Chapter 39

Waves in the Electromagnetic Field

Fundamental Concepts

- Maxwell's equation lead directly to the liner wave equation for both the electric and the magnetic field with the speed of light being the speed of the waves.
- The magnitude of the E and B fields are related in an electromagnetic wave by $E_{\max} = cB_{\max}$

39.0.1 Physical Ideas of the nature of Light

Before the 19th century (1800's) light was assumed to be a stream of particles. Newton was the chief proponent of this theory. The theory was able to explain reflection of light from mirrors and other objects and therefore explain vision. In 1678 Huygens showed that wave theory could also explain reflection and vision.

In 1801 Thomas Young demonstrated that light had attributes that were best explained by wave theory. We will study Young's experiment later today. The crux of his experiment was to show that light displayed constructive and destructive interference–clearly a wave phenomena! The theory of the nature of light took a dramatic shift

In 1805 Joseph Smith was born in Sharon, Vermont.

In September of 1832 Joseph Smith received a revelation that said in part :

> For the word of the Lord is truth, and whatsoever is truth is light, and whatsoever is light is Spirit, even the Spirit of Jesus Christ. And the Spirit giveth light to every man that cometh into the world; and

> the Spirit enlighteneth every man through the world, that hearkeneth to the voice of the Spirit. (D&C 84:45-46)

In December of 1832 Joseph Smith received another revelation that says in part:

> This Comforter is the promise which I give unto you of eternal life, even the glory of the celestial kingdom; which glory is that of the church of the Firstborn, even of God, the holiest of all, through Jesus Christ his Son—He that ascended up on high, as also he descended below all things, in that he comprehended all things, that he might be in all and through all things, the light of truth; which truth shineth. This is the light of Christ. As also he is in the sun, and the light of the sun, and the power thereof by which it was made. As also he is in the moon, and is the light of the moon, and the power thereof by which it was made; as also the light of the stars, and the power thereof by which they were made; and the earth also, and the power thereof, even the earth upon which you stand. And the light which shineth, which giveth you light, is through him who enlighteneth your eyes, which is the same light that quickeneth your understandings; which light proceedeth forth from the presence of God to fill the immensity of space—the light which is in all things, which giveth life to all things, which is the law by which all things are governed, even the power of God who sitteth upon his throne, who is in the bosom of eternity, who is in the midst of all things. (D&C 88:5-12)

Light, even real, physical light, seems to be of interest to Latter Day Saints.

In 1847 the saints entered the Salt Lake Valley.

In 1873 Maxwell published his findings that light is an electromagnetic wave (something we will try to show before this course is over!).

Planck's work in quantization theory (1900) was used by Einstein In 1905 to give an explantation of the photoelectric effect that again made light look like a particle.

Current theory allows light to exhibit the characteristics of a wave in some situations and like a particle in others. We will study both before the end of the semester.

The results of Einstein's work give us the concept of a *photon* or a quantized unit of radiant energy. Each "piece of light" or photon has energy

$$E = hf \tag{39.1}$$

where f is the frequency of the light and h is a constant

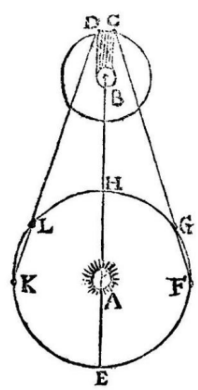
$$h = 6.63 \times 10^{-34}\,\mathrm{J\,s} \tag{39.2}$$

The nature of light is fascinating and useful both in physical and religious areas of thought.

39.1 Measurements of the Speed of Light

One of the great foundations of modern physical theory is that the speed of light is constant in a vacuum. Galileo first tried to measure the speed of light. He used two towers in town and placed a lantern and an assistant on each tower. The lanterns had shades. The plan was for one assistant to remove his shade, and then for the assistant on the other tower to remove his shade as soon as he saw the light from the first lantern. Back at the first tower, the first assistant would use a clock to determine the time difference between when the first lantern was un-shaded, and when they saw the light from the second tower. The light would have traveled twice the inter-tower distance. Dividing that distance by the time would give the speed of light. You can probably guess that this did not work. Light travels very quickly. The clocks of Galileo's day could not measure such a small time difference. Ole Rømer was the first to succeed in measuring the speed of light.

