3 Waves in One and More Dimensions

We studied waves in general last lecture. This time we will look at a specific wave, the sinusoidal wave. You might think this is terribly restrictive, but we will find that using sinusoidal waves, we can represent most any wave through an elegant mathematical trick, and the idea of superposition (that we will explain later).

Fundamental Concepts

1. The mathematical form of a sinusoidal wave is $y\left(x,t\right)=y_{\max}\cos\left(kx-\omega t+\phi_o\right)$
2. There are names for parts of a sinusoidal wave. We need to recognize the following terms: crest, trough, wavelength period frequency angular frequency, phase constant, wave number.
3. Spatial frequency is "how often" something happens along some length.
4. The phase of a sinusoidal wave is given by $\phi=kx-\omega t+\phi_o$
5. Spherical waves have the form $y=A\left(r\right)\sin\left(kx-\omega t+\phi_o\right)$
6. Sufficiently far from the source of a wave, we can treat spherical waves like plane waves.

Sinusoidal Waves

A sinusoidal graph should be familiar from our Dynamics experience. We can use what we know from oscillation to understand the equation for a sinusoidal wave. Remember that for simple harmonic oscillators we used the function

$$y\left(t\right)=A\cos\left(\omega t+\phi_o\right)\qquad SHM \tag{3.1}$$

but this only gave us a vertical displacement at one x-position. Now our sinusoidal

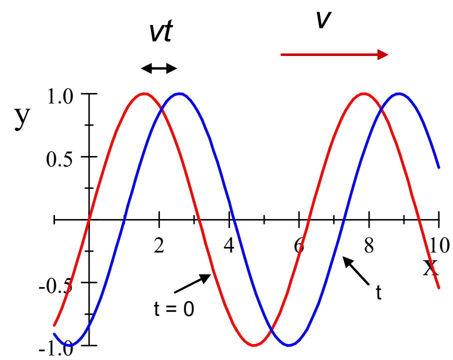


Figure 3.4.

function must also be a function of position along the wave.

$$y(x,t) = A\cos(kx - \omega t + \phi_o) \qquad waves \tag{3.2}$$

but before we study the nature of this function, lets see what we can learn from the graph of a sinusoidal wave. We will need both of our two views, the camera snapshot and the video (history) of a point. Look at figure 3.4. This is two camera snap shots superimposed. The red curve shows the wave (y position for each value of x) at $t = 0$. At some some later time t, the wave pattern has moved to the right as shown by the blue curve. The shift is by an amount $x = vt$. This reminds us of the wave function form

$$y(x - vt, 0)$$

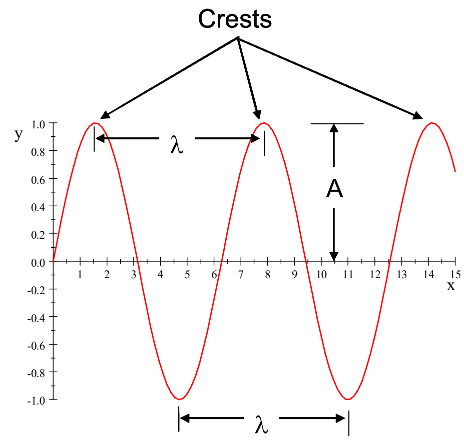
Parts of a wave

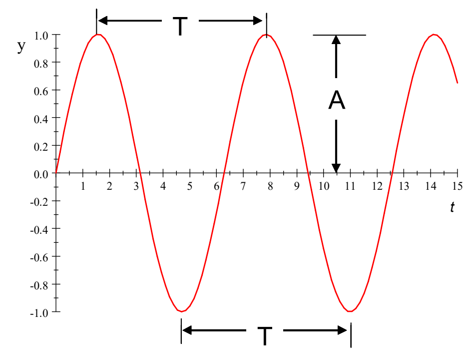
Question 123.3.1

Question 123.3.2

The peak of a wave is called the crest. For a sine wave we have a series of crests. We define the wavelength as the distance between any two nearest identical points (e.g. crests) on the wave.

Notice that this is very similar to the definition of the period, T, when we graphed SHM on a y vs. t set of axis. In fact, this similarity is even more apparent if we plot a sinusoidal wave using our two wave pictures. In the next figure, the snap-shot comes first. We can see that there will be crests. The distance between the crests is given the name *wavelength*. We give it the symbol λ. This is not the entire length of the whole wave. But is is a characteristic length of part of the wave that is easy to identify. The next figure shows all this using our snapshot and history graphs for a sine wave.





