

# 22 Electric Fields of Standard Charge Configurations Part I

## Fundamental Concepts

- Adding of vector fields for point charges
- Standard configurations of charge

## Standard Charge Configurations

Actual engineering projects or experimental designs require detailed calculations of fields using computers. These field simulations use powerful numerical techniques that are beyond this sophomore class. But we can gain some great insight by using some basic models of simple charged objects. We will often look at the following models:

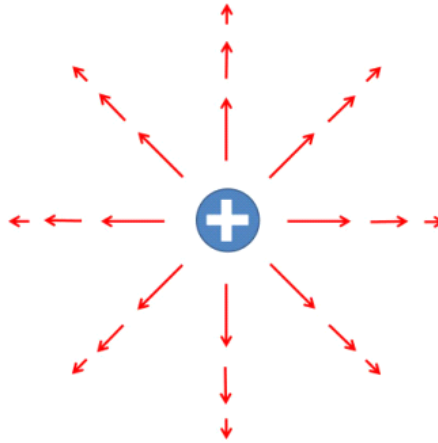
Standard Configurations of Charge
Point charge
Several point charges
Line of Charge
Semi-infinite sheet of charge
Charged sphere
Charged spherical shell
Ring of Charge

## Point Charges

We have already met one of these standard configurations, the point charge

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2} \hat{\mathbf{r}}$$

The field of the point charge is represented below



This picture requires a little explanation. The arrows are larger nearer the charge to show that the field is stronger. But note that each arrow is the magnitude and direction of the charge at one point. We really need a three dimensional picture to describe this, and even then the fact that the arrows have length can be misleading. The long arrows cover up other points, that should also have arrows. We can only draw the field at a few points, and at those points the field has both magnitude and direction. But we must remember that there is really a field magnitude and direction at every point.

The extension is a group of point charges

$$\vec{E} = \sum_i \vec{E}_i$$

where we recognize that we are summing vectors. Let's take a look at a few combinations of charges and find their fields

## Two charges

Let's go back to our idea of an environmental charge,  $Q_E$ , and a mover charge,  $q_m$ . The mover charge is considered to be small enough that its effect on  $Q_E$  is negligible. So the field due to the large charge is unaffected by this small charge.

Of course, the total field is a superposition of both fields. But recall that the mover's self-field can't move the mover. So we don't draw the field due to  $q_m$ . We can envision an environmental field that is just due to the environmental charge,  $Q_E$ , as if there are no other charges anywhere in the whole universe. Of course this is not the case, but this is how we think of the field *due to* charge  $Q_E$ .

Question 223.22.1

Question 223.22.2

We can identify that a charge  $q_m$  placed in this field due to  $Q_E$  will feel a force

$$\begin{aligned}\vec{\mathbf{F}}_e &= q_m \vec{\mathbf{E}} \\ &= \frac{1}{4\pi\epsilon_o} \frac{Q_E q_m}{r^2} \hat{\mathbf{r}}\end{aligned}$$

due to the field

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \frac{Q_E}{r^2} \hat{\mathbf{r}}$$

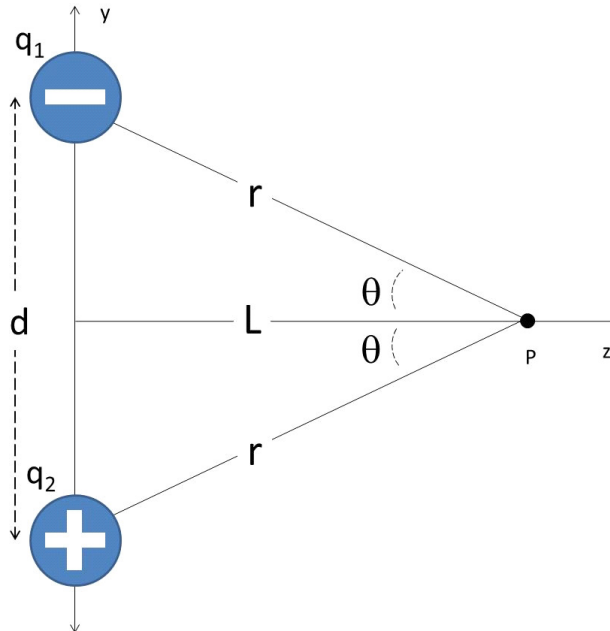
where this field is just due to  $Q_E$  and does not contain the contribution from  $q_m$ . So the charge  $q_m$  only feels a force due to the field created by charge  $Q_E$ . A third charge,  $q_{new}$  brought close to the other two would feel both  $\vec{\mathbf{E}}_Q$  and  $\vec{\mathbf{E}}_q$ . Then both  $Q_E$  and our original  $q_m$  would be environmental charges and the new charge  $q_{new}$  would be the mover. At this point, we would probably relabel  $Q_E$  and  $q_m$  as  $Q_1$  and  $Q_2$  and relabel  $q_{new}$  as  $q_m$  so we could tell that the original two charges are now the environment and the new charge is the mover.

