Chapter 20

Current loops

Fundamental Concepts

- The magnetic field due to a current in a wire is given by the integral form of the Biot-Savart law $\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi} \int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$

- The magnetic field magnitude of a long straight wire with a current is given by $B = \frac{\mu_o I}{2\pi a}$ with the direction given by the right hand rule we learned last time.

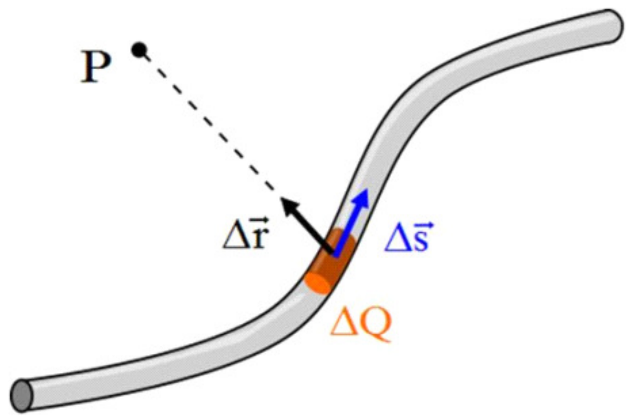
- The field due to a magnetic dipole is $\overrightarrow{\mathbf{B}} \approx \frac{\mu_o}{4\pi} \frac{2\overrightarrow{\mu}}{r^3} \hat{\imath}$ where $\overrightarrow{\mu}$ is the magnetic dipole moment $\mu = IA$ with the direction from south to north pole.

20.1 Magnetic field of a current

Last lecture, we learned the Biot-Savart law

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi} \frac{q\overrightarrow{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

now let's consider our q to be part of a current in a wire. A small amount of current moves along the wire. Let's call this small amount of charge ΔQ.



This small amount of charge will make a magnetic field, but it will be only a small part of the total field, because ΔQ is only a small part of the total amount of charge flowing in the wire. That part of the field made by ΔQ is

$$\Delta\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{\Delta Q\overrightarrow{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

Let's look at $\Delta Q\overrightarrow{\mathbf{v}}$. We can rewrite this as

$$\begin{aligned} \Delta Q\overrightarrow{\mathbf{v}} &= \Delta Q\frac{\Delta\overrightarrow{\mathbf{s}}}{\Delta t} \\ &= \frac{\Delta Q}{\Delta t}\Delta\overrightarrow{\mathbf{s}} \\ &= I\Delta\overrightarrow{\mathbf{s}} \end{aligned}$$

then our small amount of field is given by

$$\Delta\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{I\Delta\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

as usual, where there is a Δ, we can predict that we can take a limit and end up with a d

$$d\overrightarrow{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{Id\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

Some things to note about this result

1. The vector $d\overrightarrow{\mathbf{B}}$ is perpendicular to $d\overrightarrow{\mathbf{s}}$ and to the unit vector $\hat{\mathbf{r}}$ directed from $d\overrightarrow{\mathbf{s}}$ to some point P.
2. The magnitude of $d\overrightarrow{\mathbf{B}}$ is inversely proportional to r^2
3. The magnitude of $d\overrightarrow{\mathbf{B}}$ is proportional to the current
4. The magnitude of $d\overrightarrow{\mathbf{B}}$ is proportional to the length of $d\overrightarrow{\mathbf{s}}$

5. The magnitude of $d\overrightarrow{\mathbf{B}}$ is proportional to $\sin\theta$ where θ is the angle between $d\overrightarrow{\mathbf{s}}$ and $\hat{\mathbf{r}}$

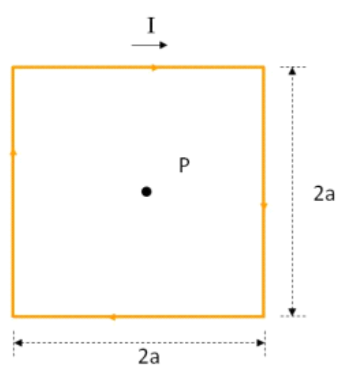
Where there is $d\overrightarrow{\mathbf{B}}$ we will surely integrate. The field $d\overrightarrow{\mathbf{B}}$ is due to just a small part of the wire $d\overrightarrow{\mathbf{s}}$. We would like the field due to all of the wire. So we take

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi}\int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

This is a case where the equation actually is as hard to deal with as it looks. The integration over a cross product is tricky. Let's do an example.

