

ORBITS AS CONIC SECTIONS
SHOWING THAT A DISTANCE SQUARED LAW GIVES AN ELIPTICAL ORBIT

by
Todd Lines

A senior thesis submitted to the faculty of
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Brigham Young University - Idaho

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DEPARTMENT APPROVAL

of a senior thesis submitted by

Todd Lines

This thesis has been reviewed by the research advisor, research coordinator,
and department chair and has been found to be satisfactory.

Date

Lance Nelson, Advisor

Date

Ryan Nielson, Committee Member

Date

Brian Tonks, Committee Member

Date

Evan Hansen, Thesis Coordinator

Date

R. Todd Lines, Chair

ABSTRACT

ORBITS AS CONIC SECTIONS

SHOWING THAT A DISTANCE SQUARED LAW GIVES AN ELIPTICAL ORBIT

Todd Lines

Department of Physics and Astronomy

Bachelor of Science

The abstract is a *summary* of the thesis, *not an introduction*. Keep in mind that abstracts are often published separately from the paper they summarize. In your abstract, give a concise synopsis of the work, emphasizing the conclusions; you need not include the supporting arguments for the conclusions. The purpose of the abstract is to help prospective readers decide whether to read your thesis, but your goal is not necessarily to persuade people to read your thesis. In fact, a successful abstract enables people to get an accurate overall view of your work without needing to read it. Usually, an abstract contains a single paragraph, but it can have more if absolutely necessary. Remember to state the subject of the paper immediately followed by a summary of the experimental or theoretical results and the methods used to obtain them. Avoid equations, graphics, and citations; if a citation is essential it must be cited fully within the abstract. Keep the abstract factual. Avoid vague statements like,

“Conclusions are drawn,” or “the significance of the experiment is discussed.”

State the conclusions and findings outright in the abstract.

ACKNOWLEDGMENTS

This page is optional. You may acknowledge whom you will—your advisor, colleagues, family members. Please keep acknowledgments in good taste. I would like to acknowledge Dr. Kristine Hansen and Dr. Elizabeth Hedengren, whose Advanced Writing Seminar motivated this project. I also wish to thank Jean-François Van Huele, Steven Turley, and Ross Spencer for reviewing this document and for ripping it to shreds as every good advisor should do to a thesis draft.

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Chapter 1

Orbits

In the following sections, we show that orbits are really conic sections. The case of a circular orbit is just a special case.

1.1 Elliptical orbits

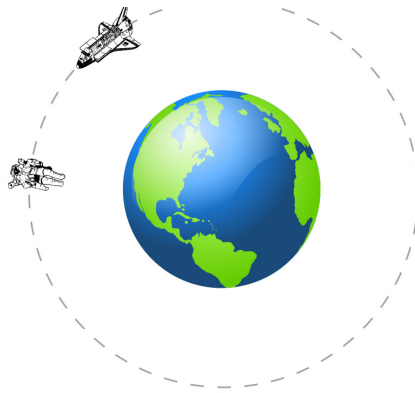


Figure 1.1 A PH121 Circular Orbit, It's not Enough!

In all our work in PH121 we used circular orbits. Kepler said orbits



Figure 1.2 A mistake

should be elliptical. And that is true. We won't go through this in class, but showing that Newton's law of gravitation implies an ellipse is a great way to show off our new mathematics of dot and cross products. So if you are comfortable with our new math, and curious to see how orbits work, read on. Let's start with Newton's second law for our orbiting satellite again.

$$\begin{aligned}
 \vec{W} &= -m_s \vec{a} \\
 &= -m_s g(r) \hat{r} \\
 &= -m_s \left(G \frac{M_E}{r_{Es}^2} \right) \hat{r}
 \end{aligned}$$

We can write this as

$$-m_s \vec{a} - m_s \left(G \frac{M_E}{r_{Es}^2} \right) \hat{r} = 0$$

The subscripts may become burdensome, so we will drop them now, but remember that $r = r_{Es}$ is the distance from the satellite to the Earth center of mass to center of mass.

$$-m_s \vec{a} - m_s \left(G \frac{M_E}{r^2} \right) \hat{r} = 0$$

In the next section we will find that conservation of energy is important 1.2. Notice that I used a marker to come up with the section number automatically.

