Chapter 46

Inductors

Fundamental Concepts

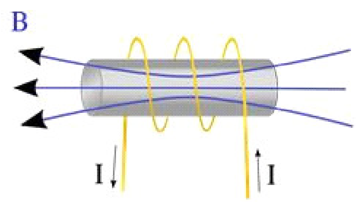
- The self inductance L has all the geometric and material properties of a coil or other inductor an it can be found using $L = N\frac{d\Phi_B}{dI}$
- The emf induced by an inductor is given by $\mathcal{E} \equiv -L\frac{\Delta I}{\Delta t}$
- For a solenoid, the inductance can be found to be $L = \mu_o n^2 V$
- The energy stored in the magnetic field is $U_L = \frac{1}{2}LI^2$ and the energy density in the magnetic field is $u_B = \frac{1}{2}\frac{1}{\mu_o}B^2$
- There is an *apparent* voltage drop across an inductor of $\Delta V_{L_{apparent}} = -L\frac{dI}{dt}$
- There is also a mutual inductance between two inductors given by $M_{12} = \frac{N_2\Phi_{12}}{I_1}$

46.1 Self Inductance

When we put capacitors and resisters in a circuit, we found that the current did not jump to it's ultimate current value all at once. There was a time dependence. But really, even if we just have a resister the current does not reach it's full value instantaneously (and we nearly always have some resistance). Think of our circuits, they are current *loops*! So as the current starts to flow, Lenz's law tells us that there will be a magnetic field that forms, and an induced emf that will oppose the flow. The potential drop across the resister in a simple battery-resister circuit is the potential drop due to the battery emf, *minus the induced emf.* Hopefully you just did problems where loops fell more slowly in a magnetic field. Some of the energy went into making an induced current. So there was less kinetic energy for the loop's fall. Since we can make an emf (a potential energy per unit charge) with a magnetic field, we need to account for this energy in our circuit. We will have

$$\Delta V_{resistor} = \mathcal{E}_{battery} - \mathcal{E}_{induced}$$

We can use this fact to control current in circuits. To see how, we can study a new case



Let's take a coil of wire wound around an iron cylindrical core.[1] We start with a current as shown in the figure above. Using our right hand rule we can find the direction of the B-field. But we now will allow the current to change. As it gets larger, we know

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

and we know that as the current changes, the magnitude of the B-field will change, so the flux through the coil will change. We will have an induced emf that will oppose the change. It will act like a resistance. We could derive this expression, but I think you can see that the induced emf is proportional to the *rate of change* of the current.

$$\mathcal{E} \equiv -L\frac{\Delta I}{\Delta t}$$

You might ask if the number of loops in the coil matters. The answer is–yes. Does the size and shape of the coil matter–yes. But we will include all these geometrical effects in the constant L called the *inductance*. It will hold all the material properties of the iron cored coil.

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \equiv -L\frac{dI}{dt}$$

so for this specific case

$$-N\frac{d\Phi_B}{dt}\frac{dt}{dI} \equiv -L$$

or

$$L = N\frac{d\Phi_B}{dI}$$

[1] Note that the wire will need to be coated with an insulator or the current will just skip through the iron core to the end. We generally consider the wire in our cases to be isolated.

If we start with no current (so no flux), then our change in flux is the current flux minus zero. We can then say that

$$L = N\frac{\Phi_B}{I}$$

It might be more useful to write the inductance as

$$L = -\frac{\mathcal{E}_L}{\frac{dI}{dt}}$$

In designing circuits, we will usually just look up the inductance of the device we choose, like we looked up the resistance of resisters or the capacitance of the capacitors we use.

But for our special case of a simple coil, we can calculate the inductance, because we know the induced emf using Faraday's law

46.1.1 Inductance of a solenoid[2]

Let's extend our inductance calculation for a coil. Really the only easy case we can do is that of a solenoid (that's probably a hint for the test). So let's do it! We will just fill our solenoid with air instead of iron (if we have iron, we have to take into account the magnetization, so it is not terribly hard, but this is not what we want to concentrate on now). If the solenoid has N turns with length L and we assume that L is much bigger than the radius r of the loops then we can use our solution for the B-field created by a solenoid

$$\begin{aligned} B &= \mu_o n I \\ &= \mu_o \frac{N}{\ell} I \end{aligned}$$

The flux through each turn is then

$$\Phi_B = BA = \mu_o \frac{N}{\ell} IA$$

[2] Think of this like the special case of a capacitor made from two flat large plates, the parallel plate capacitor. It was somewhat ideal in the way we treated it. Our treatment of the special case of a coil will likewise be somewhat ideal.

where A is the area of one of the solenoid loops. Then we use our equation for inductance for a coil

$$\begin{aligned} L &= N\frac{\Phi_B}{I} \\ &= N\frac{\left(\mu_o \frac{N}{\ell} IA\right)}{I} \\ &= \frac{\left(\mu_o N^2 A\right)}{\ell} \\ &= \frac{\left(\mu_o N^2 A\right)}{\ell}\frac{\ell}{\ell} \\ &= \frac{\mu_o N^2 A\ell}{\ell^2} \\ &= \frac{\mu_o N^2 \mathbb{V}}{\ell^2} \\ &= \mu_o n^2 \mathbb{V} \end{aligned}$$

where we used the fact that the volume of the solenoid is $\mathbb{V} = A\ell$.

