Chapter 8

Gauss' Law and its Applications

Last lecture we learned about electrical flux. In this lecture we will use electric flux to find the electric field.

Fundamental Concepts

- Gauss' Law tells us that the flux through a closed surface is equal to the charge inside the surface divided by ϵ_o:

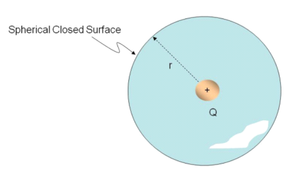
$$\Phi = \frac{Q_{in}}{\epsilon_o}$$

- Gauss' Law combined with our basic flux equation

$$\Phi_E = \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = \frac{Q_E}{\epsilon_o}$$

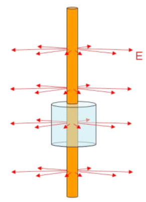
8.1 Gauss' Law

Last lecture we did two problems. We found the flux from a point charge through a spherical surface to be



$$\Phi_{sphere,point} = \frac{Q_E}{\epsilon_o}$$

and the flux from a line of charge through a cylinder to be



$$\Phi_{cylindar,line} = \frac{|\lambda|}{\epsilon_o} L$$

which we rewrote by substituting in

$$\lambda = \frac{Q}{L}$$

so that

$$\begin{aligned}\Phi_{cylindar,line} &= \frac{|Q_E/L|}{\epsilon_o} L \\ &= \frac{|Q_E|}{\epsilon_o}\end{aligned}$$

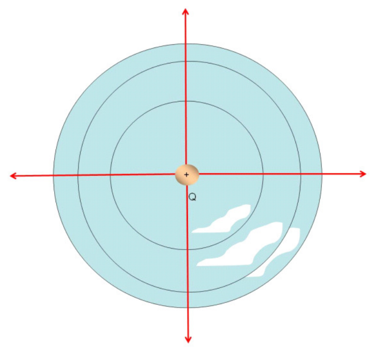
which is just what we got for the point charge and sphere! That is amazing! Think about how much work it was to find each flux, and in the end we got the same result. Wouldn't it be great if the flux through every closed surface was this simple? Then we would not have to integrate at all!

To see if we can do this, first let's think of our answer.

$$\Phi_{sphere,point} = \frac{Q_E}{\epsilon_o}$$

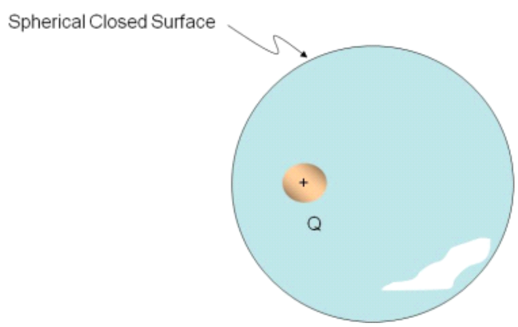
It does not depend on the radius of the spherical surface. So any spherical surface centered on the charge will do! This makes sense. No matter how big the sphere, all the field lines must leave it. Since flux gives the amount of field that penetrates an area, for our charge at the center of a sphere we see that all of the field penetrates the spherical surface no matter the size of the sphere. So the flux is the same no matter r.[1]

[1] If this still seems strange, remember that the area of a sphere is $4\pi r^2$ and that the field of a point charge is $\frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}$. The flux is like the product of these two quantities. The r^2 terms must cancel. So the fact that the flux is the same for any sphere is due to the r^2 dependence of the field.



The key to making our last lecture problems easy was that the field was always perpendicular to the surface so $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = EdA$ was easy to find.

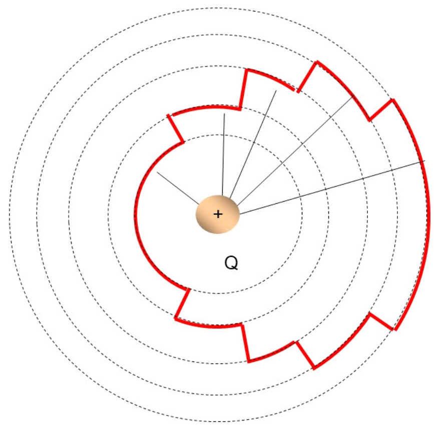
Using geometry we can arrange to make nearly all of our flux problems like this. To demonstrate, let's take the case of a point charge that is off center in a spherical surface.



