# Chapter 10

# 10 Beats and Doppler Shift 1.17.6, 1.17.7

## Fundamental Concepts

- If we mix waves of different frequencies we get a periodic change in loudness
- If we move the wave source or detector the relative motion causes a change in detected frequency

## 10.1 Beats

Beat Demo

Up till now we only superposed waves that had the same frequency. But what happens if we take waves with different frequencies? Let's take the case where $\phi_2 = \phi_1 = \phi_o$, $k_2 = k_1 = k$ and let's let the waves mix at the same location $x_2 = x_1$ so

$$y_1 = y_{\max} \sin(kx - \omega_1 t + \phi_o)$$
$$y_2 = y_{\max} \sin(kx - \omega_2 t + \phi_o)$$

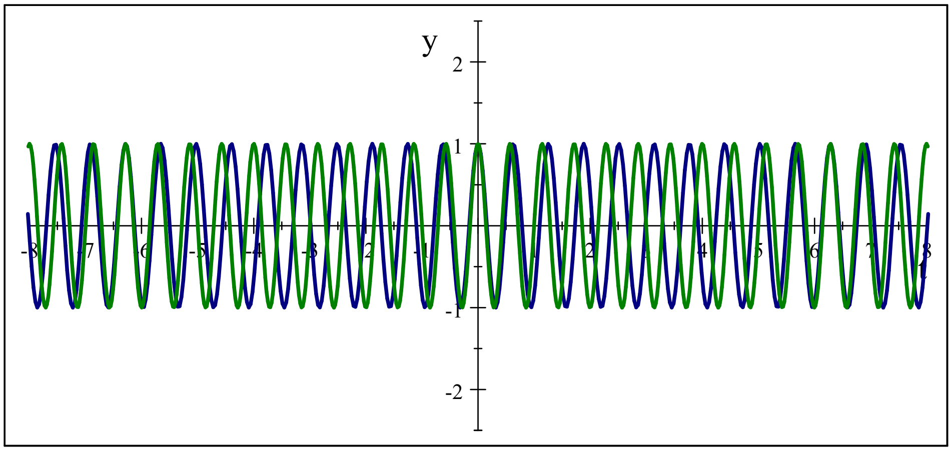
$$\begin{aligned}\Delta\phi &= (kx - \omega_1 t + \phi_o) - (kx - \omega_1 t + \phi_o) \\ &= (\omega_1 - \omega_2)\, t\end{aligned}$$

We can use our criteria for constructive and destructive interference, but before going on let's put this back into the total amplitude function for two wave mixing.

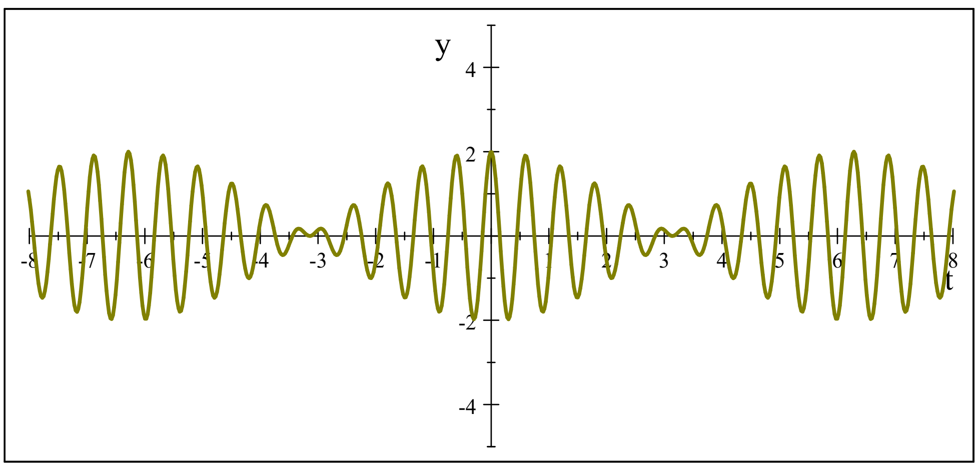
$$\begin{aligned}A &= 2y_{\max} \cos\left(\frac{1}{2}\Delta\phi\right) \\ &= 2y_{\max} \cos\left(\frac{1}{2}(\omega_1 - \omega_2)\, t\right)\end{aligned}$$

and we can see that our amplitude will change in time. Sometimes it will be constructive interference and sometimes destructive interference and often in between.

