Chapter 28

The Electromagnetic field

We started off our study of electricity and magnetism saying we would consider the environment made by a charge and how that environment affected a mover charge. Then we found that moving charges are affected by the environment created by other moving charges (currents). It is time to consider the overall environment created by both electric and magnetic fields acting together.

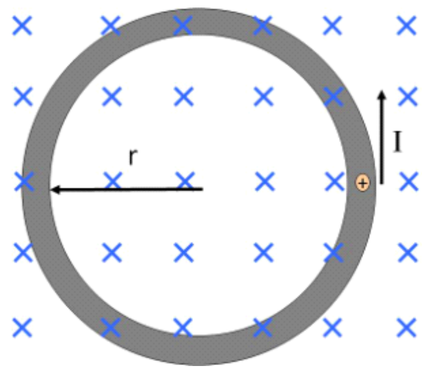
Fundamental Concepts

- The electric and magnetic fields are really different manifestations of the electromagnetic field. Which is manifest depends on our relative motion.
- The Galilean field transformations are

$$
\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in } S} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}} \\
\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right) \\
\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}'_{\text{charges in } S'} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
\overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnets in } S'} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)
\end{aligned}
$$

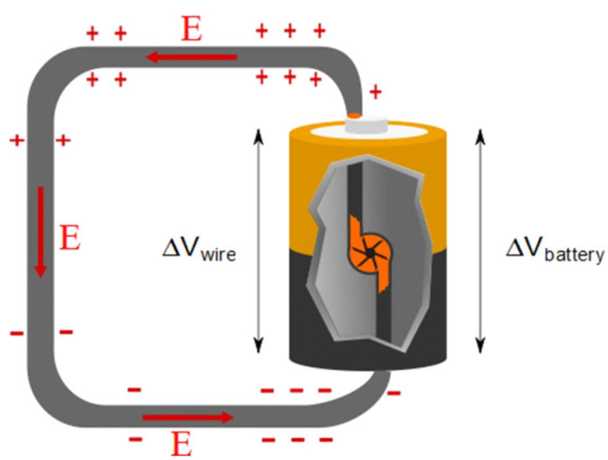
28.1 Induced Electric Fields

Consider again a magnetic field and a moving charge. If the magnetic field changes, the magnetic flux changes. Say, for example, that the magnetic field is increasing in strength.

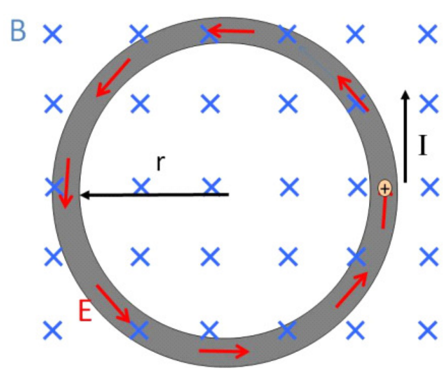


The charge will move in a circle within the wire. We now understand that this is because we have induced an emf. But think again about a battery.

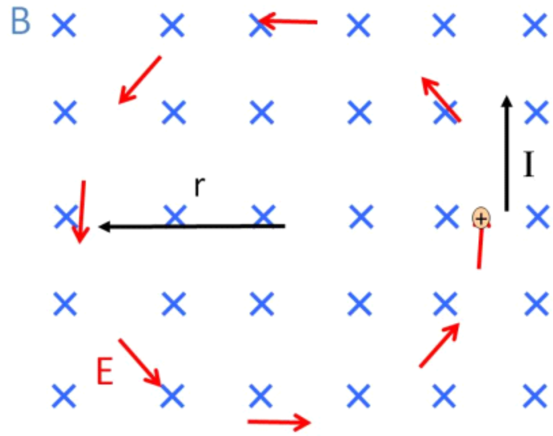
The battery makes an electric field inside a wire. Recall this figure



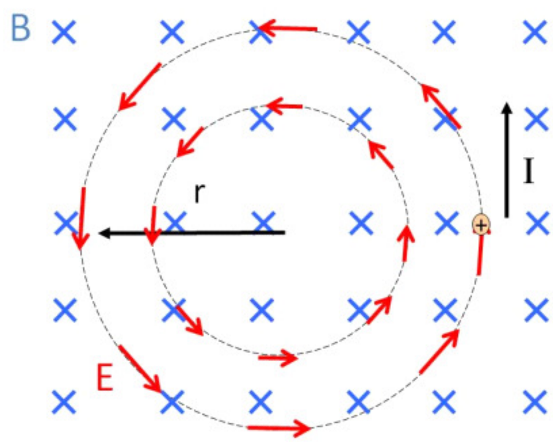
We must conclude that if we create an emf, we must have created an electric field.