Note that there are crests in the history graph view as well. That is because one marked part of the medium is being displaced as a function of time (think of our marked piece of the rope going up and down, or think of floating in the ocean at one point, you travel up and down as the waves go by). But now the horizontal axis is time. There will be a characteristic time between crests. That time is called the *period*. Like the wavelength is not the length of the whole wave, the period is not the time the whole wave exists. It is just the time it takes the part of the medium we are watching to go through one complete cycle. Notice that this video picture is exactly the same as a plot of the motion of a simple harmonic oscillator! For a sinusoidal wave, each part of the medium experiences simple harmonic motion.

We remember frequency from simple harmonic motion. But now we have a wave, and the wave is moving. We can extend our view of frequency by defining it as follows:

Definition 3.1 *The frequency of a periodic wave is the number of crests (or any other point of the wave) that pass a given point in a unit time interval.*

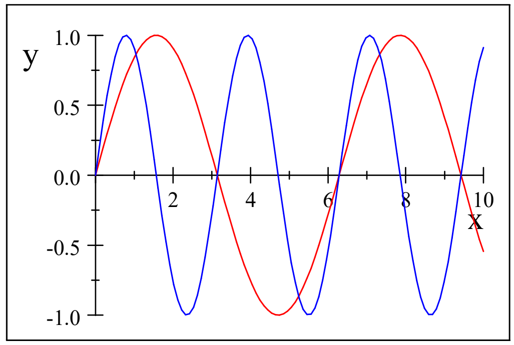


Figure 3.5.

In figure 3.5, the blue curve has twice the frequency as the red curve. Notice how it has two crests for every red crest. The maximum displacement of the wave is called the *amplitude* just as it was for simple harmonic oscillators.

Question 123.3.3

Question 123.3.4

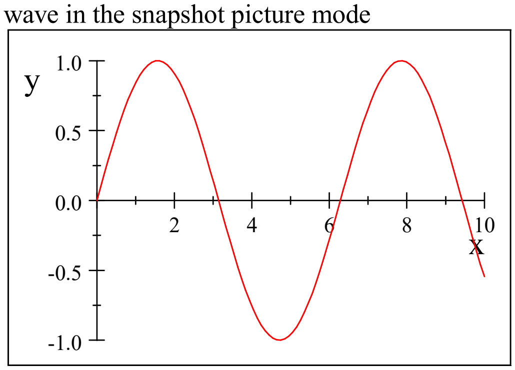
Question 123.3.5

Wavenumber and wave speed

Consider again a sinusoidal wave.

$$y(x,t) = A\cos(kx - \omega t + \phi_o) \tag{3.3}$$

We have drawn the wave in the snapshot picture mode



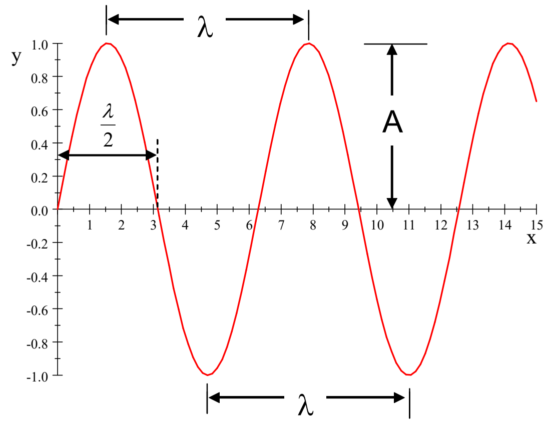
To make this graph, we set $t = 0$ and plot the resulting function

$$y(x,0) = A\sin(kx + 0) \tag{3.4}$$

A is the amplitude. I want to investigate the meaning of the constant k. Lets find k like we did for SHM when we found ω. Consider the point $x = 0$. At this point

$$y(0,0) = A\sin(k(0)) = 0 \tag{3.5}$$

The next time $y = 0$ is when $x = \frac{\lambda}{2}$



then

$$y\left(\frac{\lambda}{2},0\right)=A\sin\left(k\frac{\lambda}{2}\right)=0 \tag{3.6}$$

