46 Inductors

Fundamental Concepts

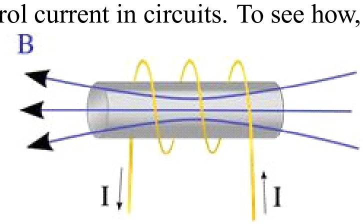
- The self inductance L has all the geometric and material properties of a coil or other inductor an it can be found using $L = N\frac{d\Phi_B}{dI}$
- The emf induced by an inductor is given by $\mathcal{E} \equiv -L\frac{\Delta I}{\Delta t}$
- For a solenoid, the inductance can be found to be $L = \mu_o n^2 V$
- The energy stored in the magnetic field is $U_L = \frac{1}{2}LI^2$ and the energy density in the magnetic field is $u_B = \frac{1}{2}\frac{1}{\mu_o}B^2$
- There is an *apparent* voltage drop across an inductor of $\Delta V_{L_{apparent}} = -L\frac{dI}{dt}$
- There is also a mutual inductance between two inductors given by $M_{12} = \frac{N_2\Phi_{12}}{I_1}$

Self Inductance

Question 223.46.1

When we put capacitors and resisters in a circuit, we found that the current did not jump to it's ultimate current value all at once. There was a time dependence. But really, even if we just have a resister (and we always have some resistance) the current does not reach it's full value instantaneously. Think of our circuits, they are current loops! So as the current starts to flow, Lenz's law tells us that there will be an induced emf that will oppose the flow. The potential drop across the resister in a simple battery-resister circuit is the potential drop due to the battery emf, *minus the induced emf*.

We can use this fact to control current in circuits. To see how, we can study a new case



Let's take a coil of wire wound around an iron cylindrical core. We start with a current as shown in the figure above. Using our right hand rule we can find the direction of the

B-field. But we now will allow the current to change. As it gets larger, we know

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

and we know that as the current changes, the magnitude of the B-field will change, so the flux through the coil will change. We will have an induced emf. We could derive this expression, but I think you can see that the induced emf is proportional to the *rate of change* of the current.

$$\mathcal{E} \equiv -L\frac{\Delta I}{\Delta t}$$

You might ask if the number of loops in the coil matters. The answer is–yes. Does the size and shape of the coil matter–yes. But we will include all these geometrical effects in the constant L called the *inductance*. It will hold all the material properties of the iron cored coil.

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \equiv -L\frac{dI}{dt}$$

so for this case

$$-N\frac{d\Phi_B}{dt}\frac{dt}{dI} \equiv -L$$

or

$$L = N\frac{d\Phi_B}{dI}$$

If we start with no current (so no flux), then our change in flux is the current flux minus zero. We can then say that

$$L = N\frac{\Phi_B}{I}$$

It might be more useful to write the inductance as

$$L = -\frac{\mathcal{E}_L}{\frac{dI}{dt}}$$

In designing circuits, we will usually just look up the inductance of the device we choose, like we looked up the resistance of resisters or the capacitance of the capacitors we use.

But for our special case of a simple coil, we can calculate the inductance, because we know the induced emf using Faraday's law

Question 223.46.2

Inductance of a solenoid[27]

Let's extend our calculation for our coil. Really the only easy case we can do is that of a

[27] Think of this like the special case of a capacitor made from two flat large plates, the parallel plate capacitor. It was somewhat ideal in the way we treated it. Our treatment of the special case of a coil will likewise be somewhat ideal.

solenoid (that's probably a hint for the test). So let's do it! We will just fill our solenoid with air instead of iron (if we have iron, we have to take into account the magnetization, so it is not terribly hard, but this is not what we want to concentrate on now). If the solenoid has N turns with length L and we assume that L is much bigger than the radius r of the loops then we can use our solution for the B-field created by a solenoid

$$\begin{aligned} B &= \mu_o n I \\ &= \mu_o \frac{N}{\ell} I \end{aligned}$$

The flux through each turn is then

$$\Phi_B = BA = \mu_o \frac{N}{\ell} IA$$

where A is the area of one of the solenoid loops. Then we use our equation for inductance for a coil

$$\begin{aligned} L &= N\frac{\Phi_B}{I} \\ &= N\frac{\left(\mu_o \frac{N}{\ell} IA\right)}{I} \\ &= \frac{\left(\mu_o N^2 A\right)}{\ell} \\ &= \frac{\left(\mu_o N^2 A\right)}{\ell}\frac{\ell}{\ell} \\ &= \frac{\mu_o N^2 A\ell}{\ell^2} \\ &= \frac{\mu_o N^2 V}{\ell^2} \\ &= \mu_o n^2 V \end{aligned}$$

where we used the fact that the volume of the solenoid is $V = A\ell$.

Many inductors built for use in electronics are just this, air filled solenoids. So this really is a somewhat practical solution.

Energy in a Magnetic Field

Question 223.46.3

An inductor, like a capacitor, stores energy in it's field. We would like to know how much energy an inductor can store. From basic circuit theory we know the power in a circuit will be

$$\mathcal{P} = I\Delta V$$

If we just have an inductor, then the power removed from the circuit is

$$\begin{aligned}\mathcal{P}_{cir} &= I\Delta V = I\mathcal{E} \\ &= I\left(-L\frac{dI}{dt}\right) \\ &= -LI\frac{dI}{dt}\end{aligned}$$

As with a resistor, we are taking power *from the circuit* so the result is negative. But unlike a resistor, this power is not being dissipated as heat. It is going into the magnetic field of the inductor. Therefore, we expect the power stored in the inductor field to be

$$\mathcal{P}_L = -\mathcal{P}_{cir} = LI\frac{dI}{dt}$$

Power is the time rate of change of energy , so we can write this power delivered to the inductor as

$$\frac{dU_L}{dt} = LI\frac{dI}{dt}$$

Multiplying by dt gives

$$dU_L = LIdI$$

To find the total energy stored in the inductor we must integrate over I.