Remember, we made up this surface. So we can place the surface anywhere we like. And this time we would like the charge to be off center. We will call these made up surfaces *Gaussian surfaces* after the mathematician that thought up this method of avoiding integrals. Having the charge off center would make for a difficult integration because $\overrightarrow{\mathbf{E}}$ and $d\overrightarrow{\mathbf{A}}$ have different directions as we go around the sphere. But let's consider, would there be less flux through the surface than there was when the charge was centered in the sphere? Every field line that is generated will still leave the surface. Flux gives us the amount of field that penetrates the surface.[2] Since flux is the amount of field penetrating our surface, it seems that the flux should be exactly the same as when the charge was in the center of the sphere. To prove this, let's take our surface and approximate it using area segments. But let's have the area segments be either along a radius of a sphere centered on the charge, or along the surface of a sphere centered on

[2] Think of water flow rate again. We could place the end of a garden hose in a wire mesh container. The water would flow out the hose end and through the wire mesh sides of the container. The flow rate tells us how much water passes through the container surface. The flow rate does not depend on the shape of the container. The hose end is like a charge. The hose is the source of water, the charge is the source of electric field.

the charge.



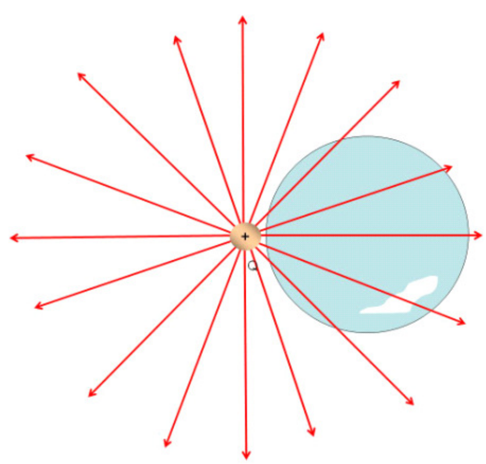
No flux goes through the radial pieces. And the rest of the pieces are all parts of spheres centered on the charge. But for the spherical segments, the field will be perpendicular to the segment no matter what sphere the segment is a part of, because we chose only spheres that were concentric with the charge. The r we have for the little spherical pieces does not matter, so on all of these surfaces $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = EdA$. Then the integration for these pieces will be easy.

Of course this surface made of little segments from other spheres is a poor approximation to the shape of the offset sphere. But we can make our small segments smaller and smaller. In the limit that they are infinitely small, our shape becomes the offset sphere. That means that once again our flux is

$$\Phi = \frac{Q_E}{\epsilon_o}$$

This is fantastic! We don't have to do the integration at all. We just count up the charge inside our surface and divide by ϵ_o.

What happens if the charge is on the outside of the surface?

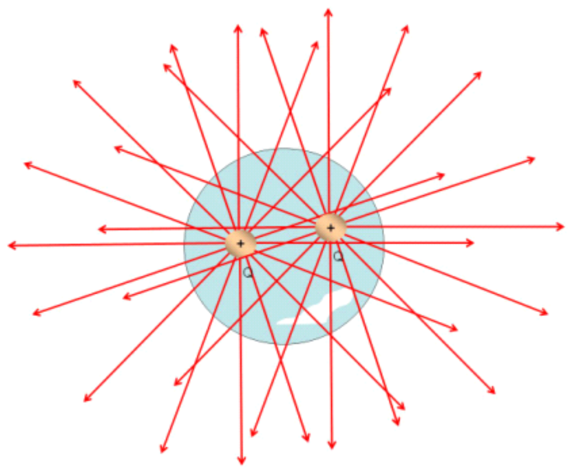


Every field line that enters goes back out. We encountered this last time. The flux going in is negative, the flux going out is positive, and they must be the same because every line leaves that enters. So the net flux must be zero. We can still write our flux as

$$\Phi = \frac{Q_{inside}}{\epsilon_o}$$

because outside charges won't contribute to the flux, and there is no charge in side. So, our expression works for charges outside our closed surface.