From our trigonometry experience, we know that this is true when

$$k\frac{\lambda}{2}=\pi \tag{3.7}$$

solving for k gives

$$k=\frac{2\pi}{\lambda} \tag{3.8}$$

Then we now have a feeling for what k means. It is 2π over the spacing between the crests. The 2π must have units of radians attached. Then

$$y\left(x,0\right)=A\sin\left(\frac{2\pi}{\lambda}x+0\right) \tag{3.9}$$

We have a special name for the quantity k. It is called the *wave number*.

$$k\equiv\frac{2\pi}{\lambda} \tag{3.10}$$

Both the name and the symbol are somewhat unfortunate. Neither gives much insight into the meaning of this quantity. But from what we have done, we can understand it better. For a harmonic oscillator, we know that

$$y\left(t\right)=A\sin\left(\omega t\right)$$

where

$$\omega=2\pi f=\frac{2\pi}{T}$$

Question 123.3.6

T is how far, in time, the crests are apart, and the inverse of this, $\frac{1}{T}$ is the frequency. The frequency tells us how often we encounter a crest as we march along in time. So $\frac{1}{T}$ must be how many crests we have in a unit amount of time.

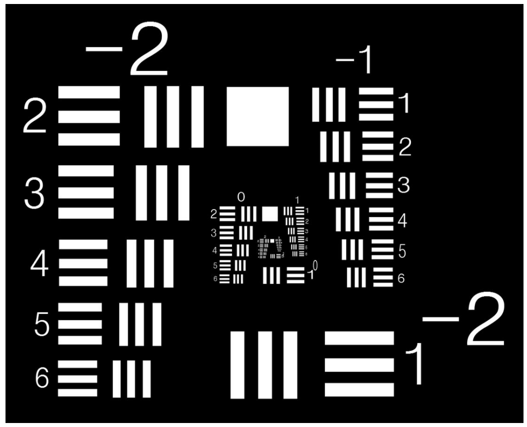
Now think of the relationship between the snapshot and the video representation for a sinusoidal wave. We have a new quantity

$$k = \frac{2\pi}{\lambda}$$

where λ is how far, in distance, the crests are apart. This implies that $\frac{1}{\lambda}$ plays the same role in the snap shot graph that f plays in the video graph. It must tell us how many crests we have, but this time it is how many crests in a given amount of distance. We found above that k told us something about how often the zeros (well, every other zero) will occur. But the crests must occur at the same rate. So k tells us how often we encounter a crest in our snapshot graph.

The frequency in the video graph is how often we encounter a crest, $\frac{1}{\lambda}$ is how often we encounter a crest in the snap shot graph. Thus $\frac{1}{\lambda}$ is playing the same role for a snap shot graph as frequency plays for a history graph. We could call $\frac{1}{\lambda}$ a *spatial frequency.* It is how often we encounter a crest as we march along in position, or how many crests we have in a unit amount of distance. And λ could be called a *spatial period*. Both $1/T$ and $1/\lambda$ answer the question "how often something happens in a unit of something" but one asks the question in time and the other in position along the wave.

My mental image for this is the set of groves on the edge of a highway. There is a distance between them, like a wavelength, and how often I encounter one as I move a distance along the road is the spatial frequency. If you are a farmer, you may think of plowed fields, with a distance between furrows as a wavelength, and how closely spaced the furrows are as a measure of spatial frequency. We use this concept in optics to test how well an optical system resolves details in a photograph. The next figure is a test image. A good camera will resolve all spatial frequencies equally well. Notice the test image has sets of bars with different spatial frequencies. By forming an image of this pattern, you can see which spacial frequencies are faithfully represented by the optical system.



Resolution test target based on the USAF 1951 Resolution Test Pattern (not drawn to exact specifications).

In class you will see that our projector does not represent all spatial frequencies equally well! You can also see this now in the copy you are reading. If you are reading on-line or an electronic copy, your screen resolution will limit the representation of some spatial frequencies. Look for the smallest set of three bars where you can still tell for sure that there are three bars. A printed version that has been printed on a laser printer will usually allow you to see even smaller sets of three bars clearly.