1.2 Conservation of Orbital Mechanical Energy

Now we are going to do something strange. For no apparent reason, lets compute the dot product of both sides of this equation

$$\vec{v} \cdot \left(m_s \vec{a} + m_s \left(G \frac{M_E}{r^2} \right) \hat{r} \right) = \vec{v} \cdot 0$$

then

$$\vec{v} \cdot m_s \vec{a} + \vec{v} \cdot m_s \left(G \frac{M_E}{r^2} \right) \hat{r} = 0$$

or

$$m_s \vec{v} \cdot \vec{a} + m_s \left(G \frac{M_E}{r^2} \right) \vec{v} \cdot \hat{r} = 0$$

Now we need to learn a little bit more about dot products mixed with derivatives. We have a position vector $\vec{r} = r\hat{r}$ The derivative of this position vector is

$$\frac{d\vec{r}}{dt} = \frac{d}{dt} (r\hat{r})$$

$$= r \frac{d\hat{r}}{dt} + \frac{dr}{dt} \hat{r}$$

so if we take

$$\frac{d\vec{r}}{dt} \cdot \hat{r} = \left(r \frac{d\hat{r}}{dt} + \frac{dr}{dt} \hat{r} \right) \cdot \hat{r}$$

$$= r \frac{d\hat{r}}{dt} \cdot \hat{r} + \frac{dr}{dt} \hat{r} \cdot \hat{r}$$

$$= 0 + \frac{dr}{dt}$$

since $d\hat{r}/dt = 0$.

So

$$\frac{d\vec{r}}{dt} \cdot \hat{r} = \frac{dr}{dt}$$

and we recognize

$$\vec{v} = \frac{d\vec{r}}{dt}$$

so we can write

$$\vec{v} \cdot \hat{r} = \frac{dr}{dt} \tag{1.1}$$

and we have this in our orbit equation. Our orbit equation becomes

$$m_s \vec{v} \cdot \vec{a} + m_s \left(G \frac{M_E}{r_{Es}^2} \right) \frac{d\vec{r}}{dt} \cdot \hat{r} = 0$$

or just

$$m_s \vec{v} \cdot \vec{a} + m_s \left(G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

We can do something similar for the first term We can recognize

$$\vec{a} = \frac{d\vec{v}}{dt}$$

and that $\vec{v} = v\hat{v}$ where \hat{v} is a unit vector in the same direction \vec{v} . Then

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{v})$$

$$= v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v}$$

and

$$\begin{aligned}\vec{v} \cdot \vec{a} &= \vec{v} \cdot \frac{d\vec{v}}{dt} \\ &= \vec{v} \cdot \left(v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v} \right) \\ &= v\hat{v} \cdot v\frac{d\hat{v}}{dt} + v\hat{v} \cdot \frac{dv}{dt}\hat{v} \\ &= 0 + v\frac{dv}{dt}\hat{v} \cdot \hat{v} \\ &= v\frac{dv}{dt}\end{aligned}$$

so our orbit equation becomes

$$m_s v \frac{dv}{dt} + m_s \left(G \frac{M_E}{r_{Es}^2} \right) \frac{dr}{dt} = 0$$

Now let's play a clever mathematical trick. Let's take the derivative of the kinetic energy with respect to time.

$$\begin{aligned}\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) &= \frac{1}{2} m \frac{d}{dt} (v^2) \\ &= \frac{1}{2} m \left(2v \frac{dv}{dt} \right) \\ &= m v \frac{dv}{dt}\end{aligned}$$

Notice that this is in our orbit equation! So then

$$m_s v \frac{dv}{dt} + m_s \left(G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

becomes

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) + m_s \left(G \frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

