Chapter 28

The Electromagnetic field

We started off our study of electricity and magnetism saying we would consider the environment made by a charge and how that environment affected a mover charge. Then we found that moving charges are affected by the environment created by other moving charges (currents). It is time to consider the overall environment created by both electric and magnetic fields acting together.

Fundamental Concepts

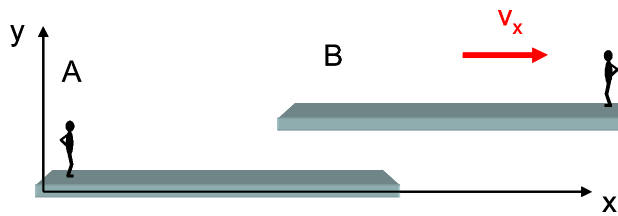
- The electric and magnetic fields are really different manifestations of the electromagnetic field. Which is manifest depends on our relative motion.
- The Galilean field transformations are

$$\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges}} + \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}} \\
\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnet}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right) \\
\overrightarrow{\mathbf{E}} &= \mathbf{E}'_{\text{charges}} - \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
\overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnet}} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)
\end{aligned}$$

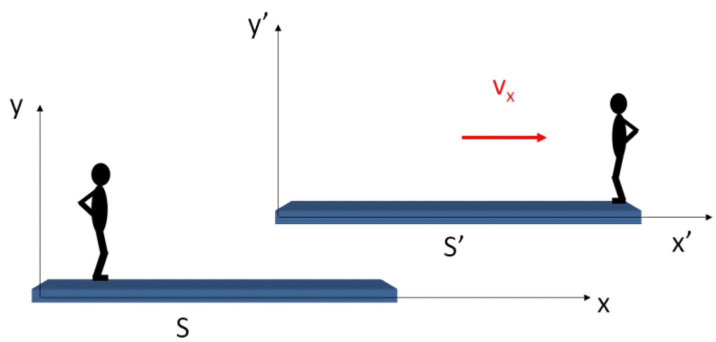
- Gauss' law for magnetic fields is $\oint \mathbf{B} \times d\mathbf{A} = 0$

28.1 Relative motion and field theory

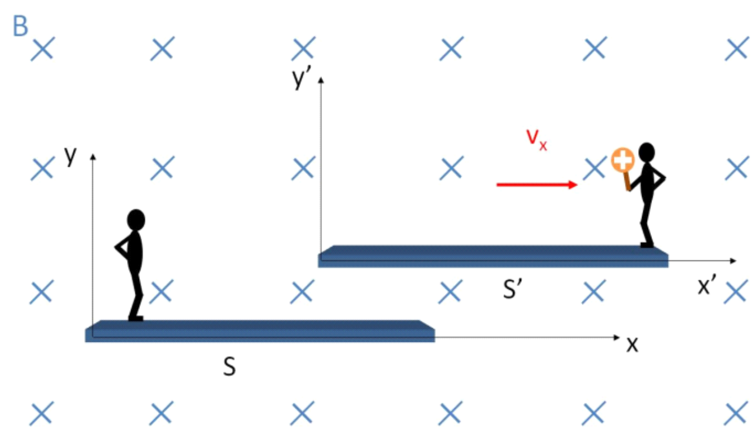
Long ago in your study of physics we talked about relative motion when we discussed moving objects and Doppler shift. We considered two reference frames with a relative velocity $v\hat{\imath}$. We called them frame A and frame B



We need to return to relative motion, considering what happens when there are fields and charged particles involved. We will need to relabel our diagram to avoid confusion because now B will represent a magnetic field. So let's call the two reference frames S and S'. We will label each axis with a prime in the S' frame.



Now let's assume we have a magnetic field in the region of space where our two reference frames exist. Let's say that the magnetic field is stationary in frame S. This will be our environment. Let's also give a charge to the person in frame S'. This will be our mover charge.



Is there a force on the charge?

If we are with the person in reference frame S, then we must say yes. The charge is moving along with frame S' with a velocity $\overrightarrow{\mathbf{v}} = v\hat{\imath}$ so there will be a force

$$\begin{aligned} \overrightarrow{\mathbf{F}} &= q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}} \\ &= qv\hat{\imath} \times B\left(-\hat{k}a\right) \\ &= qVB\hat{\jmath} \end{aligned}$$

in the $\widehat{\jmath}$ direction.

