ORBITS AS CONIC SECTIONS

SHOWING THAT A DISTANCE SQUARED LAW GIVES AN ELIPTICAL ORBIT

by

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DEPARTMENT APPROVAL

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ABSTRACT

ORBITS AS CONIC SECTIONS

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Department of Physics and Astronomy

Bachelor of Science

The abstract is a *summary* of the thesis, *not an introduction.* Keep in mind that abstracts are often published separately from the paper they summarize. In your abstract, give a concise synopsis of the work, emphasizing the conclusions; you need not include the supporting arguments for the conclusions. The purpose of the abstract is to help prospective readers decide whether to read your thesis, but your goal is not necessarily to persuade people to read your thesis. In fact, a successful abstract enables people to get an accurate overall view of your work without needing to read it. Usually, an abstract contains a single paragraph, but it can have more if absolutely necessary. Remember to state the subject of the paper immediately followed by a summary of the experimental or theoretical results and the methods used to obtain them. Avoid equations, graphics, and citations; if a citation is essential it must be cited fully within the abstract. Keep the abstract factual. Avoid vague statements like,

"Conclusions are drawn," or "the significance of the experiment is discussed." State the conclusions and findings outright in the abstract.

ACKNOWLEDGMENTS

This page is optional. You may acknowledge whom you will—your advisor, colleagues, family members. Please keep acknowledgments in good taste. I would like to acknowledge Dr. Kristine Hansen and Dr. Elizabeth Hedengren, whose Advanced Writing Seminar motivated this project. I also wish to thank Jean-François Van Huele, Steven Turley, and Ross Spencer for reviewing this document and for ripping it to shreds as every good advisor should do to a thesis draft.

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Chapter 1

Orbits

In the following sections, we show that orbits are really conic sections. The case of a circular orbit is just a special case.

1.1 Eliptical orbits

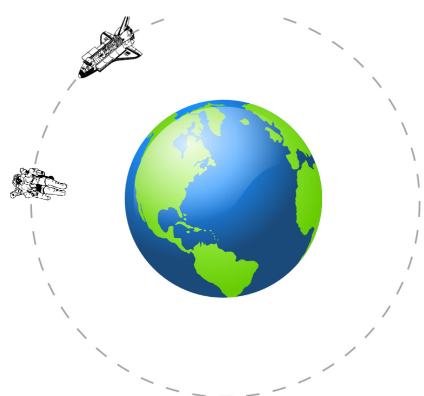

Figure 1.1 A PH121 Circular Orbit, It's not Enough!

In all our work in PH121 we used circular orbits. Kepler said orbits

Figure 1.2 A mistake

should be elliptical. And that is true. We won't go though this in class, but showing that Newton's law of gravitation implies an ellipse is a great way to show off our new mathematics of dot and cross products. So if you are comfortable with our new math, and curious to see how orbits work, read on. Let's start with Newton's second law for our orbiting satellite again.

$$\begin{aligned}\overrightarrow{W} &= -m_s \overrightarrow{a} \\ &= -m_s g\left(r\right)\hat{r} \\ &= -m_s\left(G\frac{M_E}{r_{Es}^2}\right)\hat{r}\end{aligned}$$

We can write this as

$$-m_s\overrightarrow{a} - m_s\left(G\frac{M_E}{r_{Es}^2}\right)\hat{r} = 0$$

The subscripts may become burdensome, so we will drop them now, but remember that $r = r_{Es}$ is the distance from the satellite to the Earth center of mass to center of mass.

$$-m_s \overrightarrow{a} - m_s \left(G \frac{M_E}{r^2} \right) \hat{r} = 0$$

In the next section we will find that conservation of energy is important 1.2. Notice that I used a marker to come up with the section number automatically.

1.2 Conservation of Orbital Mechanical Energy

Now we are going to do something strange. For no apparent reason, lets compute the dot product of both sides of this equation

$$\overrightarrow{v} \cdot \left(m_s \overrightarrow{a} + m_s \left(G \frac{M_E}{r^2} \right) \hat{r} \right) = \overrightarrow{v} \cdot 0$$

then

$$\overrightarrow{v} \cdot m_s \overrightarrow{a} + \overrightarrow{v} \cdot m_s \left(G \frac{M_E}{r^2} \right) \hat{r} = 0$$

or

$$m_s \overrightarrow{v} \cdot \overrightarrow{a} + m_s \left(G \frac{M_E}{r^2} \right) \overrightarrow{v} \cdot \hat{r} = 0$$

