. It's still true that by convention (meaning physicists got together and voted on this) we say that an upside down image has a negative magnification. You just have to memorize this, there is no obvious reason for this except it is mathematically convenient.

We will find that

$$m = -\frac{d_i}{d_o}$$

17.2 Virtual Images with Lenses

Lets take another case and draw a ray diagram. This time let's place the object closer than the focal distance. This is the case when we use a lens as a magnifying glass. The rays will look like this.



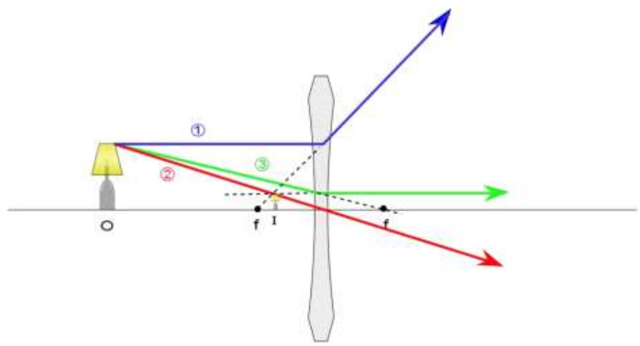
Notice that these rays never converge! We won't get an image that could project on a paper. But we know that there is an image, we can look through the lens and see it! And that is the key. The image does not really exist. There is no place where light gathers and then diverges into a pattern that we recognize. But we look through the lens, and our mind interprets the diverging rays coming from the lens as though they had only traveled in straight lines. If we extend these rays backwards along straight lines, they appear to come from a common point. This is the point they would have had to have come from if there were no lens. We believe we see an image at this location. But no light really goes there! Because this image is not really made from light diverging from this position, we call it a *virtual image*. The image we formed before that could be projected on a screen is called a *real image*. This is just like our image terminology for mirrors!

By convention, we say the distance from the lens to the virtual image is a negative value.

17.2.1 Diverging Lenses

So far our lenses have only been the sort that work as magnifying glasses. We call these *converging lenses.* These lenses are fatter in the middle and thinner on the edges. Because of this they are sometimes called *convex lenses*. By convention, we say the focal distance for this type of lens is positive. For this reason, they are often called *positive lenses.*

But what if we make a lens that is thinner in the middle and thicker on the edges. We can call this sort of lens a *concave lens*, and we will give it a negative focal length by convention, so we can also call it a *negative lens*. But what would this lens do? If we think about our three rays, ray 1 won't be bent toward the optic axis for this type of lens. In fact, if we observe an object through this lens, ray number 1 will appear to come from the focal point. Ray number 2 will still go through the middle of the lens, and if the lens is thin enough, ray 2 will pass through undeviated.



finally ray three leaves the object in the direction of the far focal point. It will hit the lens and leave parallel to the optic axis. From the figure we see that these three rays will never converge. Thiis is like our convex mirror! And from our mirror experience we expect the rays (with help from our brain) will form a virtual image. If we extend the rays backward as shown, we see that the extensions all meet at a point. The rays leaving the lens appear to come from this point. This is the location of the top of the virtual image of the object.

You might wonder what good such a lens could do, but we will find that this type of lens is used to correct vision for nearsighted people. It's likely that many people in our class have this type of lens with them either on their eye (contacts) or on their nose (eye glasses).

17.3 Thin Lenses

Lets' find an equation for a spherical surface once more. But this time, let's let it be more practical and not make the "lens" semi-infinate. We will need to deal with two sides of the lens because (usually) both will be curved.

