Chapter 11

11 Shock waves and Non-Sinusoidal Waves 1.17.8

Fundamental Concepts

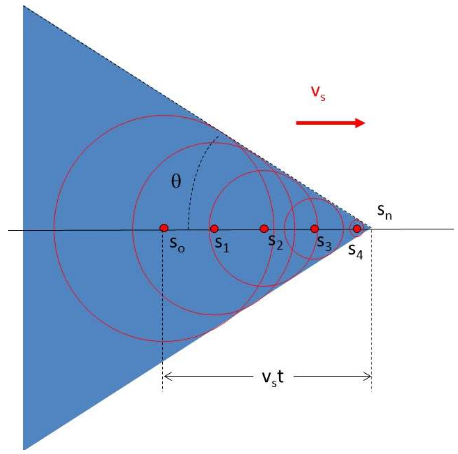
- If the source travels faster than the wave speed in the medium you get shock waves
- Actual waves can be viewed as sums of sinusoidal waves

11.1 Shock Waves

What happens when the speed of the source is greater than the wave speed?

Remember that the wave speed depends only on the medium. Let's call the crests of a wave the *wave front*. In the picture below, a point source is generating a wave and the red lines are the wave fronts.

When $v_e = v_{sound}$ the waves begin to pile up. If we allow $v_e > v_{sound}$ then the wave fronts are no longer generated within each other.



The leading edge of the wave fronts “build up” to form a cone shape. We recognize this as a superposition of the waves. This is constructive interference. The half angle of this cone is called the *Mach angle*

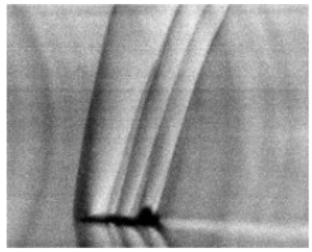
$$\sin\theta = \frac{v_{sound}t}{v_e t} = \frac{v_{sound}}{v_e} \tag{11.1}$$

This ration v_{sound}/v_e is called the Mach number and the conical wave front is called a shock wave. We see them often in water

Boat wakes as a Doppler cone. Image courtacy US Navy. Image is in the Public Domain.

Dopler Movie

and hear them when jet aircraft go supersonic. In the next figure we can see a picture of a T-38 breaking the sound barrier. You can see the Mach cones, but notice that there are several! Remember that a disturbance creates a wave. There are disturbances created by the nose of the plane, the rudder, and the wings, and perhaps the cockpit in this Schlieren photograph.



Dr. Leonard Weinstein's Schlieren photograph of a T-38 Talon at Mach 1.1, altitude 13,700 feet, taken at NASA Langley Research Center, Wallops in 1993. Image Courtacy NASA, image is in the Public Domain.

Superposition is the basis of making music and we will see how this works in our next lecture.

11.2 Non Sinusoidal Waves

You have probably wondered if all waves are sinusoidal. Can the universe really be described by such simple mathematics? The answer is both no, and yes. There are non-sinusoidal waves, in fact, most waves are not sinusoidal. But it turns out that we can use a very clever mathematical trick to make any shape wave out of a superposition of many sinusoidal waves. So our mathematics for sinusoidal waves turns out to be quite general.

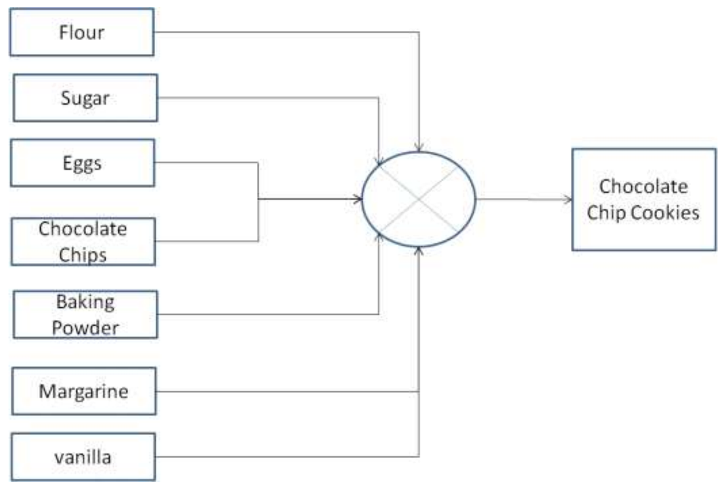
11.2.1 Music and Non-sinusoidal waves

Let's take the example of music.

