Chapter 3

Electric Fields of Standard Charge Configurations Part I

Fundamental Concepts

- Adding of vector fields for point charges
- Standard configurations of charge

3.1 Standard Charge Configurations

Actual engineering projects or experimental designs require detailed calculations of fields using computers. These field simulations use powerful numerical techniques that are beyond this sophomore class. But we can gain some great insight by using some basic models of simple charged objects. We will often look at the following models:

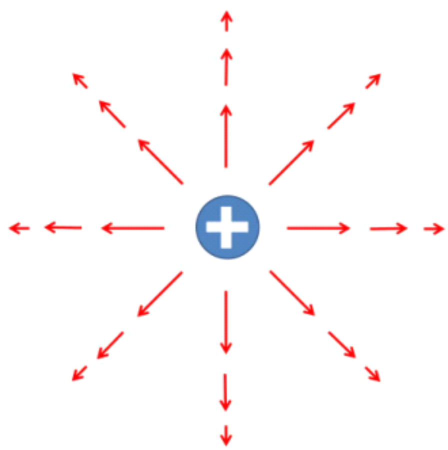
Standard Configurations of Charge
Point charge
Several point charges
Line of Charge
Semi-infinite sheet of charge
Charged sphere
Charged spherical shell
Ring of Charge

3.2 Point Charges

We have already met one of these standard configurations, the point charge

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}$$

The field of the point charge is represented below



This picture requires a little explanation. The arrows are larger nearer the charge to show that the field is stronger. But note that each arrow is the magnitude and direction of the charge at one point. We really need a three dimensional picture to describe this, and even then the fact that the arrows have length can be misleading. The long arrows cover up other points, that should also have arrows. We can only draw the field at a few points, and at those points the field has both magnitude and direction. But we must remember that there is really a field magnitude and direction at every point.

To go beyond single charges we need a group of point charges of some sort. The fields add like forces

$$\overrightarrow{\mathbf{E}} = \sum_i \overrightarrow{\mathbf{E}}_i$$

were we recognize that we are summing vectors. Let's take a look at a few combinations of charges and find their fields

3.2.1 Two charges

Let's go back to our idea of an environmental charge, Q_E, and a mover charge, q_m. The mover charge is considered to be small enough that its effect on Q_E is negligible. So the field due to the large charge is unaffected by this small charge.

Of course, the total field is a superposition of both fields. We call the field produced by the little mover charge it's *self-field*. But the mover charge can't move itself.[1] The mover's self-field can't move the mover. So we don't draw the field due to q_m. We we can envision an environmental field that is just due to the environmental charge, Q_E, as if there are no other charges any where in the whole universe. Of course this is not the case, but this is how we think of the field *due to* charge Q_E.

We can identify that a charge q_m placed in this field due to Q_E will feel a force

$$\begin{aligned}\overrightarrow{\mathbf{F}}_e &= q_m \overrightarrow{\mathbf{E}} \\ &= q_m \left(\frac{1}{4\pi\epsilon_o}\frac{Q_E}{r^2}\right) \\ &= \frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r^2}\hat{\mathbf{r}}\end{aligned}$$

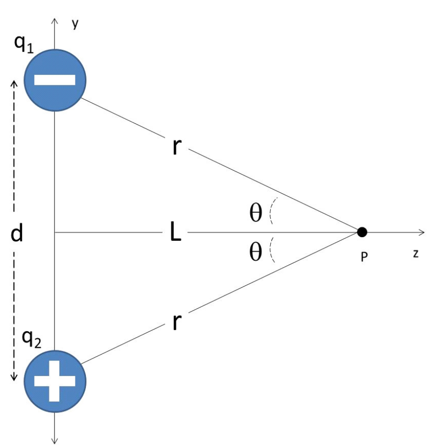
due to the field

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{Q_E}{r^2}\hat{\mathbf{r}}$$

where this field is just due to Q_E and does not contain the contribution from q_m. So the charge q_m only feels a force due to the field created by charge Q_E. A third charge, q_{new} brought close to the other two would feel both $\overrightarrow{\mathbf{E}}_Q$ and $\overrightarrow{\mathbf{E}}_q$. Then both Q_E and our original q_m would be environmental charges and the new charge q_{new} would be the mover. At this point, we would probably relabel Q_E and q_m as Q_1 and Q_2 and relabel q_{new} as q_m so we could tell that the original two charges are now the environment and the new charge is the mover.

