Chapter 35

Standing Waves

A special case of superposition is that of two waves of the same frequency traveling opposite directions. Mixing two such waves can give rise to resonant patterns. These resonant patterns are the basis of music, and are of concern in building structures, among other things.

Fundamental Concepts

- When a wave meets a boundary, it will reflect
- Reflected waves will invert if the boundary is fixed or more like a fixed boundary.
- Reflected waves will not invert if the boundary is free or more like a free boundary.
- Two waves of equal frequency but traveling opposite directions can cause resonant patterns called standing waves.
- Only certain frequencies will produce standing waves. The boundary conditions determine which frequencies will work.
- The series of frequencies that produce standing waves is called the harmonic series.

35.1 Mathematical Description of Superposition

We know what superposition is, but we don't really want to add values for millions of points in a medium to find out what a combination of waves will look like. At the very least, we want to make a computer do that (and programs like

OpenFoam do something very akin to this!). But where we can, we would like to combine wave functions algebraically. Let's see how this can work.

Let's define two wave functions

$$y_1 = y_{\max} \sin(kx - \omega t)$$

and

$$y_2 = y_{\max} \sin(kx - \omega t + \phi_o)$$

These are two waves with the same frequency and wave number traveling the same direction in the medium, but they start at different times. The graph of y_2 is shifted by an amount ϕ_o.

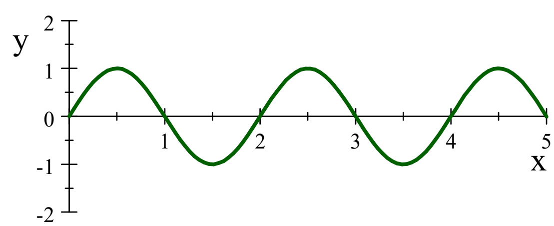
I will pick some values for the constants

$$\begin{array}{l} \lambda = 2 \\ k = \frac{2\pi}{\lambda} \\ \omega = 1 \\ \phi_o = \frac{\pi}{6} \\ t = 0 \\ y_{\max} = 1 \end{array}$$

then for y_1 we have

$$\begin{aligned} y_1 &= (1)\sin\left(\frac{2\pi}{\lambda}x - (1)\,t\right) \\ &= \sin\left(\frac{2\pi}{2}x - (1)\,t\right) \\ &= \sin(\pi x - t) \end{aligned}$$

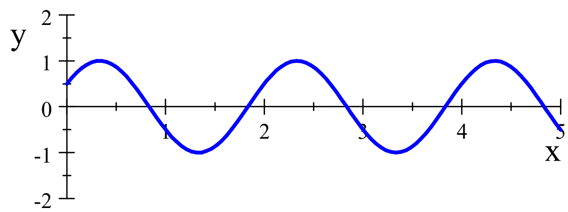
here is a snapshot plot of the wave function, y_1



Now let's consider y_2. Using the values we chose, y_2 can be written as

$$\begin{aligned} y_2 &= y_{\max} \sin(kx - \omega t + \phi_o) \\ &= \sin\left(\pi x - t + \frac{\pi}{6}\right) \end{aligned}$$

which looks like this



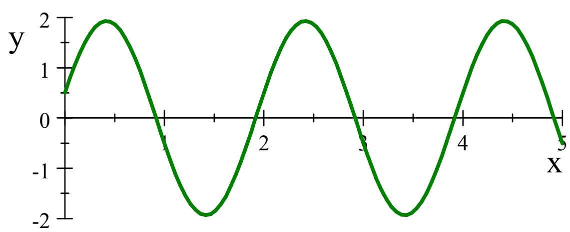
What does it look like if we add these waves using superposition? Symbolically we have

$$y_r = y_{\max} \sin(kx - \omega t) + y_{\max} \sin(kx - \omega t + \phi_o) \tag{35.1}$$

and putting in the numbers gives

$$y_r = \sin(\pi x - t) + \sin\left(\pi x - t + \frac{\pi}{6}\right)$$

which is shown in the next graph.



Notice that the wave form is taller (larger amplitude). Also notice it is shifted along the x axis.

