# Chapter 30

# Oscillation

We stopped our last lecture with waves in the electromagnetic field. But to understand these electromagnetic waves we need to understand wave motion better. And to understand wave motion we need to understand the motion that starts waves. So we will begin our study of waves with a study of simple harmonic motion.

## Fundamental Concepts

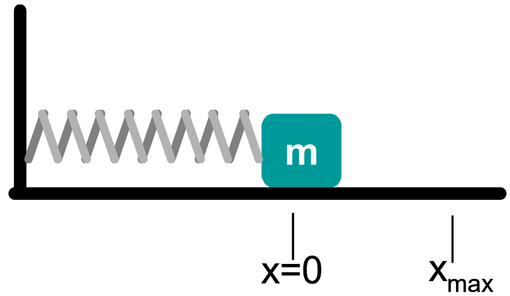
- There is a very special kind of motion called simple harmonic motion (SHM)
- SMH is a form of oscillation.
- SMH is described by sine and cosine functions
- the velocity and acceleration for SHM are also sinusoids.
- We can find equations for the kinetic and potential energy of objects in SHM

## 30.1 Simple Harmonic Motion

You are, no doubt, an expert springs and spring forces after your Principles of Physics (PH121) class. So, let's consider a mass attached to a spring resting on a frictionless surface[1]. This mass-spring system can oscillate.

In the position shown the spring is neither pushing nor pulling on the mass. We will call this position the *equilibrium position* for the mass.

[1] Yes, I know there are no actual frictionless surfaces, but we are once again starting out at freshman level physics, so we will make the math simple enough that a freshman could do it by making simplifying assumptions. In this case, that the surface if frictionless.

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Equilibrium Position: The position of the mass when the spring is neither stretched nor compressed.

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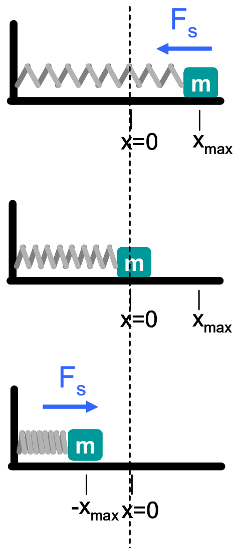
### 30.1.1 Hooke's Law

Long ago it was noticed that the pull of a spring grew in strength as the spring was pulled out of equilibrium. The mathematical expression of this is

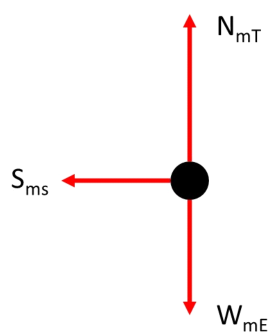
$$F_s = -k\Delta x_{eq} \tag{30.1}$$

The force, $F_s$ is directly proportional to the displacement from equilibrium, $\Delta x_{eq}$. Since a man named Hooke wrote this down, it is called Hooke's law. It should be familiar from your Principles of Physics I experience.

Hooke's Law is, strictly speaking, not a law that is always obeyed. It is a good model for most springs as long as we don't stretch them too far. We will often use the word "law" to mean *an equation that gives a basic relationship.* In that sense, Hook's law is a law.




Lets write Newton's second Law for our spring




where I have used $S_{ms} = F_S$ as the force on the mass due to the spring. Newton's second law give us

$$\Sigma F_x = ma_x = -S_{mS}$$

and

$$\Sigma F_y = ma_y = N_{mT} - W_{mE}$$

From the last of the two equations we get

$$N_{mT} = W_{mE}$$

because $a_y = 0$. But from the first we get

$$ma_x = -S_{mS}$$

If we assume no friction, we have just

$$ma_x = -k\Delta x_{eq}$$

We can write this as

$$a_x = -\frac{k}{m}\Delta x_{eq} \tag{30.2}$$

This expression says the acceleration is directly proportional to the position, and opposite the direction of the displacement from equilibrium. We can see that the spring force tries to oppose the change in displacement. We call such a force a *restoring force*. This restoring force will make the mass go back and forth once we stretch the spring and let it go. It will oscillate. This is what we call simple harmonic motion.

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Restoring force: A force that is always directed toward the equilibrium position

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## 30.2 Mathematical Representation of Simple Harmonic Motion

Recall from your Principles of Physics class that acceleration is the second derivative of position

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Hook's Law tells us

$$F_{net} = ma = -k\left(x - x_{eq}\right)$$

where $x_{eq}$ it the equilibrium location. Suppose we define our coordinate system so that $x_{eq}$ is right at the origin. Then $x_{eq} = 0$ and we would have

$$\begin{aligned} F &= ma = -k\left(x - 0\right) \\ m\frac{d^2x}{dt^2} &= -kx \end{aligned}$$

We have differential equation! If you are taking this sophomore level physics class as a... well... sophomore, you may not have seen this kind of equation so much before. But really the chances are that you are a sophomore or junior (or even a senior) and have lot of experience with differential equations. And we have already seen an equation like this for LR circuits! The solution of this equation is a function or functions that will describe the motion of our mass-spring system as a function of time. We will need to know this function, so let's see how we can find it.