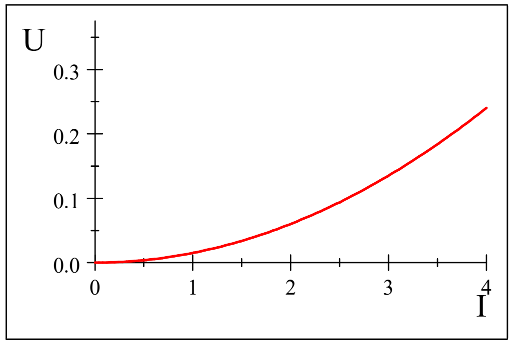
$$\begin{aligned}U_L &= \int dU_L \\ &= \int_0^I LIdI \\ &= L\int_0^I IdI \\ &= \frac{1}{2}LI^2\end{aligned}$$

Thus,

$$U_L = \frac{1}{2}LI^2$$

is the energy stored in the magnetic field of the inductor.

Suppose we have an inductor $L = 30.0 \times 10^{-3}$ H. Plotting shows us the dependence of U_L on I.



We should take a moment to see how our inductor compares to a capacitor as an energy storage device. The energy stored in the electric field of a capacitor

$$U_L = \frac{1}{2}L\left(I\right)^2$$

$$U_C = \frac{1}{2}C\left(\Delta V\right)^2$$

Notice that Remarkable similarity!

Energy Density in the magnetic field

Question 223.46.4

We found that there was energy stored in the electric field of a capacitor. Is the energy stored in the inductor really stored in the magnetic field of the inductor? We believe that this is just the case, the energy, U_L, is stored in the field. We would like to have an expression for the density of the energy in the field.

To see this, let's start with the inductance of a solenoid.

$$L = \mu_o n^2 A\ell$$

The magnetic field is given by

$$B = \mu_o nI$$

then the energy in the field is given by

$$\begin{aligned} U_B &= \frac{1}{2}LI^2 \\ &= \frac{1}{2}\mu_o n^2 A\ell I^2 \end{aligned}$$

If we rearrange this, we can see the solenoid field is found in the expression twice

$$\begin{aligned} U_B &= \frac{1}{2}\left(\mu_o n I\right) A\ell \frac{\mu_o}{\mu_o} n I \\ &= \frac{1}{2\mu_o} B^2 A\ell \end{aligned}$$

and the energy density is

$$\begin{aligned} u_B &= \frac{U_B}{A\ell} \\ &= \frac{1}{2}\frac{1}{\mu_o} B^2 \end{aligned}$$

Just like our energy density for the electric field, we derived this for a specific case, a solenoid. But this expression is general. We should compare to the energy density in the electric field.

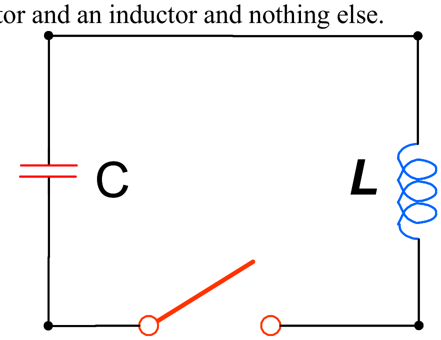
$$u_E = \frac{1}{2}\epsilon_o E^2$$

Again, note the similarity!

Oscillations in an LC Circuit

We introduce a new circuit symbol for inductors

It looks like a coil, for obvious reasons. We can place this new circuit element in a circuit. But what will it do? To investigate this, let's start with a simple case, a circuit with a charged capacitor and an inductor and nothing else.



Let us make two unrealistic assumptions (we will relax these assumptions later).

> Assumption 1: There is no resistance in our LC circuit.
> Assumption 2: There is no radiation emitted from the circuit.

Given these two assumptions, there is no mechanism for energy to escape the circuit.

Energy must be conserved. Can we describe the charge on the capacitor, the current, and the energy as a function of time?

Question 223.46.5

It may pay off to recall some details of oscillators. Energy of the Simple Harmonic Oscillator

Remember from Dynamics or PH121 that a mass-spring system will oscillate. The mass has kinetic energy because the mass is moving

$$K = \frac{1}{2}mv^2 \tag{46.1}$$

for our Simple Harmonic Oscillator we know that the position of the mass as a function of time is given by

$$x(t) = x_{\max}\cos(\omega t + \phi)$$

and the speed as a function of time is

$$v(t) = -\omega x_{\max}\sin(\omega t + \phi)$$

then the kinetic energy as a function of time is

$$\begin{aligned} K &= \frac{1}{2}m\left(-\omega x_{\max}\sin(\omega t + \phi)\right)^2 \\ &= \frac{1}{2}m\omega^2 x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}m\frac{k}{m}x_{\max}^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi) \end{aligned}$$

The spring has potential energy given by

$$U = \frac{1}{2}kx^2 \tag{46.2}$$

For our mechanical oscillator the potential as a function of time is

$$U = \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi)$$

The total energy is given by

$$\begin{aligned}
E &= K + U \\
&= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi) + \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi) \\
&= \frac{1}{2}kx_{\max}^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) \\
&= \frac{1}{2}kx_{\max}^2
\end{aligned}$$

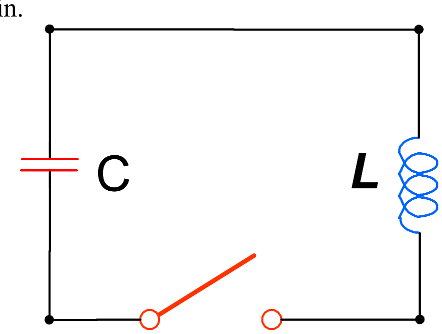
If we plot the kinetic and potential energies it looks like this

We can see that the total energy won't change, and the energy switches back and forth from kinetic to potential as the mass moves back and forth. If we plot the kinetic and potential energy at points along the mass' path we get something like this.

