

ORBITS AS CONIC SECTIONS SHOWING THAT A DISTANCE  
SQUARED LAW GIVES AN ELLIPTICAL ORBIT

by

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## ABSTRACT

### ORBITS AS CONIC SECTIONS SHOWING THAT A DISTANCE SQUARED LAW GIVES AN ELLIPTICAL ORBIT

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The abstract is a *summary* of the thesis, *not an introduction*. Keep in mind that abstracts are often published separately from the paper they summarize. In your abstract, give a concise synopsis of the work, emphasizing the conclusions; you need not include the supporting arguments for the conclusions. The purpose of the abstract is to help prospective readers decide whether to read your thesis, but your goal is not necessarily to persuade people to read your thesis. In fact, a successful abstract enables people to get an accurate overall view of your work without needing to read it. Usually, an abstract contains a single paragraph, but it can have more if absolutely necessary. Remember to state the subject of the paper immediately followed by a summary of the experimental or theoretical results and the methods used to obtain them. Avoid equations, graphics, and citations; if a citation is essential it must be cited fully

within the abstract. Keep the abstract factual. Avoid vague statements like, “Conclusions are drawn,” or “the significance of the experiment is discussed.” State the conclusions and findings outright in the abstract.

## ACKNOWLEDGMENTS

This page is optional. You may acknowledge whom you will—your advisor, colleagues, family members. Please keep acknowledgments in good taste. I would like to acknowledge Dr. Kristine Hansen and Dr. Elizabeth Hedengren, whose Advanced Writing Seminar motivated this project. I also wish to thank Jean-François Van Huele, Steven Turley, and Ross Spencer for reviewing this document and for ripping it to shreds as every good advisor should do to a thesis draft.

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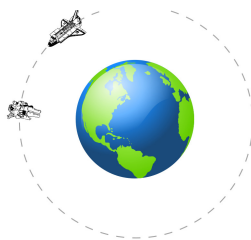
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# Chapter 1

## Orbits

In the following sections, we show that orbits are really conic sections. The case of a circular orbit is just a special case.

### 1.1 Elliptical orbits



**Figure 1.1** A PH121 Circular Orbit, It's not Enough!

In all our work in PH121 we used circular orbits. Kepler said orbits should be elliptical. And that is true. We won't go through this in class, but showing that Newton's law of gravitation implies an ellipse is a great way to show off our new mathematics of dot and cross products. So if you are



**Figure 1.2** A mistake

comfortable with our new math, and curious to see how orbits work, read on.

Let's start with Newton's second law for our orbiting satellite again.

$$\begin{aligned}
 \vec{W} &= -m_s \vec{a} \\
 &= -m_s g(r) \hat{r} \\
 &= -m_s \left( G \frac{M_E}{r_{Es}^2} \right) \hat{r}
 \end{aligned}$$

We can write this as

$$-m_s \vec{a} - m_s \left( G \frac{M_E}{r_{Es}^2} \right) \hat{r} = 0$$

The subscripts may become burdensome, so we will drop them now, but remember that  $r = r_{Es}$  is the distance from the satellite to the Earth center of mass to center of mass.

$$-m_s \vec{a} - m_s \left( G \frac{M_E}{r^2} \right) \hat{r} = 0$$

In the next section we will find that conservation of energy is important 1.2. Notice that I used a marker to come up with the section number automatically.

## 1.2 Conservation of Orbital Mechanical Energy

Now we are going to do something strange. For no apparent reason, lets compute the dot product of both sides of this equation

$$\vec{v} \cdot \left( m_s \vec{a} + m_s \left( G \frac{M_E}{r^2} \right) \hat{r} \right) = \vec{v} \cdot 0$$

then

$$\vec{v} \cdot m_s \vec{a} + \vec{v} \cdot m_s \left( G \frac{M_E}{r^2} \right) \hat{r} = 0$$

or

$$m_s \vec{v} \cdot \vec{a} + m_s \left( G \frac{M_E}{r^2} \right) \vec{v} \cdot \hat{r} = 0$$

Now we need to learn a little bit more about dot products mixed with derivatives. We have a position vector  $\vec{r} = r\hat{r}$  The derivative of this position vector is

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d}{dt}(r\hat{r}) \\ &= r \frac{d\hat{r}}{dt} + \frac{dr}{dt} \hat{r} \end{aligned}$$

so if we take

$$\frac{d\vec{r}}{dt} \cdot \hat{r} = \left( r \frac{d\hat{r}}{dt} + \frac{dr}{dt} \hat{r} \right) \cdot \hat{r}$$

$$= r \frac{d\hat{r}}{dt} \cdot \hat{r} + \frac{dr}{dt} \hat{r} \cdot \hat{r}$$

$$= 0 + \frac{dr}{dt}$$

since  $d\hat{r}/dt = 0$ .