We now know that this is how wireless chargers might work. But now let's ask ourselves, do we need the wire there for this electric field to happen? Of course, the electrical force on the charge is the same if there is no wire, so the E-field must be there whether or not there is a wire.



In fact, the electric field is there in every place the magnetic field exists so long as the magnetic field continues to increase.



This is quite a profound statement. We have said that a changing magnetic field *creates* an electric field. Before, only charges could create electric fields, but in this case, the magnetic field is creating the electric field. Of course, we know that somewhere there are moving charges that are making the magnetic field, so it is not totally surprising that the fields would be related.

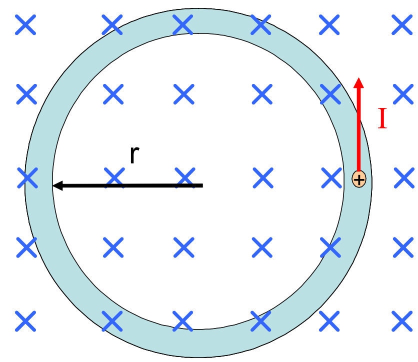
This electric field is just like a field produced by charges in that it exerts a force

$$F = q_m E$$

on a charge q_m. But the electric field source is now very different.

28.2 Relationship between induced fields

It would be nice to have a relationship between the changing B-field and the E-field that is created. It would be good to obtain the most general relationship we can that relates the electric field to the magnetic field. By understanding this relationship, we can hope to gain insight into how to build things, and into how the universe works. Let's start with a thought experiment.



Suppose we have a uniform but time varying magnetic field into the paper. In this field, we have a conducting ring. If the field strength is increasing, then the charges in the conducting loop shown will feel an induced emf, and they will form a current that is tangent to the ring.

Let's find the work required to move a charge once around the loop. The amount of potential energy difference is equal to the work done, so

$$|\Delta U| = |w|$$

but in terms of the electric potential this is

$$\Delta U = q\Delta V = q\mathcal{E}$$

so

$$|w| = |q\mathcal{E}|$$

Now let's do this another way. Let's use

$$w = \int \mathbf{F} \cdot d\mathbf{s}$$

The force making the current move is due to the induced potential difference. This is just

$$F = qE$$

which will not change as we go around the loop. The path will be along the loop, so

$$w = \int_{loop} F ds = \int_{loop} qE ds$$

and since the E-field is uniform in space at any given time as we travel around the loop,

$$w = qE \int_{loop} ds = qE2\pi r$$

So we have two expressions for the work. Let's set them equal to each other

$$q\mathcal{E} = qE2\pi r$$

The electric field is then

$$\frac{\mathcal{E}}{2\pi r} = E \tag{28.1}$$

But we know Faraday's law

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

So we can substitute in for $\mathcal{E}$ in our equation for E

$$\begin{aligned} E &= \frac{-N}{2\pi r}\frac{d\Phi_B}{dt} \\ &= \frac{-1}{2\pi r}\frac{d\Phi_B}{dt} \end{aligned}$$

So if we know how our B-field varies in time, we can find the E-field. Let's rewrite this one more time

$$2\pi r E = -\frac{d\Phi_B}{dt}$$

Since the E-field is constant as we go around the loop, we can recognize the LHS as

$$2\pi r E = \int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$$

which should be little surprise, since we found

$$\Delta V = \int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$$

to be our basic definition of the electric potential. We can write out $2\pi r E$ this way to get

$$\int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{28.2}$$

This is a more general form of Faraday's law of induction. It says that if you change the magnetic flux you get an electric field!

This electric field is fundamentally different than the E-fields we studied before in a particular way. It is not a static field. If it were, then $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$ would be zero around a ring of current. Think of conservation of energy. Around a closed loop $\Delta V = 0$ normally. Then

$$\Delta V = \oint_{loop} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = \mathbf{0} \qquad \text{no magnetic field}$$

But since $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} \neq 0$ for our induced E-field, we must recognize that this field is different from those made by static charges. We call this field that does not return the charge to the same energy state on traversing the loop a *nonconservative field*. It is still just an electric field, but we are *gaining energy* from the magnetic field, so ΔV around the loop should not be not zero because of the input energy from the magnetic field.

