29 Electric potential Energy

Fundamental Concepts

- Gravitational potential energy of point masses and binding energy
- Electrical potential energy of point charges
- Electrical potential energy of dipoles

Point charge potential energy

As we said last lecture, we want to use gravitation as an analogy for the electric potential energy. Gravitation is more intuitive. But chances are gravitation of whole planets was not stressed in Dynamics (If you took PH121 you should be fine, and this will be a review). So let's take a few moments out of a PE101 class (introductory planetary engineering) and study non-uniform gravitational fields.

Gravitational analog

Question 223.29.1

Question 223.29.2

Long, long ago you studied the potential energy of objects in what we can now call the Earth's gravitational field.

The presentation of the idea of potential energy likely started with

$$U_g = mgy$$

where m is the mass of the object, g is the acceleration due to gravity, and y is how high the object is compared to a $y = 0$ point. If you recall, we got to pick that $y = 0$ point. It could be any height.

This all works fairly well so long as we take fairly small objects near the much larger Earth. But hopefully you also considered objects farther away from the Earth's surface, or larger objects like the moon. For these objects, mgy is not enough to describe the potential energy. The reason is that if we are far away from the center of the Earth we will notice that the Earth's gravitational field is not uniform. It curves and diminishes with distance. So, if an object is large, it will feel the change in the gravitational field over its (the object's) large volume.

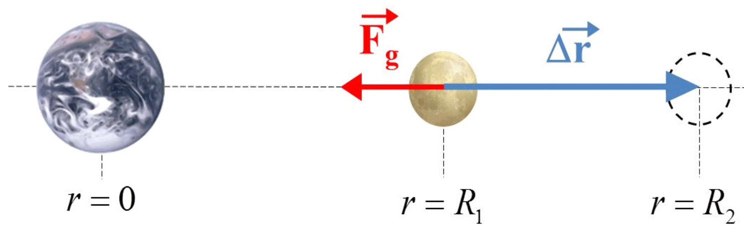
We have the tools to find the potential energy of this situation. We know that a change in potential energy is just an amount of work

$$\Delta U_g = -W_g = -\int \overrightarrow{\mathbf{F}}_g \cdot d\overrightarrow{\mathbf{r}}$$

The magnitude of the gravitational force is

$$F_g = G\frac{M_E m_m}{r_{Em}^2}$$

where M_E is the mass of the Earth, m_m is the mass of the mover object, and r_{Em} is the distance between the two. The constant, G, is the gravitational constant.



The field is radial, so $\overrightarrow{\mathbf{F}}_g \cdot d\overrightarrow{\mathbf{r}} = -F dr$ for the configuration we have shown, and we can perform the integration. Say we move the object a distance Δr away were

$$\Delta r = R_2 - R_1$$

and Δr is large, comparable to the size of the Earth or larger. Then

$$\begin{aligned} \Delta U_g &= -\int_{R_1}^{R_2} \left(-G\frac{M_E m_m}{r^2}\right) dr \\ &= G M_E m_m \int_{R_1}^{R_2} \frac{dr}{r^2} \end{aligned}$$

where R is the distance from the center of the Earth to the center of our object.

$$\begin{aligned}
\Delta U_g &= GM_E m_m \int_{R_1}^{R_2} \frac{dr}{r^2} \\
&= GM_E m_m \left[-\frac{1}{r} \right|_{R_1}^{R_2} \\
&= GM_E m_m \left[-\frac{1}{R_2} - \left(-\frac{1}{R_1} \right) \right] \\
&= -GM_E m_m \left[\frac{1}{R_2} - \frac{1}{R_1} \right] \\
&= -G\frac{M_E m_m}{R_2} + G\frac{M_E m_m}{R_1}
\end{aligned}$$

We recall that we need to set a zero point for the potential energy. Before, when we used the approximation $m_m g y$ we could choose $y = 0$ anywhere we wanted. But now we see an obvious choice for the zero point of the potential energy. If we let $R_2 \rightarrow \infty$ and then the first term in our expression will be zero. Likewise, of we let $R_1 \rightarrow \infty$ the second term will be zero. It looks like as we get infinitely far away from the Earth, the potential energy naturally goes to zero! Mathematically this makes sense. But we will have to interpret what this choice of zero point means.