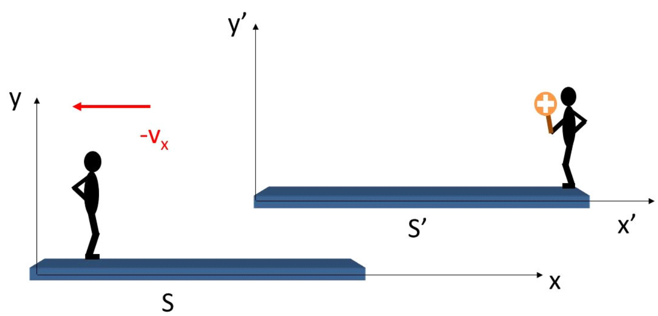
Now let's ride along with the person in frame S'. From this frame, the charge looks stationary. So $v = 0$ and

$$F = q\left(\mathbf{0}\right) \times \overrightarrow{\mathbf{B}} = 0$$

Both can't be true! So which is it? Is there a force on the charge or not? Consider that the existence of a force is something we can test. A force causes motion to change in ways we can detect. (the person in frame S' would *feel* the pull on the charge he is holding). So ultimately we can perform the experiment and see that there really is a force. But where does the force come from?

Let's consider our fields. We have come to see fields as the source of electric and magnetic forces. Electric forces come from electric fields which come from environmental charges. Magnetic forces come from environmental magnetic fields which come from moving charges.

And here is the difficulty, we are having trouble recognizing when the charge is moving. We know from our consideration of relative motion that we could view this situation as frame S' moving to the right with frame S stationary, or frame S moving to the left with frame S' stationary. There is no way to say that only one of these views is correct. Both are equally valid.

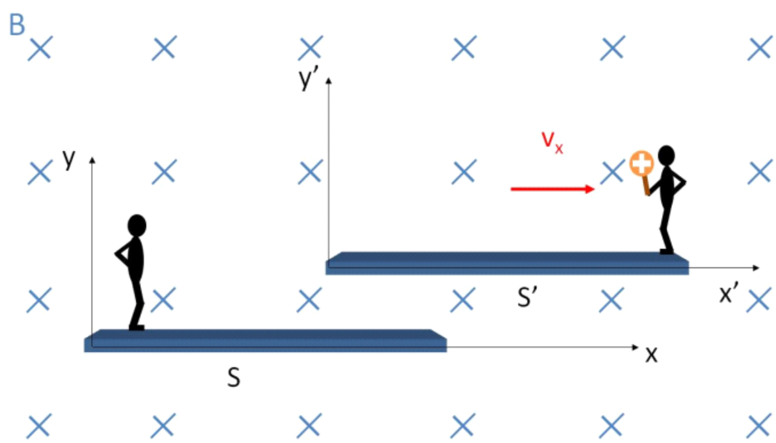


In our case, we are considering that person S sees a moving charge. We have learned that a moving charge will make *both* an electric field *and* a magnetic field; the electric field just because it is a charge, and the magnetic field because it is a charge that is moving. This is the situation from frame S. But person

S' sees a static charge. This charge will *only* make an electric field. We need a way to resolve this apparent contradiction.

28.1.1 Galilean transformation

To resolve this difficulty, let's go back to forces. Here is our case of a constant magnetic field that is stationary in frame S with a charge in frame S' again.



We can't see fields, but we can see acceleration of a particle. Since by Newton's second law

$$F = ma$$

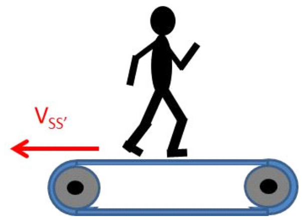
we will know if there is an acceleration, and therefore we will know if there is a force! So are the forces and accelerations of a charged particle the same in each frame? Let's find out.