We can plot both waves on the same graph and see that this will happen.




Notice that there are places where the waves are in phase, and places where they are not. The superposition looks like this




where there is constructive interference, the resulting wave amplitude is large, where there is destructive interference, the resulting amplitude is zero. We get a traveling wave who's amplitude varies. We can find the amplitude function algebraically.

We can write out the entire resultant wave in our usually way. Our two waves are

$$y_1 = y_{\max} \sin(kx - \omega_1 t + \phi_o)$$

$$y_2 = y_{\max} \sin(kx - \omega_2 t + \phi_o)$$

and the resultant

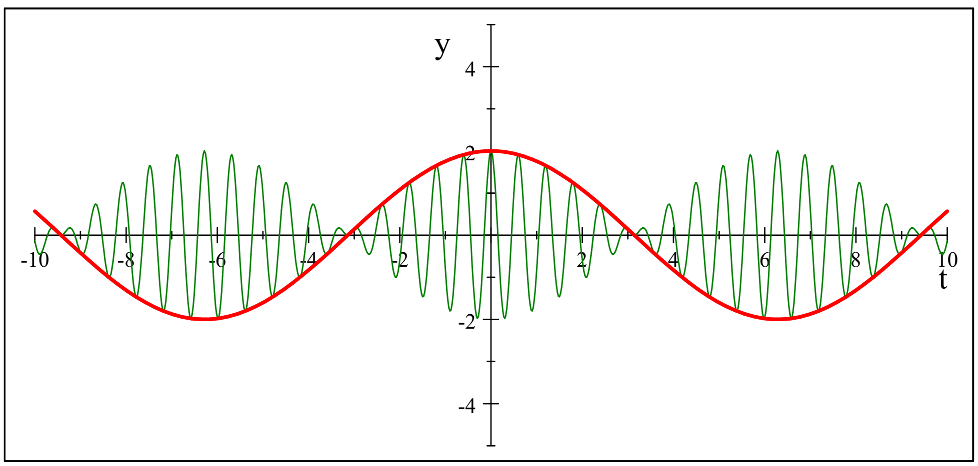
$$
\begin{aligned}
& y_r 2y_{\max}\cos\left(\frac{(kx-\omega_2 t+\phi_o)-(kx-\omega_1 t+\phi_o)}{2}\right)\sin\left(\frac{kx-\omega_2 t+\phi_o+kx-\omega_1 t+\phi_o}{2}\right) \\
= & \; 2y_{\max}\cos\left(\frac{kx-2\pi f_2 t-(kx-2\pi f_1 t)}{2}\right)\sin\left(\frac{kx-2\pi f_2 t+kx-2\pi f_1 t+2\phi_o}{2}\right) \\
= & \; 2y_{\max}\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\sin\left(kx-2\pi\frac{f_1+f_2}{2}t+\phi_o\right) \\
= & \left[2y_{\max}\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\right]\sin\left(kx-2\pi\frac{f_1+f_2}{2}t+\phi_o\right)
\end{aligned}
$$

The sine part is a wave, it is a function of position and time. We see that it has a frequency that is the average of $f_1$ and $f_2$. This is the frequency we hear. But we have another complicated amplitude term, and this time it is a function of time just as we suspected. The amplitude has its own frequency that is half the difference of $f_1$ and $f_2$.

$$A_{\text{resultant}} = 2A\cos\left(2\pi\frac{f_1-f_2}{2}t\right)$$

So the sound amplitude will vary in time for a given position in the medium.