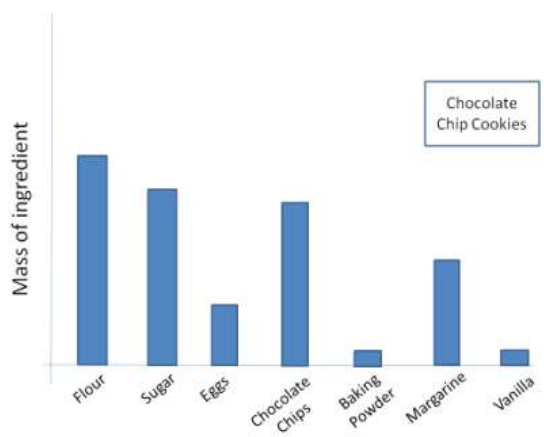
From our example of standing waves on strings, we know that a string can support a series of standing waves with discrete frequencies–the harmonic series. We have also discussed that usually we excite more than one standing wave at a time. The fundamental mode tends to give us the pitch we hear, but what are the other standing waves for?

To understand, lets take an analogy. Making cookies and cakes.

Here is the beginning of a recipe for cookies.

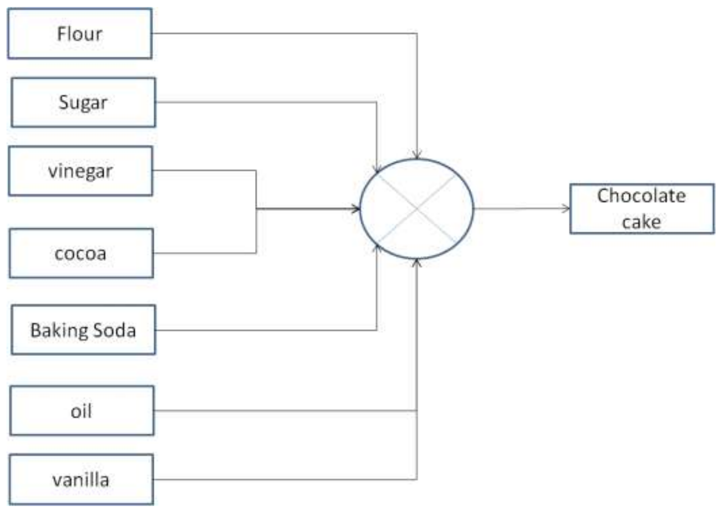


The recipe is a list of ingredients, and a symbolic instruction to mix and bake. The product is chocolate chip cookies. Of course we need more information. We need to know now much of each ingredient to use.



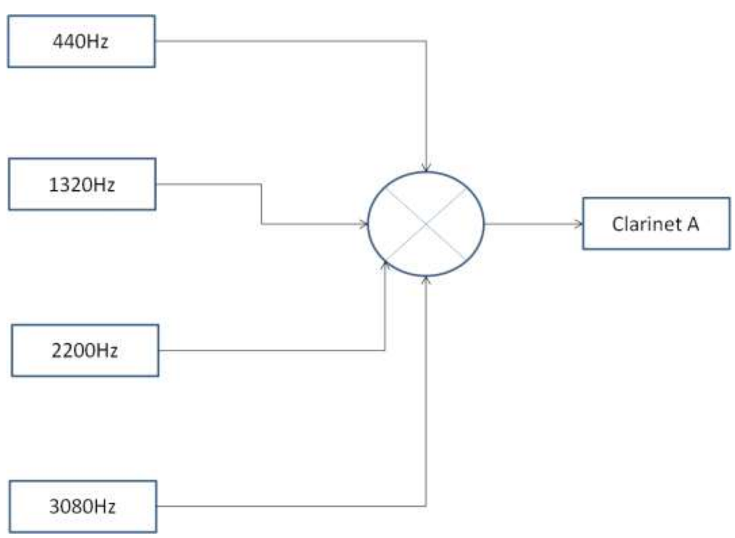
This graph gives us the amount of each ingredient by mass.

