Chapter 6

6 Power and Superposition 6 1.16.4 1.16.5

Fundamental Concepts

- Because waves are three dimensional, describing the power or energy delivered per time of the wave is not enough, We describe how spatially spread out that power is. We call this spread power *intensity*
- Waves invert on reflection from "fixed ends" and don't invert from "free ends"
- If we make more than one wave in a medium, the waves "add up" or *superimpose.*
- When two waves interfere we get a new wave with a more complicated amplitude

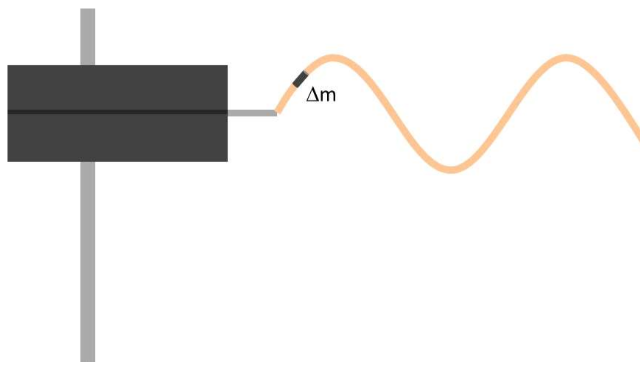
6.1 Energy and Power in waves

We have said that waves don't transport mass, they transport energy. We used the example of sound waves. If the air molecules move to another location , we call it wind. But if the air molecules just vibrate around an equilibrium position, we called that sound waves. The molecules don't end up somewhere else. But something does move.

Maybe a better example is a water wave in a swimming pool. Suppose someone jumps into a calm pool. The person is a disturbance and the water is a medium. We will get waves. But the water doesn't end up bunched together on the other side of the pool. The water is still everywhere in the pool, even around the person that jumped in. So what did move away from the person? It is energy.

For sinusoidal waves, the amount of energy in the wave is related to both it's amplitude and it's frequency. And the specifics of that relationship depend on the type of wave. We will study both sound and light waves later in our course and we will find that relationship between energy, amplitude, and frequency are different for the two different kinds of waves.

Let's look at a specific case of a rope attached to a mechanical oscillator. You can see the situation in the next figure. The oscillator (black thing) has a piece that goes up and down (silver thing sticking out to the right). The rope is attached to this silver oscillating piece. The oscillatory acts as a disturbance. A wave is formed in the rope.



Once again let's take a small part of our medium. Before we took a pice that was Δx long, but this time let's describe our piece of rope as a small bit of mass Δm. We already know about linear mass densities.

$$\mu = \frac{M}{L}$$

and since our rope is uniform, the linear mass density is the same for each part of the rope. So for our marked part

$$\mu = \frac{\Delta m}{\Delta x}$$

Then for this part of the rope medium there will be a kinetic energy

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}\Delta m v_y^2 \end{aligned}$$

using our linear mass density we can write

$$\Delta m = \mu \Delta x$$

so the kinetic energy is

$$K = \frac{1}{2}\left(\mu \Delta x\right) v_y^2$$

and from before we know that $v_y = -\omega A \cos(kx - \omega t + \phi_o)$ so our kinetic energy is

$$K = \frac{1}{2}(\mu \Delta x)\left(-\omega A \cos(kx - \omega t + \phi_o)\right)^2$$

we can clean this up a bit

$$K = \frac{1}{2}\mu \Delta x \omega^2 A^2 \cos^2(kx - \omega t + \phi_o)$$

Now let's simplify our situation by letting $\phi_o = 0$ and let's take a snap shot view with $t = 0$.

$$K = \frac{1}{2}\mu \Delta x \omega^2 A^2 \cos^2(kx)$$

Then, if we take only part of the energy associated with one very tiny part of our rope medium, our Δx becomes just dx. We don't have all the energy of the wave, just a small part of it. So we can write this as

$$dK = \frac{1}{2}\mu \omega^2 A^2 \cos^2(kx)\, dx$$

and we can integrate this up to find the energy of just one wavelength of the wave.

$$\int_0^{K_\lambda} dK = \int_0^{\lambda} \frac{1}{2}\mu \omega^2 A^2 \cos^2(kx)\, dx$$

A lot of the parts of our equation are constants, let's take them out front.

$$\int_0^{K_\lambda} dK = \frac{1}{2}\mu \omega^2 A^2 \int_0^{\lambda} \cos^2(kx)\, dx$$

The left hand side of this equation is just K_λ

$$K_\lambda = \frac{1}{2}\mu \omega^2 A^2 \int_0^{\lambda} \cos^2(kx)\, dx$$

but if you are like me, the right hand side is an integral that you have conveniently forgotten. But we can look up this integral in an integral table. My table gave the form

$$\int \left(\cos^2(ax)\right) dx = \frac{1}{2}x + \frac{1}{4a}\sin(2ax)$$

We have to match this to our equation. We see $a = k$ so

$$\begin{aligned}
K_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \left[\frac{1}{2}x + \frac{1}{4k}\sin(2kx)\right|_0^\lambda \\
&= \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda + \frac{1}{4k}\sin(2k\lambda)\right\} - \left\{\frac{1}{2}(0) + \frac{1}{4k}\sin(2k(0))\right\}\right] \\
&= \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda + \frac{1}{4\frac{2\pi}{\lambda}}\sin\left(2\frac{2\pi}{\lambda}\lambda\right)\right\} - \frac{1}{4k}\sin(2k(0))\right] \\
&= \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda + \frac{\lambda}{8\pi}\sin(4\pi)\right\}\right] \\
&= \frac{1}{4}\mu\omega^2 A^2 \lambda
\end{aligned}$$