Let's place k in the full equation for the sine wave for any time, t.

$$y(t) = A\cos(kx - \omega t + \phi_o) \tag{3.11}$$

We would like this to look like our wave function equation

$$y(x,t) = y(x - vt, 0)$$

With a little algebra we can do this

$$\begin{aligned} y(t) &= A\cos(kx - \omega t + \phi_o) \\ &= A\cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi_o\right) \\ &= A\cos\left(\frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T}t\right) + \phi_o\right) \end{aligned}$$

This is in the form of a wave function so long as

$$v = \frac{\lambda}{T} \tag{3.12}$$

then

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) \tag{3.13}$$

We can see that the wave travels one wavelength in one period. The simple relationship

$$v = \frac{\lambda}{T} \tag{3.14}$$

is of tremendous importance.

Wave speed forms

We found

$$v = \frac{\lambda}{T} \tag{3.15}$$

but it is easy to see that

$$v = \frac{2\pi\lambda}{2\pi T} = \frac{\omega}{k} \tag{3.16}$$

and

$$v = \lambda f \tag{3.17}$$

This last formula is, perhaps, the most common form encountered in our study of light.

Phase

You may be wondering about the phase constant we learned about in our study of SHM. We have ignored it up to now. But of course we can shift our sine just like we did for our plots of position vs. time for oscillation. Only now with a wave we have two graphs, a history and snapshot graphs, so we could shift along the x in a snapshot graph or along the t axes in a history graph. So the sine wave has the form.

$$y(x,t) = A\sin(kx - \omega t + \phi_o) \tag{3.18}$$

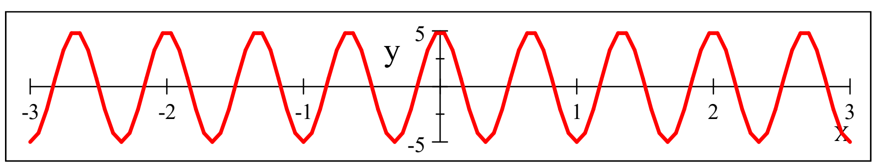
were ϕ_o will need to be determined by initial conditions just like in SHM problems and those initial conditions will include initial positions as well as initial times.

Let's consider that we have two views of a wave, the snapshot and history view. Each of these looks like sinusoids for a sinusoidal wave. Let's consider a specific wave,

$$y(x,t) = 5\sin\left(3\pi x - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$

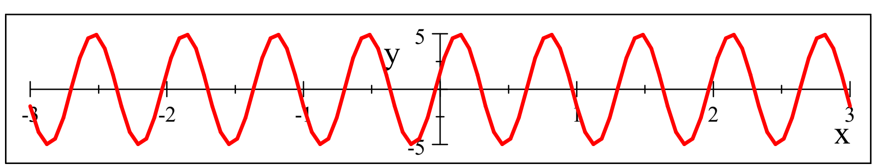
we look at a snapshot graph at $t = 0$

$$y(x,0) = 5\sin\left(3\pi x - \frac{\pi}{5}(0) + \frac{\pi}{2}\right)$$



and another at $t = 2\,\mathrm{s}$

$$y(x,2\,\mathrm{s}) = 5\sin\left(3\pi x - \frac{\pi}{5}(2\,\mathrm{s}) + \frac{\pi}{2}\right)$$



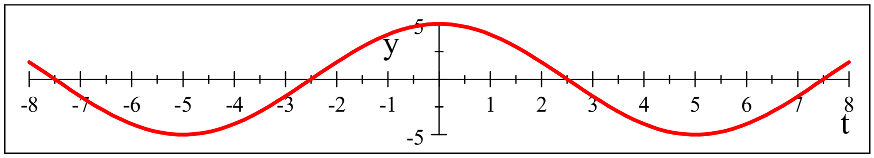
Comparing the two, we could view the latter as having a different phase constant

$$\phi_{total} = \omega\Delta t + \phi_o = -\frac{\pi}{5}\left(2\,\mathrm{s}\right) + \frac{\pi}{2}$$

that is, within the snapshot view, the time dependent part of the argument of the sign acts like an additional phase constant.

