Chapter 29

Displacement Current and Field Equations

We started this class with a study of waves. We learned about optics, and finally electromagnetic field theory. In this lecture we will take on a case study that involves all three. We will have come full circle and in the process, hopefully understand all three topics a little better.

Fundamental Concepts

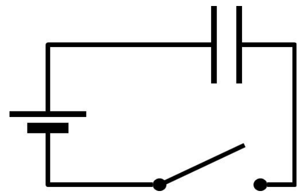
- Changing electric fields produce magnetic fields
- A changing electric flux is described as a displacement current $I_d = \varepsilon_o \frac{d\Phi_E}{dt}$
- The complete version of Ampere's law is $\oint \mathbf{B} \cdot d\ell = \mu_o \left(I + I_d\right)$
- Maxwell's equations give a complete classical picture of electromagnetic fields
- Maxwell's equations plus the Lorentz force describe all of electrodynamics.

29.1 Displacement Current

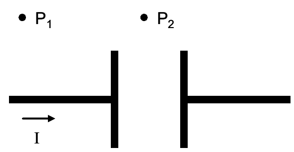
Last time we listed Ampere's law as one of the basic field equations. But we did not discuss it at all. That is because we were saving it for our discussion in this lecture We need to look deeply into Ampere's law. Here is what we have for Ampere's law so far

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I_{through}$$

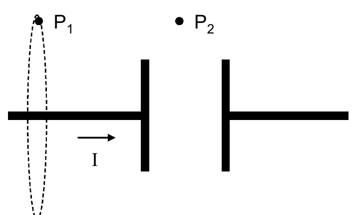
To see why we need to consider it further, let's do a hard problem with Ampere's law. Let's set up a circuit with a battery a switch and a circular plate capacitor in the wire.



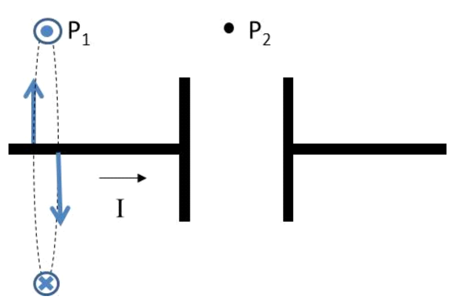
Using this circuit, let's calculate the magnetic field using Ampere's law. Here is a detailed diagram of the capacitor.



I could find the magnetic field using the Biot-Savart equation, but that would be hard. I don't know how to solve the resulting integral. So let's try Ampere's law. Let's start at P_1. We add in an imaginary surface at P_1. I will choose a simple circular surface.



We have done this before. If we choose P_1 so that it is far from the capacitor, then we know what the magnetic field will look like.



Right at P_1 it will be out of the page. We also know that for a long straight wire, the field magnitude does not change as we go around the wire, so we can write our integral as

$$\oint \mathbf{B} \cdot d\ell = B \oint d\ell = B2\pi r = \mu_o I$$

so

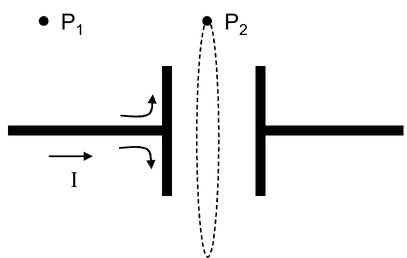
$$B2\pi r = \mu_o I$$

so the field is

$$B = \frac{\mu_o I}{2\pi r}$$

which is very familiar, just the equation for a field from a long straight wire.

Now Let's try this at P_2. What would we expect? Will the magnetic field change much as we pass by the capacitor?