20.1.1 The field due to a square current loop

Suppose we have a square current loop. Of course there would have to be a battery or some potential source in the loop to make the current, but we will just draw the loop with a current as shown. The current must be the same in all parts of the loop.



Let's find the field in the center of the loop at point P.

I will break up the integration into four parts, one for each side of the loop. For each part, we will need to find $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$ and r to find the field using

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi}\int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

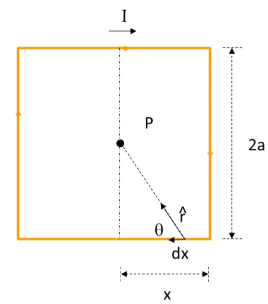
This is very like what we did to find electric fields. For electric fields we had to find dq, r, and take components to remove $\hat{\mathbf{r}}$. And we integrated using

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$$

Now we need I and r and we need to deal with $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$. For electric fields, we just needed to deal with the vector $\hat{\mathbf{r}}$. Now we need to deal with a cross product, $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$, involving $\hat{\mathbf{r}}$. Since I is constant, the part of our new equation that deals with charge comes out of the integral! So instead of starting with dq let's deal with $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$ as our first step. For the bottom part of our loop $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$ is just

$$\begin{aligned} d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}} &= d\overrightarrow{\mathbf{x}} \times \hat{\mathbf{r}} \\ &= -dx \sin\theta \hat{\mathbf{k}} \end{aligned}$$

where $+\hat{\mathbf{k}}$ is out of the page. We can see this in the figure



So our field from the bottom wire is

$$\begin{aligned} \overrightarrow{\mathbf{B}}_b &= \frac{\mu_o I}{4\pi} \int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \\ &= \frac{\mu_o I}{4\pi} \int \frac{-dx \sin\theta \hat{\mathbf{k}}}{r^2} \end{aligned}$$

Next we need to find r. We would like to not have more than one variable. So it would be good to try to pick x or θ and to put everything in terms of that one variable. Let's try θ. From trigonometry we realize

$$\sin\theta = \frac{a}{r}$$

on the right side of the wire, and

$$\sin(\pi - \theta) = \sin\theta = \frac{a}{r}$$

on the left side, So all along the bottom wire r is given by

$$r = \frac{a}{\sin\theta}$$

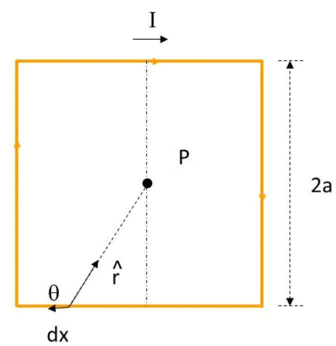
Then our field equation for the bottom wire becomes

$$\overrightarrow{\mathbf{B}}_b = \frac{\mu_o I}{4\pi} \int \frac{-(dx)\sin\theta \hat{\mathbf{k}}}{\left(\frac{a}{\sin\theta}\right)^2}$$

but now we have an integration over dx and our function is in terms of θ which depends on x. We should try to fix this. Let's find dx in terms of $d\theta$. We can pick $x = 0$ to be the middle of the wire. Then

$$\tan\theta = \frac{a}{x}$$

on the right and



$$\tan(\pi-\theta) = -\tan\theta = \frac{a}{x}$$

on the left. Since on the left x is negative, this makes sense. So we have either

$$x = \frac{a}{\tan\theta}$$

or

$$x = -\frac{a}{\tan\theta}$$

depending on which size of the dotted line we are on. We could write these as

$$x = \pm\frac{a}{\tan\theta} = \pm\frac{a\cos\theta}{\sin\theta}$$

for both cases. We really want dx and moreover we want it as a magnitude (we deal with the direction in the cross product). So we can take a derivative and then take the magnitude (absolute value).

