Chapter 6

Power and Superposition 6 1.16.4 1.16.5

Fundamental Concepts

- Because waves are three dimensional, describing the power or energy delivered per time of the wave is not enough, We describe how spatially spread out that power is. We call this spread power *intensity*
- If we make more than one wave in a medium, the waves “add up” or *superimpose.*

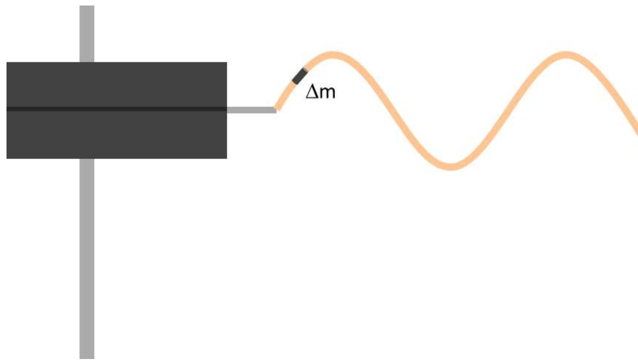
6.1 Energy and Power in waves

We have said that waves don't transport mass, they transport energy. We used the example of sound waves. If the air molecules move to another location , we call it wind. But if the air molecules just vibrate around an equilibrium position, we called that sound waves. The molecules don't end up somewhere else. But something does move.

Maybe a better example is a water wave in a swimming pool. Suppose someone jumps into a calm pool. The person is a disturbance and the water is a medium. We will get waves. But the water doesn't end up bunched together on the other side of the pool. The water is still everywhere in the pool, even around the person that jumped in. So what did move away from the person? It is energy.

For sinusoidal waves, the amount of energy in the wave is related to both it's amplitude and it's frequency. And the specifics of that relationship depend on the type of wave. We will study both sound and light waves later in our course and we will find that relationship between energy, amplitude, and frequency are different for the two different kinds of waves.

Let's look at a specific case of a rope attached to a mechanical oscillator. You can see the situation in the next figure. The oscillator (black thing) has a piece that goes up and down (silver thing sticking out to the right). The rope is attached to this silver oscillating piece. The oscillatory acts as a disturbance. A wave is formed in the rope.



Once again let's take a small part of our medium. Before we took a pice that was Δx long, but this time let's describe our piece of rope as a small bit of mass Δm. We already know about linear mass densities.

$$\mu = \frac{M}{L}$$

and since our rope is uniform, the linear mass density is the same for each part of the rope. So for our marked part

$$\mu = \frac{\Delta m}{\Delta x}$$

Then for this part of the rope medium there will be a kinetic energy

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}\Delta m v_y^2 \end{aligned}$$

using our linear mass density we can write

$$\Delta m = \mu \Delta x$$

so the kinetic energy is

$$K = \frac{1}{2}\left(\mu \Delta x\right) v_y^2$$

and from before we know that $v_y = -\omega A \cos\left(kx - \omega t + \phi_o\right)$ so our kinetic energy is

$$K = \frac{1}{2}\left(\mu \Delta x\right)\left(-\omega A \cos\left(kx - \omega t + \phi_o\right)\right)^2$$

we can clean this up a bit

$$K=\frac{1}{2}\mu\Delta x\omega^2A^2\cos^2(kx-\omega t+\phi_o)$$

Now let's simplify our situation by letting $\phi_o=0$ and let's take a snap shot view with $t=0$.

$$K=\frac{1}{2}\mu\Delta x\omega^2A^2\cos^2(kx)$$

Then, if we take only part of the energy associated with one very tiny part of our rope medium, our Δx becomes just dx. We don't have all the energy of the wave, just a small part of it. So we can write this as

$$dK=\frac{1}{2}\mu\omega^2A^2\cos^2(kx)\,dx$$

and we can integrate this up to find the energy of just one wavelength of the wave.

$$\int_0^{K_\lambda}dK=\int_0^\lambda\frac{1}{2}\mu\omega^2A^2\cos^2(kx)\,dx$$

A lot of the parts of our equation are constants, let's take them out front.