Now suppose we want chocolate cake.

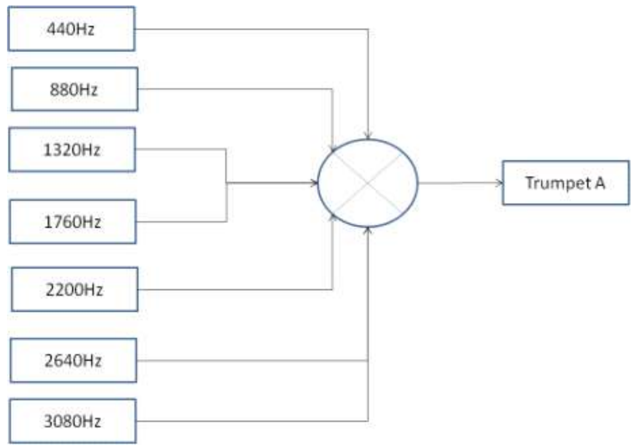


The predominant taste in each of these foods is chocolate. But chocolate cake and chocolate chip cookies don't taste exactly the same. We can easily see that the differences in the other ingredients make the difference between the "cookie" taste and the "cake" taste that goes along with the "chocolate" taste that predominates.

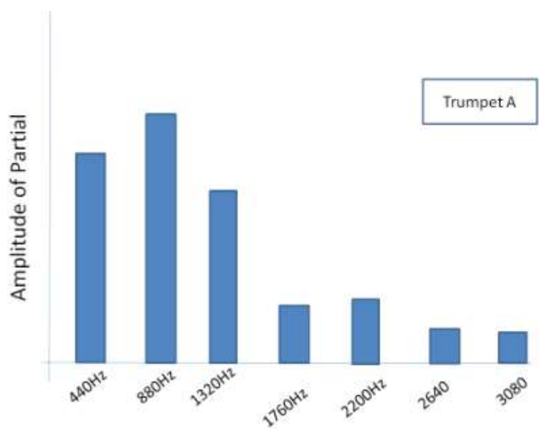
The sound waves produced by musical instruments work in a similar way. Here is a recipe for an "A" note from a clarinet.



and here is one for a trumpet playing the same "A" note.

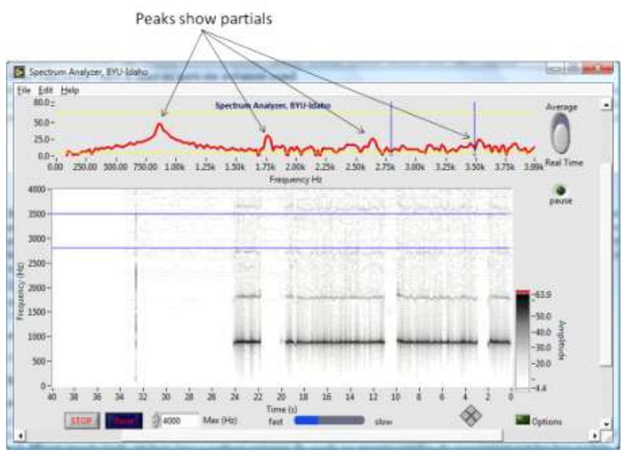


A trumpet sounds different than a clarinet, and now we see why. There are more harmonics involved with the trumpet sound than the clarinet sound. These extra standing waves make up the "brassiness" of the trumpet sound. As with our baking example, we need to know how much of each standing wave we have. Each will have a different amplitude. For our trumpet, we might get amplitudes as shown.

