Chapter 23

23 Double Slit and Grating intensity patterns 3.4.3, 3.4.4

Fundamental Concepts

- Double Slit Intensity Pattern
- Grating Intensity Pattern
- Integration over many periods to get what detectors (like eyes) see.

23.1 Double Slit Intensity Pattern

The fringes we have seen are not just points, but are patterns that fade from a maximum intensity. We can calculate the intensity pattern. We need to know a little bit about electric fields to do this.

We can represent an electromagnetic wave using just the electric field (the magnetic field pattern is very similar and can be derived from the electric field pattern) as we did at the beginning of this lecture

$$
\begin{aligned}
E_1 &= E_{\max} \sin\left(k_2 r_2 - \omega_2 t_2 + \phi_2\right) \\
E_2 &= E_{\max} \sin\left(k_1 r_1 - \omega_1 t_1 + \phi_1\right)
\end{aligned}
$$

and the resulting wave will be

$$\begin{aligned}
E_r &= E_{\max}\sin\left(k_2r_2-\omega_2t_2+\phi_2\right)+A\sin\left(k_1r_1-\omega_1t_1+\phi_1\right)\\
&= 2E_{\max}\cos\left(\frac{\left(k_2r_2-\omega_2t_2+\phi_2\right)-\left(k_1r_1-\omega_1t_1+\phi_1\right)}{2}\right)\\
&\quad\times\sin\left(\frac{\left(k_2r_2-\omega_2t_2+\phi_2\right)+\left(k_1r_1-\omega_1t_1+\phi_1\right)}{2}\right)\\
&= 2E_{\max}\cos\left(\frac{1}{2}\left[\left(k_2r_2-\omega_2t_2+\phi_2\right)-\left(k_1r_1-\omega_1t_1+\phi_1\right)\right]\right)\\
&\quad\times\sin\left(\frac{\left(k_2r_2-\omega_2t_2+\phi_2\right)+\left(k_1r_1-\omega_1t_1+\phi_1\right)}{2}\right)
\end{aligned}$$

but now we know that we can simplify this because $\omega_2=\omega_1=\omega$, $k_2=k_1=k$, $t_2=t_1=t$, and $\phi_2=\phi_1=\phi_o$.

$$\begin{aligned}
E_r &= 2E_{\max}\cos\left(\frac{1}{2}k\delta\right)\sin\left(\frac{kr_2+kr_1}{2}-\omega t+\phi_o\right)\\
&= 2E_{\max}\cos\left(\frac{1}{2}\frac{2\pi}{\lambda}d\sin\theta\right)\sin\left(k\frac{r_2+r_1}{2}-\omega t+\phi_o\right)
\end{aligned}$$

We have a combined wave at point P that is a traveling wave $\left(\sin\left(k\frac{(r_2+r_1)}{2}-\omega t+\phi_o\right)\right)$ but with amplitude $\left(2E_o\cos\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\right)$ that depends on our total phase $\Delta\phi=\frac{2\pi}{\lambda}d\sin\theta$.

But the situation is more complicated because of how we detect light. Our eyes, film, and most detectors measure the intensity of the light. We know that

$$I=\frac{\mathcal{P}}{A}$$

In PH 220 you will learn that the power is proportional to the square of the electric field wave amplitude.

$$\mathcal{P}\propto E^2 \tag{23.1}$$

Then the intensity must also be proportional to the amplitude of the electric field squared.

$$\begin{aligned}
I &= \frac{P}{A}\propto E^2\\
&\propto 4E_o^2\cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\sin^2\left(\frac{k\left(r_2+r_1\right)}{2}-\omega t+\phi_o\right)
\end{aligned}$$

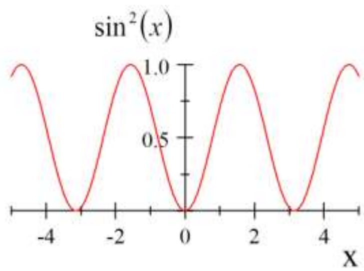
Light detectors collect power for a set amount of time. So most light detection will be a value averaged over a set *integration time*. This means that the detector sums up (or integrates) the amount of power received over the detector time. Usually the integration time is much longer than a period, so we need to time-average our intensity.