We can find out how much shift we got by rewriting y_r. And the goal is to rewrite equation (35.1) so it is easier to interpret in general. To do this we need to remember a trig identity

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

To use this trig identity, let $a = kx - \omega t$ and $b = kx - \omega t + \phi_o$. This lets us rewrite our resultant wave as.

$$\begin{aligned}
y_r &= y_{\max}\sin(kx - \omega t) + y_{\max}\sin(kx - \omega t + \phi_o) \\
&= 2y_{\max}\cos\left(\frac{(kx-\omega t) - (kx - \omega t + \phi_o)}{2}\right)\sin\left(\frac{(kx-\omega t) + (kx - \omega t + \phi_o)}{2}\right) \\
&= 2y_{\max}\cos\left(\frac{-\phi_o}{2}\right)\sin\left(\frac{2kx - 2\omega t + \phi_o}{2}\right) \\
&= 2y_{\max}\cos\left(\frac{-\phi_o}{2}\right)\sin\left(kx - \omega t + \frac{\phi_o}{2}\right) \\
&= 2y_{\max}\cos\left(\frac{\phi_o}{2}\right)\sin\left(kx - \omega t + \frac{\phi_o}{2}\right)
\end{aligned}$$

where we used the fact that $\cos(-\theta) = \cos(\theta)$.

To interpret this new form of our resultant wave equation, let's look at the parts of this expression. First take

$$\sin\left(kx - \omega t + \frac{\phi_o}{2}\right) \tag{35.2}$$

This part is a traveling wave with the same k and ω as our original waves, but it has a phase constant of $\phi_o/2$. So our combined wave is shifted by $\phi_o/2$ or half the phase shift of y_2.

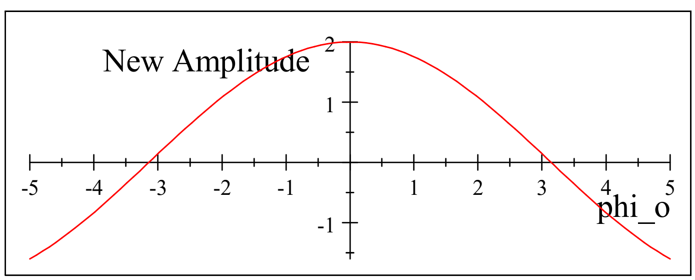
Now let's look at other factor

$$2y_{\max}\cos\left(\frac{\phi_o}{2}\right) \tag{35.3}$$

This part has no time dependence. We recognize from our basic equation

$$y(x,t) = y_{\max}\sin(kx - \omega t + \phi_o)$$

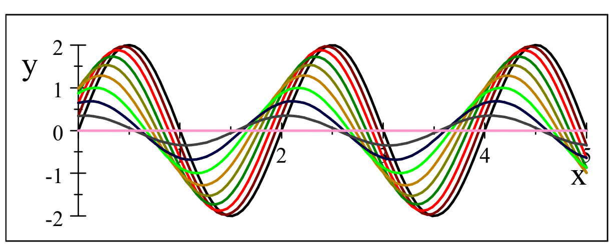
that the amplitude, $y_{\max}$ is a constant – not dependent on x or t, that multiplies the sine function. But now we have a more complex term that is not dependent on x or t that multiplies the sine function. The whole term $2y_{\max}\cos\left(\frac{\phi_o}{2}\right)$ must be the new amplitude! It has a maximum value when $\phi_o = 0$



When $\phi_o = \pi$, then

$$2y_{\max}\cos\left(\frac{\pi}{2}\right) = 0$$

so when $\phi_o = 0$ we have a new maximum amplitude of twice the original amplitude, $2y_{\max}$, and when $\phi_o = \pi$ we have no amplitude. Here is our wave for several choices of ϕ_o.



