Chapter 31

Damping, Resonance, and Waves

In our last lecture we talked about oscillators, but made the approximation that there was no friction. In real systems there is nearly always some kind of friction or dissipative force. In this lecture we will look at the consequences of friction like forces on oscillation and then consider adding a force to an oscillator. We will end by considering our oscillator as a source or disturbance that makes a wave.

Fundamental Concepts

1. A wave requires a disturbance, and a medium that can transfer energy
2. Waves are categorized as longitudinal or transverse (or a combination of the two).

31.1 Damped Oscillations

Last lecture we found from Newton's second law that the force due to a spring can cause oscillations. We turned Newton's second law

$$F_{net} = ma = -k\left(x - 0\right)$$

into a differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad (31.1)$$

where

$$\omega^2 = \frac{k}{m}$$

The solution to this equation was another equation

$$x(t) = A\cos(\omega t + \phi_o) \tag{31.2}$$

But notice in our Newton's second law we only had the spring force. There was no friction force. What if we did have a friction like force, like,

$$\mathbf{F}_d = -b\mathbf{v} \tag{31.3}$$

to our Newton's second law! This force is proportional to the velocity. This is typical of viscus fluids. So this is what we would get if we place our mass-spring system (or pendulum) in air or some other fluid. We call b the damping coefficient. Now our net force is

$$\Sigma F = -kx - bv_x = ma$$

We can write the acceleration and velocity as derivatives of the position

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is another differential equation. It is harder to guess its solution. But this is what it looks like

$$x(t) = x_{\max}e^{-\frac{b}{2m}t}\cos(\omega t + \phi_o) \tag{31.4}$$

which looks a lot like our frictionless solution. But now our angular frequency, ω, is more complicated

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{31.5}$$

We have three cases:

1. The damping force is small: ($b < \sqrt{4mk}$) The system oscillates, but the amplitude is smaller as as time goes on. We call this "underdamped"
2. When ($b = \sqrt{4mk}$) the system is critically damped (see below)
3. The damping force is large: ($b > \sqrt{4mk}$)The system does not oscillate. we call this "overdamped." We can also say that $\frac{b}{2m} > \omega_o$ (after we define ω_o below)

Let's see if we can see why these definitions make sense. If b is small then $\frac{k}{m} - \left(\frac{b}{2m}\right)^2 \approx \frac{k}{m}$ and our ω would look just like when there is no damping force. If b is a little bigger, then our freque4ncy changes a little, getting smaller as b gets bigger. At some point, b is big enough so $\frac{k}{m} - \left(\frac{b}{2m}\right)^2 = 0$. Then $\omega = 0$ and there is no oscillation. This happens when

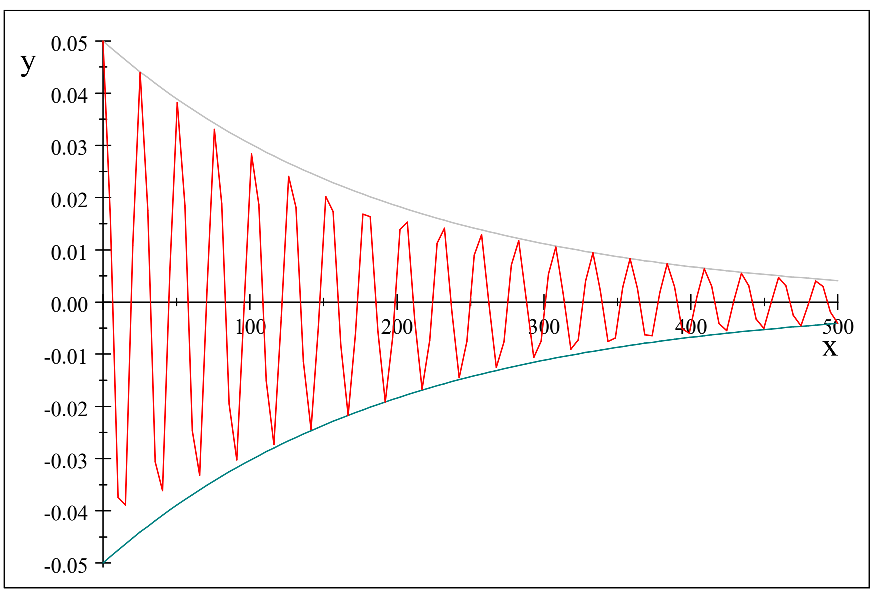
$$\begin{aligned} \frac{k}{m} - \left(\frac{b}{2m}\right)^2 &= 0 \\ \frac{k}{m} &= \left(\frac{b}{2m}\right)^2 \\ \sqrt{\frac{k}{m}} &= \frac{b}{2m} \\ b &= 2m\sqrt{\frac{k}{m}} \\ b &= \sqrt{4mk} \end{aligned}$$

At this b, there is no oscillation. A higher b would mean ω is negative, which can't be true. So with higher b there really really is no oscillation.