Now we need to learn a little bit more about dot products mixed with derivatives. We have a position vector $\overrightarrow{r} = r\hat{r}$ The derivative of this position vector is

$$\frac{d\overrightarrow{r}}{dt} = \frac{d}{dt} (r\hat{r})$$

$$= r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}$$

so if we take

$$\frac{d\overrightarrow{r}}{dt} \cdot \hat{r} = \left(r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right) \cdot \hat{r}$$

$$= r\frac{d\hat{r}}{dt} \cdot \hat{r} + \frac{dr}{dt}\hat{r} \cdot \hat{r}$$

$$= 0 + \frac{dr}{dt}$$

since $d\hat{r}/dt = 0$.

So

$$\frac{d\overrightarrow{r}}{dt} \cdot \hat{r} = \frac{dr}{dt}$$

and we recognize

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt}$$

so we can write

$$\overrightarrow{v} \cdot \hat{r} = \frac{dr}{dt} \tag{1.1}$$

and we have this in our orbit equation. Our orbit equation becomes

$$m_s\overrightarrow{v} \cdot \overrightarrow{a} + m_s\left(G\frac{M_E}{r_{Es}^2}\right)\frac{d\overrightarrow{r}}{dt} \cdot \hat{r} = 0$$

or just

$$m_s\overrightarrow{v} \cdot \overrightarrow{a} + m_s\left(G\frac{M_E}{r^2}\right)\frac{dr}{dt} = 0$$

We can do something similar for the first term We can recognize

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

and that $\overrightarrow{v} = v\hat{v}$ where $\hat{v}$ is a unit vector in

the same direction $\overrightarrow{v}$. Then

$$\frac{d\overrightarrow{v}}{dt} = \frac{d}{dt}\left(v\hat{v}\right)$$

$$= v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v}$$

and

$$\overrightarrow{v} \cdot \overrightarrow{a} = \overrightarrow{v} \cdot \frac{d\overrightarrow{v}}{dt}$$

$$= \overrightarrow{v} \cdot \left(v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v}\right)$$

$$= v\hat{v} \cdot v\frac{d\hat{v}}{dt} + v\hat{v} \cdot \frac{dv}{dt}\hat{v}$$

$$= 0 + v\frac{dv}{dt}\hat{v} \cdot \hat{v}$$

$$= v\frac{dv}{dt}$$

so our orbit equation becomes

$$m_s v\frac{dv}{dt} + m_s\left(G\frac{M_E}{r_{Es}^2}\right)\frac{dr}{dt} = 0$$

Now let's play a clever mathematical trick. Let's take the derivative of the kinetic energy with respect to time.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\frac{d}{dt}\left(v^2\right)$$
$$= \frac{1}{2}m\left(2v\frac{dv}{dt}\right)$$
$$= mv\frac{dv}{dt}$$

Notice that this is in our orbit equation! So then

$$m_s v\frac{dv}{dt} + m_s\left(G\frac{M_E}{r^2}\right)\frac{dr}{dt} = 0$$

becomes

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) + m_s\left(G\frac{M_E}{r^2}\right)\frac{dr}{dt} = 0$$

We can play this trick again for the second term

$$\frac{d}{dt}\left(G\frac{M_E}{r}\right) = GM_E\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$= GM_E\left(-\frac{1}{r^2}\frac{dr}{dt}\right)$$

which once again we recognize this as part of our orbit equation so we can

write

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) - m_s\frac{d}{dt}\left(G\frac{M_E}{r}\right) = 0$$

or

$$\frac{d}{dt}\left(\left(\frac{1}{2}mv^2\right) - m_s\left(G\frac{M_E}{r}\right)\right) = 0$$

which tells us that

$$\left(\frac{1}{2}mv^2\right) - m_s\left(G\frac{M_E}{r_{Es}}\right) = \text{constant}$$

That is, the mechanical energy is conserved since we recognize this as just

$$K + U_g = \text{constant}$$

And this makes sense. There are no energy loss mechanisms in our orbit. Our masses are particles (no tidal forces inside the objects, etc.) So we expect conservation of energy in forming our orbit.