The situation is odder still. We have a cosine function, but it is really an envelope for the higher frequency motion of the air particles. The air molecules move back and forth for both the crest and the trough of the envelope function.




So we will hear two maxima for every period! This frequency with which we hear the sound get loud at a given location as the wave goes by is called the *beat frequency*. The red envelope (solid heavy line in the last figure) has a frequency of

$$f_A = \frac{f_1-f_2}{2}$$

but it is just the envelope. We can see that the green (thin line) wave will push and pull air molecules, and therefore our ear drums, with maximum loudness at twice this frequency. So our beat frequency is

$$f_{beat} = |f_1 - f_2|$$

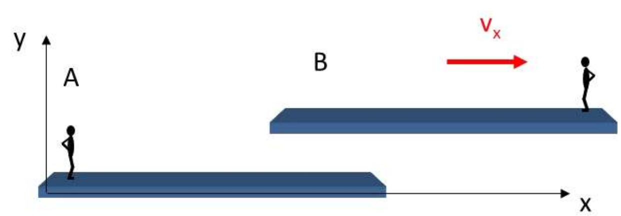
Any time we mix waves of different frequencies we get beating. Often the beat frequency is very fast, and our hearing system can't track the beats, so we don't hear them. And if we mixed more than two waves the beats might not come at perfectly regular intervals. The mixing of waves can become quite complicated. Yet even a barbershop quartet is a mixing of at least four waves. So complicated superpositions are common. In the next lecture we will try to see how we could take on these complicated combined waves.

## 10.2 Doppler Effect

Doppler Ball Demo

Let's start by considering an inertial reference frame (remember this from Dynamics/PH121?)

Suppose we pick two reference frames, one traveling with a velocity $v_r$ with respect to the other. Let's also place them far away from any other object.



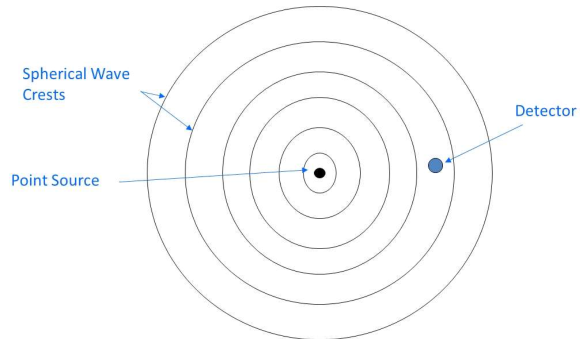

Person $A$ sees himself as stationary and sees person $B$ traveling with velocity $v_x$. Person $B$ sees himself as stationary, and person $A$ traveling with velocity $-v_x$.

In looking at this situation it is the *relative* speed $v_x$ that we must consider. We recall that we could write the speed of guy $b$ as seen from platform $A$ as

$$v_{bA} = v_{bB} + v_{BA}$$

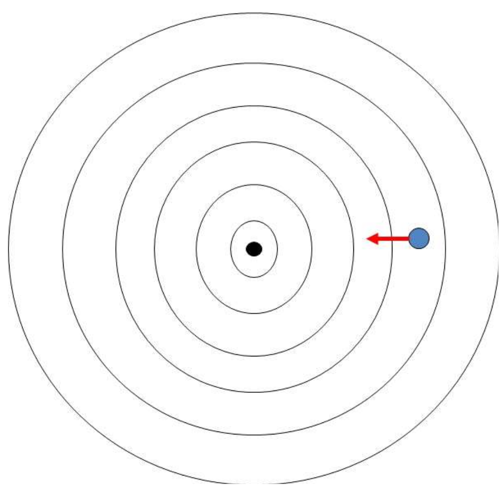
where $v_{bB}$ is guy b's speed seen from platform $B$ and $v_{BA}$ is the speed of platform $B$ as seen from platform $A$. In this case $v_{BA} = v_x$. Now suppose we have a wave generator (a point source) creating spherical waves. Let the point source be at rest. We will call this point source an emitter and use a subscript $e$ for it.