Remember from Dynamics or PH121 that the speed of a particle transforms like this

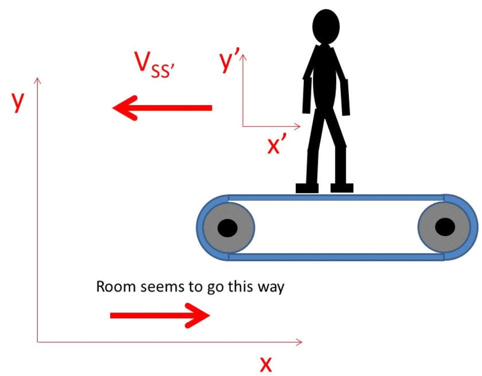
$$\begin{aligned} \overrightarrow{\mathbf{v}}' &= \overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{v}}_{S'S} \\ \overrightarrow{\mathbf{v}} &= \overrightarrow{\mathbf{v}}' + \overrightarrow{\mathbf{v}}_{S'S} \end{aligned} \tag{28.1}$$

where $V_{S'S}$ is the relative speed between the two frames. What this means is that if we have a particle moving with speed v' in frame S' and we observe this particle in frame S the speed of that particle will seem to be $\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}' + \overrightarrow{\mathbf{v}}_{S'S}$. In our case, $\overrightarrow{\mathbf{v}}_{SS'} = v_x\hat{\imath}$ so $\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}' + v_x\hat{\imath}$.

A quick example might help. Suppose we have a person in the gym running on a treadmill.

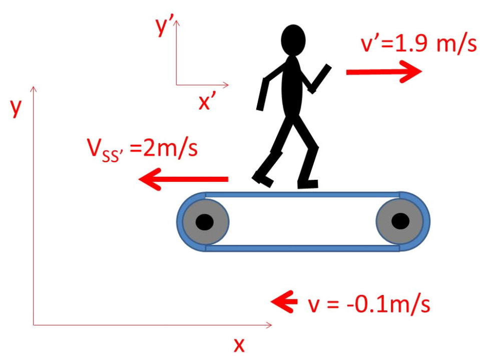


The treadmill track belt has a relative speed $\overrightarrow{\mathbf{v}}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ where the S frame in this case is the gym room and the S' frame is the treadmill belt. A person standing on the treadmill in frame S' could view themselves as not moving, and the rest of the room as moving the opposite direction.



The notation $v_{S'S}$ means the speed of the reference frame S' with respect to frame S or in our case the speed of the treadmill with respect to the room $\overrightarrow{\mathbf{v}}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$.

Now suppose the person starts running on the treadmill at speed $\overrightarrow{\mathbf{v}}' = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}'$ in the tread mill frame S'.



What is his/her speed with respect to the room? It seems obvious that we take the two speeds and add them.

$$\overrightarrow{\mathbf{v}} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}' - 2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$$

since the i and $\hat{\imath}'$ directions are the same.

The person is going to fall off the end of the treadmill unless they pick up the pace! This example just used the second equation in our transformation.

$$\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}' + \overrightarrow{\mathbf{v}}_{S'S}$$

likewise, if we want to know how fast the person is walking with respect to the treadmill frame, we take the room speed $\overrightarrow{\mathbf{v}} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ and subtract from it the treadmill/room relative speed $\overrightarrow{V}_{S'S} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ to obtain

$$\overrightarrow{\mathbf{v}}' = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - \left(-2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}\right) = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}'$$

Armed with the Galilean transform, we can find the acceleration by taking a derivative

$$\begin{aligned}\frac{d\overrightarrow{\mathbf{v}}'}{dt} &= \frac{d\overrightarrow{\mathbf{v}}}{dt} - \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \\ \frac{d\overrightarrow{\mathbf{v}}}{dt} &= \frac{d\overrightarrow{\mathbf{v}}'}{dt} + \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt}\end{aligned}$$

then

$$\begin{aligned}\overrightarrow{\mathbf{a}}' &= \overrightarrow{\mathbf{a}} - \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} \\ \overrightarrow{\mathbf{a}} &= \overrightarrow{\mathbf{a}}' + \frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt}\end{aligned}$$

but we will only consider constant relative motion in our class[1], so

$$\frac{d\overrightarrow{\mathbf{v}}_{S'S}}{dt} = 0$$

then both equations tell us

$$\overrightarrow{\mathbf{a}}' = \overrightarrow{\mathbf{a}}$$

This was a lot of work, but the end of all this talk about reference frames shows us that there *must be a force*

$$\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}} = m\overrightarrow{\mathbf{a}}'$$

in both frame S and S'. The mass is the same in both frames, and so is the acceleration.