We found that for refraction

$$\frac{n_1}{d_o} + \frac{n_2}{d_i} = \frac{(n_2 - n_1)}{R}$$

but we did this for a spherical bump on a semi-infinite piece of glass. For this new problem let's make a few assumptions:

1. We have two spherical surfaces, with R_1 and R_2 as the radii of curvature
2. We have only paraxial rays
3. The image formed by one refractive surface serves as the object for the second surface
4. The lens is not very thick (the thickness is much smaller than both R_1 and R_2)

The answer we will get is quite simple

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \tag{17.1}$$

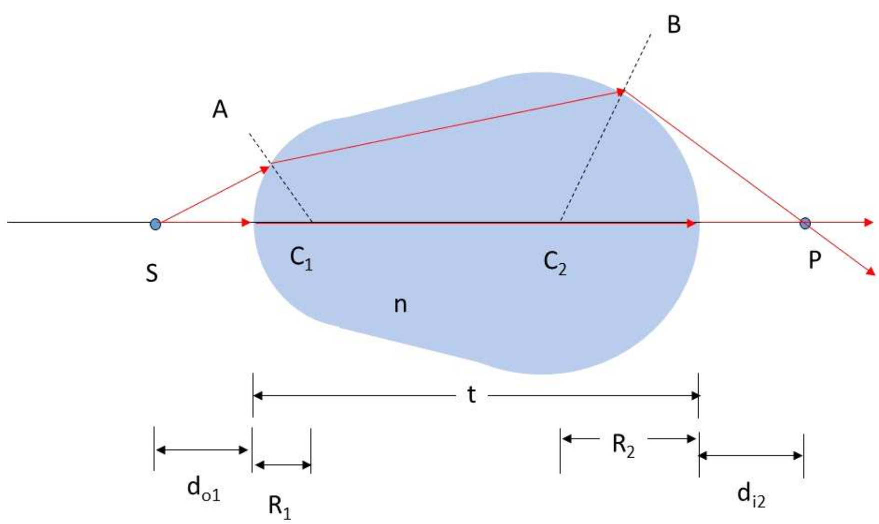
where

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{17.2}$$

but to appreciate what it means, lets find out where it comes from.

17.3.1 Derivation of the lens equation

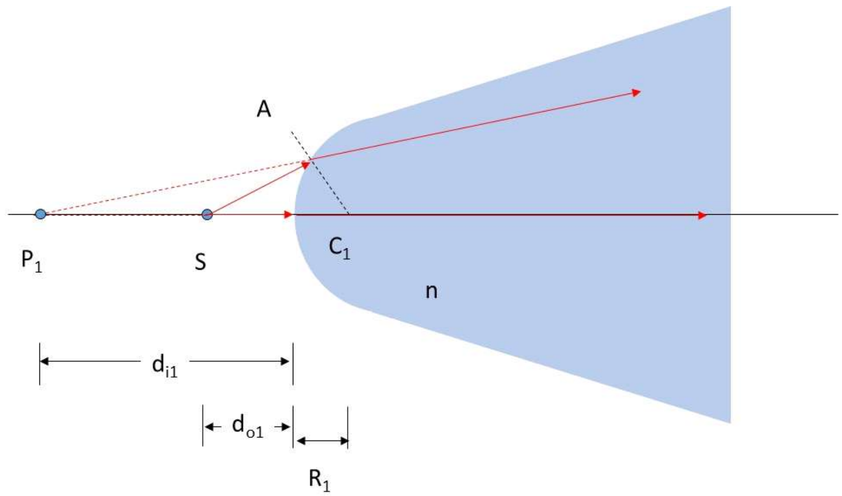
Consider the optical element in the figure below. Notice that our object is a dot, so our image will also be a dot. This is not as boring as it sounds if we consider any object can be considered as a collection of dots.



Light enters at a spherical surface on the left hand side. We use a point object located at S on the principal axis. We could put out dot object anywhere, but let's put it on the axis and trace two rays. The ray along the principal axis crosses each spherical surface at right angles, and therefore traves straight through the optic (this makes on ray very easy to trace!). The second ray hits the first spherical surface at point A. It is refracted and travels to point B. It is again refracted and travels toward the principal axis, crossing at P. The image location is the intersection of these rays, so we have an image at P.

