Chapter 9

Multiple Frequency Interference

Fundamental Concepts

- Mixing waves of different frequencies produces a time-varying amplitude called beating.
- Complex waves can be treated as a superposition of simple sinusoidal waves.
- Limited signals are multi-frequency

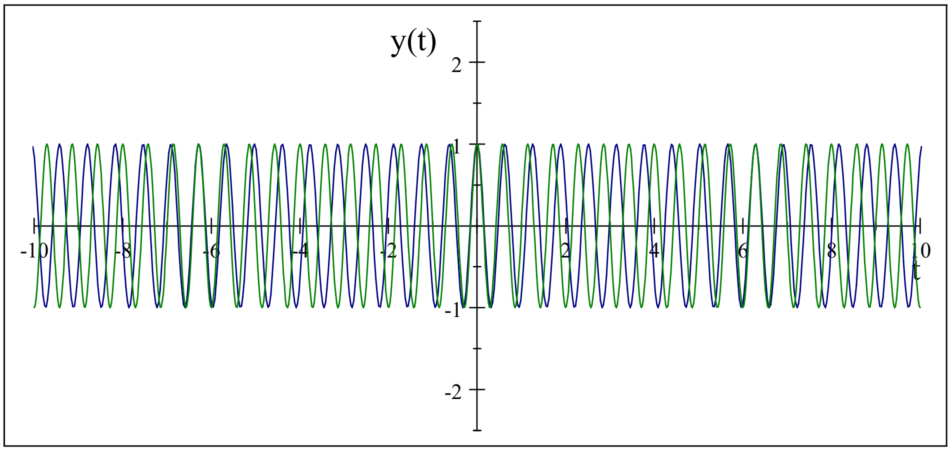
9.1 Beats

Up till now we only superposed waves that had the same frequency. But what happens if we take waves with different frequencies?

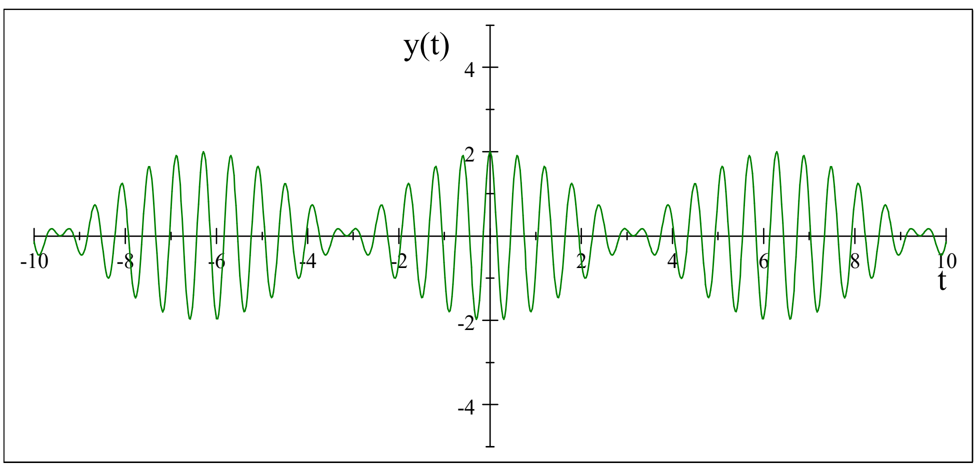
$$y_1 = y_{\max} \sin(kx - \omega_1 t)$$

$$y_2 = y_{\max} \sin(kx - \omega_2 t)$$

We can plot both waves on the same graph, in this case a history graph.



Notice that there are places where the waves are in phase, and places where they are not. The superposition looks like this



where there is constructive interference, the resulting wave amplitude is large, where there is destructive interference, the resulting amplitude is zero. We get a traveling wave who's amplitude varies. We can find the amplitude function algebraically.