Start by defining a quantity $\omega$ as

$$\omega^2 = \frac{k}{m} \tag{30.3}$$

why define $\omega^2$? Because experience has shown that it is useful to define $\omega$ this way! But you probably remember $\omega$ as having to do with rotational speed, and from trigonometry you may remember using $\omega$ to mean angular frequency

$$\omega = 2\pi f$$

so our definition of $\omega$ may hint that $k/m$ will have something to do with the frequency of oscillation of the mass-spring system.

We can write our differential equation as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{30.4}$$

To solve this differential equation we need a function who's second derivative is the negative of itself. We know a few of these

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \quad (30.5) \\ x(t) &= A\sin(\omega t + \phi_o) \end{aligned}$$

where $A$, $\omega$, and $\phi_o$ are constants that we must find. Let's choose the cosine function and explicitly take its derivatives.

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ \frac{dx(t)}{dt} &= -\omega A\sin(\omega t + \phi_o) \\ \frac{d^2x(t)}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

Let's substitute these expressions into our differential equation for the motion

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ -\omega^2 A\cos(\omega t + \phi_o) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

As long as the constant $\omega^2$ is our $\omega^2 = k/m$ we have a solution (now you know why we defined it as $\omega^2$!). Since from trig we remember $\omega$ as the angular frequency.

$$\omega = 2\pi f$$

Thus

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f \tag{30.6}$$

The frequency of oscillation depends on the mass and the stiffness of the spring.

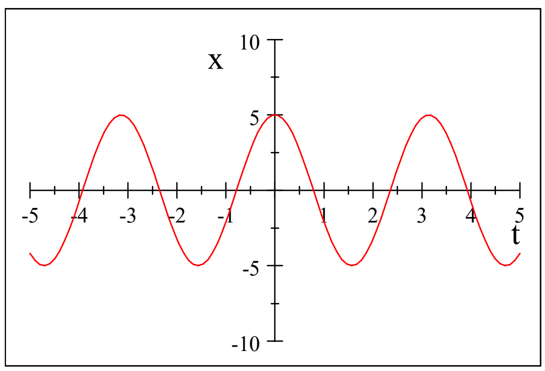
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{30.7}$$

Let's see if this is reasonable. Imagine driving along in your student car (say, a 1972 Gremlin). You go over a bump, and the car oscillates. Your car is a mass, and your shock absorbers are springs. You have an oscillation. But suppose you load your car with everyone in your apartment[2]. Now as you hit the bump the car oscillates at a different frequency, a lower frequency. That is what our frequency equation tells us. Note also that if we changed to a different set of shocks, the $k$ would change, and we would get a different frequency.

We still don't have a complete solution to our differential equation, because we don't know $A$ and $\phi_o$. From trigonometry, we recognize $\phi_o$ as the initial phase angle. We will call it the *phase constant* in this class. We will have to find this by knowing the initial conditions of the motion. We will do this in the paragraphs that follow.

$A$ is the amplitude. We can find its value when the motion has reached its maximum displacement. Let's look at a specific case

$$A = 5$$
$$\phi_o = 0$$
$$\omega = 2$$




We can see that the amplitude $A$ corresponds to the maximum displacement $x_{\max}$.

### 30.2.1 Other useful quantities we can identify

We know from trigonometry that a cosine function has a period $T$.

The period is related to the frequency

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{30.8}$$

We can write the period and frequency in terms of our mass and spring constant

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{30.9}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{30.10}$$

[2] If you are married, imagin taking two other couples with you in your car.

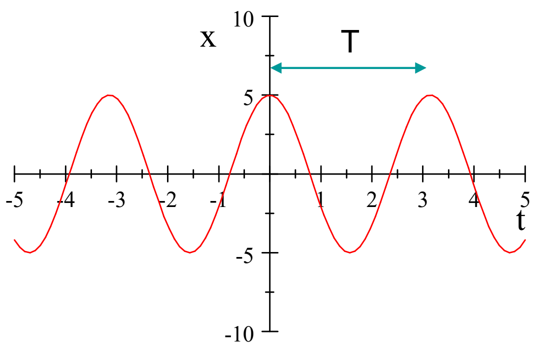



Figure 30.1:

### 30.2.2 Velocity and Acceleration

Since we know the derivatives of

$$x(t) = A\cos(\omega t + \phi_o) \tag{30.11}$$

we can identify the instantaneous velocity of the mass and its acceleration

$$v(t) = \frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o)$$

Recall that $A = x_{\max}$

$$v(t) = \frac{dx(t)}{dt} = -\omega x_{\max} \sin(\omega t + \phi_o) \tag{30.12}$$

We identify

$$v_{\max} = \omega x_{\max} = x_{\max}\sqrt{\frac{k}{m}} \tag{30.13}$$

Likewise for the acceleration

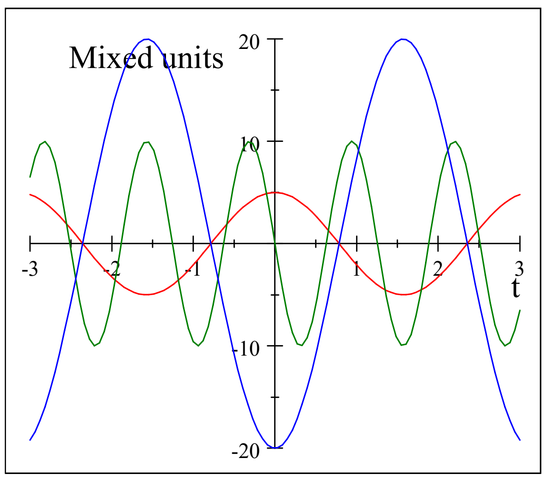
$$\begin{aligned} a(t) &= \frac{dv(t)}{dt} && (30.14) \\ &= \frac{d}{dt}(-\omega x_{\max}\sin(\omega t + \phi_o)) && (30.15) \\ &= -\omega^2 x_{\max}\cos(\omega t + \phi_o) \end{aligned}$$

where we can identify

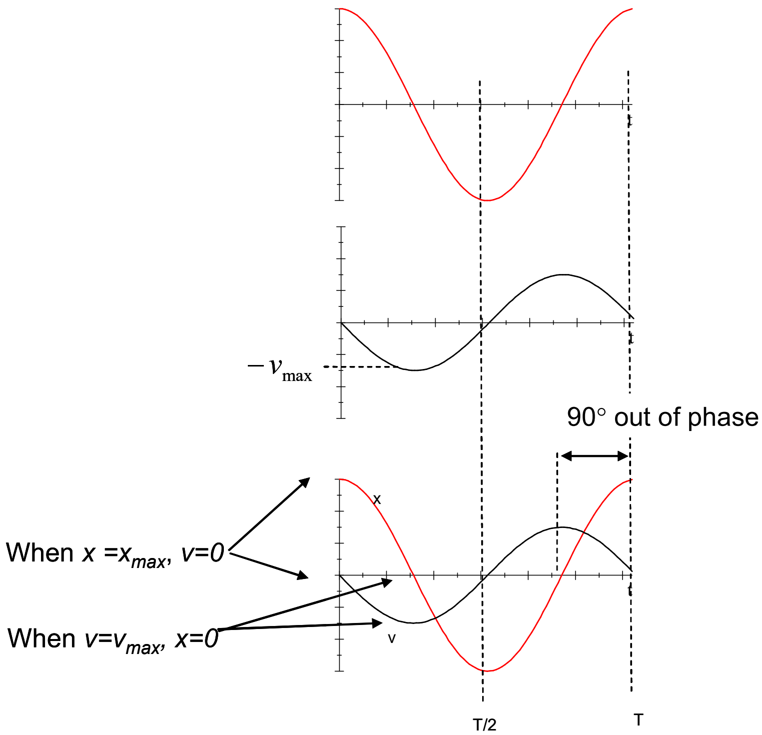
$$a_{\max} = \omega^2 x_{\max} = \frac{k}{m} x_{\max} \tag{30.16}$$

### 30.2.3 Comparison of position, velocity, acceleration

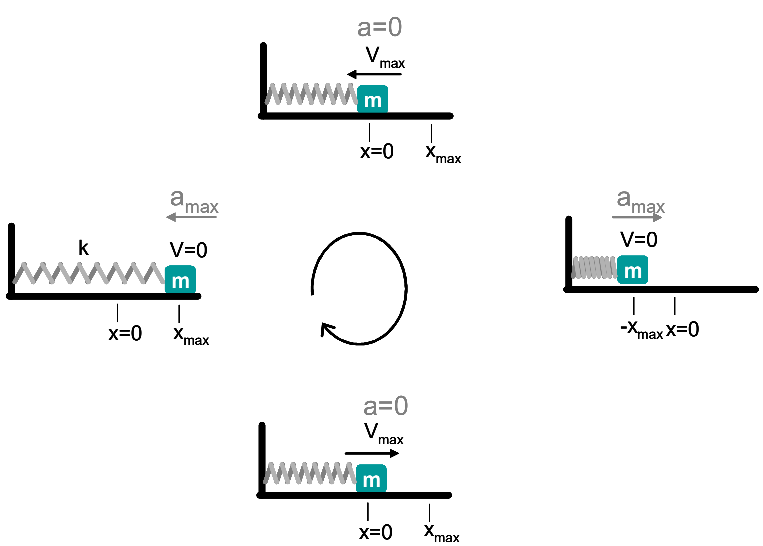
Let's plot $x(t)$, $v(t)$, and $a(t)$ for a specific case