Question 223.46.6

One of the important uses of an inductor is to create *electrical oscillations.* Having recalled what oscillations look like, we can see that a LC circuit will have an oscillating current.

here is our circuit again.



We will start with the switch open the capacitor charged to its maximum value $Q_{\max}$. For $t > 0$ the switch is closed. Recall that the energy stored in the capacitor is

$$U_C = \frac{Q^2}{2C}$$

and the energy stored in the inductor is

$$U_L = \frac{1}{2} I^2 L$$

The total energy (because of our assumptions) is

$$\begin{aligned} U &= U_C + U_L \\ &= \frac{Q^2}{2C} + \frac{1}{2} I^2 L \end{aligned}$$

The change in energy over time must be zero (again because of our assumptions) so

$$\begin{aligned} \frac{dU}{dt} &= 0 \\ &= \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} I^2 L\right) \\ &= \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \end{aligned}$$

We recall that

$$I = \frac{dQ}{dt}$$

$$\begin{aligned} 0 &= \frac{Q}{C}\left(\frac{dQ}{dt}\right) + LI\frac{dI}{dt} \\ 0 &= \frac{Q}{C}(I) + LI\frac{dI}{dt} \\ 0 &= \frac{Q}{C}I + LI\frac{d\left(\frac{dQ}{dt}\right)}{dt} \\ 0 &= \frac{Q}{C} + L\frac{d^2Q}{dt^2} \end{aligned}$$

or

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$

This is a differential equation that we recognize from M316. It looks just like the differential equation for oscillatory motion! We try a solution of the form

$$Q = A\cos(\omega t + \phi)$$

then

$$\frac{dQ}{dt} = -A\omega \sin(\omega t + \phi)$$

and

$$\frac{d^2Q}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$$

thus

$$A\omega^2 \cos(\omega t + \phi) = -\frac{1}{LC} A\cos(\omega t + \phi)$$

This is indeed a solution if

$$\omega = \frac{1}{\sqrt{LC}}$$

When $\cos(\omega t+\phi)=1$, $Q=Q_{\max}$, thus

$$Q=Q_{\max}\cos(\omega t+\phi)$$

Now recall,

$$\begin{aligned} I &= \frac{dQ}{dt} \\ &= \frac{d}{dt}\left(Q_{\max}\cos(\omega t+\phi)\right) \\ &= -\omega Q_{\max}\sin(\omega t+\phi) \end{aligned}$$

We would like to determine ϕ. We use the initial conditions $t=0$, $I=0$ and $Q=Q_{\max}$. Then

$$0=-\omega Q_{\max}\sin(\phi)$$

This is true for $\phi=0$. Then

$$\begin{aligned} Q &= Q_{\max}\cos(\omega t) \\ I &= -\omega Q_{\max}\sin(\omega t) \\ &= -I_{\max}\sin(\omega t) \end{aligned}$$

We can use the solution for the charge on the capacitor and the current in the inductor as a function of time to expand our energy equation

$$\begin{aligned} U &= U_C+U_L \\ &= \frac{Q^2}{2C}+\frac{1}{2}I^2L \\ &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t)+\frac{1}{2}LI_{\max}^2\sin^2(\omega t) \end{aligned}$$

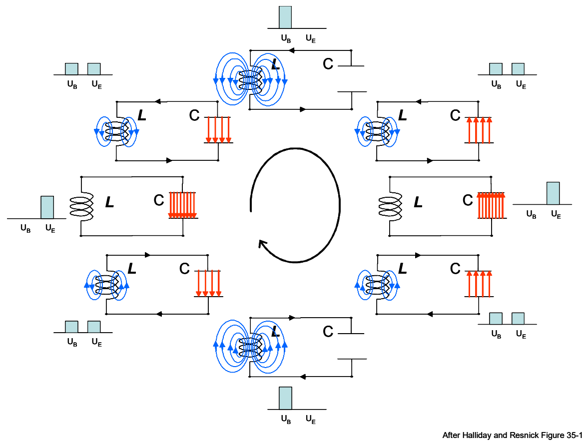
This looks a lot like our kinetic and potential energy equation for a mass-spring system. The energy shifts from the capacitor to the inductor and back like energy shifted from kinetic to potential energy for our mass-spring, with the components out of phase by $90\,^{\circ}$. By energy conservation, we know that

$$\frac{1}{2C}Q_{\max}^2=\frac{1}{2}LI_{\max}^2$$

that is, the maximum energy in the capacitor equals the maximum energy in the inductor. Then the total energy

$$\begin{aligned} U &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t)+\frac{1}{2}LI_{\max}^2\sin^2(\omega t) \\ &= \frac{1}{2C}Q_{\max}^2\cos^2(\omega t)+\frac{1}{2C}Q_{\max}^2\sin^2(\omega t) \\ &= \frac{Q_{\max}^2}{2C} \end{aligned}$$

which must be the case if energy is conserved. We can plot the capacitor and inductor energies at points in time as the current switches back and forth.