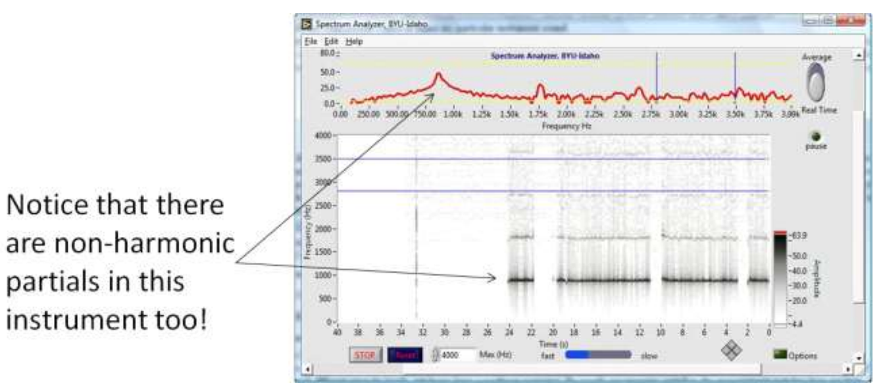


Note that the second harmonic has a larger amplitude, but we still hear the "A" as at 440 Hz. A fugal horn would still sound brassy, but would have a different mix of harmonics.

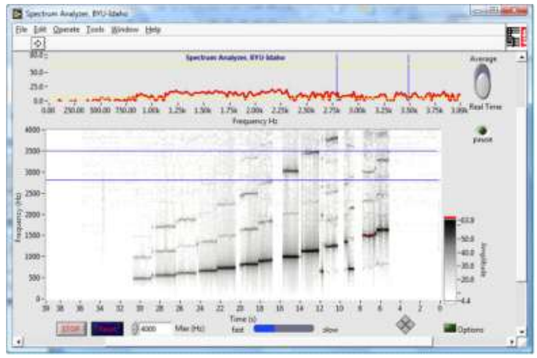
We have a tool that you can download to your PC to detect the mix of harmonics of musical instruments, or mechanical systems. In music, the different harmonics are called *partials* because they make up part of the sound. A graph that shows which harmonics are involved is called a *spectrum*. The next figure is the spectrum of a six holed bamboo flute. Note that there are several harmonics involved.



Note that our graph has two parts. One is the instantaneous spectrum, and one is the spectrum time history.

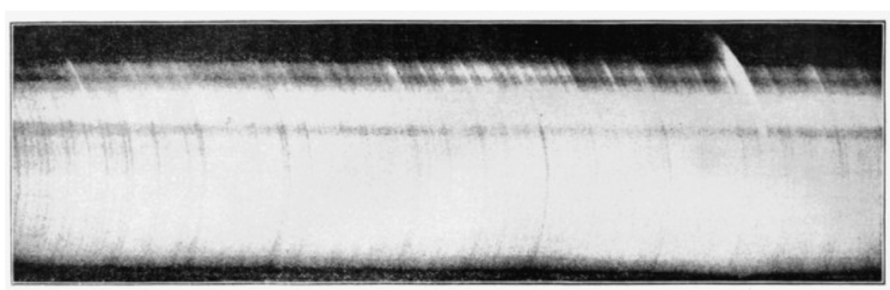


By observing the time history, we can see changes in the spectrum. We can also see that we don't have pure harmonics. The graph shows some response off the specific harmonic frequencies. This six holed flute is very "breathy" giving a lot of wind noise along with the notes, and we see this in the spectrum. In the next picture, I played a scale on the flute.



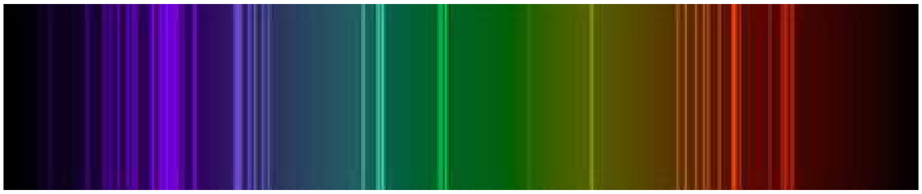
The instantaneous spectrum is not active in this figure (since it can't show more than one note at a time) but in the time history we see that as the fundamental frequency changes by shorting the length of the flute (uncovering holes), we see that every partial also goes up in frequency. The flute still has the characteristic spectrum of a flute, but shifted to new frequencies. We can use this fact to identify things by their vibration spectrum. In fact, that is how you recognize voices and musics within your auditory system!

