Chapter 26

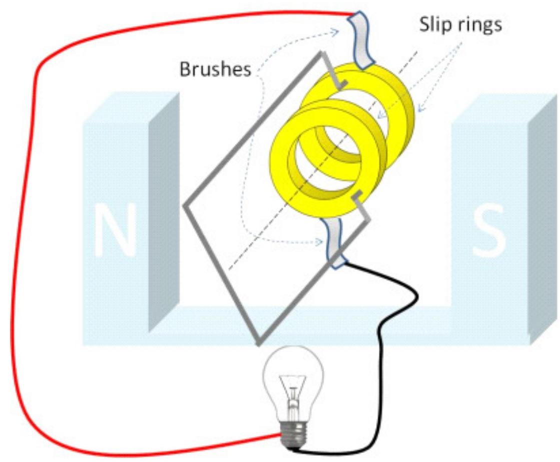
Induced Fields

Fundamental Concepts

- Changing the commutator for slip rings makes a motor into a Generators
- Using alternating current, we can build an inductive device that can change from one voltage to another. This device is called a transformer.
- A more general form of Faraday's law is $\int \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$

26.1 Generators

Whether you are just plugging in an appliance, or preparing for an emergency, you likely would think of a generator as a source of electrical energy. Our studies so far have strongly hinted on how we would build an electric generator. In this lecture, we will fill in the details.



We can learn a lot by studying this device as an example. The figure shows the important parts of the generator (and a light bulb, which is not an important part of a generator, but just represents some device that will use the electrical current we make). The generator has at least one magnet. In the figure, there is one with a north end on the left and a south end on the right. A generator also has a wire loop. Usually in real generators, there are thousands of turns of wire forming the loop. In our picture, there is just one. The wire loop is connected to two metal rings. The rings will turn as the loop turns. Metal contacts (brushes) that can slip along the rings, but maintain an electrical connection, are placed on the rings. So as the rings turn, current can still flow through the connected wires (to the light bulb in this case).

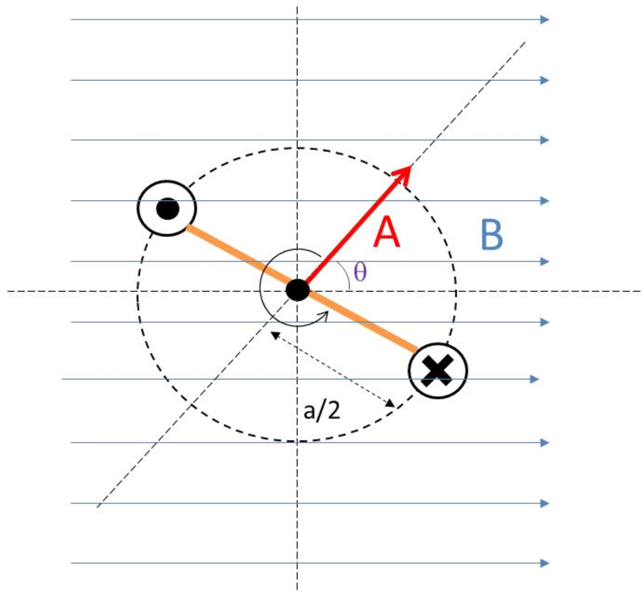
This should look familiar. This is the same basic setup as the motor, with a few exceptions. An important exception is that the commutator has been replaced by the set of rings. We will call these ring contacts *slip rings* because the wires can slip along them while still maintaining electrical contact because of the brushes. We have a current loop in a (nearly) uniform, constant field. If I look from the slip ring side of the loop, I have the same geometry we had before when we considered motors. This time I want to consider doing work to turn the loop, and find the induced emf in the loop. We start with Faraday's law

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{26.1}$$

since in our special case we only have one loop, this is just

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{26.2}$$

Here is a the view looking at the cross section of the loop facing toward the slip rings.



Let's consider the flux through the loop. The definition we have for flux is

$$\begin{aligned}\Phi_B &= \mathbf{B}\cdot\mathbf{A} \\ &= BA\cos\theta \\ &= BA_{proj}\end{aligned}$$

where θ is the angle between the loop area vector and the magnetic field direction.