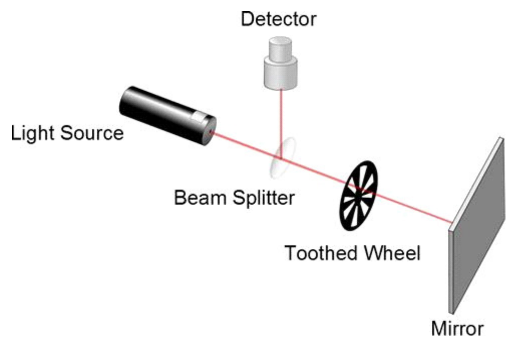
39.1.1 Rømer's Measurement of the speed of light



A diagram illustrating Rømer's determination of the speed of light. Point A is the Sun, piont B is Jupiter. Point C is the immersion of Io into Jupiter's shadow at the start of an eclipse

Rømer performed his measurement in 1675, 269 years before digital devices existed!. He used the period of revolution of Io, a moon of Jupiter, as Jupiter revolved around the sun. He first measured the period of Io's rotation about Jupiter, then he predicted an eclipse of Io three months later. But he found his calculation was off by 600 s. After careful thought, he realized that the Earth had moved in its orbit, and that the light had to travel the extra distance due to the Earth's new position. Given Rømer's best estimate for the orbital radius of the earth and his time difference, Rømer arrived at a estimate of $c = 2.3 \times 10^8 \, \frac{\text{m}}{\text{s}}$. Amazing for 1675!

39.1.2 Fizeau's Measurement of the speed of light



Hippolyte Fizeau measured the speed of light in 1849 using a toothed wheel and a mirror and a beam of light. The light passed through the open space in the wheel's teeth as the wheel rotated. Then was reflected by the mirror. The speed would be

$$v = \frac{\Delta x}{\Delta t}$$

We just need Δx and Δt.

It is easy to see that

$$\Delta x = 2d$$

because the light travels twice the distance to the mirror (d) and back. So the speed is just

$$v = \frac{2d}{\Delta t}$$

If the wheel turned just at the right angular speed, then the reflected light would hit the next tooth and be blocked. Think of angular speed

$$\omega = \frac{\Delta \theta}{\Delta t}$$

so the time difference would be

$$\Delta t = \frac{\Delta \theta}{\omega}$$

We find $\Delta \theta$ by taking the number of teeth on the wheel and dividing by 2π by that number.

$$\Delta \theta = \frac{2\pi}{N_{teeth}}$$

Then the speed of light must be

$$\begin{aligned} c &= v = \frac{2d}{\frac{\Delta\theta}{\omega}} \\ &= \frac{2d\omega}{\Delta\theta} \\ &= \frac{2d\omega N_{teeth}}{2\pi} \\ &= \frac{d\omega N_{teeth}}{\pi} \end{aligned}$$

then if we have 720 teeth and ω is measured to be $172.79\,\mathrm{rad/s}$ and $d = 7500\,\mathrm{m}$

$$\begin{aligned} c &= \frac{(7500\,\mathrm{m})\left(172.79\,\frac{\mathrm{rad}}{\mathrm{s}}\right)(720)}{\pi} \\ &= 2.97\times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

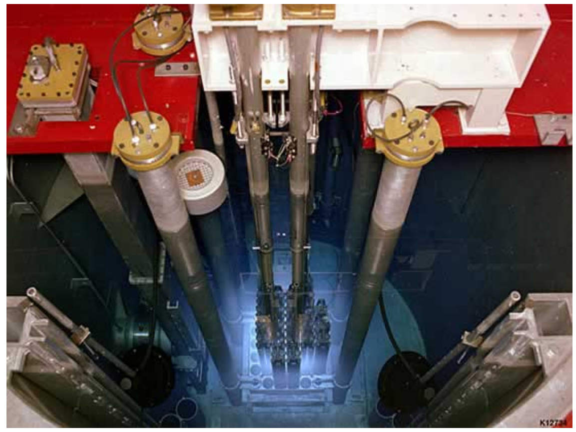
which is Fizeau's number and it is pretty good!

Modern measurements are performed in very much the same way that Fizeau did his calculation. The current value is

$$c = 2.9979\times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \tag{39.3}$$

39.1.3 Faster than light

The speed of light in a vacuum is constant, but in matter the speed of light changes. We will study this in detail when we look at refraction. But for now, a dramatic example is Cherenkov radiation. It is an eerie blue glow around the core of nuclear reactors. It occurs when electrons are accelerated past the speed of light in the water surrounding the core. The electrons emit light and the light waves form a Doppler cone or a light-sonic boom! The result is the blue glow.