$$\frac{dx}{d\theta} = \frac{\sin\theta\left(-a\sin\theta\right) - a\cos\theta\cos\theta}{\sin^2\theta} = \frac{-a}{\sin^2\theta}$$

This derivative was not obvious! We had to use the quotient rule. But once we have found it we can rewrite dx as

$$dx = \left|\frac{-a}{\sin^2\theta}d\theta\right|$$

(now with the absolute value inserted) and since neither a nor $\sin^2\theta$ can be negative we can just write this as $dx = \frac{a}{\sin^2\theta}d\theta$. When we put this in our integral equation for the bottom wire we have

$$\overrightarrow{\mathbf{B}}_b = \frac{\mu_o I}{4\pi}\int\frac{-\left(\frac{a}{\sin^2\theta}\right)d\theta\sin\theta\hat{\mathbf{k}}}{\left(\frac{a}{\sin\theta}\right)^2}$$

which we should simplify before we try to integrate.

$$\begin{aligned}\overrightarrow{\mathbf{B}}_b &= \frac{\mu_o I}{4\pi}\int\frac{-\sin\theta d\theta\hat{\mathbf{k}}}{a} \\ &= -\frac{\mu_o I}{4\pi a}\hat{\mathbf{k}}\int\sin\theta d\theta\end{aligned}$$

which is really not too bad considering the integral we had at the start. When we get to the corner of the left hand side $\theta = \frac{3\pi}{4}$ and when we start on the right hand side $\theta = \frac{\pi}{4}$ and along the bottom wire θ will be somewhere in between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Then $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are our limits of integration. We can perform this integral

$$\begin{aligned}\overrightarrow{\mathbf{B}}_b &= -\frac{\mu_o I}{4\pi a}\hat{\mathbf{k}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin\theta d\theta \\ &= -\frac{\mu_o I}{4\pi a}\hat{\mathbf{k}}\left[-\cos\theta\right|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= -\frac{\mu_o I}{4\pi a}\hat{\mathbf{k}}\left(\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\right) \\ &= -\frac{\mu_o I\sqrt{2}}{4\pi a}\hat{\mathbf{k}}\end{aligned}$$

This was just for the bottom of the loop. Now let's look at the top of the loop. There is finally some good news. The math will all be the same except for the directions. We had better work out $d\overrightarrow{\mathbf{s}}\times\hat{\mathbf{r}}$ to see how different it is.

Now the $d\overrightarrow{\mathbf{s}}$ is to the right and $\hat{\mathbf{r}}$ is downward so

$$d\overrightarrow{\mathbf{s}}\times\hat{\mathbf{r}} = -dx\sin\theta\hat{\mathbf{k}}$$

But this is just as before. So even this is the same! The integral across the top wire will have exactly the same result as the integral across the bottom wire. We can just multiply our previous result by two.

How about the sides? Again we get the same $d\overrightarrow{\mathbf{s}}\times\hat{\mathbf{r}}$ direction and all the rest is the same, so our total field is

$$\overrightarrow{\mathbf{B}} = 4\overrightarrow{\mathbf{B}}_b = -\frac{\mu_o I\sqrt{2}}{\pi a}\hat{\mathbf{k}}$$

This was a long hard, messy problem. But current loops are important! Every electric circuit is a current loop. Does this mean that every circuit is making a magnetic field? The answer is yes! As you might guess, this can have a profound effect on circuit design. If your circuit is very sensitive, adding extra fields (and therefore extra forces on the charges) can be disastrous causing the design to fail. There is some concern about "electronic noise" and possible effects on the body (cataracts are one side effect that is well known). And of course, as the circuit changes its current, the field it creates changes. this can create the opportunity for espionage. The field exists far away from the circuit. A savvy spy can determine what your circuit is doing by watching the field change!