1.3 Conservation of Orbital Angular Momentum

Now, let's do just what we did before only let's use a cross product with $\overrightarrow{r}$.

$$\overrightarrow{r} \times \left(-m_s\overrightarrow{a} - m_s\left(G\frac{M_E}{r^2}\right)\hat{r}\right) = \overrightarrow{r} \times 0$$

$$-\overrightarrow{r} \times m_s\overrightarrow{a} - \overrightarrow{r} \times m_s\left(G\frac{M_E}{r^2}\right)\hat{r} = 0$$

$$m_s\overrightarrow{r} \times \overrightarrow{a} + m_sG\frac{M_E}{r^2}\overrightarrow{r} \times \hat{r} = 0$$

$$m_s\overrightarrow{r} \times \overrightarrow{a} + m_sG\frac{M_E}{r^2}r\hat{r} \times \hat{r} = 0$$

The last term has $\hat{r} \times \hat{r}$. The angle between $\hat{r}$ and $\hat{r}$ must be zero (they are in the same direction) so

$$\hat{r} \times \hat{r} = (1)(1)\sin(0) = 0$$

and we are left with

$$m_s \overrightarrow{r} \times \overrightarrow{a} = 0$$

which really does note seem to helpful, but it is. Consider that

$$\overrightarrow{a} = \frac{d^2 \overrightarrow{r}}{dt^2}$$

so

$$\overrightarrow{r} \times \overrightarrow{a} = \overrightarrow{r} \times \frac{d^2 \overrightarrow{r}}{dt^2}$$

$$= \overrightarrow{r} \times \frac{d^2 (r\hat{r})}{dt^2}$$

Now consider the quantity

$$\frac{d}{dt}\left(\overrightarrow{r} \times \frac{d\overrightarrow{r}}{dt}\right) = \overrightarrow{r} \times \frac{d^2 \overrightarrow{r}}{dt^2} + \frac{d\overrightarrow{r}}{dt} \times \frac{d\overrightarrow{r}}{dt}$$

The second term must be zero because the angle between any vector and itself must be zero and $\sin(0) = 0$, but the first term is just what we have in our equation! so our equation becomes

$$m_s \overrightarrow{r} \times \overrightarrow{a} = m_s \frac{d}{dt}\left(\overrightarrow{r} \times \frac{d\overrightarrow{r}}{dt}\right) = 0$$

which we can write as

$$\frac{d}{dt}\left(\overrightarrow{r} \times m_s \frac{d\overrightarrow{r}}{dt}\right) = 0$$

$$\frac{d}{dt}\left(\overrightarrow{r} \times m_s \overrightarrow{v}\right)$$

$$= \frac{d}{dt}\left(\overrightarrow{L}\right) = 0$$

and, hurray! we have conservation of angular momentum for our general orbit!

1.4 Conic Section Equation

You may not be a thrilled as I was at this point, but what we have done is typical for physicists. We use the power of mathematics and some ingenuity to predict what motions will be. You might say, "but I would never think of taking cross and dot products seemingly randomly to find a result." This may be true now, but as you get used to using the mathematical tools an operation like this may become more obvious. In any case, recall that early physicists spent many years trying out ways to use their mathematical tools. So eventually someone was bound to try our cross and dot product tricks. But we have only shown conservation of energy and angular momentum. We have not reached our goal. So let's return to our basic motion equation that we started with

$$-m_s\overrightarrow{a} - m_s\left(G\frac{M_E}{r^2}\right)\hat{r} = 0$$

and now let's consider our equation for angular momentum

$$\overrightarrow{L} = \overrightarrow{r} \times m_s\overrightarrow{v}$$

and form the cross product of the first equation with $\overrightarrow{L}$

$$\left(-m_s\overrightarrow{a} - m_s\left(G\frac{M_E}{r^2}\right)\hat{r}\right) \times \overrightarrow{L} = 0 \times \overrightarrow{L}$$

$$m_s\overrightarrow{a} \times \overrightarrow{L} + m_s\left(G\frac{M_E}{r^2}\right)\hat{r} \times \overrightarrow{L} = 0$$