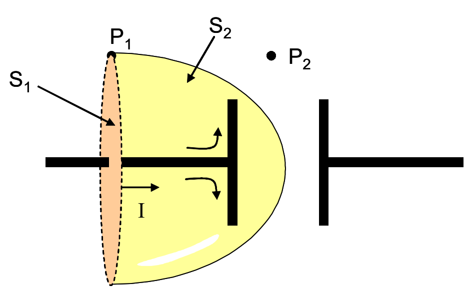
Again we could use Biot-Savart, but think about what the current does at the plate. It would be very very hard to do the integration!. So again let's try Ampere's law. If we use the same size surface

$$\oint \mathbf{B} \cdot d\ell = B \oint d\ell = B2\pi r$$

but this is equal to $\mu_o I_{through}$. There is no I going through the capacitor! so

$$B2\pi r = 0 \tag{29.1}$$

and this would give $B = 0$. But, our wires are not really ideal and infinitely long. And even if they were, would we really expect the field to be zero if we just have a small gap in our capacitor? It get's even worse!

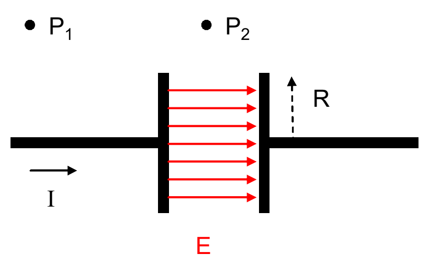


Ampere's law tells us we need a surface, but it does not say it has to be a circular surface. In fact, we could use the strange surface labeled S_2 in the figure above. This is a perfectly good surface to associate with the loop at P_1. So this gives us

$$\oint \mathbf{B} \cdot d\ell = \mu_o I = 0$$

at P_1! So we have two different results with Ampere's law for the same point. This can't be!

Ampere knew this was a problem, but did not find a solution. Maxwell solved this. He asked himself, what was different inside the capacitor that might be making a difference. Of course, there is an electric field inside the capacitor!



We know that in the limit that the plates can be considered to be very big the field is approximately

$$E = \frac{\eta}{\varepsilon_o} = \frac{Q}{\pi R^2 \varepsilon_o}$$

but we know that the charge is changing in time once the switch is thrown. We can find the rate of change of the field, then

$$\frac{dE}{dt} = \frac{1}{\pi R^2 \varepsilon_o} \frac{dQ}{dt}$$

By definition

$$I = \frac{dQ}{dt}$$

is a current, but what current? It must be the current that is supplying the charge to the capacitor. That current is what is changing the Q in the capacitor, and it is the Q separation that is making the field. So the time derivative of the electric field is

$$\frac{dE}{dt} = \frac{I}{\pi R^2 \varepsilon_o}$$

where I is the current in the wire, and only if the wire current is zero will there be no change in the electric field.

This gives us an idea. A changing electric field creates a magnetic field. Suppose this changing electric field created a magnetic field like the current does? It would as though there were a current with a value

$$I_d = \pi R^2 \varepsilon_o \frac{dE}{dt} \tag{29.2}$$

It doesn't really cause a current in the capacitor. What really happens is that the changing electric field is creating a magnetic field. But that magnetic field is just like the field that a current would create. So we can (somewhat incorrectly) say that the changing electric field has created something like a current in the capacitor. But no charge is crossing the capacitor.

Note that in this we have the area of the plate, $A_{plate} = \pi R^2$ multiplied by the time rate of change of the electric field. Also note, that in our approximation for our capacitor, there is only an electric field inside the plates. So, remembering electric flux,

$$\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$$

our flux though the surface at P_2 would be

$$\begin{aligned}\Phi_E &= EA \\ &= \pi R^2 E\end{aligned}$$

so we can identify

$$\pi R^2 dE = A_{plate} dE = d\Phi_E$$

as a small amount of *electric* flux. Then our equivalent current will be

$$I_d = \varepsilon_o \frac{d\Phi_E}{dt} \tag{29.3}$$

Maxwell decided that, since this looked like equivalent to a current, he would call it a current and include it in Ampere's law.