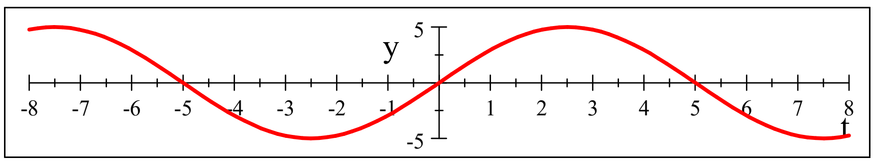
Likewise, in the history view, we can plot our wave at $x = 0$

$$y\left(0,t\right) = 5\sin\left(3\pi\left(0\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



and at $x = 1.5\,\mathrm{m}$

$$y\left(1.5\,\mathrm{m},t\right) = 5\sin\left(3\pi\left(1.5\,\mathrm{m}\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



Within the history view, the kx part of the argument acts like a phase constant.

$$\phi_{total} = k\Delta x + \phi_o = 3\pi\left(1.5\,\mathrm{m}\right) + \frac{\pi}{2}$$

Of course neither kx nor ωt are constant, But within individual views of the wave we have set them constant to form our snapshot and history representations. We can see that any part of the argument of the sine, $kx - \omega t + \phi_o$ could contribute to a phase shift, depending on the view we are taking.

Because of this, it is customary to call the entire argument of the sine function, $\phi = kx - \omega t + \phi_o$ the *phase of the wave*. Where ϕ_o is the phase constant, ϕ is the phase. Of course then, ϕ must be a function of x and t, so we have a different value for $\phi\left(x,t\right)$ for every point on the wave for every time.

Sinusoidal waves on strings

Take a jump rope, and shake one end up and down while your partner keeps his or her end stationary. You can make a sine wave in the rope. You can do a better job by attaching a wave generator to the end.

Really, as long at the wave forms are identical and periodic, the relationships

$$f = \frac{1}{T} \tag{3.19}$$

and

$$v = f\lambda \tag{3.20}$$

will hold. But we will make our device vibrate with simple harmonic motion.

Let's call an element of the rope Δx. Here the "Δ" is being used to mean "a small amount of." We are taking a small amount of the rope and calling it's length Δx.

Each element of the rope (Δx_i) will also oscillate with SHM (think of a driven SHO). Note that the elements of the rope oscillate in the y direction, but the wave travels in x. This is a transverse wave.

Let's describe the motion of an element of the string at point P.

At $t = 0$,

$$y = A \sin (kx - \omega t) \tag{3.21}$$

(where I have chosen $\phi_o = 0$ for this example). The element does not move in the x direction. So we define the *transverse speed*, v_y, and the *transverse acceleration,* a_y, as the velocity and acceleration of the element of rope in the y direction. These are not the velocity and acceleration of the wave, just the velocity and acceleration of the element Δx at a point P.

Because we are doing this at one specific x location we need partial derivatives to find the velocity

$$v_y = \left.\frac{dy}{dt}\right]_{x=\text{constant}} = \frac{\partial y}{\partial t} \tag{3.22}$$

That is, we take the derivative of y with respect to t, but we pretend that x is not a variable because we just want one x position. Then

$$v_y = \frac{\partial y}{\partial t} = -\omega A \cos (kx - \omega t) \tag{3.23}$$

and

$$a_y = \frac{\partial v_y}{\partial t} = -\omega^2 A \sin (kx - \omega t) \tag{3.24}$$

These solutions should look very familiar! We expect them to be the same as a harmonic oscillator except that we now have to specify which oscillator–which part of the rope–we are looking at. That is what the kx part is doing.

The speed of Waves on Strings

Only use if there are questions on this.

Let's work a problem together. Let's find an expression for the speed of the wave as it travels along a string.

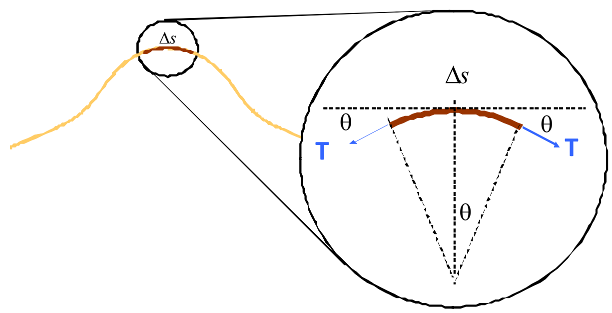


Figure 3.6.

We will use Newton's second law

$$\Sigma \overrightarrow{F} = \overrightarrow{F}_{net} = m\overrightarrow{a}$$

to do this, so we need a sum of the forces. What are the forces acting on an element of string?