BYU Demo



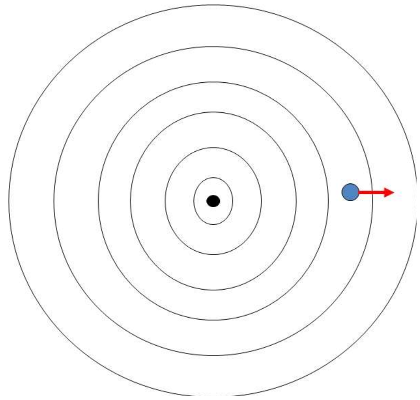

Let's also assume a detector. If the detector is stationary with respect to the emitter so the detector is in the emitter's reference frame, it sees a frequency of the wave in the emitter frame, $f_{we}$. But lets have the detector move relative to the emitter.

Move George



Remember, that the frequency is the number of crests that pass by a given point in a unit time. Does the moving detector see the same number of crests per unit time as when it was stationary?

No, the frequency appears to be higher! How about if we let the detector move the other way?



Again the frequency seen by the detector is different, but this time lower.

We can quantify this change. Take our usual variables $f_{we}$, $\lambda_{we}$, and the velocity of sound $v_{sound} = v_{we}$ for the wave speed of the sound wave because the air itself is not moving with respect to the emitter and the wave travels through the air. When the detector moves toward the source, it sees the wave velocity as

$$v_{wd} = v_{we} + v_{de} \tag{10.1}$$

where $v_{wd}$ is the velocity of the wave in the detector frame, and $v_{de}$ is the velocity of the detector in the emitter frame. The wavelength will not be changed, so

$$v_{we} = \lambda_{we} f_{we}$$

can just be written as

$$v_{we} = \lambda f_{we}$$

which tells us the frequency must change.

$$f_{wd} = \frac{v_{wd}}{\lambda} = \frac{v_{we} + v_{de}}{\lambda}$$

We can eliminate $\lambda$ from this expression for the change in $f$ by using $v_{we} = \lambda f_{we}$ again, this time solving for $\lambda$ we get

$$\lambda = \frac{v_{we}}{f_{we}}$$

and substitute this into our $f_{wd}$ equation

$$f_{wd} = \frac{v_{we} + v_{de}}{\frac{v_{we}}{f_{we}}}$$

or, after rearranging

$$f_{wd} = \frac{v_{we} + v_{de}}{v_{we}} f_{we} \qquad \text{detector moving toward the emitter} \tag{10.2}$$

or recognizing that $v_{we}$ is the speed of sound in the stationary frame of the emitter

$$f_{wd} = \frac{v_{sound} + v_{de}}{v_{sound}} f_{we} \qquad \text{detector moving toward the emitter} \tag{10.3}$$

emo

Now if the detector is going the other way

$$v_{wd} = v_{we} - v_{de}$$

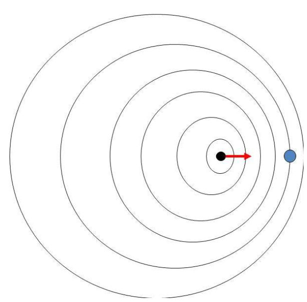
and the same reasoning gives

$$f_{wd} = \frac{v_{we} - v_{de}}{v_{we}} f_{we} \qquad \text{detector moving away from the emitter} \tag{10.4}$$

or

$$f_{wd} = \frac{v_{sound} - v_{de}}{v_{sound}} f_{we} \qquad \text{detector moving away from the emitter} \tag{10.5}$$

From our thinking about the motion of two inertial reference frames, we should expect a similar situation if the detector is stationary and the source moves. In this case the detector will see a different wavelength.