$$\begin{aligned}\int_{\text{many periods}} I dt \quad \propto \quad &= \int_{\text{many periods}} 4E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda} d \sin\theta\right)\right) \sin^2\left(\frac{k\left(r_2+r_1\right)}{2} - \omega t + \phi_o\right) dt \\ &= 4E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda} d \sin\theta\right)\right) \int_{\text{many periods}} \sin^2\left(\frac{k\left(rx_2+r_1\right)}{2} - \omega t + \phi_o\right) dt\end{aligned}$$

but the term

$$\int_{\text{many periods}} \sin^2\left(\frac{k\left(r_2+r_1\right)}{2} - \omega t + \phi o\right) dt = \frac{1}{2} \tag{23.2}$$

To convince yourself of this, think that $\sin^2(x)$ has a maximum value of 1 and a minimum of 0. Looking at the graph



should be believable that the average value over a period is 1/2. The average over many periods will still be 1/2.

So we have

$$\bar{I} = \int_{\text{many periods}} I dt \propto 2E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda} d \sin\theta\right)\right) \tag{23.3}$$

where $\bar{I}$ is the time average intensity. The important part is that the time varying part has averaged out.

So, usually in optics, we ignore the fast fluctuating parts of such calculations because we can't see them and so we write

$$I = I_{\max} \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda} d \sin\theta\right)\right)$$

where we have dropped the bar from the I, but it is understood that the intensity we report is a time average over many periods.

We should remind our selves, our intensity pattern

$$I = I_{\max} \cos^2\left(\frac{1}{2}\frac{2\pi}{\lambda} d \sin\theta\right)$$

is really

$$I = I_{\max} \cos^2\left(\frac{\Delta\phi}{2}\right)$$

Which is just our amplitude squared for the mixing of two waves. All we have done to find the intensity pattern is to find and expression for the phase difference $\Delta\phi$.

Our intensity pattern should give the same location for the center of the bright spots as we got before. Let's check that it works. We used the small angle approximation before. It is still valid, so let's use it again now. For for small angles

$$\begin{aligned} I &= I_{\max}\cos^2\left(\frac{\pi d}{\lambda}\theta\right) \\ &= I_{\max}\cos^2\left(\frac{\pi d}{\lambda}\frac{y}{L}\right) \end{aligned}$$

Then we have constructive interference when

$$\frac{\pi d}{\lambda}\frac{y}{L} = m\pi$$

or

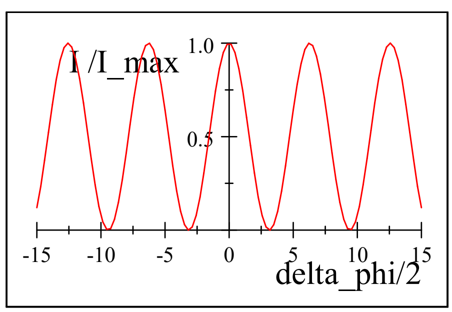
$$y = m\frac{L\lambda}{d}$$

which is what we found before.

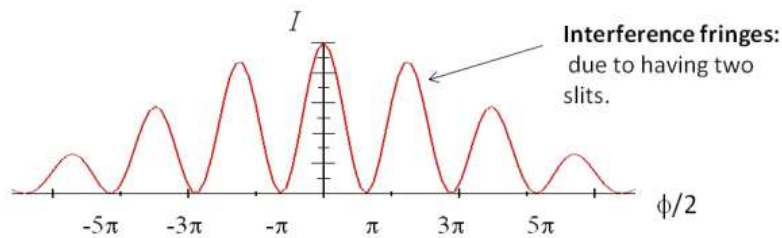
The plot of normalized intensity

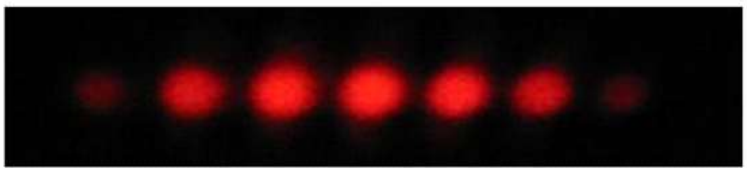
$$\frac{I}{I_{\max}} = \cos^2\left(\frac{\Delta\phi}{2}\right)$$