Let's plot our function for damped oscillation. For the following values,

$$\begin{aligned} &x_{\max} = 5\,\mathrm{cm} \\ &b = 0.005\,\tfrac{\mathrm{kg}}{\mathrm{s}} \\ &k = .5\,\tfrac{\mathrm{N}}{\mathrm{m}} \\ &m = .5\,\mathrm{kg} \end{aligned}$$

we have a graph that looks like this



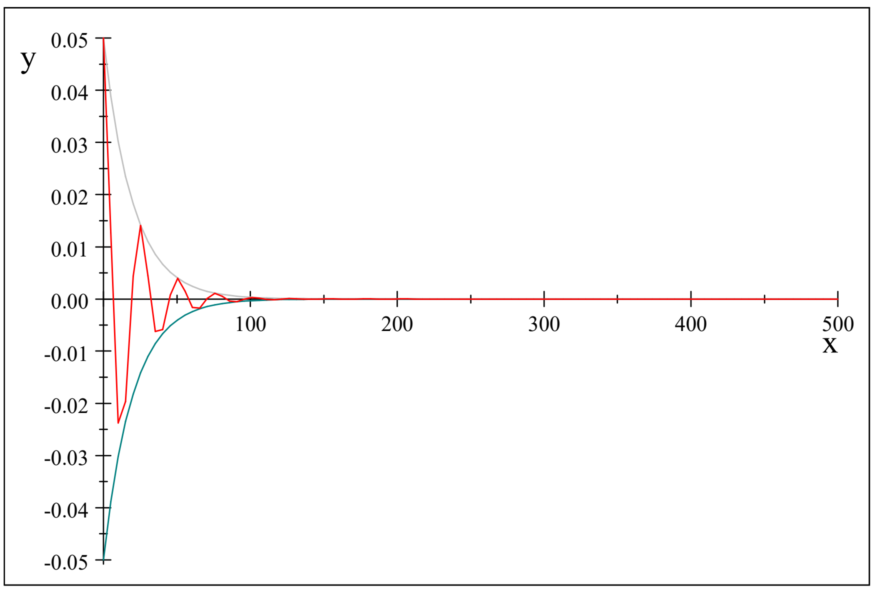
The gray lines are

$$\pm x_{\max} e^{-\frac{b}{2m}t} \tag{31.6}$$

They describe how the amplitude changes. We call this the *envelope* of the curve. Note that the entire $x_{\max}e^{-\frac{b}{2m}t}$ is now our amplitude. And that amplitude changes.