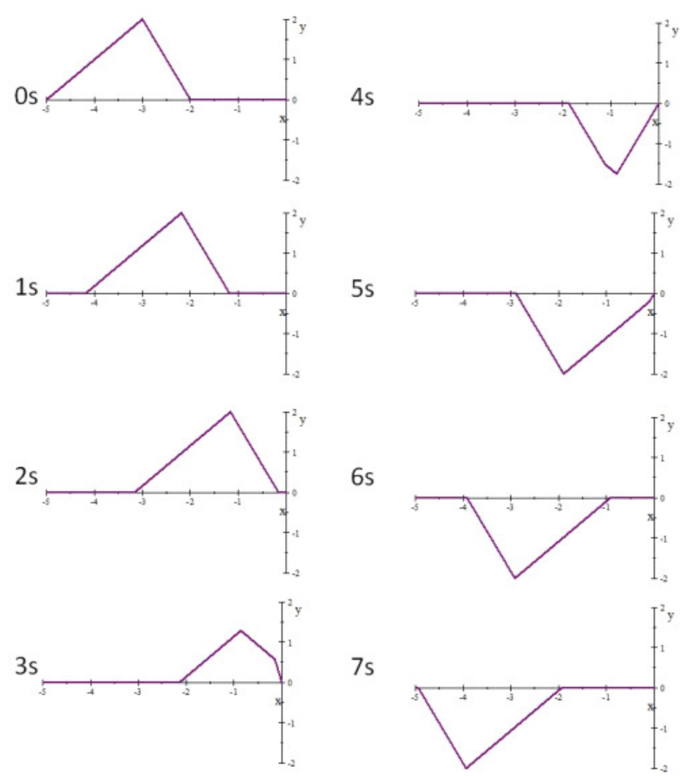
We can see that in our case the fact that the two waves added to produce a larger amplitude was just luck. We could have gotten anything from twice the single wave amplitude to no amplitude at all.

35.2 Reflection and Transmission

In our examples so far, we have not explained how we got two waves into a medium. One way is to simply reflect one wave back on top of itself.

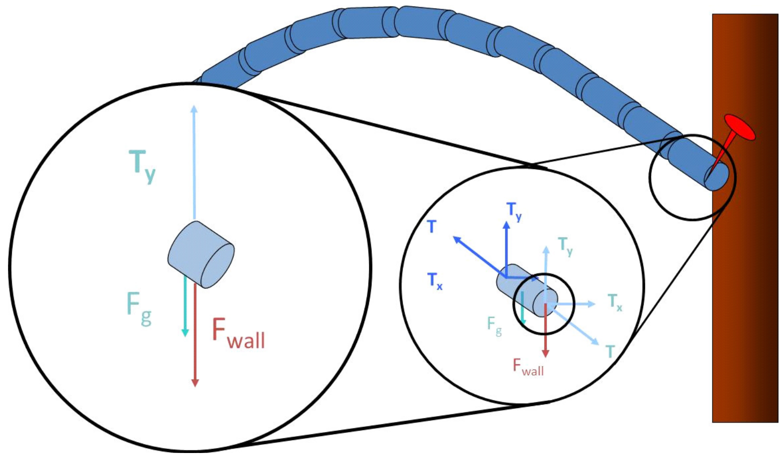
In class we made pulses on a long spring with one end of the spring fixed (held by a class member). What happens when the pulse reaches the end of the rope?

35.2.1 Case I: Fixed rope end.



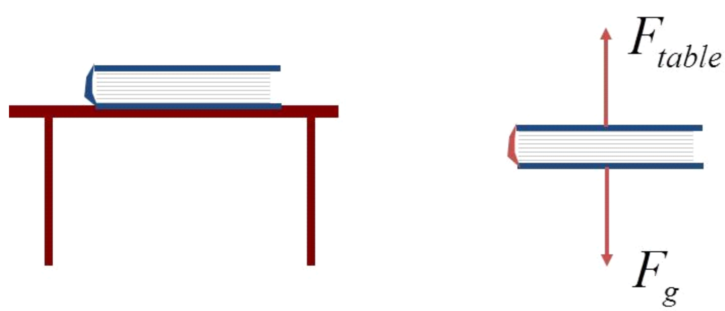
There is a big change in the medium at the end of the rope–the rope ends. This change in medium causes a reflection.

In the fixed end case, the pulse is inverted. We should consider why this inversion happens.