## Vector nature of the field

Question 223.22.3

Question 223.22.4

Remember that the field is a force per unit charge. Forces add as vectors, so we should expect fields to add as vectors too. Let's do a problem.



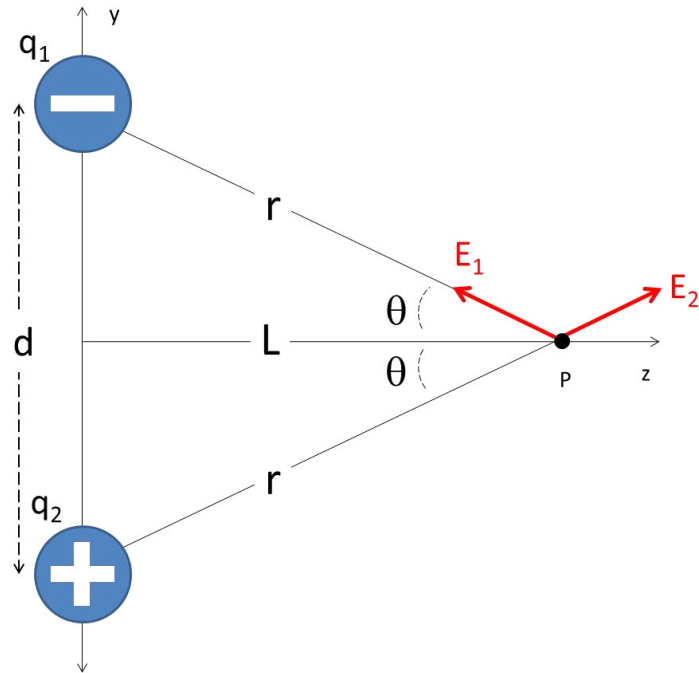
Two charges are separated by a distance  $d$ . What is the field a distance  $L$  from the

center of the two charges?

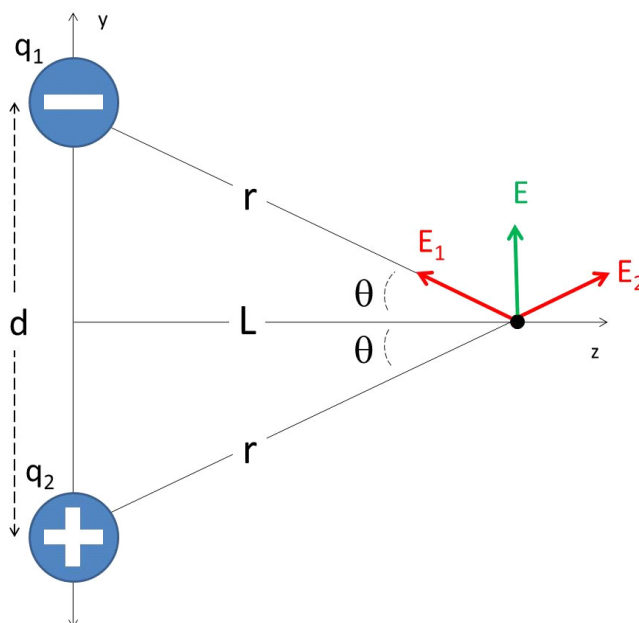
We should recognize this as our old friend, the dipole.

Note that both of these charges are environmental charges. We are asked in this problem to find the environment, the field. We don't really have a mover charge. But we could pretend we do have a mover,  $q_o$  at point  $P$  where we want to know the environment if it helps us picture the situation. But really we are calculating what the environment around the two charges will be.

We start by drawing the situation. I chose not to draw field lines. Instead I drew the field vectors at the point,  $P$ , where we want the field. The field lines would tell me about the whole environment everywhere, and that might be useful. But this problem only wants to know the field at one point,  $P$ . So it was less work to draw the field using vectors at our one point.