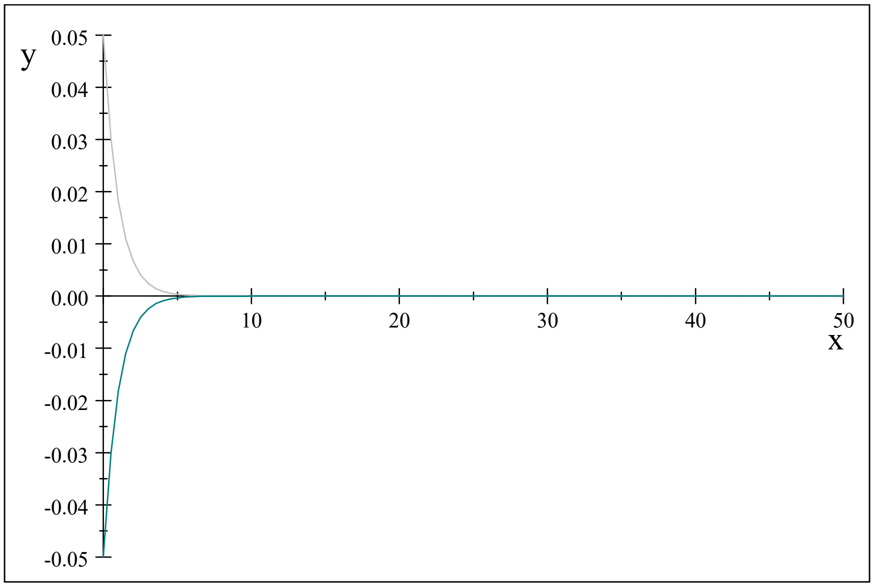
Let's try another case with larger b

$$x_{\max} = 5\,\mathrm{cm}$$
$$b = 0.05\,\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k = .5\,\frac{\mathrm{N}}{\mathrm{m}}$$
$$m = .5\,\mathrm{kg}$$



We have less oscillation. Let's try another with an even higher b

$$A = 5\,\mathrm{cm}$$
$$b = 0.5\,\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k = .5\,\frac{\mathrm{N}}{\mathrm{m}}$$
$$m = .5\,\mathrm{kg}$$



What happened to our oscillation? We already know that when the damping force gets bigger, the oscillation eventually stops. Only the exponential decay is observed. We said before that the oscillation stops when we are critically damped. This happens when

$$\frac{b}{2m} = \sqrt{\frac{k}{m}} \tag{31.7}$$

then

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0 \tag{31.8}$$

We define

$$\omega_o = \sqrt{\frac{k}{m}} \tag{31.9}$$

as the *natural frequency* of the system. Then the value of b that gives us critically damped behavior is

$$b_c = 2m\omega_o \tag{31.10}$$

Driven Oscillations and Resonance

We found in the last section that if we added a force like

$$\mathbf{F}_d = -b\mathbf{v} \tag{31.11}$$

our oscillation died out. Suppose we want to keep it going? Let's apply a periodic force like

$$F(t) = F_o \sin(\omega_f t)$$

where ω_f is the angular frequency of this new driving force and where F_o is a constant. Our Newton's second law now looks like this

$$\Sigma F = F_o \sin(\omega_f t) - kx - bv_x = ma$$

When this system starts out, the solutions is very messy. It is so messy that we will not give it in this class! But after a while, a steady-state is reached. In this state, the energy added by our driving force $F_o \sin(\omega_f t)$ is equal to the energy lost by the drag force, and we have

$$x(t) = A\cos(\omega_f t + \phi_o) \tag{31.12}$$

our old friend! BUT NOW

$$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2 - \omega_o^2\right)^2 + \left(\frac{b\omega_f}{m}\right)^2}} \tag{31.13}$$

and where

$$\omega_o = \sqrt{\frac{k}{m}} \tag{31.14}$$

as before. It is more convenient to drop the f subscripts. Then our solution is

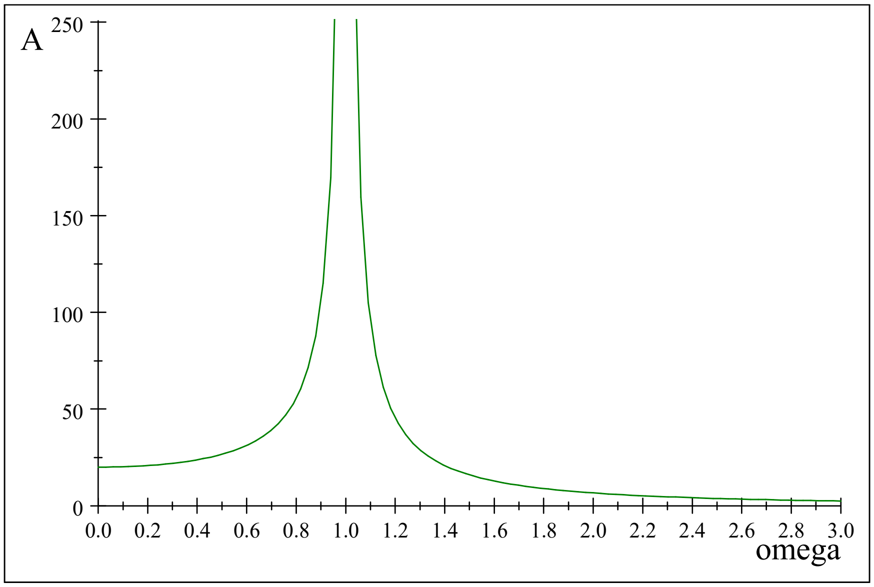
$$x(t) = A\cos(\omega t + \phi_o) \tag{31.15}$$

$$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega^2 - \omega_o^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}} \tag{31.16}$$

So, now our solution looks more like our original SHM solution (except for the wild formula for A).