We can play this trick again for the second term

$$\begin{aligned}\frac{d}{dt} \left(G \frac{M_E}{r} \right) &= G M_E \frac{d}{dt} \left(\frac{1}{r} \right) \\ &= G M_E \left(-\frac{1}{r^2} \frac{dr}{dt} \right)\end{aligned}$$

which once again we recognize this as part of our orbit equation so we can write

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) - m_s \frac{d}{dt} \left(G \frac{M_E}{r} \right) = 0$$

or

$$\frac{d}{dt} \left(\left(\frac{1}{2} m v^2 \right) - m_s \left(G \frac{M_E}{r} \right) \right) = 0$$

which tells us that

$$\left(\frac{1}{2}mv^2\right) - m_s \left(G \frac{M_E}{r_{Es}}\right) = \text{constant}$$

That is, the mechanical energy is conserved since we recognize this as just

$$K + U_g = \text{constant}$$

And this makes sense. There are no energy loss mechanisms in our orbit. Our masses are particles (no tidal forces inside the objects, etc.) So we expect conservation of energy in forming our orbit.

1.3 Conservation of Orbital Angular Momentum

Now, let's do just what we did before only let's use a cross product with \vec{r} .

$$\vec{r} \times \left(-m_s \vec{a} - m_s \left(G \frac{M_E}{r^2}\right) \hat{r}\right) = \vec{r} \times 0$$

$$-\vec{r} \times m_s \vec{a} - \vec{r} \times m_s \left(G \frac{M_E}{r^2}\right) \hat{r} = 0$$

$$m_s \vec{r} \times \vec{a} + m_s G \frac{M_E}{r^2} \vec{r} \times \hat{r} = 0$$

$$m_s \vec{r} \times \vec{a} + m_s G \frac{M_E}{r^2} r \hat{r} \times \hat{r} = 0$$

The last term has $\hat{r} \times \hat{r}$. The angle between \hat{r} and \hat{r} must be zero (they are in the same direction) so

$$\hat{r} \times \hat{r} = (1)(1) \sin(0) = 0$$

and we are left with

$$m_s \vec{r} \times \vec{a} = 0$$

which really does not seem to be helpful, but it is. Consider that

$$\vec{a} = \frac{d^2 \vec{r}}{dt^2}$$

so

$$\begin{aligned} \vec{r} \times \vec{a} &= \vec{r} \times \frac{d^2 \vec{r}}{dt^2} \\ &= \vec{r} \times \frac{d^2 (r \hat{r})}{dt^2} \end{aligned}$$

Now consider the quantity

$$\frac{d}{dt} \left(\vec{r} \times \frac{d \vec{r}}{dt} \right) = \vec{r} \times \frac{d^2 \vec{r}}{dt^2} + \frac{d \vec{r}}{dt} \times \frac{d \vec{r}}{dt}$$

The second term must be zero because the angle between any vector and itself must be zero and $\sin(0) = 0$, but the first term is just what we have in our equation! so our equation becomes

$$m_s \vec{r} \times \vec{a} = m_s \frac{d}{dt} \left(\vec{r} \times \frac{d \vec{r}}{dt} \right) = 0$$

which we can write as

$$\frac{d}{dt} \left(\vec{r} \times m_s \frac{d \vec{r}}{dt} \right) = 0$$

$$\frac{d}{dt} (\vec{r} \times m_s \vec{v})$$

$$= \frac{d}{dt} (\vec{L}) = 0$$

and, hurray! we have conservation of angular momentum for our general orbit!

