Chapter 4

Electric Fields of Standard Charge Configurations Part II

Fundamental Concepts

- Integrating vector fields for continuous distributions of charge
  - Start with $\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \int \frac{dq}{r^2} \hat{\mathbf{r}}$
  - Find an expression for dq
  - Use geometry to find expressions for r and to eliminate $\hat{\mathbf{r}}$
  - Solve the integral

4.1 Fields from Continuous Charge Distributions

Suppose we have a continuous distribution of charge with some mover charge q_m fairly far away. You might ask, how do we get a continuous distribution of charge? After all, charge seems to be quantized. Well, if we have a collection of charges where the distances between the individual charge carriers are much smaller that the distance from the whole collection of charges to some point where we want to measure the field (where the mover charge might be), then in our field calculations at this distant point we can model the charge distribution causing the field as continuous. As an analogy, think of your computer screen. It is really a collection of dots of light. But if we are a few feet away, we see a continuous picture. We can treat the dots as though there were no space in between them. For our continuous charge model, it is the same. We are supposing we are observing from far enough away that we won't notice the effects of the charges being separated by small distances.

We should remember, though, that this is a macroscopic view. At some point it must break down, since charge is carried in discrete amounts. If we want the field very close to a distribution of charges, we must treat our charge distribution as a collection of individual charges like we did in the last lecture. Notice in our last lecture that we found that the field infinitely far from the charges was always zero. That is too far away for our continuous charge model to be useful. But if we went far enough away–but not too far, the three charge configuration looked like a point charge with a total charge that was the sum of the individual charges. At such distances, the separation between the charges become unimportant. This is the sort of large distance we are talking about in our continuous charge distribution model.

To find the field due to a continuous charge distribution, we break up the charged object into small pieces in a calculus sense. Each small piece is still a continuous distribution of charge. It will have an amount of charge Δq, where here the Δ means "a small amount of." Then we calculate the field due to this element of charge. We repeat the process for each element using the superposition principle to sum up all the individual field contributions. This is very like our method of finding the field from individual charges, only instead of a sum we want to let Δq become very small and use an integral. The field due to this bit of charge is

$$\Delta\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{\Delta q}{r^2}\hat{\mathbf{r}}$$

Recall that here Δ means "a small bit of" and is not a difference between two charge values or two field values. We learned that we can sum up the fields from each piece

$$\begin{aligned}\overrightarrow{\mathbf{E}} &\approx \sum_i \Delta\overrightarrow{\mathbf{E}}_i \\ &\approx \frac{1}{4\pi\epsilon_o}\sum_i \frac{\Delta q_i}{r_i^2}\hat{\mathbf{r}}_i\end{aligned}$$

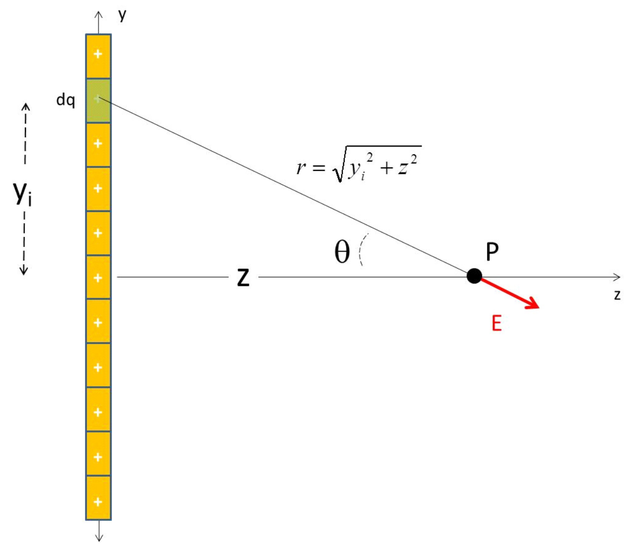
and now we use our M215 (or M113) tricks to convert this into an integral. We let our small element of charge become very small (but not so small that we violate our assumption that the charge distribution of Δq is continuous).

$$\begin{aligned}\overrightarrow{\mathbf{E}} &= \lim_{\Delta q_i \to 0}\frac{1}{4\pi\epsilon_o}\sum_i \frac{\Delta q_i}{r_i^2}\hat{\mathbf{r}}_i \\ &= \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}\end{aligned}$$

The limits of the integration must include the entire distribution of charge if we want the total field. This will be our basic equation for finding the field for continuous distributions of charge.