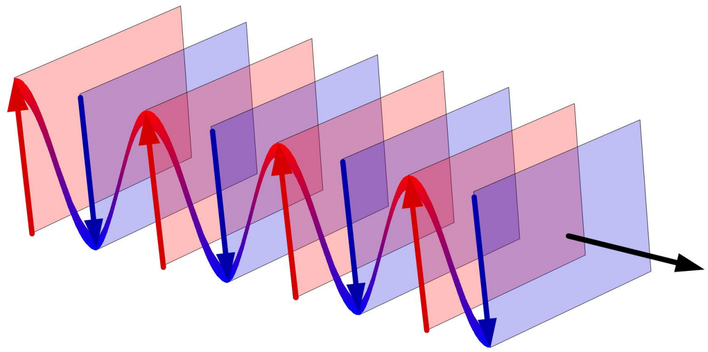
Cherenkov radiation from a 250kW TRIGA reactor. (Image in the Public Domain, courtesy US Department of Energy)

This does bring up a problem in terminology. What does the word "medium" mean? We have used it to mean the substance through which a wave travels. This substance must have the property of transferring energy between it's parts, like the coils of a spring can transfer energy to each other, or like air molecules can transfer energy by collision. For light the wave medium is the electromagnetic field. This field can store and transfer energy (we will see this later in the course). But many books on physics call materials like glass a "medium" through which light travels. The water in our last example is such a medium. Are glass and water wave mediums for light? The answer is no. Light does not need any matter to form it's wave. The wave medium is the electromagnetic field. So we will have to keep this in mind as we allow light to travel through matter. We may call the matter a "medium," but it is not the wave medium.

39.2 From Maxwell to Waves

We left our study of electricity and magnetism with Maxwell's equations. But how do we know that electromagnetic fields can carry waves, and what is the hint that these waves are light?

Fundamental Concepts



A representation of a plane wave. Remember that the planes are really of infinite extent. Image is public domain.

Let's picture our wave front far from the source. No matter what the total shape, if we look at a small patch of the fields far away, they will look like the plane wave in the last figure. Since this is a useful and common situation (except if you use lasers), we will perform some calculations assuming plane wave geometry.

We will assume we are in empty space, so the charge q and current I will both be zero. Then our Maxwell Equations become

$$\begin{array}{ll} \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = 0 & \text{Gauss's law for electric fields} \\ \oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{A}} = 0 & \text{Gauss's law for magnetic fields} \\ \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} & \text{Faraday's law} \\ \oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = \varepsilon_o \mu_o \frac{d\Phi_E}{dt} & \text{Ampere-Maxwell Law} \end{array} \tag{39.4}$$

Our goal is to show that these equations tell us that we can have waves in the field. To do this, we will show that Maxwell's equations really contain the linear wave equation within them. As a reminder, here is the linear wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

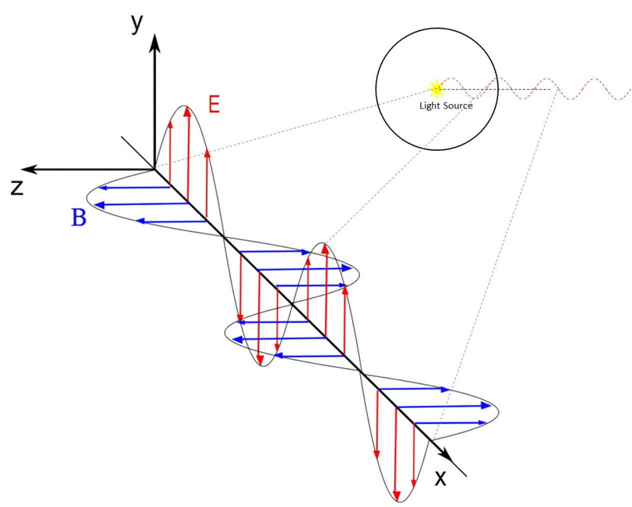
it is a second order differential equation where the left side derivatives are take with respect to position, and the right side derivatives are taken with respect to time. The quantity, v, is the wave speed. In this form of the equation y is the displacement of a medium. Our medium will be the electromagnetic field.

39.2.1 Rewriting of Faraday's law

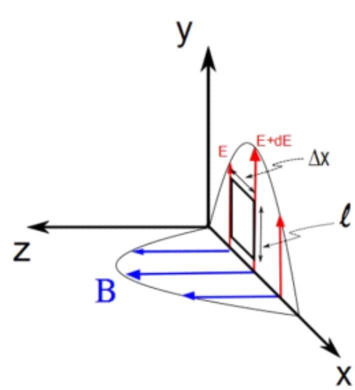
Let's start with Faraday's law

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{39.5}$$

Given our geometry, we can say the wave is traveling in the x direction with the $\overrightarrow{\mathbf{E}}$ field positive in the y direction. From our discussion of displacement currents we have a strong hint that the $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{B}}$ fields will be perpendicular. So let's take the magnetic field as positive in the z direction. So as the light wave moves from the source along a line we could draw the $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{B}}$ fields something like this.