We know that fields superimpose, that is, they add up, so we would expect that if we have two charges inside a surface,



we would add up their contributions to the total flux

$$\Phi_{total} = \Phi_1 + \Phi_2$$

which means that Q_{inside} is the sum of all the charges inside. We recognize that if some charges are negative, they will cancel equal amounts of charge that are positive.

This leaves us with a fantastic time savings law

> The electric flux Φ through any closed surface is equal to the net charge inside the surface multiplied by $4\pi k_e$. The closed surface is often called a *Gaussian Surface.*

$$\Phi_E = \oint \overrightarrow{E} \cdot d\overrightarrow{A} = \frac{Q_{in}}{\epsilon_o}$$

This was first expressed by Gauss, and therefore this expression is called Gauss' law.

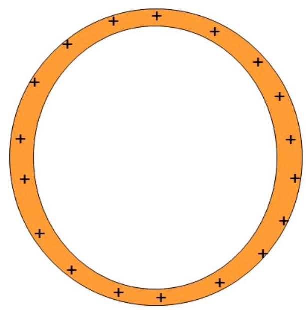
8.2 Examples of Gauss' Law

But why do we get so excited about flux? The reason is that we can use the idea of flux combined with Gauss' law gives us an easy way to calculate the electric field from a distribution of charge if we can find a suitable symmetric surface! If we can find the field, we can find forces, and we can predict motion.

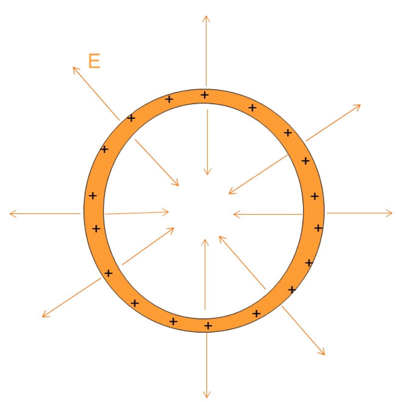
Let's show how to do this by working some examples.

Charged Spherical Shell

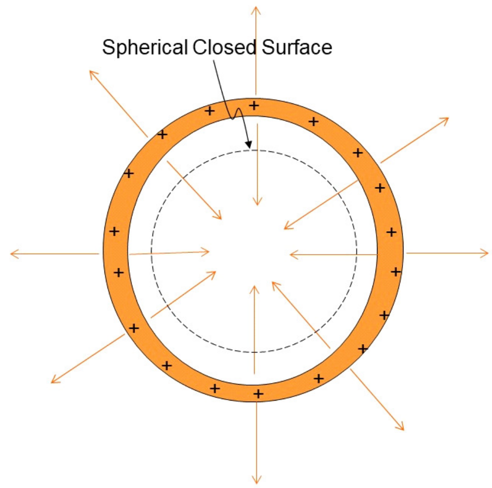
First let's take a charged spherical shell and find the field inside.



To use Gauss' law we need to be able to guess the shape of the field. We use symmetry to make our guess. We can guess that the field will be radial both inside and outside of the shell. If it were not so, then our symmetry tests would fail.