Many inductors built for use in electronics are just this, air filled solenoids. So this really is a somewhat practical solution.

46.2 Energy in a Magnetic Field

An inductor, like a capacitor, stores energy in it's field. And in a transformer that energy is shared into another circuit. You can see us weaving our way toward wireless chargers and wireless power connections!.

We would like to know how much energy an inductor can store. From our previous study of power we know the power in a circuit will be

$$\mathcal{P} = I\Delta V$$

If we just have an inductor, then the power removed from the circuit is

$$\begin{aligned} \mathcal{P}_{cir} &= I\Delta V = I\mathcal{E}_{induced} \\ &= I\left(-L\frac{dI}{dt}\right) \\ &= -LI\frac{dI}{dt} \end{aligned}$$

As with a resistor, we are taking power *from the circuit* so the result is negative. But unlike a resistor, this power is not being dissipated as heat. It is going into the magnetic field of the inductor. Therefore, we expect the power stored in the inductor field to be

$$\mathcal{P}_L = -\mathcal{P}_{cir} = LI\frac{dI}{dt}$$

Power is the time rate of change of energy, so we can write this power delivered to the inductor as

$$\frac{dU_L}{dt} = LI\frac{dI}{dt}$$

Multiplying by dt gives

$$dU_L = LIdI$$

To find the total energy stored in the inductor we must integrate over I.

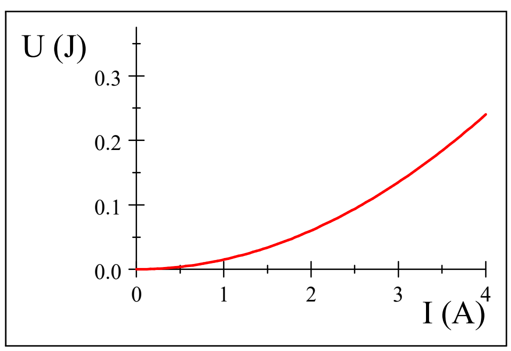
$$\begin{aligned} U_L &= \int dU_L \\ &= \int_0^I LIdI \\ &= L\int_0^I IdI \\ &= \frac{1}{2}LI^2 \end{aligned}$$

Thus,

$$U_L = \frac{1}{2}LI^2$$

is the energy stored in the magnetic field of the inductor.

Suppose we have an inductor $L = 30.0 \times 10^{-3}\,\mathrm{H}$. Plotting shows us the dependence of U_L on I.



We should take a moment to see how our inductor compares to a capacitor as an energy storage device. The energy stored in the electric field of a capacitor is

$$U_C = \frac{1}{2}C\left(\Delta V\right)^2$$

and for our inductor we found

$$U_L = \frac{1}{2}L\left(I\right)^2$$

Notice that Remarkable similarity!

46.2.1 Energy Density in the magnetic field

We found that there was energy stored in the electric field of a capacitor. Is the energy stored in the inductor really stored in the magnetic field of the inductor? We hope so, because then we can transfer that energy through the magnetic field. We can start with the energy, U_L, is stored in the field and find the density of the energy in the field.

Let's take a specific case, the field inside a solenoid. And start with the inductance of a solenoid.

$$L = \mu_o n^2 A\ell$$

The magnetic field is given by

$$B = \mu_o nI$$

then the energy in the field is given by

$$\begin{aligned} U_B &= \frac{1}{2}LI^2 \\ &= \frac{1}{2}\mu_o n^2 A\ell I^2 \end{aligned}$$

If we rearrange this, we can see the solenoid field is found in the expression twice

$$\begin{aligned} U_B &= \frac{1}{2}\left(\mu_o nI\right) A\ell \frac{(\mu_o nI)}{\mu_o} \\ &= \frac{1}{2\mu_o} B^2 A\ell \end{aligned}$$

and the energy density is

$$\begin{aligned} u_B &= \frac{U_B}{A\ell} \\ &= \frac{1}{2}\frac{1}{\mu_o} B^2 \end{aligned}$$

Just like our energy density for the electric field, we derived this for a specific case, a solenoid. But this expression is general. We should compare to the energy density in the electric field.

$$u_E = \frac{1}{2}\epsilon_o E^2$$

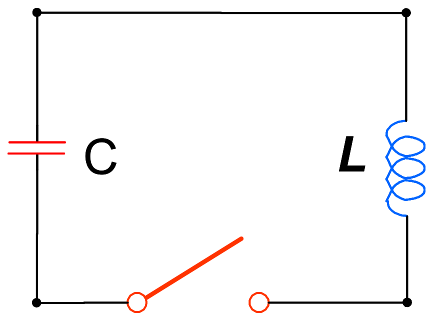
Again, note the similarity!

So we have energy stored in our magnetic field. We should see what we can do with that energy.

46.2.2 Oscillations in an LC Circuit

We introduce a new circuit symbol for inductors

It looks like a coil, for obvious reasons. We can place this new circuit element in a circuit. But what will it do? To investigate this, let's start with a simple case, a circuit with a charged capacitor and an inductor and nothing else.



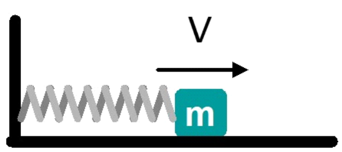
Let us make two unrealistic assumptions (we will relax these assumptions later).

