Chapter 15

Capacitance

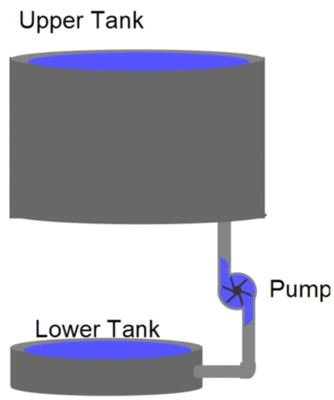
So far we have concentrated on electric field theory. We have mentioned practical things from time to time. But it is time to take a break from field theory and look at a few practical devices that use charge and field. You will use some of these devices in your PH250 class to build instruments or in your mechatronics class to design controllers. And of course you have every day devices like computers and cell phones that use practical applications of electric fields and charge. We have studied the capacitor, and it turns out that capacitors are one of these practical devices that are used to build controllers or instruments. Let's start by looking at practical uses of capacitors.

Fundamental Concepts

- The charge on a capacitor is proportional to the potential difference $Q = C\Delta V$
- The constant of proportionality is called the capacitance and for a parallel plate capacitor, it is given by $C = \frac{A}{d}\epsilon_o$
- In parallel capacitors capacitances add $C_{eq} = C_1 + C_2$
- In series capacitors capacitances combine as $\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2}$

15.1 Capacitance and capacitors

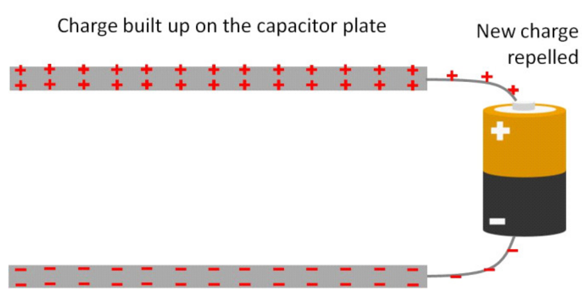
Consider the following design for a pump-tank system.



This is may not be an optimal design. At first there is no problem, water flows into the upper tank just fine. But once the upper tank begins to fill, the water already in the upper tank will make it harder to pump in more water. As the tank fills, the pressure at the bottom increases, and it takes more work for the pump to overcome the increasing pressure.

Something analogous happens when a capacitor is connected to a battery.

At first the charge is free to flow to the plates, but as the charge builds, it takes more work to bring on successive charges.



The charges repel each other, so the charge already on a capacitor plate repels the new charge arriving from the battery. The repelling force gets larger until finally the force repelling the charge balances the force driving the charge from the battery and the charge stops flowing onto the capacitor.

A capacitor is made from two plates. For us, let's assume they are semi-infinte sheets of charge. Of course this is not exactly true, but it is not too wrong near the center of the plates. And we know quite a lot about semi-infinite sheets of charge because they are one of our standard change configurations. We know the field for each sheet is

$$E = \frac{\eta}{2\epsilon_o}$$

and that for two sheets, one with $+\eta$ and one with $-\eta$ the field in between will be

$$E = \frac{\eta}{\epsilon_o}$$

We also know the potential difference between the two plates is just

$$\Delta V = E\Delta s = Ed$$

where E is our electric field and d is the capacitor spacing.

We can guess that we will build up charge until the potential energy difference of the capacitor is equal to the potential energy difference of the battery

$$\Delta V_{capacitor} = \Delta V_{battery}$$

because at that point the forces causing the potential energy will be equal.

We can write our electric field between the two plates as

$$E = \frac{\eta}{\epsilon_o} = \frac{Q}{A\epsilon_o}$$

so

$$\Delta V = \frac{Q}{A\epsilon_o}d$$

Then the potential difference is directly proportional to the charge. I want to switch this around, and solve for the amount of charge.

$$Q = \left(\frac{A\epsilon_o}{d}\right)\Delta V$$

Since all the terms in the parenthesis are constants, we could replace them with a constant, C.

$$Q = C\Delta V \tag{15.1}$$

where

$$C = \frac{A}{d}\epsilon_o \tag{15.2}$$

is a constant that depends on the geometry and construction of the plates. This equation tells us that if we build two different sets of plates, say, one circular and one triangular, and we give them the same potential difference (say, connect them both to 12 V batteries) then, if both have the same construction constant C, they will carry the same charge even though their size and shape are different. We can reduce the burden of calculation of how much charge a capacitor can hold but asking the person who manufactured it to calculate the construction constant and mark the value on the outside of the capacitor.