Let's take a small rectangle of area to find $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$



The top and bottom of the rectangle don't contribute because $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = \mathbf{0}$ along these paths. On the sides, the field is either in the $d\overrightarrow{\mathbf{s}}$ or it is in the opposite direction. So

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = \oint E ds$$

or

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\oint E ds$$

along the sides. Let's say we travel counter-clockwise along the loop. Then the left side will be negative and the right side will be positive.

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = \int_{right} E ds - \int_{left} E ds$$

On the left side, we are at a position x away from the axis, and on the right side we are a position $x + \Delta x$ away from the axis. Then the field of the left side is $E(x,t)$ and the field on the right hand side is approximately

$$E(x + \Delta x, t) \approx E(x,t) + \frac{\partial E}{\partial x}\Delta x \tag{39.6}$$

so if our loop is small, then ℓ is small and E won't change much so we can write approximately

$$\begin{aligned}
\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} &= \int_{right} E ds - \int_{left} E ds & (39.7)\\
&\approx E(x + \Delta x, t)\,\ell - E(x,t)\,\ell & (39.8)\\
&= \left(E(x,t) + \frac{\partial E}{\partial x}\Delta x\right)\ell - E(x,t)\,\ell \\
&= \left(E(x,t) + \frac{\partial E}{\partial x}\Delta x\right)\ell - E(x,t)\,\ell \\
&= \ell\frac{\partial E}{\partial x}\Delta x & (39.9)
\end{aligned}$$

So far then, Faraday's law [1]

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

becomes

$$\ell\frac{\partial E}{\partial x}\Delta x = -\frac{d\Phi_B}{dt}$$

Let's move on to the right hand side of Faraday's law. We need to find Φ_B so that we can find the time rate of change of the flux. We can say that B is nearly constant over such a small area, so

$$\begin{aligned}
\Phi_B &= \mathbf{B} \cdot \mathbf{A} \\
&= BA\cos\theta \\
&= BA \\
&= B\ell\Delta x
\end{aligned}$$

[1] We need ds to be very small, much smaller than the wavelength of the wave.

where here Δx means "a small distance" as it did above. Then

$$\begin{aligned} \frac{d\Phi_B}{dt} &= \frac{d}{dt}\left(B\ell\Delta x\right) \\ &= \ell\Delta x \left.\frac{dB}{dt}\right|_{x \text{ constant}} \\ &= \ell\Delta x\frac{\partial B}{\partial t} \end{aligned}$$

where we have held x constant because we are not changing our small area, so Faraday's law

$$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

becomes

$$\begin{aligned} \ell\frac{\partial E}{\partial x}\Delta x &= -\ell\Delta x\frac{\partial B}{\partial t} \\ \frac{\partial E}{\partial x} &= -\frac{\partial B}{\partial t} \end{aligned} \tag{39.10}$$

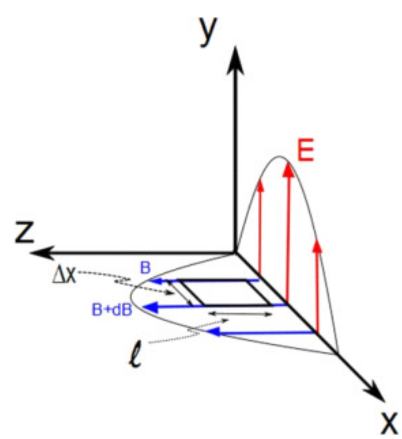
We have made some progress, we have a differential equation relating the fields, but it is a mixed equation containing both the electric and magnetic fields. We are only half way there.

39.2.2 Rewriting of the Maxwell-Ampere Law

We have used one field equation so far and that took us part of the way. We have the Maxwell-Ampere law as well. We can use this to modify our result from Faraday's law to find the linear wave equation that we expect. The Maxwell-Ampere law with no sources (charges or currents) states

$$\oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = \varepsilon_o\mu_o\frac{d\Phi_E}{dt}$$

This time we must consider the magnetic field path integral



We can do the same thing we did with Faraday's law with an area, but this time we will use the area within the magnetic field (shown in the figure above). Again, let's start with the left hand side of the equation. We see that the sides of our area that are parallel to the x-axis do not matter because $\overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = 0$ along these sides, but the other two are in the direction (or opposite direction) of the field. They do contribute to the line integral.