We can write these as

$$y_1 = y_{\max} \sin(kx - 2\pi f_1 t)$$

$$y_2 = y_{\max} \sin(kx - 2\pi f_2 t)$$

The sum is just

$$y = y_{\max} \sin(kx - 2\pi f_1 t) + y_{\max} \sin(kx - 2\pi f_2 t)$$

We use another trig identity

$$\sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

which allows us to write tis as

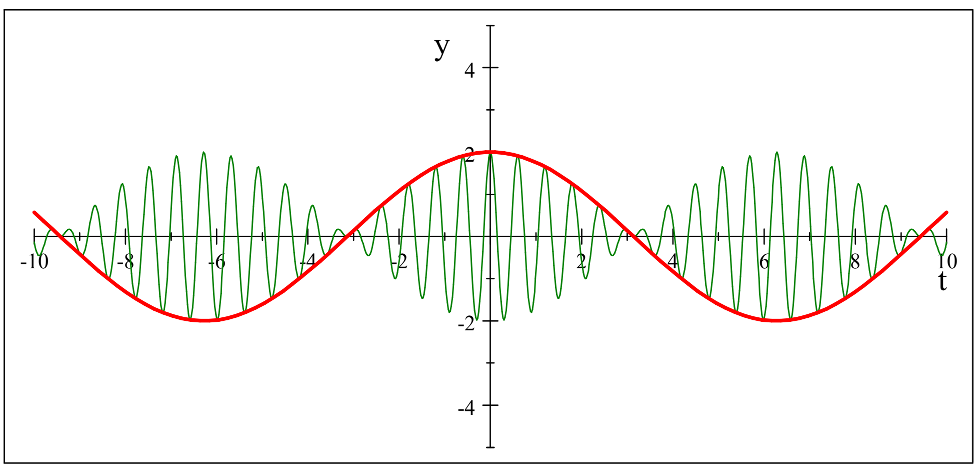
$$\begin{aligned} y &= 2y_{\max}\cos\left(\frac{kx - 2\pi f_2 t - (kx - 2\pi f_1 t)}{2}\right)\sin\left(\frac{kx - 2\pi f_2 t + kx - 2\pi f_1 t}{2}\right) \\ &= 2y_{\max}\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)\sin\left(-2\pi\frac{f_1 + f_2}{2}t\right) \\ &= \left[2y_{\max}\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)\right]\sin\left(kx - 2\pi\frac{f_1 + f_2}{2}t\right) \end{aligned}$$

We see that we have a part that has a frequency that is the average of f_1 and f_2. This is the frequency we hear. But we have another complicated amplitude term, and this time it is a function of time (just to be confusing). The amplitude has its own frequency that is half the difference of f_1 and f_2.

$$A_{\text{resultant}} = 2y_{\max}\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)$$

So the sound amplitude will vary in time for a given spatial location.

The situation is odder still. We have a cosine function, but it is really an envelope for the higher frequency motion of the air particles.



Our ear drum does not care which way the envelope function goes. We can see that the green (thin line) wave will push and pull air molecules, and therefore our ear drums, with maximum loudness at twice this frequency. So we will hear two maxima for every envelope period!

This frequency with which we hear the sound get loud at a given location as the wave goes by is called the *beat frequency*. The red envelope (solid heavy line in the last figure) has a frequency of

$$f_A = \frac{f_1 - f_2}{2}$$

So our beat frequency is

$$f_{beat} = |f_1 - f_2|$$

9.2 Non Sinusoidal Waves

You have probably wondered if all waves are sinusoidal. Can the universe really be described by such simple mathematics? The answer is both no, and yes. There are non-sinusoidal waves, in fact, most waves are not sinusoidal. But it turns out that we can use a very clever mathematical trick to make any shape wave out of a superposition of many sinusoidal waves. So our mathematics for sinusoidal waves turns out to be quite general.

9.2.1 Music and Non-sinusoidal waves

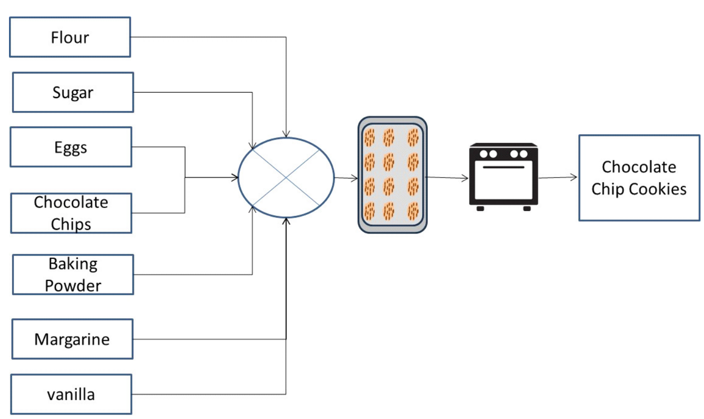
Let's take the example of music.

From our example of standing waves on strings, we know that a string can support a series of standing waves with discrete frequencies–the harmonic series. We have also discussed that usually we excite the waves with a pluck or some discrete event, not with an oscillator. Only the harmonic series of frequencies will resonate, creating standing waves. Other frequencies waves die out quickly. But there is no reason to suppose that we get energy in only one standing wave at a time. Most sounds are a combination of harmonics.

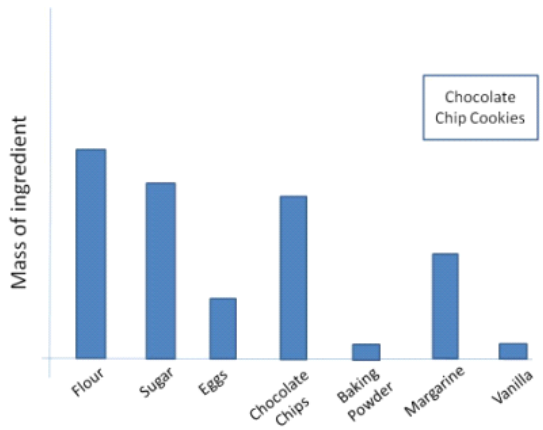
The fundamental mode tends to give us the pitch we hear, but what are the other standing waves for?

To understand, lets take an analogy. Making cookies and cakes.

Here is the beginning of a recipe for cookies.