I want to write the flux in terms of the lengths of the wire. When the loop is standing up straight along the y-direction the projected area is just the area

$$A = \ell a$$

Then the flux is

$$\Phi_B = B\ell a\cos\theta$$

Let's think about what this means. When the loop is standing up straight along the y-direction $\theta = 0^\circ$, and $\cos\theta = 1$ so

$$\Phi_B = B\ell a\cos\theta = B\ell a = \Phi_{\max}$$

and when we turn the loop a quarter turn we get the loop being in the x-direction so $\theta = 90^\circ$, and $\cos\theta = 0$ so

$$\Phi_B = B\ell a\cos\theta = 0 = \Phi_{\min}$$

If we turn the loop so it has $\theta = 180^\circ$ then

$$\Phi_B = B\ell a\cos(180^\circ) = -B\ell a = -\Phi_{\max}$$

and we see the flux has changed directions.

To find the emf generated, we need

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

We recognize that θ changes as the loop turns Since B and A are not changing.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -B\ell a \frac{d}{dt}(\cos\theta)$$

We remember from PH121 that we can use $\theta = \omega t$ where ω is the angular speed of the rotating loop. Then

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -B\ell a \frac{d}{dt}(\cos\omega t)$$

or simply

$$\mathcal{E} = B\ell a\omega \sin(\omega t)$$

Look at what we got! it is a sinusoidal emf. This will make a sinusoidal current!

$$\begin{aligned} I &= \frac{\mathcal{E}}{R} \\ &= \frac{B\ell a\omega \sin(\omega t)}{R} \end{aligned}$$

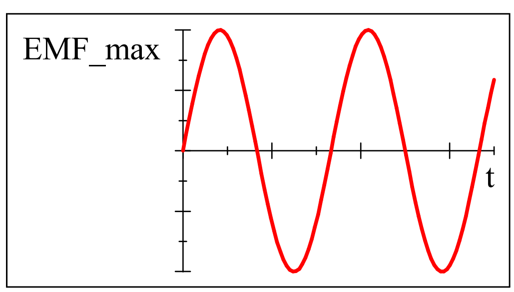
for a circuit. Our emf looks like

$$\mathcal{E} = \mathcal{E}_{\max} \sin(\omega t) \tag{26.3}$$

where

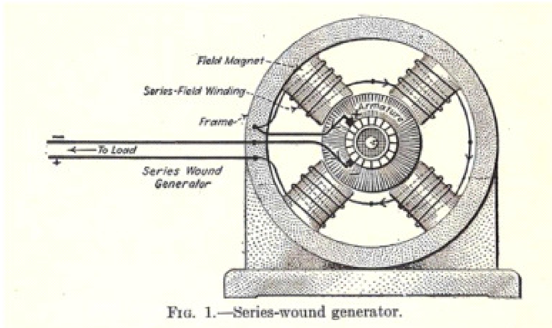
$$\mathcal{E}_{\max} = B\ell a\omega \tag{26.4}$$

Here is a plot of the emf function



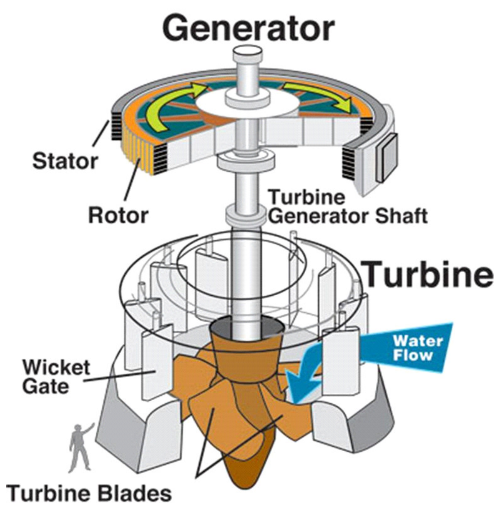
Of course this sinusoidal emf will create what we call an *alternating current*. This is how the current in the outlets in your house is generated.

Of course, our generator only has one coil. Actual generators have multiple coils.



Double Armature Generator (Public Domain Image)

and we need a source of work to turn the generator. A water turbine is an example,

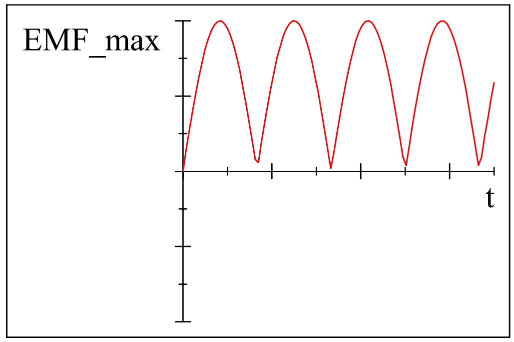


Water Turbine driven Generator (Public Domain Image courtesy U.S. Army Corps of Engineers)

or for emergencies, you might have a gasoline powered generator, or in a nuclear reactor you might have a steam driven generator.