Assumption 1: There is no resistance in our LC circuit.
Assumption 2: There is no radiation emitted from the circuit.

Given these two assumptions, there is no mechanism for energy to escape the circuit. Energy must be conserved. Can we describe the charge on the capacitor, the current, and the energy as a function of time?

It may pay off to recall some details of oscillators.

Energy of the Simple Harmonic Oscillator



Remember from Dynamics or PH121 that a mass-spring system will oscillate. The mass has kinetic energy because the mass is moving

$$K = \frac{1}{2}mv^2 \tag{46.1}$$

for our Simple Harmonic Oscillator we know that the position of the mass as a function of time is given by

$$x(t) = x_{\max}\cos(\omega t + \phi)$$

and the speed as a function of time is

$$v(t) = -\omega x_{\max} \sin(\omega t + \phi)$$

then the kinetic energy as a function of time is

$$\begin{aligned} K &= \frac{1}{2} m \left(-\omega x_{\max} \sin(\omega t + \phi)\right)^2 \\ &= \frac{1}{2} m \omega^2 x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} m \frac{k}{m} x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} k x_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

The spring has potential energy given by

$$U = \frac{1}{2} k x^2 \tag{46.2}$$

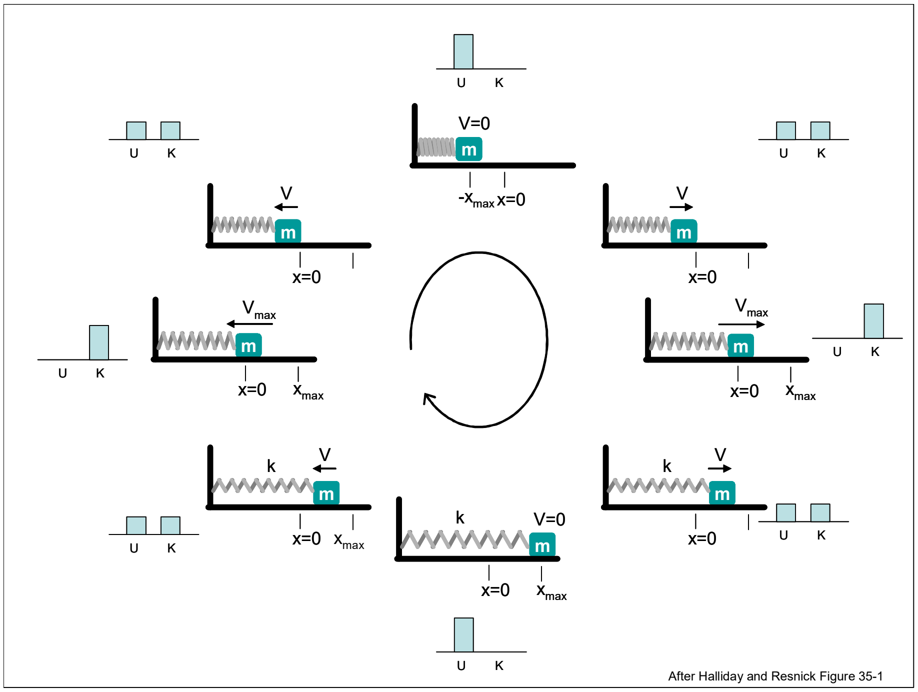
For our mechanical oscillator the potential as a function of time is

$$U = \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi)$$

The total energy is given by

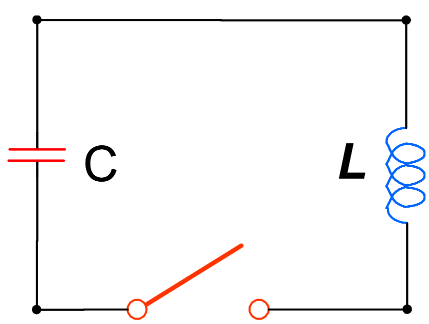
$$\begin{aligned} E &= K + U \\ &= \frac{1}{2} k x_{\max}^2 \sin^2(\omega t + \phi) + \frac{1}{2} k x_{\max}^2 \cos^2(\omega t + \phi) \\ &= \frac{1}{2} k x_{\max}^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) \\ &= \frac{1}{2} k x_{\max}^2 \end{aligned}$$

We can see that the total energy won't change, and the energy switches back and forth from kinetic to potential as the mass moves back and forth. If we plot the kinetic and potential energy at points along the mass' path we get something like this.



One of the important uses of an inductor is to create *electrical oscillations.* Having recalled what oscillations look like, we can see that a LC circuit will have an oscillating current.

Here is our circuit again.