In fact, if we measure the distance between the crests we must account for the fact that the source moved by an amount

$$\Delta x_{ed} = v_{ed} T_{we} = \frac{v_{ed}}{f_{we}}$$

during one period of oscillation of the emitter before making the next crest. We can see that in this case, $v_{ed} = v_{sound}$ because this time the air is not moving with respect to the detector, so they are in the same reference frame. Then the wavelength will be shorter by this amount! That is, if we take the wavelength that we would get if the emitter were at rest, and subtract $\Delta x_{ed}$ we should have our new wavelength at the detector. Let's start by finding the wavelength we expect if the emitter were at rest.

$$\lambda_{we,rest} = \frac{v_{we}}{f_{we}} = \frac{v_{sound}}{f_{we}}$$

This is because if the emitter were at rest, then $v_{we} = v_{sound.}$ But in our case the emitter is moving, so we must subtract $\Delta x_{ed}$ from this

$$\lambda_{wd} = \lambda_{we,rest} - \Delta x_{ed}$$

Then the wavelength we would see at the detector would be

$$\lambda_{wd} = \lambda_{we,rest} - \frac{v_{ed}}{f_{we}}$$

Using

$$\lambda_{wd} = \frac{v_{wd}}{f_{wd}}$$

once more, we can write the frequency in the detector frame as

$$\begin{aligned} f_{wd} &= \frac{v_{wd}}{\lambda_{wd}} = \frac{v_{wd}}{\lambda_{we,rest} - \frac{v_{ed}}{f_{we}}} \\ &= \frac{v_{wd}}{\frac{v_{sound}}{f_{we}} - \frac{v_{ed}}{f_{we}}} \\ & \frac{v_{wd}}{v_{sound} - v_{ed}} f_{we} \end{aligned}$$

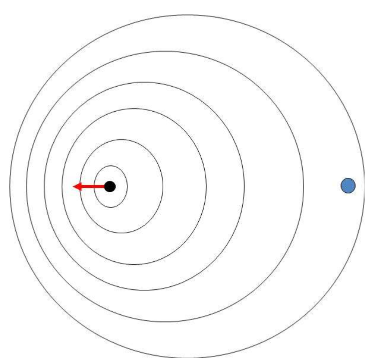
or, again with a little rearranging

$$f_{wd} = \frac{v_{wd}}{v_{sound} - v_{ed}} f_{we} \qquad \text{emitter moving toward detector} \tag{10.6}$$

Now recall that it is the detector that is stationary this time so $v_{sound} = v_{wd}$

$$f_{wd} = \frac{v_{sound}}{v_{sound} - v_{ed}} f_{we} \qquad \text{emitter moving toward detector} \tag{10.7}$$

When the source is moving away from the detector,



we expect the wavelength to be larger. The same reasoning gives

$$f' = \frac{v_{wd}}{v_{wd} + v_{ed}} f_{we} \qquad \textbf{emitter moving away from detector} \qquad (10.8)$$

**or**

$$f' = \frac{v_{sound}}{v_{sound} + v_{ed}} f_{we} \qquad \textbf{emitter moving away from detector} \qquad (10.9)$$

### Combined Doppler Equation

We can combine these formulae to make one expression, but to do so we need to remember what $v_{de}$ and $v_{ed}$ mean. The first was the speed of the detector when the emitter was not moving. The second was the speed of the emitter when the detector was not moving. But we are experienced with relative motion. We should ask, "not moving with respect to what?" Let's envision a reference frame that is not tied to either the emitter or detector. In this reference frame $v_{dR}$ is the speed of the detector, and $v_{eR}$ is the speed of the emitter. In this $R$ reference frame our first Doppler equation for a moving detector with a stationary emitter might be written as

$$f_{wd} = \frac{v_{sound} \pm v_{dR}}{v_{sound}} f_{we} \qquad \text{detector moving emitter stationary in R frame} \qquad (10.10)$$

and our second equation for a moving emitter with a stationary detector might be written as

$$f' = \frac{v_{wd}}{v_{wd} \mp v_{eR}} f_{we} \qquad \text{emitter moving detector stationary in R frame} \qquad (10.11)$$

We could, of course have both the detector and emitter moving in the $R$ frame. This would combine both of our previous scenarios

$$f_{wd} = \frac{v_{sound} \pm v_{dR}}{v_{sound} \mp v_{eR}} f_{we} \qquad (10.12)$$

where we use the top sign for the speed when the mover is going toward the non-mover.