3.2.2 Vector nature of the field

Remember that the field is a force per unit charge. Forces add as vectors, so we should expect fields to add as vectors too. Let's do a problem.



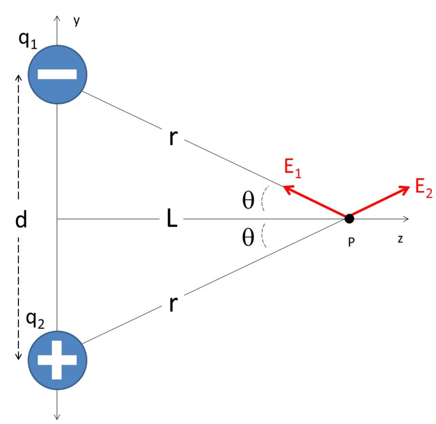
[1] This would allow perpetual motion, breaking the second law of thermodynamics.

Two charges are separated by a distance d. What is the field a distance L from the center of the two charges?

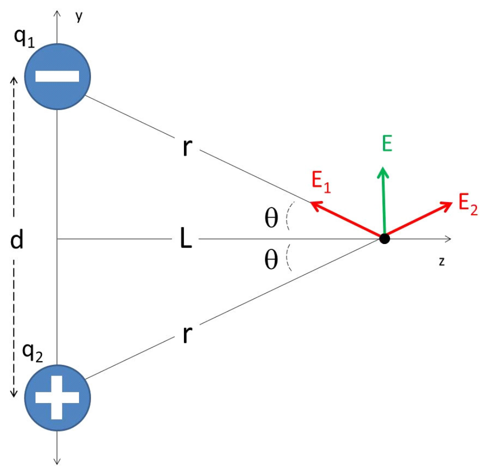
We should recognize this as our old friend, the dipole.

Note that both of these charges are environmental charges. We are asked in this problem to find the environment, the field. We don't really have a mover charge. But we could pretend we do have a mover, q_m at point P where we want to know the environment if it helps us picture the situation. But really we are calculating what the environment around the two charges will be.

We start by drawing the situation. I chose not to draw field lines. Instead I drew the field vectors at the point, P, where we want the field. The field lines would tell me about the whole environment everywhere, and that might be useful. But this problem only wants to know the field at one point, P. So it was less work to draw the field using vectors at our one point.



Note that I need a vector for each of the environmental charges. Each contributes to the environment. The contribution to the field due to environmental charge q_1 is labeled E_1 and likewise the contribution to the field from environmental charge q_2 is labeled E_2.



The net environment is the superposition of the fields due to each of the environmental charges.

$$\overrightarrow{\mathbf{E}}_{net} = \overrightarrow{\mathbf{E}}_1 + \overrightarrow{\mathbf{E}}_3$$

From the figure, we see that if we had a small mover charge, q_o on the axis a distance at point p then we would get two forces, one from each of the environmental charges q_1 and q_2. We can use Newton's second law to find the net force on our imaginary q_o.

$$\begin{aligned} F_{net_z} &= ma_z = -F_1 \cos\theta + F_2 \cos\theta \\ F_{net_y} &= ma_y = F_1 \sin\theta + F_2 \sin\theta \end{aligned}$$

We are going to need the magnitudes F_1 and $F_{2.}$ Both will be of the form

$$F = q_m \left(\frac{1}{4\pi\epsilon_o} \frac{Q_E}{r^2} \right)$$

so that

$$F_1 = q_m \left(\frac{1}{4\pi\epsilon_o} \frac{q_1}{r_1^2} \right)$$

$$F_2 = q_m \left(\frac{1}{4\pi\epsilon_o} \frac{q_2}{r_2^2} \right)$$

so we need to know q_1 and q_2 and r_1 and $r_{2.}$ Remember that r is the distance from the environmental charge to the mover charge. Our mover charge would be at point P. And from the figure we can see that the distance from each environmental charge to point P is

$$r_1 = r_2 = \sqrt{\frac{d^2}{4} + L^2}$$

so

$$\begin{aligned} F_{net_z} &= ma_z = -\left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_1}{\frac{d^2}{4}+L^2}\right)\cos\theta + \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_2}{\frac{d^2}{4}+L^2}\right)\cos\theta \\ F_{net_y} &= ma_y = \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_1}{\frac{d^2}{4}+L^2}\right)\sin\theta + \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_2}{\frac{d^2}{4}+L^2}\right)\sin\theta \end{aligned}$$

and now we need the $\cos\theta$ and $\sin\theta$ parts. But look, the forces will cancel in the z-direction. So the net force in the z-direction should be zero.

$$0 = -\left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_1}{r_1^2}\right)\cos\theta + \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_2}{r_2^2}\right)\cos\theta$$

and since q_1 and q_2 are the same in magnitude, we can see that indeed everything does cancel in the z-direction.