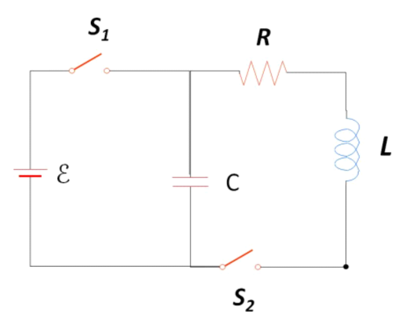


This is very much like our harmonic oscillator picture. We can see that we have, indeed made an electronic oscillator.

This type of circuit is a major component of radios which need a local oscillatory circuit to operate.

The RLC circuit

As fascinating as the last section was, we know there really is some resistance in the wire. So the restriction of no resistance needs to be relaxed in our analysis.



We can use the circuit in the picture to imagine an LRC circuit. At first, we will keep S_2 open and close S_1 to charge up the capacitor. Then we will close S_1 and open S_2. What will happen?

It is easier to find the current and charge on the capacitor as a function of time by using energy arguments. The resistor will remove energy from the circuit by dissipation (getting hot). The circuit has energy

$$U = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \tag{46.3}$$

so from the work energy theorem,

$$W_{nc} = \Delta U$$

the energy lost will be related to a change in the energy in the capacitor and the inductor. Let's look at the rate of energy loss again

$$\begin{aligned} \frac{dU}{dt} &= \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) \\ &= \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \end{aligned} \tag{46.4}$$

but this must be equal to the loss rate. The power lost will be $P = I^2 R$

$$-I^2 R = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \tag{46.5}$$

This is a differential equation we can solve, let's first rearrange, remembering that

$$I = \frac{dQ}{dt}$$

then

$$\begin{aligned} -I^2 R &= \frac{Q}{C}I + LI\frac{dI}{dt} \\ -IR &= \frac{Q}{C} + L\frac{dI}{dt} \end{aligned}$$

again using $I = \frac{dQ}{dt}$

$$+L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C} = 0 \tag{46.6}$$

This is a good exercise for those of you who have taken math 316. This is just like the equation governing a damped harmonic oscillator. The solution is

$$Q = Q_{\max}e^{-\frac{Rt}{2L}}\cos\omega_d t \tag{46.7}$$

where the angular frequency, ω_d is given by

$$\omega_d = \left(\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right)^{\frac{1}{2}} \tag{46.8}$$

Remember that for a damped harmonic oscillator

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

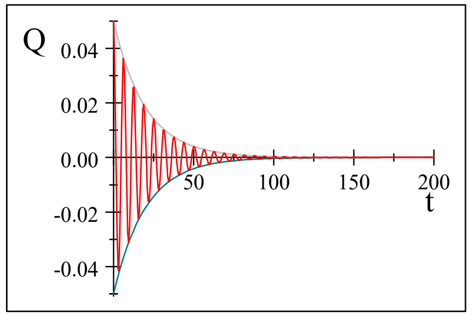
and

$$\omega = \left(\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right)^{\frac{1}{2}}$$

The resistance acts like a damping coefficient! Suppose

$$
\begin{aligned}
&Q_{\max} = 0.05\,\mathrm{C}\\
&R = 5\,\Omega\\
&L = 50\,\mathrm{H}\\
&C = 0.02\,\mathrm{F}
\end{aligned}
$$

we have a graph that looks like this.



The gray lines are

$$\pm Q_{\max} e^{-\frac{Rt}{2L}} \tag{46.9}$$

They describe how the amplitude changes. We call this the *envelope* of the curve.

Let's look at

$$\omega_d = \left(\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right)^{\frac{1}{2}} \tag{46.10}$$

If $\omega_d = 0$ then

$$
\begin{aligned}
0 &= \frac{1}{LC} - \left(\frac{R}{2L}\right)^2\\
\frac{1}{LC} &= \left(\frac{R}{2L}\right)^2\\
2L\sqrt{\frac{1}{LC}} &= R
\end{aligned}
$$

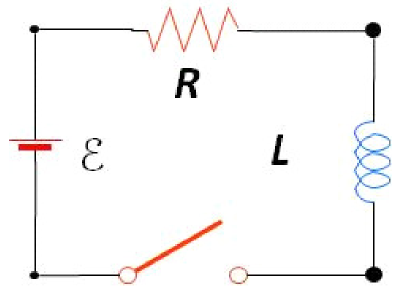
or

$$R = \sqrt{\frac{4L}{C}} \tag{46.11}$$

We know that if $\omega_d = 0$ there is no oscillation. We will call this the critical resistance, R_c. When the resistance is $R \geq R_c$ there will be no oscillation. These represent the cases of being critically damped $(R = R_c)$ and overdamped $(R > R_c)$. If $R < R_c$ we are underdamped, and the circuit will oscillate.

We don't know how to make electromagnetic waves yet, but we will in a few lecture. Those waves carry what we call radio signals. To make the waves, we often use circuits with resisters, capacitors, and inductors to provide the oscillation. You can guess that if Q on the capacitor oscillates, so does the current. This oscillating current is what we use to drive the radio antenna.

Now that we have some resistance, we could consider a circuit with just an inductor and a resistor and a battery.



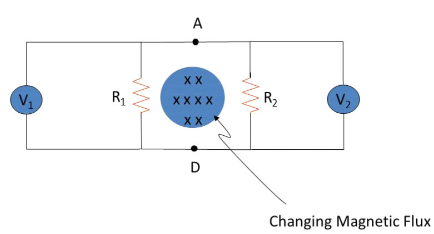
This is a little harder than our author indicates. We will examine the difficulties in thinking about such a circuit in the next section.