Let's do some examples.

4.1.1 Line of charge



Let's try this for a line of charge. This could be a long wire that has been charged up by a battery. This may seem like a simple charge configuration, but this problem is really quite challenging. Remember in our last chapter we added the contributions to the environmental field from three charges in a line. We want to do the same thing here, but for many charges. The first thing we need to do is to divide up our line into individual charge segments that we will call dq. And we will have to choose the amount of charge that we put into our dq segments. Then we treat each dq like the individual charges using

$$\overrightarrow{E}_{segment} = \frac{1}{4\pi\epsilon_o}\frac{dq}{r^2}\hat{\mathbf{r}}$$

as the contribution from that dq segment to the total environmental electric field. And we sum up all these contributions from each dq segment to find the total $\overrightarrow{E}$ from the entire line. If we make dq small, we can use our integral formula to do the summation.

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int\frac{dq}{r^2}\hat{\mathbf{r}}$$

Then, let's start by finding an amount of charge for our dq units. Let's say that the charge is evenly distributed along the line. Then we can use the linear charge density

$$\lambda = Q/\ell$$

to define our small amount of charge dq. The quantity Q is the total amount of charge on the wire and ℓ is the length of the wire. Then

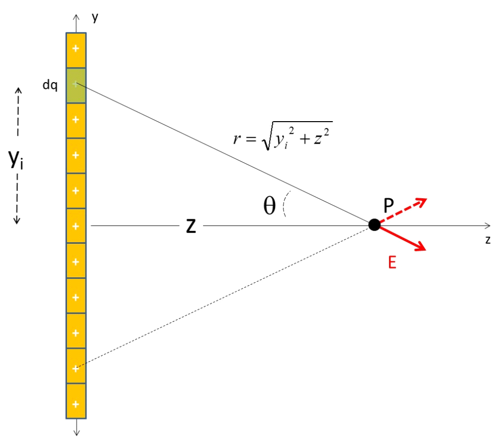
$$dq = \frac{Q}{\ell} dy = \lambda dy$$

Of course, we may not always have a constant density, then we need to have and element of charge that varies with position. For a line charge, we would have

$$dq = \lambda \left(y \right) dy$$

but for now, let's assume the linear charge density is constant. Our basic formula tells us that we should add up all the field contributions from all the dq elements. But we have an obstacle. We need a different $\hat{\mathbf{r}}_i$ for every dq_i. How do we deal with this?

Just like with last lecture, we only need the component of the part of the field that does not cancel. Here we need to have drawn a good picture.



From our drawing we can tell that, for each dq on the top half of the line there is a dq on the bottom half that will make another E contribution like the dotted line in the drawing. And notice that this new contribution will cancel out the y-component of the first contribution from dq. So in the end only the z component will survive (the y-components cancel). Then we only need to find the z component. We change our basic equation into one for just the z component by dotting $\overrightarrow{\mathbf{E}}$ with $\hat{\mathbf{k}}$

$$E_z = \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{k}}$$

Even if the y-component didn't cancel, we would just make y and z-components and combine them together at the end of our problem. So taking components is once again how we deal with two dimensional problems in physics. Then to eliminate the $\hat{\mathbf{r}}$ in our equation

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \int \frac{dq}{r^2} \hat{\mathbf{r}}$$

we take components, in this case the z-component so we can integrate over the while wire from $-y_{\max}$ to $y_{\max}$

$$\begin{aligned} E_z &= \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_o} \int_{--y_{\max}}^{y_{\max}} \frac{dq}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{k}} \end{aligned}$$

And we recognize

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = \cos\theta$$

so we are left with just

$$E_z = \frac{1}{4\pi\epsilon_o} \int_{-y_{\max}}^{y_{\max}} \frac{dq}{r^2} \cos\theta$$

which is much more likely be be integrable with what we know from M113 or M215.