Lets study the surfaces separately

Surface 1:



Let's treat surface 1 as though surface two did not exist. The light would bend at point A and head off into the lens material. This is just our semi-infinite bump problem so we know that

$$\frac{n_1}{d_{o1}} + \frac{n_2}{d_{i1}} = \frac{(n_2 - n_1)}{R}$$

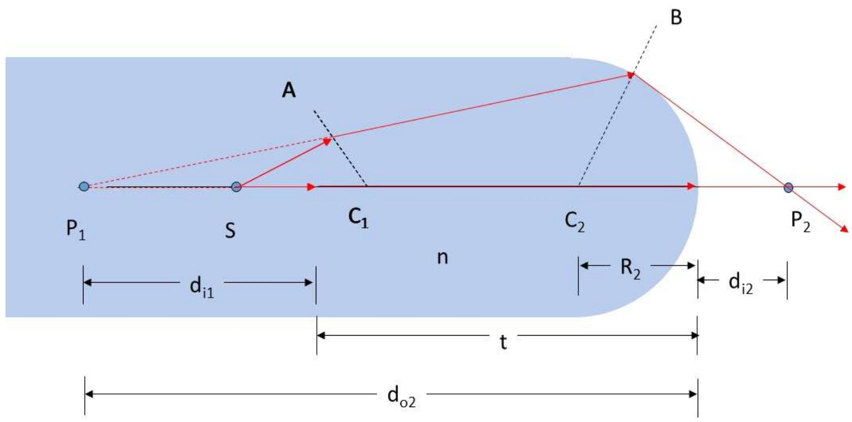
We can consider $n_1 = 1$ and $n_2 = n$ for a air-glass interface and noting that d_{i1} is negative (by convention). Then

$$\frac{1}{d_{o1}} - \frac{n}{d_{i1}} = \frac{(n-1)}{R_1} \tag{17.3}$$

Note that our rays are *not* converging in the glass this time. We can find the image formed by this side of our lens by tracing the diverging rays backward as we did for the fish tank. The image formed from the first side of the lens is virtual.

Surface 2

Now consider the second surface.



The second surface sees light diverging as though it came from a semi-infinite piece of glass with the object at P_1. It's not true, the light came from S. But the second surface can't tell the difference. It only knows light is incident in a particular pattern, and that pattern appears to come from point P_1. So we can treat the situation at surface 2 as though the object is the virtual image formed by surface 1. So

$$d_{o2} = d_{i1} + t$$

We again use our refractive equation

$$\frac{n_1}{d_{o2}} + \frac{n_2}{d_{i2}} = \frac{(n_2 - n_1)}{R}$$

but we identify $n_1 = n$ and $n_2 = 1$. We have for surface 2

$$\frac{n}{d_{o2}} + \frac{1}{d_{i2}} = \frac{(1-n)}{R_2} \tag{17.4}$$

or

$$\frac{n}{d_{i1} + t} + \frac{1}{d_{i2}} = \frac{(1-n)}{R_2} \tag{17.5}$$

Now we take our thin lens approximation. Let $t \to 0$. Then equations (17.3) and (17.5) become

$$\frac{1}{d_{o1}} - \frac{n}{d_{i1}} = \frac{(n-1)}{R_1}$$

$$\frac{n}{d_{i1}} + \frac{1}{d_{i2}} = \frac{(1-n)}{R_2}$$

We would prefer a nice equation that has how far away the object is from the lens and how far away the image is formed like our mirror equation. We can get such an equation by adding these two equations for the two surfaces

$$\frac{1}{d_{o1}} - \frac{n}{d_{i1}} + \frac{n}{d_{i1}} + \frac{1}{d_{i2}} = \frac{(n-1)}{R_1} + \frac{(1-n)}{R_2}$$

or

$$\frac{1}{d_{o1}} + \frac{1}{d_{i2}} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

This equation is very useful. It reminds us of the mirror equation (well, a little) . If we again let $d_{o1} = \infty$ (put the object at ∞ so the rays enter surface 1 parallel) we find

$$\frac{1}{d_{i2}} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

The spot where the rays gather if the object is infinitely far away is the focal point, f. so for parallel rays we can identify $d_{i2} = f$ as the focal length of the optic