This wavelength sized piece of tope will also have some potential energy because the fibers that make the rope are stretched to make the wave. They aren't really quite like little springs, but it's not to far wrong to model them as though they were. So let's take the spring potential energy

$$U = \frac{1}{2}k_s(y - y_e)^2$$

and let $y_e = 0$ for our piece of rope. We know that our piece of rope will go up and down in SHM. The natural frequency of our rope piece is

$$\omega = \sqrt{\frac{k_s}{m}}$$

We can solve this for the spring constant

$$k_s = m\omega^2$$

or for our little piece of the rope

$$k_s = \Delta m\omega^2$$

Let's put this into our potential energy equation

$$U = \frac{1}{2}\Delta m\omega^2(y)^2$$

and substitute $\Delta m = \mu\Delta x$ again

$$U = \frac{1}{2}\mu\Delta x\omega^2(y)^2$$

and of course, let our Δx get small so we have a small bit of potential energy from a small bit of rope.

$$dU = \frac{1}{2}\mu\omega^2(y)^2\,dx$$

and integrate this to find the potential energy in the stretchy bonds of one wavelength worth of rope.

$$\int_0^{U_\lambda} dU = \int_0^{\lambda} \frac{1}{2}\mu\omega^2 (y)^2 \, dx$$

We need to put in our equation for the position for our wave.

$$y(x,t) = A\sin(kx - \omega t + \phi_o)$$

but once again let's choose $\phi_o = 0$ and $t = 0$ so that

$$y(x,0) = A\sin(kx)$$

then

$$\int_0^{U_\lambda} dU = \int_0^{\lambda} \frac{1}{2}\mu\omega^2 (A\sin(kx))^2 \, dx$$

The left hand side is easy. And there are, once again, constants

$$U_\lambda = \frac{1}{2}\mu\omega^2 A^2 \int_0^{\lambda} (\sin(kx))^2 \, dx$$

and we have another integral I don't remember how to do. But I still have my table of integrals and my table tells me

$$\int \left(\sin^2(ax)\right) dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Which feels very familiar. Lets let $a = k$ again. Then

$$\begin{aligned} U_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \left[\frac{1}{2}x - \frac{1}{4k}\sin(2kx)\right|_0^\lambda \\ & \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda - \frac{1}{4k}\sin(2k\lambda)\right\} - \left\{\frac{1}{2}0 - \frac{1}{4k}\sin(2k(0))\right\}\right] \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda \end{aligned}$$

The total energy in the λ-sized rope piece would be

$$\begin{aligned} E_\lambda &= K_\lambda + U_\lambda \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda + \frac{1}{4}\mu\omega^2 A^2 \lambda \\ &= \frac{1}{2}\mu\omega^2 A^2 \lambda \end{aligned}$$

We know that energy is being transferred by the wave, whether it is a light or sound wave or any other mechanical wave. We should wonder, how fast is energy transferring. This can mean the difference between a warm ray of sin on a cool spring day and being burned by a laser beam. We will start by considering the rate of energy transfer, *power*. The concept of power should be familiar to us from PH121. We can find the power as the rate at which energy is transferred.

$$\mathcal{P} = \frac{\Delta E}{\Delta t}$$

Since we picked the amount of energy in one wavelength, and we know the time it takes for one wavelength of the wave to pass by is T. Then the power is

$$P = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T}$$

but remember

$$v = \frac{\lambda}{T}$$

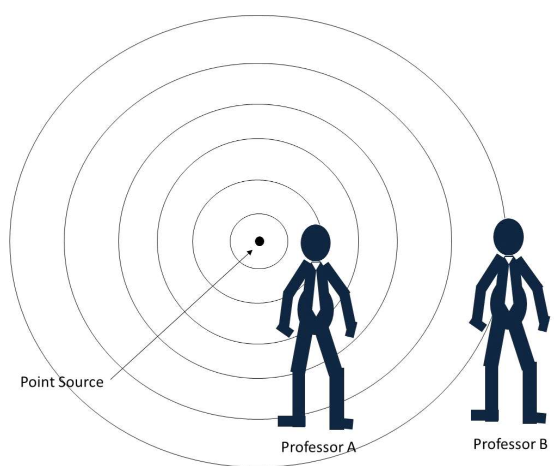
so we can write our power as

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

Power is important. Most detectors (like our ears and eyes) really detect the power delivered by a wave. And we can see that the power is proportional to the amplitude squared and the frequency squared for the wave. More on this as we study light and sound.

6.2 Power and Intensity

What we have done works fine for linear waves. But if we consider our spherical wave from a point source, we can see that this description isn't good enough. In the next figure we have two professors.