The quantity r is still the distance from the environmental charge to the point P where we want to know the field just like it was in our last lecture. We can use the same geometry technique to get an expression for r.

$$r = \sqrt{y^2 + z^2}$$

And like in last lecture in the three charge problem, it makes it easier if we write

$$\cos\theta = \frac{z}{\sqrt{y^2 + z^2}}$$

Then our integral can be written as

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_o} \int_{-y_{\max}}^{y_{\max}} \frac{\lambda dy}{y^2 + z^2} \frac{z}{\sqrt{y^2 + z^2}} \\ &= \frac{\lambda z}{4\pi\epsilon_o} \int_{-y_{\max}}^{y_{\max}} \frac{dy}{(y^2 + z^2)^{\frac{3}{2}}} \end{aligned}$$

This now looks like a M215 or M113 problem. We can find this integral in an integral table or you can use your calculator, or a symbolic math package, or you can remember your M215 or M113 techniques and prove that

$$\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}$$

This is what an integral table entry would look like. We just need to match it to our variables. Using it looks like this:

$$\begin{aligned}
E_z &= \frac{\lambda z}{4\pi\epsilon_o}\int_{-y_{\max}}^{y_{\max}}\frac{dy}{\left(y^2+z^2\right)^{\frac{3}{2}}} \\
&= \frac{\lambda z}{4\pi\epsilon_o}\left[\frac{y}{z^2\sqrt{y^2+z^2}}\right|_{-y_{\max}}^{y_{\max}} \\
&= \frac{\lambda z}{4\pi\epsilon_o}\left[\frac{y_{\max}}{z^2\sqrt{\left(y_{\max}\right)^2+z^2}}-\frac{-y_{\max}}{z^2\sqrt{\left(-y_{\max}\right)^2+z^2}}\right] \\
&= \frac{\lambda}{4\pi z\epsilon_o}\frac{2y_{\max}}{\sqrt{\left(y_{\max}\right)^2+z^2}}
\end{aligned}$$

and we probably should already have noted that

$$\ell = 2y_{\max}$$

so

$$\begin{aligned}
E_z &= \frac{\frac{Q}{2y_{\max}}}{4\pi z\epsilon_o}\frac{2y_{\max}}{\sqrt{\left(y_{\max}\right)^2+z^2}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{z\sqrt{\left(y_{\max}\right)^2+z^2}}
\end{aligned}$$

This is the field due to a charged rod of length $\ell = 2y_{\max}$.

Note that there are only a few integrals that we can solve in closed form to find electric fields. It might be a good idea to build your own integral table for our exams, including the integrals from the problems and examples we work.

An infinitely long line of charge is one of our basic charge models. So far our line of charge is not infinitely long. We can find the field due to an infinite line of charge by letting L become large. To do this it is tradition to go back to expressing our field in terms of λ so let's go back a few equations, and take the limit as $y_{\max}\rightarrow\infty$

$$E_z = \lim_{y_{\max}\rightarrow\infty}\frac{\lambda}{4\pi z\epsilon_o}\frac{2y_{\max}}{\sqrt{\left(y_{\max}\right)^2+z^2}}$$