20.1.2 Long Straight wires

In our last example, we found that the magnitude of the field due to a wire is

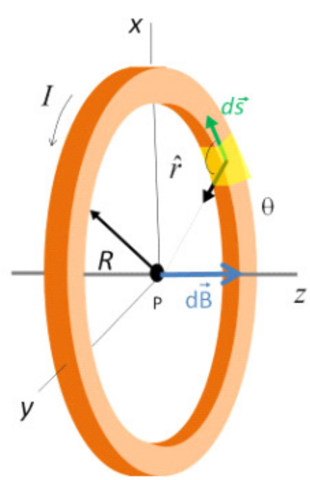
$$B = \left| -\frac{\mu_o I}{4\pi a} \int \sin\theta d\theta \right|$$

Of course, we would like to relate this to our standard charge configuration, in this case an infinite line of (now moving) charge. If the wire is infinitely long, then the limits of integration are just from $\theta = 0$ to $\theta = \pi$

$$\begin{aligned} B &= \left| -\frac{\mu_o I}{4\pi a} \int_0^\pi \sin\theta d\theta \right| \\ &= \left| -\frac{\mu_o I}{4\pi a} \left(-\cos\theta\right)\Big|_0^\pi \right| \\ &= \frac{\mu_o I}{2\pi a} \end{aligned}$$

This is an important result. We can add a new geometry to our list of special cases, a long straight wire that is carrying a current I. The direction of the magnetic field, we already know, is given by our right-hand-rule. Of course, if our wire is not infinitely long, we now know how to find the actual field. It is all a matter of finding the right limits of integration.

20.2 Magnetic dipoles



As a second example, let's find the magnetic field due to a round loop at the center of the loop. We start again with

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi} \int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

We need to find $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$ and r, to do the integration. Our steps are:

1. Find an expression for $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$
2. Find an expression for r
3. Assemble the integral, including limits of integration
4. Solve the integral.

Let's start with the first step. As we go around the loop $d\overrightarrow{\mathbf{s}}$ and $\hat{\mathbf{r}}$ will be perpendicular to each other, so

$$d\mathbf{s} \times \hat{\mathbf{r}} = ds\hat{\mathbf{k}}$$

For the second step, we realize that r is just the radius of the loop, R. Then the integration is quite easy (much easier to set up than the last case!)

$$B = \frac{\mu_o I}{4\pi} \int \frac{ds}{R^2}\hat{\mathbf{k}}$$

The limits of integration will be 0 to $2\pi R$. We can perform this integral

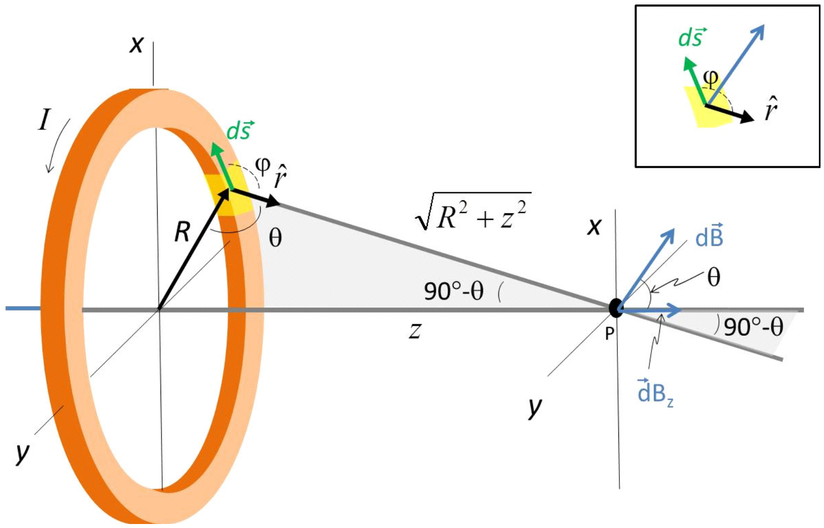
$$\begin{aligned} B &= \frac{\mu_o I}{4\pi} \int_0^{2\pi R} \frac{ds}{R^2}\hat{\mathbf{k}} \\ &= \frac{\mu_o I}{4\pi} \frac{2\pi R}{R^2}\hat{\mathbf{k}} \end{aligned}$$