We will start with the switch open the capacitor charged to its maximum value $Q_{\max}$. For $t > 0$ the switch is closed. Recall that the energy stored in the capacitor is

$$U_C = \frac{Q^2}{2C}$$

and the energy stored in the inductor is

$$U_L = \frac{1}{2} I^2 L$$

The total energy (because of our assumptions) is

$$\begin{aligned} U &= U_C + U_L \\ &= \frac{Q^2}{2C} + \frac{1}{2} I^2 L \end{aligned}$$

The change in energy over time must be zero (again because of our assumptions) so

$$\begin{aligned} \frac{dU}{dt} &= 0 \\ &= \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} I^2 L\right) \\ &= \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \end{aligned}$$

We recall that

$$I = \frac{dQ}{dt}$$

$$\begin{aligned} 0 &= \frac{Q}{C}\left(\frac{dQ}{dt}\right) + LI\frac{dI}{dt} \\ 0 &= \frac{Q}{C}(I) + LI\frac{dI}{dt} \\ 0 &= \frac{Q}{C} I + LI\frac{d\left(\frac{dQ}{dt}\right)}{dt} \\ 0 &= \frac{Q}{C} + L\frac{d^2Q}{dt^2} \end{aligned}$$

or

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$

This is a differential equation that we recognize from M316. It looks just like the differential equation for oscillatory motion! We try a solution of the form

$$Q = A\cos(\omega t + \phi)$$

then

$$\frac{dQ}{dt} = -A\omega \sin(\omega t + \phi)$$

and

$$\frac{d^2Q}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$$

thus

$$A\omega^2 \cos(\omega t + \phi) = -\frac{1}{LC} A \cos(\omega t + \phi)$$

This is indeed a solution if

$$\omega = \frac{1}{\sqrt{LC}}$$

When $\cos(\omega t + \phi) = 1$, $Q = Q_{\max}$, thus

$$Q = Q_{\max} \cos(\omega t + \phi)$$

Now recall,

$$\begin{aligned} I &= \frac{dQ}{dt} \\ &= \frac{d}{dt}\left(Q_{\max}\cos(\omega t + \phi)\right) \\ &= -\omega Q_{\max} \sin(\omega t + \phi) \end{aligned}$$

We would like to determine ϕ. We use the initial conditions $t = 0$, $I = 0$ and $Q = Q_{\max}$. Then

$$0 = -\omega Q_{\max} \sin(\phi)$$

This is true for $\phi = 0$. Then

$$\begin{aligned} Q &= Q_{\max} \cos(\omega t) \\ I &= -\omega Q_{\max} \sin(\omega t) \\ &= -I_{\max} \sin(\omega t) \end{aligned}$$

We can use the solution for the charge on the capacitor and the current in the inductor as a function of time to expand our energy equation

$$\begin{aligned} U &= U_C + U_L \\ &= \frac{Q^2}{2C} + \frac{1}{2} I^2 L \\ &= \frac{1}{2C} Q_{\max}^2 \cos^2(\omega t) + \frac{1}{2} L I_{\max}^2 \sin^2(\omega t) \end{aligned}$$

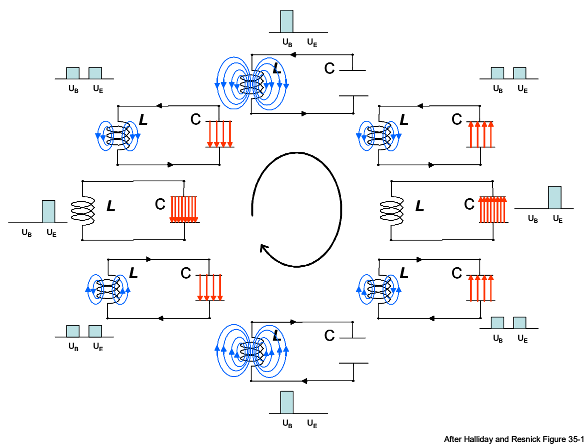
This looks a lot like our kinetic and potential energy equation for a mass-spring system. The energy shifts from the capacitor to the inductor and back like energy shifted from kinetic to potential energy for our mass-spring, with the components out of phase by $90\,^{\circ}$. By energy conservation, we know that

$$\frac{1}{2C} Q_{\max}^2 = \frac{1}{2} L I_{\max}^2$$

that is, the maximum energy in the capacitor equals the maximum energy in the inductor. Then the total energy

$$\begin{aligned} U &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t)+\frac{1}{2}LI_{\max}^2\sin^2(\omega t) \\ &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t)+\frac{1}{2C}Q_{\max}^2\sin^2(\omega t) \\ &= \frac{Q_{\max}^2}{2C} \end{aligned}$$

which must be the case if energy is conserved. We can plot the capacitor and inductor energies at points in time as the current switches back and forth.

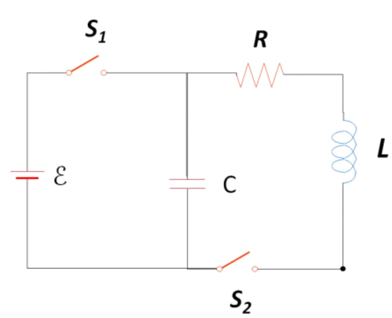


This is very much like our harmonic oscillator picture. We can see that we have, indeed made an electronic oscillator.

This type of circuit is a major component of radios which need a local oscillatory circuit to operate.

46.2.3 The RLC circuit

As fascinating as the last section was, we know there really is some resistance in the wire. So the restriction of no resistance needs to be relaxed in our analysis.



We can use the circuit in the picture to imagine an LRC circuit. At first, we will keep S_2 open and close S_1 to charge up the capacitor. Then we will close S_1 and open S_2. What will happen?