Lets look at A for some values of ω. I will pick some nice numbers for the other values.

$$\begin{aligned}
F_o &= 2\,\mathrm{N} \\
b &= 0.5\,\tfrac{\mathrm{kg}}{\mathrm{s}} \\
k &= 0.5\,\tfrac{\mathrm{N}}{\mathrm{m}} \\
m &= 0.5\,\mathrm{kg}
\end{aligned}$$



now let's calculate ω_o

$$\begin{aligned}\omega_o &= \sqrt{\frac{0.5\,\frac{\mathrm{N}}{\mathrm{m}}}{0.5\,\mathrm{kg}}} \\ &= \frac{1.0}{\mathrm{s}}\end{aligned}$$

Notice that right at $\omega = \omega_o$ our solution gets very big. This is called *resonance*. To see why this happens, think of the velocity

$$\frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o) \tag{31.17}$$

note that our driving force is

$$F(t) = F_o \sin(\omega t) \tag{31.18}$$

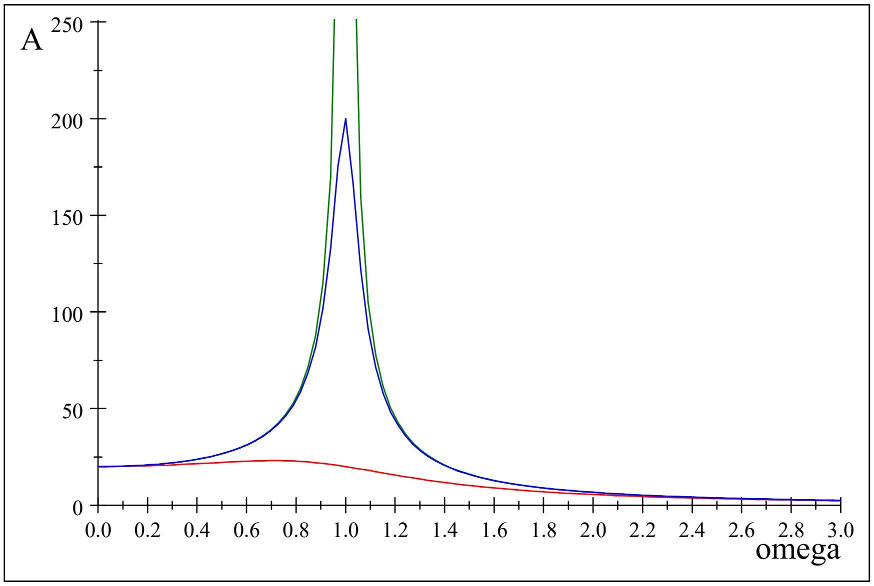
The rate at which work is done (power) is

$$\begin{aligned}\mathcal{P} &= \frac{w}{\Delta t} = \frac{\mathbf{F}\cdot\Delta\mathbf{x}}{\Delta t} && (31.19)\\ &= \mathbf{F}\cdot\mathbf{v} && (31.20)\\ &= -\left(F_o \sin(\omega t)\right)\left(\omega A \sin(\omega t + \phi_o)\right)\cos\theta && (31.21)\end{aligned}$$

if F and v are in phase, and in the same direction the power will be at a maximum!

$$\begin{aligned}\mathcal{P} &= \frac{w}{\Delta t} = \frac{\mathbf{F}\cdot\Delta\mathbf{x}}{\Delta t} && (31.22)\\ &= \mathbf{F}\cdot\mathbf{v} && (31.23)\\ &= -F_o \omega A \sin^2(\omega t) && (31.24)\end{aligned}$$

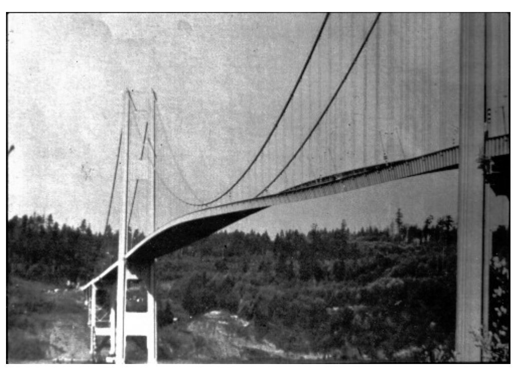
We can plot A for several values of b



Green: b=0.005kg/s; Blue: b=0.05kg/s; Red b=0.01 kg/s

As $b \rightarrow 0$ we see that our resonance peak gets larger. In real systems b can never be zero, but sometimes it can get small. As $b \rightarrow$large, the resonance dies down and our A gets small.