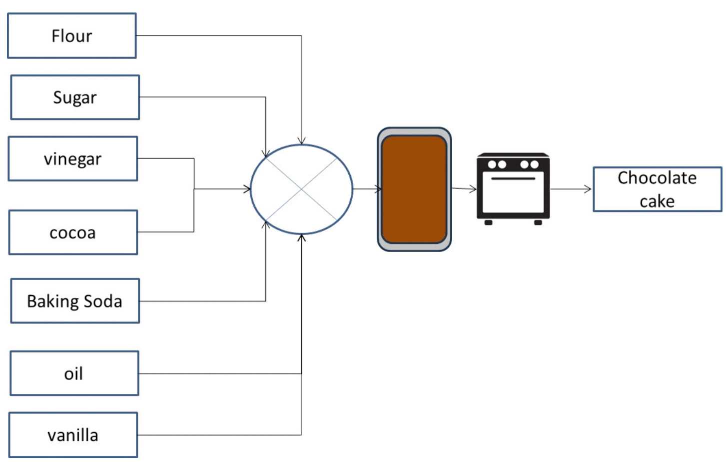


The recipe is a list of ingredients, and a symbolic instruction to mix and bake. The product is chocolate chip cookies. Of course we need more information. We need to know now much of each ingredient to use.



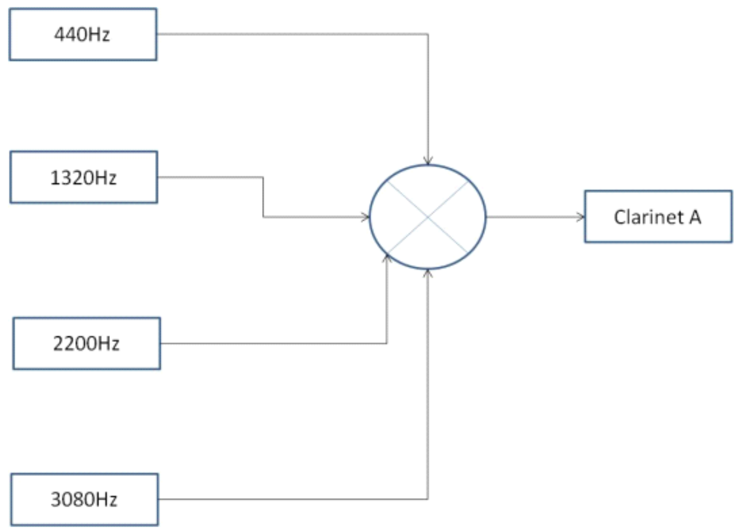
This graph gives us the amount of each ingredient by mass.

Now suppose we want chocolate cake.

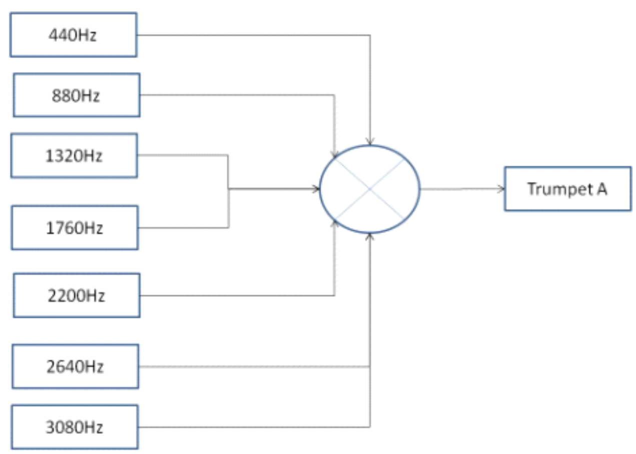


The predominant taste in each of these foods is chocolate. But chocolate cake and chocolate chip cookies don't taste exactly the same. We can easily see that the differences in the other ingredients make the difference between the "cookie" taste and the "cake" taste that goes along with the "chocolate" taste that predominates.

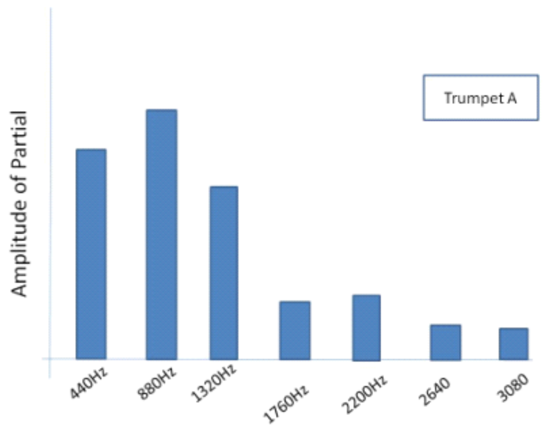
The sound waves produced by musical instruments work in a similar way. Here is a recipe for an "A" note from a clarinet.



and here is one for a trumpet playing the same "A" note.

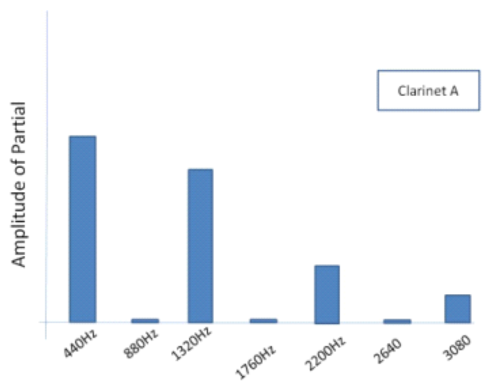


A trumpet sounds different than a clarinet, and now we see why. There are more harmonics involved with the trumpet sound than the clarinet sound. These extra standing waves make up the "brassiness" of the trumpet sound. As with our baking example, we need to know how much of each standing wave we have. Each will have a different amplitude. For our trumpet, we might get amplitudes as shown.



