25 Dipole motion, Symmetry

Fundamental Concepts

- Force and torque on a dipole in a uniform field
- Force on a dipole in a non-uniform field
- Drawing the shape of a field using symmetry

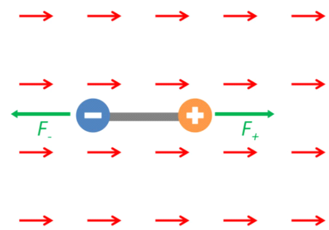
This lecture combines two topics that might be better separated. The first relates to forces on charges in uniform fields. This is what we discussed last lecture. The next is the beginning of the ideas that will allow us to use symmetry and geometry to avoid integration over charges. But because our lecture times are only an hour, and we can only do so much at once, they are combined here together. But they form a nice transition between the two topics this way. We will first study the motion of dipoles in uniform, and not so uniform fields. We will find symmetry and geometry plays a part in our solutions. Then we will study the fields of standard symmetric objects.

Dipole motion in an electromagnetic field

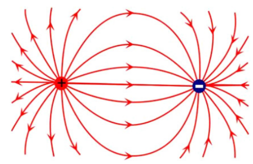
We remember dipoles, a pair of charges of equal magnitude but opposite in charge, bound together at set separation distance. Let's take our environment to be a constant electric field, and our mover to be a dipole.

Question 223.25.1

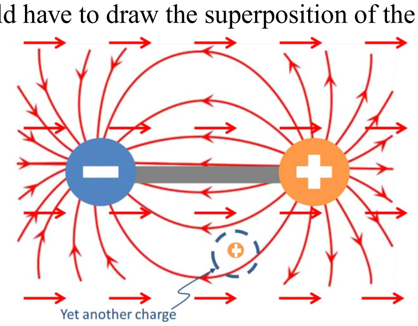
Here is a diagram of the situation.



Notice that as usual, just the environmental field is drawn. There is a field from the dipole, too,

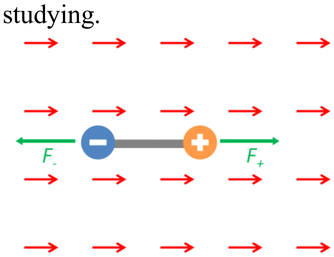


but this is the mover's self-field and it cannot create a force on the dipole, so we will not draw it. Of course, if we introduce yet another charge, q_{new}, the environmental field this new charge would feel would be a combination of both the dipole field and the uniform field! We would have to draw the superposition of the two fields.

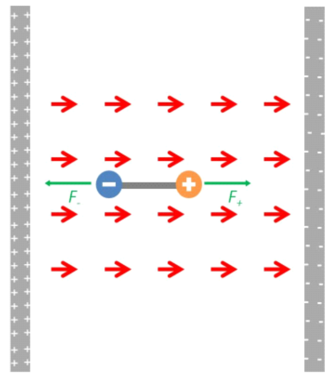


But that is a different problem!

Here is our case again. We only draw the environmental field that will cause the motion of the mover object we are studying.



To understand these figures,we have to remember that the red field arrows are an *external field* that is, the dipole is not making this field, so something else must be. We did not draw that something else. Since it is a uniform field, it is probably a capacitor. Here is what it might look like



The positive side must be to the left, because the red external field arrows come from the left. The negative side must be to the right, because the field arrows are pointed that direction. We can get away with not drawing the source of the external field because the force on the dipole charges is just

$$\overrightarrow{\mathbf{F}} = q_m \overrightarrow{\mathbf{E}}$$

If we know $\overrightarrow{\mathbf{E}}$, then we don't need any information about it's source to find the force. Since the field is the environment that the mover charges feel, the field is enough. Let's find the net force on the dipole due to the environmental field.