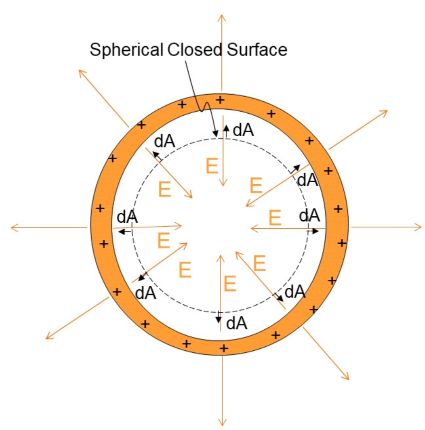
Suppose the shell has a total charge of $+Q$. If we place an imaginary Gaussian spherical surface inside the shell



Then we can use Gauss's law.

$$\Phi = \frac{Q_{inside}}{\epsilon_o}$$

We can tell from the symmetry of the situation that $\overrightarrow{\mathbf{E}}$ is everywhere colinear with (but in the opposite direction as) $d\overrightarrow{\mathbf{A}}$ when we are inside the shell,



so

$$\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = EdA\cos\left(180\,^{\circ}\right)$$

then our flux is

$$\Phi = \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = -\oint EdA$$

We can even make a guess that the field must be constant on this surface, because all along the spherical Gaussian surface there is extreme symmetry. No change in reflection, or rotation etc. will change the shape of the charge distribution, so around the spherical surface the field must have the same value. Then we can take out E.

$$\Phi = -E \oint dA = -EA$$

Equating our flux equations gives

$$-EA = \frac{Q_{inside}}{\epsilon_o}$$

or

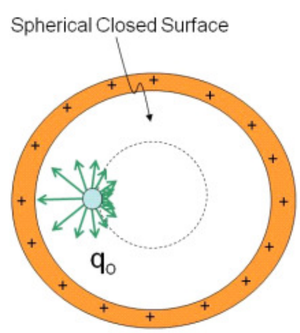
$$E = -\frac{Q_{inside}}{A\epsilon_o}$$

but what is Q_{inside}? It is zero! The inside of the shell is empty. Then

$$E = -\frac{0}{A\epsilon_o} = 0$$

There is no net field inside!

This may seem surprising, but think of placing a tiny moving charge (sometimes called a test charge, q_o), inside the sphere. The next figure shows the forces acting on such a test charge. The force is stronger between the charge and the near surface, but there is more of the surface tugging the other way.



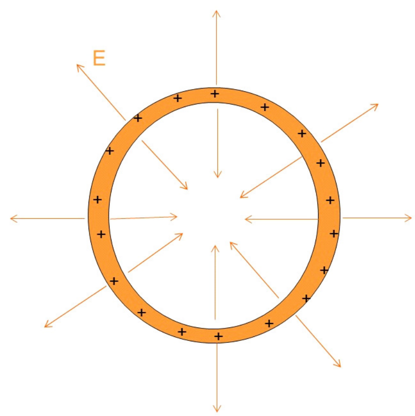
The forces just balance. Since

$$F = qE$$

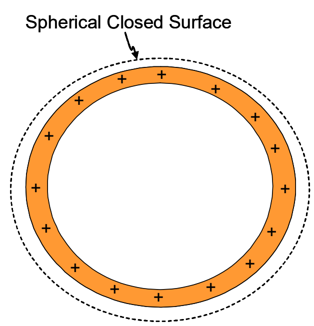
if the net force is zero, then the field must be zero too.

That was inside the spherical shell, but is there a field outside of the spherical shell? Let's repeat our process to find out.

We can use the same picture of the field that we drew before



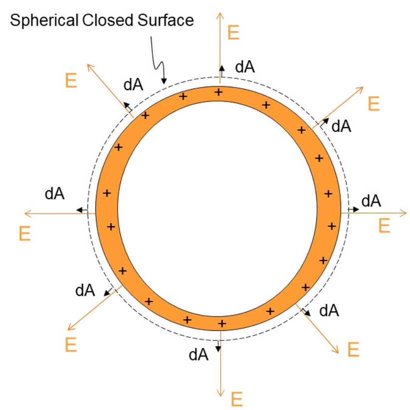
but now we want an imaginary Gaussian surface that is outside the shell. We want our Gaussian surface to match the symmetry of the charge distribution so is seems like a spherical surface might be a good choice.



We need to draw the field and the $d\overrightarrow{\mathbf{A}}$ vectors to see if it is still true that

$$\Phi = \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = \oint E dA$$

but this time we have a positive sign on the last integral because $\overrightarrow{\mathbf{E}}$ and $d\overrightarrow{\mathbf{A}}$ are in the same direction.