$$\begin{aligned}\oint \mathbf{B} \cdot d\ell &= \mu_o \left(I + I_d\right) \\ &= \mu_o \left(I + \varepsilon_o \frac{d\Phi_E}{dt}\right)\end{aligned}$$

but remember it is not really a current. What we have is a changing electric field that is making a magnetic field *as though there were a current* I_d. We can try this on or capacitor problem. We have done our capacitor problem for S_1 where we expect $\frac{d\Phi_E}{dt} \approx 0$ so our original calculation stands

$$B_{S_1} = \frac{\mu_o I}{2\pi r}$$

but now we know that if we use S_2 we have $\frac{d\Phi_E}{dt} \neq 0$, and we realize that at P_2 the current $I = 0$ so

$$\oint \mathbf{B} \cdot d\ell = \mu_o \left(0 + \varepsilon_o \frac{d\Phi_E}{dt}\right)$$

and for our geometry we found $\frac{d\Phi_E}{dt}$

$$\oint \mathbf{B} \cdot d\ell = \mu_o \left(0 + \pi R^2 \varepsilon_o \frac{dE}{dt}\right)$$

and we calculated $\frac{dE}{dt}$ so we can substitute it in

$$\oint \mathbf{B} \cdot d\ell = \mu_o \left(0 + \pi R^2 \varepsilon_o \frac{I}{\pi R^2 \varepsilon_o} \right)$$

where we remember that the current I is the current making the electric field–the current in the wire. Then we have

$$B 2\pi r = \mu_o (0 + I)$$

and our field is

$$B = \frac{\mu_o I}{2\pi r}$$

which is just what we found using S_1. Maxwell seems to have saved the day! There is no contradiction in our field calculation at P_1 if we use S_1 or S_2. We get the same answer!

There is one more fix we will have to do to Ampere's law eventually. We found this form of Ampere's law with the capacitor empty–not even containing air. But we could do the same derivation with a dielectric filled capacitor. We also could have magnetic materials involved.

But what we have done so far is really a momentous result. We have shown that, indeed, we should have an equation that provides symmetry with Faraday's law. We suspected that

$$\oint \mathbf{B} \cdot d\mathbf{s} \propto \frac{d\Phi_E}{dt}$$

and we can write the constants of proportionality as

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o \epsilon_o \frac{d\Phi_E}{dt}$$

but because we have $\oint \mathbf{B} \cdot d\mathbf{s}$ also in Ampere's law, we can combine the two to yield

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{s} &= \mu_o (I + I_d) \\ &= \mu_o \left(I + \varepsilon_o \frac{d\Phi_E}{dt} \right) \end{aligned}$$

This is the last of our field equations. It is called the Maxwell-Ampere law.

Let's use this to solve for the magnetic field inside the capacitor. A changing electric field will make a magnetic field.

Take a surface inside the plates that is a circle of radius $r < R$. Then

$$\oint \mathbf{B} \cdot d\mathbf{s} = B 2\pi r$$

from our modified Ampere's equation

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{s} &= \mu_o (I + I_d) \\ &= \mu_o \left(I + \varepsilon_o \frac{d\Phi_E}{dt} \right) \end{aligned}$$

so

$$\begin{aligned} B2\pi r &= \mu_o\left(0+\varepsilon_o\frac{d\Phi_E}{dt}\right) \\ &= \pi r^2\mu_o\varepsilon_o\frac{dE}{dt} \\ &= \pi r^2\mu_o\varepsilon_o\frac{I}{\pi R^2\varepsilon_o} \\ &= \mu_o\frac{r^2 I}{R^2} \end{aligned}$$

so

$$B=\mu_o\frac{rI}{2\pi R^2} \tag{29.4}$$

We should pause to realize what we have just done. We have shown that, indeed, a changing electric field can produce a magnetic field. This statement is a profound look at the way the universe works!