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

which is known as the *lens makers' equation.*

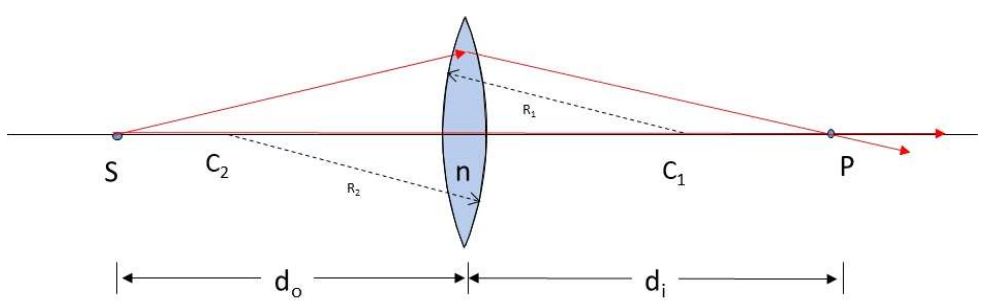
Then we have a relationship between the object distance in front of the lens, and the final image in back of the lens:

$$\begin{aligned}\frac{1}{d_{o1}} + \frac{1}{d_{i2}} &= (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ &= \frac{1}{f}\end{aligned}$$

We can drop the subscripts (which we can do now that we let $t = 0$ since the internal distances for the inside points are not important).

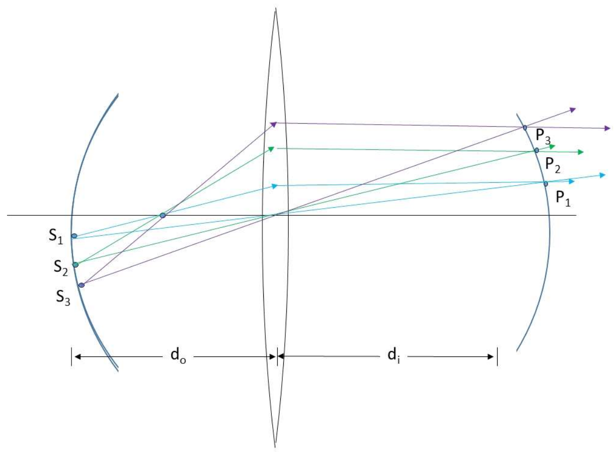
$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

This is called the *thin lens equation.* The resulting approximate geometry is shown below.



And this is fantastic! It is the same equation as the thin mirror equation! Only we have a different equation for finding the focal length.

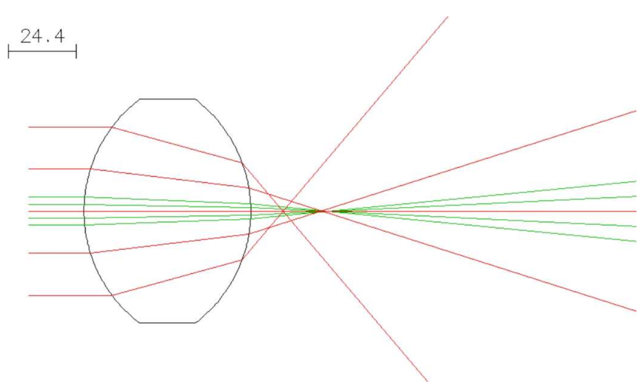
We did our math for just one point of light at S. Of course any real object is made of lots of points, and not all of the points are on an axis. But each point will be imaged to a corresponding point on the image. Here is an example with three points in the object (the s_i points) and where their images are (the p_i points).



but it would work for millions of points. Our simple analysis explains the formation of actual images and not just point images.