In the y-direction both the F_{1y} and F_{2y} components point the same way. These won't cancel. To find our field in the y-direction we could use an expression for $\sin\theta$. From the figure,

$$\sin\theta = \frac{d}{2\sqrt{\frac{d^2}{4}+L^2}}$$

which we can put into our Newton's second law for the y-direction.

$$\begin{aligned} F_{net_y} &= ma_y = \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_1}{\frac{d^2}{4}+L^2}\right)\left(\frac{d}{2\sqrt{\frac{d^2}{4}+L^2}}\right) + \left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_2}{\frac{d^2}{4}+L^2}\right)\left(\frac{d}{2\sqrt{\frac{d^2}{4}+L^2}}\right) \\ &= 2\left(q_m \frac{1}{4\pi\epsilon_o}\frac{q_1 d}{2\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}}\right) \\ &= q_m \frac{1}{4\pi\epsilon_o}\frac{q_1 d}{\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}} \end{aligned}$$

where we have used the fact that $q_1 = q_2$. This is our net force. But we wanted the electric field! We know that

$$\overrightarrow{F}_{net} = q_m \overrightarrow{E}$$

Our Newton's second law becomes an equation for the components of the combined electric field if we just divide by q_m.

$$\overrightarrow{E} = \frac{\overrightarrow{F}_{net}}{q_m} = \frac{1}{4\pi\epsilon_o}\frac{q_1 d}{\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}}\hat{\jmath}$$

We should check to make sure that $\overrightarrow{E}$ depends on only one of the environmental charges. And it does.

But now we realize that we could have done all of this by dividing by q_m from the moment we set up our Newton's second law.

$$\begin{aligned}
\frac{F_{net_z}}{q_m} &= -\frac{F_1}{q_m}\cos\theta + \frac{F_2}{q_M}\cos\theta \\
\frac{F_{net_y}}{q_o} &= \frac{F_1}{q_o}\sin\theta + \frac{F_2}{q_o}\sin\theta
\end{aligned}$$

This would give

$$\begin{aligned}
E_{net_z} &= -E_1\cos\theta + E_2\cos\theta \\
E_{net_y} &= E_1\sin\theta + E_2\sin\theta
\end{aligned}$$

Which is really just

$$\overrightarrow{E}_{net} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$

We can find the combined field by adding the individual fields like vectors. Note that if the charges are equal in magnitude it is customary to define that magnitude $q = q_1 = q_2$ and to write our field equation as

$$\overrightarrow{E} = \frac{1}{4\pi\epsilon_o}\frac{qd}{\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}}\hat{\jmath}$$

Note that we pretended that we had a mover, q_m, but in finding the field the q_m canceled out, so indeed we are left with just the environment in our calculation, we just have the field.

Now suppose our mover charge is very far away. That is, suppose we make L very large. So large that $L \gg d$ then

$$\lim_{L\gg d}\frac{1}{\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}} = \frac{1}{L^3}$$

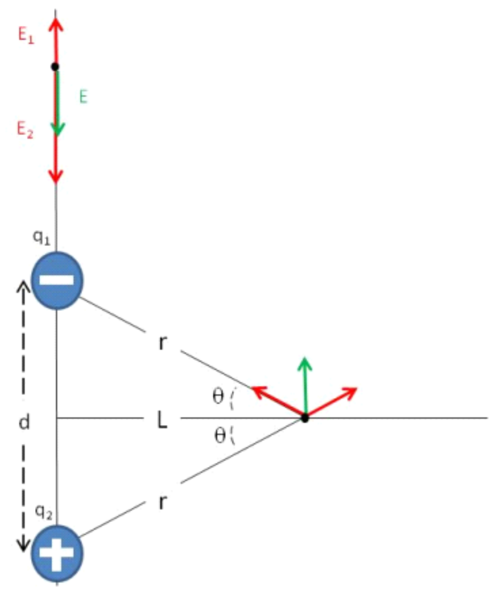
Then our field becomes

$$\begin{aligned}
E &= E_y = \lim_{L\gg d}\frac{2}{4\pi\epsilon_o}\frac{qd}{\left(\frac{d^2}{4}+L^2\right)^{\frac{3}{2}}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{qd}{L^3}
\end{aligned}$$

Since many charged particles are small, like atoms or molecules, this limit is often useful.