26.1.1 DC current from a generator

We can also make a non-alternating current with a generator, but we have to get tricky to do it. We use the same idea we used to make a motor. We cut slots in the slip rings, so the current will switch directions every half turn. We get a kind of poor quality current from this because the emf still varies a lot.



Clever engineers design generators for non-alternating or *direct current* generators by overlapping several current loops at different angles. Each loop has it's own cut slip rings. The combined currents smooth out the ripples we see in the previous figure. For semiconductor devices, special circuits are used to make the current very smooth.

26.1.2 Back emf

Now that we know how a generator works we can see that a motor is really just a DC generator run backwards. I want to mention that when we talk about motors, we have to realize that as we send current into the motor coils, there will be an induced emf that will try to maintain the existing flux as the motor's loops turn. This emf will be in the opposite direction of the applied current! So it reduces the amount of work the motor can do. This is like the resistive force we encountered when we pulled a loop from a magnetic field last lecture. This resistive force is called the *back emf* and must be accounted for in motor design.

26.1.3 rms voltage

We can realize that we have a slight problem in talking about alternating voltages. The voltage constantly changes. How do we describe what the voltage is?

We could give the maximum voltage–the amplitude of our $\mathcal{E}(t)$ curve. But the voltage is at the maximum only a small percentage of the time. We can't take the average. That is zero. And zero really doesn't describe our voltage well!.

The average doesn't work because our generators make the emf go negative. We could fix this by squaring the emf before we average it

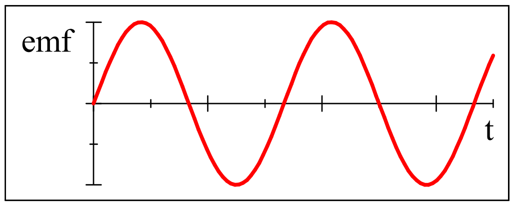
$$\begin{aligned} \overline{\mathcal{E}(t)} &= 0 \\ \overline{\mathcal{E}^2(t)} &\neq 0 \end{aligned}$$

and this could work. But then we have the average voltage squared, and we really want just the voltage. No problem, let's take a square root.

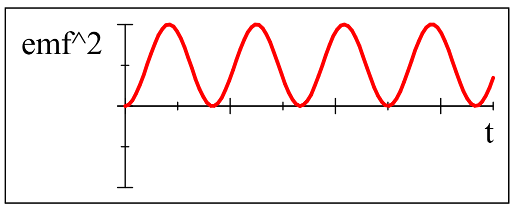
$$\sqrt{\overline{\mathcal{E}^2\left(t\right)}}$$

This has units of volts, is like an average of the emf, but doesn't cancel out because $\mathcal{E}\left(t\right)$ goes negative. Here is the process graphically.

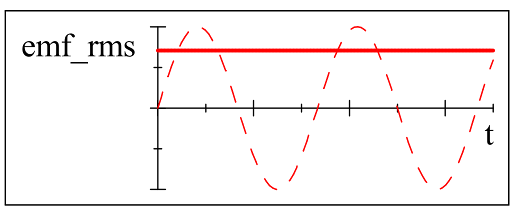
first $\mathcal{E}\left(t\right)$



now $\mathcal{E}^2\left(t\right)$



and finally $\sqrt{\overline{\mathcal{E}^2\left(t\right)}}$



What we did is take the square Root of the Mean of the Square of the emf. We can call this process the root-mean-square process or rms for short. This is more useful than the mean voltage–which is zero for alternating voltages. It is a better estimate of the overall potential than the peak voltage value. So although it is still not ideal, we often use rms voltages to describe sources of alternating current. Many multimeters give rms values as their default for alternating voltages.

