33 Calculating fields from potentials

Fundamental Concepts

- To find the field knowing the potential, we use $\overrightarrow{\mathbf{E}} = -\left(\frac{d}{dx}\hat{\imath} + \frac{d}{dy}\hat{\jmath} + \frac{d}{dz}\hat{\mathbf{k}}\right) V$
- The gradient shows the direction of steepest change
- The potential of conductors in equilibrium

Finding electric field from the potential

We did part-one of relating fields to potentials in the last lecture. Now it is time for part two, obtaining the electric field from a known potential. Starting with

$$\Delta V = -\int_A^B \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$$

we realize that we should be able to write the integrand as a small bit of potential

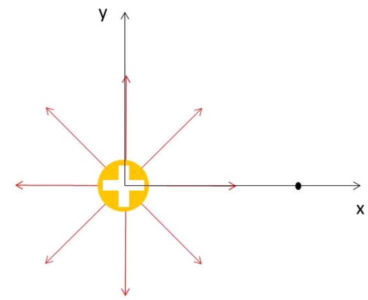
$$\begin{aligned} dV &= -\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} \\ &= -E_s ds \end{aligned}$$

where E_s is the component of the electric field in the $\hat{\mathbf{s}}$ direction. We can rearrange this

$$E_s = -\frac{dV}{ds}$$

This tells us that the magnitude of our field is the change in electric potential. Of course, $\overrightarrow{\mathbf{E}}$ is a vector and V is not. So the best we can do is to get the magnitude of the component in the $\overrightarrow{\mathbf{s}}$ direction.

We can try this out on a geometry we know, say, a point charge along the x-axis



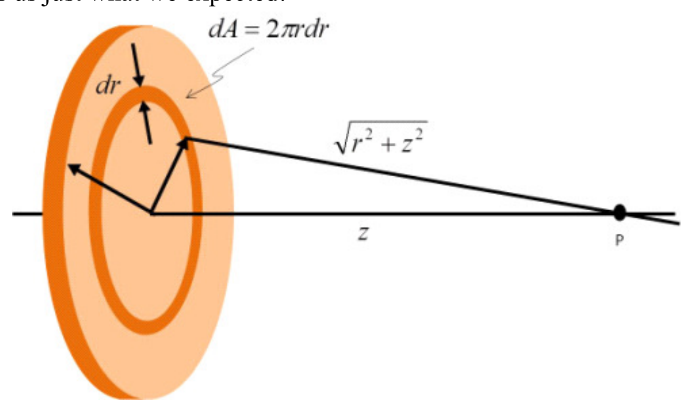
We know the potential will be

$$V = \frac{1}{4\pi\epsilon_o}\frac{q}{x}$$

then we can try

$$\begin{aligned} E_s &= -\frac{dV}{dx} = -\frac{d}{dx}\frac{1}{4\pi\epsilon_o}\frac{q}{x} \\ &= \frac{1}{4\pi\epsilon_o}\frac{q}{x^2} \end{aligned}$$

which gives us just what we expected!



Let's try another. Let's find the electric field due to a disk of charge along the axis. We have done this problem before. We know the field should be

$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \tag{33.1}$$

and in the previous lectures we found the potential to be

$$V = \frac{\eta}{2\epsilon_o}\left(\sqrt{a^2+z^2} - z\right) \tag{33.2}$$

Now can we find the electric field at P from V? Let's start by finding the $z-$component of the field, E_z

$$\begin{aligned} E_z &= -\frac{dV}{dz} && (33.3)\\ &= -\frac{d}{dz}\left(\frac{\eta}{2\epsilon_o}\left(\sqrt{a^2+z^2}-z\right)\right) && (33.4)\\ &= -\frac{d}{dz}\frac{\eta}{2\epsilon_o}\sqrt{a^2+z^2}+\frac{d}{dz}\frac{\eta}{2\epsilon_o}z && (33.5)\\ &= -\frac{\eta}{2\epsilon_o}\frac{d}{dz}\sqrt{a^2+z^2}+\frac{\eta}{2\epsilon_o} && (33.6)\\ &= -\frac{\eta}{2\epsilon_o}\frac{z}{\sqrt{a^2+z^2}}+\frac{\eta}{2\epsilon_o} && (33.7)\\ E_z &= \frac{\eta}{2\epsilon_o}\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) && (33.8) \end{aligned}$$

or

$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \tag{33.9}$$

But remember that this situation is highly symmetric. We can see by inspection that all the x and y components will all cancel out. So this is our field! And it is just what we found before.