Red is the displacement, green is the velocity, and blue is the acceleration. Note that each has a different maximum amplitude. That is a bit confusing until we recognize that they each have different units. We have just plotted them on the same graph to make it easy to compare their shapes. Note that the peaks are not in the same places! We use a phrase to say the peaks happen at different times. We say they are *not in phase*.




The acceleration is 90 ° out of phase from the velocity.



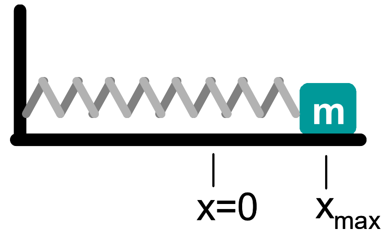



Figure 30.2:

## 30.3 An example of oscillation

We want to see how to find $A$, $\omega$, and especially $\phi_o$. These quantities will be important in our study of waves. So let's do a problem.

Let's take as our system a horizontal mass-spring system where the mass is on a frictionless surface.

### Initial Conditions

Now let's find $A$ and $\phi_o$. To do this we need to know how we started the mass-spring motion. We call the information on how the system starts it's motion the *initial conditions.*

Suppose we start the motion by pulling the mass to $x = x_{\max}$ and releasing it at $t = 0$. These our our initial conditions. Let's see if we can find the phase. Our initial conditions require

$$\begin{aligned} x(0) &= x_{\max} \\ v(0) &= 0 \end{aligned} \tag{30.17}$$

and we have our new equations for simple harmonic motion

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t + \phi_o) \end{aligned}$$

Using our formula for $x(t)$ and $v(t)$ we have

$$\begin{aligned} x(0) &= x_{\max} = x_{\max}\cos(0 + \phi_o) \\ v(0) &= 0 = -v_{\max}\sin(0 + \phi_o) \end{aligned} \tag{30.18}$$

From the first equation we get

$$1 = \cos(\phi_o)$$

which is true if

$$\phi_o = 0, 2\pi, 4\pi, \cdots$$

from the second equation we have

$$0 = \sin \phi_o$$

which is true for

$$\phi_o = 0, \pi, 2\pi, \cdots$$

If we choose $\phi_o = 0$, these conditions are both met. Of course we could choose $\phi_o = 2\pi$, or $\phi_o = 4\pi$, but we will follow the rule to take the smallest value for $\phi_o$ that meets the initial conditions.

### 30.3.1 A second example

Using the same equipment, let's start with

$$\begin{aligned} x(0) &= 0 \\ v(0) &= +v_i \end{aligned} \tag{30.19}$$

that is, the mass is moving, and we start watching just as it passes the equilibrium point. And once again we have our simple harmonic motion equations

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max} \sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t + \phi_o) \end{aligned}$$

we can use our zeros and substitute in $v_i$

$$\begin{aligned} x(0) &= 0 = x_{\max} \cos(0 + \phi_o) \\ v(0) &= v_i = -v_{\max} \sin(0 + \phi_o) \end{aligned} \tag{30.20}$$

Then, from the first equation

$$0 = x_{\max} \cos(\phi_o)$$

We see that[3]

$$\phi_o = \pm \frac{\pi}{2}$$

but we don't know the sign. Using our initial velocity condition

$$\begin{aligned} v_i &= -v_{\max} \sin\left(\pm\frac{\pi}{2}\right) \\ v_i &= -\omega x_{\max} \sin\left(\pm\frac{\pi}{2}\right) \end{aligned}$$

We defined the initial velocity as positive, and we insist on having positive amplitudes, so $x_{\max}$ is positive. Thus we need a minus sign from $\sin(\phi_o)$ to make $v_i$ positive. This tells us to choose

$$\phi_o = -\frac{\pi}{2}$$

with a minus sign.

[3] Really there are more possibilities, but we are taking the smallest value for $\phi_o$ as we discussed above.