The technique of taking apart a wave into its components is very powerful. With light waves, the spectrum is an indication of the chemical composition of the emitter. For example, the spectrum of the sun looks something like this



Solar coronal spectrum taken during a solar eclipse. The successive curved lines are each different wavelengths, and the dark lines are wavelengths that are absorbed. The pattern of absorbed wavelengths allows a chemical analysis of the corona. (Image in the Public Domain, orignally published in Bailey, Solon, L, Popular Science Monthly, Vol 60, Nov. 1919, pp 244)

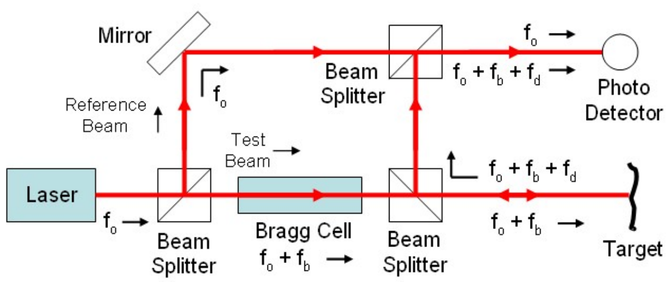
The lines in this graph show the amplitude of each harmonic component of the light. Darker lines have larger amplitudes. The harmonics come from the excitation of electrons in their orbitals. Each orbital is a different energy state, and when the electrons jump from orbital to orbital, they produce specific wave frequencies. By observing the mix of dark lines in pervious figure, and comparing to laboratory measurements from each element (see next figure) we can find the composition of the source. This figure shows the emission spectrum for Calcium. because it is an emission spectrum the lines are bright instead of dark. We can even see the color of each line!



Emission spectrum of Calcium (Image in the Public Domain, courtesy NASA)

11.2.2 Vibrometry

Just like each atom has a specific spectrum, and each instrument, each engine, machine, or anything that vibrates has a spectrum. We can use this to monitor the health of machinery, or even to identify a piece of equipment. Laser or acoustic vibrometers are available commercially.



Laser Vibrometer Schematic (Public Domain Image from Laderaranch: http://commons.wikimedia.org/wiki/File:LDV_Schematic.png)

They provide a way to monitor equipment in places where it would be dangerous or even impossible to send a person. The equipment also does not need to be shut down, a great benefit for factories that are never shut down, or for a satellite system that cannot be reached by anyone.

Fourier Series: Mathematics of Non-Sinusoidal Waves

We should take a quick look at the mathematics of non-sinusoidal waves.

Let' start with a superposition of many sinusoidal waves. The math looks like this

$$y(t) = \sum_n \left(A_n \sin(2\pi f_n t) + B_n \cos(2\pi f_n t)\right)$$

where A_n and B_n are a series of coefficients and f_n are the harmonic series of frequencies.

Example: Fourier representation of a square wave.

For example, we could represent a function $f(x)$ with the following series

$$\begin{aligned} f(x) &= C_o + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) && (11.2)\\ &\quad + C_2 \cos\left(\frac{2\pi}{\frac{\lambda}{2}}x + \varepsilon_2\right) && (11.3)\\ &\quad + C_3 \cos\left(\frac{2\pi}{\frac{\lambda}{3}}x + \varepsilon_3\right) && (11.4)\\ &\quad + \dots && (11.5)\\ &\quad + C_n \cos\left(\frac{2\pi}{\frac{\lambda}{n}}x + \varepsilon_n\right) && (11.6)\\ &\quad + \dots && (11.7) \end{aligned}$$

where we will let $\varepsilon_i = \omega_i t + \phi_i$

The $C's$ are just coefficients that tell us the amplitude of the individual cosine waves. The more terms in the series we take, the better the approximation we will have, with the series exactly matching $f(x)$ when the number of terms, $N \to \infty$.