Question 223.25.2

We use Newton's second law to find that

$$F_{net_x} = -F_- + F_+ = ma_x$$

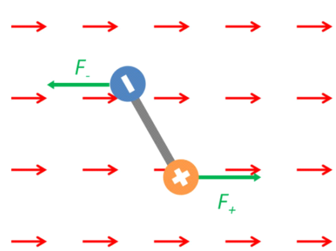
and our definition of the electric field to find

$$\begin{aligned} F_- &= q_- E \\ F_+ &= q_+ E \end{aligned}$$

so, since $|q_-| = |q_+| = q$

$$-qE + qE = ma_x$$

which tells us that there is no acceleration, no net force. The center of mass of a dipole does not accelerate in a uniform field. But we remember from PH121 that we can make things rotate.



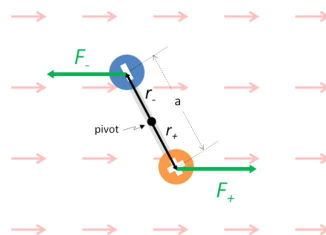
If the dipole is not aligned with it's axis in the field direction, then the forces will cause a torque.

Question 223.25.3

We remember that torque is given by

$$\overrightarrow{\boldsymbol{\tau}} = \overrightarrow{\mathbf{r}} \times \overrightarrow{\mathbf{F}}$$

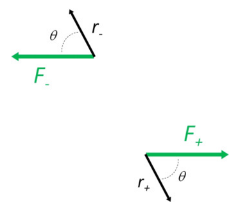
substituting in our force and defining the distance between the charges to be a we can write this out



The magnitude of the torque is given by

$$|\tau| = rF\sin\theta$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. It is easier to find that angle if we redraw each displacement vector from the pivot and each force with their tails together



Then for one charge, say, $q\_$

$$\boldsymbol{\tau} = \frac{a}{2}qE\sin\theta$$

We use the right-hand-rule that you learned in Dynamics or PH121 to find the direction. We can see that the direction will be out of the page. But we have two charges, so we have a torque from each charge. A quick check with the right-hand-rule for torques will convince us that the direction for the torque due to q_+ is also out of the page, and the magnitude is the same, so our total torque is

$$\begin{aligned} \tau_{net} &= \tau_+ + \tau_- \\ &= aqE\sin\theta \end{aligned}$$

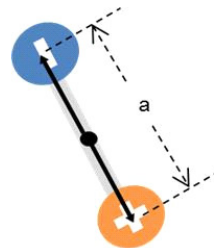
Question 223.25.4 which we can write as

$$\tau_{net} = pE\sin\theta$$

or the *dipole moment*, p, multiplied by $E\sin\theta$. Recalling the form of a cross product

$$\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}} = AB\sin\theta\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is perpendicular to both $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$, we have a hint that we could write our torque as a cross product. We would have to make p a vector, though. So let's define $\overrightarrow{\mathbf{p}}$ as a vector with magnitude aq and make its direction along the line connecting the charge centers, with the direction from negative to positive.



Then we can write the torque as

$$\overrightarrow{\boldsymbol{\tau}} = \overrightarrow{\mathbf{p}} \times \overrightarrow{\mathbf{E}} \tag{25.1}$$

which is our form for the torque on a dipole.

Let's try a problem. Let's find the maximum angular acceleration for a dipole.

Recall that Newton's second law for rotational motion is

$$\Sigma\tau = I\alpha$$

where I is the moment of inertia and α is the angular acceleration. Then we can find how the dipole will accelerate

$$\alpha = \frac{\tau_{net}}{I}$$

For a dipole, I is simple

$$\begin{aligned} I &= m_{-}r_{-}^{2} + m_{+}r_{+}^{2} \\ &= m\left(\frac{a}{2}\right)^{2} + m\left(\frac{a}{2}\right)^{2} \\ &= \frac{1}{2}ma^{2} \end{aligned}$$

so our acceleration is

$$\begin{aligned} \alpha &= \frac{pE\sin\theta}{\frac{1}{2}ma^{2}} \\ &= \frac{2pE\sin\theta}{ma^{2}} \end{aligned}$$

Suppose we look at this for a water molecule in a microwave oven. What is the maximum angular acceleration experienced by the water molecule if the oven has a field strength of $E = 200\,\mathrm{V/\,m}$?