We can come up with a convenient way to find the rms emf. Consider that our alternating emf is given by

$$\mathcal{E} = \mathcal{E}_{\max}\sin\left(\omega t\right)$$

but we squared this

$$\mathcal{E}^2 = \mathcal{E}_{\max}^2\sin^2\left(\omega t\right)$$

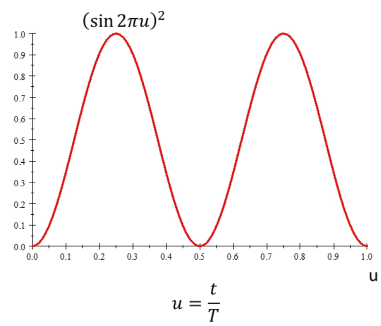
and then took an average. Suppose we average $\sin^2(\omega t)$ over a long time so that ωt gets large. And we want the case where $\theta_{\max}$ is large enough that our alternating voltage make many cycles. We would have

$$\begin{aligned}\overline{\mathcal{E}^2} &= \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathcal{E}_{\max}^2 \sin^2(\omega t)\, dt \\ &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\int_{t_i}^{t_f} \sin^2(\omega t)\, dt\end{aligned}$$

but what is the integral of $\sin^2(\omega t)$ over a period? It turns out that

$$\int_T \sin^2(\theta) = \frac{1}{2} \tag{26.5}$$

To convince yourself of this, think that $\sin^2(\omega t)$ has a maximum value of 1 and a minimum of 0. Looking at the graph of $\sin^2\left(2\pi\frac{t}{T}\right)$



Note that if we cut off the peaks and put them upside down in the valleys we would get the area under the curve to be 1/2. And the integral is the area under the curve! So it should be convincing that the average value over a period is just 1/2.

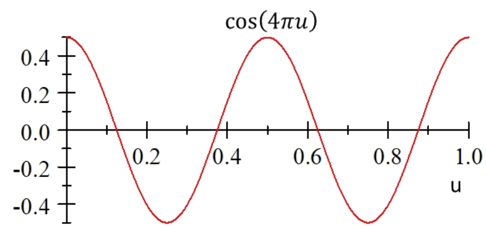
We can see this is true in our case by using a trig identity

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$$

Then

$$\begin{aligned}\overline{\mathcal{E}^2} &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\int_{t_i}^{t_f} \sin^2(\omega t)\, dt \\ &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\int_{t_i}^{t_f} \left(\frac{1}{2}-\frac{1}{2}\cos 2t\omega\right) dt \\ &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\left(\int_{t_i}^{t_f} \frac{1}{2}dt - \int_{t_i}^{t_f} \frac{1}{2}\cos 2t\omega\, dt\right) \\ &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\left(\frac{1}{2}\Delta t - \int_{t_i}^{t_f} \frac{1}{2}\cos 2t\omega\, dt\right)\end{aligned}$$

The second integral over $\cos(2\omega t)$ will be zero.



because it goes negative as well as positive, so the area under the curve is zero.

So we have

$$\begin{aligned}\overline{\mathcal{E}^2} &= \frac{\mathcal{E}_{\max}^2}{\Delta t}\frac{1}{2}\Delta t \\ &= \frac{1}{2}\mathcal{E}_{\max}^2\end{aligned}$$

Now we need to take a square root to finish the rms process.

$$\begin{aligned}\mathcal{E}_{rms} &= \sqrt{\overline{\mathcal{E}^2}} \\ &= \sqrt{\frac{1}{2}\mathcal{E}_{\max}^2} \\ &= \frac{1}{\sqrt{2}}\mathcal{E}_{\max}\end{aligned}$$

So the rms emf can be found from the maximum emf by dividing by the square root of 2.

This gives a pretty good idea of the nature of the voltage for alternating current. For example an *rms* emf value of 120 V would have a peak emf of

$$\begin{aligned}\mathcal{E}_{\max} &= \sqrt{2}\mathcal{E}_{rms} \\ &= \sqrt{2}\,(120\,\mathrm{V}) \\ &= 169.71\,\mathrm{V}\end{aligned}$$

The *rms* value isn't the peak, it isn't the average (0) but it gives us a measure of how much voltage (and risk) we have.