- Tension on the right hand side (RHS) of the element from the rest of the string on the right, T_r
- Tension on the left hand side (LHS) of the element from the rest of the string on the left, T_l
- The force due to gravity on our element of string, F_g

Lets assume that the element of string, Δs, at the crest is approximately an arc of a circle with radius R.

There is a force pulling left on the left end of the element that is tangent to the arc, there is a force pulling right at the right end of the element which is also tangent to the arc. The horizontal components of the forces cancel $(T \cos\theta)$. The vertical component, $(T \sin(\theta))$ is directed toward the center of the arc. Then these forces must be a mass times a n acceleration and because they are center seeking we can call these accelerations centripetal accelerations

$$a = \frac{v^2}{R} \tag{3.25}$$

If the rope is not moving in the x direction, then

$$\Sigma F_x = 0 = -T_l \cos\theta + T_r \cos\theta$$
$$T_l = T_r$$

Then, the radial force F_r will have matching components from each side of the element that together are $2T\sin(\theta)$. Since the element is small,

$$\Sigma F_r = 2T\sin(\theta) \approx 2T\theta \tag{3.26}$$

The element has a mass m.

$$m = \mu\Delta s \tag{3.27}$$

where μ is the mas per unit length. Using the arc length formula

$$\Delta s = R(2\theta) \tag{3.28}$$

so

$$m = \mu\Delta s = 2\mu R\theta \tag{3.29}$$

and finally we use the formula for the radial acceleration

$$F_r = ma = (2\mu R\theta)\frac{v^2}{R} \tag{3.30}$$

Combining these two expressions for F_r

$$2T\theta = (2\mu R\theta)\frac{v^2}{R} \tag{3.31}$$

$$T = (\mu R)\frac{v^2}{R} \tag{3.32}$$

$$\frac{T}{\mu} = v^2 \tag{3.33}$$

and we find that

$$v = \sqrt{\frac{T}{\mu}} \tag{3.34}$$

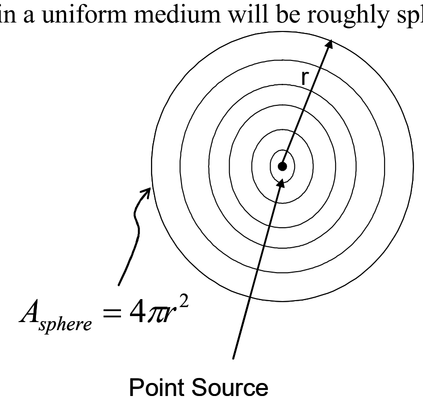
Note that we made many assumptions along the way. Despite this, the approximation is quite good.

Waves in two and three dimensions

So far we have written expressions for waves, but our experience tells us that waves don't usually come as one dimensional phenomena. In the next figure, we see the disturbance (a drop) creating a water wave.

Picture of a water drop (Jon Paul Johnson, used by permision)

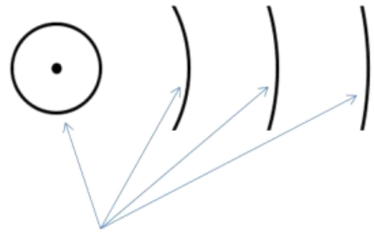
The wave is clearly not one dimensional. It appears nearly circular. In fact, it is closer to hemispherical, and this limit is only true because the disturbance is at the air-water boundary. Most waves in a uniform medium will be roughly spherical.



As such a wave travels away from the source, the energy traveling gets more spread out. This causes the amplitude to decrease. Think of a sound wave, it gets quieter the farther you are from the source. We change our equation to account for this by making the amplitude a function of the distance, r, from the source

$$y = A(r)\sin\left(\overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{r}} - \omega t + \phi_o\right) \tag{3.35}$$

Of course, if we look at a very large wave, but we only look at part of the wave, we see that our part looks flatter as the wave expands.



Portion of a Spherical Wave: Wave becomes more flat as it expands

Very far from the source, our wave is flat enough that we can ignore the curvature across it's wave fronts. We call such a wave a *plane wave*. There are no true plane waves in nature, but this idealization makes our mathematical solutions simpler and many waves come close to this approximation. We will usually stick with the plane wave approximation in this class.