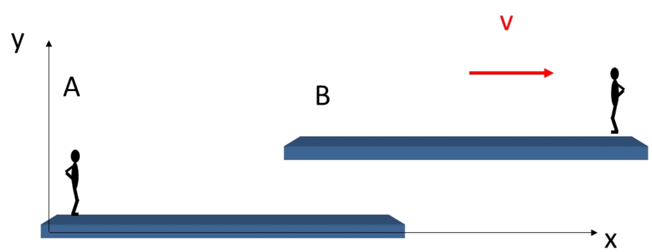
The equation

$$\oint_{loop} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{28.3}$$

is the most general form of Faraday's equation, but it is hard to use in calculation for normal circuits where there is no magnetic field or where the fields are weak. So we won't use it as we design normal circuits. We will use the idea of inductance instead. But Faraday's law plays a large part in the electromagnetic theory of optics which we will study later.[1]

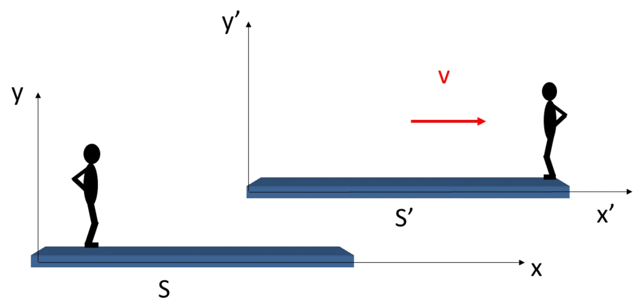
28.3 Relative motion and field theory

Long ago in your study of physics we talked about relative motion when we discussed moving objects. We considered two reference frames with a relative velocity $v\hat{\imath}$. We called them frame A and frame B

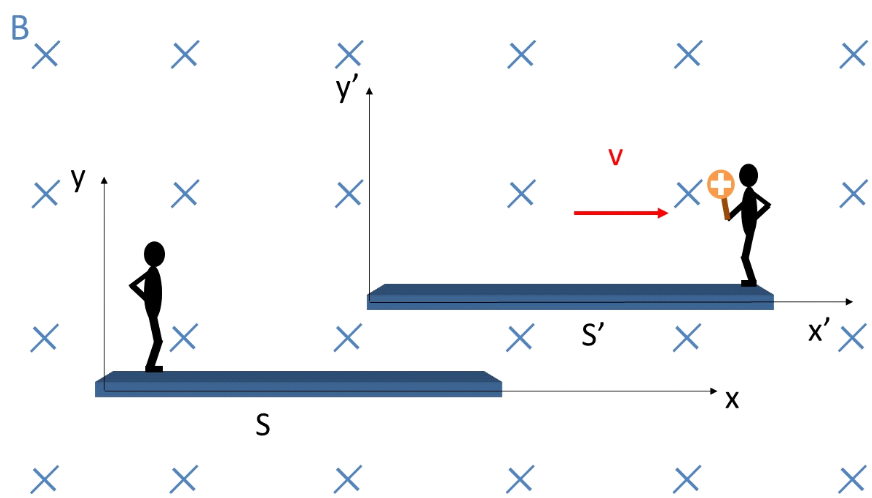


We need to return to relative motion, considering what happens when there are fields and charged particles involved. We will need to relabel our diagram to avoid confusion because now B will represent a magnetic field. So let's call the two reference frames S and S'. We will label each axis with a prime in the S' frame.

[1] We actually have an upper division class on Optics, PH375.



Now let's assume we have a magnetic field in the region of space where our two reference frames exist. Let's say that the magnetic field is stationary in frame S. This will be our environment. Let's also give a charge to the person in frame S'. This will be our mover charge.



Is there a force on the charge?

If we are with the person in reference frame S, then we must say yes. The charge is moving along with frame S' with a velocity $\overrightarrow{\mathbf{v}} = v\hat{\imath}$ so there will be a force

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}} \\ &= qv\hat{\imath} \times B\left(-\hat{k}a\right) \\ &= qVB\hat{\jmath}\end{aligned}$$

in the $\widehat{\jmath}$ direction.