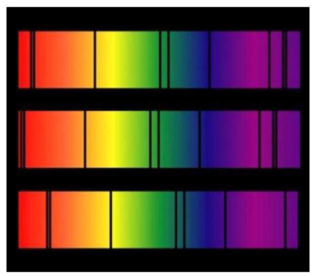
With this view, that we are neither in the $e$ frame nor the $d$ frame but in a separate $R$ frame, we could even drop the $R$ subscript without too much confusion

$$f_{wd} = \frac{v_{sound} \pm v_d}{v_{sound} \mp v_e} f_{we} \qquad (10.13)$$

where now $v_d$ is the speed of the detector in the reference frame of the observer, and $v_e$ is the speed of the emitter in the reference frame of the observer. The quantity $f_{we}$ is still the frequency of the wave as seen by the emitter and $f_{wd}$ is still the frequency of the wave recorded by the detector.

### 10.2.1 Doppler effect in light

Light is also a wave, and so we would expect a Doppler shift in light. Indeed we do see a Doppler shift when we look at moving objects. Here is an optical spectrum of the Sun on the top and a spectrum of a similar start moving away from us in the middle. The final spectrum is for a star moving toward us.

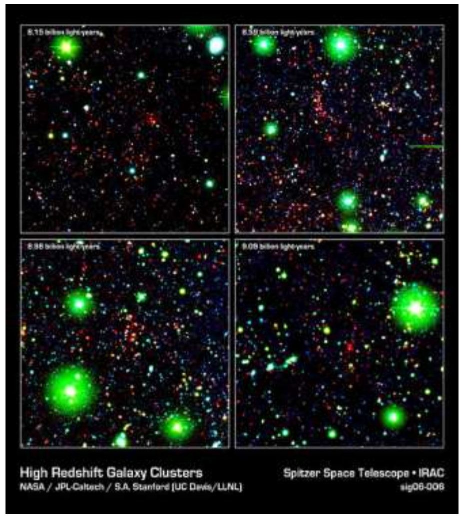


Top: Normal 'dark' spectral line positions at rest. Middle: Source moving away from observer. Bottom: Source moving towards observer. (Public domain image courtesy NASA: http://www.jwst.nasa.gov/education/7Page45.pdf)

Note that the wavelength of the lines is shifted toward the red part of the spectrum when the glowing object moves away from us. This is equivalent to lowering of the frequency of a truck engine noise as it goes away from us. The larger wavelengths indicate a lower frequency of light because

$$f = \frac{c}{\lambda}$$

This gives us a way to determine if distant stars and galaxies are moving toward or away from us. We look for the chemical signature pattern of lines, then see whether they are shifted to the red (moving away from us) or blue (moving toward us) compared to the position in their spectrum of the Sun. This photo is of some of the most distant galaxies that are moving very fast away from us. Their redshift is very large.




High Redshift Galaxy Cluster shown here in false color from the Spitzer Space Telescope. (Public domain image courtesy NASA/JPL-Caltech/S.A. Stanford (UC Davis/LLNL)

Deriving the Doppler equation for light is more tricky because the speed of light is constant and the same in every reference frame. We really tackle this in our PH279 class. So I will just quote the result here.

$$\lambda_{-} = \lambda_o \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \qquad \text{receding source} \tag{10.14}$$

$$\lambda_{-} = \lambda_o \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad \text{Approaching source} \tag{10.15}$$