Note that I need a vector for each of the environmental charges. Each contributes to the environment. The contribution to the field due to environmental charge  $q_1$  is labeled  $E_1$  and likewise the contribution to the field from environmental charge  $q_2$  is labeled  $E_2$ .



The net environment is the superposition of the fields due to each of the environmental charges.

$$\vec{E}_{net} = \vec{E}_1 + \vec{E}_2$$

From the figure, we see that if we had a small mover charge,  $q_o$  on the axis a distance at point  $p$  then we would get two forces, one from each of the environmental charges  $q_1$  and  $q_2$ . We can use Newton's second law to find the net force on our imaginary  $q_o$ .

$$F_{net_z} = ma_z = -F_1 \cos \theta + F_2 \cos \theta$$

$$F_{net_y} = ma_y = F_1 \sin \theta + F_2 \sin \theta$$

we can see that the distance from each charge to point  $P$  is

$$r = \sqrt{\frac{d^2}{4} + L^2}$$

so

$$\sin \theta = \frac{d}{2\sqrt{\frac{d^2}{4} + L^2}}$$

we also know from Coulomb's law that

$$F_1 = F_2 = \frac{1}{4\pi\epsilon_o} \frac{qq_o}{r^2}$$

but we want the field, so we need to divide all of this by  $q_o$

$$E_1 = E_2 = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2}$$

Our Newton's second law becomes an equation for the components of the combined

electric field.

$$\begin{aligned}\frac{F_{net_z}}{q_o} &= -\frac{F_1}{q_o} \cos \theta + \frac{F_2}{q_o} \cos \theta \\ \frac{F_{net_y}}{q_o} &= \frac{F_1}{q_o} \sin \theta + \frac{F_2}{q_o} \sin \theta\end{aligned}$$

or just

$$\begin{aligned}E_{net_z} &= -E_1 \cos \theta + E_2 \cos \theta \\ E_{net_y} &= E_1 \sin \theta + E_2 \sin \theta\end{aligned}$$

We can see from the figure that in the  $x$ -direction we will have no net field,

$$E_z = -E_1 \cos(\theta) + E_1 \cos \theta = 0$$

But in the  $y$ -direction we have

$$\begin{aligned}E_y &= E_1 \sin \theta + E_2 \sin \theta \\ &= 2E_1 \sin \theta \\ &= \frac{2}{4\pi\epsilon_o} \frac{q}{r^2} \sin \theta\end{aligned}$$

and since we found that

$$\sin \theta = \frac{d}{2\sqrt{\frac{d^2}{4} + L^2}}$$

we can write our field as

$$\begin{aligned}E_y &= \frac{2}{4\pi\epsilon_o} \frac{q}{\frac{d^2}{4} + L^2} \frac{d}{2\sqrt{\frac{d^2}{4} + L^2}} \\ &= \frac{1}{4\pi\epsilon_o} \frac{qd}{\left(\frac{d^2}{4} + L^2\right)^{\frac{3}{2}}}\end{aligned}$$

This is our total field at the distance  $L$  away on the axis. This is the environment that a mover charge could move through.

Note that we pretended that we had a mover,  $q_o$ , but in finding the field the  $q_o$  canceled out, so indeed we are left with just the environment in our calculation, we just have the field.

Now suppose our mover charge is very far away. That is, suppose we make  $L$  very large. So large that  $L \gg d$  then

$$\lim_{L \gg d} \frac{1}{\left(\frac{d^2}{4} + L^2\right)^{\frac{3}{2}}} = \frac{1}{L^3}$$

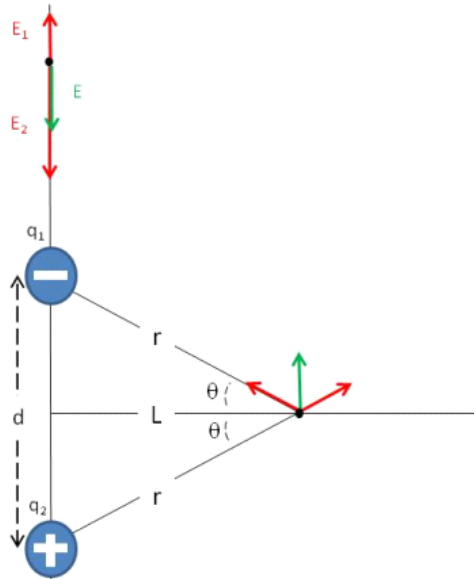
Then our field becomes

$$\begin{aligned}E &= E_y = \frac{2}{4\pi\epsilon_o} \frac{qd}{\left(\frac{d^2}{4} + L^2\right)^{\frac{3}{2}}} \\ &= \frac{1}{4\pi\epsilon_o} \frac{qd}{L^3}\end{aligned}$$

Since many charged particles are small, like atoms or molecules, this limit is often useful.