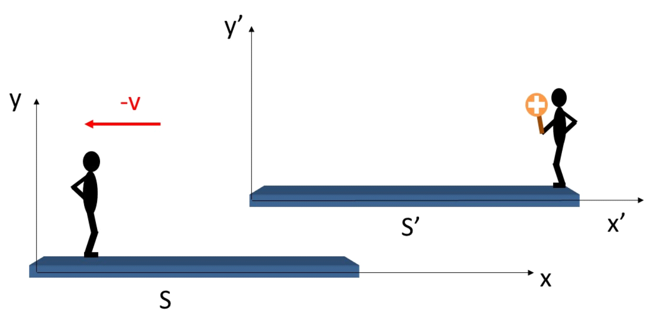
Now let's ride along with the person in frame S'. From this frame, the charge looks stationary. So $v = 0$ and

$$F = q\left(\mathbf{0}\right) \times \overrightarrow{\mathbf{B}} = 0$$

Both can't be true! So which is it? Is there a force on the charge or not? Consider that the existence of a force is something we can test. A force causes motion to change in ways we can detect. (the person in frame S' would *feel* the pull on the charge he is holding). So ultimately we can perform the experiment and see that there really is a force. But where does the force come from?

Let's consider our fields. We have come to see fields as the source of electric and magnetic forces. Electric forces come from electric fields which come from environmental charges. Magnetic forces come from environmental magnetic fields which come from moving charges.

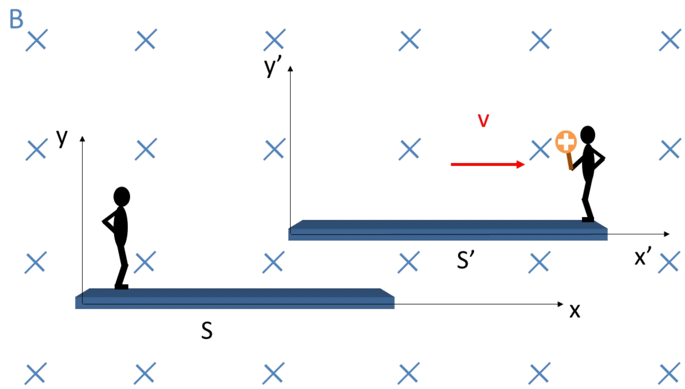
And here is the difficulty, we are having trouble recognizing when the charge is moving. We know from our consideration of relative motion that we could view this situation as frame S' moving to the right with frame S stationary, or frame S moving to the left with frame S' stationary. There is no way to say that only one of these views is correct. Both are equally valid.



In our case, we are considering that person S sees a moving charge. We have learned that a moving charge will make *both* an electric field *and* a magnetic field; the electric field just because it is a charge, and the magnetic field because it is a charge that is moving. This is the situation from frame S. But person S' sees a static charge. This charge will *only* make an electric field. We need a way to resolve this apparent contradiction.

28.3.1 Galilean transformation

To resolve this difficulty, let's go back to forces. Here is our case of a constant magnetic field that is stationary in frame S with a charge in frame S' again.



We can't see fields, but we can see acceleration of a particle. Since by Newton's second law

$$F = ma$$

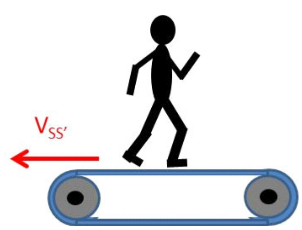
we will know if there is an acceleration, and therefore we will know if there is a force! So are the forces and accelerations of a charged particle the same in each frame? Let's find out.

Remember from Dynamics or PH121 that the speed of a particle transforms like this

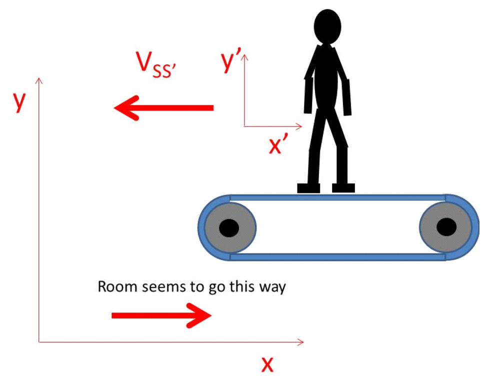
$$\begin{aligned} \vec{\mathbf{v}}' &= \vec{\mathbf{v}} - \vec{\mathbf{v}}_{S'S} \\ \vec{\mathbf{v}} &= \vec{\mathbf{v}}' + \vec{\mathbf{v}}_{S'S} \end{aligned} \tag{28.4}$$

where $V_{S'S}$ is the relative speed between the two frames. What this means is that if we have a particle moving with speed v' in frame S' and we observe this particle in frame S the speed of that particle will seem to be $\vec{\mathbf{v}} = \vec{\mathbf{v}}' + \vec{\mathbf{v}}_{S'S}$. In our case, $\vec{\mathbf{v}}_{SS'} = v_x\hat{\imath}$ so $\vec{\mathbf{v}} = \vec{\mathbf{v}}' + v_x\hat{\imath}$.