so

$$B = \frac{\mu_o I}{2R}\hat{\mathbf{k}} \qquad \text{loop}$$

The field is perpendicular to the plane of the loop, which agrees with our square loop problem.

Let's extend this problem to a point along the axis a distance z away from the loop.



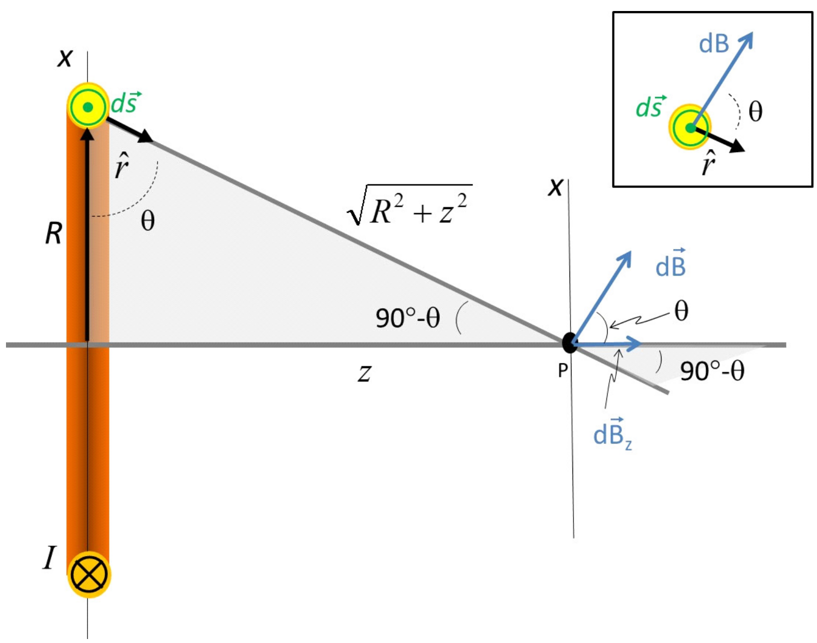
We need to go back to our basic equation again.

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi} \int \frac{d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

Starting with step 1, we realize that, in general, our value of $d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}}$ is

$$\left| d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}} \right| = ds \sin\phi$$

where ϕ is the angle between $d\overrightarrow{\mathbf{s}}$ and $\hat{\mathbf{r}}$. We can see that for this case ϕ will still be $90\,^{\circ}$.



We have tipped $\hat{\mathbf{r}}$ toward our point P, but tipping $\hat{\mathbf{r}}$ from pointing to the center of the hoop to pointing to a point on the axis just rotated $\hat{\mathbf{r}}$ about part of the hoop. We still have $\phi = 90\,^\circ$. So

$$\left| d\overrightarrow{\mathbf{s}} \times \hat{\mathbf{r}} \right| = ds$$

with a direction shown in the figure. We have used symmetry to argue that we can take just x or y-components in the past because all the others clearly canceled out. We can also do that again here. Using symmetry we see that only the z-component of the magnetic field will survive. So we can take the projection onto the z-axis.