Different capacitors may be constructed differently (different A or d values) but so long as the construction constant, C, is the same, the charge amount for a given voltage will be the same.

The electronics field gives this construction constant a name, *capacitance.*

$$C = \frac{Q}{\Delta V} \tag{15.3}$$

The capacitance will have units of $\mathrm{C/V}$ but we give this a name all it's own, the *Farad* (F). A Farad is a very large capacitance. Many capacitors in electronic devices are measured in microfarads.

15.1.1 Capacitors and sources of potential

Consider what happens when we connect our two parallel plates to the terminals of a battery. Assuming the plates are initially uncharged, charge flows from the battery through the conducting wires and onto the plates. Recall that for a metal, the entire surface will be at the same potential under electrostatic conditions. The charge carriers supplied by the battery will try to achieve electrostatic equilibrium, so we expect the plate that is connected to the positive terminal of the battery to eventually be at the same potential as the positive battery terminal. Likewise for the negative terminal and the plate connected to it.

We can even use our capacitor as a source of electrical power. A camera flash uses capacitors to make the burst of light that illuminates the subject of your photo.

Camera flash unit (Public Domain image by Julo)

15.1.2 Single conductor capacitance

Physicists can't leave a good thing alone. We often calculate the capacitance of a single conductor! If the geometry is simple we can easily do this. It is not immediately obvious that a single conductor should even have a capacitance, so it might be a problem if you forget this in a design problem for an unusual device.

As an example, let's take a sphere. We will assume there is a spherical conduction shell that is infinitely far away. This configuration gives exactly the same field lines that the charged sphere gives on it's own, but the mental picture is helpful. The imaginary shell will give $V = 0$ (we set our zero potential at $r = \infty$). The potential of the little sphere we know must be just like the potential of a point charge if we are outside of the sphere

$$V = k_e \frac{Q}{r}$$

for $r = R$, the radius of our little sphere. Then

$$\Delta V = k_e \frac{Q}{R} - 0 = k_e \frac{Q}{R}$$

so

$$C = \frac{Q}{\Delta V} = \frac{Q}{k_e \frac{Q}{R}} = \frac{R}{k_e} = 4\pi\varepsilon_o R \tag{15.4}$$

This is the capacitance of a single sphere. Note that C only depends on geometry! not on Q, just as we would expect.

But why would we care? This says that even if we just connect a ball to, say, the positive terminal of a battery, that there will be some capacitance. This capacitance will limit the flow of charge to the ball. So it will take time to charge even a single conductor. This is always true when a device is initially connected to a power source. Often we can ignore such "transient" effects because the charging times are still small. But in special cases, this may not be possible because the changing voltage or charge could damage sensitive equipment. So although this is rarely a problem, it is good to keep in the back of our minds.

15.1.3 Capacitance of two parallel plates

The capacitance of single conductors is profound, but more useful to us in understanding common electronic components is the parallel plate capacitor. We found that for parallel plates we also had only geometry factors in the capacitance. Of course, there are other shapes possible. Let's see if we can reason out how the capacitance depends on the geometry.

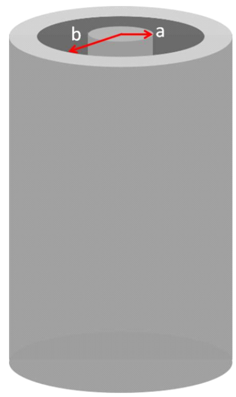
Since the charge will tend to separate to the surface of a conductor, we might expect that if the surface area increases, the amount of charge that the capacitor can hold might increase as well. We see this in our equation for the parallel plate capacitor.

$$C = \frac{A}{d}\epsilon_o$$

We also see that it matters how far apart the plates are placed. The greater the distance, the less the capacitance. This makes some sense. If the plates are farther apart, the Coulomb force is weaker, and less charge can be held in the capacitor, because the force attracting the charges (the force between the charges on the opposite plates) is weaker.