Usually we rewrite the terms of the series as

$$C_m \cos(mkx + \varepsilon_m) = A_m \cos(mkx) + B_m \sin(mkx) \tag{11.8}$$

where k is the wavenumber

$$k = \frac{2\pi}{\lambda} \tag{11.9}$$

and λ is the wavelength of the complicated but still periodic function $f(x)$. Then we identify

$$\begin{aligned} A_m &= C_m \cos(\varepsilon_m) && (11.10)\\ B_m &= -C_m \sin(\varepsilon_m) && (11.11) \end{aligned}$$

then

$$f(x) = \frac{A_o}{2} + \sum_{m-1}^{\infty} A_m \cos(mkx) + \sum_{m-1}^{\infty} B_m \sin(mkx) \tag{11.12}$$

where we separated out the $A_o/2$ term because it mikes things nicer later.

Fourier Analysis

The process of finding the coefficients of the series is called *Fourier analysis.* We start by integrating equation (11.12)

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx + \int_0^\lambda \sum_{m-1}^{\infty} A_m \cos(mkx)\,dx + \int_0^\lambda \sum_{m-1}^{\infty} B_m \sin(mkx)\,dx \tag{11.13}$$

We can see immediately that all the sine and cosine terms integrate to zero (we integrated over a wavelength) so

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx = \frac{A_o}{2}\lambda \tag{11.14}$$

We solve this for A_o

$$A_o = \frac{2}{\lambda}\int_0^\lambda f(x)\,dx \tag{11.15}$$

To find the rest of the coefficients we need to remind ourselves of the orthogonality of sinusoidal functions

$$\int_0^\lambda \sin(akx)\cos(bkx)\,dx = 0 \tag{11.16}$$

$$\int_0^\lambda \cos(akx)\cos(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{11.17}$$

$$\int_0^\lambda \sin(akx)\sin(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{11.18}$$

where δ_{ab} is the Kronecker delta.

To find the coefficients, then, we multiply both sides of equation (11.12) by $\cos(lkx)$ where l is a positive integer. Then we integrate over one wavelength.

$$\int_0^\lambda f(x)\cos(lkx)\,dx = \int_0^\lambda \frac{A_o}{2}\cos(lkx)\,dx \tag{11.19}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} A_m\cos(mkx)\cos(lkx)\,dx \tag{11.20}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} B_m\sin(mkx)\cos(lkx)\,dx \tag{11.21}$$

which gives

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \int_0^\lambda A_m\cos(mkx)\cos(mkx)\,dx \tag{11.22}$$

that is, only the term with two cosine functions where $l = m$ will be non zero. So

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \frac{\lambda}{2}A_m \tag{11.23}$$

solving for A_m we have

$$A_m = \frac{2}{\lambda}\int_0^\lambda f(x)\cos(mkx)\,dx \tag{11.24}$$

We can perform the same steps to find B_m only we use $\sin(lkx)$ as the multiplier. Then we find

$$B_m = \frac{2}{\lambda}\int_0^\lambda f(x)\sin(mkx)\,dx \tag{11.25}$$

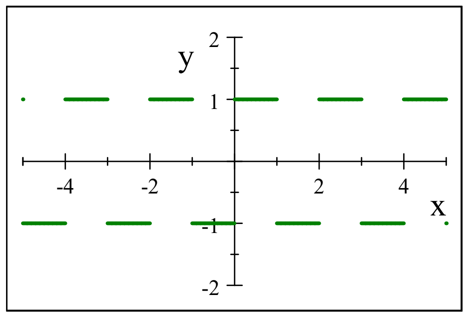
11.2.3 Square wave

Let's find the series for a square wave using our Fourier analysis technique.