Note that the second harmonic has a larger amplitude, but we still hear the musical not as “A” at 440 Hz. A Flugelhorn horn would still sound brassy, but would have a different mix of harmonics.

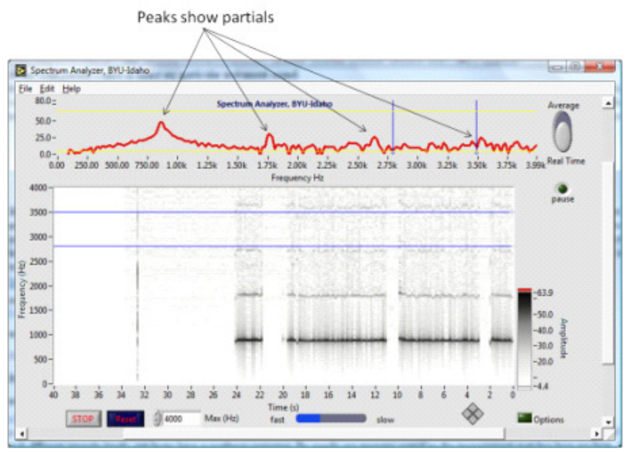
The clarinet graph would look quite different, perhaps something like this



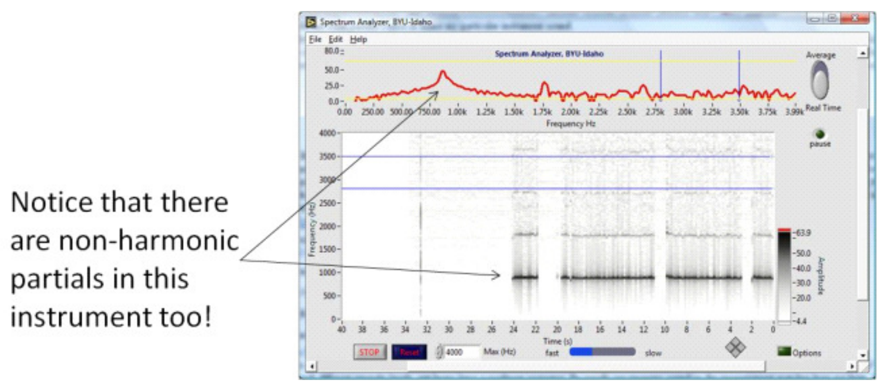
because it does not have as many “ingredients” as the trumpet.

All of this should remind you of our analysis of open and closed pipes. Remember when we closed a pipe, we lost all the even multiples the fundamental frequency. A similar thing is happening with our instruments. The rich sound of the brass instrument includes more harmonics and this is achieved by the shape of the instrument (the flared bell is a big part of making these extra harmonics and providing the rich trumpet sound).

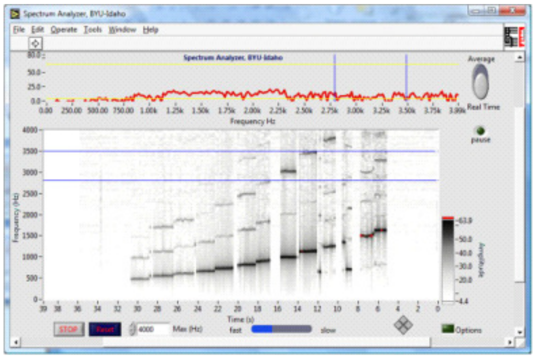
We have a tool that you can download to your PC to detect the mix of harmonics of musical instruments, or mechanical systems. In music, the different harmonics are called *partials* because they make up part of the sound. A graph that shows which harmonics are involved is called a *spectrum*. The next figure is the spectrum of a six holed bamboo flute. Note that there are several harmonics involved.



Note that our software display has two parts. One is the instantaneous spectrum, and one is the spectrum time history.

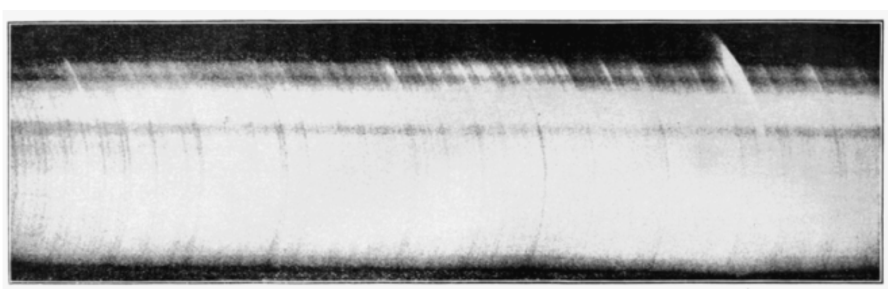


By observing the time history, we can see changes in the spectrum. We can also see that we don't have pure harmonics. The graph shows some response off the specific harmonic frequencies. This six holed flute is very "breathy" giving a lot of wind noise along with the notes, and we see this in the spectrum. In the next picture, I played a scale on the flute.