But first, let's see how much work it would take to move the moon out of obit and move it farther away. Say, from $R_{1,}$ the present orbit radius, to $R_2 = 2R_1$, or twice the original orbit distance. Then

$$\begin{aligned}
\Delta U_g &= U_2 - U_1 = -G\frac{M_E m_m}{2R_1} + G\frac{M_E m_m}{R_1} \\
&= G\frac{M_E m_m}{R_1}\left(-\frac{1}{2} + 1 \right) \\
&= \left(\frac{1}{2} \right) G\frac{M_E m_m}{R_1}
\end{aligned}$$

The change is positive. We gained potential energy as we went farther from the Earth's surface. That makes sense! That is analogous to increasing y in mgy. The potential energy also gets larger if the mass of our object (like the moon or a satellite) gets larger. Again that makes sense because in our more familiar approximation the potential energy increases with mass. So this new form for our equation for potential energy seems to work.

But what does it mean that the potential energy is zero infinitely far away? Recall that a change in potential energy is an amount of work

$$W = -\Delta U$$

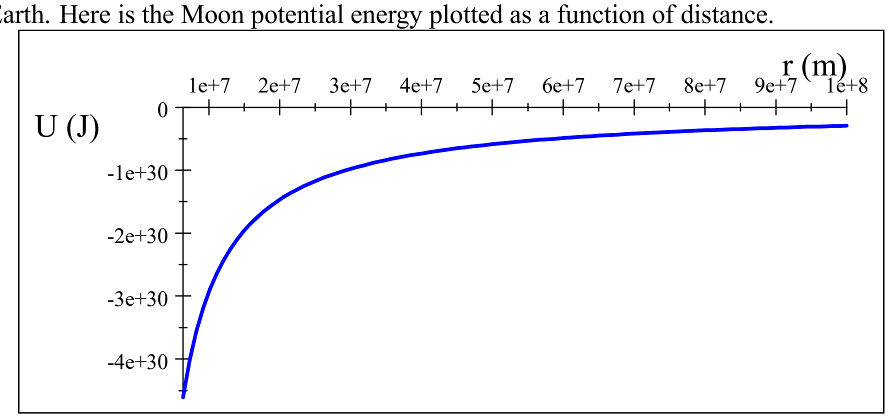
Usually we will consider the potential energy to be the amount of work it takes to bring the test mass m_m from infinitely far away (our zero point!) to the location where we want it. It is how much energy is stored by having the object in that position. Like how much energy is stored by putting a mass high on a shelf. For example we could bring the moon in from infinitely far away. Then

$$\begin{aligned}\Delta U_g &= U_2 - U_1 = -G\frac{M_E m_m}{R_2} + G\frac{M_E m_m}{\infty} \\ U_2 &= -G\frac{M_E m_m}{R_2}\end{aligned}$$

This is how much potential energy the moon has as it orbits the Earth because it is high, above the Earth. But notice, this is a negative number! What can it mean to have a

Question 223.29.3 negative potential energy?

We use this convention to indicate that the test mass, m_m is bound to the Earth. It would take an input of energy to get the moon free from the gravitational pull of the Earth. Here is the Moon potential energy plotted as a function of distance.



We can see that you have to go an infinite distance to overcome the Earth's gravity completely. That makes sense from our force equation. The force only goes to zero infinitely far away. When we finally get infinitely far away, there will be no potential energy due to the gravitational force because the gravitational force will be zero.

Of course, there are more than just two objects (Earth and Moon) in the universe, so as we get farther away from the Earth, the gravitational pull of, say, a galaxy, might dominate. So we might not notice the weak pull of the Earth as we encounter other objects.

We should show that this form for the potential energy due to gravity becomes the more familiar mgh if our distances are small compared to the Earth's radius.