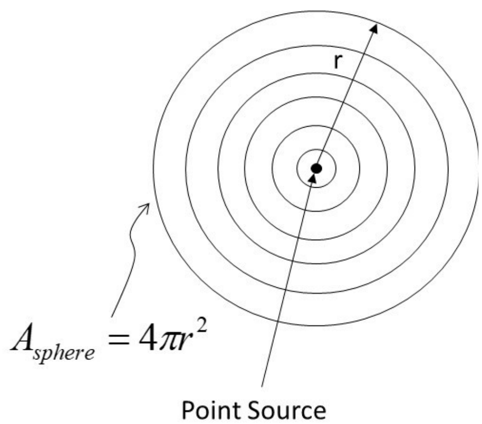
Thinking from our experience we would say that the sound will seem louder for professor A. This is because the energy in the sound wave is being spread over the surface of the wave, and that surface is getting bigger as the wave moves outward. The energy is more spread out by the time it gets to professor B.

To be able to describe how much energy we get from our wave we need define something new.

$$\mathcal{I} \equiv \frac{\mathcal{P}}{A} \tag{6.1}$$

that is, the power divided by an area. But what does it mean?

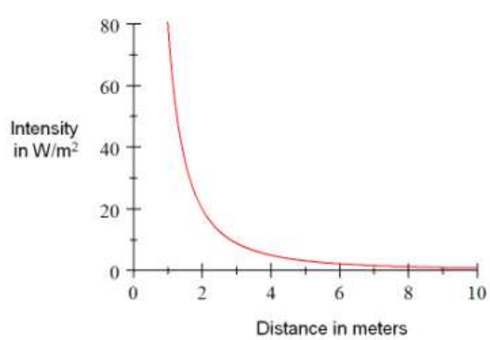
Consider a point source. This could be a loud speaker, or a buzzer, or a baby crying, etc.



it sends out waves in all directions. The wave crests will define a sphere around the points source (the figure shows a cross section but remember it is a wave from a point source, so we are really drawing concentric spheres like balloons inside of balloons.). Then form our point source

$$\mathcal{I} = \frac{\mathcal{P}}{4\pi r^2} \tag{6.2}$$

As the wave travels, its the power per unit area decreases with the square of the distance (think gravity) because the area is getting larger.



This quantity that tells us how spread out our power has become is called the *intensity* of the wave. Professor A would agree with us that the wave he heard was more intense than the wave heard by Professor B. That is because the wave was less spread out for Professor A.

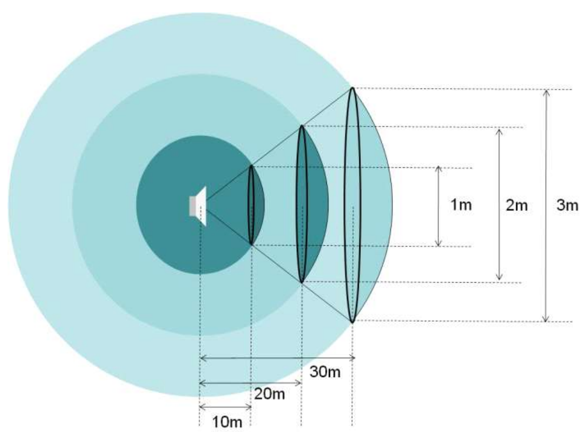
Suppose we cup our hand to our ear. What are we doing? We are increasing the area of our ear. Our ears work by transferring the energy of the sound wave to a mechanical-electro-chemical device that creates a nerve signal.[1] The more energy, the stronger the signal. If we are a distance r away from the source of the sound then the intensity is

$$\mathcal{I} = \frac{\mathcal{P}_{source}}{A_{wave}}$$

But we are collecting the sound wave with another area, the area of our hand. The power received is

$$\begin{aligned} \mathcal{P}_{received} &= \mathcal{I} A_{hand} \\ &= \frac{A_{hand}}{A_{wave}} \mathcal{P}_{source} \end{aligned}$$

and we can see that, indeed, the larger the hand, the more power, and therefore more energy we collect. This is the idea behind a dish antenna for communications and the idea behind the acoustic dish microphones we see at sporting events. In next figure, we can see that it would take an increasingly larger dish to maintain the same power gathering capability as we get farther from the source.

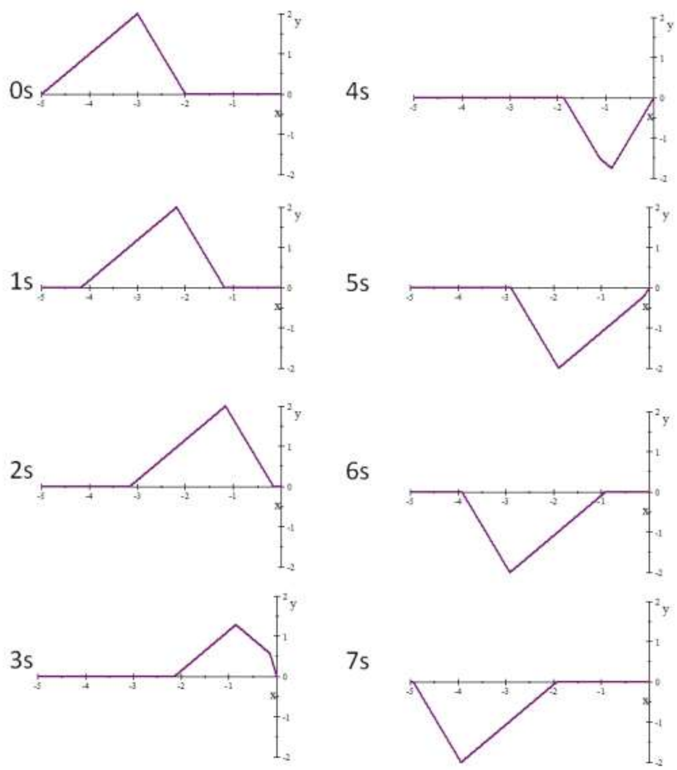


[1] The inner hair cells in the organ of Corti in the cochlea.