1.4 Conic Section Equation

You may not be as thrilled as I was at this point, but what we have done is typical for physicists. We use the power of mathematics and some ingenuity to predict what motions will be. You might say, “but I would never think of taking cross and dot products seemingly randomly to find a result.” This may be true now, but as you get used to using the mathematical tools an operation like this may become more obvious. In any case, recall that early physicists spent many years trying out ways to use their mathematical tools. So eventually someone was bound to try our cross and dot product tricks. But we have only shown conservation of energy and angular momentum. We have not reached our goal. So let’s return to our basic motion equation that we started with

$$-m_s \vec{a} - m_s \left(G \frac{M_E}{r^2} \right) \hat{r} = 0$$

and now let’s consider our equation for angular momentum

$$\vec{L} = \vec{r} \times m_s \vec{v}$$

and form the cross product of the first equation with \vec{L}

$$\left(-m_s \vec{a} - m_s \left(G \frac{M_E}{r^2} \right) \hat{r} \right) \times \vec{L} = 0 \times \vec{L}$$

$$m_s \vec{a} \times \vec{L} + m_s \left(G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L} = 0$$

Again this may not seem like an obvious thing to do! But we find that

$$m_s \vec{a} \times \vec{L} = - \left(G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L}$$

and it is time for another mathematical trick. Consider the quantity

$$\begin{aligned} \frac{d}{dt} (\vec{v} \times \vec{L}) &= \vec{v} \times \frac{d\vec{L}}{dt} + \frac{d\vec{v}}{dt} \times \vec{L} \\ &= \vec{v} \times \frac{d}{dt} (\vec{r} \times m_s \vec{v}) + \vec{a} \times \vec{L} \\ &= \vec{v} \times (0) + \vec{a} \times \vec{L} \\ &= \vec{a} \times \vec{L} \end{aligned}$$

for our situation because we have already shown that angular momentum is conserved.

So we have

$$m_s \frac{d}{dt} (\vec{v} \times \vec{L}) = - \left(G \frac{M_E}{r^2} \right) \hat{r} \times \vec{L}$$

Now let's look at the right hand side. Writing out the angular momentum gives

$$\hat{r} \times \vec{L} = \hat{r} \times (\vec{r} \times m_s \vec{v})$$

and I will use a vector product identity that I will let the math department teach you

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

so for us

$$\begin{aligned} \vec{r} \times \vec{L} &= m_s \hat{r} \times (\vec{r} \times \vec{v}) \\ &= m_s (\vec{r} (\hat{r} \cdot \vec{v}) - \vec{v} (\hat{r} \cdot \vec{r})) \\ &= m_s (\vec{r} (\hat{r} \cdot \vec{v}) - \vec{v} r) \end{aligned}$$

We already know from equation (1.1) that

$$\vec{v} \cdot \hat{r} = \frac{dr}{dt}$$

then

$$\hat{r} \times \vec{L} = m_s \left(r \hat{r} \left(\frac{dr}{dt} \right) - \vec{v} r \right)$$

then finally

$$\begin{aligned} m_s \frac{d}{dt} (\vec{v} \times \vec{L}) &= - \left(G \frac{M_E m_s}{r^2} \right) \left(r \hat{r} \left(\frac{dr}{dt} \right) - \vec{v} r \right) \\ &= - (GM_E m_s) \left[\left(\frac{dr}{dt} \right) \frac{\hat{r}}{r} - \frac{\vec{v}}{r} \right] \end{aligned}$$

Let's employ one more mathematical trick

$$\begin{aligned} \frac{d}{dt} \left(\frac{\vec{r}}{r} \right) &= - \vec{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \frac{d\vec{r}}{dt} \\ &= - \vec{r} \frac{1}{r^2} \frac{dr}{dt} + \frac{1}{r} \vec{v} \\ &= - \left(\hat{r} \frac{1}{r} \frac{dr}{dt} - \frac{1}{r} \vec{v} \right) \end{aligned}$$

and this is the part of our equation that I wrote in square brackets, so with a substitution our equation becomes

$$m_s \frac{d}{dt} (\vec{v} \times \vec{L}) = (GM_E m_s) \left(\frac{d}{dt} \left(\frac{\vec{r}}{r} \right) \right)$$

or, canceling the dt factors from both sides

$$m_s d(\vec{v} \times \vec{L}) = (GM_E m_s) \left(d \left(\frac{\vec{r}}{r} \right) \right)$$