The dipole moment for a water molecule is something like

$$p_w = 6.2 \times 10^{-30}\,\mathrm{C\,m}$$

and the separation between the charge centers is something like

$$a = 3.9 \times 10^{-12}\,\mathrm{m}$$

and the molecular mass of water is

$$M = 18\frac{\mathrm{g}}{\mathrm{mol}}$$

which is

$$M = mN_A$$

so the mass of a water molecule is

$$\begin{aligned} m &= \frac{M}{N_A} = \frac{18\frac{\mathrm{g}}{\mathrm{mol}}}{6.022 \times 10^{23}\frac{1}{\mathrm{mol}}} \\ &= 2.989 \times 10^{-26}\,\mathrm{kg} \end{aligned}$$

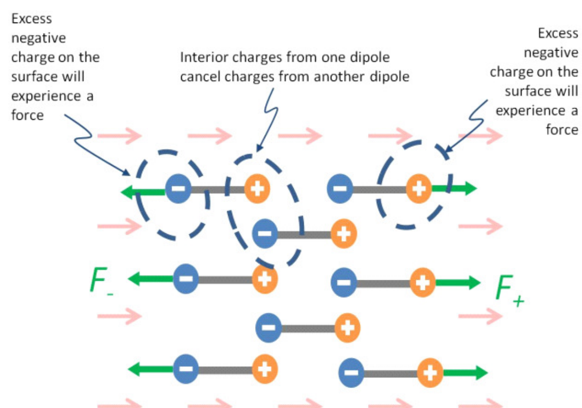
then when $\sin\theta = 1$ we will have a maximum

$$\begin{aligned} \alpha &= \frac{2\left(6.2 \times 10^{-30}\,\mathrm{C\,m}\right)(200\,\mathrm{V/\,m})}{(2.989 \times 10^{-26}\,\mathrm{kg})\left(3.9 \times 10^{-12}\,\mathrm{m}\right)^2} \\ &= 5.455 \times 10^{21}\frac{\mathrm{rad}}{\mathrm{s}^2} \end{aligned}$$

Our numbers were kind of rough estimates, but still the result is amazing. Imagine if this happened inside of you! which is why we really should be careful with microwave ovens and microwave equipment.

Induced dipoles

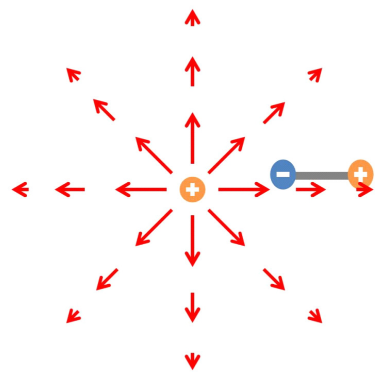
Suppose that we place a large insulator in a uniform electric field.



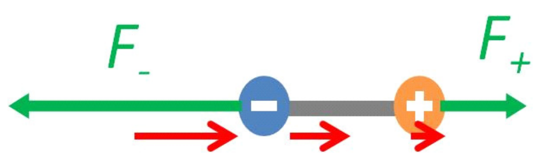
The atoms tend to polarize and become dipoles. We say we have *induced* dipoles within the material. Notice that in the middle of the conductor there is still no net charge. But because we have made the atoms into dipoles, one side of the insulator becomes negatively charged and the other side becomes positively charged. This does not create a net force, but we will find that separating the charges like this can be useful in building capacitors.

Non-uniform fields and dipoles

Suppose we place our dipole in a non-uniform field? Of course the result will depend on the field, so let's take an example. Let's place a dipole in the field due to a point charge.



We can see that the field is much weaker at the location of the positive charge than it is at the negative charge location. If we zoom in on the location near our dipole we can see that now we will have an acceleration!



$$\Sigma F_x = -F_- + F_+ = ma_x$$

so

$$-qE_{\text{large}} + qE_{\text{small}} = ma_x$$

Let's go back to our charged balloon from many lectures ago. We found that the charge "leaked off" our balloon. We can see why now. The water molecules in the air are attracted to the charges, and stick to them. When the water molecules float off, they will take our charge with them. For this problem, the dipole is the environment and our balloon electron is the moving charge. We can calculate the net force easily with our field from a dipole that we found earlier,

$$\vec{\mathbf{E}}_y = \frac{2}{4\pi\epsilon_o}\frac{\vec{\mathbf{p}}}{L^3}$$

then the force on the electron on the balloon is

$$\begin{aligned} F &= q_e E \\ &= \frac{2q_e}{4\pi\epsilon_o}\frac{p}{L^3} \end{aligned}$$