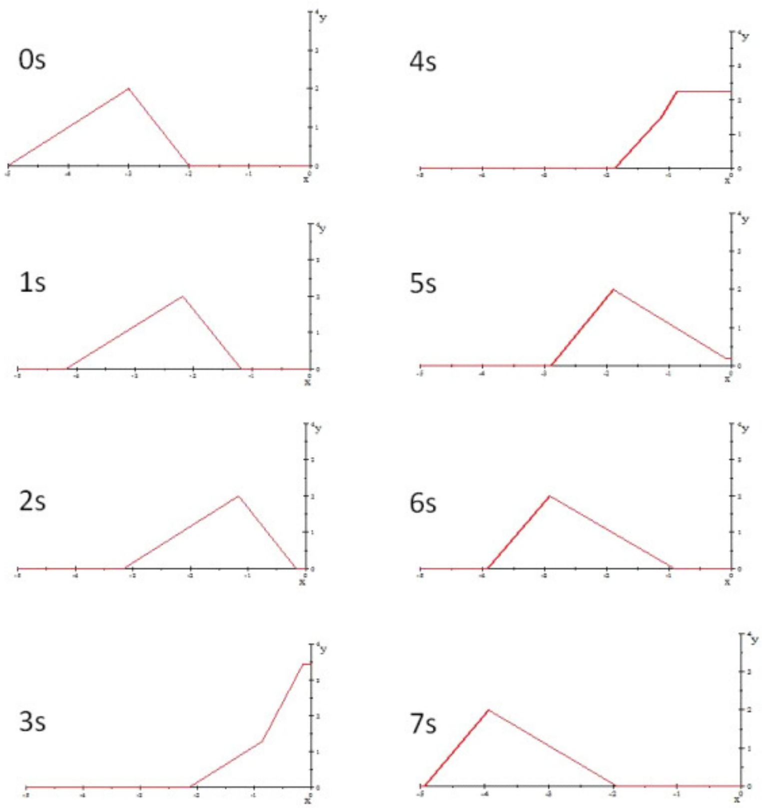
The end of the rope pushes up on the support (person, or nail or whatever). By Newton's third law the support must push back in an equal, but opposite direction, on the rope. This force sends the pieces of rope near it downward. We could think of the squashed nail atoms as having been given an amount of spring potential energy. They will transfer this energy back to the rope by pushing the end of the rope down. This downward motion becomes a new pulse that is an inversion of the original pulse traveling the opposite direction.

Does this seem reasonable? Remember studying normal forces? Consider a book on a table. The book has a force due to gravity. The table exerts a force equal to $m_{book}g$ on the book, or else the book smashes through the table.



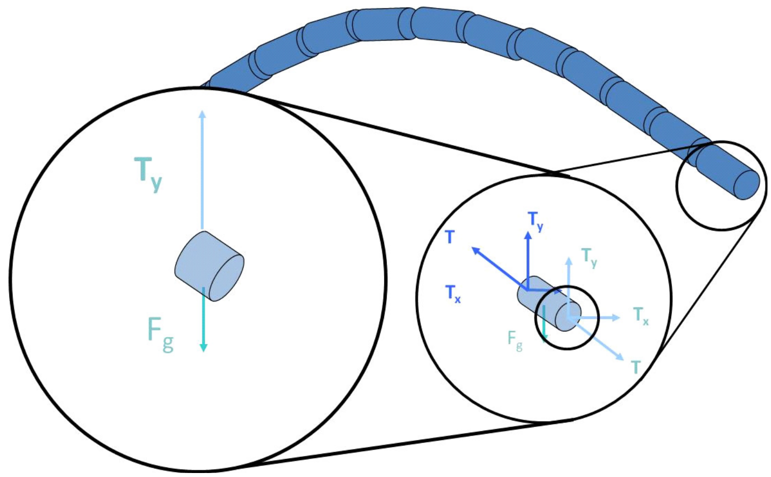
The normal force exerted by the atoms of the table keep the book up. It is this same type of force that keeps the rope on the nail. The atoms of the nail must push down on the end of the rope. They exert a force and this causes the inversion.

35.2.2 Case II: Loose rope end.



But what happens if the rope end is not fixed?

The rope end rises, and therefore there is no force exerted. The pulse (or at least part of the pulse energy) is still reflected, but there is no inversion!