Our solutions are

$$x(t) = \frac{v_i}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$v(t) = v_i \sin\left(\omega t - \frac{\pi}{2}\right)$$

Generally to have a complete solution to a differential equation, you must find all the constants (like $A$ and $\phi_o$) based on the initial conditions.

### 30.3.2 A third example

So far we have concentrated on finding $\phi_o$. Let's do a more complete example where we find $\phi_o$, $A$, and $\omega$.

A particle moving along the $x$ axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is $4.00\,\mathrm{cm}$, and the frequency is $1.50\,\mathrm{Hz}$.

a) show that the position of the particle is given by

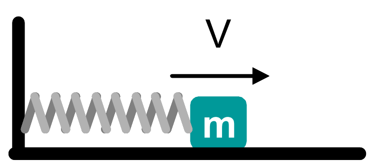
$$x = (4.00\,\mathrm{cm})\sin(3.00\pi t)$$

determine

b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed,

c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration, and

d) the total distance traveled between $t = 0$ and $t = 1.00\,\mathrm{s}$




**Type of problem**

We can recognize this as an oscillation problem. This leads us to a set of basic equations

**Basic Equations**

$$x(t) = A\cos(\omega t + \phi_o)$$
$$v(t) = -\omega x_{\max}\sin(\omega t + \phi_o)$$
$$a(t) = -\omega^2 A\cos(\omega t + \phi_o)$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

We should write down what we know and give a set of variables

**Variables**

| | | |
|---|---|---|
| $t$ | time, initial time =0 | $t_o = 0$ |
| $x$ | Position, Initial position =0 | $x(0) = 0$ |
| $v$ | | |
| $a$ | | |
| $x_m$ | $x$ amplitude | $x_m = 4.00\,\mathrm{cm}$ |
| $v_m$ | $v$ amplitude | |
| $a_m$ | $a$ amplitude | |
| $\omega$ | angular frequency | |
| $\phi_o$ | phase | |
| $f$ | frequency | $f = 1.50\,\mathrm{Hz}$ |

Now we are ready to start solving the problem. Of course we do this with algebraic symbols first

**Symbolic Solution**

**Part (a)**

We can start by recognizing that we can find $\omega$ because we know the frequency. We just use the basic equation.

$$\omega = 2\pi f$$

We also know the amplitude $A = x_{\max}$ which is given. Knowing that

$$x(0) = 0 = A\cos(0 + \phi_o)$$

we can guess that

$$\phi_o = \pm\frac{\pi}{2}$$

Using

$$v(0) = -\omega x_{\max} \sin\left(0 \pm \frac{\pi}{2}\right)$$

again and demanding that amplitudes be positive values, and noting that at $t = 0$ the velocity is positive from the initial conditions:

$$\phi_o = -\frac{\pi}{2}$$

We also note from trigonometry that

$$x(t) = x_{\max} \cos\left(2\pi f t - \frac{\pi}{2}\right)$$

which is a perfectly good answer. However, if we remember our trig, we could write this using

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

Then we have

$$\begin{aligned} x(t) &= x_{\max} \cos\left(2\pi f t - \frac{\pi}{2}\right) \\ &= x_{\max} \sin(2\pi f t) \end{aligned}$$

**Part (b)**

We have a basic equation for $v_{\max}$

$$\begin{aligned} v_m &= \omega x_{\max} \\ &= 2\pi f x_{\max} \end{aligned}$$

to find when this happens, take

$$v(t) = v_{\max} = -\omega x_{\max} \sin\left(2\pi f t - \frac{\pi}{2}\right)$$

and recognize that $\sin(\theta) = 1$ is at a maximum when $\theta = \pi/2$ so the entire argument of the sine function must be $\pi/2$ when we are at the maximum displacement, so

$$\frac{\pi}{2} = \left(2\pi f t - \frac{\pi}{2}\right)$$

or

$$\pi = 2\pi f t$$

then the time is

$$\frac{1}{2f} = t$$

**Part (c)**

Like with the velocity we must use a basic formula, this time

$$a(t) = -\omega^2 A \cos(\omega t + \phi_o)$$

but recognize that the maximum is achieved when $\cos(\omega t + \phi_o) = 1$ or when $\omega t + \phi_o = 0$

$$\begin{aligned} t &= \frac{\phi_o}{\omega} \\ &= \frac{-\frac{\pi}{2}}{2\pi f} \\ &= \frac{-1}{4f} \end{aligned}$$

The formula for $a_{\max}$ is

$$\begin{aligned} a_{\max} &= -\omega^2 x_{\max} \\ &= -(2\pi f)^2 x_m \end{aligned}$$

**Part (d)**

We know the period is

$$T = \frac{1}{f}$$

We should find the number of periods in $t = 1.00\,\mathrm{s}$

$$N_{periods} = \frac{t}{T}$$

and find the distance traveled in one period, and multiply them together. In one period the distance traveled is

$$d = 4x_m$$

$$d_{tot} = d * \frac{t}{T} = 4f x_m t$$

**Numerical Solutions**

We found algebraic answers (or symbolic answers) to the parts of our problem above. We will always do this first. Then substitute in the numbers to find numeric answers.