$$\int_0^{K_\lambda}dK=\frac{1}{2}\mu\omega^2A^2\int_0^\lambda\cos^2(kx)\,dx$$

The left hand side of this equation is just K_λ

$$K_\lambda=\frac{1}{2}\mu\omega^2A^2\int_0^\lambda\cos^2(kx)\,dx$$

but if you are like me, the right hand side is an integral that you have conveniently forgotten. But we can look up this integral in an integral table. My table gave the form

$$\int\left(\cos^2(ax)\right)dx=\frac{1}{2}x+\frac{1}{4a}\sin(2ax)$$

We have to match this to our equation. We see $a=k$ so

$$\begin{aligned}
K_\lambda &= \frac{1}{2}\mu\omega^2A^2\left[\frac{1}{2}x+\frac{1}{4k}\sin(2kx)\right|_0^\lambda \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{1}{4k}\sin(2k\lambda)\right\}-\left\{\frac{1}{2}(0)+\frac{1}{4k}\sin(2k(0))\right\}\right] \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{1}{4\frac{2\pi}{\lambda}}\sin\left(2\frac{2\pi}{\lambda}\lambda\right)\right\}-\frac{1}{4k}\sin(2k(0))\right] \\
&= \frac{1}{2}\mu\omega^2A^2\left[\left\{\frac{1}{2}\lambda+\frac{\lambda}{8\pi}\sin(4\pi)\right\}\right] \\
&= \frac{1}{4}\mu\omega^2A^2\lambda
\end{aligned}$$

This wavelength sized piece of tope will also have some potential energy because the fibers that make the rope are stretched to make the wave. They aren't really quite like little springs, but it's not to far wrong to model them as though they were. So let's take the spring potential energy

$$U = \frac{1}{2}k_s\left(y - y_e\right)^2$$

and let $y_e = 0$ for our piece of rope. We know that our piece of rope will go up and down in SHM. The natural frequency of our rope piece is

$$\omega = \sqrt{\frac{k_s}{m}}$$

We can solve this for the spring constant

$$k_s = m\omega^2$$

or for our little piece of the rope

$$k_s = \Delta m\omega^2$$

Let's put this into our potential energy equation

$$U = \frac{1}{2}\Delta m\omega^2\left(y\right)^2$$

and substitute $\Delta m = \mu\Delta x$ again

$$U = \frac{1}{2}\mu\Delta x\omega^2\left(y\right)^2$$

and of course, let our Δx get small so we have a small bit of potential energy from a small bit of rope.

$$dU = \frac{1}{2}\mu\omega^2\left(y\right)^2 dx$$

and integrate this to find the potential energy in the stretchy bonds of one wavelength worth of rope.

$$\int_0^{U_\lambda} dU = \int_0^{\lambda} \frac{1}{2}\mu\omega^2\left(y\right)^2 dx$$

We need to put in our equation for the position for our wave.

$$y\left(x,t\right) = A\sin\left(kx - \omega t + \phi_o\right)$$

but once again let's choose $\phi_o = 0$ and $t = 0$ so that

$$y\left(x,0\right) = A\sin\left(kx\right)$$

then

$$\int_0^{U_\lambda} dU = \int_0^{\lambda} \frac{1}{2}\mu\omega^2\left(A\sin\left(kx\right)\right)^2 dx$$

The left hand side is easy. And there are, once again, constants

$$U_\lambda = \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda (\sin(kx))^2 \, dx$$

and we have another integral I don't remember how to do. But I still have my table of integrals and my table tells me

$$\int \left(\sin^2(ax)\right) dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Which feels very familiar. Lets let $a = k$ again. Then

$$\begin{aligned} U_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \left[\frac{1}{2}x - \frac{1}{4k}\sin(2kx)\right|_0^\lambda \\ & \frac{1}{2}\mu\omega^2 A^2 \left[\left\{\frac{1}{2}\lambda - \frac{1}{4k}\sin(2k\lambda)\right\} - \left\{\frac{1}{2}0 - \frac{1}{4k}\sin(2k(0))\right\}\right] \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda \end{aligned}$$

The total energy in the λ-sized rope piece would be

$$\begin{aligned} E_\lambda &= K_\lambda + U_\lambda \\ &= \frac{1}{4}\mu\omega^2 A^2 \lambda + \frac{1}{4}\mu\omega^2 A^2 \lambda \\ &= \frac{1}{2}\mu\omega^2 A^2 \lambda \end{aligned}$$

We know that energy is being transferred by the wave, whether it is a light or sound wave or any other mechanical wave. We should wonder, how fast is energy transferring. This can mean the difference between a warm ray of sin on a cool spring day and being burned by a laser beam. We will start by considering the rate of energy transfer, *power*. The concept of power should be familiar to us from PH121. We can find the power as the rate at which energy is transferred.