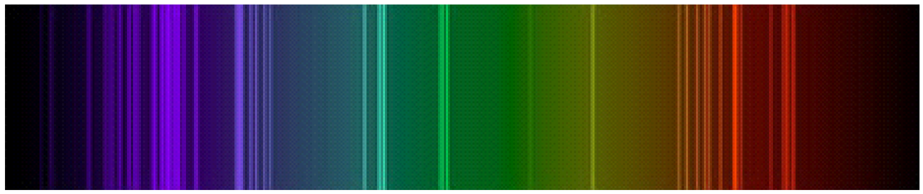
The instantaneous spectrum is not active in this figure (since it can't show more than one note at a time on the instantaneous graph) but in the time history we see that as the fundamental frequency changes by shorting the length of the flute (uncovering holes), we see that every partial also goes up in frequency. The flute still has the characteristic spectrum of a flute, but shifted to new set of frequencies. We can use this fact to identify things by their vibration spectrum. In fact, that is how you recognize voices and instruments within your auditory system!

The technique of taking apart a wave into its components is very powerful. With light waves, the spectrum is an indication of the chemical composition of the emitter. For example, the spectrum of the sun looks something like this



Solar coronal spectrum taken during a solar eclipse. The successive curved lines are each different wavelengths, and the dark lines are wavelengths that are absorbed. The pattern of absorbed wavelengths allows a chemical analysis of the corona. (Image in the Public Domain, orignally published in Bailey, Solon, L, *Popular Science Monthly*, Vol 60, Nov. 1919, pp 244)

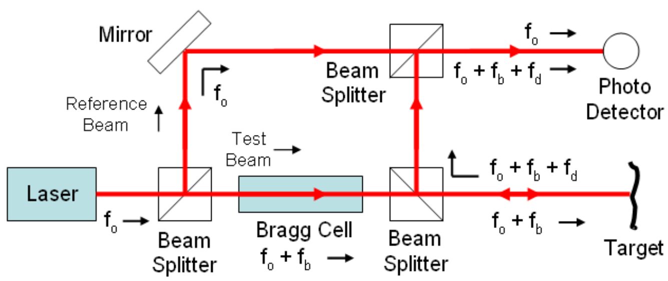
The lines in this graph show the amplitude of each harmonic component of the light. Darker lines have larger amplitudes. The harmonics come from the excitation of electrons in their orbitals. Each orbital is a different energy state, and when the electrons jump from orbital to orbital, they produce specific wave frequencies. By observing the mix of dark lines in pervious figure, and comparing to laboratory measurements from each element (see next figure) we can find the composition of the source. This figure shows the emission spectrum for Calcium. Because it is an emission spectrum the lines are bright instead of dark. We can even see the color of each line!



Emission spectrum of Calcium (Image in the Public Domain, courtesy NASA)

9.2.2 Vibrometry

Just like each atom has a specific spectrum, and each instrument, each engine, machine, or anything that vibrates has a spectrum. We can use this to monitor the health of machinery, or even to identify a piece of equipment. Laser or acoustic vibrometers are available commercially.



Laser Vibrometer Schematic (Public Domain Image from Laderaranch: http://commons.wikimedia.org/wiki/File:LDV_Schematic.png)

They provide a way to monitor equipment in places where it would be dangerous or even impossible to send a person. The equipment also does not need to be shut down, a great benefit for factories that are never shut down, or for a satellite system that cannot be reached by anyone.

9.2.3 Fourier Series: Mathematics of Non-Sinusoidal Waves

We should take a quick look at the mathematics of non-sinusoidal waves.

Let' start with a superposition of many sinusoidal waves. The math looks like this

$$y(t) = \sum_n \left(A_n \sin(2\pi f_n t) + B_n \cos(2\pi f_n t)\right)$$

where A_n and B_n are a series of coefficients and f_n are the harmonic series of frequencies. The coefficients are amplitudes for the many individual waves making up the complicated wave.

9.2.4 Example: Fourier representation of a square wave.

For example, we could represent a function $f(x)$ with the following series

$$\begin{aligned} f(x) &= C_o + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) && (9.1)\\ &+ C_2 \cos\left(\frac{2\pi}{\frac{\lambda}{2}}x + \varepsilon_2\right) && (9.2)\\ &+ C_3 \cos\left(\frac{2\pi}{\frac{\lambda}{3}}x + \varepsilon_3\right) && (9.3)\\ &+ \ldots && (9.4)\\ &+ C_n \cos\left(\frac{2\pi}{\frac{\lambda}{n}}x + \varepsilon_n\right) && (9.5)\\ &+ \ldots && (9.6) \end{aligned}$$

where we will let $\varepsilon_i = \omega_i t + \phi_i$

The $C's$ are just coefficients that tell us the amplitude of the individual cosine waves. The more terms in the series we take, the better the approximation we will have, with the series exactly matching $f(x)$ when the number of terms, $N \to \infty$.