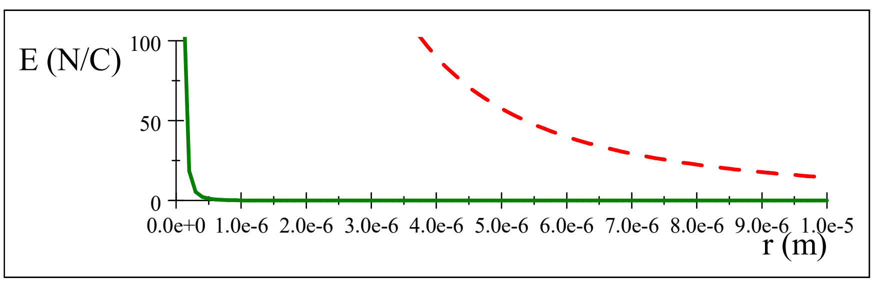
Suppose we repeat the calculation, but this time we chose a point that is L away, but that is on the y-axis above the charges, we would find

$$E = E_y = \frac{2}{4\pi\epsilon_o}\frac{qd}{L^3}$$



The result is similar, but the field is a little stronger in this direction.

Let's look at one of these cases by graphing it.

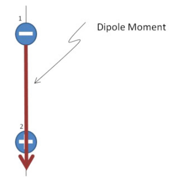


We can see that the dipole field (solid green line) falls off much faster than a point charge field (dashed red line). This makes sense because the farther away we get, the more it looks like the two charges are right next to each other, and since they are opposite in sign, they are essentially neutral when viewed together from far away. We can see why atoms don't exhibit a significant charge forces at normal distances.

This arrangement of charges we already know as a dipole. We are treating the two charges as a unit making the environment in which other charges might move. Since we are treating the two charges as one unit, it is customary to define a quantity

$$p = qd$$

and to make this a vector by defining the direction of p to be from the negative to the positive charge along the axis.

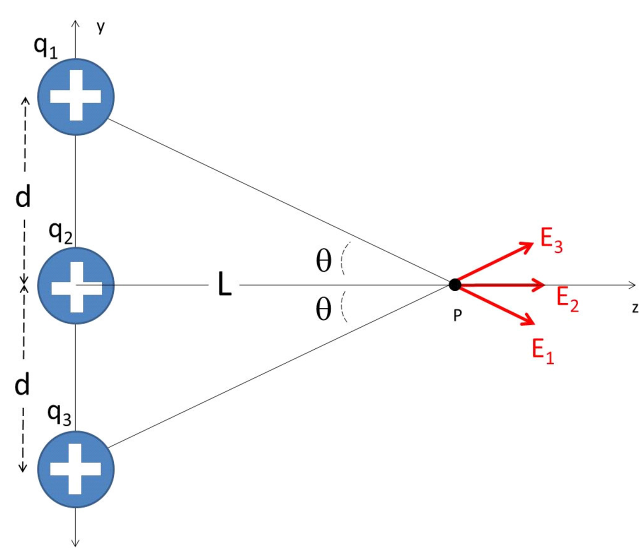


Then we can write the dipole field as

$$\overrightarrow{\mathbf{E}}_y = \frac{2}{4\pi\epsilon_o}\frac{\overrightarrow{\mathbf{p}}}{L^3}$$