$$\mathcal{P} = \frac{\Delta E}{\Delta t}$$

Since we picked the amount of energy in one wavelength, and we know the time it takes for one wavelength of the wave to pass by is T. Then the power is

$$P = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T}$$

but remember

$$v = \frac{\lambda}{T}$$

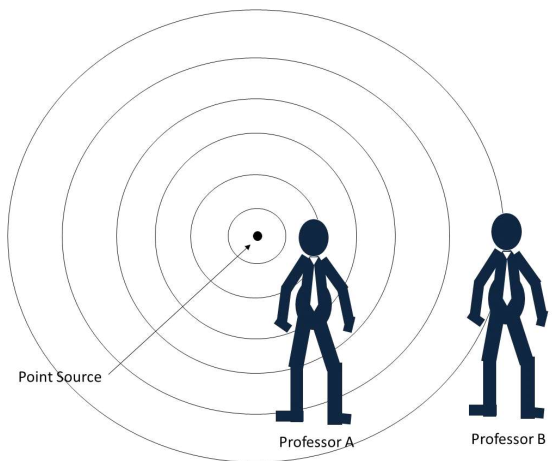
so we can write our power as

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

Power is important. Most detectors (like our ears and eyes) really detect the power delivered by a wave. And we can see that the power is proportional to the amplitude squared and the frequency squared for the wave. More on this as we study light and sound.

6.2 Power and Intensity

What we have done works fine for linear waves. But if we consider our spherical wave from a point source, we can see that this description isn't good enough. In the next figure we have two professors.



Thinking from our experience we would say that the sound will seem louder for professor A. This is because the energy in the sound wave is being spread over the surface of the wave, and that surface is getting bigger as the wave moves outward. The energy is more spread out by the time it gets to professor B.

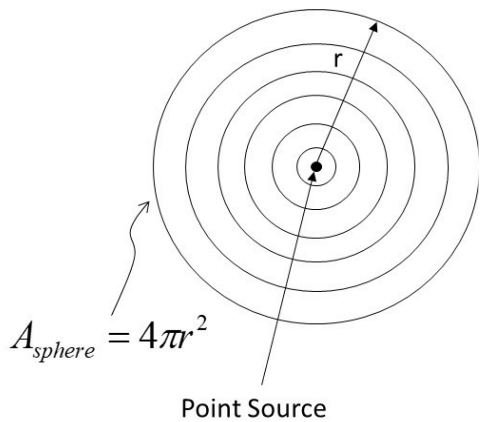
Tuning Fork Demo

To be able to describe how much energy we get from our wave we need define something new.

$$\mathcal{I} \equiv \frac{\mathcal{P}}{A} \tag{6.1}$$

that is, the power divided by an area. But what does it mean?

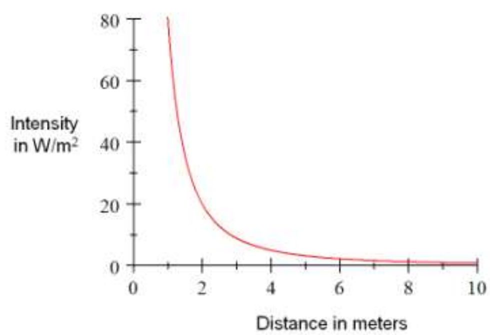
Consider a point source. This could be a loud speaker, or a buzzer, or a baby crying, etc.



it sends out waves in all directions. The wave crests will define a sphere around the points source (the figure shows a cross section but remember it is a wave from a point source, so we are really drawing concentric spheres like balloons inside of balloons.). Then form our point source

$$\mathcal{I} = \frac{\mathcal{P}}{4\pi r^2} \tag{6.2}$$

As the wave travels, its the power per unit area decreases with the square of the distance (think gravity) because the area is getting larger.



This quantity that tells us how spread out our power has become is called the *intensity* of the wave. Professor A would agree with us that the wave he heard was more intense than the wave heard by Professor B. That is because the wave was less spread out for Professor A.

Suppose we cup our hand to our ear. What are we doing? We are increasing the area of our ear. Our ears work by transferring the energy of the sound wave to a mechanical-electro-chemical device that creates a nerve signal.[1] The more

[1] The inner hair cells in the organ of Corti in the cochlea.