Usually we rewrite the terms of the series as

$$C_m \cos(mkx + \varepsilon_m) = A_m \cos(mkx) + B_m \sin(mkx) \tag{9.7}$$

where k is the wavenumber

$$k = \frac{2\pi}{\lambda} \tag{9.8}$$

and λ is the wavelength of the complicated but still periodic function $f(x)$. Then we identify

$$\begin{aligned} A_m &= C_m \cos(\varepsilon_m) && (9.9)\\ B_m &= -C_m \sin(\varepsilon_m) && (9.10) \end{aligned}$$

then

$$f(x) = \frac{A_o}{2} + \sum_{m-1}^{\infty} A_m \cos(mkx) + \sum_{m-1}^{\infty} B_m \sin(mkx) \tag{9.11}$$

where we separated out the $A_o/2$ term because it makes things nicer later.

Fourier Analysis

The process of finding the coefficients of the series is called *Fourier analysis.* We start by integrating equation (9.11)

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx + \int_0^\lambda \sum_{m-1}^{\infty} A_m \cos(mkx)\,dx + \int_0^\lambda \sum_{m-1}^{\infty} B_m \sin(mkx)\,dx \tag{9.12}$$

We can see immediately that all the sine and cosine terms integrate to zero (we integrated over a wavelength) so

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx = \frac{A_o}{2}\lambda \tag{9.13}$$

We solve this for A_o

$$A_o = \frac{2}{\lambda}\int_0^\lambda f(x)\,dx \tag{9.14}$$

To find the rest of the coefficients we need to remind ourselves of the orthogonality of sinusoidal functions

$$\int_0^\lambda \sin(akx)\cos(bkx)\,dx = 0 \tag{9.15}$$

$$\int_0^\lambda \cos(akx)\cos(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{9.16}$$

$$\int_0^\lambda \sin(akx)\sin(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{9.17}$$

where δ_{ab} is the Kronecker delta.

To find the coefficients, then, we multiply both sides of equation (9.11) by $\cos(lkx)$ where l is a positive integer. Then we integrate over one wavelength.

$$\int_0^\lambda f(x)\cos(lkx)\,dx = \int_0^\lambda \frac{A_o}{2}\cos(lkx)\,dx \tag{9.18}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} A_m\cos(mkx)\cos(lkx)\,dx \tag{9.19}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} B_m\sin(mkx)\cos(lkx)\,dx \tag{9.20}$$

which gives

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \int_0^\lambda A_m\cos(mkx)\cos(mkx)\,dx \tag{9.21}$$

that is, only the term with two cosine functions where $l = m$ will be non zero. So

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \frac{\lambda}{2}A_m \tag{9.22}$$

solving for A_m we have

$$A_m = \frac{2}{\lambda}\int_0^\lambda f(x)\cos(mkx)\,dx \tag{9.23}$$

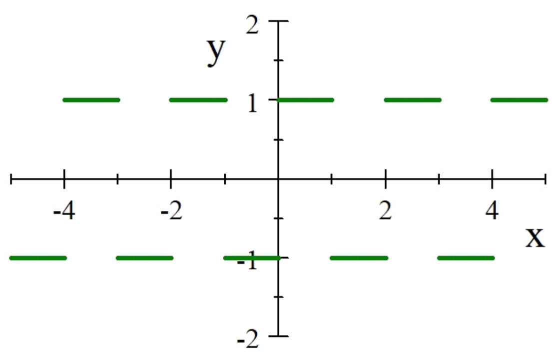
We can perform the same steps to find B_m only we use $\sin(lkx)$ as the multiplier. Then we find

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(mkx)\,dx \tag{9.24}$$

Let's find the series for a square wave using our Fourier analysis technique. Let's take

$$\lambda = 2 \tag{9.25}$$

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \frac{\lambda}{2} \\ -1 & \text{if} \quad \frac{\lambda}{2} < x < \lambda \end{cases} \tag{9.26}$$



since $f(x)$ *is* odd, $A_m = 0$ for all m. We have

$$B_m = \frac{2}{\lambda}\int_0^{\frac{\lambda}{2}} (1)\sin(mkx)\,dx + \frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda} (-1)\sin(mkx)\,dx \tag{9.27}$$

so

$$B_m = \frac{1}{m\pi}\left(-\cos(mkx)\right|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi}\left(\cos(mkx)\right|_{\frac{\lambda}{2}}^{\lambda} \tag{9.28}$$

Which is

$$B_m = \frac{1}{m\pi}\left(1\cos\left(m\frac{2\pi}{\lambda}x\right)\right|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi}\left(\cos\left(m\frac{2\pi}{\lambda}x\right)\right|_{\frac{\lambda}{2}}^{\lambda} \tag{9.29}$$

so

$$\begin{aligned} B_m &= \frac{1}{m\pi}\left(\left(-\cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right) + \cos\left(m\frac{2\pi}{\lambda}(0)\right)\right) && (9.30)\\ &+\frac{1}{m\pi}\left(\left(\cos\left(m\frac{2\pi}{\lambda}\lambda\right) - \cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right)\right) && (9.31)\end{aligned}$$