It is easier to find the current and charge on the capacitor as a function of time by using energy arguments. The resistor will remove energy from the circuit by dissipation (getting hot). The circuit has energy

$$U = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \tag{46.3}$$

so from the work energy theorem,

$$W_{nc} = \Delta U$$

the energy lost will be related to a change in the energy in the capacitor and the inductor. Let's look at the rate of energy loss again

$$\begin{aligned} \frac{dU}{dt} &= \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) \qquad (46.4) \\ &= \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \end{aligned}$$

but this must be equal to the loss rate. The power lost will be $P = I^2R$

$$-I^2R = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \tag{46.5}$$

This is a differential equation we can solve, let's first rearrange, remembering that

$$I = \frac{dQ}{dt}$$

then

$$\begin{aligned} -I^2R &= \frac{Q}{C}I + LI\frac{dI}{dt} \\ -IR &= \frac{Q}{C} + L\frac{dI}{dt} \end{aligned}$$

again using $I = \frac{dQ}{dt}$

$$+L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C} = 0 \tag{46.6}$$

This is a good exercise for those of you who have taken math 316. This is just like the equation governing a damped harmonic oscillator. The solution is

$$Q = Q_{\max}e^{-\frac{Rt}{2L}}\cos\omega_d t \tag{46.7}$$

where the angular frequency, ω_d is given by

$$\omega_d = \left(\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right)^{\frac{1}{2}} \tag{46.8}$$

Remember that for a damped harmonic oscillator

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

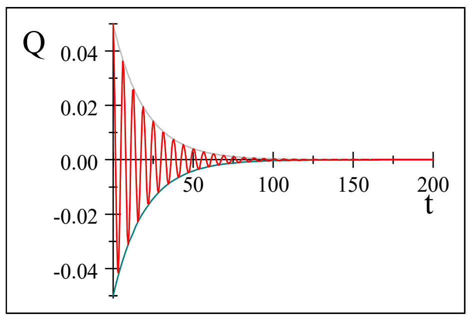
and

$$\omega = \left(\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right)^{\frac{1}{2}}$$

The resistance acts like a damping coefficient! Suppose

$$\begin{aligned}
&Q_{\max} = 0.05\,\mathrm{C} \\
&R = 5\,\Omega \\
&L = 50\,\mathrm{H} \\
&C = 0.02\,\mathrm{F}
\end{aligned}$$

we have a graph that looks like this.



The gray lines are

$$\pm Q_{\max} e^{-\frac{Rt}{2L}} \tag{46.9}$$

They describe how the amplitude changes. We call this the *envelope* of the curve.

Let's look at

$$\omega_d = \left(\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right)^{\frac{1}{2}} \tag{46.10}$$

If $\omega_d = 0$ then

$$\begin{aligned}
0 &= \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 \\
\frac{1}{LC} &= \left(\frac{R}{2L}\right)^2 \\
2L\sqrt{\frac{1}{LC}} &= R
\end{aligned}$$

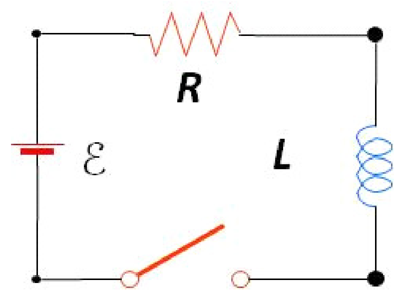
or

$$R = \sqrt{\frac{4L}{C}} \tag{46.11}$$

We know that if $\omega_d = 0$ there is no oscillation. We will call this the critical resistance, R_c. When the resistance is $R \geq R_c$ there will be no oscillation. These represent the cases of being critically damped ($R = R_c$) and overdamped $(R > R_c)$. If $R < R_c$ we are underdamped, and the circuit will oscillate.

We don't know how to make electromagnetic waves yet, but we will in a few lecture. Those waves carry what we call radio signals. To make the waves, we often use circuits with resisters, capacitors, and inductors to provide the oscillation. You can guess that if Q on the capacitor oscillates, so does the current. This oscillating current is what we use to drive the radio antenna.

Now that we have some resistance, we could consider a circuit with just an inductor and a resistor and a battery.



This is a little harder to deal with than it might appear. Let's examine the difficulties in thinking about such a circuit in the next section.

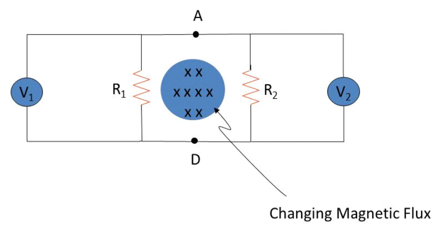
46.3 Return to Non-Conservative Fields

A few decades ago, we could have stopped here in an engineering class in considering and LRC circuit. But as electrical devices become ever more complicated, it might be good if we examine circuits with inductors and resistors more carefully. After all, we have wireless chargers and bluetooth. We now have to be more careful. We actually

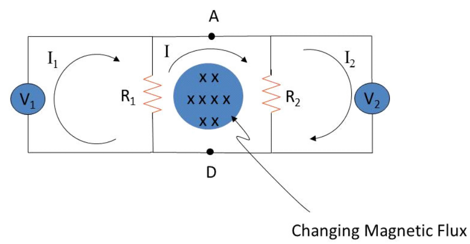
A few lectures ago we found that

$$\int \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

implies a non-conservative electric field. We should take a moment to see what this means. We should also investigate mutual inductance, which has become a major engineering technique for wireless power. First let's consider the following circuit.[**?**]



Notice that there is no battery. If the field flux changes, will there be a potential difference measured by the voltmeters? Let's use conservation of energy to analyze the circuit. I can draw in guesses for the currents.