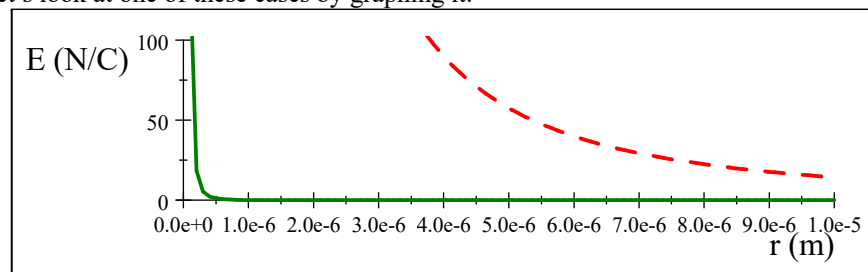
Now suppose we repeat the calculation, but this time we chose a point that is  $L$  away, but that is on the  $y$ -axis above the charges, we would find

$$E = E_y = \frac{2}{4\pi\epsilon_0} \frac{qd}{L^3}$$



The result is similar, but the field is a little stronger in this direction.

Let's look at one of these cases by graphing it.

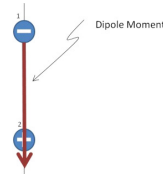


We can see that the dipole field (solid green line) falls off much faster than a point charge field (dashed red line). This makes sense because the farther away we get, the more it looks like the two charges are right next to each other, and since they are opposite in sign, they are essentially neutral when viewed together from far away. We can see why atoms don't exhibit a significant charge forces at normal distances.

This arrangement of charges we already know as a dipole. We are treating the two charges as a unit making the environment that other charges might move in. Since we are treating the two charges as one unit, it is customary to define a quantity

$$p = qd$$

and to make this a vector by defining the direction of  $p$  to be from the negative to the positive charge along the axis.



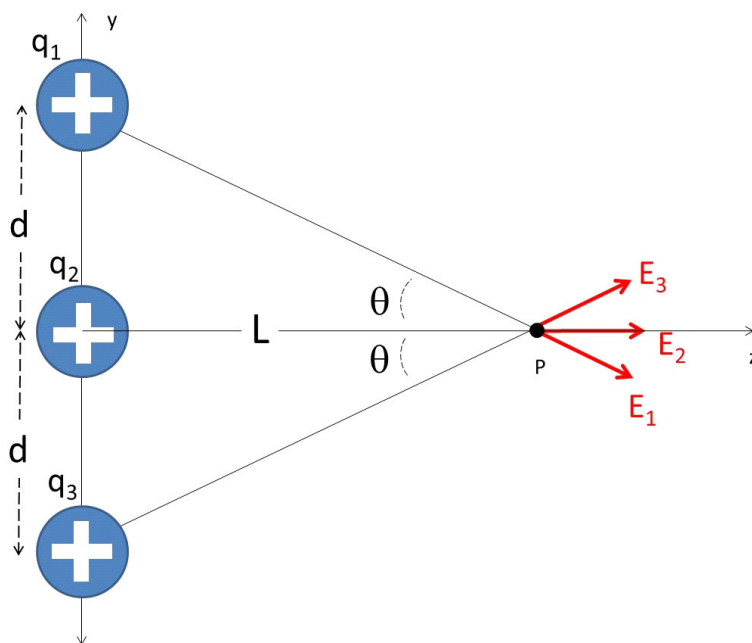
Then we can write the dipole field as

$$\vec{\mathbf{E}}_y = \frac{2}{4\pi\epsilon_0} \frac{\vec{\mathbf{p}}}{L^3}$$

We could also treat this dipole as a complicated mover charge in some other environmental field!. Then this quantity  $\vec{\mathbf{p}}$  will help us understand how a dipole will move when placed in an environmental electric field. For example, we know that water molecules are dipoles. A microwave oven creates a strong environmental electric field that makes the water molecules rotate. When we studied rotational motion we found a mass-like term that helped us to know how difficult something was to make rotate. That was the moment of inertia. This dipole term,  $\vec{\mathbf{p}}$ , will tell us how likely a dipole is to spin, so we will call  $\vec{\mathbf{p}}$  the *dipole moment*.