17.3.2 Lens Aberrations

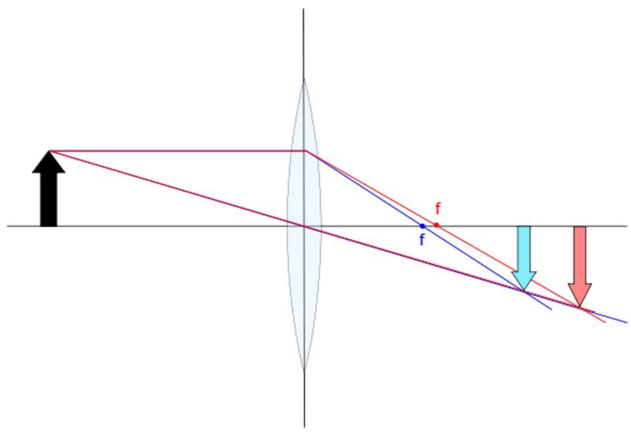
We found that for mirrors if we used a spherical shape rather than a parabola we had a problem with our images. We called the problem spiracle aberration. The reason we used a sphere is that spheres are easy to manufacture. The same problem happens with lenses if we make their sides from parts of spheres. The correct shape for a lens is a hyperbola, and hyperbole are hard to manufacture. Spheres are so much easier.



Example of Spherical Aberration for Lenses. Optical Ray Trace done in the OSLO Optical Design Package.

Again non-paraxial and paraxial rays focus at different spots because we have the wrong shape for our lens. Spherical aberration was made famous as the main problem with the Hubble Telescope.

We get another image problem or aberration with lenses. Remember that the index of refraction is different for different wavelengths of light. If we send white light into our lens, different colors will bend slightly differently (think Snell's law). So we will get a different focus point for different colors of light. I the next figure blue light makes an image at a shorter focal distance than red light. This problem is called *chromatic aberration*.



Generally mirrors can't have chromatic aberration because they don't refract the light. But Lenses always will.

There are many aberrations that come from making lenses that are easy to manufacture, but that are not the perfect shape. We won't study all these in this class. If you are curious, we cover these aberrations in PH375.

17.3.3 Sign Convention

As we have been working we used our sign convention but we need to add to our sign convention table a second radius, and the focal length.

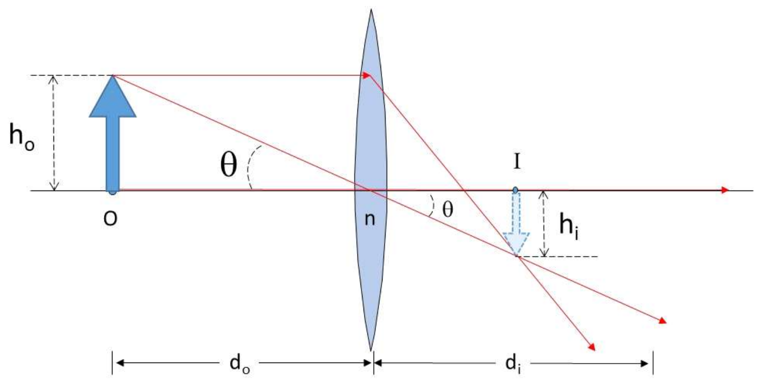
Quantity	**Positive if**	**Negative if**
Object location (d_o)	Object is in front of surface	Object is in back of surface (virtual object)
Image location (d_i)	Image is in back of surface (real image)	Image is in front of surface (virtual image)
Image height (h')	Image is upright	Image is inverted
Radius (R_1 and R_2)	Center of curvature is in back of surface	Center of curvature is in front of surface
Focal length (f)	Converging lens	Diverging lens

17.3.4 Magnification

We defined the magnification earlier as a comparison of the image height to the object height.

$$m = \frac{h_i}{h_o} \tag{17.6}$$

And we found we can put this into terms of d_o and d_i for mirrors. But we will this work for lenses?