15.1.4 Capacitance of a cylindrical capacitor

We should try some harder geometries. A cylindrical capacitor is a good case to start with



(you will likely do a sphere in the homework problems). We want to find the capacitance of the cylindrical capacitor. Our strategy will be to find the voltage difference for the capacitor and the amount of charge on the capacitor, and then divide to find C.

$$C = \frac{Q}{\Delta V}$$

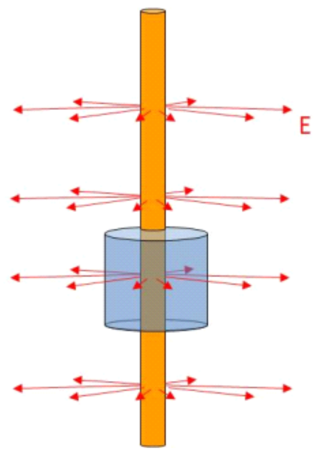
We need ΔV to find C, but we need the field to find ΔV. Let's begin with our equation relating potential change to field.

$$V_b - V_a = -\int_a^b \tilde{\mathbf{E}} \cdot d\tilde{\mathbf{s}} \tag{15.5}$$

Let's assume that there is a linear charge density, λ, along the cylinder with the center positive and the outside negative. Then

$$\Phi_E = \oint \tilde{\mathbf{E}} \cdot d\tilde{\mathbf{A}} \tag{15.6}$$

where I will choose a Gaussian surface that is cylindrical around the central conductor.



This is nice, since the field will be radially out from the conductor (ignoring the end effects) and so no field will pass through the end caps of the Gaussian surface ($\tilde{\mathbf{E}} \cdot d\tilde{\mathbf{A}} = \mathbf{0}$ on the end caps). Moreover, the field strikes the surface at right angles ($\tilde{\mathbf{E}} \cdot d\tilde{\mathbf{A}} = EdA$ on the side of the cylinder), and will have the same magnitude all the way around so

$$\begin{aligned} \Phi_E &= E \oint dA \\ &= EA \end{aligned}$$

Now we know from Gauss' law that

$$\Phi_E = \frac{Q_{in}}{\varepsilon_o}$$

where

$$Q_{in} = \lambda h$$

and where h is the height of our Gaussian surface, so

$$\Phi_E = \frac{\lambda h}{\varepsilon_o} = E 2\pi r h$$

$$\frac{\lambda}{2\pi r \varepsilon_o} = E$$

Now, knowing our field, and taking a radial path from a to b, we can take

$$\begin{aligned} V_b - V_a &= -\int_{r_a}^{r_b} \frac{\lambda}{2\pi r \varepsilon_o} d\mathbf{r} \\ &= -\frac{\lambda}{2\pi \varepsilon_o} \int_{r_a}^{r_b} \frac{1}{r} d\mathbf{r} \\ &= -\frac{\lambda}{2\pi \varepsilon_o} \ln\left(\frac{r_b}{r_a}\right) \end{aligned}$$

Using this, we can find the capacitance, We have a negative value for ΔV, but this is just due to our choice of making the center of the concentric cylinders positive and the outside negative. We chose the zero point on the positive center. The amount of potential change going from r_a to r_b is just $|\Delta V|$. Then in finding the capacitance using

$$Q = C\Delta V$$

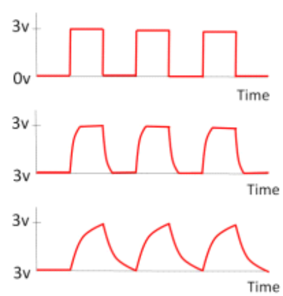
We want just the value of ΔV so we will plug in the absolute value of our result.

$$|\Delta V| = \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{b}{a}\right)$$

Then, solving for C gives

$$\begin{aligned} C &= \frac{Q}{\Delta V} \\ &= \frac{Q}{\frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)} \\ &= \frac{Q}{\frac{Q}{2\pi h\varepsilon_o}\ln\left(\frac{b}{a}\right)} \\ &= \frac{2\pi h\varepsilon_o}{\ln\left(\frac{b}{a}\right)} \end{aligned}$$

Wow! That was fun! But more importantly, this is a coaxial cable geometry, and we can see that coaxial cable will have some capacitance and that that capacitance will depend on the geometry of the cable including its length and width. This capacitance can affect signals sent through the cable. Later in our course we will see why. But for now just know that if I combine a resistor and a capacitor together it takes more time for the charge to move. So in our signal cable the signal will get distorted.



Increasing amounts of distortion in a signal due to increasing cable capacitance.

The nice square pulses that represent digital data will be distorted, and in extreme cases, undetectable. When designing data lines, this capacitance of the cable must be taken into account.