At the junction, we can use conservation of charge to see how the currents combine or divide. This will allow us to find the voltages.

But recall that

$$\oint \mathbf{E} \cdot ds = 0$$

is a statement of conservation of energy. In electronics, we sometimes call this Kirchhoff's loop rule. And we learned that this is not true for induced emfs. So in the middle loop Kirchhoff's loop rule–conservation of energy–is not true! Some energy is transferred into or out of the circuit. We now know that is because of the changing magnetic field,

$$\oint \mathbf{E} \cdot ds = \mathcal{E} = -\frac{d\Phi_B}{dt}$$

for the middle loop. In this case, $\mathcal{E}$ comes just from the changing external flux. It does *not* depend on R_1 or on R_2.

We can write a conservation of energy equation (per unit charge) for each loop.

$$\begin{aligned} I_1 R_i - I R_1 &= 0 \\ -I R_1 - I R_2 + \mathcal{E} &= 0 \\ I_2 R_i - I R_2 &= 0 \end{aligned}$$

where R_i is the internal resistance of the voltmeters. If there were no $\mathcal{E}$, then the volt meters would not read anything, but now we see that

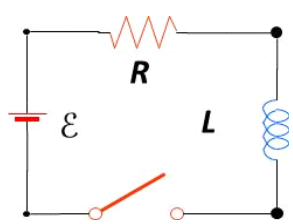
$$\begin{aligned} |V_1| &= I_1 R_i \approx I R_1 \\ |V_2| &= I_2 R_i \approx I R_2 \end{aligned}$$

This seems crazy. Each volt meter reads a different voltage. But really it is isn't so strange because there *is* energy coming into the circuit from outside. Energy for the circuit should *not* be conserved. This is just what we want for a wireless charger or a Bluetooth communication device. We will leave this non-conservative field issue for now. If you are interested in pursuing it a little further see Appendix . But for now we can say that the additional energy from the field could be given by calculating the induced emf

$$\mathcal{E}_{induced} = L\frac{dI}{dt} \tag{46.12}$$

that is coming from the changing magnetic field.

46.3.1 RL Circuits: Solving for the current as a function of time



Let's consider a simple battery, resistor, and inductor circuit. We can call this an RL circuit because of the resistor and inductor. We can try to account for all the changes in potential energy using Kirchhoff's loop rule. The battery gives us $\Delta V_{bat} = +\mathcal{E}$ and from the resistor we get a loss $\Delta V_{ristor} = -IR$. And from the inductor we will have another loss because we are removing energy from the circuit and putting it in a magnetic field $\Delta V_{inductor} = L\frac{dI}{dt}$. Note that this last ΔV is different. It is a loss of potential energy $\Delta U = q\Delta V_{inductor}$ but it does not come from an electric field like a true voltage would. It comes from moving energy out of the circuit and into a magnetic field. But we can treat it as a voltage loss because voltage losses are potential energy losses.

Putting this all together we have.

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \tag{46.13}$$

This is a differential equation. We can solve it for the current. To do so, let's define a variable

$$x = \frac{\mathcal{E}}{R} - I$$

and then we see that

$$dx = -dI$$

Then we can write our differential equation as

$$\begin{aligned} \frac{\mathcal{E}}{R} - I - \frac{L}{R}\frac{dI}{dt} &= 0 \\ x + \frac{L}{R}\frac{dx}{dt} &= 0 \end{aligned}$$

and so

$$x = -\frac{L}{R}\frac{dx}{dt}$$

You might be able to guess the solution at this point from your M316 experience if you have had the class. But let's work it out as a review. We see that our x equation separates into

$$\frac{dx}{x} = -\frac{R}{L}dt$$

Integration yields

$$\int_{x_o}^{x} \frac{dx}{x} = -\int_0^t \frac{R}{L}dt$$

$$\ln\left(\frac{x}{x_o}\right) = -\frac{R}{L}t$$

exponentiating both sides gives

$$\left(\frac{x}{x_o}\right) = e^{-\frac{R}{L}t}$$

Now we replace x with $\frac{\mathcal{E}}{R} - I$

$$\left(\frac{\frac{\mathcal{E}}{R} - I}{\frac{\mathcal{E}}{R} - I_o}\right) = e^{-\frac{R}{L}t}$$

And because at $t = 0$, $I = 0$

$$\left(\frac{\frac{\mathcal{E}}{R} - I}{\frac{\mathcal{E}}{R}}\right) = e^{-\frac{R}{L}t}$$

rearranging gives

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \tag{46.14}$$

or, defining another time constant

$$\tau = \frac{L}{R} \tag{46.15}$$

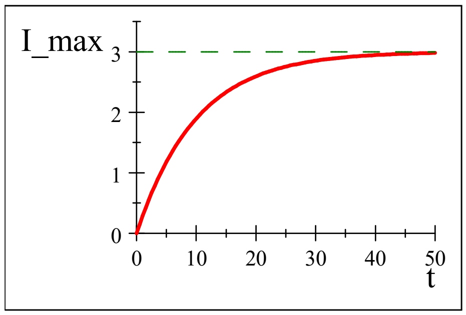
we have

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \tag{46.16}$$

We can see that

$$\frac{\mathcal{E}}{R} = I_{\max} \tag{46.17}$$

comes from Ohm's law. So just like with our capacitor-resister circuit, we have a current that grows in time, approaching the maximum value we get after a time t which is much longer than τ.