6.3 Reflection and Transmission

In our examples so far, we have not explained how we got two waves into a medium. One way is to simply reflect one wave back on top of itself.

In class we will made pulses on a spring with one end of the rope fixed (held by a class member). What happened when the pulse reached the end of the rope?

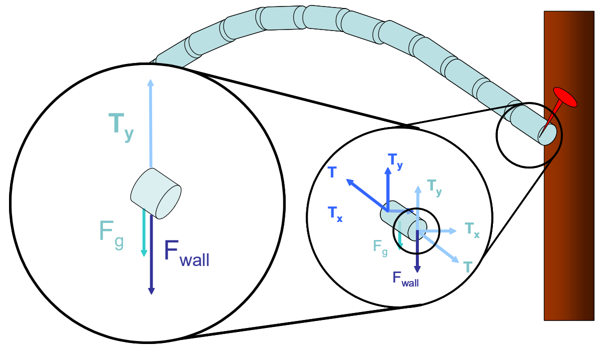


6.3.1 Case I: Fixed rope end.

There is a big change in the medium at the end of the rope. The rope ends. There is a person or (as in the next figure) some thing holding the rope in place. This change in medium causes a reflection.

In the fixed end case, the pulse is inverted. Why?

In the next figure I have envisioned a rope made of small rope segments.



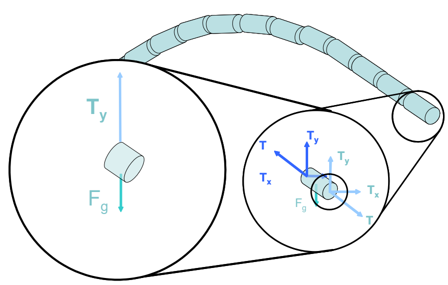
The end of the rope pushes up on the support (person in class, or nail in the figure). By Newton's third law there must be a normal force downward on the rope end. Compressing the molecules in the support stores potential energy in those compressed atoms. They will release that potential energy and create kinetic energy in the rope end. That kinetic energy will have a rope end velocity

$$K = \frac{1}{2}mv^2$$

and that velocity will be downward. This will pull the rope down, inverting the wave.

But what happens if the rope end is not fixed?

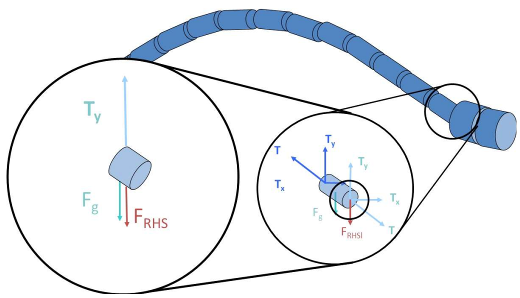
The rope end rises, and therefore there is no force exerted. The pulse (or at least part of the pulse energy) is still reflected, but there is no inversion because there was not downward force or no stored potential energy in the support!



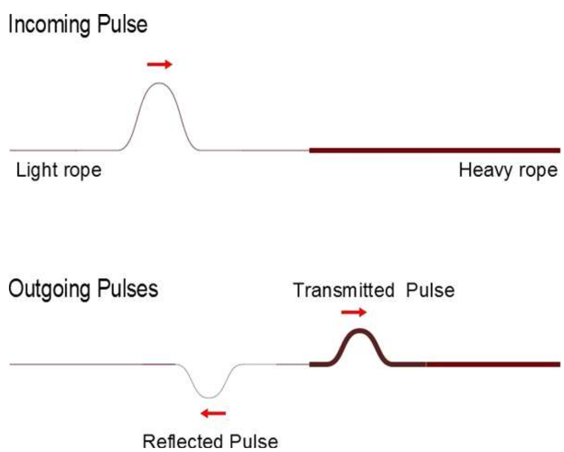
6.3.2 Case III: Partially attached rope end

Now lets tie the rope to another rope that is larger, more dense, than the rope we have been using, what will happen?

The light end of the rope exerts a force on the heavy beginning of the new rope. The atoms in the heavy rope will be pulled and compressed. The heavy rope will move, but the heaviness of the rope will prevent them from going very far. The fibers on the end of the heavy rope will build up potential energy because their bonds will be compressed. They will push downward on the light end of the rope. Once again the potential energy from the compressed heavy rope atoms will transfer to kinetic energy in the light rope end. Once again that light rope end will move downward.



We expect part of the energy to reflect back along the light rope, and this pulse will be inverted. Notice that the heavy rope did move upward a little, so there will also be a pulse on the heavy rope. Since this pulse formed from the light rope pulling up the heavy rope, and the light rope atoms were not compressed, this pulse will not be inverted.



This is all great for pulses. But suppose a sinusoidal wave approaches the boundary. We can envision the crests like pulses, and we expect the first crest to slow down when it reaches the boundary, letting the other crests catch up. Once the wave passes the boundary, the crests will be closer together. The wavelength changes as we move to the slower medium.