So

$$\frac{d\vec{r}}{dt} \cdot \hat{r} = \frac{dr}{dt}$$

and we recognize

$$\vec{v} = \frac{d\vec{r}}{dt}$$

so we can write

$$\vec{v} \cdot \hat{r} = \frac{dr}{dt} \tag{1.1}$$

and we have this in our orbit equation. Our orbit equation becomes

$$m_s \vec{v} \cdot \vec{a} + m_s \left( G \frac{M_E}{r_{Es}^2} \right) \frac{d\vec{r}}{dt} \cdot \hat{r} = 0$$

or just

$$m_s \vec{v} \cdot \vec{a} + m_s \left( G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

We can do something similar for the first term We can recognize

$$\vec{a} = \frac{d\vec{v}}{dt}$$

and that  $\vec{v} = v\hat{v}$  where  $\hat{v}$  is a unit vector in

the same direction  $\vec{v}$ . Then

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{v})$$

$$= v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v}$$

and

$$\vec{v} \cdot \vec{a} = \vec{v} \cdot \frac{d\vec{v}}{dt}$$

$$= \vec{v} \cdot \left( v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v} \right)$$

$$= v\hat{v} \cdot v\frac{d\hat{v}}{dt} + v\hat{v} \cdot \frac{dv}{dt}\hat{v}$$

$$= 0 + v\frac{dv}{dt}\hat{v} \cdot \hat{v}$$

$$= v\frac{dv}{dt}$$

so our orbit equation becomes

$$m_s v \frac{dv}{dt} + m_s \left( G \frac{M_E}{r_{Es}^2} \right) \frac{dr}{dt} = 0$$

Now let's play a clever mathematical trick. Let's take the derivative of the kinetic energy with respect to time.

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) &= \frac{1}{2} m \frac{d}{dt} (v^2) \\ &= \frac{1}{2} m \left( 2v \frac{dv}{dt} \right) \end{aligned}$$

$$= mv \frac{dv}{dt}$$

Notice that this is in our orbit equation! So then

$$m_s v \frac{dv}{dt} + m_s \left( G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

becomes

$$\frac{d}{dt} \left( \frac{1}{2} m v^2 \right) + m_s \left( G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

We can play this trick again for the second term

$$\begin{aligned} \frac{d}{dt} \left( G \frac{M_E}{r} \right) &= G M_E \frac{d}{dt} \left( \frac{1}{r} \right) \\ &= G M_E \left( -\frac{1}{r^2} \frac{dr}{dt} \right) \end{aligned}$$

which once again we recognize this as part of our orbit equation so we can

write

$$\frac{d}{dt} \left( \frac{1}{2} m v^2 \right) - m_s \frac{d}{dt} \left( G \frac{M_E}{r} \right) = 0$$

or

$$\frac{d}{dt} \left( \left( \frac{1}{2} m v^2 \right) - m_s \left( G \frac{M_E}{r} \right) \right) = 0$$

which tells us that

$$\left( \frac{1}{2} m v^2 \right) - m_s \left( G \frac{M_E}{r_{Es}} \right) = \text{constant}$$

That is, the mechanical energy is conserved since we recognize this as just

$$K + U_g = \text{constant}$$

And this makes sense. There are no energy loss mechanisms in our orbit. Our masses are particles (no tidal forces inside the objects, etc.) So we expect conservation of energy in forming our orbit.