Let our distance from the center of the Earth be $R_2 = R_E + y$ where R_E is the radius of the Earth and $y \ll R_E$. Then

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_2} \\ &= -G\frac{M_E m_m}{R_E + y} \end{aligned}$$

We can rewrite this as

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_E\left(1+\frac{y}{R_E}\right)} \\ &= -G\frac{M_E m_m}{R_E}\left(1+\frac{y}{R_E}\right)^{-1} \end{aligned}$$

Since y is small y/R_E is very small and we can approximate the therm in parenthesis using the binomial expansion

$$(1 \pm x)^n \approx 1 \mp nx \qquad \text{if } x \ll 1$$

then we have

$$\left(1+\frac{y}{R_E}\right)^{-1} \approx 1-(-1)\frac{y}{R_E} \qquad \text{if } \frac{y}{R_E} \ll 1$$

and our potential energy is

$$U = -G\frac{M_E m_m}{R_E}\left(1+\frac{y}{R_E}\right)$$

then

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_E} + G\frac{M_E m_m y}{R_E^2} \\ &= U_o + m_m\left(G\frac{M_E}{R_E^2}\right)y \end{aligned}$$

If we realize that U_o is the potential energy of the object at the surface of the Earth, then the change in potential energy as we lift the object from the surface to a height y is

$$\begin{aligned} \Delta U &= \left(U_o + m_m\left(G\frac{M_E}{R_E^2}\right)y - \left(U_o + m_m\left(G\frac{M_E}{R_E^2}\right)(0)\right)\right) \\ &= m_m\left(G\frac{M_E}{R_E^2}\right)y \end{aligned}$$

All that is left is to realize that

$$\left(G\frac{M_E}{R_E^2}\right)$$

has units if acceleration. This is just g

$$g = \left(G\frac{M_E}{R_E^2}\right)$$

so we have

$$\Delta U = m_m g y$$

and there is no contradiction. But we should realize that this is an approximation. The more accurate version of our potential energy is

$$U_2 = -G\frac{M_E m_m}{R_2}$$

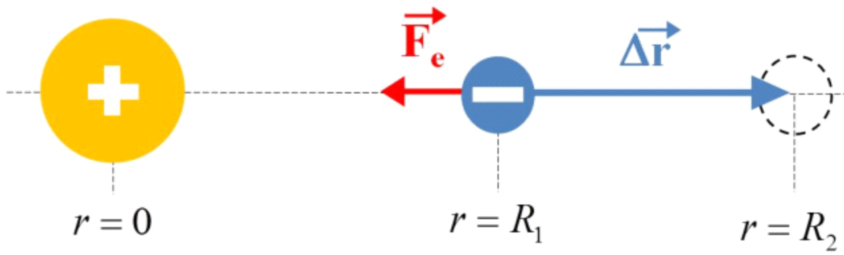
Likewise we should expect that for charges

$$\Delta U_C = -q_m E d$$

is an approximation that is only good when the field, E, can be approximated as a constant magnitude and direction and that the distribution of charge, q_m, is not spatially too big. With this understanding, we can understand electrical potential energy of point charges.

Point charges potential

Suppose we now take a positive charge and define it's position as $r = 0$ and place a negative mover charge near the positive charge.



The work it would take to move the charge a distance $\Delta r = R_2 - R_1$ would be

$$\Delta U_e = -W_e = -\int \overrightarrow{\mathbf{F}}_e \cdot d\overrightarrow{\mathbf{r}}$$

The magnitude of the electrical force is

$$F_e = \frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r^2}$$

once again $\overrightarrow{\mathbf{F}}_e \cdot d\overrightarrow{\mathbf{r}} = -F_e dr$ and

$$\begin{aligned}
\Delta U_e &= -\int_{R_1}^{R_2}\left(-\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r^2}\right)dr \\
&= \frac{Q_E q_m}{4\pi\epsilon_o}\int_{R_1}^{R_2}\frac{dr}{r^2}
\end{aligned}$$

and we realize that this is exactly the same integral we faced in the gravitational case. The answer must be

$$\Delta U_e = -\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{R_2} + \frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{R_1}$$

The similarity is hardly a surprise since the force equation for the Coulomb force is really just like the force equation for gravity.