verses $\Delta\phi/2$ is given next,

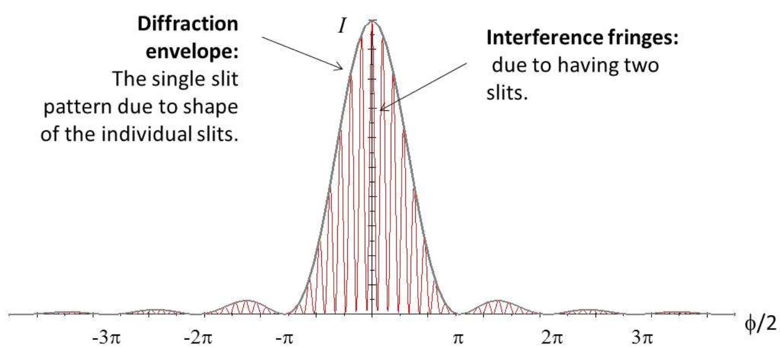


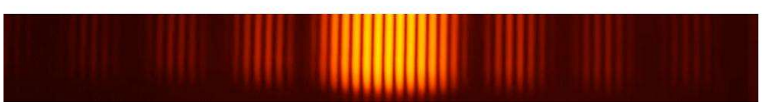
This is really an interesting result. You might wonder why, when we found the two slit interference pattern, there was no evidence of the single slit fringing that we discovered in this chapter. After all, a double slit system is made from single slits. Shouldn't there be some effect due to the fact that the slits are individually single slits? The answer is that we did see some hint of the single slit pattern. Remember the figure below.





The intensity of the peaks seems to fall off with distance from the center. We dealt with only the center-most part of the pattern. If we draw the pattern for larger angles, we see the following.





It takes a bright laser or dark room to see the secondary groups of fringes easily, but we can do it. We can also graph the intensity pattern. It is the combination of the two slit and single slit pattern with the single slit pattern acting and an envelope.

$$I = I_{\max} \cos^2\left(\frac{\pi d \sin(\theta)}{\lambda}\right)\left(\frac{\sin\left(\frac{\pi a \sin(\theta)}{\lambda}\right)}{\frac{\pi a \sin(\theta)}{\lambda}}\right)^2 \tag{23.4}$$

Note that one of the double slit maxima is clobbered by a minimum in the single slit pattern. We can find out the order of the missing maximum. Recall that

$$d \sin(\theta) = m\lambda$$

describes the maxima from the double slit. But

$$a \sin(\theta) = \lambda$$

describes the minimum from the single slit. Dividing these yields

$$\begin{aligned} \frac{d \sin(\theta)}{a \sin(\theta)} &= \frac{m\lambda}{\lambda} \\ \frac{d}{a} &= m \end{aligned}$$

so the

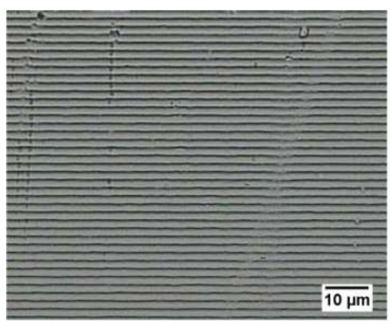
$$m = \frac{d}{a} \tag{23.5}$$

double slit maximum will be missing.

23.2 Diffraction Gratings

Rainbow Glasses

A diffraction grating is an optical element with many many parallel slits spaced very close together. Here is a typical diffraction grating created by etching lines in a piece of glass. The etchings scatter the light, but the un-etched part allows the light to pass through. The un-etched parts are essentially a series of slits.