Again this may not seem like an obvious thing to do! But we find that

$$m_s \overrightarrow{a} \times \overrightarrow{L} = -\left(G\frac{M_E}{r^2}\right)\hat{r} \times \overrightarrow{L}$$

and it is time for another mathematical trick. Consider the quantity

$$\frac{d}{dt}\left(\overrightarrow{v} \times \overrightarrow{L}\right) = \overrightarrow{v} \times \frac{d\overrightarrow{L}}{dt} + \frac{d\overrightarrow{v}}{dt} \times \overrightarrow{L}$$

$$= \overrightarrow{v} \times \frac{d}{dt}\left(\overrightarrow{r} \times m_s \overrightarrow{v}\right) + \overrightarrow{a} \times \overrightarrow{L}$$

$$= \overrightarrow{v} \times (0) + \overrightarrow{a} \times \overrightarrow{L}$$

$$= \overrightarrow{a} \times \overrightarrow{L}$$

for our situation because we have already shown that angular momentum is conserved. So we have

$$m_s \frac{d}{dt}\left(\overrightarrow{v} \times \overrightarrow{L}\right) = -\left(G\frac{M_E}{r^2}\right)\hat{r} \times \overrightarrow{L}$$

Now let's look at the right hand side. Writing out the angular momentum gives

$$\hat{r} \times \overrightarrow{L} = \hat{r} \times \left(\overrightarrow{r} \times m_s \overrightarrow{v}\right)$$

and I will use a vector product identity that I will let the math department teach you

$$\overrightarrow{A} \times \left(\overrightarrow{B} \times \overrightarrow{C}\right) = \overrightarrow{B}\left(\overrightarrow{A} \cdot \overrightarrow{C}\right) - \overrightarrow{C}\left(\overrightarrow{A} \cdot \overrightarrow{B}\right)$$

so for us

$$\overrightarrow{r} \times \overrightarrow{L} = m_s \hat{r} \times \left(\overrightarrow{r} \times \overrightarrow{v}\right)$$

$$= m_s \left(\overrightarrow{r}\left(\hat{r} \cdot \overrightarrow{v}\right) - \overrightarrow{v}\left(\hat{r} \cdot \overrightarrow{r}\right)\right)$$

$$= m_s \left(\overrightarrow{r}\left(\hat{r} \cdot \overrightarrow{v}\right) - \overrightarrow{v} r\right)$$

We already know from equation (1.1) that

$$\overrightarrow{v} \cdot \hat{r} = \frac{dr}{dt}$$

then

$$\hat{r} \times \overrightarrow{L} = m_s \left(r\hat{r} \left(\frac{dr}{dt} \right) - \overrightarrow{v} r \right)$$

then finally

$$\begin{aligned} m_s \frac{d}{dt} \left(\overrightarrow{v} \times \overrightarrow{L} \right) &= - \left(G \frac{M_E m_s}{r^2} \right) \left(r\hat{r} \left(\frac{dr}{dt} \right) - \overrightarrow{v} r \right) \\ &= - \left(G M_E m_s \right) \left[\left(\frac{dr}{dt} \right) \frac{\hat{r}}{r} - \frac{\overrightarrow{v}}{r} \right] \end{aligned}$$

Let's employ one more mathematical trick

$$\begin{aligned} \frac{d}{dt} \left(\frac{\overrightarrow{r}}{r} \right) &= - \overrightarrow{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \frac{d\overrightarrow{r}}{dt} \\ &= - \overrightarrow{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \overrightarrow{v} \\ &= - \left(\hat{r} \frac{1}{r} \frac{dr}{dt} - \frac{1}{r} \overrightarrow{v} \right) \end{aligned}$$

and this is the part of our equation that I wrote in square brackets, so with a substitution our equation becomes

$$m_s \frac{d}{dt} \left(\overrightarrow{v} \times \overrightarrow{L} \right) = \left(G M_E m_s \right) \left(\frac{d}{dt} \left(\frac{\overrightarrow{r}}{r} \right) \right)$$

or, canceling the dt factors from both sides

$$m_s d \left(\overrightarrow{v} \times \overrightarrow{L} \right) = \left(G M_E m_s \right) \left(d \left(\frac{\overrightarrow{r}}{r} \right) \right)$$

and we can integrate both sides

$$m_s \int d \left(\overrightarrow{v} \times \overrightarrow{L} \right) = - \left(G M_E m_s \right) \int \left(d \left(\frac{\overrightarrow{r}}{r} \right) \right)$$

to find

$$m_s \overrightarrow{v} \times \overrightarrow{L} = \left(G M_E m_s \right) \left(\frac{\overrightarrow{r}}{r} \right) + \overrightarrow{B}$$

where $\overrightarrow{B}$ is a vector constant of integration. Once again for no apparent reason let's take the dot product of this equation with $\overrightarrow{r}$

$$\overrightarrow{r} \cdot \left(m_s \overrightarrow{v} \times \overrightarrow{L} \right) = - \left(G M_E m_s \right) \overrightarrow{r} \cdot \left(\frac{\overrightarrow{r}}{r} \right) + \overrightarrow{r} \cdot \overrightarrow{B}$$

and use another vector product identity

$$\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B} \cdot \overrightarrow{C}$$