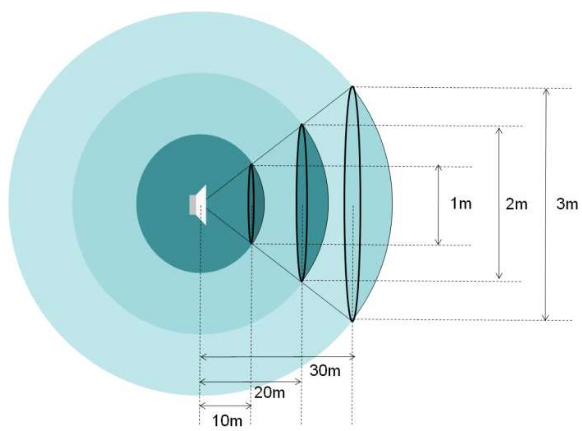
energy, the stronger the signal. If we are a distance r away from the source of the sound then the intensity is

$$\mathcal{I} = \frac{\mathcal{P}_{source}}{A_{wave}}$$

But we are collecting the sound wave with another area, the area of our hand. The power received is

$$\begin{aligned}\mathcal{P}_{received} &= \mathcal{I}A_{hand} \\ &= \frac{A_{hand}}{A_{wave}}\mathcal{P}_{source}\end{aligned}$$

and we can see that, indeed, the larger the hand, the more power, and therefore more energy we collect. This is the idea behind a dish antenna for communications and the idea behind the acoustic dish microphones we see at sporting events. In next figure, we can see that it would take an increasingly larger dish to maintain the same power gathering capability as we get farther from the source.



6.3 Superposition Principle

Wave Machine Demo

What happens if we have more than one wave propagating in a medium? If you remember being a little child in a bath tub, you will probably remember making waves in the water. If you made a wave with each hand, the two waves seemed to “pile up” in the middle and make a big splash. We should expect something like this for any kind of wave. We call the “piling up” of waves *superposition.* The word literally means putting one wave on top of another.

Superposition: If two or more traveling waves are moving through a medium, the resultant wave formed at any point is the algebraic sum of the values of the individual wave forms.

So if we have

$$y_1(x,t) \tag{6.3}$$

and

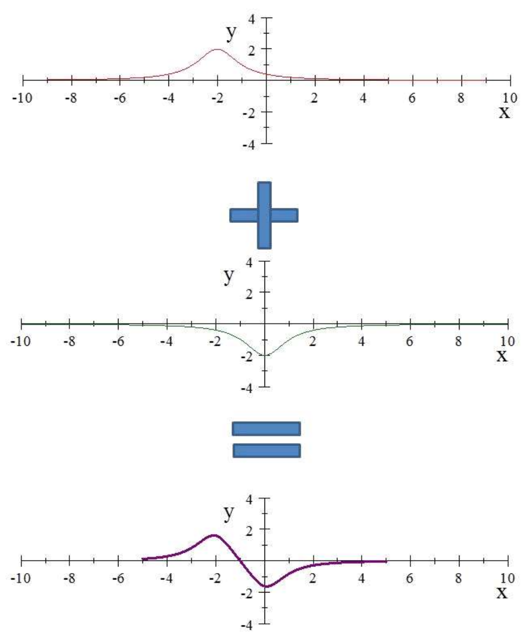
$$y_2(x,t) \tag{6.4}$$

both propagating on a string, then we would see

$$y_r(x,t) = y_1(x,t) + y_2(x,t) \tag{6.5}$$

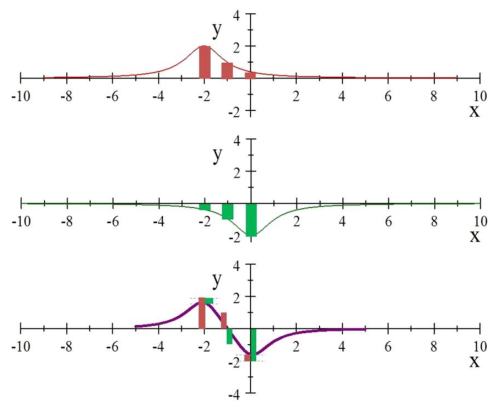
This is a fantastically simple way for the universe to act!

Let's look at an example. let's add the top wave (red) to the middle wave(green). We get the bottom wave (purple)



Of course we are adding these in the snapshot view. So this is all done for just one instant of time.

Let's see how to do this.