We can gain some insight into finding the mysterious missing force in frame S' by considering the net force in the case of both an electric and a magnetic field

$$\overrightarrow{\mathbf{F}}_{net} = q\overrightarrow{\mathbf{E}} + q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}$$

[1] Accelerating reference frames are treated by General Relatively and are treated with the notation of contravariant and covariant vectors, which are beyond this course. They are taken up in a graduate level electricity and magnetism course.

This was first written by Lorentz, so it is called the *Lorentz force*, and is usually written as

$$\overrightarrow{\mathbf{F}}_{net} = q\left(\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}\right)$$

Using this, let's consider the view point of each frame.

Going back to our two guys on different frames, In frame S, the person sees

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= q\left(0 + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}\right) = qv_x\hat{\imath} \times B\left(-\hat{k}\right) \\ &= qVB\hat{\jmath}\end{aligned}$$

and in frame S' the person sees

$$\overrightarrow{\mathbf{F}}' = q\left(\overrightarrow{\mathbf{E}}' + 0 \times \overrightarrow{\mathbf{B}}'\right) = q\overrightarrow{\mathbf{E}}'$$

It seems that the only way that $\overrightarrow{\mathbf{F}} = \overrightarrow{\mathbf{F}}'$ is that $\overrightarrow{\mathbf{E}}' \neq 0$ in the primed frame! So in frame S' our person must conclude that there is an *external* electric field that produces the force $\overrightarrow{\mathbf{F}}'$. In frame S the person is convinced that the magnetic field, $\overrightarrow{\mathbf{B}}$, is making the force. In frame S' the person is convinced that the electric field $\overrightarrow{\mathbf{E}}'$ is making the force.

We can find the strength of this electric field by setting the forces equal

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= \overrightarrow{\mathbf{F}}' \\ q\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}} &= q\overrightarrow{\mathbf{E}}'\end{aligned}$$

so

$$\overrightarrow{\mathbf{E}}' = \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}$$

and the direction must be

$$\overrightarrow{\mathbf{E}}' = v_{S'S}B\hat{\jmath}$$

Our interpretation of this result is mind-blowing. It seems that whether we see a magnetic field or an electric field causing the force depends on our reference frame! The implication is that the electric and magnetic fields are not really two different things. They are one field viewed from different reference frames!

Anther way to say what we have found might be that moving magnetic fields show up as electric fields.

So far we have been talking about external fields only. The field $\overrightarrow{\mathbf{B}}$ in our case study is created by some outside agent. So the field $\overrightarrow{\mathbf{E}}'$ observed in frame S' is also an environmental field. But the charge, itself, creates a field. So the total electric field in frame S' is the environmental field $\overrightarrow{\mathbf{E}}'$ plus the field due to the charge, itself $\overrightarrow{\mathbf{E}}_{\text{self}}$, or

$$\begin{aligned}\overrightarrow{\mathbf{E}}'_{tot} &= \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{E}}'_{\text{environment}} \\ &= \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}\end{aligned}$$

which we usually just write as

$$\overrightarrow{\mathbf{E}}' = \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}$$

We would predict that if we had a charge that is stationary in frame S and we rode along with frame S' that we would see a field

$$\overrightarrow{\mathbf{E}} = \mathbf{E}'_{\text{self}} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

Of course, $\overrightarrow{\mathbf{E}}_{\text{self}}$ can't create a force on the moving charge, because it is a self-field. The field made by a charge can't move that same charge. So we only need to be concerned with $\overrightarrow{\mathbf{E}}_{\text{self}}$ if we have other charges that could move. We could actually have a group of charges riding along with frame S'. In that case we would have an additional field E'_{charges} . We could write this as