Return to Non-Conservative Fields

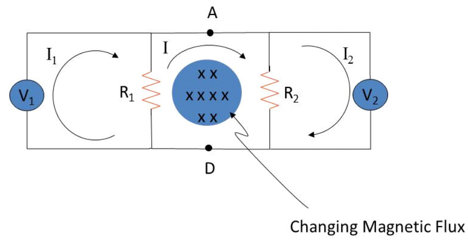
A few decades ago, we could have stopped here in an engineering class in considering and LRC circuit. But as electrical devices become every more complicated, it might be good if we examine circuits with inductors and resistors more carefully. A few lectures ago we found that

$$\int \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

implies a non-conservative electric field. We should take a moment to see what this means. We should also, if we have time, investigate mutual inductance, which has become a major engineering technique for wireless power. First let's consider the following circuit.[**?**]



notice that there is no battery. If the field flux changes, will there be a potential difference measured by the voltmeters? Let's use Kirchhoff's rules to analyze the circuit. I can draw in guesses for the currents.



and now we use the junction and loop rules to find the voltages.

But recall that

$$\oint \mathbf{E} \cdot ds = 0$$

was the basis for Kirchhoff's loop rule. And we learned that this is not true for induced emfs. So in the middle loop Kirchhoff's loop rule is not true! We now know that because of the changing magnetic field,

$$\oint \mathbf{E} \cdot ds = \mathcal{E} = -\frac{d\Phi_B}{dt}$$

for this middle loop. Now then, $\mathcal{E}$ comes just from the changing external flux. It does *not* depend on R_1 or on R_2.

We can write a Kirchhoff's law-like equation for each loop.

$$\begin{aligned} I_1 R_i - I R_1 &= 0 \\ -I R_1 - I R_2 + \mathcal{E} &= 0 \\ I_2 R_i - I R_2 &= 0 \end{aligned}$$

where R_i is the internal resistance of the voltmeters. If there were no $\mathcal{E}$, then the volt meters would certainly not read anything, but now we see that

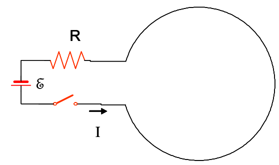
$$\begin{aligned} |V_1| &= I_1 R_i \approx I R_1 \\ |V_2| &= I_2 R_i \approx I R_2 \end{aligned}$$

This seems crazy. Each volt meter reads a different voltage.

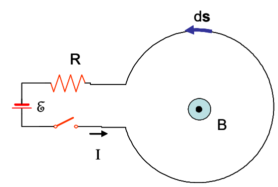
To understand this, remember that our induced field is not a conservative field. As we go around the loop we no longer expect to get back to our starting voltage. We have lost some energy in making a magnetic field. And for non-conservative fields, $\oint \mathbf{E} \cdot ds$ *is path dependent*.

So as crazy as it seems, this is actually what we would find, each volt meter reads a different voltage.

To try to make this idea of inductance make some sense, let's take another strange circuit.



There is a battery, and resister, and a single loop inductor. When the switch is thrown, the current will flow as shown. The current will create a magnetic field that is out of the page in the center of the loop. Since the loop, itself, is creating this field, let's call this field a *self field*.



Consider this self-field for a moment. When we studied charge, we found that charge created an electric field. That electric field could make *another* charge accelerate. But the electric field crated by a charge does not make the charge that created it accelerate. This is an instance of a self-field, an electric self-field. Now with this background, let's return to our magnetic self-field.

Let's take Faraday's law and apply it to this circuit. Let me choose an area vector **A** that is the area of the big loop and positive out of the page. Again, let's use Kirchhoff's loop law. Let's find $\oint \mathbf{E} \cdot ds$ for the entire circuit. We can start with the battery. Since there is an electric field inside the battery we will have a component of $\oint_{bat} \mathbf{E} \cdot ds$ as we cross it. The battery field goes from positive to negative. If we go counter-clockwise, our ds direction traverses this from negative to positive, so the electric field is up and the ds direction is down, we have

$$\oint_{bat} \mathbf{E} \cdot ds = -\mathcal{E}_{bat}$$

for this section of the circuit. Suppose we have ideal wires. If the wire has no resistance, then it takes no work to move the charges through the wire. In this case, an electron launched by the electric in the battery just coasts from the battery to the resister. There is no need to have an acceleration in the ideal wire. The electric potential won't change from the battery to the resister. So there won't be a field in this ideal wire part. But let's next we consider the resister. There is a potential change as we go across it. And if there is a change in potential, there must be an electric field. So the resister also has an electric field inside of it. We have a component of $\oint_{R} \mathbf{E} \cdot ds$ that is equal to $\mathcal{E}_R = IR$

from this field.

$$\oint_R \mathbf{E} \cdot ds = IR$$

Now we come to the big loop part. Since we have ideal wire, there is no resistance in this part so there is no voltage drop for this part of the circuit. All the energy that was given to the electrons by the battery was lost in the resister. They just coast back to the other terminal of the battery. Since there is no voltage drop in the big loop,

$$\mathcal{E}_{\text{big loop}} = 0$$

there is no electric field in the big loop either. Along the big loop, ds is certainly not zero. so

$$\mathcal{E}_{\text{big loop}} = \oint_{\text{big loop}} \mathbf{E} \cdot ds = 0$$

For the total loop we would have

$$\oint \mathbf{E} \cdot ds = -\mathcal{E}_{batt} + IR + 0 \tag{46.12}$$

Normally, Kirchhoff's loop rule would say that all this must be zero, since the sum of the energy changes around the loop must be zero if no energy is lost. But now we know energy *is* lost in making a magnetic field.