Start at $x = -2$. In the figure, I drew a red bar to show the y value at $x = -2$ for the red curve. Likewise, I have a green bar sowing the value of y at $x = -2$ for the green wave. Note that this is negative. On the bottom graph, the bars have been repeated, and we can see that the red bar minus the green bar brings us to the value for the resulting wave at the point $x = -2$. We need to do this at every point along all the waves for this instant of time.

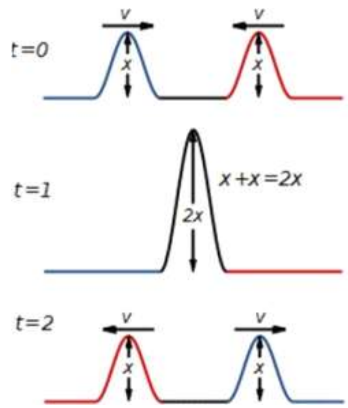
This is tedious by hand, so we won't generally do this calculation by hand. But a computer can do it easily.

Note that this is really only true for *linear* systems. Let's take the example of a slinky. If we form two waves in the slinky, they behave according to the superposition principle most of the time. But suppose we make the amplitude of the individual waves large. They may travel individually OK, but when the amplitudes add we may overstretch the slinky. Then it would never return to it's original shape. The wave form would have to change. Such a wave is not linear. There is a good rule of thumb for when waves are linear.

A wave is generally linear when its amplitude is much smaller than its wavelength.

6.4 Consequences of superposition

Linear waves traveling in media can pass through each other without being destroyed or altered!

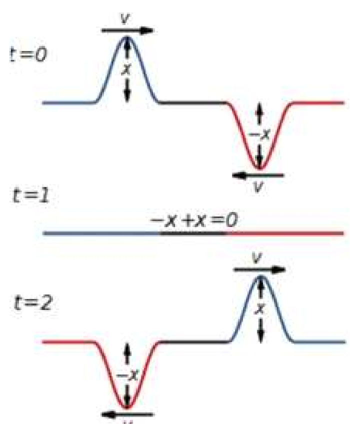


Constructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Constructive_interference.svg)

Our wave on the string makes the string segments move in the y direction. Both waves do this. So putting the two waves together just makes the string segments move more! There is a special name for what we observe

1. *interference*: The combination of separate waves in the same region of space to produce a resultant wave.

What happens if one of the pulses is inverted?



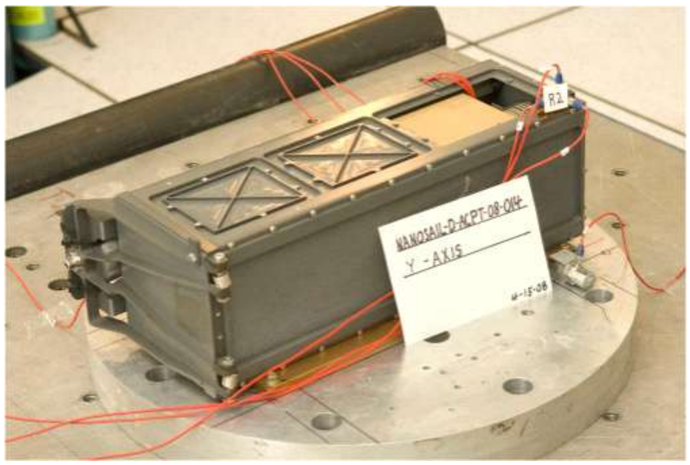
Destructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Destructive_interference1.svg)

When the two pulses meet, they "cancel each other out." But do they go away? No! the energy is still there, the string segment motions have just summed vectorially to zero, the energy carried by each wave is still there. We have a few more definitions. The type of interference we have just seen is the first

1. *Destructive Interference*: Interference between waves when the displacements caused by the two waves are opposite in direction

2. *Constructive Interference*: interference between waves when the displacements caused by the two waves are in the same direction

The combination of waves is important for both scientists and engineers. In engineering this is the hart of vibrometry.



Marshall and Cal Poly technicians wired the NanoSail-D spacecraft to accelerometers, instruments which measure vibration response during simulated launch conditions. Image couracy NASA, image in the Public Domain.

Mechanical systems have moving parts. These moving parts can be the disturbance that creates a wave. If more than one wave crest arrives at a location in the device, the amplitude at that location could become large. The oscillation of this part of the device could rattle apart welds or bolts, destroying the device. Later, as we study spectroscopy, we will see how to diagnose such a problem and hint at how to correct it.