**Part (a)**

$$\begin{aligned} x(t) &= x_{\max} \sin(2\pi f t) \\ &= (4.00\,\mathrm{cm}) \sin(3.00\pi t) \end{aligned}$$

**Part (b)**

$$\begin{aligned} v_m &= 2\pi\,(1.50\,\mathrm{Hz})\,(4.00\,\mathrm{cm}) \\ & \quad 0.377\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

$$\frac{1}{2f} = t$$

$$\begin{aligned} \frac{1}{2\,(1.50\,\mathrm{Hz})} &= t \\ &= 0.333\ \mathrm{s} \end{aligned}$$

**Part (c)**

$$
\begin{aligned}
t &= \frac{-1}{4f} \\
&= -0.166\,67\,\mathrm{s}
\end{aligned}
$$

$$
\begin{aligned}
a_{\max} &= (2\pi f)^2 x_m \\
&= (2\pi 1.5\,\mathrm{Hz})^2\,(4.00\,\mathrm{cm}) \\
&= 3.\,553\,1\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}
$$

**Part (d)**

$$
\begin{aligned}
d_{tot} &= 4 f x_m t \\
&= 4 \times 4.00\,\mathrm{cm} * 1.50\,\mathrm{Hz} * 1.00\,\mathrm{s} \\
&= 0.24\,\mathrm{m}
\end{aligned}
$$

We should make sure the units check. We put in units along the way, so we can be confident that they do.

We should also make sure our answers are reasonable. They seem to be something a mass on a spring could do.

## 30.4 Energy of the Simple Harmonic Oscillator

If there is motion, there is energy. We can find the energy in a harmonic oscillator. Let's start with kinetic energy. Recall that

$$K = \frac{1}{2} m v^2$$

For our Simple Harmonic Oscillator (SHO) we have

$$
\begin{aligned}
K &= \frac{1}{2} m \left(-\omega x_{\max} \sin\left(\omega t + \phi_o\right)\right)^2 \\
&= \frac{1}{2} m \omega^2 x_{\max}^2 \sin^2\left(\omega t + \phi_o\right) \\
&= \frac{1}{2} m \frac{k}{m} x_{\max}^2 \sin^2\left(\omega t + \phi_o\right) \\
&= \frac{1}{2} k x_{\max}^2 \sin^2\left(\omega t + \phi_o\right)
\end{aligned}
$$

The potential energy due to a spring is given by (from your Principles of Physics I)

$$U = \frac{1}{2} k x^2 \tag{30.21}$$

Again for our SHO we have

$$U = \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi_o) \tag{30.22}$$

The total energy is given by

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi_o) + \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi_o) \\ &= \frac{1}{2}kx_{\max}^2 \left(\sin^2(\omega t + \phi_o) + \cos^2(\omega t + \phi_o)\right) \\ &= \frac{1}{2}kx_{\max}^2 \end{aligned} \tag{30.23}$$

The amount of energy at any given time is equal to the amount of energy we started with. We are not changing how much energy we have.

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The total mechanical energy of a SHO is a constant of motion

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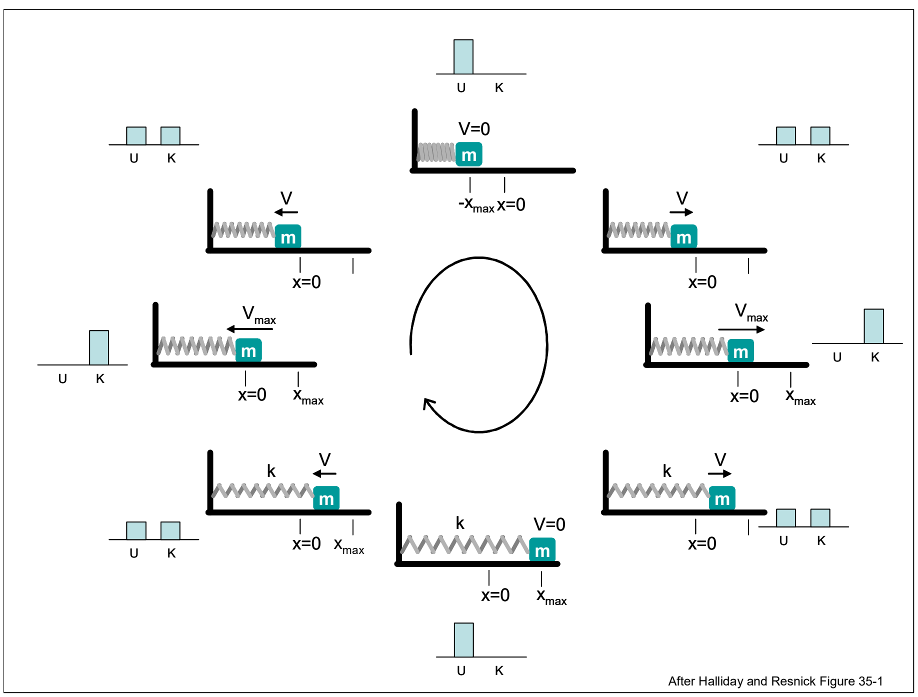