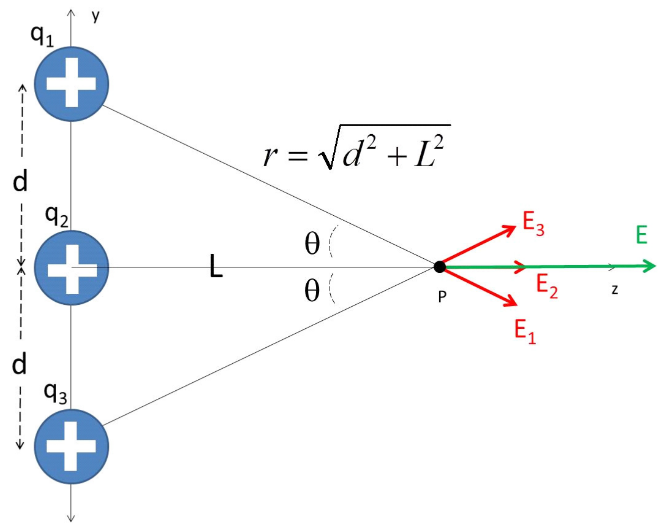
We could also treat this dipole as a complicated mover charge in some other environmental field! Then this quantity $\overrightarrow{\mathbf{p}}$ will help us understand how a dipole will move when placed in an environmental electric field. For example, we know that water molecules are dipoles. A microwave oven creates a strong environmental electric field that makes the water molecules rotate. When we studied rotational motion we found a mass-like term that helped us to know how difficult something was to make rotate. That was the moment of inertia. This dipole term, $\overrightarrow{\mathbf{p}}$, will tell us how likely a dipole is to spin, so we will call $\overrightarrow{\mathbf{p}}$ the *dipole moment*.

3.2.3 Three charges



We are working our way toward many charges that will require using integration to sum up the contributions to the field. But let's make this transition slowly. Next let's add just one more environmental charge, for a total of three.

Let's just start with the fields this time. From our picture, we expect in this case to have only z-components. Since all the charges are the same sign,



then

$$E_{net_z} = E_1 \cos(-\theta) + E_2 + E_3 \cos(\theta)$$

We can guess from symmetry that

$$E_1 = E_3 = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2}$$

But this time, since we have redefined d, the distance from q_1 and q_3 to the point P where we want to know the field is

$$r = \sqrt{d^2 + L^2}$$

so

$$E_1 = E_3 = \frac{1}{4\pi\epsilon_o} \frac{q}{(d^2 + L^2)}$$

and

$$E_2 = \frac{1}{4\pi\epsilon_o} \frac{q}{L^2}$$

and observing the triangles formed and remembering our trigonometry, we have

$$\cos\theta = \frac{L}{\sqrt{d^2 + L^2}}$$

so

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_o} \frac{q}{(d^2 + L^2)} \left(\frac{L}{\sqrt{d^2 + L^2}} \right) \\ &\quad + \frac{1}{4\pi\epsilon_o} \frac{q}{L^2} \\ &\quad + \frac{1}{4\pi\epsilon_o} \frac{q}{(d^2 + L^2)} \left(\frac{L}{\sqrt{d^2 + L^2}} \right) \end{aligned}$$

or

$$E_z = \frac{q}{4\pi\epsilon_o}\left(\frac{2L}{\left(d^2+L^2\right)^{\frac{3}{2}}} + \frac{1}{L^2}\right)$$

This is our answer.

Once again let's consider the limit $L \gg d$. If our answer is right, when we get very far from the group of charges they should look like a single charge with the amount of charge being the sum of all three environmental charges. In this limit

$$\lim_{L \gg d} \frac{1}{\left(d^2+L^2\right)^{\frac{3}{2}}} = \frac{1}{L^3}$$

so our limit becomes

$$\begin{aligned} E_z &\approx \frac{q}{4\pi\epsilon_o}\left(\frac{2L}{L^3} + \frac{1}{L^2}\right) \\ &= \frac{1}{4\pi\epsilon_o}\left(\frac{3q}{L^2}\right) \end{aligned}$$

so on the central axis

$$\overrightarrow{\mathbf{E}} \approx \frac{1}{4\pi\epsilon_o}\left(\frac{3q}{L^2}\right)\hat{\mathbf{k}}$$

And indeed, this is very like one charge that is three times as large as our actual charges if we get far enough away.

This shows us a pattern we will often see. Far away, our field looks like what we would expect if the net charge were all congregated in a point. Near the charges, we must calculate the superposition of the fields. But far away we can treat the distribution as a point charge. This is very like what we did with mass in PH121 or Dynamics. We could often treat masses as point masses at the center of mass, if the distances involved were larger than the mass sizes.

Of course, in these last two problems we picked nice places along axes to find the electric field. If we picked less convenient places we would have both y and z-components.

3.3 Combinations of many charges

We have found the field from a point charge.