An example of this is well known to mechanical engineers. The next picture is of the Tacoma Narrows Bridge. As a steady wind blew across the bridge it formed turbulent wind gusts.



Tacoma Narrows Bridge (Image in the Public Domain)

The wind gusts formed a periodic driving force that allowed a driving harmonic oscillation to form. Since the bridge was resonant with the gust frequency, the amplitude grew until the bridge materials broke.

31.2 What is a Wave?

But we started out in this section of our class trying to understand electromagnetic waves. What is a wave? Waves are organized motions in a medium. Waves are a transfer of energy through space without transfer of matter.

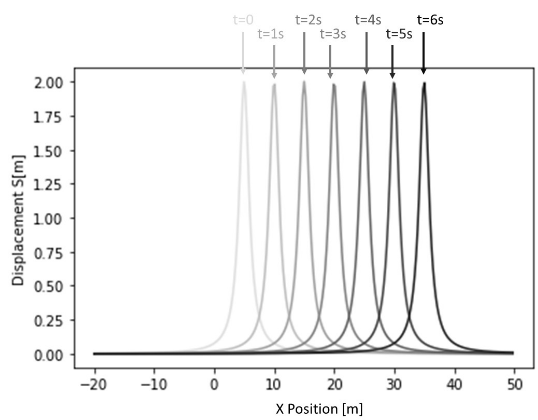
Waves require:

1. some source of disturbance
2. a medium that can be disturbed
3. some physical mechanism by which the elements of the medium can influence each other

Let's consider a wave on a string (like a guitar string). In the limit that the string mass is negligible we represent a one-dimensional wave mathematically as a function of two variables, position and time, $y(x,t)$. There are two ways to look at waves, we call them "snapshot" and "history" (or video) views. The snapshot view freezes time at some specific instant, t_o and gives us the physical shape of the wave at that time $y(x,t_o)$. The next graph is a series of snapshot views of the wave[1]

$$y = \frac{2}{(x-5t)^2+1}$$

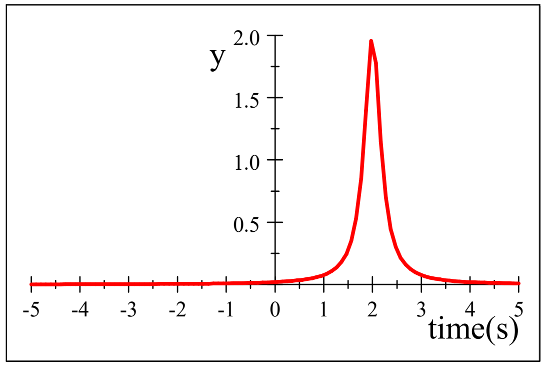
with $t = 0\,\mathrm{s}$, then $t = 1\,\mathrm{s}$ and then $t = 2\,\mathrm{s}$ up to $t = 6\,\mathrm{s}$.



The history view gives us a history of the wave at just one point x_o. It is like we have a movie of just the x_o part of the medium, $y(x_o,t)$. For our example wave we could pick $x_0 = 10\,\mathrm{m}$. Then our wave equation would be

$$y = \frac{2}{(10-5t)^2+1}$$

[1] This is not an important wave function, just one I picked because it makes a nice graphic example.



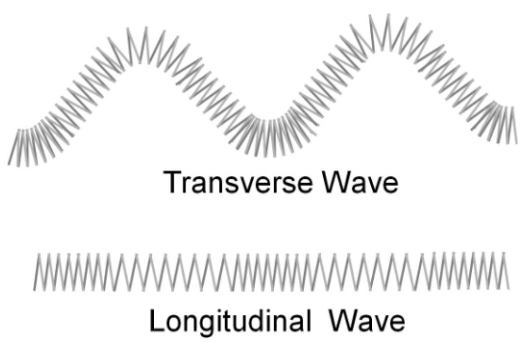
We can view this as though we were standing in the water on a beach when a wave crest went by. We would go up and own with the wave. This last graph is us going up and down as the wave passes. We will see more examples of these views in our next chapter.