In the figure you can see that the kinetic and potential energies trade back and forth, but the total amount of energy does not change. Note that the kinetic

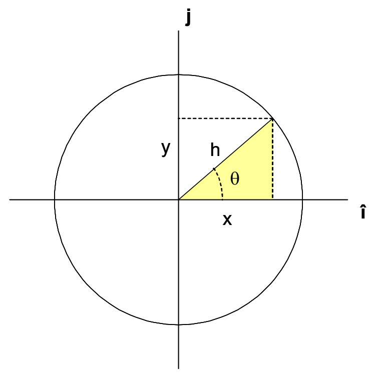
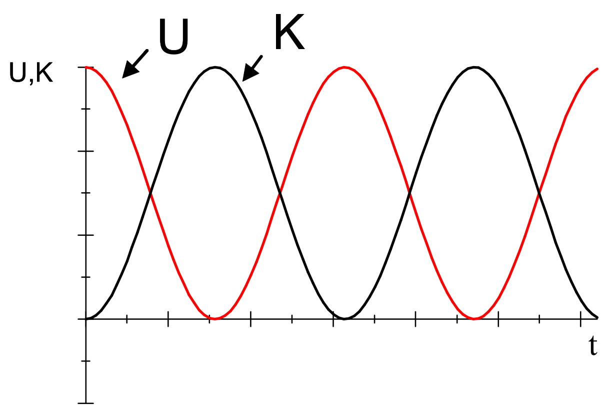



Figure 30.3:

and potential energy are out of phase with each other. If we plot them on the same scale ( for the case $\phi_o = 0$) we have




## 30.5 Circular Motion and SHM

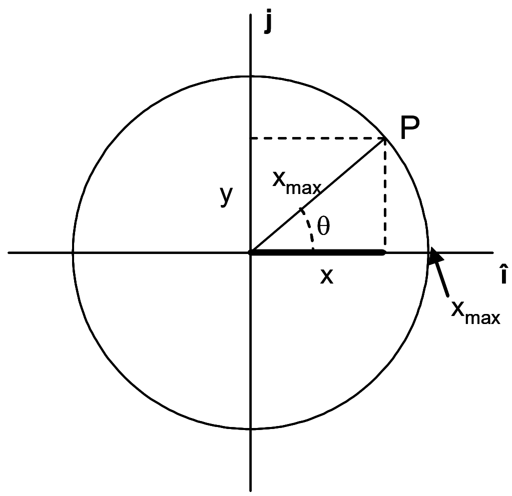
That circular motion and SHM are related should not be a surprise once we found the solutions to the equations of motion were trig functions. Recall that the trig functions are defined on a unit circle

$$\tan\theta = \frac{x}{y} \tag{30.24}$$

$$\cos\theta = \frac{x}{h} \tag{30.25}$$

$$\sin\theta = \frac{y}{h} \tag{30.26}$$

Let's relate this to our equations of motion.




Look at the projection $x$ of the point $P$ on the $x$ axis. Lets follow this projection as $P$ travels around the circle. We find it ranges from $-x_{\mathrm{max}}$ to $x_{\mathrm{max}}$. If we watch closely we find its velocity is zero at the extreme points and is a maximum in the middle. This projection is given as the cosine of the vector from the origin to $P$. This model, indeed fits our SHO solution.

Now lets define a projection of $P$ onto the $y$ axis. Again we have SHM, but this time the projection is a sin function. Because

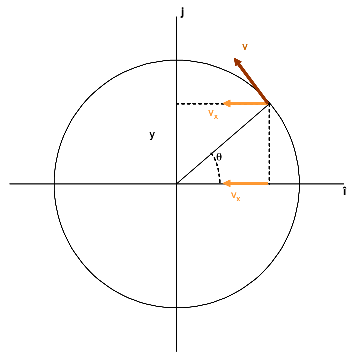
$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta) \tag{30.27}$$

we can see that this is just a SHO that is $90^\circ$ out of phase.