which is

$$B_m = \frac{2}{m\pi}\left(1 - \cos\left(m\pi\right)\right) \tag{9.32}$$

Our series is then just

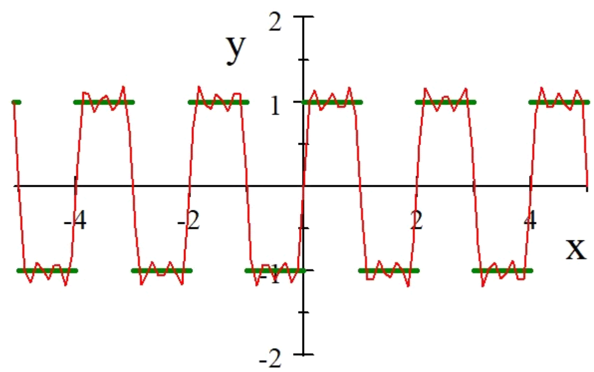
$$f\left(x\right) = \sum_{m-1}^{\infty} \frac{2}{m\pi}\left(1 - \cos\left(m\pi\right)\right)\sin\left(mkx\right) \tag{9.33}$$

and we can write a few terms

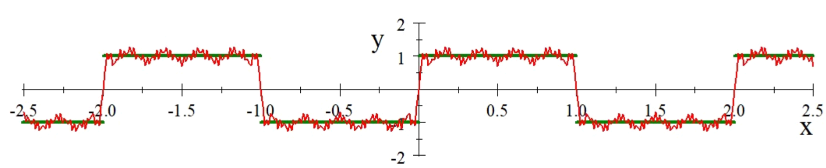
$Term$	
1	$\frac{4}{\pi}\sin\left(kx\right)$
2	0
3	$\frac{4}{3\pi}\sin\left(3kx\right)$
4	0
5	$\frac{4}{5\pi}\sin\left(5kx\right)$

(9.34)

then the partial sum up to $m = 5$ looks like



With twenty terms we would have



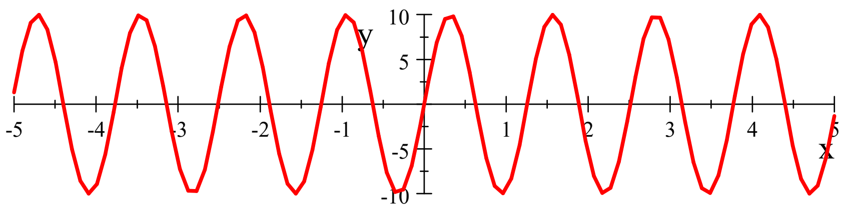
In the limit of infinitely many waves, the match would be perfect. But we don't usually need an infinite number of terms. we can pick the part of the spectrum that best represents the phenomena we desire to observe. For example, oil based compounds all have specific spectral signatures in the wavelength range between $3 - 5$ micrometers. If you wish to tell the difference between gasoline and crude oil, you can restrict your study to these wavelengths alone.

9.3 Frequency Uncertainty for Signals and Particles

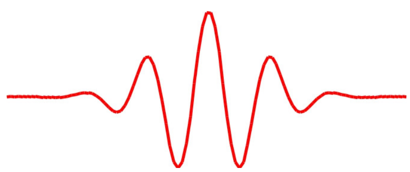
Up to this lecture, when we thought of a wave we have mostly thought of something like this

$$y = y_{\max} \sin(kx - \omega t - \phi)$$

which in practice might look like this

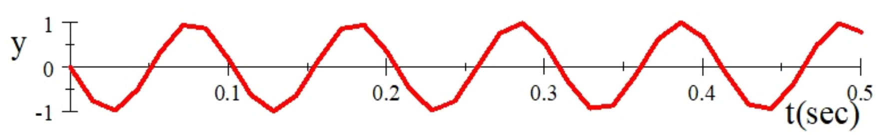


And we have noticed that there is no start or stop to this kind of wave. Our figure starts at $x = -5$ and ends at $x = 5$, but the equation does not! There is a value of y for every x from $-\infty$ to $+\infty$. But many signals are not such waves. They may be very limited in size.

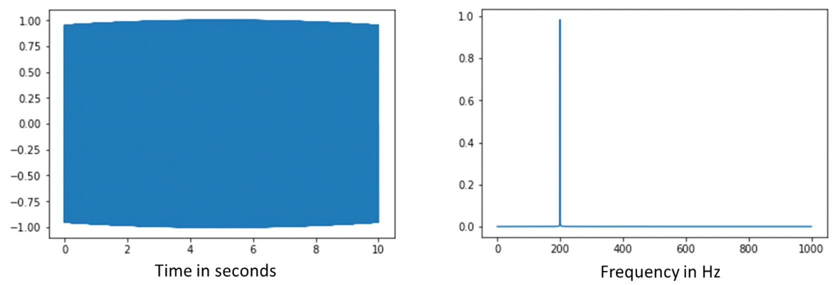


We should investigate what happens when you have a limited wave. I did this investigation using Python. Suppose we have a sine wave with $f = 200\,\text{Hz}$, but I limit this wave's existence by making it start at $t_i = 0$ and then make it end at $t_f = 10\,\text{s}$. I could do this in practice by turning on a radio transmission or even an acoustic speaker, and then turning the device off ten seconds later. Our screen resolution is terrible for plotting such a function, but in the figure below you can see that our signal only exists from $t = 0$ to $t = 10\,\text{s}$.

If I zoom in on a part of the graph we can see that it is really a sine wave.