So if the dipole is about a $0.01\,\mathrm{cm}$ away

$$\begin{aligned} F &= \frac{2\left(1.602\times10^{-19}\right)}{4\pi\left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)}\frac{6.2\times10^{-30}\,\mathrm{C\,m}}{(0.01\,\mathrm{cm})^3} \\ &= 1.7862\times10^{-26}\,\mathrm{N} \end{aligned}$$

But wait! we used the dipole as the environmental object and the single charge as the mover. So this is the force on the single charge! But by Newton's third law, the force on the dipole due to the electron must have the same magnitude and opposite direction so

$$F_{dipole} = -1.7862\times10^{-26}\,\mathrm{N}$$

We could do this problem the other way, thinking of the point charge as the environment and the dipole as the moving object. We know Coulomb's law for a point charge. So we use it to find the force on the individual parts of the dipole. We have to be careful because the minus charge is at a different r value than the positive charge.

$$\begin{aligned} -qE_- + qE_+ &= ma_x \\ -q\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{r_-^2}\right) + q\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{r_+^2}\right) &= ma_x \end{aligned}$$

or

$$\frac{Qq}{4\pi\epsilon_o}\left(\frac{1}{r_+^2}-\frac{1}{r_-^2}\right) = ma_x = F_{net}$$

this is the net force on a dipole due to the point charge.

The effective charge on one side of the water molecule is

$$\begin{aligned} q &= \frac{p}{a} = \frac{6.2\times10^{-30}\,\mathrm{C\,m}}{3.9\times10^{-12}\,\mathrm{m}} \\ &= 1.5897\times10^{-18}\,\mathrm{A\,s} \end{aligned}$$

(how can this be true?) so if the dipole is abut a $0.01\,\mathrm{cm}$ away then

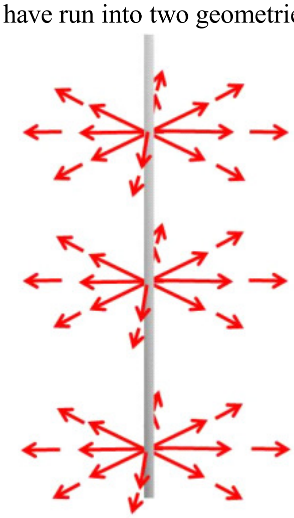
$$\begin{aligned} F_{net} &= \frac{\left(1.0\times10^{-19}\right)\left(1.5897\times10^{-18}\,\mathrm{A\,s}\right)}{4\pi\left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)} \\ &\quad\times\left(\frac{1}{\left(0.01\,\mathrm{cm}+\frac{3.9\times10^{-12}\,\mathrm{m}}{2}\right)^2}-\frac{1}{\left(0.01\,\mathrm{cm}-\frac{3.9\times10^{-12}\,\mathrm{m}}{2}\right)^2}\right) \\ &= -1.1150\times10^{-26}\,\mathrm{N} \end{aligned}$$

We expect the negative sign, both forces should be to the left. The answers are different, but within one order of magnitude. This is pretty good since for our dipole field we assumed that the distance from the dipole is very large and $0.01\,\mathrm{cm}$ is a somewhat shorter version of very large!

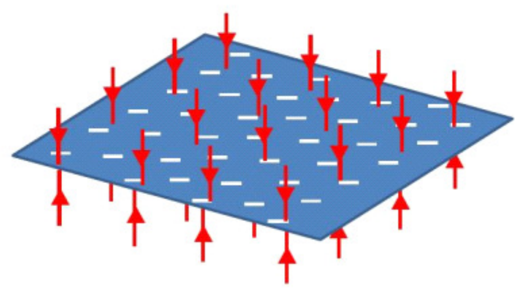
Symmetry

The symmetry of the uniform field figured strongly in the dipole problem. When the shape of the field changed, so did the resulting motion. This suggests that we could solve some problems just knowing the symmetry, or at least that symmetry might help us do simple predictions to help get a problem started. We need to be able to predict the field lines of a geometry to draw a picture to start solving a problem.