A quick example might help. Suppose we have a person in the gym running on a treadmill.

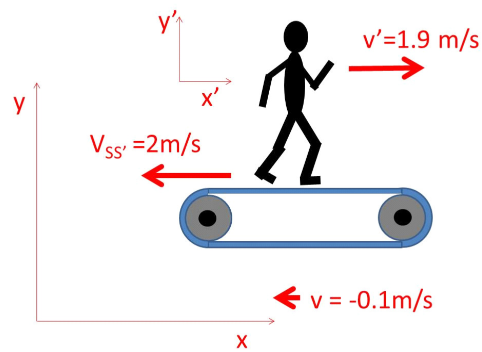


The treadmill track belt has a relative speed $\vec{\mathbf{v}}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ where the S frame in this case is the gym room and the S' frame is the treadmill belt. A person standing on the treadmill in frame S' could view themselves as not moving, and the rest of the room as moving the opposite direction.



The notation $v_{S'S}$ means the speed of the reference frame S' with respect to frame S or in our case the speed of the treadmill with respect to the room $\overrightarrow{\mathbf{v}}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$.

Now suppose the person starts running on the treadmill at speed $\overrightarrow{\mathbf{v}}' = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}'$ in the tread mill frame S'.



What is his/her speed with respect to the room? It seems obvious that we take the two speeds and add them.

$$\overrightarrow{\mathbf{v}} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}' - 2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$$

since the i and $\hat{\imath}'$ directions are the same.

The person is going to fall off the end of the treadmill unless they pick up the pace! This example just used the second equation in our transformation.

$$\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}' + \overrightarrow{\mathbf{v}}_{S'S}$$

likewise, if we want to know how fast the person is walking with respect to the treadmill frame, we take the room speed $\overrightarrow{\mathbf{v}} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ and subtract from it the treadmill/room relative speed $\overrightarrow{V}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ to obtain

$$\overrightarrow{\mathbf{v}}' = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - \left(-2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}\right) = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}'$$

Armed with the Galilean transform, we can find the acceleration by taking a derivative

$$\begin{aligned} \frac{d\overrightarrow{\mathbf{v}}'}{dt} &= \frac{d\overrightarrow{\mathbf{v}}}{dt} - \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \\ \frac{d\overrightarrow{\mathbf{v}}}{dt} &= \frac{d\overrightarrow{\mathbf{v}}'}{dt} + \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \end{aligned}$$

then

$$\begin{aligned} \overrightarrow{\mathbf{a}}' &= \overrightarrow{\mathbf{a}} - \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \\ \overrightarrow{\mathbf{a}} &= \overrightarrow{\mathbf{a}}' + \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \end{aligned}$$

but we will only consider constant relative motion in our class[2], so

$$\frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} = 0$$

then both equations tell us

$$\overrightarrow{\mathbf{a}}' = \overrightarrow{\mathbf{a}}$$

This was a lot of work, but the end of all this talk about reference frames shows us that there *must be a force*

$$\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}} = m\overrightarrow{\mathbf{a}}'$$

in both frame S and S'. The mass is the same in both frames, and so is the acceleration.

We can gain some insight into finding the mysterious missing force in frame S' by considering the net force in the case of both an electric and a magnetic field

$$\overrightarrow{\mathbf{F}}_{net} = q\overrightarrow{\mathbf{E}} + q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}$$

This was first written by Lorentz, so it is called the *Lorentz force*, and is usually written as

$$\overrightarrow{\mathbf{F}}_{net} = q\left(\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}\right)$$

Using this, let's consider the view point of each frame.

Going back to our two guys on different frames, In frame S, the person sees

$$\begin{aligned} \overrightarrow{\mathbf{F}} &= q\left(0 + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}\right) = qv_x\hat{\imath} \times B\left(-\hat{k}\right) \\ &= qVB\hat{\jmath} \end{aligned}$$

and in frame S' the person sees

$$\overrightarrow{\mathbf{F}}' = q\left(\overrightarrow{\mathbf{E}}' + 0 \times \overrightarrow{\mathbf{B}}'\right) = q\overrightarrow{\mathbf{E}}'$$

[2] Accelerating reference frames are treated by General Relatively and are treated with the notation of contravariant and covariant vectors, which are beyond this course. They are taken up in a graduate level electricity and magnetism course.