This looks hard, so let's do some algebra to try to make it easier. Let's take a

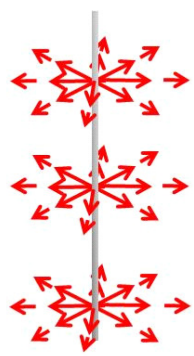
$y_{\max}$ out of the square root

$$\begin{aligned} E_z &= \lim_{y_{\max}\to\infty} \frac{\lambda}{4\pi z\epsilon_o} \frac{2y_{\max}}{y_{\max}\sqrt{1+\frac{z^2}{y_{\max}}}} \\ &= \lim_{y_{\max}\to\infty} \frac{\lambda}{4\pi z\epsilon_o} \frac{2}{\sqrt{1+\frac{z^2}{y_{\max}}}} \\ &= \frac{\lambda}{4\pi z\epsilon_o} \frac{2}{\sqrt{1+0}} \\ &= \frac{1}{4\pi\epsilon_o} \frac{2\lambda}{z} \end{aligned}$$

or if we use r now in place of z to define the distance from the center of the line of charge (so it is easier to compare to our point charge formula), we have

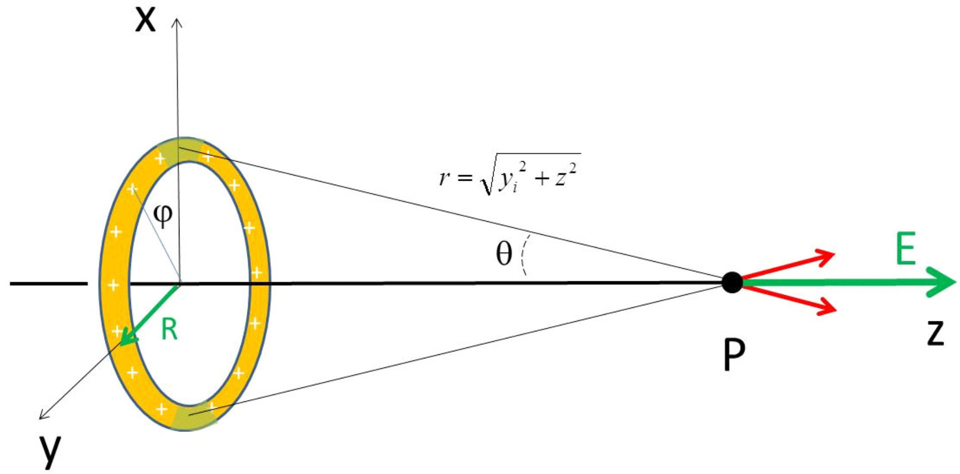
$$\overrightarrow{\mathbf{E}}_z = \frac{1}{4\pi\epsilon_o} \frac{2\lambda}{r} \hat{\mathbf{k}}$$

We should get a mental picture of what this means.



The field around a long line of charge only depends on the distance away from the line, and on the linear charge density. As we would expect, the field gets weaker as we get farther away. But it does not get weaker as fast as the point charge case. That makes some sense, because our infinite line of charge is, well, really big. You are never really too far away from something that is infinitely big. So we should not expect such a charge configuration to look very like a point charge no matter how far away we go. Of course an infinite line of charge is not something we can really build. It is just a useful approximation near, say, a charged wire. But farther from the wire the approximation would not be so good and we would have to go back to our finite line solution.

4.1.2 Ring of charge



Using what we have learned from the line of charge, we can find the axial field of a ring of charge. Again, our picture is critically important. We will need to divide up the charge into dq segments, find and expression for r, and solve the problem of eliminating $\hat{\mathbf{r}}$ by taking components. Then we can integrate to find the total field.

From the picture, we can see that we will only have a z-component again. So we can eliminate $\hat{\mathbf{r}}$ the same way as in the last problem. We model the ring as a line of charge of length $2\pi R$ that has been bent into a circle. Again we have the basic equation

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$$

Since the ring of charge is like a line of charge bent into a hoop. So we can plan to work this problem very like the the line charge. Start again with

$$dq = \lambda dy$$

but now we know that for the hoop

$$dq = \lambda ds$$

where s is the arc length. Recall that

$$\begin{aligned} s &= R\phi \\ ds &= Rd\phi \end{aligned}$$

where R is the radius of the ring and ϕ is an angle measured from the x-axis. So our dq expression becomes

$$dq = \lambda R d\phi$$

For the whole ring

$$
\begin{aligned}
Q &= \lambda R 2\pi \\
&= 2\pi R\lambda
\end{aligned}
$$

We also need to use geometry to find r, the distance from our dq segments to our point were we want to know the field.