We can graph these functions to compare them (what would you expect?). To do this we really need values, but instead, let's play a clever trick that some of you will see in advanced or older books. I am going to substitute in place of z the variable $u = \frac{z}{a}$. Then

$$\begin{aligned} V &= \frac{\eta}{2\epsilon_{o\,e}}\left(\sqrt{a^2+z^2}-z\right)\\ &= \frac{\eta a}{2\epsilon_o}\left(\sqrt{1+\frac{z^2}{a^2}}-\frac{z}{a}\right)\\ &= \frac{\eta a}{2\epsilon_o}\left(\sqrt{1+u^2}-u\right) \end{aligned}$$

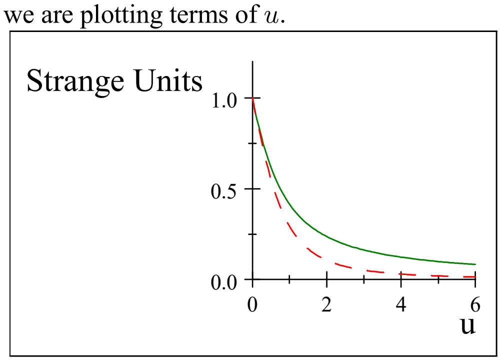
and

$$
\begin{aligned}
E_z &= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \qquad (33.10)\\
&= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{z}{a\sqrt{1+\frac{z^2}{a^2}}}\right)\\
&= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{z}{a\sqrt{1+\frac{z^2}{a^2}}}\right)\\
&= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{\frac{z}{a}}{\sqrt{1+\frac{z^2}{a^2}}}\right)\\
&= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{u}{\sqrt{1+u^2}}\right)
\end{aligned}
$$

Both my equation for V and for E_z now are in the form of a set of constants times a function of u.

$$
\begin{aligned}
V &= \frac{\eta a}{2\epsilon_o}\left(\sqrt{1+u^2}-u\right)\\
&= \frac{\eta a}{2\epsilon_o} f(u)\\
E_z &= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-\frac{u}{\sqrt{1+u^2}}\right)\\
&= \frac{2\pi\eta}{4\pi\epsilon_o} g(u) \qquad (33.11)
\end{aligned}
$$

If I plot V in units of $\frac{\eta a}{2\epsilon_o}$ (the constants out in front) I can see the shape of the curve. It is the function of $f(u)$. I can compare this to E_z in units of $\frac{2\pi\eta}{4\pi\epsilon_o}$. The shape of E_z will be $g(u)$. Of course we are plotting terms of u.



Now we can ask, is this reasonable? Does it look like the $E-$field (red dashed line) is the right shape for the derivative of the potential (solid green line)? It is also comforting to see that as u (a function of z) gets larger the field falls off to zero and so does the potential as we would expect. When V (green solid curve) has a large slope, E_z is a large number (positive because of the negative sign in the equation

$$E_s = -\frac{dV}{ds}$$

and when V is fairly flat, E_z is nearly zero. Our strategy for finding E from V seems to work.

Geometry of field and potential

You should probably worry that so far our equation

$$E_s = -\frac{dV}{ds}$$

is only one dimensional. We know the electric field is a three dimensional vector field. We may find situations where we need two or three dimensions. But this is easy to fix. Our equation

$$E_s = -\frac{dV}{ds}$$

gives us the field magnitude along the $\hat{\mathbf{s}}$ direction. Let's choose this to be the $\hat{\mathbf{x}}$ direction. Then

$$E_x = -\frac{dV}{dx}$$

is the x-component of the electric field. Likewise

$$\begin{aligned} E_y &= -\frac{dV}{dy} \\ E_z &= -\frac{dV}{dz} \end{aligned}$$

The total field will be the vector sum of it's components

$$\begin{aligned} \overrightarrow{\mathbf{E}} &= E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{\mathbf{k}} \\ &= -\frac{dV}{dx}\hat{\imath} - \frac{dV}{dy}\hat{\jmath} - \frac{dV}{dz}\hat{\mathbf{k}} \end{aligned}$$