We can write this as to write our dot product equation as

$$(m_s \overrightarrow{r} \times \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB \cos\theta_{rB}$$

or

$$\frac{1}{m_s} (\overrightarrow{r} \times m_s \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB \cos\theta_{rB}$$

$$(\overrightarrow{r} \times m_s \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB \cos\theta_{rB}$$

$$\overrightarrow{L} \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB \cos\theta_{rB}$$

$$L^2 = (GM_E m_s)\, r + rB \cos\theta_{rB}$$

and now we can solve for r

$$L^2 = r\left((GM_E m_s) + B\cos\theta_{rB}\right)$$

then

$$r = \frac{L^2}{\left((GM_E m_s) + B\cos\theta_{rB}\right)}$$

or, rearranging slightly,

$$r = \frac{L^2 / (GM_E m_s)}{\left(1 + (B/GM_E m_s)\cos\theta_{rB}\right)}$$

If we compare this to the parametric equation for a conic section (straight out of your calculus text book),

$$r = \frac{p}{1 + e\cos\nu}$$

we can see that our orbit must be a conic section with a semi-latus rectum,

$$p = L^2 / (GM_E m_s)$$

and an eccentricity,

$$e = B/GM_E m_s$$

and an angle

$$\nu = \theta_{rB}$$

This means our orbit could be any conic section, circle, ellipse, parabola, or hyperbola. For satellites we most often choose ellipses. But the other conic sections are possible.So Kepler was partially right. An ellipse is a general form for an orbit, but it might even be better to write Kepler's law to say that orbits are conic sections.

If you are a normal PH121 student, your reaction to this problem might be "Agh, maybe I should change my major to horticulture!" But don't worry, This was really a junior level problem, and for us physics majors we have many classes (both physics and math classes) to take before we would be expected to do a problem like this. Still it is fun to see that we *can* do a problem like this with the math we learned in lowly PH121 if we are very persistent! Interested students can read more in the the book *Fundamentals of Astrodynamics* by Bate *et. al.* [1][2][3][4]

Bibliography

[1] R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics* (Dover, New York, 1971).

[2] C. Barnatt, https:/www.explainingcomputers.com/, 2021-05-15.

[3] BYUI Web Team, https:/www.byui.edu/physics/, 2021-11-05.

[4] C. Acquista, "Light Scattering by Tenuous Particles: A Generalization of the Rayleigh-Gans-Rocard Approach," Applied Optics **15,** 2932–2936 (1976).

Appendix A

Things that belong in an appendix

The purpose of an appendix is to provide supplementary information which would distract if included in the main body of the thesis. Items appearing as an appendix might include lengthy derivations. If students feel compelled to include a brief tutorial on relevant background information (not new research), it should appear as an appendix. An appendix might consist of portions of unique computer code that was developed as part of the project.