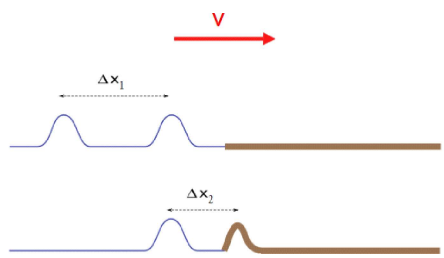
But does the frequency change? We know that

$$v = \lambda f$$

so

$$f = \frac{v}{\lambda}$$

both the speed and the wavelength have changed, but did they change proportionately so f is constant? This must be so. Think that the change in wavelength is due to the relative speed of the wave in the two media. If Δv is small the change in λ will be small because the crests are not delayed too long. If Δv is large, the crests are delayed by a large amount and so the change in λ is large. We won't derive the fact that f is constant, but we can see that is is very believable that it is true.



This is true for all waves, even light. When a wave crosses a boundary from a fast to a slow or a slow to a fast medium, λ will change and f will remain constant.

6.4 Superposition Principle

Wave Machine Demo

What happens if we have more than one wave propagating in a medium? If you remember being a little child in a bath tub, you will probably remember making waves in the water. If you made a wave with each hand, the two waves seemed to "pile up" in the middle and make a big splash. We should expect something like this for any kind of wave. We call the "piling up" of waves *superposition.* The word literally means putting one wave on top of another.

Superposition: If two or more traveling waves are moving through a medium, the resultant wave formed at any point is the algebraic sum of the values of the individual wave forms.

So if we have

$$y_1(x,t) \tag{6.3}$$

and

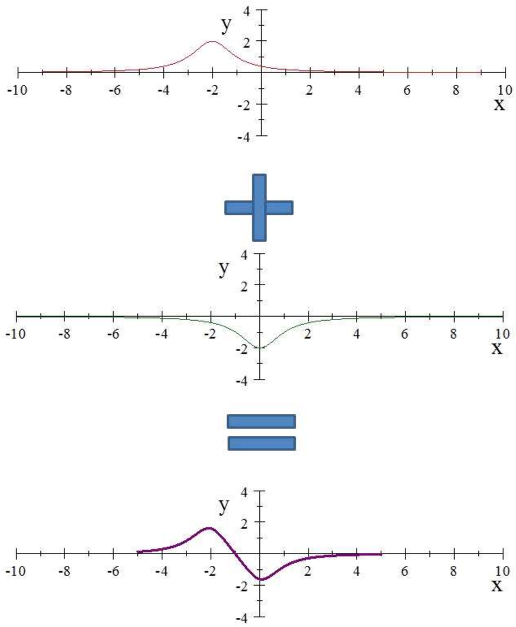
$$y_2(x,t) \tag{6.4}$$

both propagating on a string, then we would see

$$y_r(x,t) = y_1(x,t) + y_2(x,t) \tag{6.5}$$

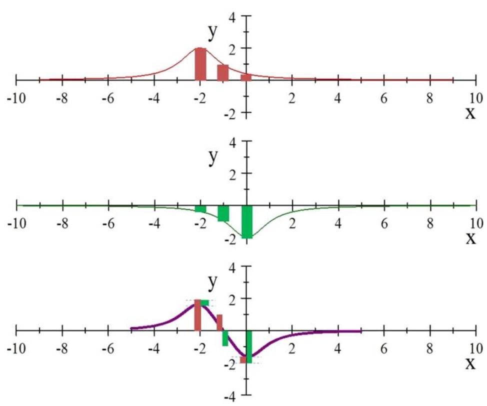
This is a fantastically simple way for the universe to act!

Let's look at an example. let's add the top wave (red) to the middle wave(green). We get the bottom wave (purple)



Of course we are adding these in the snapshot view. So this is all done for just one instant of time.

Let's see how to do this.



Start at $x = -2$. In the figure, I drew a red bar to show the y value at $x = -2$ for the red curve. Likewise, I have a green bar sowing the value of y at $x = -2$ for the green wave. Note that this is negative. On the bottom graph, the bars have been repeated, and we can see that the red bar minus the green bar brings us to the value for the resulting wave at the point $x = -2$. We need to do this at every point along all the waves for this instant of time.

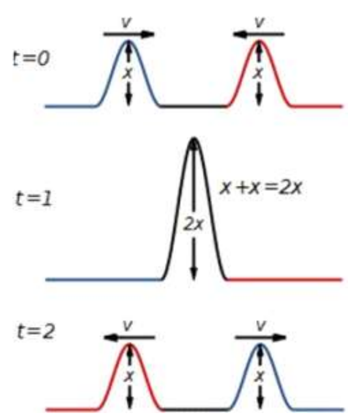
This is tedious by hand, so we won't generally do this calculation by hand. But a computer can do it easily.

Note that this is really only true for *linear* systems. Let's take the example of a slinky. If we form two waves in the slinky, they behave according to the superposition principle most of the time. But suppose we make the amplitude of the individual waves large. They may travel individually OK, but when the amplitudes add we may overstretch the slinky. Then it would never return to it's original shape. The wave form would have to change. Such a wave is not linear. There is a good rule of thumb for when waves are linear.

A wave is generally linear when its amplitude is much smaller than its wavelength.