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{q_E}{r^2}\hat{\mathbf{r}} \tag{3.1}$$

where the field is in the same direction as $\hat{r}$ if the charge is positive, and in the opposite direction if the charge is negative (think of our field lines, they go toward the negative charge). This will become one of a group of standard charge configurations that we will use to gain a mental picture of complex configurations of charge. We have done this already for combinations of point charges. We can combine the point charge fields to get the total field.

The other standard models are combinations of many, many charges.

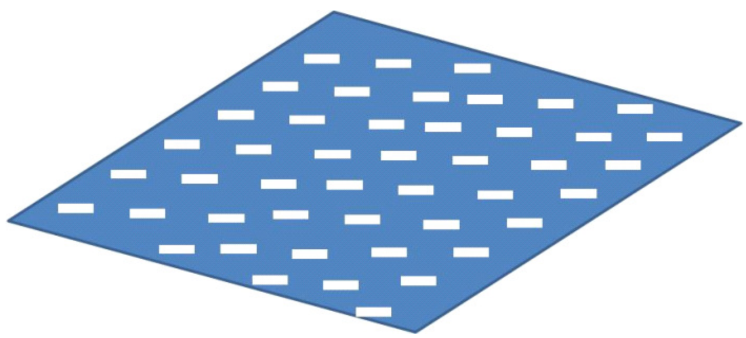
3.3.1 Line of Charge

Another is an infinitely long line of charge, or a infinite charged wire. Since this long line of charge is infinite, it must have an infinite amount of charge. But we can describe “how much” charge it has with a linear charge density

$$\lambda = \frac{Q}{L}$$

3.3.2 Semi-infinite sheet of charge

A sheet or plane of charge,usually a semi-infinite sheet of charge is also useful



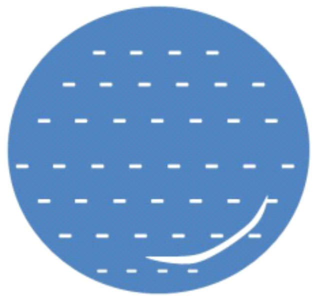
We have the same problem of having infinite charge, but if we define an amount of charge per unit area

$$\eta = \frac{Q}{A}$$

we can compare sheets that are more charge rich than others.

3.3.3 Sphere of charge

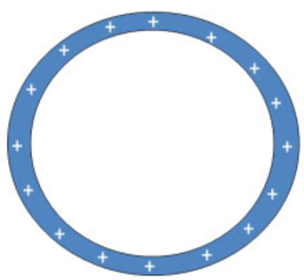
Finally, we have drawn a sphere of charge already



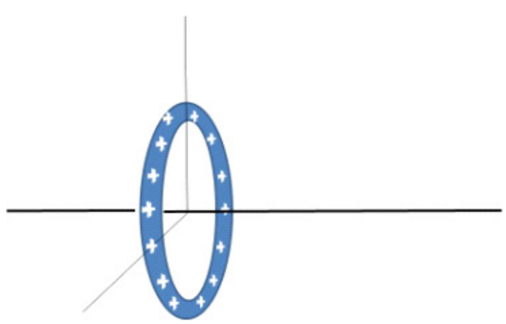
We can define an amount of charge per unit volume to help describe this distribution

$$\rho = \frac{Q}{\mathbb{V}}$$

The spherical shell of charge is related to a sheet of charge, so we will include it here



This configuration of charge is drawn in cross section like the others. From your calculus experience you can guess that a spherical shell of charge with a certain volume charge density might be useful in integration, but we also can easily produce such a configuration of charge by charging a round balloon or a spherical conductor.



The ring of charge is similar to the spherical shell, but is also much like the line of charge.

In our next lecture, we will take on the job of finding the fields that result from these last few charge configurations except the spherical shell, which will have to wait a few lectures.

3.4 On-Line Visualizations

For a 2D visualization of the field try:

https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_all.html

https://icphysweb.z13.web.core.windows.net/simulation.html

http://www.falstad.com/emstatic/index.html

And here is a 3D visualization:

http://www.falstad.com/vector3de/

Basic Equations

It's time to put Newton's second law back on your equation sheet

$$\begin{aligned} F_{net_z} &= ma_z \\ F_{net_y} &= ma_y \end{aligned}$$

Charge densities:

$$\lambda = \frac{Q}{L}$$

$$\eta = \frac{Q}{A}$$

$$\rho = \frac{Q}{V}$$

Electriec Field

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{Q_E}{r^2}\hat{\mathbf{r}}$$