Question 223.33.1

Question 223.33.2

which we can cryptically write as

$$\overrightarrow{\mathbf{E}} = -\left(\frac{d}{dx}\hat{\imath} + \frac{d}{dy}\hat{\jmath} + \frac{d}{dz}\hat{\mathbf{k}}\right)V$$

The odd group of operations in the parenthesis is call a *gradient* and is written as

$$\overrightarrow{\boldsymbol{\nabla}} = \left(\frac{d}{dx}\hat{\imath} + \frac{d}{dy}\hat{\jmath} + \frac{d}{dz}\hat{\mathbf{k}}\right)$$

using this we have

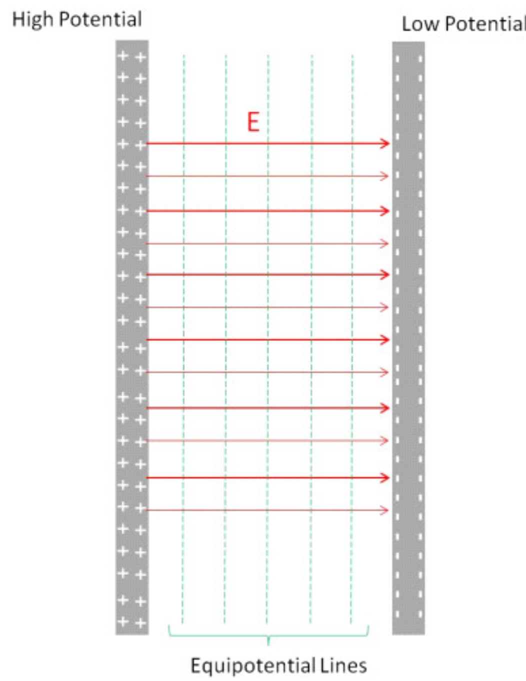
$$\overrightarrow{\mathbf{E}} = -\overrightarrow{\boldsymbol{\nabla}}V$$

which is how the relationship is stated in higher level electrodynamics books. But what does it mean?

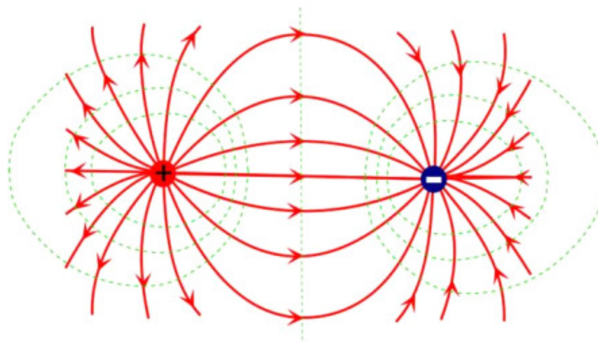
The gradient is really kind of what it sounds like. If you go down a steep grade, you will notice you are going down hill and will notice if you are going down the steepest part of the hill. The gradient finds the direction of steepest decent. That is, the direction where the potential changes fastest. This is like looking from the top of the hill and taking the steepest way down! Our relationship tells us that the electric field points in this steepest direction, and the minus sign tells us that the electric field points down hill away from a positive charge, never up hill (think of the acceleration due to gravity being negative). Let's see if this makes sense for our geometries that we know.

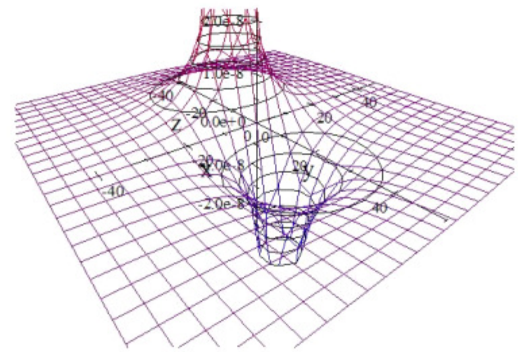
Stamp in a circle: mimic a blindfolded person swiveling on one foot and testing the slope with the other

Here is our capacitor. We see that indeed the field points from the high potential to the low potential. The steepest way "down the hill" is perpendicular to the equipotential lines.