### Three charges

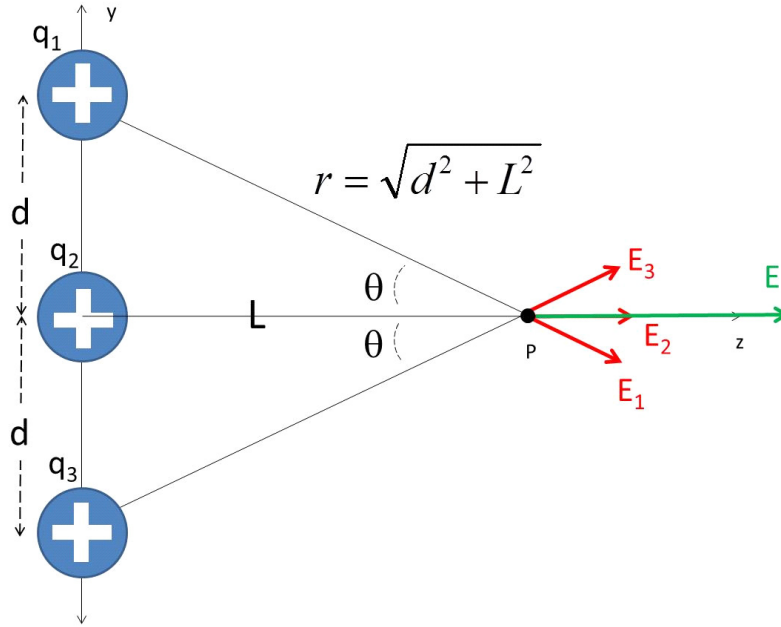




Question 223.22.5

We are working our way toward many charges that will require using integration to sum up the contributions to the field. But let's make this transition slowly. Next let's add just one more environmental charge, for a total of three.

Let's just start with the fields this time. From our picture, we expect in this case to have only  $z$ -components. Since all the charges are the same sign,



then

$$E_{net_z} = E_1 \cos(-\theta) + E_2 + E_3 \cos(\theta)$$

We can guess from symmetry that

$$E_1 = E_3 = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2}$$

But this time, since we have redefined  $d$ , the distance from  $q_1$  and  $q_3$  to the point  $P$  where we want to know the field is

$$r = \sqrt{d^2 + L^2}$$

so

$$E_1 = E_3 = \frac{1}{4\pi\epsilon_o} \frac{q}{d^2 + L^2}$$

and

$$E_2 = \frac{1}{4\pi\epsilon_o} \frac{q}{L^2}$$

and observing the triangles formed and remembering our trigonometry, we have

$$\cos \theta = \frac{L}{\sqrt{d^2 + L^2}}$$

so

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_o} \frac{q}{d^2 + L^2} \frac{L}{\sqrt{d^2 + L^2}} \\ &\quad + \frac{1}{4\pi\epsilon_o} \frac{q}{L^2} \\ &\quad + \frac{1}{4\pi\epsilon_o} \frac{q}{d^2 + L^2} \frac{L}{\sqrt{d^2 + L^2}} \end{aligned}$$

or

$$E_z = \frac{q}{4\pi\epsilon_o} \left( \frac{2L}{(d^2 + L^2)^{\frac{3}{2}}} + \frac{1}{L^2} \right)$$

This is our answer.

Once again let's consider the limit  $L \gg d$ . If our answer is right, when we get very far from the group of charges they should look like a single charge with the amount of charge being the sum of all three environmental charges. In this limit

$$\lim_{L \gg d} \frac{1}{(d^2 + L^2)^{\frac{3}{2}}} = \frac{1}{L^3}$$

so

$$\begin{aligned} E_z &\approx \frac{q}{4\pi\epsilon_o} \left( \frac{2L}{L^3} + \frac{1}{L^2} \right) \\ &= \frac{1}{4\pi\epsilon_o} \left( \frac{3q}{L^2} \right) \end{aligned}$$

so on the central axis

$$\vec{\mathbf{E}} \approx \frac{1}{4\pi\epsilon_o} \left( \frac{3q}{L^2} \right) \hat{\mathbf{k}}$$

And indeed, this is very like one charge that is three times as large as our actual charges if we get far enough away.