This time we see that $\overrightarrow{\mathbf{E}}$ and $d\overrightarrow{\mathbf{A}}$ are always in the same direction.

$$\begin{aligned}\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} &= EdA\cos\left(0^{\circ}\right) \\ &= EdA\end{aligned}$$

Then our flux is

$$\Phi = \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = \oint EdA$$

Once again from the symmetry we expect E to be constant on our Gaussian surface. So

$$\Phi = E\oint dA = EA$$

and we can combine this with Gauss' law

$$EA = +\frac{Q_{inside}}{\epsilon_o}$$

All of our analysis is the same as in our last problem, except now Q_{inside} is not zero

$$E = \frac{Q_{inside}}{A\epsilon_o}$$

The area is the area of our imaginary sphere is $4\pi r^2$

$$E = \frac{Q_{inside}}{\left(4\pi r^2\right)\epsilon_o}$$

and since $Q_{inside} = +Q$, then

$$E = \frac{+Q}{4\pi\epsilon_o r^2}$$

and we have found the field outside our spherical shell.

So for a spherical shell there is no net field inside and outside the field is

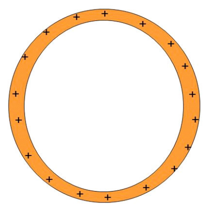
$$\overrightarrow{\mathbf{E}} = \frac{+Q}{4\pi\epsilon_o r^2}\hat{\mathbf{r}}$$

Note that this outside field looks very like a point charge at the center of the spherical shell (at the center of charge), but by now that is not much of a surprise!

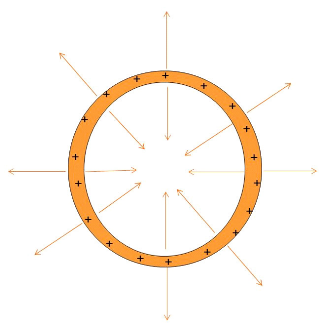
Strategy for Gauss' law problems

Let's review what we have done before we go on to our last example. For each Gauss' law problem, we

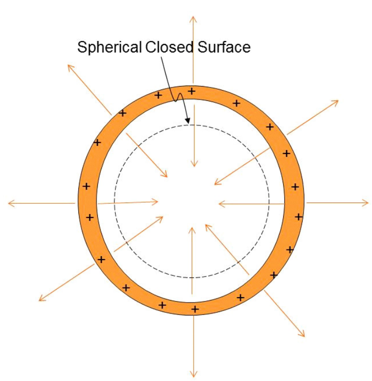
1. draw the charge distribution



2. Draw the field lines using symmetry



3. Choose (make up, invent) a closed surface that makes $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}}$ either just EdA or 0.

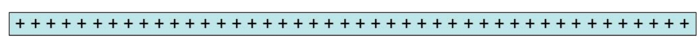


4. Find Q_{in}.
5. Solve $\oint EdA = \frac{Q_{inside}}{\epsilon_o}$ for the non, zero parts

The integral should be trivial now due to our use of symmetry.

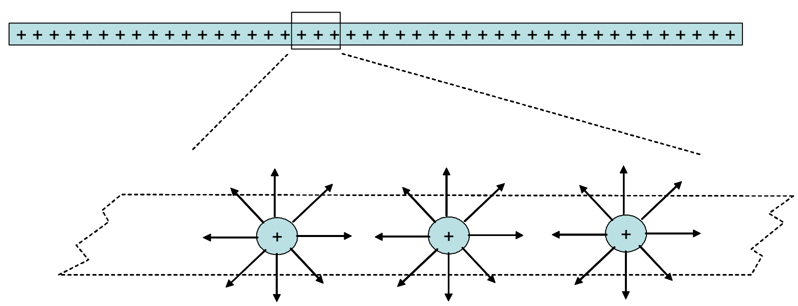
An infinite sheet of charge.