It seems that the only way that $\overrightarrow{\mathbf{F}} = \overrightarrow{\mathbf{F}}'$ is that $\overrightarrow{\mathbf{E}}' \neq 0$ in the primed frame! So in frame S' our person must conclude that there is an *external electric field* that produces the force $\overrightarrow{\mathbf{F}}'$. In frame S the person is convinced that the magnetic field, $\overrightarrow{\mathbf{B}}$, is making the force. In frame S' the person is convinced that the electric field $\overrightarrow{\mathbf{E}}'$ is making the force.

We can find the strength of this electric field by setting the forces equal

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= \overrightarrow{\mathbf{F}}' \\ q\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}} &= q\overrightarrow{\mathbf{E}}'\end{aligned}$$

so

$$\overrightarrow{\mathbf{E}}' = \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}$$

and the direction must be

$$\overrightarrow{\mathbf{E}}' = v_{S'S}B\hat{\jmath}$$

Our interpretation of this result is mind-blowing. It seems that whether we see a magnetic field or an electric field causing the force depends on our reference frame! The implication is that the electric and magnetic fields are not really two different things. They are one field viewed from different reference frames!

Anther way to say what we have found might be that moving magnetic fields show up as electric fields.

So far we have been talking about external fields only. The field $\overrightarrow{\mathbf{B}}$ in our case study is created by some outside agent. So the field $\overrightarrow{\mathbf{E}}'$ observed in frame S' is also an environmental field. But the charge, itself, creates a field. So the total electric field in frame S' is the environmental field $\overrightarrow{\mathbf{E}}'$ plus the field due to the charge, itself $\overrightarrow{\mathbf{E}}_{\text{self}}$, or

$$\begin{aligned}\overrightarrow{\mathbf{E}}'_{tot} &= \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{E}}'_{\text{environment}} \\ &= \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}\end{aligned}$$

which we usually just write as

$$\overrightarrow{\mathbf{E}}' = \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}$$

We would predict that if we had a charge that is stationary in frame S and we rode along with frame S' that we would see a field

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}'_{\text{self}} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

Of course, $\overrightarrow{\mathbf{E}}_{\text{self}}$ can't create a force on the moving charge, because it is a self-field. The field made by a charge can't move that same charge. So we only need to be concerned with $\overrightarrow{\mathbf{E}}_{\text{self}}$ if we have other charges that could move. We could actually have a group of charges riding along with frame S'. In that case we would have an additional field E'_{charges} . We could write this as

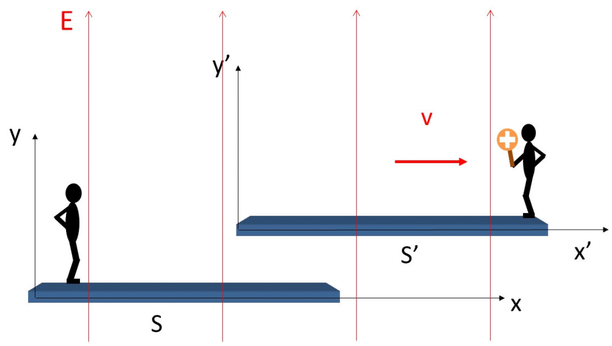
$$\overrightarrow{\mathbf{E}}_{total} = \overrightarrow{\mathbf{E}}'_{\text{charges in } S'} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

or just

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

What we have developed is important! We have an equation that let's us determine the electric field in a frame, given the fields measured in another frame.

We would expect that a similar thing would happen if we replaced the magnetic fields with electric fields. Suppose we have an electric field in the region of our frames and that this electric field is stationary with respect to frame S' this time. Will frame S see a magnetic field?



To see that this is true, let's examine the case where we have no external fields, and we just have a charge moving along with frame S'. Then in frame S' we have the fields

$$\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}} \\
\overrightarrow{\mathbf{B}}' &= 0
\end{aligned}$$

in frame S the electric field is

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}} + \overrightarrow{\mathbf{v}}_{S'S} \times \mathbf{0} \\
&= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}
\end{aligned}$$

so

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}'_{\text{charges in } S\prime} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}$$

We see the same electric field due to the point charge being there in both frames.