## 1.3 Conservation of Orbital Angular Momentum

Now, let's do just what we did before only let's use a cross product with  $\vec{r}$ .

$$\vec{r} \times \left( -m_s \vec{a} - m_s \left( G \frac{M_E}{r^2} \right) \hat{r} \right) = \vec{r} \times 0$$

$$-\vec{r} \times m_s \vec{a} - \vec{r} \times m_s \left( G \frac{M_E}{r^2} \right) \hat{r} = 0$$

$$m_s \vec{r} \times \vec{a} + m_s G \frac{M_E}{r^2} \vec{r} \times \hat{r} = 0$$

$$m_s \vec{r} \times \vec{a} + m_s G \frac{M_E}{r^2} r \hat{r} \times \hat{r} = 0$$

The last term has  $\hat{r} \times \hat{r}$ . The angle between  $\hat{r}$  and  $\hat{r}$  must be zero (they are in the same direction) so

$$\hat{r} \times \hat{r} = (1)(1) \sin(0) = 0$$

and we are left with

$$m_s \vec{r} \times \vec{a} = 0$$

which really does not seem to help, but it is. Consider that

$$\vec{a} = \frac{d^2 \vec{r}}{dt^2}$$

so



$$\begin{aligned}\vec{r} \times \vec{a} &= \vec{r} \times \frac{d^2 \vec{r}}{dt^2} \\ &= \vec{r} \times \frac{d^2 (r \hat{r})}{dt^2}\end{aligned}$$

Now consider the quantity

$$\frac{d}{dt} \left( \vec{r} \times \frac{d \vec{r}}{dt} \right) = \vec{r} \times \frac{d^2 \vec{r}}{dt^2} + \frac{d \vec{r}}{dt} \times \frac{d \vec{r}}{dt}$$

The second term must be zero because the angle between any vector and itself must be zero and  $\sin(0) = 0$ , but the first term is just what we have in our equation! so our equation becomes

$$m_s \vec{r} \times \vec{a} = m_s \frac{d}{dt} \left( \vec{r} \times \frac{d \vec{r}}{dt} \right) = 0$$

which we can write as

$$\frac{d}{dt} \left( \vec{r} \times m_s \frac{d \vec{r}}{dt} \right) = 0$$

$$\frac{d}{dt} (\vec{r} \times m_s \vec{v})$$

$$= \frac{d}{dt} (\vec{L}) = 0$$

and, hurrah! we have conservation of angular momentum for our general orbit!

## 1.4 Conic Section Equation

You may not be as thrilled as I was at this point, but what we have done is typical for physicists. We use the power of mathematics and some ingenuity to predict what

motions will be. You might say, “but I would never think of taking cross and dot products seemingly randomly to find a result.” This may be true now, but as you get used to using the mathematical tools an operation like this may become more obvious. In any case, recall that early physicists spent many years trying out ways to use their mathematical tools. So eventually someone was bound to try our cross and dot product tricks. But we have only shown conservation of energy and angular momentum. We have not reached our goal. So let’s return to our basic motion equation that we started with

$$-m_s \vec{a} - m_s \left( G \frac{M_E}{r^2} \right) \hat{r} = 0$$

and now let’s consider our equation for angular momentum

$$\vec{L} = \vec{r} \times m_s \vec{v}$$

and form the cross product of the first equation with  $\vec{L}$

$$\begin{aligned} \left( -m_s \vec{a} - m_s \left( G \frac{M_E}{r^2} \right) \hat{r} \right) \times \vec{L} &= 0 \times \vec{L} \\ m_s \vec{a} \times \vec{L} + m_s \left( G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L} &= 0 \end{aligned}$$

Again this may not seem like an obvious thing to do! But we find that

$$m_s \vec{a} \times \vec{L} = - \left( G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L}$$

and it is time for another mathematical trick. Consider the quantity

$$\begin{aligned} \frac{d}{dt} (\vec{v} \times \vec{L}) &= \vec{v} \times \frac{d\vec{L}}{dt} + \frac{d\vec{v}}{dt} \times \vec{L} \\ &= \vec{v} \times \frac{d}{dt} (\vec{r} \times m_s \vec{v}) + \vec{a} \times \vec{L} \\ &= \vec{v} \times (0) + \vec{a} \times \vec{L} \\ &= \vec{a} \times \vec{L} \end{aligned}$$

for our situation because we have already shown that angular momentum is conserved.