Spherical cases were easy. Let's try a harder one. Let's try our infinite sheet of charge. It is a little hard to draw. So we will draw it looking at it from the side from within the sheet of charge (somewhere in it's middle, if an infinite sheet can have a middle).

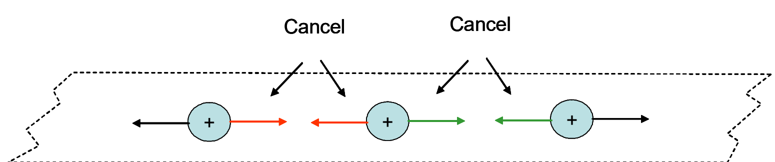


This completes step 1).

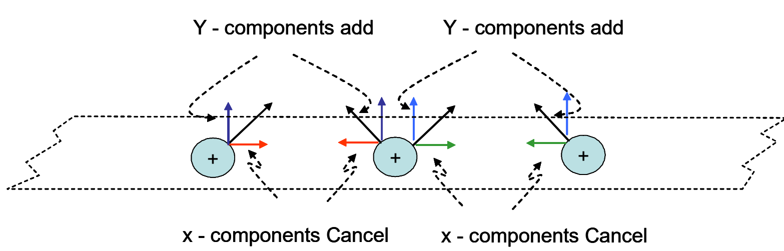
For step 2), let's think about what the electric field will look like.



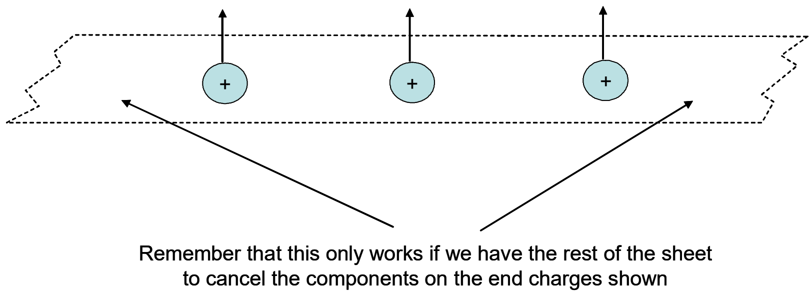
In the figure above I have blown up the view on three charge carriers and drawn some field lines. Notice that in the $x-$direction the fields will cancel.



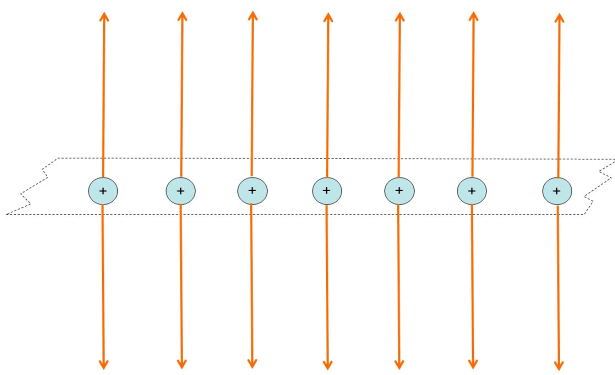
The $y-$components add



So we have only a field in the y direction

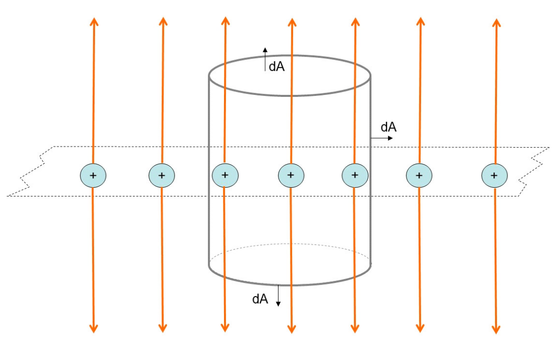


Now if we had edges of our sheet of charge, not all the $x-$components would cancel and the problem would be harder, but we won't do that problem now. Also note that there is a field in the $-y$-direction, I only drew some of the field lines in the figures.



This is step 2).