and we can integrate both sides

$$m_s \int d(\vec{v} \times \vec{L}) = - (GM_E m_s) \int \left(d \left(\frac{\vec{r}}{r} \right) \right)$$

to find

$$m_s \vec{v} \times \vec{L} = (GM_E m_s) \left(\frac{\vec{r}}{r} \right) + \vec{B}$$

where \vec{B} is a vector constant of integration. Once again for no apparent reason let's take the dot product of this equation with \vec{r}

$$\vec{r} \cdot (m_s \vec{v} \times \vec{L}) = - (GM_E m_s) \vec{r} \cdot \left(\frac{\vec{r}}{r} \right) + \vec{r} \cdot \vec{B}$$

and use another vector product identity

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$$

We can write this as to write our dot product equation as

$$(m_s \vec{r} \times \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

or

$$\frac{1}{m_s} (\vec{r} \times m_s \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$(\vec{r} \times m_s \vec{v}) \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$\vec{L} \cdot \vec{L} = (GM_E m_s) r + r B \cos \theta_{rB}$$

$$L^2 = (GM_E m_s) r + r B \cos \theta_{rB}$$

and now we can solve for r

$$L^2 = r ((GM_E m_s) + B \cos \theta_{rB})$$

then

$$r = \frac{L^2}{((GM_E m_s) + B \cos \theta_{rB})}$$

or, rearranging slightly,

$$r = \frac{L^2 / (GM_E m_s)}{(1 + (B / GM_E m_s) \cos \theta_{rB})}$$

If we compare this to the parametric equation for a conic section (straight out of your calculus text book),

$$r = \frac{p}{1 + e \cos \nu}$$

we can see that our orbit must be a conic section with a semi-latus rectum,

$$p = L^2 / (GM_E m_s)$$

and an eccentricity,

$$e = B/GM_E m_s$$

and an angle

$$\nu = \theta_{rB}$$

This means our orbit could be any conic section, circle, ellipse, parabola, or hyperbola. For satellites we most often choose ellipses. But the other conic sections are possible. So Kepler was partially right. An ellipse is a general form for an orbit, but it might even be better to write Kepler's law to say that orbits are conic sections.

If you are a normal PH121 student, your reaction to this problem might be "Agh, maybe I should change my major to horticulture!" But don't worry, This was really a junior level problem, and for us physics majors we have many classes (both physics and math classes) to take before we would be expected to do a problem like this. Still it is fun to see that we *can* do a problem like this with the math we learned in lowly PH121 if we are very persistent! Interested students can read more in the the book *Fundamentals of Astrodynamics* by Bate *et. al.* [1][2][3][4]

Chapter 2

Numerical Verification of the Orbit Equation

Let's start with a long meaningless quotation so we can see how quotations are done.

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal.

Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure. We are met on a great battle-field of that war. We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this.

But, in a larger sense, we can not dedicate – we can not consecrate – we can not hallow – this ground. The brave men, living and dead, who struggled here, have consecrated it, far above our poor power to add or detract. The world will little note, nor long remember what we say here,

but it can never forget what they did here. It is for us the living, rather, to be dedicated here to the unfinished work which they who fought here have thus far so nobly advanced. It is rather for us to be here dedicated to the great task remaining before us – that from these honored dead we take increased devotion to that cause for which they gave the last full measure of devotion – that we here highly resolve that these dead shall not have died in vain – that this nation, under God, shall have a new birth of freedom – and that government of the people, by the people, for the people, shall not perish from the earth.

Now some text outside the quotation. There is very little O₂ in space.

2.1 Some section names are way to long to appear in the table of contents so you need to include a short title for the TOC

This is an example of a section name that is too long. You should shorten it if you can, but if you can't this long name will mess up the Table of Contents. So there is an option to include a short title. Put the short section name version in square brackets before the actual (long) section name (in curly brackets).