6.5 Consequences of superposition

Linear waves traveling in media can pass through each other without being destroyed or altered!

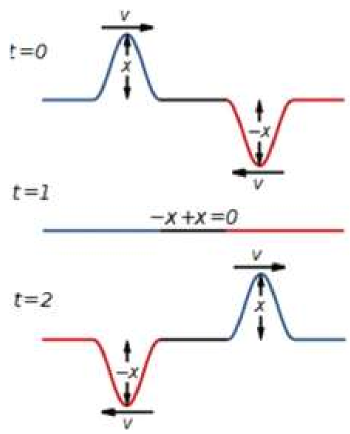


Constructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Constructive_interference.svg)

Our wave on the string makes the string segments move in the y direction. Both waves do this. So putting the two waves together just makes the string segments move more! There is a special name for what we observe

1. *interference*: The combination of separate waves in the same region of space to produce a resultant wave.

What happens if one of the pulses is inverted?



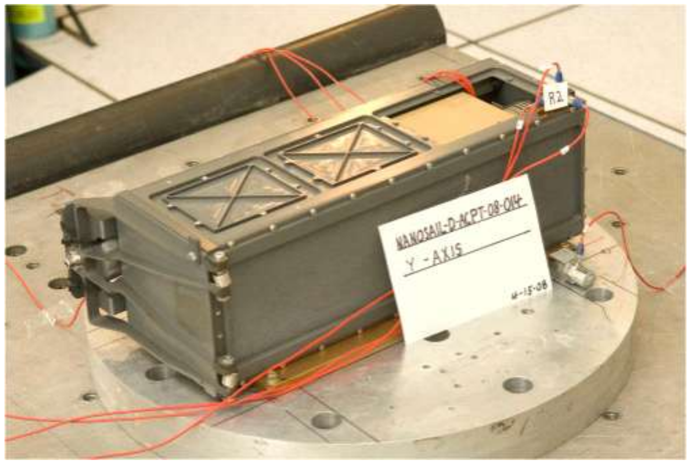
Destructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Destructive_interference1.svg)

When the two pulses meet, they "cancel each other out." But do they go away? No! the energy is still there, the string segment motions have just summed vectorially to zero, the energy carried by each wave is still there. We have a few more definitions. The type of interference we have just seen is the first

1. *Destructive Interference*: Interference between waves when the displacements caused by the two waves are opposite in direction

2. *Constructive Interference*: interference between waves when the displacements caused by the two waves are in the same direction

The combination of waves is important for both scientists and engineers. In engineering this is the hart of vibrometry.



Marshall and Cal Poly technicians wired the NanoSail-D spacecraft to accelerometers, instruments which measure vibration response during simulated launch conditions. Image couracy NASA, image in the Public Domain.

Mechanical systems have moving parts. These moving parts can be the disturbance that creates a wave. If more than one wave crest arrives at a location in the device, the amplitude at that location could become large. The oscillation of this part of the device could rattle apart welds or bolts, destroying the device. Later, as we study spectroscopy, we will see how to diagnose such a problem and hint at how to correct it.

6.6 Single Frequency Interference in one dimension

Two speaker demo

We have seen interference from superimposing two waves in the same medium. Whether we get constructive or destructive interference really depends on the total phase difference. Let's review this, but let's make our two waves as general as possible.

$$\begin{aligned} y_1 &= y_{\max} \sin (k_1 x_1 - \omega_1 t_1 + \phi_1) \\ y_2 &= y_{\max} \sin (k_2 x_2 - \omega_2 t_2 + \phi_2) \end{aligned}$$

where we have allowed the wavelengths to be different (so then the wave numbers, k_1 and k_2 could be different), the distances traveled by the waves could be different (x_2 might not be equal to x_1), the frequencies might be different, the times observed might be different, and the phase constants might be different. We have not made the amplitudes different (but it is likely to happen in an upcoming homework problem).

Then the resulting wave would be

$$\begin{aligned} y_r &= y_{\max}\sin\left(k_2x_2 - \omega_2 t_2 + \phi_2\right) + A\sin\left(k_1x_1 - \omega_1 t_1 + \phi_1\right) \\ &= 2y_{\max}\cos\left(\frac{(k_2x_2 - \omega_2 t_2 + \phi_2) - (k_1x_1 - \omega_1 t_1 + \phi_1)}{2}\right) \\ &\quad \times\sin\left(\frac{(k_2x_2 - \omega_2 t_2 + \phi_2) + (k_1x_1 - \omega_1 t_1 + \phi_1)}{2}\right) \end{aligned}$$

just as before. We can rewire this as

$$\begin{aligned} y_r &= 2y_{\max}\cos\left(\frac{1}{2}\left[(k_2x_2 - \omega_2 t_2 + \phi_2) - (k_1x_1 - \omega_1 t_1 + \phi_1)\right]\right) \\ &\quad \times\sin\left(\frac{(k_2x_2 - \omega_2 t_2 + \phi_2) + (k_1x_1 - \omega_1 t_1 + \phi_1)}{2}\right) \end{aligned}$$

and recognize that the sine part is still a function of position and time (just a complicated one) so it must be a wave. The cosine part must be part of the amplitude. Let's write out this amplitude

$$A = 2y_{\max}\cos\left(\frac{1}{2}\left[(k_2x_2 - \omega_2 t_2 + \phi_2) - (k_1x_1 - \omega_1 t_1 + \phi_1)\right]\right)$$