29.2 Maxwell Equations

We have developed a powerful set of understanding equations for electricity and magnetism. Maxwell summarized our knowledge in a series of four equations

$$\begin{array}{ll} \oint \mathbf{E}\cdot d\mathbf{A}=\frac{Q_{in}}{\varepsilon_o} & \text{Gauss's law for electric fields} \\ \oint \mathbf{B}\cdot d\mathbf{A}=0 & \text{Gauss's law for magnetic fields} \\ \oint \mathbf{E}\cdot d\mathbf{s}=-\frac{d\Phi_B}{dt} & \text{Faraday's law} \\ \oint \mathbf{B}\cdot d\mathbf{s}=\boldsymbol{\mu}_o I+\varepsilon_o\mu_o\frac{d\Phi_E}{dt} & \text{Ampere-Maxwell Law} \end{array} \tag{29.5}$$

If we have a dielectric, we might see these written as[**?**]

$$\begin{array}{ll} \oint \mathbf{E}\cdot d\mathbf{A}=\frac{Q_{in}}{\varepsilon_o\kappa} & \text{Gauss's law for electric fields} \\ \oint \mathbf{B}\cdot d\mathbf{A}=0 & \text{Gauss's law for magnetic fields} \\ \oint \mathbf{E}\cdot d\mathbf{s}=-\frac{d\Phi_B}{dt} & \text{Faraday's law} \\ \oint \mathbf{B}\cdot d\mathbf{s}=\mu_o\kappa_m\left(I+\varepsilon_o\kappa\frac{d\Phi_E}{dt}\right) & \text{Ampere-Maxwell Law} \end{array} \tag{29.6}$$

Since we have all had multivariate calculus, we may also see these written as

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o} & \text{Gauss's law for electric fields} \\ \boldsymbol{\nabla}\cdot\mathbf{B}=0 & \text{Gauss's law for magnetic fields} \\ \boldsymbol{\nabla}\times\mathbf{E}=-\frac{d\mathbf{B}}{dt} & \text{Faraday's law} \\ c^2\boldsymbol{\nabla}\times\mathbf{B}=\frac{\mathbf{J}}{\varepsilon_o}+\frac{d\mathbf{E}}{dt} & \text{Ampere-Maxwell Law} \end{array} \tag{29.7}$$

I'll let you remember the process to do the translation from $\oint \mathbf{B} \cdot d\mathbf{A}$ to $\boldsymbol{\nabla} \cdot \mathbf{B}$.

But we are familiar with all of these equations now. These four equations are the basis of all of classical electrodynamics. In an electromagnetic problem, we find the fields using the Maxwell equations to find the fields, and then apply the fields to find the Lorentz forces

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{29.8}$$

It turns out that these four equations strongly imply that there can be waves in the fields. Maxwell took the hint that $\mu_o \epsilon_o$ was related to c, the speed of light and he thought that light might be a wave in the electromagnetic field. We know about waves. We can describe a wave by looking for a surface of constant amplitude–a wave crest. We already know from our study of optics that these waves are what we call light. A point source will cause spherical surfaces of constant amplitude. A half-wave antenna makes a toroidal shaped wave front. We will not deal with spherical or worse wave shapes. Unfortunately, many antennas send out complicated wave patterns that take spherical harmonics to describe well. That is beyond the math we want to do in this course. We will stick to simple shapes. But we will see how waves in the electromagnetic field describe light in a future lecture. To prepare for this, we need to understand wave motion. Let's take on oscillation and waves next.

Basic Equations

$$I_d = \varepsilon_o \frac{d\Phi_E}{dt}$$

$$\oint \mathbf{B} \cdot d\ell = \mu_o \left(I + I_d \right)$$

$$\begin{array}{ll}
\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{in}}{\varepsilon_o} & \text{Gauss's law for electric fields} \\
\oint \mathbf{B} \cdot d\mathbf{A} = 0 & \text{Gauss's law for magnetic fields} \\
\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} & \text{Faraday's law} \\
\oint \mathbf{B} \cdot d\mathbf{s} = \boldsymbol{\mu}_o I + \varepsilon_o \mu_o \frac{d\Phi_E}{dt} & \text{Ampere-Maxwell Law}
\end{array} \tag{29.9}$$