From the diagram, we can see that

$$\begin{aligned} \tan\theta &= \frac{h_o}{d_o} \\ \tan\theta &= \frac{h_i}{d_i} \end{aligned}$$

then

$$\frac{h_o}{d_o} = \frac{h_i}{d_i}$$

or

$$\frac{h_i}{h_o} = \frac{d_i}{d_o}$$

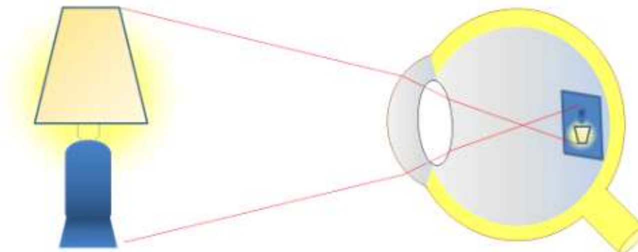
And once again we can write the magnification for a converging lens as minus the ratio of the image distance over the object distance.

$$m = -\frac{d_i}{d_o}$$

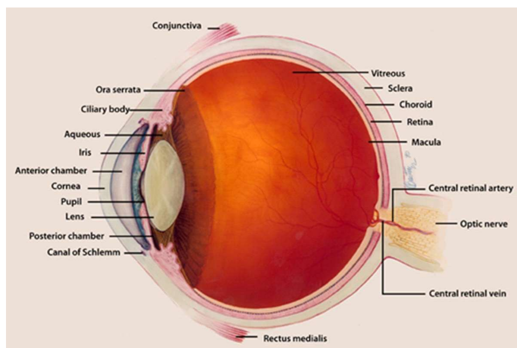
Just like before the minus sign comes from our sign convention because the image will be upside down.

We found the thin lens formula using converging lenses, but it works for diverging lenses as well, so long as the thin lens approximation is valid.

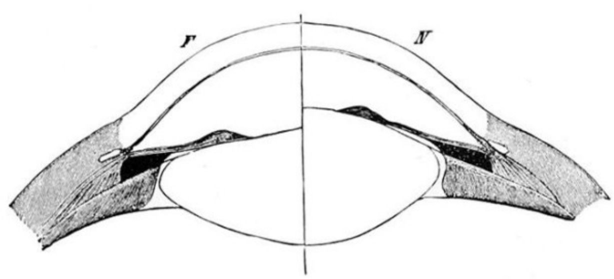
17.4 The Eye



The figure above shows the parts of the eye. The eye is like a camera in its operation, but is much more complex. It is truly a marvel. The parts that concern us are the cornea, crystalline lens, pupil, and the retina.



The Cornea-lens system refracts the light onto the retina, which detects the light. The lens is focused by a set of mussels that flatten the lens to change it's focal length. The focusing process is different from a standard camera. The camera moves the lens to achieve a different image distance. Our eye can't change the distance between the lens system and the retina. So our eye changes the shape of the lens, changing it's focal length.



The crystaline lens becomes thicker, and therefore more curved when the cilliary musscle flexes. Austin Flint, "The Eye as an Optical Instrument," *Popular Science Monthly,* Vol. 45, p203, 1894 (Image in the public domain)

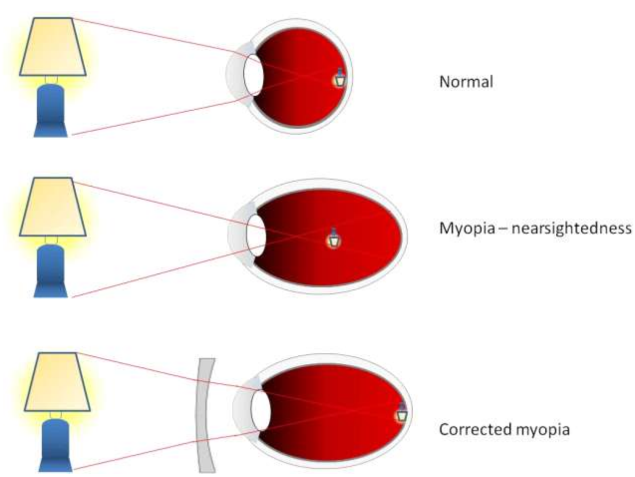
The focusing system is called accommodation. This system becomes less effective at about the time you reach an age of 40 years because the lens becomes less flexible. The closest point that can be focused by accommodation is called the near point. It is about 25 cm on average. There is, of course, no such thing as an average person, all of us are a little bit different. You young students probably have a much shorter near point than 25 cm. For those of us that are a little older, 25 cm or more is more likely.