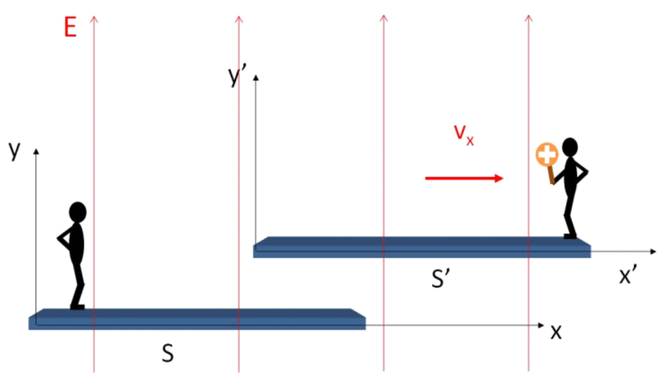
$$\overrightarrow{\mathbf{E}}_{total} = \mathbf{E}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

or just

$$\overrightarrow{\mathbf{E}} = \mathbf{E}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}}$$

What we have developed is important! We have an equation that let's us determine the electric field in a frame, given the fields measured in another frame.

We would expect that a similar thing would happen if we replaced the magnetic fields with electric fields. Suppose we have an electric field in the region of our frames and that this electric field is stationary with respect to frame S' this time. Will frame S see a magnetic field?



To see that this is true, let's examine the case where we have no external fields, and we just have a charge moving along with frame S'. Then in frame S' we have the fields

$$\begin{aligned} \overrightarrow{\mathbf{E}}' &= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}} \\ \overrightarrow{\mathbf{B}}' &= 0 \end{aligned}$$

in frame S the electric field is

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}'_{\text{self}} - \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}} + \overrightarrow{\mathbf{v}}_{S'S} \times \mathbf{0} \\
&= \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}
\end{aligned}$$

so

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}'_{\text{self}} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}$$

We see the same electric field due to the point charge being there in both frames.

But in frame S we are expecting the person to see a magnetic field because to person S the charge is moving. Using the Biot-Savart law

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\overrightarrow{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

since our charge is moving along with the S' frame $\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{V}}_{S'S}$ so

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q}{r^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \hat{\mathbf{r}}\right)$$

but we can rewrite this by rearranging terms

$$\begin{aligned}
\overrightarrow{\mathbf{B}} &= \frac{\mu_o}{4\pi}\frac{q}{r^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \hat{\mathbf{r}}\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \frac{\mu_o}{4\pi}\frac{q}{r^2}\hat{\mathbf{r}}\right)
\end{aligned}$$

which looks vaguely familiar. Let's multiply top and bottom by ϵ_o

$$\begin{aligned}
\overrightarrow{\mathbf{B}} &= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \frac{\mu_o\epsilon_o}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \mu_o\epsilon_o\left(\frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\mathbf{r}}\right)\right) \\
&= \left(\overrightarrow{\mathbf{v}}_{S'S} \times \mu_o\epsilon_o\left(\overrightarrow{\mathbf{E}}'\right)\right) \\
&= \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'\right)
\end{aligned}$$

which is really quite astounding! Our B-fields have apparently always just been due to moving electric fields after all! Of course, we could have an additional magnet riding along with frame S'. To allow for that case, let's include a term $\overrightarrow{\mathbf{B}}'_{\text{magnet}}$.

$$\overrightarrow{\mathbf{B}}_{total} = \overrightarrow{\mathbf{B}}'_{\text{magnets in }S'} + \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)$$

or just

$$\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{B}}'_{\text{magnets in }S'} + \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)$$

and we would expect that if we worked this problem from the other frame's point of view we would likewise find

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \mu_o\epsilon_o\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right)$$

where the minus sign comes from the relative velocity being in the other direction. Again $\overrightarrow{\mathbf{B}}_{\text{magnet}}$ is a self-field. It won't move the magnet creating it, but it might be important if we have a second magnet in our experiment. Then $\overrightarrow{\mathbf{B}}_{\text{magnet}}$ would cause a force on this second magnet.