We could find an *rms* current by using Ohm's law

$$\Delta V = IR$$

This would be true for

$$\mathcal{E}_{\max} = I_{\max}R$$

then we could find

$$I_{\max} = \frac{\mathcal{E}_{\max}}{R}$$

then if we divide by $\sqrt{2}$ we have

$$\frac{I_{\max}}{\sqrt{2}} = \frac{\mathcal{E}_{\max}}{\sqrt{2}R}$$

The right hand side is just

$$\frac{I_{\max}}{\sqrt{2}} = \frac{\mathcal{E}_{rms}}{R}$$

so let's take

$$I_{rms} = \frac{I_{\max}}{\sqrt{2}}$$

then Ohm's law for alternating currents becomes

$$I_{rms} = \frac{\mathcal{E}_{rms}}{R}$$

or

$$\mathcal{E}_{rms} = I_{rms}R$$

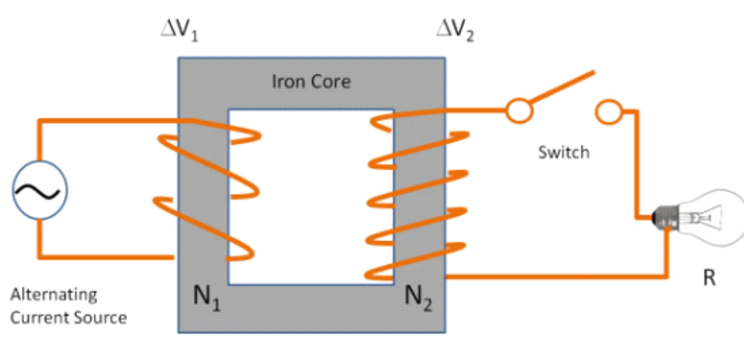
26.2 Transformers

The power comes into our houses at about $120V$. Your iPhone probably requires 3 V to 5 V. How do we get the voltage we want out of what the power company delivers? You know the answer is to plug in your phone using a special adaptor. Lets see how it works.

Let's consider Faraday's law again. We know that

$$\mathcal{E} = \Delta V(t) = -N\frac{\Delta\Phi}{\Delta t}$$

Suppose we use Faraday's idea and hook two coils up next to each other.



One side we will hook to an alternating emf. We will call this side coil 1. The other side we will hook a second coil with some resistive load like a light bulb. We will call this coil 2.The iron core keeps the magnetic field inside, so the flux through coil 1 ends up going through coil 2. (think of all the little domains in the iron lining up along the field lines, and enhancing the field lines with their own induced fields).

The alternating potential from the source will create a change in flux in coil 1.

$$\mathcal{E}_1(t) = -N_1 \frac{\Delta \Phi_1}{\Delta t}$$

If little flux is lost in the iron, then we will retrieve most of the flux in coil 2 and an emf will be induced in the resister (light bulb in our case).

$$\mathcal{E}_2(t) = -N_2 \frac{\Delta \Phi_2}{\Delta t}$$

we just convinced ourselves that

$$\frac{\Delta \Phi_1}{\Delta t} \approx \frac{\Delta \Phi_2}{\Delta t}$$

so we can solve each equation for the change in flux term, and set them equal.

$$\begin{aligned} \frac{\mathcal{E}_1(t)}{N_1} &= -\frac{\Delta \Phi_1}{\Delta t} \\ \frac{\mathcal{E}_2(t)}{N_2} &= -\frac{\Delta \Phi_2}{\Delta t} \end{aligned}$$

so we have

$$\frac{\mathcal{E}_1(t)}{N_1} = \frac{\mathcal{E}_2(t)}{N_2} \tag{26.6}$$

If we solve for $\mathcal{E}_2(t)$ we can find the emf in coil 2.

$$\frac{N_2}{N_1}\mathcal{E}_1(t) = \mathcal{E}_2(t) \tag{26.7}$$

You have probably already guessed how we make $\mathcal{E}_2$ to be some emf amount we want. We take, say, our wall current that has a rms value of $\mathcal{E}_1 = 120\,\mathrm{V}$. We pass it through this device we have built. We design the device so that $\frac{N_2}{N_1}\mathcal{E}_1$ gives just the potential that we want for $\mathcal{E}_2$. If we want a lower emf, say $12\,\mathrm{V}$, then we make $\frac{N_2}{N_1} = 0.1$ so

$$\frac{N_2}{N_1}\mathcal{E}_1 = 0.1\,(120\,\mathrm{V}) = 12\,\mathrm{V} \tag{26.8}$$

This is part of what the wall adaptor does. Usually wall adapters also have some circuitry to make the alternating current into direct current.