So we have

$$m_s \frac{d}{dt} (\vec{v} \times \vec{L}) = - \left( G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L}$$

Now let's look at the right hand side. Writing out the angular momentum gives

$$\hat{r} \times \vec{L} = \hat{r} \times (\vec{r} \times m_s \vec{v})$$

and I will use a vector product identity that I will let the math department teach you

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

so for us

$$\begin{aligned} \vec{r} \times \vec{L} &= m_s \hat{r} \times (\vec{r} \times \vec{v}) \\ &= m_s (\vec{r} (\hat{r} \cdot \vec{v}) - \vec{v} (\hat{r} \cdot \vec{r})) \\ &= m_s (\vec{r} (\hat{r} \cdot \vec{v}) - \vec{v} r) \end{aligned}$$

We already know from equation (1.1) that

$$\vec{v} \cdot \hat{r} = \frac{dr}{dt}$$

then

$$\hat{r} \times \vec{L} = m_s \left( r \hat{r} \left( \frac{dr}{dt} \right) - \vec{v} r \right)$$

then finally

$$\begin{aligned} m_s \frac{d}{dt} (\vec{v} \times \vec{L}) &= - \left( G \frac{M_E m_s}{r^2} \right) \left( r \hat{r} \left( \frac{dr}{dt} \right) - \vec{v} r \right) \\ &= - (G M_E m_s) \left[ \left( \frac{dr}{dt} \right) \frac{\hat{r}}{r} - \frac{\vec{v}}{r} \right] \end{aligned}$$

Let's employ one more mathematical trick

$$\frac{d}{dt} \left( \frac{\vec{r}}{r} \right) = - \vec{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
&= -\vec{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \vec{v} \\
&= -\left( \hat{r} \frac{1}{r} \frac{dr}{dt} - \frac{1}{r} \vec{v} \right)
\end{aligned}$$

and this is the part of our equation that I wrote in square brackets, so with a substitution our equation becomes

$$m_s \frac{d}{dt} (\vec{v} \times \vec{L}) = (GM_E m_s) \left( \frac{d}{dt} \left( \frac{\vec{r}}{r} \right) \right)$$

or, canceling the  $dt$  factors from both sides

$$m_s d(\vec{v} \times \vec{L}) = (GM_E m_s) \left( d \left( \frac{\vec{r}}{r} \right) \right)$$

and we can integrate both sides

$$m_s \int d(\vec{v} \times \vec{L}) = - (GM_E m_s) \int \left( d \left( \frac{\vec{r}}{r} \right) \right)$$

to find

$$m_s \vec{v} \times \vec{L} = (GM_E m_s) \left( \frac{\vec{r}}{r} \right) + \vec{B}$$

where  $\vec{B}$  is a vector constant of integration. Once again for no apparent reason let's take the dot product of this equation with  $\vec{r}$

$$\vec{r} \cdot (m_s \vec{v} \times \vec{L}) = - (GM_E m_s) \vec{r} \cdot \left( \frac{\vec{r}}{r} \right) + \vec{r} \cdot \vec{B}$$

and use another vector product identity

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$$

We can write this as to write our dot product equation as

$$(m_s \vec{r} \times \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

or

$$\frac{1}{m_s} (\vec{r} \times m_s \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$(\vec{r} \times m_s \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$\vec{L} \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$L^2 = (GM_E m_s) r + r B \cos \theta_{rB}$$

and now we can solve for  $r$

$$L^2 = r ((GM_E m_s) + B \cos \theta_{rB})$$

then

$$r = \frac{L^2}{((GM_E m_s) + B \cos \theta_{rB})}$$

or, rearranging slightly,

$$r = \frac{L^2 / (GM_E m_s)}{(1 + (B/GM_E m_s) \cos \theta_{rB})}$$

If we compare this to the parametric equation for a conic section (straight out of your calculus text book),

$$r = \frac{p}{1 + e \cos \nu}$$

we can see that our orbit must be a conic section with a semi-latus rectum,

$$p = L^2 / (GM_E m_s)$$

and an eccentricity,

$$e = B/GM_E m_s$$

and an angle

$$\nu = \theta_{rB}$$

This means our orbit could be any conic section, circle, ellipse, parabola, or hyperbola. For satellites we most often choose ellipses. But the other conic sections are

possible. So Kepler was partially right. An ellipse is a general form for an orbit, but it might even be better to write Kepler's law to say that orbits are conic sections.

If you are a normal PH121 student, your reaction to this problem might be "Agh, maybe I should change my major to horticulture!" But don't worry, This was really a junior level problem, and for us physics majors we have many classes (both physics and math classes) to take before we would be expected to do a problem like this. Still it is fun to see that we *can* do a problem like this with the math we learned in lowly PH121 if we are very persistent! Interested students can read more in the the book *Fundamentals of Astrodynamics* by Bate *et. al.* [1][2][3] [4]

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