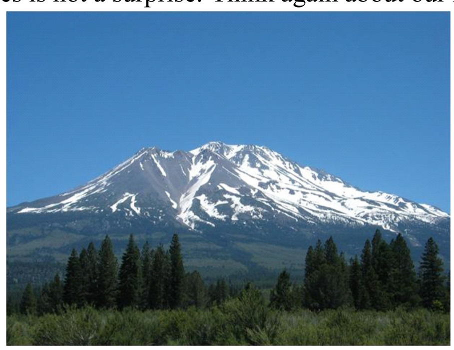


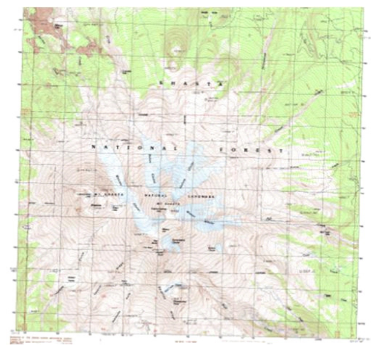
We also know the shape of the field for a dipole. The equipotential lines we have seen before.





But now we can see that the field lines and equipotential lines are always perpendicular and the field points “down hill.” The meeting of the field and equipotential lines at right angles is not a surprise. Think again about our mountain





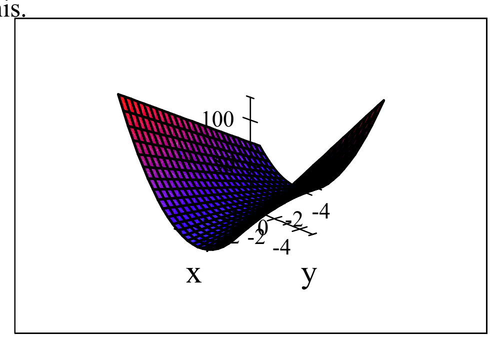
Map courtesy USGS, Picture is in the Public Domain.

The steepest path is always perpendicular to lines of equal potential energy.

We should try another example of finding the field from the gradient. Suppose we have a potential that varies as

$$V = 3x^2 + 2xy$$

I don’t know what is making this potential, but let’s suppose we have such a potential. It would look like this.



what is the electric field?

$$\overrightarrow{\mathbf{E}}=-\overrightarrow{\boldsymbol{\nabla}}V$$

or

$$\overrightarrow{\mathbf{E}}=-\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)V$$

so

$$\overrightarrow{\mathbf{E}}=-\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)\left(3x^2+2xy\right)$$

$$\begin{aligned}\overrightarrow{\mathbf{E}} &= -\left(\hat{\imath}\frac{d}{dx}\left(3x^2+2xy\right)+\hat{\jmath}\frac{d}{dy}\left(3x^2+2xy\right)+\hat{\mathbf{k}}\frac{d}{dz}\left(3x^2+2xy\right)\right)\\ &= -\left(\hat{\imath}\left(6x+2y\right)+\hat{\jmath}\frac{d}{dy}\left(2xy\right)+0\right)\end{aligned}$$

This example shows how to perform the operation, but it does not give much insight. We have learned to work with our standard charge configurations, and this is really not one of them. So we don't have much intuitive feel for this electric field that we found.

To gain more insight, let's return to finding the point charge field from the point charge potential. The potential for a point charge is

$$V=\frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

And of course we know that the field is

$$E=\frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}$$

but we want to show this using

$$\overrightarrow{\mathbf{E}}=-\overrightarrow{\boldsymbol{\nabla}}V$$

So

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &= -\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)V \\
&= -\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)\frac{1}{4\pi\epsilon_o}\frac{Q}{r} \\
&= -\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)\frac{1}{4\pi\epsilon_o}\frac{Q}{\sqrt{x^2+y^2+z^2}} \\
&= -\frac{Q}{4\pi\epsilon_o}\left(\frac{d}{dx}\hat{\imath}+\frac{d}{dy}\hat{\jmath}+\frac{d}{dz}\hat{\mathbf{k}}\right)\frac{1}{\sqrt{x^2+y^2+z^2}} \\
&= -\frac{Q}{4\pi\epsilon_o}\left(-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\imath}-\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\jmath}-\frac{k}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\mathbf{k}}\right) \\
&= \frac{Q}{4\pi\epsilon_o}\frac{\left(x\hat{\imath}+y\hat{\jmath}+k\hat{\mathbf{k}}\right)}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
&= \frac{Q}{4\pi\epsilon_o}\frac{\left(x\hat{\imath}+y\hat{\jmath}+k\hat{\mathbf{k}}\right)}{(x^2+y^2+z^2)\sqrt{(x^2+y^2+z^2)}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\frac{\left(x\hat{\imath}+y\hat{\jmath}+k\hat{\mathbf{k}}\right)}{\sqrt{(x^2+y^2+z^2)}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}
\end{aligned}$$