Note that there is a cost to doing this. The power must be the same on both sides (or a little less on side 2). So

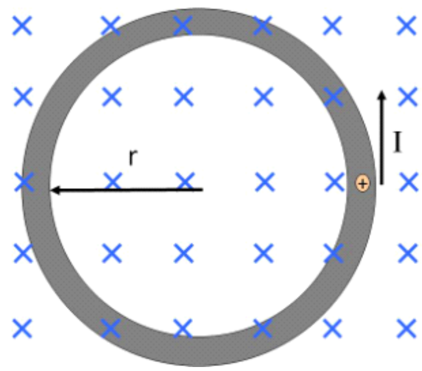
$$\mathcal{P}_{av} = I_{1,rms}\mathcal{E}_{1,rms} = I_{2,rms}\mathcal{E}_{2,rms}$$

We can change the emf, but it will effect our current.

This device is called a transformer. Real transformers do lose power. Some loss is due to the fact that not all the B-field from coil 1 makes it inside coil 2. But real transformers are not too bad with efficiencies ranging from 90% to 99%.

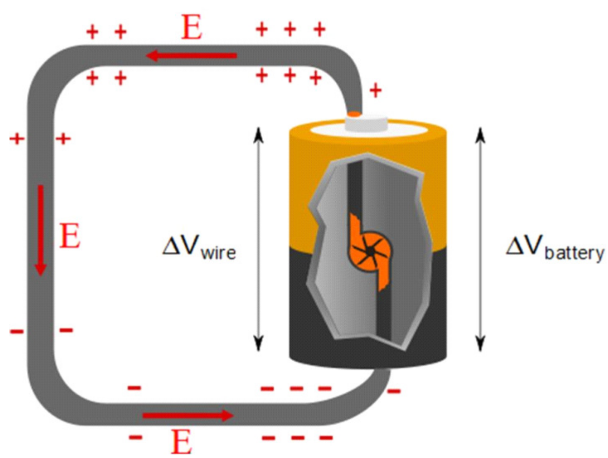
26.3 Induced Electric Fields

Consider again a magnetic field and a moving charge. If the field changes, the flux changes. Say, for example, that the field is increasing in strength.

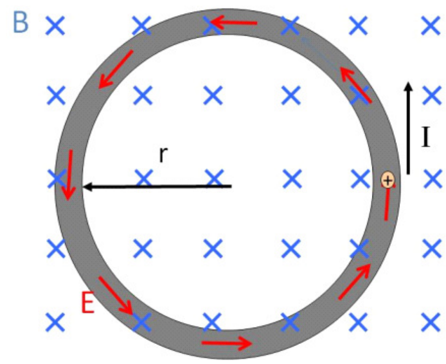


The charge will move in a circle within the wire. We now understand that this is because we have induced an emf. But think again about a battery.

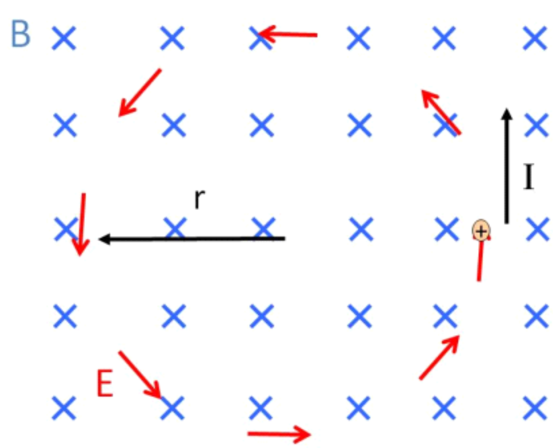
The battery makes an electric field inside a wire. Recall this figure



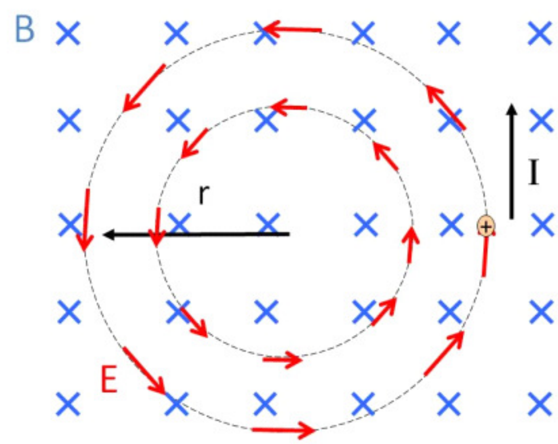
We must conclude that if we create an emf, we must have created an electric field.



This is really interesting. We now have a hint at how wireless chargers might work (we will return to this later). But now let's ask ourselves, do we need the wire there for this electric field to happen? Of course, the force on the charge is the same if there is no wire, so the E-field must be there whether or not there is a wire.