Bibliography

- [1] R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics* (Dover, New York, 1971).
- [2] C. Barnatt, <https://www.explainingcomputers.com/>, 2021-05-15.
- [3] BYU Web Team, <https://www.byui.edu/physics/>, 2021-11-05.
- [4] C. Acquista, “Light Scattering by Tenuous Particles: A Generalization of the Rayleigh-Gans-Rocard Approach,” *Applied Optics* **15**, 2932–2936 (1976).

Appendix A

Things that belong in an appendix

The purpose of an appendix is to provide supplementary information which would distract if included in the main body of the thesis. Items appearing as an appendix might include lengthy derivations. If students feel compelled to include a brief tutorial on relevant background information (not new research), it should appear as an appendix. An appendix might consist of portions of unique computer code that was developed as part of the project.

Appendix B

Including Your Code

You might want to include some code in your thesis. This is done with a verbatim environment so L^AT_EX doesn't try to interpret the code as L^AT_EX commands. I want to make sure the code fits on the page, so I am going to reduce the font size with a small environment as well.

```
#####  
# second order Runge Cutta for a projectile with expansion  #  
#####  
# Todd Lines  
# 2020 08 14  
#####  
#  
#####  
# Load Libraries  
import numpy as np  
#import matplotlib  
#matplotlib.use('tkagg')  
import matplotlib.pyplot as plt
```

```
#####

# Initial conditions and physial setup constants
R=6371E3      # radius of Earth in m
M=5.9E24      # Mass of plannet in Kg
m=0.020       # mass of sattelite in kilograms
G=5.11E-11    # N*m**2/kg**2
x0=2*R        # initial x position in m
y0=0.5*R      # initial y position in m
v0=6000.0     # initial velocity in m/s
thetadeg=100  # launch angle in degrees
Rexp=0.007    # expansion coefficient in m/s/m This is really
# a type of Hubble constant, but I am not using
# the right value yet.

#####

## Set up the time steps and number of calcualtions
deltat=1.0     # Time steps of 0.01 seconds
ti=0          # starting at t=0
tf=200000     # final time
N=int((tf-ti)/deltat) # calcualte how many time steps are in 20 seconds

#####

# Preliminary calculations
pi=np.arccos(-1.0)      # calculate pi to machine percision
theta=thetadeg*pi/180   # calcualte theta in radians
vx1=v0*np.cos(theta)    # calculte the x component of the initial velocity
vy1=v0*np.sin(theta)    # calculte the y component of the initial velocity

#####
```

```

# define and zero arrays
t=np.zeros((N))
x=np.zeros((N))
y=np.zeros((N))
z=np.zeros((N))
vx=np.zeros((N))
vy=np.zeros((N))
xnd=np.zeros((N))
ynd=np.zeros((N))
xE=np.zeros((N))
yE=np.zeros((N))

#####

## make an array of times to use
t=np.linspace(0,tf,num=N);

#####

# Draw The Earth
r=R      # distance from center in m
ThetaE0=0.00      # initial angle in degrees
delta_ThetaE=5  # change in angle in degrees
Ne= int(360/delta_ThetaE)+1      # number of points
ye=np.zeros((Ne))
xe=np.zeros((Ne))

#####

# Draw Circle
ye[0]=0      # initial y positoin
xe[0]=r

```

```

thetaE=ThetaE0
for i in range (0,Ne):
    thetaE=thetaE+delta_ThetaE
    xe[i]=r*np.cos(thetaE*pi/180)
    ye[i]=r*np.sin(thetaE*pi/180)
    plt.plot(xe,ye)

#####
# now perform an Euler's Method Calculation
# now recalling that vx(i) already has a cos(theta) in it,
# we can use this to calculate the x part of the resistive
# force and likewise use vy(i) in calculating the y part of
# the resistive force. No explicit calculation of theta is
# necessary this way, and we save lots of computation time.
x=np.zeros((N))
y=np.zeros((N))
vx=np.zeros((N))
vy=np.zeros((N))

x[0]=x0                # initial x position
y[0]=y0                # initial y positoin
vx[0]=vx1               # initial x velocity
vy[0]=vy1              # initial y velocity, what we give it

# plus the expansion sudo velocity
print('working')
for i in range (0,N-1):
    r=np.sqrt(x[i]**2+y[i]**2)
    if x[i] != 0:
        theta=np.arctan(y[i]/x[i])