The farthest point that can be focused is a long way away. It is called the far point. Both the near and far points degrade with years leading to bifocal glasses (and much irritation because you can't see as well, but I'm really not complaining because at least I can see).

The iris changes the area of the pupil (the aperture of the eye). The pupil is, on average, about 7 mm in diameter. This acts like the aperture adjustment of a camera.

17.4.1 Nearsightedness

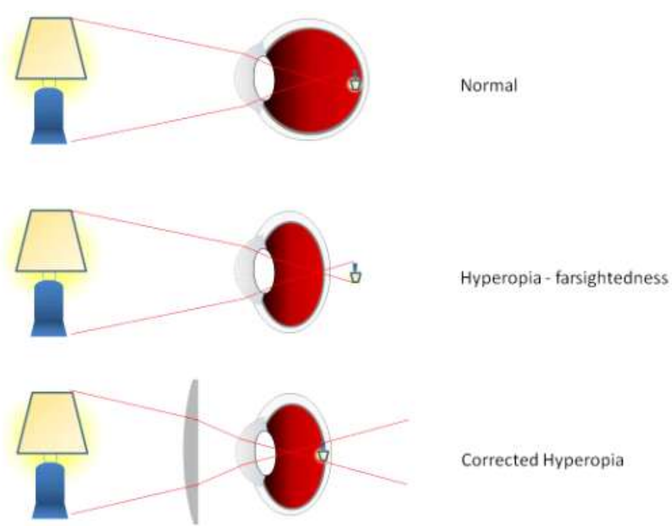
In some people the cornea-lens system focuses in front of the retina. Usually this is because the shape of their eyes is not spherical but is elongated along the optic axis of the eye. This is called nearsightedness or myopia.



So their perfectly good cornea-lens system makes a great image in the vitreous humor (the jellylike stuff that fills the eye) and not on the retina where it would be detected. From your experience with lenses you know this would produce a blurry image. And that is what nearsighted people see much of the time. We can help nearsighted people by effectively changing the cornea-lens system of the eye by adding another lens. We want a diverging lens that makes the light more spread apart so that the lens system of the eye can make it focus where the retina actually is. Alternately we could flatten the cornea, itself, with laser ablation.

17.4.2 Farsightedness

Sometimes the cornea-lens system focuses in back of the retina. This is usually because the eye grew too flat or "oblate." This is called farsightedness or hyperopia. Once again we can fix the problem by adding an additional lens. This time a converging lens.



The converging lens will make the effective focal length of the system shorter, so that it can form an image where the retina actually is.

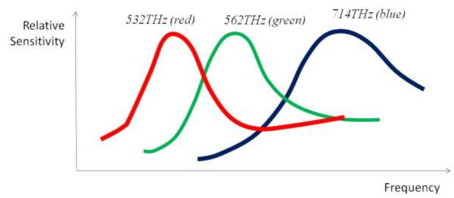
It would be convenient and very understandable if medical professionals prescribed eye glasses by telling us what focal length we needed to correct our vision. But that would not be the medical way! Eye glasses use a different unit of measure to describe how they bend light. The unit is the *diopter* and it is equal one over the focal length, but with the focal length measured in meters.

$$\text{diopter} = \frac{1}{f\,(\text{ m})} \tag{17.7}$$

This measurement is called the *power* of the lens. It is just the same as giving the focal length, but less clear for science students.

17.4.3 Color Perception

The eye detects different colors. The respecters called cones can detect red, green, and blue light.



The eye combines the red, green, and blue response to allow us to perceive many different colors.

Most digital cameras also have red, green, and, blue pixels to provide color to images. The detectors in digital cameras are often have much narrower frequency bands than the eye. Likewise, television displays and monitors have red, green, and blue pixels. By targeting the eye receptors, power need not be wasted in creating light that is not detected well by the eye. The difference in band-width can cause problems in color mixing. Yellow school busses (perceived as different amounts of green and red light) may be reddish or green if the bandwidths are chosen poorly.

The science of human visual perception of imagery is called *image science*. There are many applications for this field, from forensics to intelligence gathering.