31.2.1 Longitudinal vs. transverse

We divide the various kinds of waves that occur into two basic types:

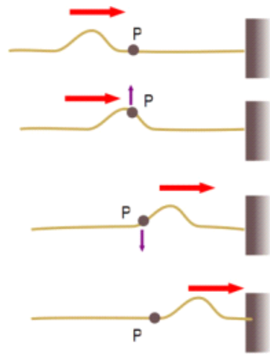
1. transverse wave: a traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation

2. Longitudinal wave: a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation



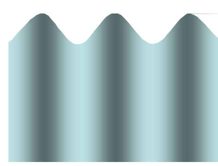
31.3 Examples of waves

Let's look at some common waves. We can start with a pulse on a rope.



In the picture above, you see a wave that has just one peak traveling to the right. We call such a wave a *pulse*. Notice how the piece of the rope marked P moves up and down, but the wave is moving to the right. This pulse is a transverse wave because the parts of the medium (observe point P) move perpendicular to the direction the wave is moving.

Of course, some waves are a combination of these two basic types[2]. Water waves, for example, are transverse at the surface of the water, but are longitudinal throughout the water.



Earthquakes produce both transverse and longitudinal waves. The two types of waves even travel at different speeds! P waves are longitudinal and travel faster, S waves are transverse and slower.

We know can make waves on strings (like guitar strings). We can perform an experiment with a rope or a long spring to see how such waves work. Suppose we make a wave on the rope or spring. Then pull the rope or spring tighter and make another wave. We see that the wave on the tighter spring travels faster.

It is harder to do, but we can also experiment with two different ropes, one light and one heavy. We would find that the heaver the rope, the slower the wave. We can express this as

$$v = \sqrt{\frac{T_s}{\mu}}$$

[2]You may have noticed that in Physics we tend to define basic types of things, and then use these basic types to define more complex objects.

where T_s is the tension in the rope, and μ is the linear mass density

$$\mu = \frac{m}{L}$$

where m is the mass of the rope, and L is the length.

The term μ might need an analogy to make it seem helpful. So suppose I have an iron bar that has a mass of 200 kg and is 2 m long. Further suppose I want to know how much mass there would be in a 20 cm section cut of the end of the rod. How would I find out?

This is not very hard, We could say that there are 200 kg spread out over 2 m, so each meter of rod has 100 kg of mass, that is, there is 100 kg/ m of mass per unit length. Then to find how much mass there is in a 0.20 m section of the rod I take

$$m = 100\frac{\text{kg}}{\text{m}} \times 0.20\,\text{m} = 20.0\,\text{kg}$$

The 100 kg/ m is μ. It is how much mass there is in a unit length segment of something In this example, it is a unit length of iron bar, but for waves on string, we want the mass per unit length of string.

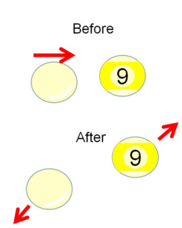
If you are buying stock steel bar, you might be able to buy it by specifying its mass per unit length. If the mass per unit length is higher then the bar is more massive for a given length. The same is true with string. The larger μ, the more massive equal string segments will be.

This works if the rope is horizontal, but what if the rope is vertical? Say we make a wave in a heavy cable that is suspended. The mass at the lower end of the cable pulls down on the upper part of the cable. The tension will actually change along the length of the cable, and so will the wave speed. Such a situation can't be represented by a single wave speed. But if the mass of the rope of cable isn't too large the change in wave speed won't be large and we can ignore it. We will do this for our class, we will assume that any such changes in tension along the rope are small enough to be ignored.

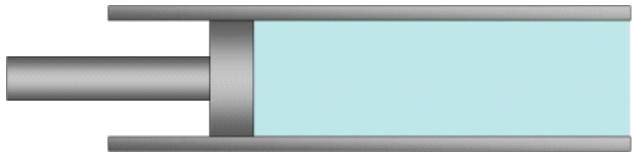
Sound is a wave, let's look at sound in more detail.