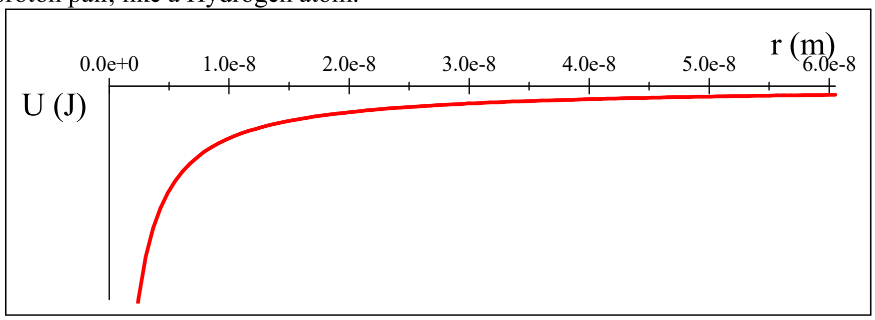
It makes sense to choose the zero point of the electric potential energy the same way we did for the gravitational potential energy since the equations is the same. We will pick $U = 0$ at $r = \infty$. Then we expect that

$$U_e = -\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r}$$

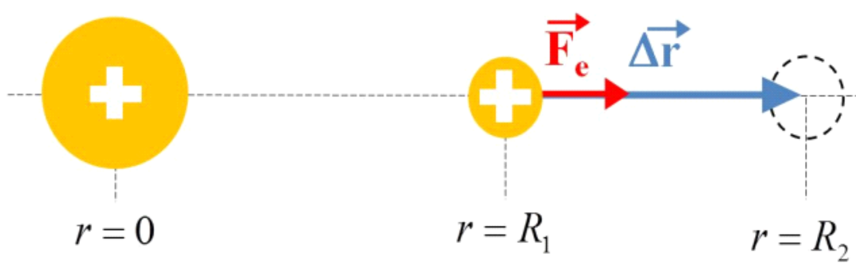
Question 223.29.4

is the electrical potential energy stored by having the charges in this configuration.

Again the negative sign shows that the two opposite charges will be bound together by the attractive force. Here is a graph of the electrical potential energy of an electron and a proton pair, like a Hydrogen atom.



Of course we remember that there is a large difference between electrical and gravitational forces. If the two charges are the same sign, then they will repel and the potential must be different for that situation. If we redraw our diagram for this case, we realize that the sign of the force must change.



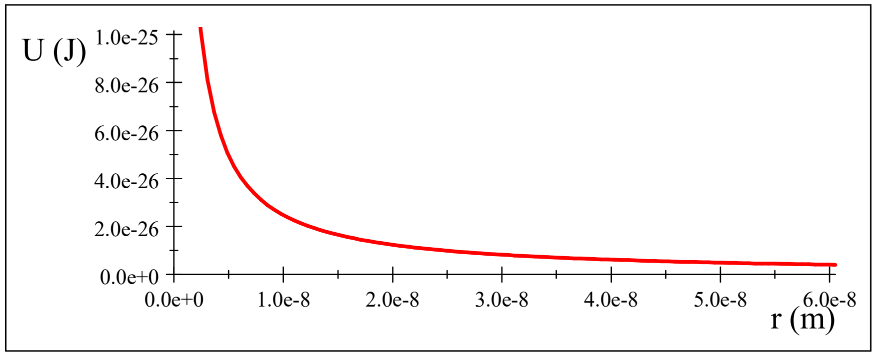
$$\Delta U_e = -W_e = -\int_{R_1}^{R_2}\left(+\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r^2}\right)dr$$

this will change all the signs in our solution

$$\Delta U_e = +\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{R_2} - \frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{R_1}$$

then

$$U_e = +\frac{1}{4\pi\epsilon_o}\frac{Q_E q_o}{r}$$



Now we can see that the potential energy gets larger as the two like charges get nearer. It takes energy to make them get closer. This is clearly not a bound situation.

Three point charges.

Question 223.29.5

Suppose we have three like charges. What will the potential energy of the three-charge system be?

Let's consider the charges one at a time. If I move one charge, q_1, from infinitely far away. there is no environemntal field electric, so there is no force, since we need two charges for there to be a force. Then there is no potential energy. This is like a rock floating in deep space far away from anything else in the universe. It just sits there, there is no potential for movement, so no potential energy. But when we bring in another charge, q_2, then q_1 is an environmental charge making a field and q_2 is our mover charge. Then q_2 will take an amount of work equal to

$$U_{12} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r_{12}}$$

to move in the charge because the two charges repeal each other. There is a force, so now there is an amount of potential energy associated with the work done to move the charges together.