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi} \int \frac{ds}{r^2} \cos\theta \hat{\mathbf{k}}$$

We know how to deal with such a situation, since we have done this before. From the diagram we can see that

$$\cos\theta = \frac{R}{\sqrt{R^2 + z^2}}$$

And our value of r we recognize as just

$$r = \sqrt{R^2 + z^2}$$

so our field becomes

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi} \hat{\mathbf{k}} \int \frac{R ds}{\left(R^2 + z^2\right)^{\frac{3}{2}}}$$

Fortunately this integral is also not too hard to do. Let's take out all the terms that don't change with ds

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I R}{4\pi \left(R^2 + z^2\right)^{\frac{3}{2}}} \hat{\mathbf{k}} \int_0^{2\pi R} ds$$

The limits of integration are 0 to $2\pi R$, the circumference of the circle

$$\begin{aligned} \overrightarrow{\mathbf{B}} &= \frac{\mu_o I R 2\pi R}{4\pi \left(R^2 + z^2\right)^{\frac{3}{2}}} \hat{\mathbf{k}} \\ &= \frac{\mu_o I R^2}{2 \left(R^2 + z^2\right)^{\frac{3}{2}}} \hat{\mathbf{k}} \end{aligned}$$

Let's take some limiting cases to see if this makes sense. Suppose $z = 0$, then

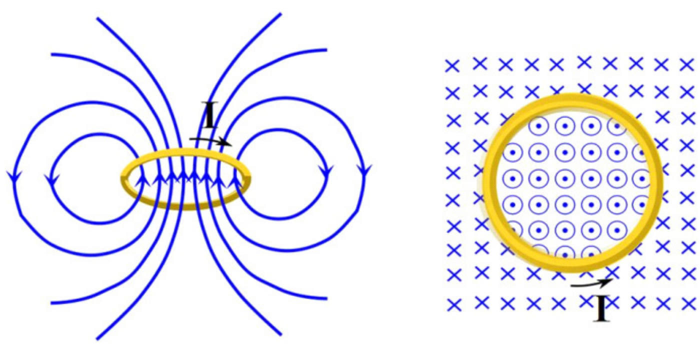
$$\begin{aligned} \overrightarrow{\mathbf{B}} &= \frac{\mu_o I R^2}{2 \left(R^2 + 0\right)^{\frac{3}{2}}} \hat{\mathbf{k}} \\ &= \frac{\mu_o I R^2}{2R^3} \hat{\mathbf{k}} \\ &= \frac{\mu_o I}{2R} \hat{\mathbf{k}} \end{aligned}$$

which is what we got before for the field at the center of the loop. That is comforting.

Now suppose that $z \gg R$. In that case, we can ignore the R^2 in the denominator.

$$\begin{aligned}\overrightarrow{\mathbf{B}} &\approx \frac{\mu_o I R^2}{2\left(z^2\right)^{\frac{3}{2}}}\hat{\mathbf{k}} \\ &= \frac{\mu_o I R^2}{2z^3}\hat{\mathbf{k}}\end{aligned}$$

We have just done this for on-axis positions because the math is easy there. But we could find the field at other locations. The result looks something like this.



The figure on the left was taken from the pattern in iron filings that was created by an actual current loop field. The figure to the right is a top down look. We will use the symbol $\odot$ to mean "coming out of the page at you" and the symbol $\otimes$ "going into the page away from you." Imagine these as parts of an arrow. The dot in the circle is the arrow tip coming at you, and the cross is the fletching going away from you. Notice that the field is up through the loop, and down on the outside.