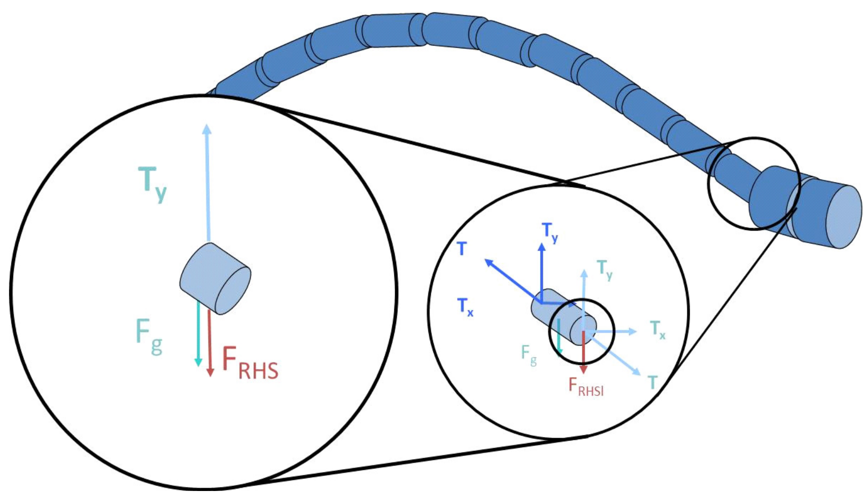


The end of the rope will come down, but the reason is that the force due to gravity acts on the mass of the rope end. The energy of the wave was made into potential energy of the rope end. As the rope end loses potential energy, that energy is put back into a wave going the opposite direction.

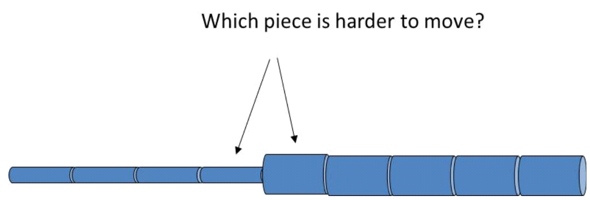
35.2.3 Case III: Partially attached rope end

Now lets tie the rope to another rope that is larger, more dense, than the rope we have been using, what will happen when we make waves in this combined rope?

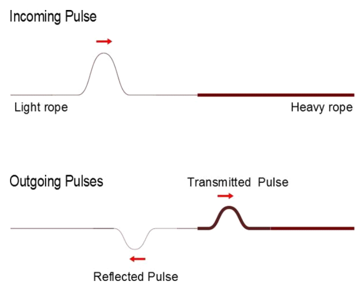
The light end of the rope exerts a force on the heavy beginning of the new rope



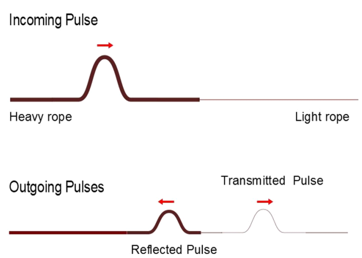
In this case consider momentum



The heavier rope resists being moved because of its larger mass. This resistance to motion is a little like our fixed end case. It is harder to transfer energy to the heavy rope, and the heavy rope resists the pull of the light rope. This resistance pushes back on the end of the light rope. This is a downward push. So once again we will have a reflected pulse in the light rope that is inverted.



We could also make a pulse in the heavy rope. What would happen then when the pulse reached the interface? You might be able to guess that the light rope won't have much effect on the end of the heavy rope. The light rope will cause a reflection, but it's weak downward push is not enough to cause the reflected pulse to invert.



Going from a heavy rope to a light rope makes an interface that is more like a free end.

Notice that in both cases there is a *transmitted* pulse. The transmitted pulse is what is left of the energy from the original pulse that has not been reflected. So we would not expect it to be inverted, and, indeed, it never is. We have split the amount of energy traveling along the rope

35.3 Mathematical description of standing waves

Now that we have a way to make two waves to superimpose, we can study the special case of a reflected wave. We will find that this special case can produce interesting patterns of constructive and destructive interference.