31.4 Sound

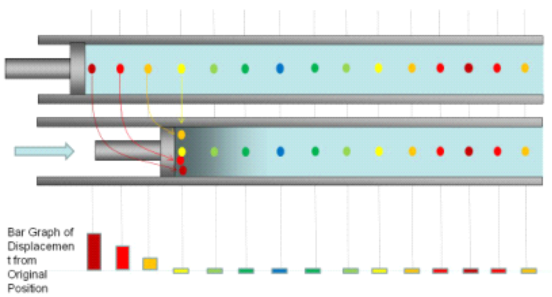
Sound is a wave. The medium is air particles. The transfer of energy is done by collision.



The wave will be a longitudinal wave. Let's see how it forms. We can take a tube with a piston in it.

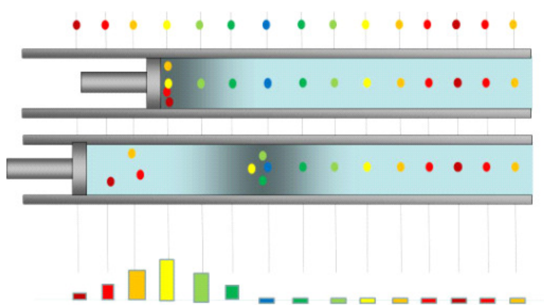


As we exert a force on the piston, the air molecules are compressed into a group. In the next figure, each dot represents a group of air molecules. In the top picture, the air molecules are not displaced. But when the piston moves, the air molecules receive energy by collision. They bunch up. We see this in the second picture.

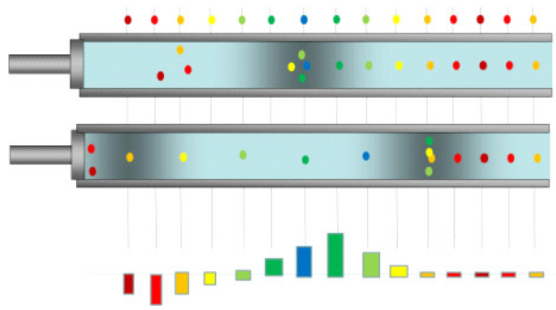


The graph below the two pictures shows how much displacement each molecule group experiences.

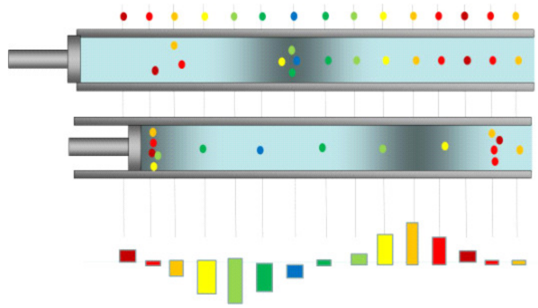
Suppose we now pull the piston back. This would allow the molecules to bounce back to the left, but the molecules that they have collided with will receive some energy and go to the right. This is shown in the next figure. Color coded dots are displayed above the before and after picture so you can see where the molecule groups started.



If we pull the piston back further, the molecules can pass their original positions.



Then we can push inward again and compress the gas.



This may seem like a senseless thing to do, but it is really what a speaker does to produce sound. In particular, a speaker is a harmonic oscillator. The simple harmonic motion of the speaker is the disturbance that makes the sound wave.

31.5 One dimensional waves

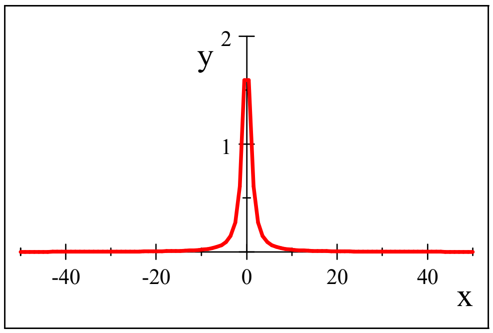
To mathematically describe a wave we will define a function of both time and position.

$$y(x,t) \tag{31.25}$$

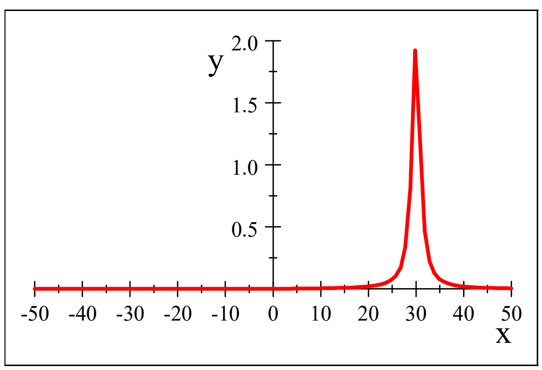
let's take a specific example[3]

$$y = \frac{2}{(x - 3.0t)^2 + 1} \tag{31.26}$$

Let's plot this for $t = 0$



what will this look like for $t = 10$?