We have run into two geometries so far that have been helpful



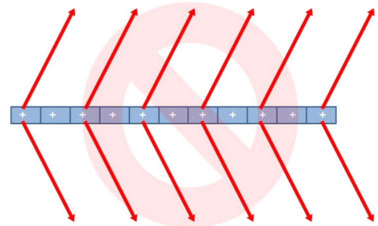
Infinite Line of Charge



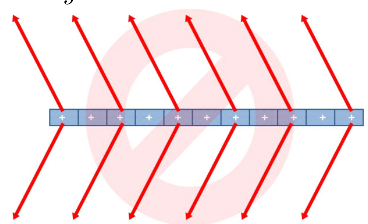
Semi-infinite Sheet of Charge

The infinite line of charge and the semi-infinite sheet of charge. We have found for the sheet that the field is constant everywhere. This is strongly symmetric. We could envision translating the sheet within the plane right or left. The field would look the same. We could envision reflecting the sheet so the left side is now the right side. That would also not change the field. We can say that the field of the sheet would be symmetric about translation within the plane of the sheet and symmetric on reflection.

Suppose we look at the sheet side-on. Suppose that we thought the field came off the sheet at an angle as shown.



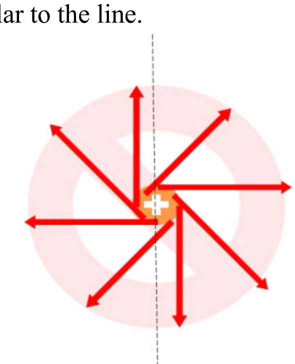
Notice that if we shift the sheet right or left, the field would still look the same, but if we reflected the sheet about the y-axis. Then we would have



But (and here is the important part) the shape of the charge distribution did not change on reflection. The sheet really looks just the same. It dose not make sense that we should change the shape of the field if the shape of the charge distribution did not change. So we can tell that this can't be the right field shape.

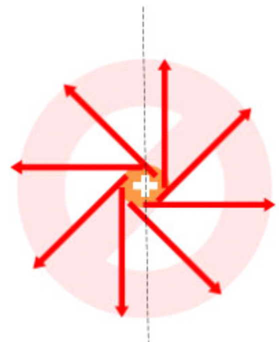
Question 223.25.5

We can do this with any symmetric distribution of charge. Think of the infinite line of charge. If we move it left or right the field definitely changes. So it is not symmetric about translation along, say, the x-axis. But if we move the wire along it's own axis, (for my coordinate system, along the y-axis) it should be symmetric because the charge distribution won't look different. We can guess from the last example that the field must come straight out perpendicular to the line of charge. It must be perpendicular, but what direction? Look at this end view. The field lines do come straight out, so this meets our criteria for being perpendicular to the line.



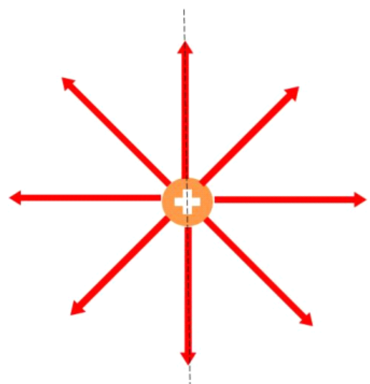
Line of Charge, End View

We could rotate the line about the axis of the line. Then the charge distribution would look just the same. The field would also look just the same on rotation. But if we reflect the charge distribution across the axis shown, the charge distribution looks just the same, but the field would change.



Line of Charge, Reflection

We can tell that this is not the right field. We can tell that the field should look more like this.