Consider the magnetic flux through the circuit. The magnetic field is made by the current in the circuit. Note that we arranged the circuit so the battery and resister are in a part that has very little area, so we can ignore the flux through that part of the circuit. Most of the flux will go through the big loop part. The magnetic field is out of the paper inside of the loop. The flux is

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{A} \tag{46.13}$$

and $\mathbf{B}$ and $\mathbf{A}$ are in the same direction. Φ_B is positive.

Then from Biot-Savart

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \tag{46.14}$$

Let me write this as

$$\begin{aligned} \mathbf{B} &= I \left(\frac{\mu_o}{4\pi} \oint \frac{d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \right) \\ &= I \,(\text{geometry factor}) \end{aligned} \tag{46.15}$$

If the geometry of the situation does not change, then B and I are proportional. Since $B \propto I$, then $\Phi_B \propto I$ since the integral in Biot-Savart is the surface integral of $\mathbf{B}$, and $\mathbf{B}$ is everywhere proportional to I. Instead of using Biot-Savart, I wish to just define a constant of proportionality that will contain all the geometric factors. I will simply say that

$$\Phi_B = LI \tag{46.16}$$

where L is my geometry factor. This geometry factor is just our inductance! This is what inductance is. It is all the geometry factors that make up our loop that will make the magnetic field if we put a current through it.

Assuming I don't change the geometry, then the inductance won't change and we have

$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt} \tag{46.17}$$

and Faraday's law gives us

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} \tag{46.18}$$

Which says that we should not have expected $\oint \mathbf{E} \cdot ds = 0$ for our case as we traverse the entire circuit. Integrating $\oint \mathbf{E} \cdot ds$ around the whole circuit including the big loop should not bring us back to zero voltage. We have lost energy in making the field. Instead it gives

$$\oint \mathbf{E} \cdot ds = -L\frac{dI}{dt}$$

We are dealing with non-conservative fields. So we have some energy loss like we would with a frictional force. It took some energy to make the magnetic field!

With this insight, we can now make a Kirchhoff-like loop like rule for such a situation. Integrating around the whole circuit gives

$$\oint \mathbf{E} \cdot ds = -\mathcal{E}_{bat} + \mathcal{E}_R$$

Which we now realize should give $-L\frac{dI}{dt}$ so

$$\oint \mathbf{E} \cdot ds = -\mathcal{E}_{bat} + \mathcal{E}_R = -L\frac{dI}{dt}$$

or more succinctly

$$-\mathcal{E}_{batt} + IR = -L\frac{dI}{dt}$$

Now I can take the RHS to the left and find

$$\mathcal{E}_{batt} - IR - L\frac{dI}{dt} = 0 \tag{46.19}$$

which certainly *looks* like Kirchhoff's rule with $-L\frac{dI}{dt}$ being a voltage drop across the single loop inductor. Under most conditions we can just treat $-L\frac{dI}{dt}$ as a voltage drop and it works fine. Most of the time thinking this way does not cause much of a problem. But technically it is not right!

We should consider where our magnetic flux came from. The magnetic flux was created by the current. It is a self-field. The current can't make a magnetic flux that would then modify that current. This self-flux won't make an electric field So there is no electric field in the big loop, so there is no potential drop in that part of the circuit. It is just that $\oint \mathbf{E} \cdot ds \neq 0$ because our field is not conservative. We had to take some energy to create the magnetic field.

Now, if you are doing simple circuit design, you can pretend you don't know about Faraday's law and this complication and just treat $-L\frac{dI}{dt}$ as though it were a voltage drop. But really it is just that going around the loop we should expect

$$\oint \mathbf{E} \cdot ds = L\frac{dI}{dt}$$

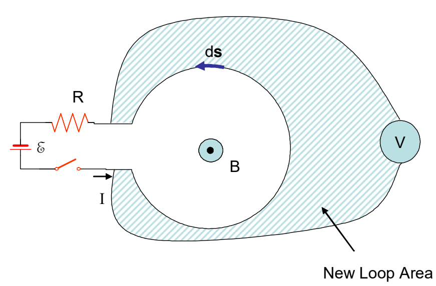
not

$$\oint \mathbf{E} \cdot ds = 0$$

The danger is that if you are designing a complicated device that depends on there being an electric field in the inductor, your device will not work. We have *no external magnetic field,* our only magnetic field is the *self-field* which will not produce an electric field (or at least will form a very small electric field compared to the electric fields in the resistor and the battery, due to the small resistance in the real wire we use to make the big loop).

This is very subtile, and I struggle to remember this! Fortunately in most circuit design it does not matter. We just treat the inductor as though it were a true voltage drop.

I can make it even more exasperating by asking what you will see if you place a voltmeter across the inductor. What I measure is a "voltage drop" of LdI/dt, so maybe the there is a voltage drop after all! But no, that is not right. The problem is that in introducing the voltmeter, we have created a new loop. For this loop, the field from our big loop *is* an external field. .

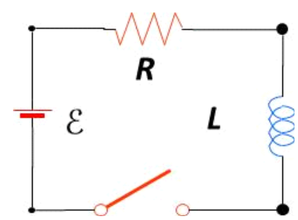


So the changing magnetic field through this voltmeter loop will produce an emf that will just match LdI/dt. And there will be an electric field–but it will be in the internal resistor in the voltmeter. And that is what you will measure!