The pulse travels along the x-axis as a function of time. We denote the speed of the pulse as v, then we can define a function

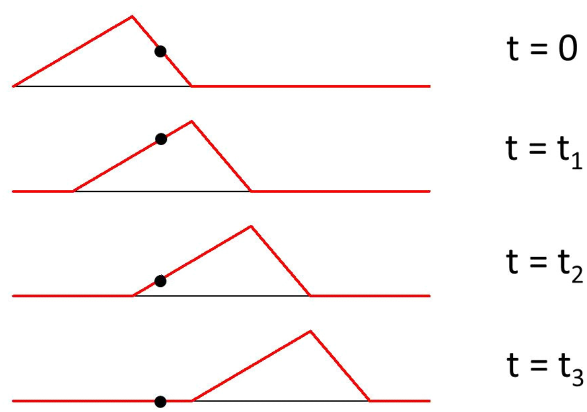
$$y(x, t) = y(x - vt, 0) \tag{31.27}$$

that describes a pulse as it travels. An element of the medium (rope, string, etc.) at position x at some time t, will have the displacement that an element had earlier at $x - vt$ when $t = 0$.

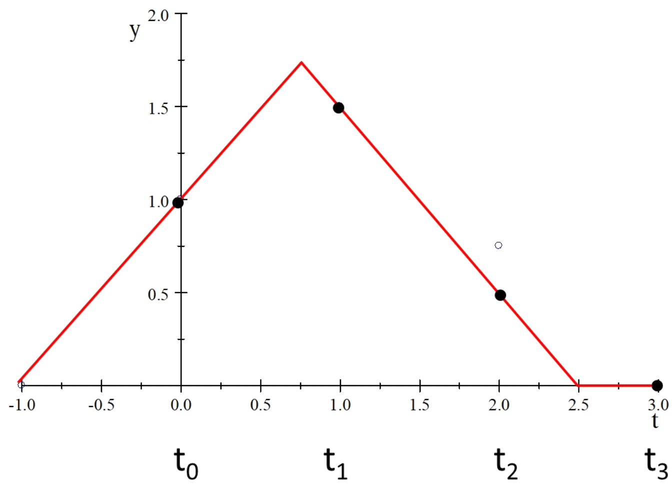
We will give $y(x - vt, 0)$ a special name, the *wave function*. It represents the y position, the transverse position in our example, of any element located at a position x at any time t.

[3] This is not an important wave function, just one I picked because it makes a nice graphic example.

Now that we have some math for waves, let's go back to our snapshot and history graphs. Notice that wave functions depend on two variables, x, and t. It is hard to draw a wave so that this dual dependence is clear. Often we draw two different graphs of the same wave so we can see independently the position and time dependence. So far we have used one of these graphs. A graph of our wave at a specific time, t_o. This gives $y(x, t_o)$. This representation of a wave is very like a photograph of the wave taken with a digital camera. It gives a picture of the entire wave, but only for one time, the time at which the photograph was taken. Of course we could take a series of photographs, but still each would be a picture of the wave at just one time.



The second representation is to observe the wave at just one point in the medium, but for many times. This is very like taking a video camera and using it to record the displacement of just one part of the medium for many times. You could envision marking just one part of a rope, and then using the video recorder to make a movie of the motion of that single part of the rope. We could then go frame by frame through the video, and plot the displacement of our marked part of the rope as a function of time. Such a graph is sometimes called a history graph of the wave.



Basic Equations

Damped Oscillation

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi_o)$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Forces Oscillation

$$x(t) = A\cos(\omega_f t + \phi_o)$$

$$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2 - \omega_o^2\right)^2 + \left(\frac{b\omega_f}{m}\right)^2}}$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

Speed of waves on string

$$v = \sqrt{\frac{T_s}{\mu}}$$

$$\mu = \frac{m}{L}$$