In fact, the electric field is there in every place the magnetic field exists so long as the magnetic field continues to increase.



This is quite a profound statement. We have said that a changing magnetic field *creates* an electric field. Before, only charges could create electric fields, but in this case, the magnetic field is creating the electric field. Of course, we know that somewhere there are moving charges that are making the magnetic field, so it is not totally surprising that the fields would be related.

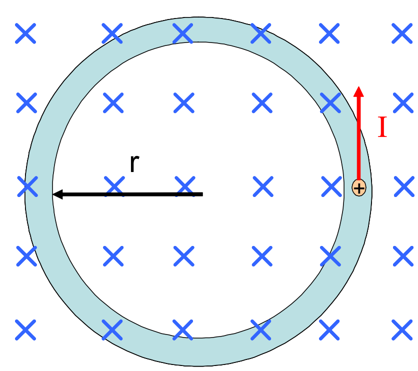
This electric field is just like a field produced by charges in that it exerts a force

$$F = q_o E$$

on a charge q_o. But the electric field source is now very different.

26.4 Relationship between induced fields

It would be nice to have a relationship between the changing B-field and the E-field that is created. It would be good to obtain the most general relationship we can that relates the electric field to the magnetic field. By understanding this relationship, we can hope to gain insight into how to build things, and into how the universe works. Let's start with a thought experiment.



Suppose we have a uniform but time varying magnetic field into the paper. In this field, we have a conducting ring. If the field strength is increasing, then the charges in the conducting loop shown will feel an induced emf, and they will form a current that is tangent to the ring.

Let's find the work required to move a charge once around the loop. The amount of potential energy difference is equal to the work done, so

$$|\Delta U| = |w|$$

but in terms of the electric potential this is

$$\Delta U = q\Delta V = q\mathcal{E}$$

so

$$|w| = |q\mathcal{E}|$$

Now let's do this another way. Let's use

$$w = \int \mathbf{F} \cdot d\mathbf{s}$$

The force making the current move is due to the induced potential difference. This is just

$$F = qE$$

which will not change as we go around the loop. The path will be along the loop, so

$$w = \int_{loop} F ds$$

and since the E-field is uniform in space at any given time as we travel around the loop,

$$w = F \int_{loop} ds = qE2\pi r$$

So we have two expressions for the work. Let's set them equal to each other

$$q\mathcal{E} = qE2\pi r$$

The field is then

$$\frac{\mathcal{E}}{2\pi r} = E \tag{26.9}$$

but

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

so

$$\begin{aligned} E &= \frac{-N}{2\pi r}\frac{d\Phi_B}{dt} \\ &= \frac{-1}{2\pi r}\frac{d\Phi_B}{dt} \end{aligned}$$

So if we know how our B-field varies in time, we can find the E-field. Let's rewrite this one more time

$$2\pi r E = -\frac{d\Phi_B}{dt}$$

Since the E-field is constant as we go around the loop, we can recognize the LHS as

$$2\pi r E = \int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$$

which should be little surprise, since we found

$$\Delta V = \int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$$

to be our basic definition of the electric potential. So

$$\int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{26.10}$$

This is a more general form of Faraday's law of induction.

This electric field is fundamentally different than the E-fields we studied before. It is not a static field. If it were, then $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$ would be zero around a ring of current. Think of conservation of energy. Around a closed loop $\Delta V = 0$ normally. Then

$$\Delta V = \oint_{loop} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = \mathbf{0} \qquad \text{no magnetic field}$$

But since $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} \neq 0$ for our induced E-field, we must recognize that this field is different from those made by static charges. We call this field that does not return the charge to the same energy state on traversing the loop a *nonconservative field*. It is still just an electric field, but we are *gaining energy* from the magnetic field, so ΔV around the loop should not be not zero because of the input energy from the magnetic field.