Pick it up here

The bottom line is that for non-conservative fields you need to be careful. If you are just designing simple circuits, you can just treat LdI/dt *as though it were a voltage drop*, but you may be badly burned by this if your system is more complicated, depending on the existence of a real electric field. You can see that if you are designing complicated sensing devices, you may need to deeply understand the underlying physics to get them to work.

RL Circuits: Solving for the current as a function of time



The equation we found from Faraday's law or incorrectly from Kirchhoff's rule is

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \tag{46.20}$$

This is a differential equation. We can solve it for the current. To do so, let's define a variable

$$x = \frac{\mathcal{E}}{R} - I$$

and then we see that

$$dx = -dI$$

Then we can write our differential equation as

$$\begin{aligned} \frac{\mathcal{E}}{R} - I - \frac{L}{R}\frac{dI}{dt} &= 0 \\ x + \frac{L}{R}\frac{dx}{dt} &= 0 \end{aligned}$$

and so

$$x = -\frac{L}{R}\frac{dx}{dt}$$

You might be able to guess the solution at this point from your M316 experience. But let's work it out as a review. We see that our x equation separates into

$$\frac{dx}{x} = -\frac{R}{L}dt$$

Integration yields

$$\int_{x_o}^{x} \frac{dx}{x} = -\int_{0}^{t} \frac{R}{L}dt$$

$$\ln\left(\frac{x}{x_o}\right) = -\frac{R}{L}t$$

exponentiating both sides gives

$$\left(\frac{x}{x_o}\right) = e^{-\frac{R}{L}t}$$

Now we replace x with $\frac{\mathcal{E}}{R} - I$

$$\left(\frac{\frac{\mathcal{E}}{R} - I}{\frac{\mathcal{E}}{R} - I_o}\right) = e^{-\frac{R}{L}t}$$

And because at $t = 0$, $I = 0$

$$\left(\frac{\frac{\mathcal{E}}{R} - I}{\frac{\mathcal{E}}{R}}\right) = e^{-\frac{R}{L}t}$$

rearranging gives

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \tag{46.21}$$

or, defining another time constant

$$\tau = \frac{L}{R} \tag{46.22}$$

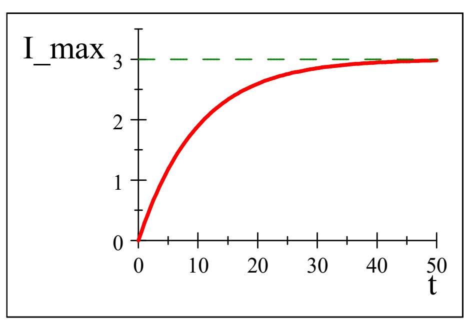
we have

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \tag{46.23}$$

We can see that

$$\frac{\mathcal{E}}{R} = I_{\max} \tag{46.24}$$

comes from Ohm's law. So just like with our capacitor-resister circuit, we have a current that grows in time, approaching the maximum value we get after a time t which is much longer than τ.



You might expect that, like for a capacitor, there is an equation for an inductor who has a maximum current flowing but for which the current source is shorted (disconnected, and replaced with a resistanceless wire). The equation is

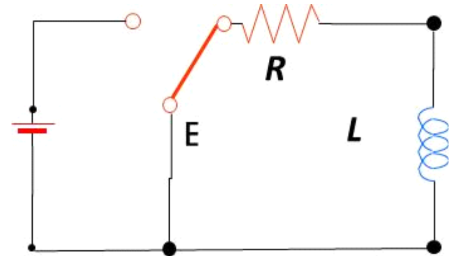
$$I = I_o e^{-\frac{t}{\tau}} \tag{46.25}$$

Magnetic Field Energy in Circuits

We found last lecture that just like with a RC circuit, we should expect there to be energy stored in a RL circuit.

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}C\left(\Delta V\right)^2$$

Consider once again the RL circuit shown below.



Recall that the current in the right-hand loop decays exponentially with time according to the expression

$$I = I_o e^{-\frac{t}{\tau}}$$

where $I_o = \mathcal{E}/R$ is the initial current in the circuit and $\tau = L/R$ is the time constant. As an example problem, let's show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

Recall that energy is delivered to the resister

$$\frac{dU}{dt} = P = I^2R$$

where I is the instantaneous current.

$$\begin{aligned}
\frac{dU}{dt} &= I^2R \\
\frac{dU}{dt} &= \left(I_o e^{-\frac{t}{\tau}}\right)^2 R \\
\frac{dU}{dt} &= I_o^2 e^{-2\frac{t}{\tau}} R
\end{aligned}$$

To find the total energy delivered to the resister we integrate

$$dU = I_o^2 e^{-2\frac{t}{\tau}} R dt$$

$$\int dU = \int_0^\infty I_o^2 e^{-2\frac{t}{\tau}} R dt$$
$$U = \int_0^\infty I_o^2 e^{-2\frac{t}{\tau}} R dt$$
$$U = I_o^2 R \int_0^\infty e^{-2\frac{t}{\tau}} dt$$