You might expect that, like for a capacitor, there is an equation for an inductor who has a maximum current flowing but for which the current source is shorted (disconnected, and replaced with a resistanceless wire). The equation is

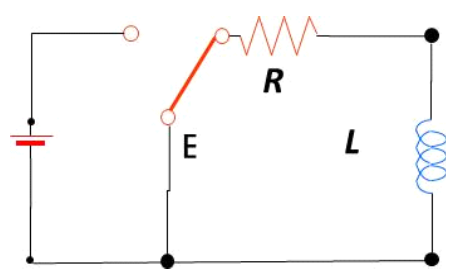
$$I = I_o e^{-\frac{t}{\tau}} \tag{46.18}$$

46.4 Magnetic Field Energy in Circuits

Let's try a problem. We found that just like with a RC circuit, we should expect there to be energy stored in a RL circuit.

$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} C \left(\Delta V\right)^2$$

Consider once again the RL circuit shown below.



Recall that the current in the right-hand loop decays exponentially with time according to the expression

$$I = I_o e^{-\frac{t}{\tau}}$$

where $I_o = \mathcal{E}/R$ is the initial current in the circuit and $\tau = L/R$ is the time constant. Let's show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

Recall that energy is delivered to the resister

$$\frac{dU}{dt} = P = I^2 R$$

where I is the instantaneous current.

$$\begin{aligned} \frac{dU}{dt} &= I^2 R \\ \frac{dU}{dt} &= \left(I_o e^{-\frac{t}{\tau}}\right)^2 R \\ \frac{dU}{dt} &= I_o^2 e^{-2\frac{t}{\tau}} R \end{aligned}$$

To find the total energy delivered to the resister we integrate

$$dU = I_o^2 e^{-2\frac{t}{\tau}} R dt$$

$$\int dU = \int_0^\infty I_o^2 e^{-2\frac{t}{\tau}} R dt$$

$$U = \int_0^\infty I_o^2 e^{-2\frac{t}{\tau}} R dt$$

$$U = I_o^2 R \int_0^\infty e^{-2\frac{t}{\tau}} dt$$

Use your calculator, or an integral table, or Mathamatica, or your very good memory to recall that

$$\int e^{-ax} dx = -\frac{1}{a} e^{-ax}$$

If we let

$$a = -\frac{2}{\tau}$$

then we can obtain

$$U = -\frac{L}{2R} I_o^2 R e^{-2\frac{t}{\tau}} \Big|_0^\infty$$

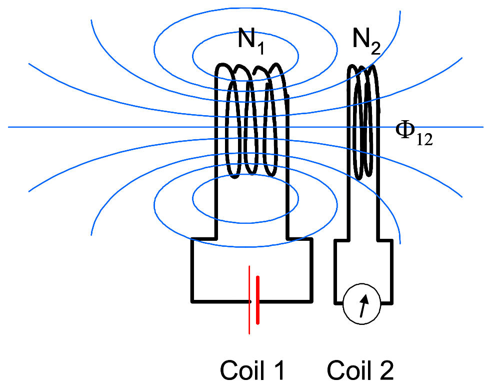
$$U = \frac{-L}{2} I_o^2 (0 - 1)$$

$$U = \frac{1}{2} I_o^2 L \tag{46.19}$$

which is the initial energy stored in the magnetic field. All of the energy that started in the inductor was delivered to the resistor.

46.5 Mutual Induction

Finely, let's consider wireless transfer of energy. Suppose we have two coils near each other. If either of the coils carries a current, will there be an induced current in the other coil?



We define Φ_{21} as the flux through coil 2 due to the current in coil 1. Likewise if the battery is placed on coil 2 we would have Φ_{12}, the flux through coil 1 due to the current in coil 2. Since we have two coils, the inductance is more complicated. Let's define a new term, the mutual inductance

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1} \tag{46.20}$$

BE CAREFUL! Not all books write the subscripts in the same order!

Like the inductance, this term contains all the geometry and design considerations of our two coil system. We can write the flux as

$$\Phi_{21} = \frac{M_{21} I_1}{N_2}$$

Then, using Faraday's law, we find the induced emf in coil 2

$$\begin{aligned}
\mathcal{E}_2 &= -N_2 \frac{d\Phi_B}{dt} \\
&= -N_2 \frac{d}{dt}\left(\frac{M_{21} I_1}{N_2}\right) \\
&= -M_{21} \frac{d}{dt}\left(I_1\right)
\end{aligned}$$

We won't prove this, but it makes sense that the coil system should work backwards. So $M_{12} = M_{21}$. Then

$$\mathcal{E}_2 = -M \frac{dI_1}{dt}$$

Example : "Wireless" battery charger

Rechargeable Toothbrush with an inductive charger (Public Domain Image courtesy Jonas Bergsten)

A rechargeable toothbrush needs a connection that is not affected by water. We can use induction to form this connection. We need two coils. One coil is the base, the other the handle. The base carries current I. The base has length l and area A and N_B turns. The handle has N_H turns and completely covers the base solenoid. What is the mutual inductance?