Let's take

$$\lambda = 2 \tag{11.26}$$

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \frac{\lambda}{2} \\ -1 & \text{if} \quad \frac{\lambda}{2} < x < \lambda \end{cases} \tag{11.27}$$



since $f(x)\, is$ odd, $A_m = 0$ for all m. We have

$$B_m = \frac{2}{\lambda} \int_0^{\frac{\lambda}{2}} (1) \sin(mkx)\, dx + \frac{2}{\lambda} \int_{\frac{\lambda}{2}}^{\lambda} (-1) \sin(mkx)\, dx \tag{11.28}$$

so

$$B_m = \frac{1}{m\pi} \left(-\cos(mkx)\right|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi} \left(\cos(mkx)\right|_{\frac{\lambda}{2}}^{\lambda} \tag{11.29}$$

Which is

$$B_m = \frac{1}{m\pi} \left(1 \cos\left(m\frac{2\pi}{\lambda}x\right)\right|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi} \left(\cos\left(m\frac{2\pi}{\lambda}x\right)\right|_{\frac{\lambda}{2}}^{\lambda} \tag{11.30}$$

so

$$B_m = \frac{1}{m\pi}\left(\left(-\cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right) + \cos\left(m\frac{2\pi}{\lambda}(0)\right)\right) \tag{11.31}$$

$$+\frac{1}{m\pi}\left(\left(\cos\left(m\frac{2\pi}{\lambda}\lambda\right) - \cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right)\right) \tag{11.32}$$

which is

$$B_m = \frac{2}{m\pi}\left(1 - \cos(m\pi)\right) \tag{11.33}$$

Our series is then just

$$f(x) = \sum_{m-1}^{\infty} \frac{2}{m\pi}\left(1 - \cos(m\pi)\right) \sin(mkx) \tag{11.34}$$

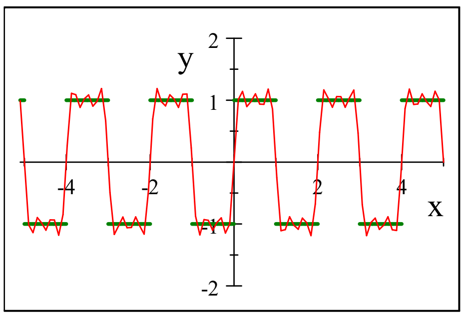
and we can write a few terms

Term	
1	$\frac{4}{\pi}\sin(kx)$
2	0
3	$\frac{4}{3\pi}\sin(3kx)$
4	0
5	$\frac{4}{5\pi}\sin(5kx)$

(11.35)

then the partial sum up to $m = 5$ looks like

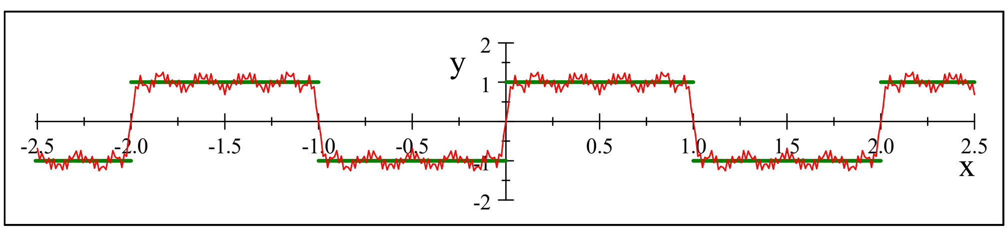
$$f(x) = \frac{4}{\pi}\sin(kx) + \frac{4}{3\pi}\sin(3kx) + \frac{4}{5\pi}\sin(5kx) \tag{11.36}$$



If we take many terms,

$$\begin{aligned} f(x) &= \frac{4}{\pi}\sin(kx) + \frac{4}{3\pi}\sin(3kx) + \frac{4}{5\pi}\sin(5kx) + \frac{4}{7\pi}\sin(7kx) + \frac{4}{9\pi}\sin(95kx) \\ &+ \frac{4}{11\pi}\sin(11kx) + \frac{4}{13\pi}\sin(13kx) + \frac{4}{15\pi}\sin(15kx) + \frac{4}{17\pi}\sin(17kx) + \frac{4}{19\pi}\sin(19kx) \end{aligned} \tag{11.37}$$

We see the function get closer and closer to a square wave.



In the limit of infinitely many waves, the match would be perfect. But we don't usually need an infinite number of terms. we can pick the part of the spectrum that best represents the phenomena we desire to observe. For example, oil based compounds all have specific spectral signatures in the wavelength range between $3 - 5$ micrometers. If you wish to tell the difference between gasoline and crude oil, you can restrict your study to these wavelengths alone.