Once again we have found a way to find a field, the magnetic field this time, in one frame if we know the fields on another frame! We call this sort of equation a *transformation.*

We should take a moment to look at the constants $\mu_o\epsilon_o$. Let's put in their values

$$\begin{aligned}\mu_o\epsilon_o &= \left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)\left(4\pi\times10^{-7}\frac{\mathrm{T\,m}}{\mathrm{A}}\right)\\ &= 1.1121\times10^{-17}\frac{\mathrm{s}^2}{\mathrm{m}^2}\end{aligned}$$

This is a very small number, and it may not appear to be interesting. We can see that the additional magnetic fields due to the movement of the charges can be quite small unless the electric field is large or the relative speed is large (or both). So much of the time this additional field due to the moving charge is negligible. But let's calculate

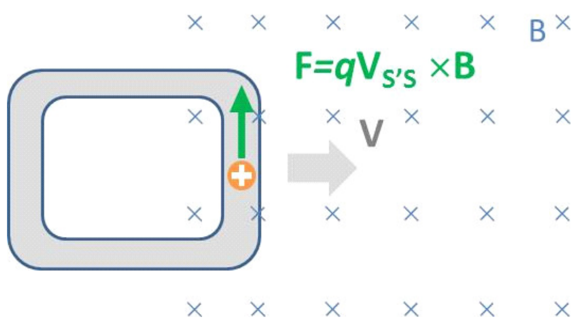
$$\begin{aligned}\frac{1}{\sqrt{\mu_o\epsilon_o}} &= \frac{1}{\sqrt{\left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)\left(4\pi\times10^{-7}\frac{\mathrm{T\,m}}{\mathrm{A}}\right)}}\\ &= 2.9986\times10^{8}\frac{\mathrm{m}}{\mathrm{s}}\\ &= c\end{aligned}$$

This is the speed of light! It even has units of m/s. This seems an amazing coincidence–too amazing. And this was one of the clues that Maxwell used to discover that light is a wave in what we will now call the *electromagnetic field* (because they are different aspects of one thing).

We can write the transformation equations for the fields for a single charged particle as

$$\begin{aligned}\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in } S} + \overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}_{\text{environment}}\\ \overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnets in } S} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{E}}_{\text{environment}}\right)\\ \overrightarrow{\mathbf{E}} &= \mathbf{E}'_{\text{charges in } S\prime} - \overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{B}}'_{\text{environment}}\\ \overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnets in } S'} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S}\times\overrightarrow{\mathbf{E}}'_{\text{environment}}\right)\end{aligned}$$

Let's do a problem. Suppose we have a metal loop moving into an area where there is a magnetic field as shown. Let's show that there is a force on charges in this loop no matter what frame we consider. First, lets consider the frame where the magnetic field is stationary and the loop moves. Let's call that the S frame.



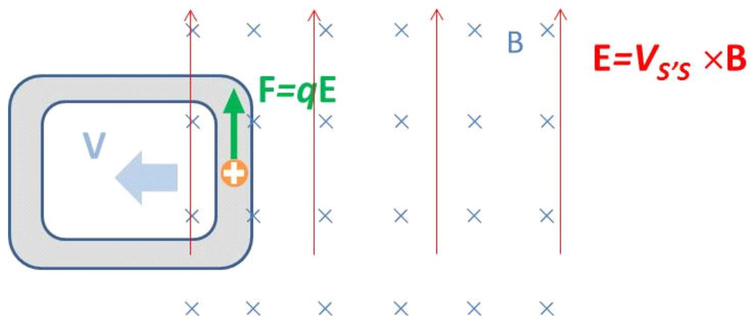
There should be an upward force on the positive charge because the charge is moving in a magnetic field. Let's say that "up" is the $\hat{\jmath}$ direction and that "to the right" is the $\hat{\imath}$ direction. Then The Lorentz force is

$$\begin{aligned}
\overrightarrow{\mathbf{F}} &= q\left(\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}\right) \\
&= q\left(\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}\right)
\end{aligned}$$