But in frame S we are expecting the person to see a magnetic field because to person S the charge is moving. Using the Biot-Savart law

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\overrightarrow{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

since our charge is moving along with the S' frame $\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{V}}_{S'S}$ so

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q}{r^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \hat{\mathbf{r}}\right)$$

but we can rewrite this by rearranging terms

$$\begin{aligned}
\overrightarrow{\mathbf{B}} &= \frac{\mu_o}{4\pi}\frac{q}{r^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \hat{\mathbf{r}}\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \frac{\mu_o}{4\pi}\frac{q}{r^2}\hat{\mathbf{r}}\right)
\end{aligned}$$

which looks vaguely familiar. Let's multiply top and bottom by ϵ_o

$$\begin{aligned}
\overrightarrow{\mathbf{B}} &= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \frac{\mu_o\epsilon_o}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \mu_o\epsilon_o\left(\frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}\right)\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \mu_o\epsilon_o\left(\overrightarrow{\mathbf{E}}'\right)\right) \\
&= \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'\right)
\end{aligned}$$

which is really quite astounding! Our B-fields have apparently always just been due to moving electric fields after all! Of course, we could have an additional magnet riding along with frame S'. To allow for that case, let's include a term $\overrightarrow{\mathbf{B}}'_{\text{magnet}}$.

$$\overrightarrow{\mathbf{B}}_{total} = \overrightarrow{\mathbf{B}}'_{\text{magnets in } S'} + \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)$$

or just

$$\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{B}}'_{\text{magnets in } S'} + \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)$$

and we would expect that if we worked this problem from the other frame's point of view we would likewise find

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right)$$

where the minus sign comes from the relative velocity being in the other direction. Again $\overrightarrow{\mathbf{B}}_{\text{magnet}}$ is a self-field. It won't move the magnet creating it, but it might be important if we have a second magnet in our experiment. Then $\overrightarrow{\mathbf{B}}_{\text{magnet}}$ would cause a force on this second magnet.

Once again we have found a way to find a field, the magnetic field this time, in one frame if we know the fields on another frame! We call this sort of equation a *transformation.*

We should take a moment to look at the constants $\mu_o\epsilon_o$. Let's put in their values

$$\begin{aligned}
\mu_o\epsilon_o &= \left(8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)\left(4\pi \times 10^{-7}\frac{\mathrm{T\,m}}{\mathrm{A}}\right) \\
&= 1.1121 \times 10^{-17}\frac{\mathrm{s}^2}{\mathrm{m}^2}
\end{aligned}$$

This is a very small number, and it may not appear to be interesting. We can see that the additional magnetic fields due to the movement of the charges can be quite small unless the electric field is large or the relative speed is large (or both). So much of the time this additional field due to the moving charge is negligible. But let's calculate

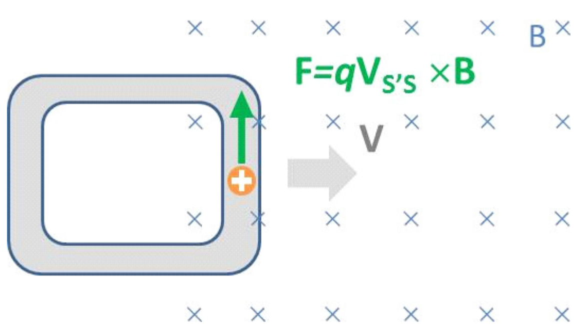
$$\begin{aligned}
\frac{1}{\sqrt{\mu_o \epsilon_o}} &= \frac{1}{\sqrt{\left(8.85\times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)\left(4\pi\times 10^{-7}\frac{\mathrm{T\,m}}{\mathrm{A}}\right)}} \\
&= 2.998\,6\times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \\
&= c
\end{aligned}$$

This is the speed of light! It even has units of m/s. This seems an amazing coincidence–too amazing. And this was one of the clues that Maxwell used to discover that light is a wave in what we will now call the *electromagnetic field* (because they are different aspects of one thing).

We can write the transformation equations for the fields for a single charged particle as

$$\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in } S} + \overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}_{\text{environment}} \\
\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{E}}_{\text{environment}}\right) \\
\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}'_{\text{environment}} \\
\overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnets in } S'} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{E}}'_{\text{environment}}\right)
\end{aligned}$$

Let's do a problem. Suppose we have a metal loop moving into an area where there is a magnetic field as shown. Let's show that there is a force on charges in this loop no matter what frame we consider. First, lets consider the frame where the magnetic field is stationary and the loop moves. Let's call that the S frame.