Now we need to choose an imaginary surface over which to integrate $\oint \vec{E} \cdot d\vec{A}$. We want $\vec{E} \cdot d\vec{A} = EdA$ or $\vec{E} \cdot d\vec{A} = 0$ over all parts of the surface. I suggest a cylinder.



Note that along the top of the cylinder, $E \parallel A$ so $\vec{E} \cdot d\vec{A} = EdA\cos\theta = EdA$. Along the side of the cylinder $E \perp A$ so $\vec{E} \cdot d\vec{A} = EdA\cos\theta = 0$. We have a surface that works! This completes step 3).

Now we need to solve the integral. The flux is just

$$\begin{aligned}
\Phi &= \oint \vec{E} \cdot d\vec{A} \\
\Phi &= \oint_{side} \vec{E} \cdot d\vec{A} + \oint_{ends} \vec{E} \cdot d\vec{A} \\
&= 0 + \oint_{ends} EdA = 2EA
\end{aligned}$$

where the factor of 2 comes because we have two caps and field in the $+y$ and $-y$ directions and where A is the area of one end cap. If we know that the sheet of charge has a surface charge density of η, then we can write the charge enclosed by the cylinder as

$$Q_{inside} = \eta A$$

so

$$\Phi_E = \frac{\eta A}{\varepsilon_o}$$

by Gauss' law. Equating the two expressions for the flux gives

$$2EA = \frac{\eta A}{\varepsilon_o}$$

or

$$E = \frac{\eta}{2\varepsilon_o} \tag{8.1}$$

which is what we found before for an infinite sheet of charge, but this way was *much* easier. If we can find a suitable surface, Gauss' law is very powerful!

8.2.1 Gauss's law strategy

In each of our problems today, we found the electric field without a nasty integration. Usually we want the electric field at a specific point. To make Gauss' law work we need to do the following for each problem:

1. Draw the charge distribution

2. Draw the field using symmetry

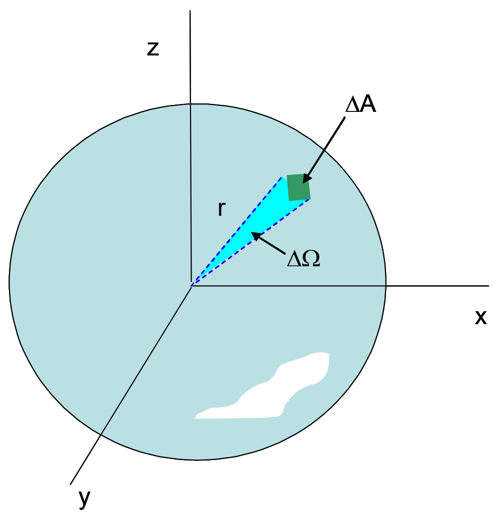
3. Invent a Gaussian surface that takes advantage of the field symmetry and that includes our point where we want the field. We will want $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = EdA$ or $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = 0$ for each part of the surface we invent.

4. Find the flux by finding the enclosed charge, Q_{in}

5. use $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = \frac{Q_{in}}{\epsilon_o}$ integrating over our carefully invented surface to find the field. If our surface that we imagined was good, then $\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}}$ will be very easy.

8.3 Derivation of Gauss' Law

A formal derivation of Gauss' Law is instructive, and it gives us the opportunity to introduce the idea of solid angle.