Now $\overrightarrow{\mathbf{v}}_{S'S}$ means the speed of the reference frame S' (the loop frame) with respect to frame S. That is $+v\hat{\imath}$. And there is no external electric field in frame S, so

$$\begin{aligned}
\overrightarrow{\mathbf{F}} &= q\left(\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}\right) \\
&= q\left(0 + v\hat{\imath} \times B\left(-\hat{k}\right)\right) \\
&= q\left(v\hat{\imath} \times \mathbf{B}\left(-\hat{k}\right)\right) \\
&= qvB\hat{\jmath}
\end{aligned}$$

Now suppose we change reference frames so we are riding along with the loop in frame, S'. In this frame, the loop is not moving, and the magnetic field is moving by us the opposite direction. We'll call this the "prime frame." We should get the same force if we change frames to ride along with the loop.



Let's use our transformations to find the E and B-fields in the new reference frame. Then

$$\begin{aligned}\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}} \\ \overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{self}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right)\end{aligned}$$

so in the prime frame we have an electric field

$$\overrightarrow{\mathbf{E}}' = \overrightarrow{\mathbf{E}}_{\text{self}} + \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}$$

The self term can't created a force on the charge that makes it so we can drop it.

Then we have an external electric field in the prime frame that is

$$\overrightarrow{\mathbf{E}}'_{\text{environment}} = \overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}}$$

To be very clear, the field $\overrightarrow{\mathbf{E}}_{\text{charge}}$ exists and would affect a different charge. But we left off the $\overrightarrow{\mathbf{E}}_{\text{charge}}$ because it can't move the charge that made it, so it is not part of the force we are looking for.

Note that $\overrightarrow{\mathbf{v}}_{S'S}$ is the speed of the primed frame (loop) as viewed from the unprimed frame. So $\overrightarrow{\mathbf{v}}_{S'S} = +v\hat{\imath}$

$$\begin{aligned}\overrightarrow{\mathbf{E}}' &= v\left(\hat{\imath}\right) \times B\left(-\hat{k}\right) \\ &= vB\hat{\jmath}\end{aligned}$$

That is our electric field in the primed frame.

The magnetic field in the primed frame is given by

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}_{\text{magnet}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{v}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right)$$

but there is no external electric field in the unprimed frame, so

$$\begin{aligned}\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnet}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times 0\right) \\ &= \overrightarrow{\mathbf{B}}_{\text{magnet}}\end{aligned}$$

where here “magnet” means what ever is making the magnetic field in the unprimed frame. It is not our moving charge that is creating this field. But something must be there making the field, and it is not our charge. It could be an electromagnet, or a permanent magnet, we have not been told. But it is not our charge, so we know $\overrightarrow{\mathbf{B}}_{\text{magnet}}$ must be there and can act on our charge. So

$$\overrightarrow{\mathbf{B}}' = \overrightarrow{\mathbf{B}}$$

The magnetic field in the primed frame is just the same as the magnetic field we see in the unprimed frame. Then in the primed frame the Lorentz force is

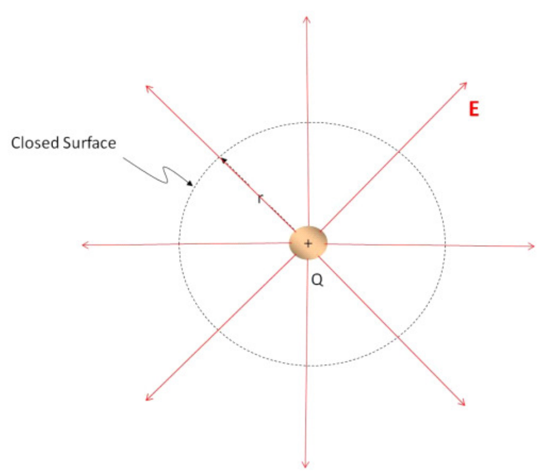
$$\begin{aligned}
\overrightarrow{\mathbf{F}}' &= q\left(\overrightarrow{\mathbf{E}}' + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}'\right) \\
&= q\left(VB\hat{\jmath} + \mathbf{0} \times \overrightarrow{\mathbf{B}}\right) \\
&= qVB\hat{\jmath}
\end{aligned}$$

Which is exactly the same force (magnitude and direction) as we got in the unprimed frame.