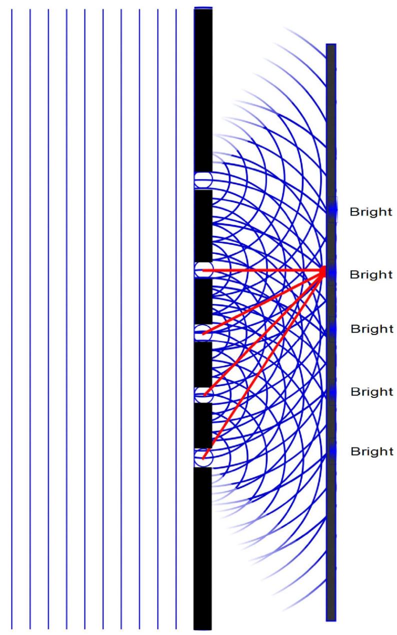
Surface of a diffraction grating (600 lines/mm). Image taken with optical transmission microscope. (Image in the public domain courtesy Scapha)

A typical grating might have 5000 slits per unit centimeter. You have probably used a diffraction grating to see rainbow colors in a beginning science class. Gratings are usually made by cutting parallel grooves in a flat surface.

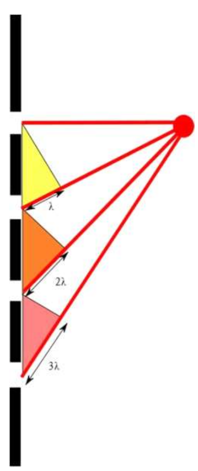
If we use $5000 \frac{grooves}{\mathrm{cm}}$ for an example, we see that the slit spacing is

$$\begin{aligned} d &= \frac{1}{5000}\,\mathrm{cm} && (23.6) \\ &= 2.0 \times 10^{-6}\,\mathrm{m} && (23.7) \end{aligned}$$

Take a section of diffraction grating as shown below



At some point, two of the slits will have a path difference that is a whole wavelength, and we would expect a bright spot. But what about the other slits? If we have a slit spacing such that each of the succeeding slits has a path difference that is just an additional wavelength, then each of the slits will contribute to the constructive interference at our point, and the point will become a bright spot.

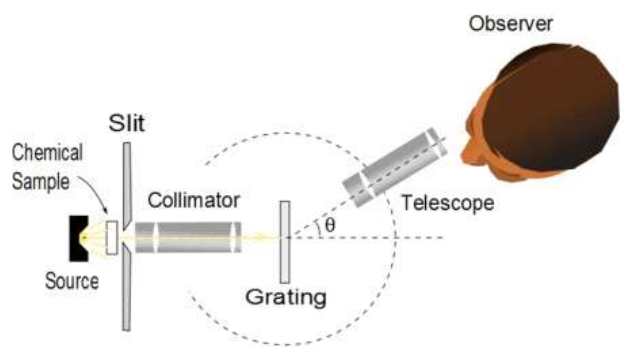


Let's look at just two slits. The light leaves each slit in phase with the light from the rest of the slits, but at some distance L away and at some angle θ we will have a path difference

$$\delta = d\sin\left(\theta_{bright}\right) = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots \tag{23.8}$$

because the path lengths are not all the same.

This equation tells us that each wavelength, λ, will experience constructive interference at a slightly different angle θ_{bright}. Different frequencies will create bright spots at different angles. We have found a way to create a spectrum with light waves. We often call a spectrum made with visible light frequencies a rainbow. Knowing d and θ allows an accurate calculation of λ. This may seem a silly thing to do, but suppose we add into our system a sample of a chemical to identify



We could then record the intensity of the transmitted light as a function of angle, which is equivalent to λ. We can again generate a spectrum. This is a traditional way to build a spectrometer and many such devices are available today.

Demo a student spectrometer with a gas tube

23.2.1 Resolving power of diffraction gratings

We noticed that with two slits, we got a bright spot for a particular wavelength, but we didn't just get one bright spot. We got several. The same is true for diffraction gratings. So we expect to get a rainbow, but we really expect to get a series of rainbows. The integer m tells us which rainbow we have in the series. The integer m is called the order number.