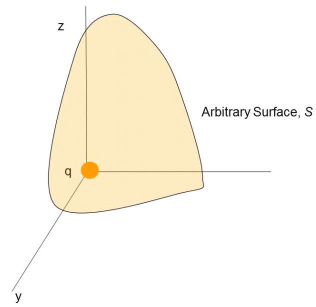


$$\Delta\Omega = \frac{\Delta A}{r^2} \tag{8.2}$$

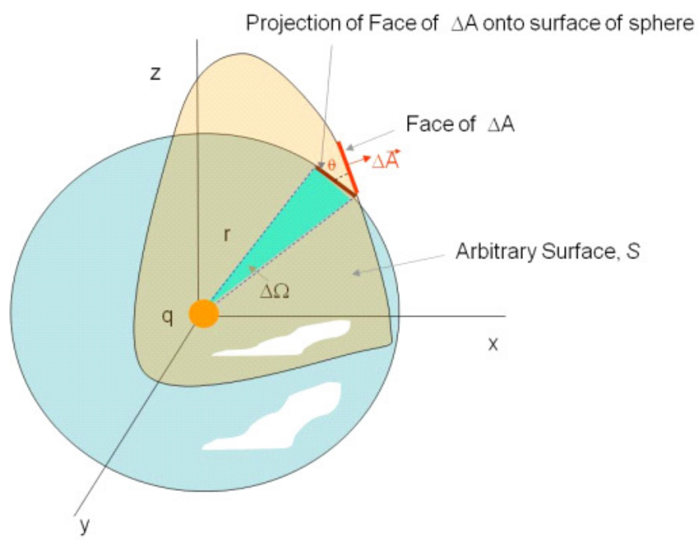
This is like a two dimensional angle. And just like an angle, it really does not have dimensions. Note that ΔA is a length squared, but so is r^2. The (dimensionless) unit for solid angle is the *steradian*. We can see that for a sphere we would have a total solid angle of

$$\Omega_{sphere} = \frac{4\pi r^2}{r^2} = 4\pi \,\mathrm{sr} \tag{8.3}$$

Now let's see why this is useful. Consider a point charge in an arbitrary closed surface.



If we look at a particular element of surface ΔA we can find the flux through that surface element. We can use our idea of solid angle to do this



$$\Delta \Phi_E = \tilde{\mathbf{E}} \cdot \mathbf{\Delta} \tilde{\mathbf{A}}$$

Since the field lines are symmetric about q and the surface is arbitrary, the element $\mathbf{\Delta}\tilde{\mathbf{A}}$ will be at some angle θ from the field direction so

$$\tilde{\mathbf{E}} \cdot \mathbf{\Delta} \tilde{\mathbf{A}} = E \Delta A \cos \theta$$

this is no surprise. But now notice that the projection of ΔA puts it onto a spherical surface of just about the same distance from q. The projected area is

$$\Delta A_P = \Delta A \cos \theta$$

At this point we should remember that we know the field due to a point charge

$$E = \frac{1}{4\pi\varepsilon_o} \frac{q}{r^2}$$

so our flux through the area element is

$$\begin{aligned}\Delta\Phi_E &= \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\Delta A\cos\theta \\ &= \frac{q}{4\pi\varepsilon_o}\frac{\Delta A\cos\theta}{r^2}\end{aligned}$$

but

$$\frac{\Delta A\cos\theta}{r^2} = \Delta\Omega$$

is the solid angle subtended by the projected area. Then

$$\Delta\Phi_E = \frac{q}{4\pi\varepsilon_o}\Delta\Omega$$

The total flux though the oddly shaped closed surface is then

$$\Phi_E = \frac{q}{4\pi\varepsilon_o}\oint d\Omega$$

where we integrate over the entire arbitrary surface, S.

$$\Phi_E = \frac{q}{4\pi\varepsilon_o}\oint_S d\Omega$$

but by definition

$$\oint_S d\Omega = 4\pi\,\mathrm{sr}$$

so

$$\begin{aligned}\Phi_E &= \frac{q}{4\pi\varepsilon_o}\oint_S d\Omega \\ &= \frac{q}{4\pi\varepsilon_o}4\pi\,\mathrm{sr} \\ &= \frac{q}{\varepsilon_o}\end{aligned}$$

which is just Gauss' law.

So far we have used mostly charged insulators to find fields. But we know we will be interested in conductors and their fields in building electronics. We will take up the study of charged conductors and their fields next.

Basic Equations

Gauss' law

$$\Phi = \frac{Q_{inside}}{\epsilon_o}$$

Gauss' law combined with our equation for flu

$$\Phi = \oint \overrightarrow{\mathbf{E}}\cdot d\overrightarrow{\mathbf{A}} = \frac{Q_{inside}}{\epsilon_o}$$