Uniform circular motion can be thought of as the combination of two SHOs, with a phase difference of $90^\circ$.

The angular velocity is given by

$$\omega = \frac{v}{r} \tag{30.28}$$




A particle traveling on the $x$-axis in SHM will travel from $x_{\max}$ to $-x_{\max}$ and from $-x_{\max}$ to $x_{\max}$ (one complete period, $T$) while the particle traveling with $P$ makes one complete revolution. Thus, the angular frequency $\omega$ of the SHO and the angular velocity of the particle at $P$ are the same. (Now we know why we used the same symbol). The magnitude of the velocity is then

$$v = \omega r = \omega x_{\max} \tag{30.29}$$

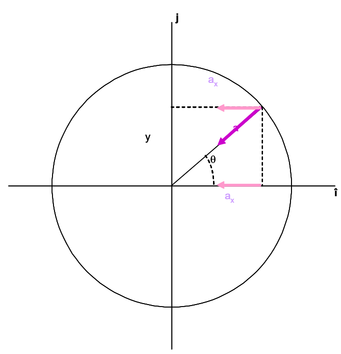
and the projection of this velocity onto the $x$-axis is

$$v_x = -\omega x_{\max} \sin(\omega t + \phi_o) \tag{30.30}$$

Just what we expected!

The angular acceleration of a particle at $P$ is given by

$$\frac{v^2}{r} = \frac{v^2}{x_{\max}} = \frac{\omega^2 x_{\max}^2}{x_{\max}} = \omega^2 x_{\max} \tag{30.31}$$

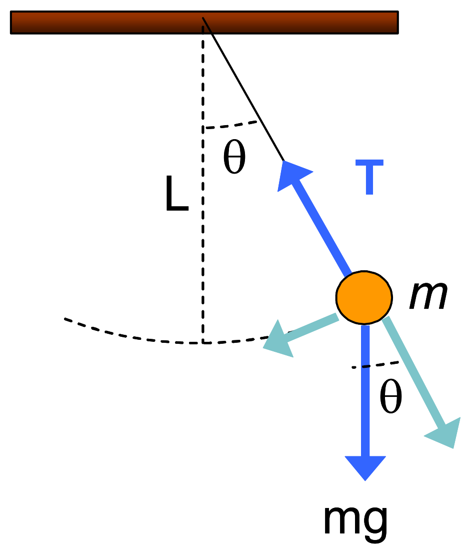



The direction of the acceleration is inward toward the origin. If we project this onto the $x$-axis we have

$$a_x = -\omega^2 x_{\max} \cos(\omega t + \phi_o) \tag{30.32}$$

We get the same equations of motion for the projection of circular motion onto the $x$-axis.

## 30.6 The Pendulum




A simple pendulum is a mass on a string. The mass is called a "bob."

A simple pendulum exhibits periodic motion, but not exactly simple harmonic motion. The forces on the bob, $m$, are the gravitational force, $F_g = \mathbf{W}$, $\mathbf{T}$ the tension on the string. The tangential component of $W$ is always directed toward $\theta = 0$. This is a restoring force! Let's call the path the bob takes $s$. The path, $s$, is along an arc, then we can use the arc-length formula to describe $s$

$$s = L\theta \tag{10.01a}$$

and we can write an equation for the restoring force that brings the bob back to its equilibrium position as

$$\begin{aligned} W_t &= -mg\sin\theta \\ &= m\frac{d^2s}{dt^2} \\ &= mL\frac{d^2\theta}{dt^2} \end{aligned} \tag{30.33}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is a harder differential equation to solve. But suppose we are building a grandfather clock with our pendulum, and we won't let the pendulum swing very far. Then we can take $\theta$ as a very small angle, then

$$\sin(\theta) \approx \theta \tag{30.34}$$

In this approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

and we have a differential equation we recognize! Compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{30.35}$$

if

$$\omega^2 = \frac{g}{L} \tag{30.36}$$

we have all the same solutions for $s$ that we found for $x$. Since $\omega$ changed, the frequency and period will now be in terms of $g$ and $L$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{30.37}$$

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The period and frequency for a pendulum with small angular displacements depend only on $L$ and $g$!

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We have learned a lot about springs and oscillation in a hurry. In our next lecture we will consider what would happen if we didn't have a frictionless surface for our mass.

# Basic Equations

$$m\frac{d^2x}{dt^2} = -kx$$

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

For springs

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$K = \frac{1}{2}kx_{\max}^2\sin^2(\omega t + \phi_o)$$

$$U = \frac{1}{2}kx_{\max}^2\cos^2(\omega t + \phi_o)$$

For pendula

$$\omega^2 = \frac{g}{L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$