Suppose we had chosen to bring in the other charge, q_3, instead. Charge q_1 forms an envirnmental field. It takes an amount of energy

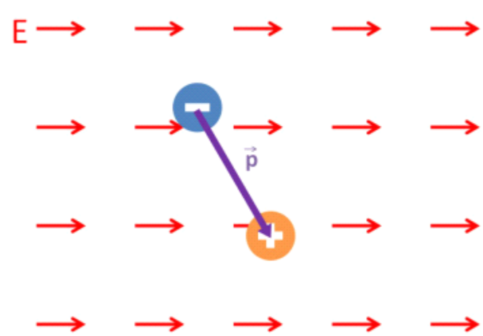
$$U_{13} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_3}{r_{13}}$$

to bring in the third charge charge But if the second charge were already there, the second charge also creates an environmental field, so it also creates a force on the third charge. So it will take more work to bring in the third charge.

$$U_3 = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_o}\frac{q_2 q_3}{r_{23}}$$

So the total amount of work involved in bringing all three charges together
$$U=\frac{1}{4\pi\epsilon_o}\frac{q_1q_2}{r_{12}}+\frac{1}{4\pi\epsilon_o}\frac{q_1q_3}{r_{13}}+\frac{1}{4\pi\epsilon_o}\frac{q_2q_3}{r_{23}}$$
then the potential energy difference would be
$$\begin{aligned}\Delta U &= U_f-U_i=-W\\ &= U_f-0\\ &= \frac{1}{4\pi\epsilon_o}\frac{q_1q_2}{r_{12}}+\frac{1}{4\pi\epsilon_o}\frac{q_1q_3}{r_{13}}+\frac{1}{4\pi\epsilon_o}\frac{q_2q_3}{r_{23}}\end{aligned}$$
which we can generalize as
$$U=\frac{1}{4\pi\epsilon_o}\sum_{i<j}\frac{q_iq_j}{r_{ij}}$$
for any number of charges. We simply add up all the potential energies. This is one reason to use electric potential energy in solving problems. The electric potential energies just add, and they are not vectors, so the addition is simple.

Dipole potential energy



Let's try out our new idea of potential energy for point charges on a dipole. We will try to keep this easy, so let's consider the dipole to be in a constant, uniform electric field. We know there will be no net force. The work done to move a charge we have stated to be
$$W=\int\overrightarrow{\mathbf{F}}_e\cdot d\overrightarrow{\mathbf{r}}$$
but in this case, we know the net force on the dipole is zero.

However, we can also do some work in rotating something
$$W_{rot}=\int\tau_e d\theta$$

we know from before that the magnitude of the torque is

$$\tau = pE\sin\theta$$

so

$$\begin{aligned} W_{rot} &= \int_{\theta_1}^{\theta_2} pE\sin\theta d\theta \\ &= pE\left(\cos\theta_2 - \cos\theta_1\right) \end{aligned}$$

this must give

$$\begin{aligned} \Delta U &= -W_{rot} = U_f - U_i \\ &= -pE\left(\cos\theta_2 - \cos\theta_1\right) \end{aligned}$$

then we can write as

$$U = -pE\cos\theta$$

This is the rotational potential energy for the dipole. We can write this as an inner product

$$U = -\overrightarrow{\mathbf{p}} \cdot \overrightarrow{\mathbf{E}}$$

What does this mean? It tells us that we have to do work to turn the dipole.

Let's go back to our example of a microwave oven. If the field is $E = 200\,\mathrm{V/\,m}$, then how much work does it take to turn the water molecules?