but really, this is a bit of a mess, we don't want to do such a problem in rectangular coordinates. We could write $\mathbf{\nabla}$ in spherical coordinates (something we won't derive here, but you should have seen in M215 or M316).

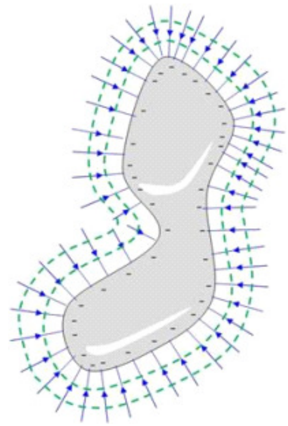
$$\overrightarrow{\mathbf{\nabla}} = \hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Let's try this out on our point charge potential. We have

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)V \\
&= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\frac{1}{4\pi\epsilon_o}\frac{Q}{r} \\
&= -\frac{Q}{4\pi\epsilon_o}\left(-\frac{1}{r^2}\hat{\mathbf{r}}+0+0\right) \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}
\end{aligned}$$

just as we expected. But this time the math was much easier. If we can, it is a good idea to match our expression for $\overrightarrow{\mathbf{\nabla}}$ to the geometry of the system. A good vector calculus book or a compendium of math functions will have various versions of $\overrightarrow{\mathbf{\nabla}}$ listed.

Conductors in equilibrium again



Question 223.33.3

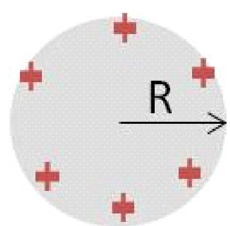
We know that there is no field inside a conductor in electrostatic equilibrium, but we should ask what that means for the electric potential. To build circuits or electronic actuators, we will need to know this. Let's start again with

$$\Delta V = -\int_A^B \mathbf{E} \cdot d\mathbf{s} \tag{33.12}$$

and since the field $E = 0$ inside the conductor, then inside

$$\Delta V_{inside} = 0 \tag{33.13}$$

On the surface we see that there is a potential, since there is a field. If we take our spherical case,



and observe the potential as we go away from the center, we expect the potential to be constant up to the surface. Then as we reach the surface, we know from Gauss' law that the field will be

$$E = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}$$

like a point charge, so the potential at the surface must be

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{R}$$

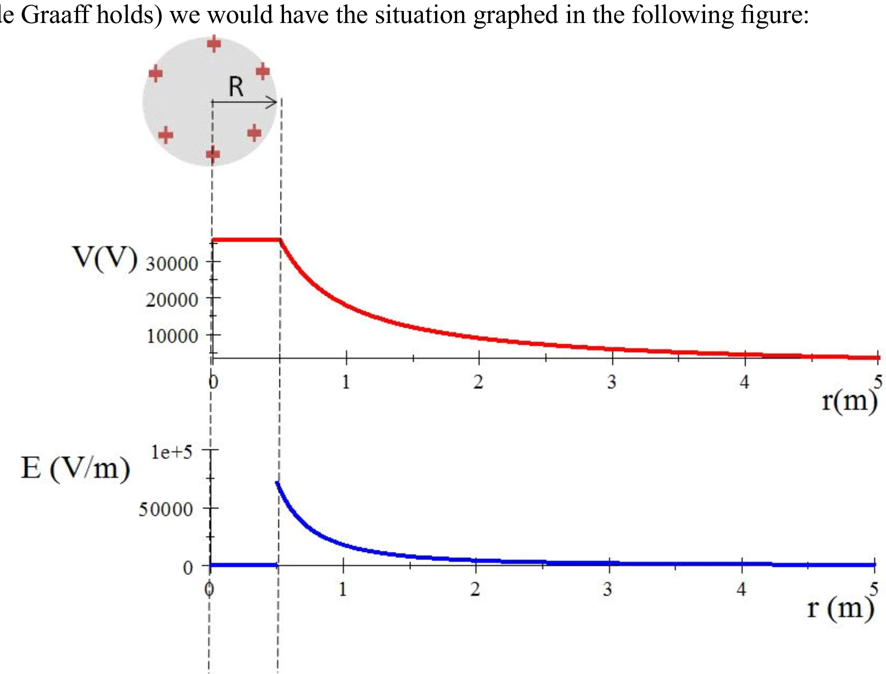
where $r = R$, the radius of our sphere. As we move into the sphere from the surface, the potential must not change. The interior will have the potential