The part in square brackets is what we have been calling the phase difference

$$\Delta\phi = (k_2x_2 - \omega_2 t_2 + \phi_2) - (k_1x_1 - \omega_1 t_1 + \phi_1)$$

Notice that to make constructive interference we want the amplitude to be as big as possible. This happens when the cosine is ± 1. We can find when this happens by recalling that cosine is ± 1 when the argument of the cosine is a multiple of π. Notice that the amplitude has the cosine of half the phase difference

$$A = 2y_{\max}\cos\left(\frac{1}{2}\Delta\phi\right)$$

so

$$\frac{1}{2}\Delta\phi = \pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

or

$$\Delta\phi = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Constructive}$$

For total destructive interference we want the amplitude to be zero. To achieve this the cosine must be zero and this happens at odd multiples of $\pi/2$.

$$\frac{1}{2}\Delta\phi = \frac{\pi}{2}m \qquad m = \pm 1, \pm 3, \pm 5, \cdots$$

Then we could say that our phase difference

$$\Delta\phi = \pi m \qquad m = \pm 1, \pm 3, \pm 5, \cdots$$

another way to say this is to allow m to be any integer but to write the condition as

$$\Delta\phi = \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Destructive}$$

Let's write these out in detail

$$\begin{aligned}
[(k_2 x_2 - \omega_2 t_2 + \phi_2) - (k_1 x_1 - \omega_1 t_1 + \phi_1)] &= 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \\
[(k_2 x_2 - \omega_2 t_2 + \phi_2) - (k_1 x_1 - \omega_1 t_1 + \phi_1)] &= \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots
\end{aligned}$$

We can see that there are at least four sources of phase difference here. We can change the phase difference by changing wavelength ($k_2 \neq k_1$), changing frequency ($\omega_2 \neq \omega_1$), having the waves travel different distances ($x_2 \neq x_1$), having different phase constants ($\phi_2 \neq \phi_1$) or even consider somehow mixing the waves at different times ($t_2 \neq t_1$). This last one is not as interesting, we want our waves mixed together at the same time so let's set

$$t_2 = t_1 = t$$

but let's let any of the other variables be changeable. This gives us many ways to produce constructive or destructive interference. In our next lecture we will use this analysis to solve specific problems. The pattern will be to use our two criteria

$$\begin{aligned}
\Delta\phi &= 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Constructive} \\
\Delta\phi &= \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Destructive}
\end{aligned}$$

to determine if we have constructive, destructive, or partially constructive or destructive interference.

If both waves have the same frequency, we can simplify this.

$$\begin{aligned}
\Delta\phi &= [(k_2 x_2 - \omega t_2 + \phi_2) - (k_1 x_1 - \omega t_1 + \phi_1)] = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \\
\Delta\phi &= [(k_2 x_2 - \omega t_2 + \phi_2) - (k_1 x_1 - \omega t_1 + \phi_1)] = \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots
\end{aligned}$$

becomes

$$\begin{aligned}
\Delta\phi &= (k_2 x_2 - k_1 x_1) + (\phi_2 - \phi_1) = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \\
\Delta\phi &= (k_2 x_2 - k_1 x_1) + (\phi_2 - \phi_1) = \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots
\end{aligned}$$

and our wave equation becomes

$$\begin{aligned}
y_r &= 2 y_{\max} \cos\left(\frac{1}{2}\left[(k_2 x_2 - k_1 x_1) + (\phi_2 - \phi_1)\right]\right) \\
&\quad \times \sin\left(\frac{k_2 x_2 + k_1 x_1}{2} - \frac{(\omega t_2 + \omega_1 t_1)}{2} + \frac{(\phi_2 + \phi_1)}{2}\right)
\end{aligned}$$

6.7 Single frequency interference in more than one dimension

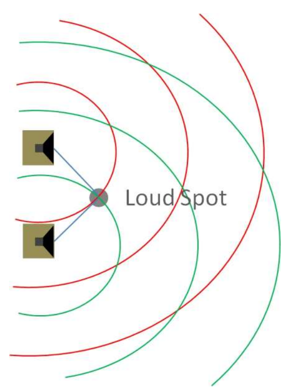
But what happens if our waves don't travel along the same line? Suppose you are at a dance, and there are two speakers. Further suppose that you are testing the system with a constant tone so $\omega_2 = \omega_1$ (either that or you have somewhat boring music with long, sustained tones). Suppose the two speakers make waves in phase ($\phi_1 = \phi_2$). There is no slower medium, so we expect $k_2 = k_1$. If you are equal distance from the two speakers, you would expect constructive interference because both $\Delta\phi_o = 0$ and $\Delta x = 0$ for this case.

$$\begin{aligned} \Delta\phi &= (k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1) \\ &= (kx_2 - \omega t + \phi) - (kx_1 - \omega t + \phi) \\ &= k\,(\Delta x) \end{aligned}$$

If we pick a spot where $x_1 = x_2$ we would expect

$$\Delta\phi = 0$$

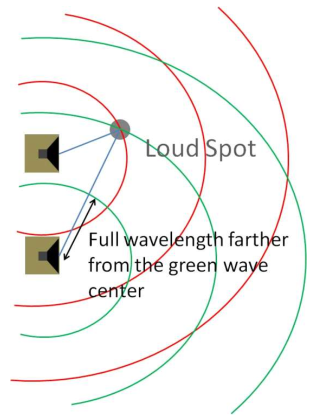
and since 0 is a multiple of 2π in our criteria, then we expect this will give us constructive interference.