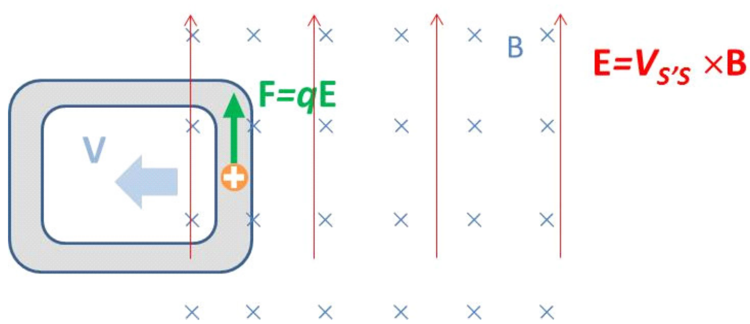
There should be an upward force on the positive charge because the charge is moving in a magnetic field. Let's say that "up" is the $\hat{\jmath}$ direction and that "to the right" is the $\hat{\imath}$ direction. Then The Lorentz force is

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= q\left(\overrightarrow{\mathbf{E}}+\overrightarrow{\mathbf{v}}\times\overrightarrow{\mathbf{B}}\right)\\ &= q\left(\overrightarrow{\mathbf{E}}+\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}\right)\end{aligned}$$

Now $\overrightarrow{\mathbf{v}}_{S'S}$ means the speed of the reference frame S' (the loop frame) with respect to frame S. That is $+v\hat{\imath}$. And there is no external electric field in frame S, so

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= q\left(\overrightarrow{\mathbf{E}}+\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}\right)\\ &= q\left(0+v\hat{\imath}\times B\left(-\hat{k}\right)\right)\\ &= q\left(v\hat{\imath}\times\mathbf{B}\left(-\hat{k}\right)\right)\\ &= qvB\hat{\jmath}\end{aligned}$$

Now suppose we change reference frames so we are riding along with the loop in frame, S'. In this frame, the loop is not moving, and the magnetic field is moving by us the opposite direction. We'll call this the "prime frame." We should get the same force if we change frames to ride along with the loop.



Let's use our transformations to find the E and B-fields in the new reference frame. Then

$$\begin{aligned}\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in } S}+\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}_{\text{environment}}\\ \overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S}-\frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{E}}_{\text{environment}}\right)\end{aligned}$$

so in the prime frame we have an electric field

$$\overrightarrow{\mathbf{E}}'=\overrightarrow{\mathbf{E}}_{\text{charges in } S}+\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}_{\text{environment}}$$

but the only charge in the S frame is our charge we are considering so we could write this as

$$\overrightarrow{\mathbf{E}}'=\overrightarrow{\mathbf{E}}_{\text{self in } S}+\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}_{\text{environment}}$$

The self term can't created a force on the charge that makes it so we can drop it.

Then we have an external electric field in the prime frame that is

$$\overrightarrow{\mathbf{E}}'_{\text{environment}} = \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}$$

To be very clear, the field $\overrightarrow{\mathbf{E}}_{\text{self in } S}$ exists and would affect a different charge. But we left off the $\overrightarrow{\mathbf{E}}_{\text{self in } S}$ because it can't move the charge that made it, so it is not part of the force we are looking for.

Note that $\overrightarrow{\mathbf{v}}_{S'S}$ is the speed of the primed frame (loop) as viewed from the unprimed frame. So $\overrightarrow{\mathbf{v}}_{S'S} = +v\hat{\imath}$

$$\begin{aligned}\overrightarrow{\mathbf{E}}' &= v(\hat{\imath}) \times B\left(-\hat{k}\right) \\ &= vB\hat{\jmath}\end{aligned}$$

That is our electric field in the primed frame.

The magnetic field in the primed frame is given by

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right)$$

but there is no external electric field in the unprimed frame, so

$$\begin{aligned}\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times 0\right) \\ &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - 0\end{aligned}$$

where here "magnets in S" means what ever is making the magnetic field in the unprimed frame. It is not our moving charge that is creating this field. But something must be there making the field, and it is not our charge. It could be an electromagnet, or a permanent magnet, we have not been told. But it is not our charge, so we know $\overrightarrow{\mathbf{B}}_{\text{magnets in } S}$ must be there and can act on our charge. So

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}$$

The magnetic field in the primed frame is just the same as the magnetic field we see in the unprimed frame. Then in the primed frame the Lorentz force is

$$\begin{aligned}\overrightarrow{\mathbf{F}}' &= q\left(\overrightarrow{\mathbf{E}}' + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}'\right) \\ &= q\left(VB\hat{\jmath} + \mathbf{0} \times \overrightarrow{\mathbf{B}}\right) \\ &= qVB\hat{\jmath}\end{aligned}$$

Which is exactly the same force (magnitude and direction) as we got in the unprimed frame.

Basic Equations

Rules for finding fields in different coordinate systems

$$\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in RF}} + \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}} \\
\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnet in RF}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right) \\
\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}'_{\text{charges in RF}} - \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
\overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnet in RF}} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)
\end{aligned}$$