```

```
else:
theta=pi/2

if y[i]>0 and x[i]<0:
theta=pi+theta
if y[i]<0 and x[i]<=0:
theta=theta+pi
if y[i]<0 and x[i]>0:
theta=2*pi+theta
if r<R:
print("hit surface")
break    #hit surface

fx=vx[i]
gx=-np.cos(theta) * G*M/r**2

fy=vy[i]
gy=-np.sin(theta) * G*M/r**2

x[i+1]=x[i]+deltat*fx
y[i+1]=y[i]+deltat*fy
vx[i+1]=vx[i]+deltat*gx
vy[i+1]=vy[i]+deltat*gy    # once again I have added in the
# expansion term
#print(i, theta, x[i],y[i],vx[i],vy[i], gx, gy)

print('done')
```

```

plt.plot(x,y)

#xEu=np.copy(x)

#yEu=np.copy(y)

#plt.plot(xEu,yEu, label='expansion2')

#####

##now perform an RK2 Method Calculation

#x[0]=x0                # initial x position
#y[0]=y0                # initial y positoin
#vx[0]=vx1              # inintal x velocity
#vy[0]=vy1+Rexp*y[0]    # initial y velocity, what we give it
#                        # plus the expansion sudo velocity

#for i in range (0,N-1):
#    v=np.sqrt(vx[i]**2+vy[i]**2)
#    # first the Euler step, This is tricky because I want just the x
#    # component and the y component of the drag force, but the drag force
#    # depends on  $v^2$ . Remembering that  $vx=v*\cos(\theta)$ , we can then multiply
#    # the speed,  $v$ , by  $vx$  to get  $v^2*\cos(\theta)$ . This way we don't have to
#    # calculate theta explicitly
#    Rx=0.5*D*rho*pi*r*r*v*vx[i]
#    Ry=0.5*D*rho*pi*r*r*v*vy[i]
#    fx=vx[i]
#    gx=-Rx/m
#    fy=vy[i]
#    gy=-g-Ry/m
#    kx1=deltat*fx
#    ky1=deltat*fy

```

```

#   kvx1=deltat*gx
#   kvy1=deltat*gy
#   #now the RK step, What to do with the  $v^2$ ? I think we can do the same
#   #thing as above to find the x abd y components of the velocity.
#   v=np.sqrt((vx[i]+kvx1/2)**2+(vy[i]+kvy1/2)**2)
#   Rx2=0.5*D*rho*pi*r*r*v*(vx[i]+kvx1/2)
#   Ry2=0.5*D*rho*pi*r*r*v*(vy[i]+kvy1/2)
#   fx2=vx[i]+kvx1/2
#   gx2=-Rx/m
#   fy2=vy[i]+kvy1/2
#   gy2=-g-Ry/m
#   #finally take the RK step.
#   x[i+1]=x[i]+deltat*fx2
#   y[i+1]=y[i]+deltat*fy2
#   vx[i+1]=vx[i]+deltat*gx2
#   vy[i+1]=vy[i]+deltat*gy2+Rexp*y[i]      # once again I have added in the
#                                           # expansion term
#   #if y(i+1)<=0.0 , break, end
#
#xRK=np.copy(x)
#yRK=np.copy(y)
#
#plt.plot(xnd,ynd, label='kinematic')
#plt.plot(xEu,yEu, label='expansion')
#plt.plot(xRK,yRK,label='RK')
#plt.xlabel('x [m]')
#plt.ylabel('y [m]')
#leg=plt.legend();

```



```
plt.show()
```