Python did equally bad at plotting this. All we see is a blue band.



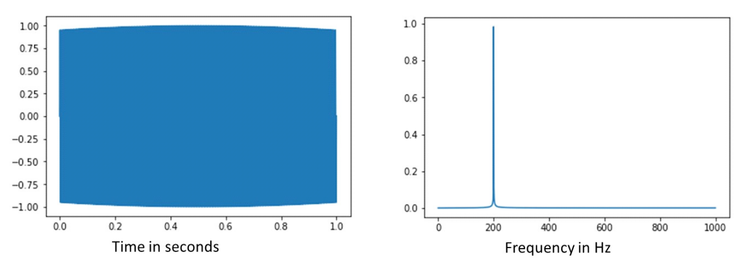
10s, 200Hz

But in the second graph, notice that we have plotted frequency. Python and most scientific programing languages have functions to find a digital version of a Fourier transform to produce a spectrum. It takes in a signal and finds which frequencies are in a signal. It performs the job of a spectrometer, so we would call the figure to the right a spectrogram (or just a spectrum).

We expect only one frequency, 200 Hz, and that is mostly what we get. Since our period for our wave is

$$T = \frac{1}{200\,\mathrm{Hz}} = 0.005 \text{ s}$$

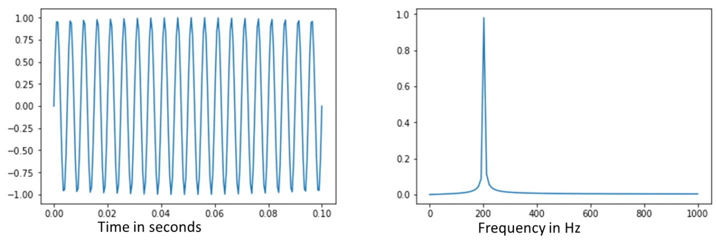
and we have 10 s of data, that is four orders of magnitude more signal than a period. The whole signal seems very long compared to a period. We expect this to look kind of like an infinite signal. But suppose we take the same wave, but for less time. We limit the wave more.



1sec, 200hz

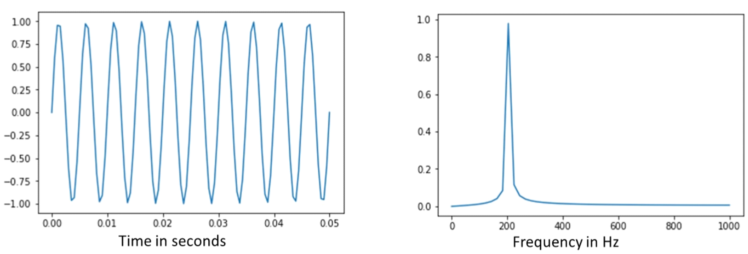
So we still get a blue blur for our wave picture, but now the wave only exists for one instead of ten seconds. If you look closely at the frequency graph, you will notice that the 200 Hz peak representing our wave is a bit wider right at the bottom.

We could limit our wave more, say, so it only lasts $t_f = 0.1$ s. We would get a set of graphs that look like this.

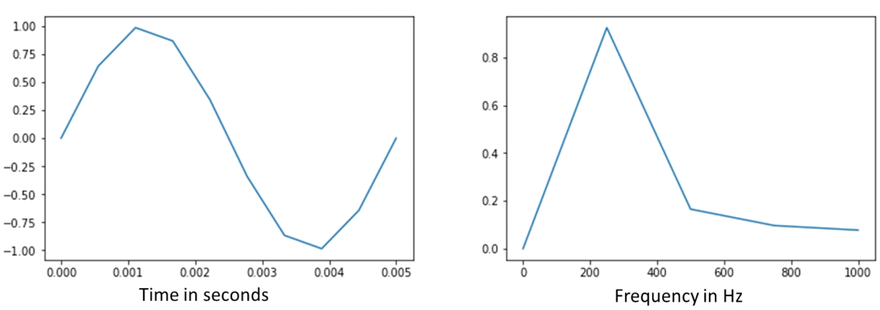


0.1 sec, 200hz

Notice that not only can Python render the wave now, but more importantly the 200 Hz frequency peak is noticeable wider. This is profound! It means that by limiting the wave, we no longer have just one frequency! The graph tells us we have mostly 200 Hz but we also have some 199 Hz and some 201 Hz and some 190 Hz and some 210 Hz, etc. The very fact that the wave does not go on so long requires that we have more than one frequency in the wave. We could say that as Δt gets smaller, our Δf is getting bigger. Here are two more examples with smaller Δt values.



0.05 sec, 200hz



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The cost of limiting our waves is that we can't have a single frequency for the wave. For an engineer, this means that if you only measure a short segment of the signal, you have an increased uncertainty in the frequency you will find from that signal. For chemists, it means that when we look at quantum wave functions we can expect uncertainty in the frequency (or wavelength) because the quantum particle (like an electron) is limited. It is important to know what we mean by uncertainty in this case. In our example above, when we say that Δf increased we really mean that we have more than one frequency. We don't just mean that we don't know the frequency well. We really are mixing more than one frequency. This idea of limiting waves creating uncertainty shows up in physical chemistry as Heisenberg's uncertainty principle.