The patterns of constructive and destructive interference are the result of the superposition of two traveling waves with the same frequency going in opposite directions. Let's start with two standing waves with the same phase constant for simplicity.

$$\begin{aligned} y_1 &= y_{\max} \sin(kx - \omega t) \\ y_2 &= y_{\max} \sin(kx + \omega t) \end{aligned}$$

The sum is

$$y = y_1 + y_2 = y_{\max} \sin(kx - \omega t) + y_{\max} \sin(kx + \omega t)$$

To gain insight into what these two waves produce, we use another of our favorite trig identities

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

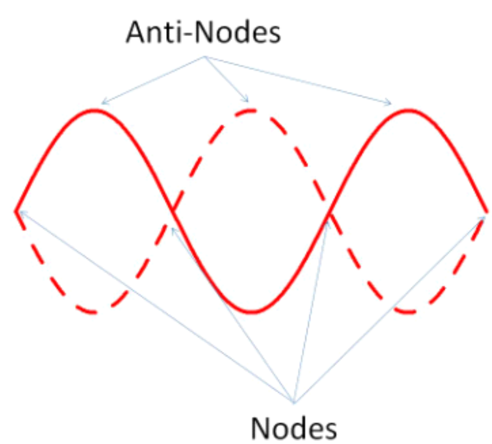
to get

$$\begin{aligned} y &= y_{\max} \sin(kx - \omega t) + y_{\max} \sin(kx + \omega t) \\ &= y_{\max} \sin(kx)\cos(\omega t) - y_{\max} \cos(kx)\sin(\omega t) + y_{\max} \sin(kx)\cos(\omega t) + y_{\max} \cos(kx)\sin(\omega t) \\ &= 2y_{\max} \sin(kx)\cos(\omega t) \\ &= (2y_{\max} \sin(kx))\cos(\omega t) \end{aligned}$$

This looks like the harmonic oscillator equation

$$y = y_{\max} \cos(\omega t + \phi_o)$$

with $\phi_o = 0$. The factor $2y_{\max} \sin(kx)$ has no time dependence, so it could be considered the amplitude of the harmonic oscillator. But this is a very odd amplitude. It depends on position. That is, we can view the rope as a set of harmonic oscillators who's amplitudes are different for each value of x.



But this is just what we see in our standing wave! We can identify spots along the x axis where the amplitude is always zero! we will call these spots *nodes.* These happen when $\sin(kx) = 0$ or when

$$kx = n\pi$$

By using

$$k = \frac{2\pi}{\lambda}$$

we have

$$\begin{aligned}
\frac{2\pi}{\lambda}x &= n\pi \\
\frac{2}{\lambda}x &= n \\
x &= n\frac{\lambda}{2}
\end{aligned}$$

We can also find the places along x where the oscillation amplitude will be largest. this occurs when $\sin(kx) = 1$ or when

$$kx = n\frac{\pi}{2}$$

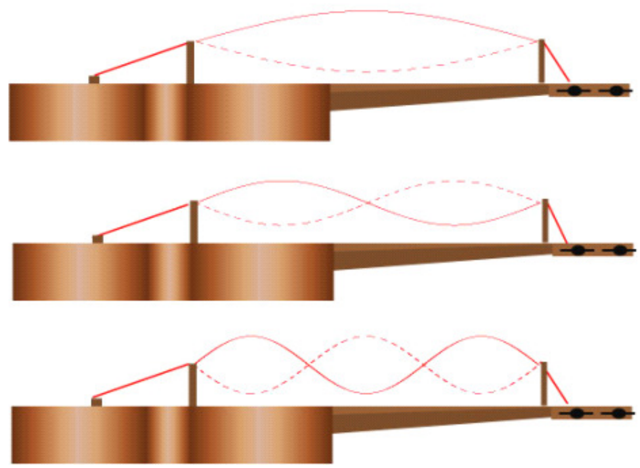
but only for odd n. Then

$$\begin{aligned}
\frac{2\pi}{\lambda}x &= n\frac{\pi}{2} \\
x &= n\frac{\lambda}{4} \qquad \text{for odd } n
\end{aligned}$$

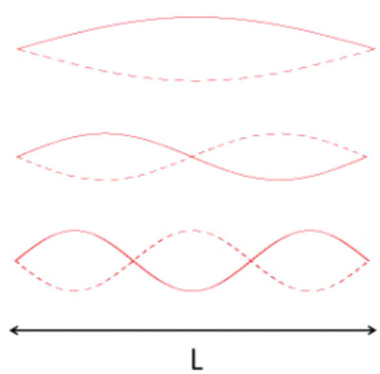
these are called *antinodes.*

This combination of two waves does not look like it goes anywhere. It seems to "stand" in place. We call it a *standing wave.* We can also create standing waves with sound or even light waves! But let's look at standing waves in some detail first.