As we generalize our solution for the magnetic field far from the loop we have

$$\overrightarrow{\mathbf{B}} \approx \frac{\mu_o I R^2}{2r^3}\hat{\mathbf{k}}$$

This looks a lot like the electric field from a dipole

$$\overrightarrow{\mathbf{E}} = \frac{2}{4\pi\epsilon_o}\frac{\overrightarrow{\mathbf{p}}}{r^3}$$

which gives us an idea. We have a dipole moment for the electric dipole. This magnetic field has the same basic form as the electric dipole. We can rewrite our field as

$$\begin{aligned}\overrightarrow{\mathbf{B}} &\approx \frac{\mu_o I\left(\pi R^2\right)}{2\left(\pi\right)r^3}\hat{\imath} \\ &= \frac{\mu_o\left(2\right)I\left(A\right)}{\left(2\right)2\left(\pi\right)r^3}\hat{\imath} \\ &= \frac{\mu_o}{4\pi}\frac{2IA}{r^3}\hat{\imath}\end{aligned}$$

where $A = \pi R^2$ is the area of the loop.

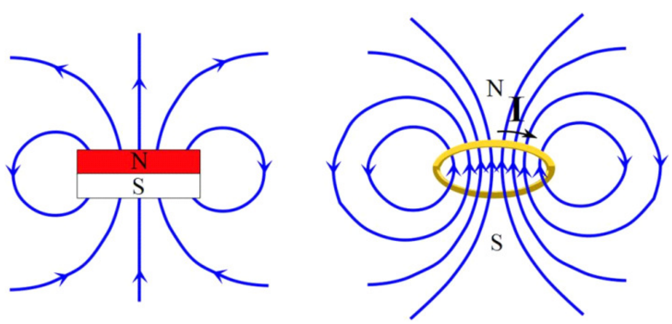
The electric dipole moment is the charge multiplied by the charge separation

$$p = qa$$

we have something like that in our magnetic field, The terms IA describe the amount of charge and the geometry of the charges. We will call these terms together the *magnetic dipole moment*

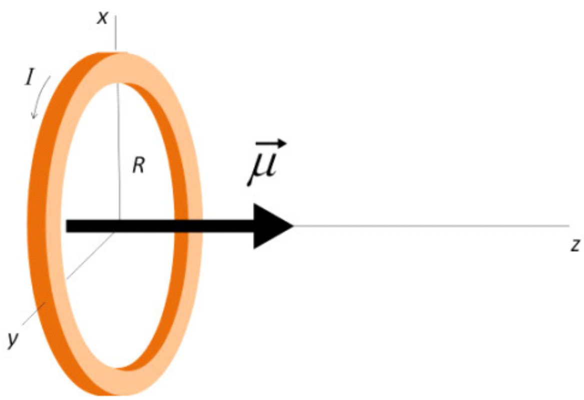
$$\mu = IA$$

and give them a direction so that μ is a vector. The direction will be from south to north pole



where we can find the south and north poles by comparison to the field of a bar magnet.

$$\overrightarrow{\mu} = IA \qquad \text{from South to North}$$



This is a way to characterize an entire current loop.

As we get farther from a loop, the exact shape of the loop becomes less important. So as long as r is much larger than R, we can write

$$\overrightarrow{\mathbf{B}} \approx \frac{\mu_o}{4\pi}\frac{2\overrightarrow{\mu}}{r^3}\hat{\mathbf{k}}$$

for any shaped current loop.

The integral for of the Biot-Savart law is very powerful. We can use computers to calculate the field do to any type of current configuration. But by hand there are only a few cases we can do because the integration becomes difficult. With electrostatics, we found ways to use geometry to eliminate or at least make the integration simpler. We will do the same thing for magnetostatics starting with the next lecture. Our goal will be to use geometry to avoid using Biot-Savart when we can.

Basic Equations

$$\overrightarrow{\mathbf{B}} = \frac{\mu_o I}{4\pi}\int\frac{d\overrightarrow{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}$$

$$\overrightarrow{\mu} = IA \qquad \text{from South to North}$$

$$\overrightarrow{\mathbf{B}} \approx \frac{\mu_o}{4\pi}\frac{2\overrightarrow{\mu}}{r^3}\hat{\mathbf{k}}$$