Use your calculator, or an integral table, or Maple, or Scientific Workplace or your very good memory to recall that

$$\int e^{-ax} dx = -\frac{1}{a} e^{-ax}$$

If we let

$$a = -\frac{2}{\tau}$$

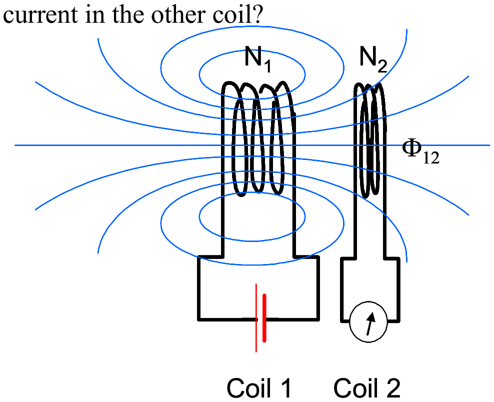
then we can obtain

$$U = -\frac{L}{2R} I_o^2 R e^{-2\frac{t}{\tau}} \Big|_0^\infty$$
$$U = \frac{-L}{2} I_o^2 (0 - 1)$$
$$U = \frac{1}{2} I_o^2 L \tag{46.26}$$

which is the initial energy stored in the magnetic field. All of the energy that started in the inductor was delivered to the resistor.

Mutual Induction

Suppose we have two coils near each other. If either of the coils carries a current, will there be an induced current in the other coil?



We define Φ_{12} as the flux through coil 2 due to the current in coil 1. Likewise if the battery is placed on coil 2 we would have Φ_{21}, the flux through coil 1 due to the current in coil 2.

We define the mutual inductance

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1} \tag{46.27}$$

BE CAREFUL! Not all books write the subscripts in the same order!

We can write the flux as

$$\Phi_{12} = \frac{M_{12} I_1}{N_2}$$

Then, using Faraday's law, we find the induced emf in coil 2

$$\begin{aligned} \mathcal{E}_2 &= -N_2 \frac{d\Phi_B}{dt} \\ &= -N_2 \frac{d}{dt} \left(\frac{M_{12} I_1}{N_2} \right) \\ &= -M_{12} \frac{d}{dt} (I_1) \end{aligned}$$

We state without proof the $M_{12} = M_{21}$. Then

$$\mathcal{E}_2 = -M \frac{dI_1}{dt}$$

Example : "Wireless" battery charger

Rechargeable Toothbrush with an inductive charger (Public Domain Image courtesy Jonas Bergsten)

A rechargeable toothbrush needs a connection that is not affected by water. We can use induction to form this connection. We need two coils. One coil is the base, the other the handle. The base carries current I. The base has length l and area A and N_B turns. The handle has N_H turns and completely covers the base solenoid. What is the mutual inductance?

Solution:

The magnetic field in the base solenoid is

$$\oint \mathbf{B}\cdot d\mathbf{s} = \mathbf{B}\,\boldsymbol{\ell} = \mu_o N_B I$$

or

$$B = \frac{\mu_o N_B I_B}{\ell}$$

Because the handle surrounds the base, the flux through the handle is the interior field of the base. The flux is

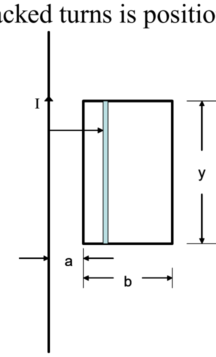
$$\Phi_{BH} = BA$$

The mutual inductance is

$$\begin{aligned} M &= \frac{N_H \Phi_{BH}}{I_B} \\ &= \frac{N_H B A}{I_B} \\ &= \frac{N_H \left(\frac{\mu_o N_B I_B}{\ell}\right) A}{I_B} \\ &= \mu_o \frac{N_H N_B A}{\ell} \end{aligned}$$

Example: Rectangular Loop and a coil

A rectangular loop of N close-packed turns is positioned near a long straight wire.



What is the coefficient of mutual inductance M for the loop-wire combination?

The basic equations are

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I$$
$$\oint \mathbf{B} \cdot d\mathbf{A} = \mathbf{\Phi}_B$$

The field from the wire

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I$$

Take the path to be a circle surrounding the wire then $\mathbf{B}$ is constant along the path and the direction of $\mathbf{B}$ is tangent to the path.

$$\begin{aligned} B \oint ds &= \mu_o I \\ B 2\pi r &= \mu_o I \end{aligned}$$

or

$$B = \frac{\mu_o I}{2\pi r}$$

The flux through the rectangular loop is then perpendicular to the plane of the loop

$$\oint \mathbf{B} \cdot d\mathbf{A} = \mathbf{\Phi}_B$$

$$\begin{aligned} \mathbf{\Phi}_B &= \int B y dr \\ &= \int_a^{b+a} \frac{\mu_o I}{2\pi r} y dr \\ &= \frac{\mu_o I y}{2\pi} \ln \frac{b+a}{a} \end{aligned}$$

then

$$M = N \frac{\mu_o y}{2\pi} \ln \frac{b+a}{a}$$

Suppose the loop has $N = 100$ turns, $a = 1\,\mathrm{cm}$, $b = 8\,\mathrm{cm}$, $y = 30\,\mathrm{cm}$, $\mu_o = 4\pi \times 10^{-7} \frac{\mathrm{T\,m}}{\mathrm{A}}$ what is the value of the mutual inductance?

$$M = N \frac{\mu_o y}{2\pi} \ln \frac{b+a}{a} = \frac{1.318\,3 \times 10^{-3}}{\mathrm{A}} \mathrm{T\,m\,cm} = \frac{1.318\,3 \times 10^{-5}}{\mathrm{A}^2} \frac{\mathrm{m}^2}{\mathrm{s}^2} \mathrm{kg}$$

$$\mathrm{H} = \frac{1}{\mathrm{A}^2} \frac{\mathrm{m}^2}{\mathrm{s}^2} \mathrm{kg}$$

Basic Equations