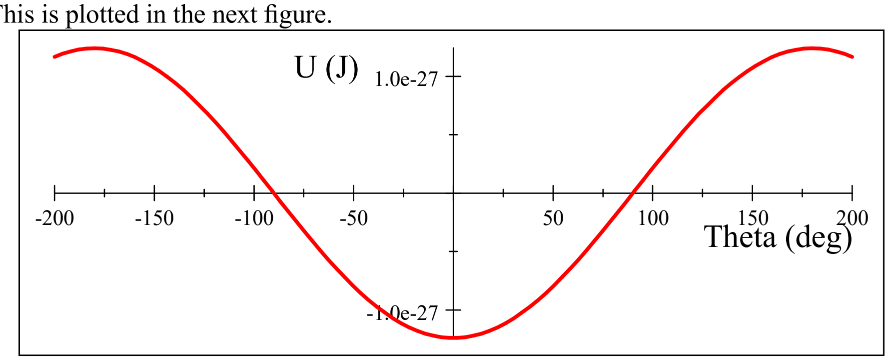
Remember that the dipole moment for a water molecule is something like

$$p_w = 6.2 \times 10^{-30}\,\mathrm{C\,m}$$

so we have

$$\begin{aligned} U &= -\left(6.2 \times 10^{-30}\,\mathrm{C\,m}\right)\left(200\,\mathrm{V/\,m}\right)\cos\theta \\ &= -1.24 \times 10^{-27}\,\mathrm{J}\cos\theta \end{aligned}$$

This is plotted in the next figure.



At zero degrees we can see that it takes energy (work) to make the dipole spin. It will try to stay at zero degrees and a small displacement from zero degrees will

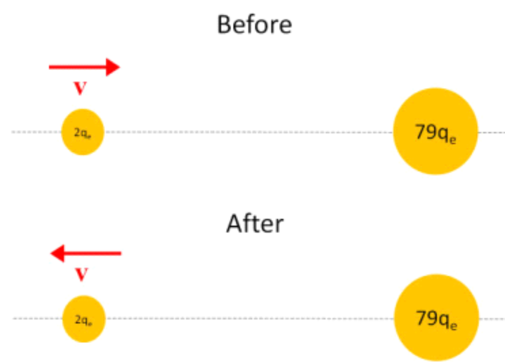
will cause the dipole to oscillate around $\theta = 0$ but it will return to $\theta = 0$ as the added energy is dissipated. then $\theta = 0\,\mathrm{rad}$ is a stable equilibrium. Conversely, at $\theta = \pi\,\mathrm{rad}$ we are at a maximum potential energy. We get rotational kinetic energy if we cause any small displacement $\Delta\theta$. The dipole will angularly accelerate. $\theta = \pm\pi\,\mathrm{rad}$ is an unstable equilibrium.

Shooting α-particles

Let's use electric potentials to think about a famous experiment. Ernest Rutherford shot α-particles, $q = +2q_e$ at gold nuclei, $q = +79q_e$. How close will the α-particles get if the collision is head-on and the initial speed of the α-particles is $3 \times 10^6\,\mathrm{m/s}$?

The easiest way to approach this is to use conservation of energy. The energies before and after must be the same because we have no frictional or dissipative forces.

The before and after pictures are as shown. The α-particle, of course, is our mover.



We can write

$$K_i + U_i = K_f + U_f$$

when the α-particles are at their closest distance to the gold nuclei, then $K_f = 0$. We can envision starting the α-particles from effectively an infinite distance away. Then $U_i \approx 0$. so

$$\frac{1}{2}m_\alpha v^2 = \frac{1}{4\pi\epsilon_o}\frac{Q_{Au}q_\alpha}{r}$$

Solving for r gives

$$\begin{aligned} r &= \frac{1}{4\pi\epsilon_o}\frac{Q_{Au}q_\alpha}{\frac{1}{2}m_\alpha v^2} \\ &= \frac{1}{2\pi\epsilon_o}\frac{(79q_e)(4q_e)}{m_\alpha v^2} \end{aligned}$$

then

$$\begin{aligned} r &= \frac{1}{2\pi\left(8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\right)}\frac{158\left(1.602\times10^{-19}\,\mathrm{C}\right)^2}{\left(6.642\,2\times10^{-27}\,\mathrm{kg}\right)\left(3\times10^{6}\,\mathrm{m/\,s}\right)^2} \\ &= 1.219\,8\times10^{-12}\,\mathrm{m} \end{aligned}$$

This is a very small number! and it sets a bound on how large the nucleus of the gold atom can be.

Next lecture, we will try to make our use of electrical potential energy more practical by defining the electrical potential energy per unit charge, and applying this to problems involving moving charges (like those in electric circuits).

Basic Equations