$$V_{inside} = \frac{1}{4\pi\epsilon_o}\frac{Q}{R} \tag{33.14}$$

Outside, of course, the potential will drop like the potential due to a point charge. We expect

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{r} \tag{33.15}$$

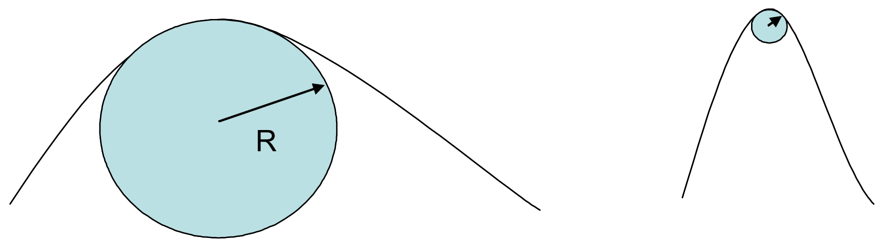
For a sphere of radius $R = 0.5\,\mathrm{m}$ carrying a charge of $0.000002\,\mathrm{C}$ (about what our van de Graaff holds) we would have the situation graphed in the following figure:



This is an important point. For a conductor, the electric potential everywhere inside the conductive material is exactly the same once we reach equilibrium. This is just what we want for capacitors or electrodes or electrical contacts in circuits.

Non spherical conductors

The field is stronger where the field lines are closer together. One way to describe this is to use a radii of curvature. That is, suppose we try to fit a small circle into a bump on the surface of a conductor.



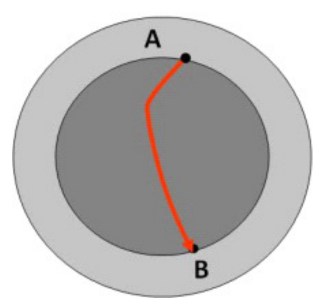
In the figure there are two bumps shown with circles fit into them. The bump on the right has a much smaller radius circle than the one on the left. The radius of the circle that fits into the bump is the radius of curvature of the bump. From what we have said, the bump on the right will have a much stronger field strength near it than the bump on the left.

Where there is a lot of charge on a conductor, and the field is very high, electrons from random ionizations of air molecules near the conductor are accelerated away from the conductor. These electrons hit other atoms, ionizing them as well. We get a small avalanche of electrons. Eventually the electrons recombine with ionized atoms, producing an eerie glow. This is called *corona discharge*. It can be used to find faults in high tension wires and other high voltage situations.

Coronal Discharge Clips

Cavities in conductors

Suppose we have a hollow conductor with no charges in the cavity. What is the field? We know from using Gauss' law what the answer should be, but let's do this using potentials.



All the parts of the conductor will be at the same potential. So let's take two points, A and B, and compute

$$V_A - V_B = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

We know that $V_A - V_B = 0$ because V_A must be the same as V_B. So for every path, s, we must have

$$-\int_A^B \mathbf{E} \cdot d\mathbf{s} = \mathbf{0}$$

We can easily conclude that E must equal zero.

So as long as there are no charges inside the cavity, the cavity is a net field free zone.

It is often much easier to find the potential, and from the potential, find the field. Much of the study of electrodynamics uses this approach. This is because it is more straight-forward to differentiate than it is to integrate. Some of you may use massive computational programs to predict electric fields. They often use differential equations in the potential to find the field rather than integral equations to find the field directly.

Basic Equations