Appendix B

Including Your Code

You might want to include some code in your thesis. This is done with a verbatim environment. I want to make sure the code fits on the page, so I am going to reduce the font size with a small environment as well.

```
##############################################################
# second order Runga Cutta for a projectile with drag        #
##############################################################
# Lab 9
# PH150
# Brother Lines
# 2010 FEB 26
##############################################################
# This code will perform two functions
#  1) It will calculate the exact solution for a projectile
#     system using the theoretical result with no drag
#  2) It will calculate the result using Euler's method
##############################################################
# Load Libraries
```

```
import numpy as np
#import matplotlib
#matplotlib.use('tkagg')
import matplotlib.pyplot as plt

##############################################################
# Initial conditions and physial setup constants
D=0.2          # drag coefficient in kg/s
m=0.020        # mass in kilograms
r=0.02         # radius of particle m
g=9.8          # acceleration due to gravity m/s^2
rho=1.23       # air density kg/m^3
x0=0           # initial x position in m
y0=70.0        # initial y position in m
v0=30          # initial velocity
thetadeg=45    # launch angle in degrees

##############################################################
## Set up the time steps and number of calcualtions
deltat=0.1         # Time steps of 0.01 seconds
ti=0               # starting at t=0
tf=6.7             # final time
N=int((tf-ti)/deltat)  # calcualte how many time steps are in 20 seconds
##############################################################
# Preliminary calculations
pi=np.arccos(-1.0)           # calculate pi to machine percision
theta=thetadeg*pi/180   # calcualte theta in radians
vx1=v0*np.arccos(theta)       # calculte the x component of the initial velocity
```

```
vy1=v0*np.sin(theta)       # calculte the y component of the initial velocity

##############################################################

# define and zero arrays

t=np.zeros((N))

x=np.zeros((N))

y=np.zeros((N))

z=np.zeros((N))

vx=np.zeros((N))

vy=np.zeros((N))

xnd=np.zeros((N))

ynd=np.zeros((N))



##############################################################

## make an array of times to use

t=np.linspace(0,tf,num=N);


##############################################################

# calcuate the postion of the mass using our known solution

# note that we can only do this for our no-friction case


for i in range (0,N):

xnd[i]=x0+vx1*t[i]+(1/2)*(0.0)*t[i]**2.0

ynd[i]=y0+vy1*t[i]+(1/2)*(-g)*t[i]**2.0
```

```
##############################################################
# now perform an Euler's Method Calculation
# now recalling that vx(i) already has a cos(theta) in it,
# we can use this to calculate the x part of the resistive
# force and likewise use vy(i) in calculating the y part of
# the resistive force. No explicit calculation of theta is
# necessary this way, and we save lots of computation time.
x[0]=x0;        # initial x position
y[0]=y0;        # initial y positoin
vx[0]=vx1;
vy[0]=vy1;
for i in range (0,N-1):
v=np.sqrt(vx[i]*vx[i]+vy[i]*vy[i]);
#remember that our theta is the angle v makes with the
#x-axis, we want the angle R makes with the x-axis. To achieve
#this we use cos(phi)=-cos(theta) and sin(phi)=-sin(theta)
#where phi is the angel R makes with the x axis.
Rx=-0.5*D*rho*pi*r*r*v*vx[i];
Ry=-0.5*D*rho*pi*r*r*v*vy[i];
fx=vx[i];
gx=+Rx/m;
fy=vy[i];
gy=-g+Ry/m;
x[i+1]=x[i]+deltat*fx;
y[i+1]=y[i]+deltat*fy;
vx[i+1]=vx[i]+deltat*gx;
vy[i+1]=vy[i]+deltat*gy;
#if y(i+1)<=0.0 , break, end
```

```
xE=x;
yE=y;


##############################################################
#now perform an RK2 Method Calculation
x[0]=x0;        # initial x position
y[0]=y0;        # initial y positoin
vx[0]=vx1;
vy[0]=vy1;
for i in range (0,N-1):
v=np.sqrt(vx[i]**2+vy[i]**2);
# first the Euler step, This is tricky because I want just the x
# component and the y component of the drag force, but the drag force
# depends on v^2. Remembering that vx=v*cos(theta), we can then multiply
# the speed, v, by vx to get v^2*cos(theta). This way we don't have to
# calculate theta explicitly
Rx=0.5*D*rho*pi*r*r*v*vx[i];
Ry=0.5*D*rho*pi*r*r*v*vy[i];
fx=vx[i];
gx=-Rx/m;
fy=vy[i];
gy=-g-Ry/m;
kx1=deltat*fx;
ky1=deltat*fy;
kvx1=deltat*gx;
kvy1=deltat*gy;
```

```
#now the RK step, What to do with the v^2? I think we can do the same
#thing as above to find the x abd y components of the velocity.
v=np.sqrt((vx[i]+kvx1/2)**2+(vy[i]+kvy1/2)**2);
Rx2=0.5*D*rho*pi*r*r*v*(vx[i]+kvx1/2);
Ry2=0.5*D*rho*pi*r*r*v*(vy[i]+kvy1/2);
fx2=vx[i]+kvx1/2;
gx2=-Rx/m;
fy2=vy[i]+kvy1/2;
gy2=-g-Ry/m;
#finally take the RK step.
x[i+1]=x[i]+deltat*fx2;
y[i+1]=y[i]+deltat*fy2;
vx[i+1]=vx[i]+deltat*gx2;
vy[i+1]=vy[i]+deltat*gy2;
#if y(i+1)<=0.0 , break, end

xRK=x;
yRK=y;



plt.plot(xnd,ynd, label='kinematic')
plt.plot(xE,yE, label='Euler')
plt.plot(xRK,yRK,label='RK')
plt.xlabel('x [m]')
plt.ylabel('y [m]')
leg=plt.legend();
plt.show()
```