Solution:

The magnetic field in the base solenoid is

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mathbf{B}\, \boldsymbol{\ell} = \mu_o N_B I$$

or

$$B = \frac{\mu_o N_B I_B}{\ell}$$

Because the handle surrounds the base, the flux through the handle is the interior field of the base. The flux is

$$\Phi_{HB} = BA$$

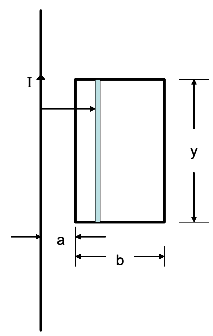
The mutual inductance is

$$\begin{aligned} M &= \frac{N_H \Phi_{HB}}{I_B} \\ &= \frac{N_H B A}{I_B} \\ &= \frac{N_H \left(\frac{\mu_o N_B I_B}{\ell}\right) A}{I_B} \\ &= \mu_o \frac{N_H N_B A}{\ell} \end{aligned}$$

Of course the end result is that the coil in the handle of the toothbrush will have a current much like we saw in the other coil of a transformer. And that current can charge the toothbrush battery.

46.5.1 Example: Rectangular Loop and a coil

A rectangular loop of N close-packed turns is positioned near a long straight wire.



What is the coefficient of mutual inductance M for the loop-wire combination?

The basic equations are

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = \Phi_B$$

We can find the field from the wire with Ampere's law

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I$$

Take the path to be a circle surrounding the wire then $\mathbf{B}$ is constant along the path and the direction of $\mathbf{B}$ is tangent to the path.

$$\begin{aligned} B \oint ds &= \mu_o I \\ B 2\pi r &= \mu_o I \end{aligned}$$

or

$$B = \frac{\mu_o I}{2\pi r}$$

The flux through the rectangular loop is then perpendicular to the plane of the loop

$$\oint \mathbf{B} \cdot d\mathbf{A} = \mathbf{\Phi}_B$$

Using the little blue rectangular area in the loop figure we have a $dA = ydr$ so our flux is

$$\begin{aligned}\mathbf{\Phi}_B &= \int Bydr \\ &= \int_a^{b+a} \frac{\mu_o I}{2\pi r} ydr \\ &= \frac{\mu_o I y}{2\pi} \ln \frac{b+a}{a}\end{aligned}$$

then

$$\begin{aligned}M &= \frac{N_2 \Phi_{21}}{I_1} \\ &= \frac{N_2}{I_1} \frac{\mu_o I_i y}{2\pi} \ln \frac{b+a}{a} \\ &= N_2 \frac{\mu_o y}{2\pi} \ln \frac{b+a}{a}\end{aligned}$$

Suppose the loop has $N_2 = 100$ turns, $a = 1\,\mathrm{cm}$, $b = 8\,\mathrm{cm}$, $y = 30\,\mathrm{cm}$, $\mu_o = 4\pi \times 10^{-7} \frac{\mathrm{T\,m}}{\mathrm{A}}$ what is the value of the mutual inductance?

$$\begin{aligned}M &= N \frac{\mu_o y}{2\pi} \ln \frac{b+a}{a} = \frac{1.3183 \times 10^{-3}}{\mathrm{A}} \,\mathrm{T\,m\,cm} \\ &= \frac{1.3183 \times 10^{-5}}{\mathrm{A}^2} \frac{\mathrm{m}^2}{\mathrm{s}^2} \,\mathrm{kg} \\ &= 1.3183 \times 10^{-5}\,\mathrm{H}\end{aligned}$$

where H stands for the Henry, our unit of mutual inductance.

$$\mathrm{H} = \frac{1}{\mathrm{A}^2} \frac{\mathrm{m}^2}{\mathrm{s}^2} \,\mathrm{kg}$$

From here we can calculate the induced emf.

Basic Equations

$$L = N \frac{d\Phi_B}{dI}$$

$$\mathcal{E} \equiv -L \frac{\Delta I}{\Delta t}$$

$$L = \mu_o n^2 V$$

$$U_L = \frac{1}{2} L I^2$$

$$u_B = \frac{1}{2} \frac{1}{\mu_o} B^2$$

$$\begin{aligned} Q &= Q_{\max}\cos(\omega t) \\ &= -I_{\max}\sin(\omega t) \end{aligned}$$

$$\begin{aligned} U &= U_C + U_L \\ &= \frac{Q^2}{2C} + \frac{1}{2}I^2 L \\ &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t) + \frac{1}{2}LI_{\max}^2\sin^2(\omega t) \end{aligned}$$

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

$$\omega = \left(\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right)^{\frac{1}{2}}$$

$$R = \sqrt{\frac{4L}{C}} \tag{46.21}$$

$$\Delta V_L = -L\frac{dI}{dt}$$

$$\tau = \frac{L}{R}$$

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$I = I_o e^{-\frac{t}{\tau}} \tag{46.22}$$

$$U = \frac{1}{2}I_o^2 L$$

$$M_{21} = \frac{N_2\Phi_{21}}{I_1}$$