35.4 Standing Waves in a String Fixed at Both Ends



If we attach a string to something on both ends, we find something interesting in the standing wave pattern. Not all imaginable standing waves can be realized. Some frequencies are preferred, and some never show up. These non-preferred frequencies will make waves, but not standing waves. We say that the standing wave pattern is *quantized*, meaning that only certain frequency values will make a standing wave pattern. The patterns that are allowed are called *normal modes*. We will see this any time a wave confined by boundary conditions (light in a resonant cavity, radio waves in a wave guide, electrons in an atom, etc.). In the last figure we saw some standing waves on a ukulele string. But we can draw the standing waves without the instrument.



The figure shows three normal modes for a string. Of course there are many more.

We find which modes are allowed by first imposing the boundary condition that each end must be a node. We start with our standing wave equation

$$y = 2y_{\max} \sin(kx) \cos(\omega t)$$

and recognize that we have one condition met because $y = 0$ when $x = 0$. We need $y = 0$ when $x = L$. That happens when

$$kL = n\pi$$

I will write this as

$$k_n L = n\pi$$

to indicate there are many values of k that could make a standing wave pattern. Solving this for λ_n gives

$$\begin{aligned} \frac{2\pi}{\lambda_n} L &= n\pi \\ \frac{2L}{n} &= \lambda_n \end{aligned}$$

Let's see how this works, the first mode will have

$$\lambda_1 = 2L$$

where L is the length of the string. Looking at the figure, we can see that this is true. The first normal mode has a length that is half the first mode wavelength.

The second mode has three nodes (one on each end and one in the middle). This gives

$$\lambda_2 = L$$

We can keep going, the third mode will have five nodes

$$\lambda_3 = \frac{2L}{3}$$

and so forth to give

$$\lambda_n = \frac{2L}{n}$$

We use our old friend

$$v = f\lambda$$

to find the frequencies of the modes

$$f = \frac{v}{\lambda}$$

Thus

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

or, in general

$$\begin{aligned} f_n &= \frac{v}{\lambda_n} = n\frac{v}{2L} \\ &= \frac{n}{2L} v \\ &= \frac{n}{2L}\sqrt{\frac{T}{\mu}} \end{aligned}$$

The lowest frequency that works has a special name, the *fundamental frequency*. The higher frequencies are integer multiples of the fundamental. When this happens we say that the frequencies form a *harmonic series,* and the modes are called *harmonics*.

35.4.1 Starting the waves

So, suppose we do not have a harmonic oscillator to make the wave patterns for a string that is fixed at both ends. Can we just pluck it to make it vibrate on a natural frequency?

Yes! only the normal modes will be excited by the pluck, any other frequencies will die out quickly (we won't show this mathematically in this class). So the only allowed frequencies (the ones that will result from a pluck) are the natural frequencies or harmonics. The frequency of waves on the string is *quantized*! That is, only some values are allowed. This idea is the basis behind Quantum mechanics (which views light and even matter as waves).

35.4.2 Musical Strings

So how do we get different notes on a guitar or Piano?

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{35.4}$$

A guitar uses tension to change the frequency or pitch (tuning) and length of string (your fingers pressing on the strings) to change notes.

A Piano uses both tension and length of string (and mass per unit length as well!). What do you expect and organ will do?

Basic Equations

Interference from two waves

$$y_r = 2y_{\max}\cos\left(\frac{\phi_o}{2}\right)\sin\left(kx - \omega t + \frac{\phi_o}{2}\right)$$

Standing waves

$$y = 2y_{\max}\sin(kx)\cos(\omega t)$$

Harmonic series

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

Nodes

$$x = n\frac{\lambda}{2}$$

Antinodes

$$x = n\frac{\lambda}{4}$$