This shows us a pattern we will often see. Far away, our field looks like what we would expect if the net charge were all congregated in a point. Near the charges, we must calculate the superposition of the fields. But far away we can treat the distribution as a point charge. This is very like what we did with mass in PH121 or Dynamics. We could often treat masses as point masses at the center of mass, if the distances involved were larger than the mass sizes.

Question 223.22.6

## Combinations of many charges

We have found the field from a point charge.

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \frac{qE}{r^2} \hat{\mathbf{r}} \quad (22.1)$$

where the field is in the same direction as  $\hat{\mathbf{r}}$  if the charge is positive, and in the opposite direction if the charge is negative (think of our field lines, they go toward the negative charge). This will become one of a group of standard charge configurations that we will use to gain a mental picture of complex configurations of charge. We have done this already for combinations of point charges. We can combine the point charge fields to get the total field.

The other standard models are combinations of many, many charges.

### Line of Charge

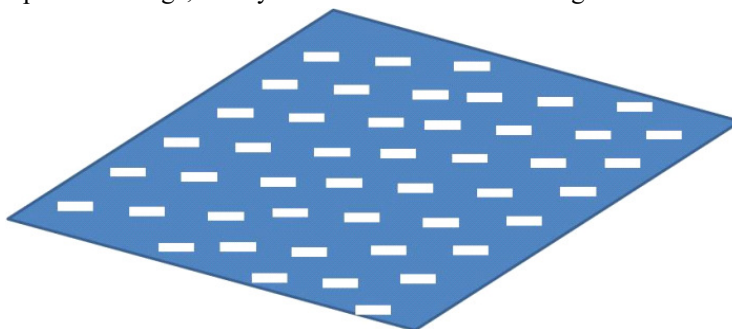
Another is an infinitely long line of charge, or a infinite charged wire. Since this long line of charge is infinite, it must have an infinite amount of charge. But we can describe “how much” charge it has with a linear charge density

$$\lambda = \frac{Q}{L}$$



### Semi-infinite sheet of charge

A sheet or plane of charge, usually a semi-infinite sheet of charge is also useful



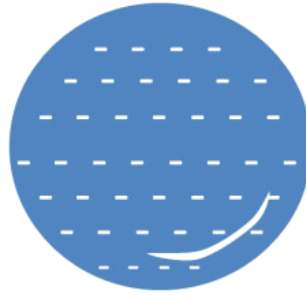
We have the same problem of having infinite charge, but if we define an amount of charge per unit area

$$\eta = \frac{Q}{A}$$

we can compare sheets that are more charge rich than others.

### Sphere of charge

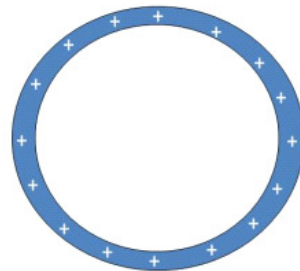
Finally, we have drawn a sphere of charge already



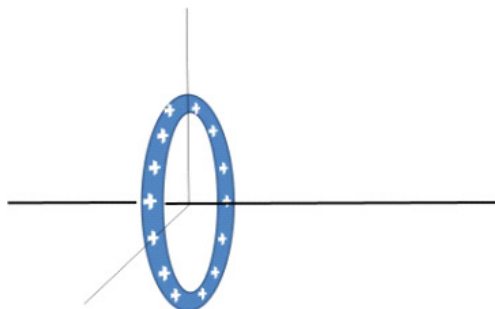
We can define an amount of charge per unit volume to help describe this distribution

$$\rho = \frac{Q}{V}$$

The spherical shell of charge is related to a sheet of charge, so we will include it here



This configuration of charge is drawn in cross section like the others. From your calculus experience you can guess that a spherical shell of charge with a certain volume charge density might be useful in integration, but we also can easily produce such a configuration of charge by charging a round balloon or a spherical conductor.



The ring of charge is similar to the spherical shell, but is also much like the line of charge.

In our next lecture, we will take on the job of finding the fields that result from these last few charge configurations except the spherical shell, which will have to wait a few lectures.

### On-Line Fun

A point charge field at a distant point visualization applet can be found at following this link:

<http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/electrostatics/03-ChargeField3d/03-chargeField320.html>

For a 2D visualization of the field try:

<http://www.falstad.com/emstatic/index.html>

And here is a 3D visualization:

<http://www.falstad.com/vector3de/>

### Basic Equations