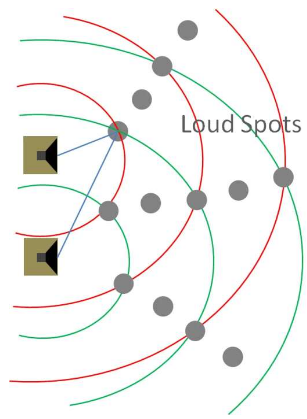
But there are more places where we expect constructive interference, because we know the sound wave is really spherical. Any time the path difference, $\Delta x = n\lambda$, then

$$\Delta\phi = \frac{2\pi}{\lambda}\,(n\lambda) = n2\pi$$

and we will have constructive interference The next figure shows an example where the path difference is one wavelength.



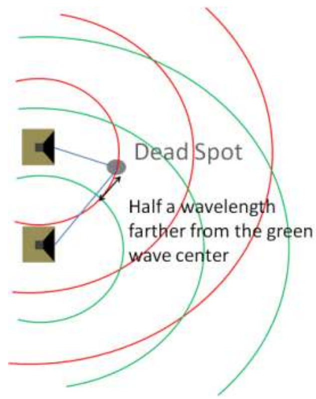
But any of the spots in the next figure will experience constructive interference. Note the loud spots are where there are two crests or two troughs together.



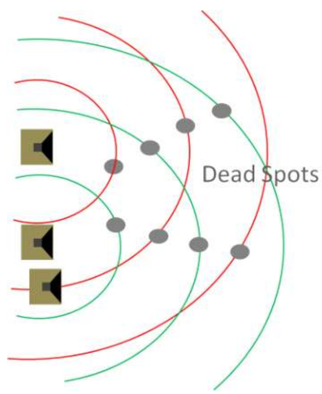
We also expect to see destructive interference. This should occur where path differences are multiples of $\lambda/2$ so that

$$\begin{aligned}\Delta\phi &= \frac{2\pi}{\lambda}\Delta x \\ &= \frac{2\pi}{\lambda}\left(n\frac{\lambda}{2}\right) \\ &= n\pi\end{aligned}$$

The next figure shows a case where $\Delta x = \lambda/2$



and the next figure shows many places where there will be destructive interference because the two waves are out of phase by half a wavelength.



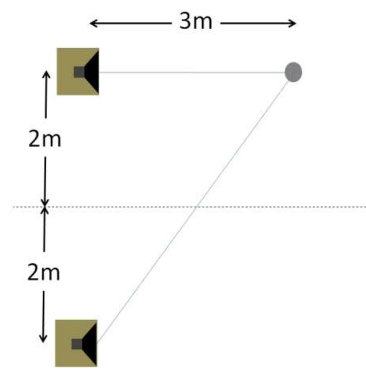
When you moved from one dimension to two dimensions in PH121, we changed from the variables x and y to the variable r where

$$r = \sqrt{x^2 + y^2}$$

Thus for this case our phase becomes

$$\Delta\phi = \left(2\pi\frac{\Delta r}{\lambda} + \Delta\phi_o\right)$$

In our dance example, suppose we have speakers that are 4 m apart and we are standing 3 m directly in front of one of the speakers. Further suppose that we play an A just above middle C which has a frequency of 440 Hz. The speed of sound is 343 m/ s. Our speakers are connected to the same stereo with equal length wires. What is the phase difference at this spot?



From the geometry we can tell that the path from the second speaker must be 5 m. So

$$\begin{aligned}\Delta x &= 5\,\mathrm{m} - 3\,\mathrm{m}\\ &= 2\,\mathrm{m}\end{aligned}$$

We can tell that the wavelength is

$$\begin{aligned}\lambda &= \frac{v}{f}\\ &= \frac{343\,\mathrm{m/s}}{440\,\mathrm{Hz}}\\ &= 0.779\,55\,\mathrm{m}\end{aligned}$$

Since the speakers are connected to the same stereo with equal length wires, $\Delta\phi_o = 0$. Then

$$\begin{aligned}\Delta\phi &= \frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\\ &= \frac{2\pi}{0.779\,55\,\mathrm{m}}(2\,\mathrm{m}) + 0\\ &= 5.\,131\,2\pi\\ &= 2\pi + 3.\,131\,2\pi\end{aligned}$$

We should ask, is this constructive or destructive interference? Well, it is neither purely constructive interference nor total destructive interference. Our amplitude would be

$$2A\cos\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right)\right)$$

so in this case we get

$$2A\cos\left(\frac{1}{2}\left(2\pi + 3.\,131\,2\pi\right)\right) = -0.409\,27A$$

which is smaller (in magnitude) than A, so it is partial destructive interference. It would be quieter at this spot than if we just had one speaker playing.

You might guess that this sort of analysis plays a large part in design of concert halls. It also is important in mechanical designs. But you should have seen a deficit in what we have learned so far. Up to this point, we have only mixed waves that have the same frequency. Can we mix waves that have different frequencies?