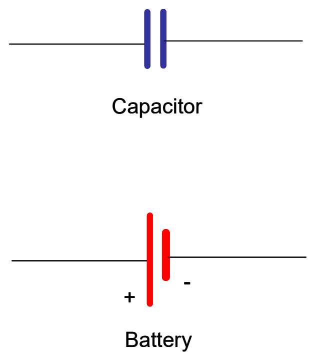
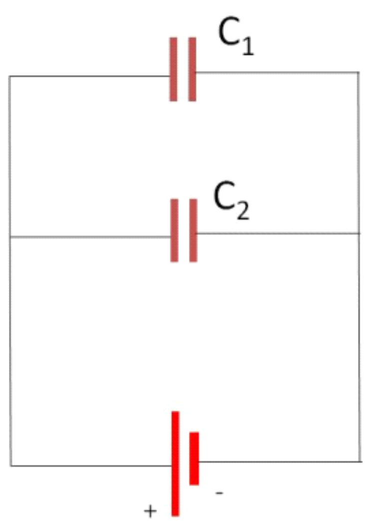


Figure 15.1:

15.2 Combinations of Capacitors

We don't want to have to do long calculations to combine capacitors that we buy from an electronics store. It would be convenient to come up with a way to combine capacitors using a simple rule.

We need a simple way to write capacitors in our homework problem drawings, here are the usual symbols for capacitor and battery. Using these symbols, let's consider two capacitors as shown below.



Remember that a conductor will be at the same potential over all of its surface. If we connect the capacitors as shown then all of the left half of this diagram will be at the positive potential of the battery terminal. Likewise, the right side will all be at the same potential. It is like we increased the area of the capacitor

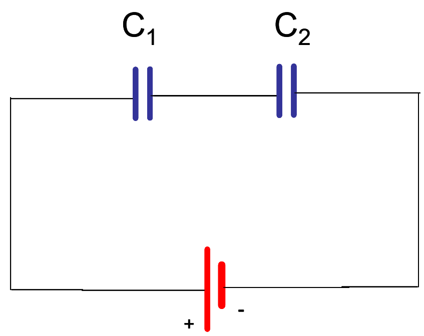


Figure 15.2:

C_1 buy adding in the area of capacitor C_2.

$$C = \frac{A_1 + A_2}{d}\epsilon_o = \frac{A_1}{d}\epsilon_o + \frac{A_2}{d}\epsilon_o$$

So we may write a combined capacitance for this set up of

$$C_{eq} = C_1 + C_2 \tag{15.7}$$

We call this set up a *parallel* circuit. This means that each of the capacitors are hooked directly to the terminals of the battery.

But suppose we hook up the capacitors as in the next drawingNow we expect the left hand side of C_1 to be at the positive potential of the positive terminal of the battery. We expect the right side of C_2 to be at the same potential as the negative side of the battery. What happens in the middle?

We can see that we will have negative charge on the right hand plate of C_2 and positive charge on the left plate of C_1. This must cause there to be a positive charge on the right plate of C_1 and a negative charge on the left plate of $C_{2.}$ Moreover, all the charges will have the same magnitude. That means each of the plates will have a potential difference

$$\Delta V_1 = \frac{Q}{C_1}$$

and

$$\Delta V_2 = \frac{Q}{C_1}$$

Now think of conservation of energy. As we go around the circuit the battery gives ΔV_{bat} of potential energy to the circuit. We will loose this same amount of potential energy as we go from the positive side of the battery back to the negative side of the battery (Kirchhoff's loop rule!). Then as we go around the loop

$$\Delta V_{bat} - \Delta V_1 - \Delta V_2 = 0$$

or

$$\Delta V_{bat} = \Delta V_1 + \Delta V_2$$

We can again define an equivalent capacitance for the combination of capacitors.

$$\Delta V = \frac{Q}{C_{equ}}$$

then

$$\begin{aligned} \Delta V_{bat} &= \Delta V_1 + \Delta V_2 \\ \frac{Q}{C_{equ}} &= \frac{Q}{C_1} + \frac{Q}{C_2} \end{aligned}$$

The Q values are all the same. So

$$\frac{1}{C_{equ}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{15.8}$$

We call this type of set up a *series circuit* because the capacitors came one after the other as you go from one side of the battery to the other.

Now after all this you might ask yourself how to know the capacitance of the parts you buy to build things. They are designed by engineers and tested at the factory, and the capacitance is usually printed on the side of the device using a special code. You can, of course, devise a test circuit based on what we have learned that could test the capacitance. Many multimeters have such a circuit in them for testing capacitors.

Basic Equations

The charge on a capacitor is proportional to the potential difference

$$Q = C\Delta V$$

The constant of proportionality is called the capacitance and for a parallel plate capacitor, it is given by

$$C = \frac{A}{d}\epsilon_o$$

In parallel capacitors capacitances add

$$C_{eq} = C_1 + C_2$$

In series capacitors capacitances combine as

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$