$$\begin{aligned} \oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} &= B\left(x,t\right)\ell - B\left(x+\Delta x,t\right)\ell \\ &\approx -\ell\frac{\partial B}{\partial x}\Delta x \end{aligned} \tag{39.11}$$

Now for the left hand side, we need the electric flux. For such a small area, the field is nearly constant so

$$\begin{aligned} \Phi_E &\approx EA\cos\theta \\ &= EA \\ &= E\ell\Delta x \end{aligned}$$

so

$$\frac{\partial \Phi_E}{\partial t} = \ell\Delta x\frac{\partial E}{\partial t} \tag{39.12}$$

Combining both sides

$$\begin{aligned} \oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} &= \varepsilon_o\mu_o\frac{d\Phi_E}{dt} \\ -\ell\frac{\partial B}{\partial x}\Delta x &= \varepsilon_o\mu_o\ell\Delta dx\frac{\partial E}{\partial t} \\ \frac{\partial B}{\partial x} &= -\varepsilon_o\mu_o\frac{\partial E}{\partial t} \end{aligned} \tag{39.13}$$

We now have a second differential equation relating B and E. But it is also a mixed differential equation.

39.3 Wave equation for plane waves

This leaves us with two equations to work with

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{39.14}$$

$$\frac{\partial B}{\partial x} = -\varepsilon_o\mu_o\frac{\partial E}{\partial t} \tag{39.15}$$

Remember that these are all partial derivatives. Taking the derivative of the first equation with respect to x gives

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial E}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\partial B}{\partial t}\right) \\
\frac{\partial^2 E}{\partial x^2} &= -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}B\right) \\
\frac{\partial^2 E}{\partial x^2} &= -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right)
\end{aligned}$$

In the last equation we swapped the order of differentiation for the right hand side. In parenthesis, we have $\partial B/\partial x$ on the right hand side. But we know what $\partial B/\partial x$ is from our second equation. We substitute from our second equation to obtain

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial t}\left(-\varepsilon_o\mu_o\frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_o\mu_o\frac{\partial^2 E}{\partial t^2} \tag{39.16}$$

We can do the same thing, but taking derivatives with respect to time to give

$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_o\mu_o\frac{\partial^2 B}{\partial t^2} \tag{39.17}$$

You will recognize both of these last equations as being in the form of the linear wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This means that both the E field and the B field are governed by the linear wave equation with the speed of the waves given by

$$v = \frac{1}{\sqrt{\varepsilon_o\mu_o}} \tag{39.18}$$

We have studied waves, so we know the solution to this equation is a sine or cosine function

$$E = E_{\max}\cos(kx - \omega t) \tag{39.19}$$
$$B = B_{\max}\cos(kx - \omega t) \tag{39.20}$$

with

$$k = \frac{2\pi}{\lambda}$$

and

$$\omega = 2\pi f$$

then

$$\frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f$$

which is the wave speed.

We can show that the magnitude of E is related to B.

Lets take derivatives of E and B with respect to x and t.

$$\begin{aligned}\frac{\partial E}{\partial x} &= -kE_{\max}\sin(kx-\omega t)\\ \frac{\partial B}{\partial t} &= \omega B_{\max}\sin(kx-\omega t)\end{aligned}$$

then we can use one of our half-way-point equations from above

$$\frac{\partial E}{\partial x}=-\frac{\partial B}{\partial t}$$

and by substitution obtain

$$\begin{aligned}-kE_{\max}\sin(kx-\omega t) &= -\omega B_{\max}\sin(kx-\omega t)\\ -kE_{\max} &= -\omega B_{\max}\end{aligned}$$

or

$$\frac{E_{\max}}{B_{\max}}=\frac{\omega}{k}=v$$

The speed is the speed of light, c, so

$$\frac{E_{\max}}{B_{\max}}=c \tag{39.21}$$

It is one of the odd things about the universe that speed of electromagnetic waves is a constant. It does not vary in vacuum, and the in-vacuum value, c is the maximum speed. It was a combination of Maxwell's work in predicting c and the observations confirming the predictions that launched Einstein to form the Special Theory of Relativity!

Note that the last equation shows why we often only deal with the electric field wave when we do optics. Since the magnetic field is proportional to the electric field, we can always find it from the electric field.

Basic Equations

$$\frac{\partial^2 E}{\partial x^2}=\varepsilon_o\mu_o\frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 B}{\partial x^2}=\varepsilon_o\mu_o\frac{\partial^2 B}{\partial t^2}$$

$$\frac{E_{\max}}{B_{\max}}=c$$