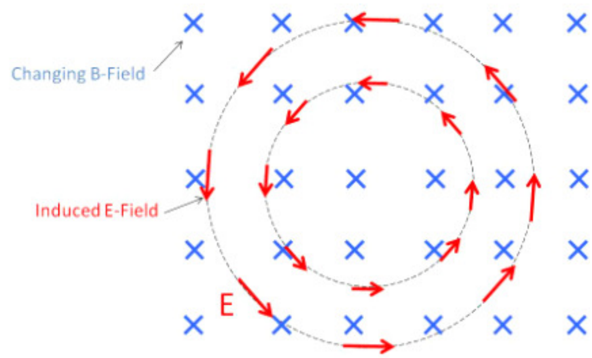
The equation

$$\oint_{loop} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{26.11}$$

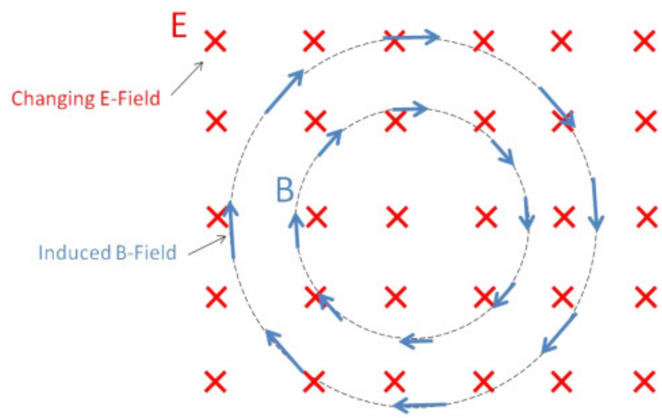
is the most general form of Faraday's equation, but it is hard to use in calculation for normal circuits where there is no magnetic field or where the fields are weak. So we won't use it as we design normal circuits. We will use the idea of inductance instead (next lecture). But Faraday's law plays a large part in the electromagnetic theory of optics (PH375). We will just get a taste of this here.

26.5 Electromagnetic waves

Let's return to the idea that a changing magnetic field makes an electric field.



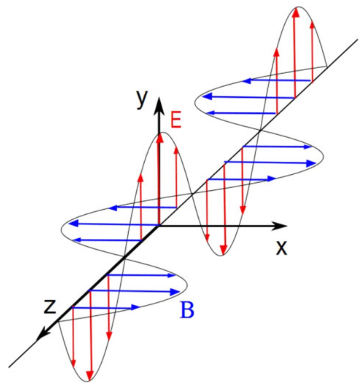
But what about a changing electric field?



For the electric and magnetic field equations to be symmetric, the changing electric field must create a magnetic field. There is no requirement that the universe display such symmetry, but we have found that it usually does. Indeed, a changing electric field creates a magnetic field.

This foreshadows our final study of light. It turns out that light is a wave in the *electromagnetic* field. What this means is that light is a wave in both the electric *and* magnetic fields.

Maxwell first predicted that such a wave could exist. The electric field of the wave changes in time like a sinusoid. But this change will produce a magnetic field that will also change in time. This changing magnetic field recreates the electric field–which recreates the magnetic field, etc. Thus the electromagnetic wave is *self-sustaining*. It can break off from the charges that create it and keep going forever because the electric field and magnetic field of the wave create each other. You often see the electromagnetic wave drawn like this:



Where you can see the electric and magnetic fields being created and recreated to make the wave self sustaining.

This is a direct result of Maxwell's study of electromagnetic field theory. Our more complete version of Faraday's law is one of the fundamental equations describing electromagnetic waves known as *Maxwell's Equations.*

$$\int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

You might guess that the symmetry we have observed would give another similar equation relating the magnetic field and the electric flux.

$$\int \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = +\frac{d\Phi_E}{dt}$$

and we will find that this is true! But we have yet to show that is so. Note that $\int \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}}$ shows up in Ampere's law,

$$\int \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = \boldsymbol{\mu}_o I$$

so this last equation is not complete, but we are guessing that there is also the possibility of an induced magnetic field from a changing electric field, so we can predict that we need to modify Ampere's law to be

$$\int \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = \boldsymbol{\mu}_o I + \frac{d\Phi_E}{dt}$$

but we will have to show this later.

In the next lecture, we will take a break from this deep theoretical discussion, and learn how to use induction to make useful circuit devices.

Basic Equations

$$\mathcal{E} = B\ell a\omega \sin(\omega t)$$

$$\mathcal{E}_{rms} = \sqrt{\overline{\mathcal{E}^2(t)}}$$

$$I_{rms} = \frac{I_{\max}}{\sqrt{2}}$$

$$\mathcal{E}_{\max} = \sqrt{2}\mathcal{E}_{rms}$$

$$\frac{N_2}{N_1}\mathcal{E}_1(t) = \mathcal{E}_2(t) \tag{26.12}$$

$$\int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$