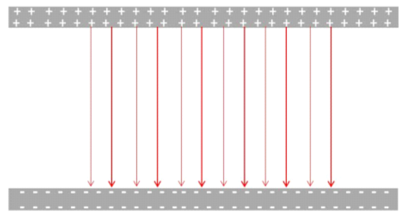


Line of Charge, End View

Combinations of symmetric charge distributions

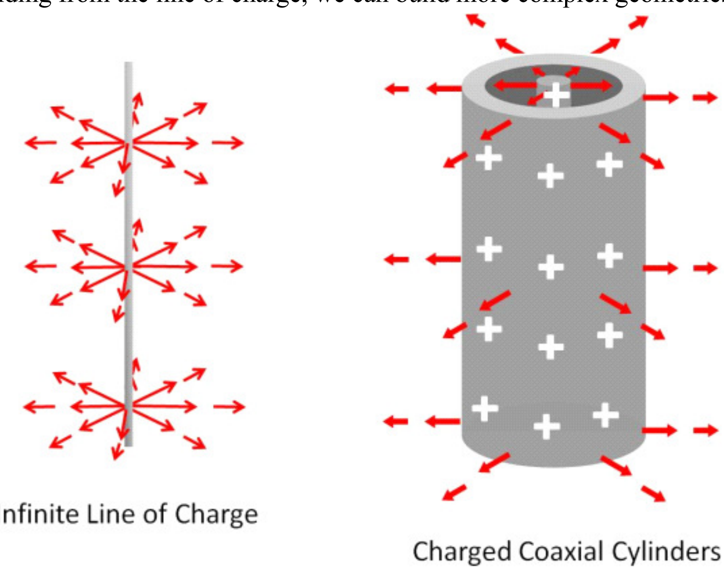
Question 223.25.6

We can combine sheets or lines of charge to build more complex systems. We did this to form a capacitor



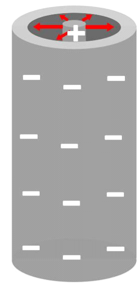
The field lines follow our symmetry guidelines. Because of the symmetry of the sheet of the field lines must be perpendicular to the sheets.

Again building from the line of charge, we can build more complex geometries



In the figure we have two positively charged concentric cylinders. The field is very reminiscent of a line charge field, and we can see that it must be using the same symmetry rules.

Of course the cylinders don't have to have the same charge.

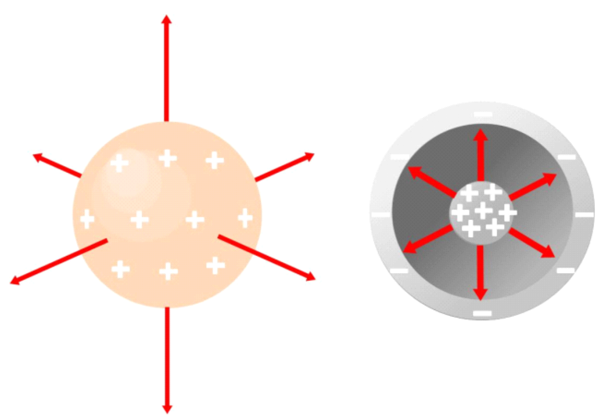


Oppositely Charged Coaxial Cylinders

If the interior cylinder is positively charged and the exterior cylinder is negatively charged, we have a situation much like the capacitor. Each cylinder has a field outside the system, but those fields cancel out if there are equal charges on each cylinder. This situation is similar to a coaxial cable, and we will revisit it later in the course.[19]

For the charge configurations we have drawn so far, we must keep in mind that they are infinite in at least one dimension. Finite configurations of charge in lines or sheets will have curved fields at the ends. The fields will be symmetric on reflection about their centers, but not on translation of any sort. Still, we will continue to use semi-infinite approximations in this class, and these constructs are good mental images under many circumstances.

Of course we can have a sphere. Spheres are very symmetrical, so we can guess using our symmetry ideas that the field from a charged sphere should be perpendicular to the surface of the sphere everywhere.



19 Indeed, this coaxial cables have a capacitance!

We can see that this is true for both the sphere and for concentric spheres or any configuration of charge that is spherical.

Basic Equations

$$\overrightarrow{\boldsymbol{\tau}} = \overrightarrow{\mathbf{p}} \times \overrightarrow{\mathbf{E}}$$