28.2 Field Laws

A “law” in physics is a mathematical statement of a physical principal or theory. We have been collecting laws for what we will now call the *electromagnetic field theory*. Let’s review:

28.2.1 Gauss’ law



We found that the electric flux through an imaginary closed surface that incloses some charge is

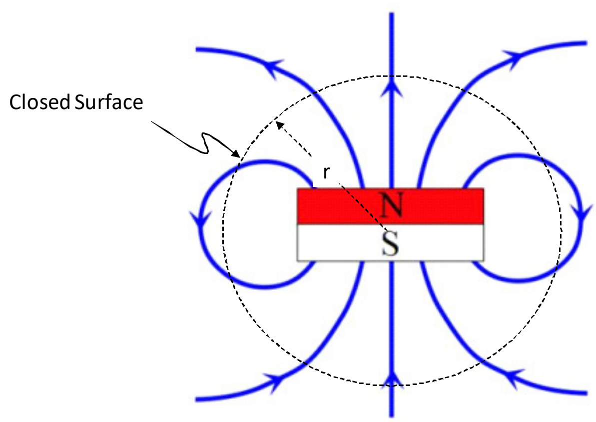
$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{in}}{\epsilon_o}$$

We called this Gauss' law.

But consider the situation with a magnet. We can define a magnetic flux just like we defined the electric flux. And now we know they must be related. Is there a Gauss' law for magnetism? Let's consider the magnetic flux.

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{A}$$

This should be proportional to the number of "magnetic charges" inclosed in the surface.

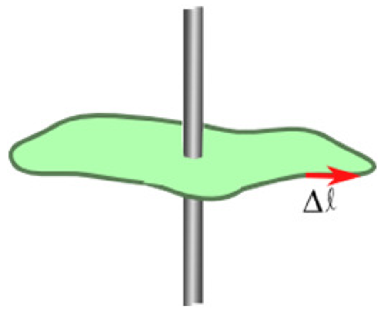


We can see that every field line that leaves comes back in. That is how we defined zero net flux, so

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{A} = 0$$

Which would tell us that there are no free "magnetic charges" or no single magnetic poles. A single magnetic pole is called a *monopole* and indeed we have never discovered one. These two forms of Gauss' law form the first two of our electromagnetic field equations.

The differences between them have to do with the fact that magnetic fields are due to moving charges.



We have a third electromagnetic field law, Ampere's law. We found Ampere's law by integrating around a closed loop with a current penetrating the loop.

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I_{through}$$

We also know Faraday's law

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

which told us that changing magnetic fields created an electric field. We have found that the opposite must be true, that a changing electric field must create a magnetic field. We express this as

$$\oint \mathbf{B} \cdot d\mathbf{s} \propto \frac{d\Phi_E}{dt}$$

Which gives two expressions for $\oint \mathbf{B} \cdot d\mathbf{s}$. But we have yet to show that this equation is true. That is the subject of our next lecture. If we can accomplish this, we will have a complete set of field equations that describe how the electromagnetic field works. In the following lecture we will complete the set of field equations, and then in the next lecture we will show that we get electromagnetic waves from these equations.

Basic Equations

Rules for finding fields in different coordinate systems

$$\begin{aligned}
\overrightarrow{\mathbf{E}}' &= \overrightarrow{\mathbf{E}}_{\text{charges in RF}} + \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}_{\text{environment}} \\
\overrightarrow{\mathbf{B}}' &= \overrightarrow{\mathbf{B}}_{\text{magnet in RF}} - \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}_{\text{environment}}\right) \\
\overrightarrow{\mathbf{E}} &= \mathbf{E}'_{\text{charges in RF}} - \overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{B}}'_{\text{environment}} \\
\overrightarrow{\mathbf{B}} &= \overrightarrow{\mathbf{B}}'_{\text{magnet in RF}} + \frac{1}{c^2}\left(\overrightarrow{\mathbf{V}}_{S'S} \times \overrightarrow{\mathbf{E}}'_{\text{environment}}\right)
\end{aligned}$$

Gauss' law for magnetic fields

$$\Phi_B = \oint \mathbf{B} \times d\mathbf{A} = 0$$