$$r = \sqrt{y_i^2 + z^2}$$

but since this is a ring, our $y_i = R$ for all i. So

$$r = \sqrt{R^2 + z^2}$$

and using the same reasoning as in our last problem,

$$\cos\theta = \frac{z}{\sqrt{R^2 + z^2}}$$

Then we can set up our integral.

$$
\begin{aligned}
E_z &= \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{k}} \\
&= \frac{1}{4\pi\epsilon_o} \int \frac{dq}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{k}}
\end{aligned}
$$

Putting in all the parts we have found yields

$$
\begin{aligned}
E_z &= \frac{1}{4\pi\epsilon_o} \int \frac{dq}{R^2 + z^2} \frac{z}{\sqrt{R^2 + z^2}} \\
&= \frac{1}{4\pi\epsilon_o} \int \frac{z\lambda R d\phi}{\left(R^2 + z^2\right)^{\frac{3}{2}}} \\
&= \frac{z\lambda R}{4\pi\epsilon_o \left(R^2 + z^2\right)^{\frac{3}{2}}} \int_0^{2\pi} d\phi
\end{aligned}
$$

This is an easy integral to do! and we see that the axial field is

$$E_z = \frac{z 2\pi R\lambda}{4\pi\epsilon_o \left(R^2 + z^2\right)^{\frac{3}{2}}}$$

or, using our form for $\lambda = Q/L$

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} \frac{zQ}{\left(R^2 + z^2\right)^{\frac{3}{2}}} \hat{\mathbf{k}}$$

Once again we should check to see if this is a reasonable result. If we take the limit as z goes to infinity, we get zero. That is comforting. Our ring of charge looks like a point charge from infinitely far away. But if we just let z be much larger than R, but not too big

$$\begin{aligned}
\lim_{z\gg R}\overrightarrow{\mathbf{E}} &= \lim_{z\gg R}\frac{1}{4\pi\epsilon_o}\frac{zQ}{\left(R^2+z^2\right)^{\frac{3}{2}}}\hat{\mathbf{k}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{zQ}{\left(z^2\right)^{\frac{3}{2}}}\hat{\mathbf{k}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{zQ}{z^3}\hat{\mathbf{k}} \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{z^2}\hat{\mathbf{k}}
\end{aligned}$$

which looks exactly like a point charge field with total charge Q! Since a ring of charge should look like a point charge if we get far enough away, this is reasonable.

We have worked two problems for continuous charge distributions. The pattern for solving both problems was the same. And we will follow the same pattern for solving for the field from continuous charge distributions in all our problems:

- Start with $\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$
- Find an expression for dq
- Use geometry to find an expressions for r, the distance from dq to the point, P, where we want to know the field.
- Eliminate $\hat{\mathbf{r}}$
- Solve the integral

If you have a harder problem, one where you need the field from a continuous charge distribution at a point that is not on an axis, or your problem has little symmetry, you can go back to

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &\approx \sum_i \Delta\overrightarrow{\mathbf{E}}_i \\
&\approx \frac{1}{4\pi\epsilon_o}\sum_i \frac{\Delta q_i}{r_i^2}\hat{\mathbf{r}}_i
\end{aligned}$$

and perform the sum numerically. We won't do this in our class, but you might in practice or in a higher level electrodynamics course.

Basic Equations

The basic equation from this chapter is the equation for finding the field from a distribution of charge

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$$

The process for using this equation is

- Start with $\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$
- Find an expression for dq
- Use geometry to find an expressions for r
- Eliminate $\hat{\mathbf{r}}$ in the usual way by turning a two or three-dimensional problem into two or three one-dimensional problems (using vector components, etc.)
- Solve the integral(s) (Don't forget to report the direction)