For two nearly equal wavelengths λ_1 and λ_2, we say that the diffraction grating can resolve the wavelengths if we can distinguish the two using the grating. The *resolving power* of the grating is defined as

$$R = \frac{\left(\lambda_1 + \lambda_2\right)}{2\left(\lambda_1 - \lambda_2\right)} = \frac{\bar{\lambda}}{\Delta\lambda} \tag{23.9}$$

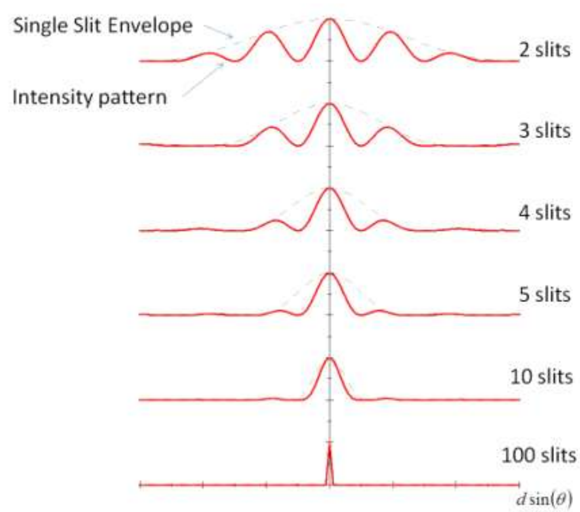
We can show that for the m-th order diffraction, the resolving power is

$$R = Nm \tag{23.10}$$

where N is the number of slits. So our ability to distinguish wavelengths increases with the number of slits and with the order (which is related to how far off-axis we look).

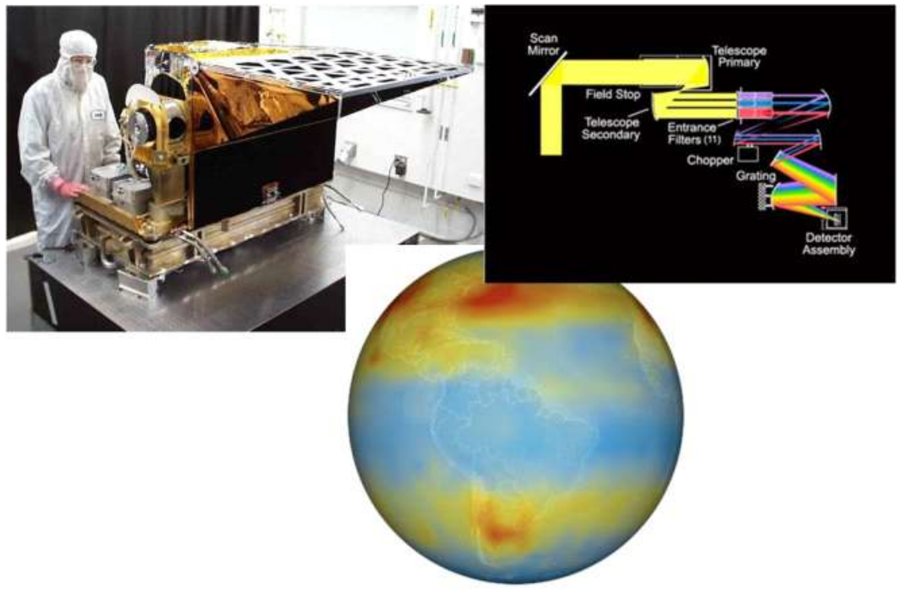
Note that for $m = 0$ we have no ability to resolve wavelengths. The central peak is a mix of all wavelengths and usually looks white for normal illumination.

That the resolution depends on the number of slits, N, means that we can improve our spectrometer by using more lines. Here is a representation of what happens as we increase N



we can see that the peaks get narrower as N increases. These graphs are for a particular λ. If the peaks for a particular λ get narrower, then there will be less overlap with adjacent $\lambda's$ which means that each wavelength can more easily be resolved.

Spectrometers are used in many places. On that has some public interest today is monitoring the atmosphere. Instruments like the one shown below detect the amount of special gasses in the atmosphere using IR spectrometers.



AIRS sensor, spectrometer design, and global CO_2 data. (Images in the Public Domain courtesy NASA)

The instrument shown is the AIRS spectrometer. You can see in the diagram that it uses a grating spectrometer. The picture of the Earth is a composite of AIRS data showing the northern and southern bands of CO_2.