Chapter 12

Electric potential of charges and groups of charges

Now that we have a new representation of the environment created by environmental charges, we will need to be able to calculate values for that representation for different configurations of charge like we did for electrical fields. But there is a huge benefit in using the electric potential representation, electric potentials are not vectors! So we don't have to deal with the vector nature of the field environment. The vector nature is still there, but we will ignore it. This means we will give up being able to give up vector directions for movement of our mover charges in many cases. But we can know much about the movement and the equations will be much simpler. We will take on the usual cases of environments from a point charge, a collection of point charges, and a continuous distribution of charges.

Fundamental Concepts

- Finding the electric potential of a point charge
- Finding the electric potential of two point charges
- Finding the electric potential of many point charges
- Finding the electric potential of continuous distributions of point charges.

12.1 Point charge potential

The capacitor was an easy electric potential to describe. Let's go back to a slightly harder one, the potential due to just one point charge. The potential energy depends on two charges

$$U_e = -\frac{1}{4\pi\epsilon_o}\frac{Qq}{r}$$

but the potential just depends on one.

$$V = \frac{U}{q}$$

where U is a function of Q, so a charge will cancel. But which charge do we divide by?

We need two charges to make a force,

$$F = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}$$

but when we defined the electric field we said the field from charge 1 would be there whether or not charge 2 was present. We wrote our equation like this to differentiate.

$$F = \frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r_{Em}^2}$$

so we would know that the environment is created by the charge we labeled Q_E and the moving charge is q_m.The situation is the same for electric potential.

$$U_e = -\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r_{Em}}$$

We say we have an electric potential due to the environmental charge even if the mover charge is not there. This is like saying there is a potential energy per unit rock, even if haven't yet picked up a rock to throw down the hill. The hill is there whether or not we are throwing rocks down it.

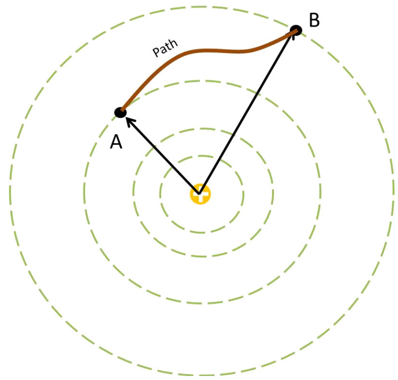
For electric potential, the potential is due to the field, and the field is there whether another charge is there or not.

Let's find this potential due to just one charge, but let's find it in a way that demonstrates how to find potentials in any situation. After all, from what we know about point charges, we can predict that

$$V = \frac{U}{q_m} = \frac{\frac{1}{4\pi\epsilon_o}\frac{Q_E q_m}{r}}{q} = \frac{1}{4\pi\epsilon_o}\frac{Q_E}{r}$$

But not not all situations come so easily. We only know forms for U for capacitors and point charges so far. So let's see how to do this in general, and compare our answer for the point charge with what we have guessed from knowing U.

Symmetry tells us the field will be radial, so the equipotential surfaces must be concentric spheres. Here is our situation:



We wish to follow the marked path from A to B finding the potential difference $\Delta V = V_B - V_A$.

Remember that the field due to a charge q is radially outward from the charge. To find the potential for this case we start again with a constant field. Our point charge is really not producing a constant field, but let's consider the constant field case (like from a capacitor) as a step in finding the actual potential for a point charge. So for our constant field case we start again with

$$w = \int F_e \cdot ds$$

but this time let's say that we have previously moved the charge across the capacitor (like in last lecture) and now we are ready to let it go. This is like carrying the rock up the hill before throwing it back down. Once the charge has been moved across the capacitor, the electric force will accelerate the charge back to the other side. The field will do work. And this time the force is in the same direction as the displacement.

$$\begin{aligned} w &= \int F_e \cdot ds \\ &= \int F_e \cos\left(\theta_{Fs}\right) ds \\ &= \int F_e \left(1\right) ds \\ &= F_e \Delta s \\ &= q_m E \Delta s \end{aligned}$$

where we have used the idea that F_e was constant as our charge moves. But from PH121 we remember that

$$w = -\Delta U$$

so we can find the electric potential energy difference for allowing the charge to move in the constant electric field

$$\Delta U = -q_m E \Delta s$$

and we now can find the electric potential difference

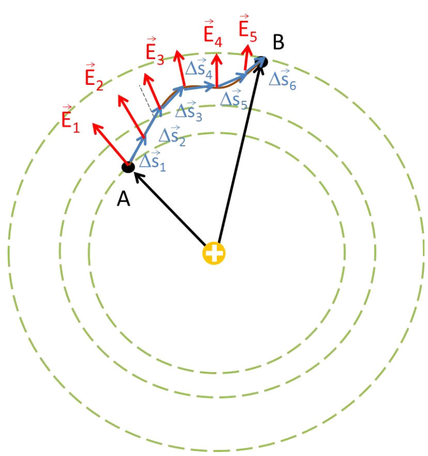
$$\Delta V = \frac{\Delta U}{q_m} = \frac{-q_m E \Delta s}{q_m} = -E\Delta s$$

where s is the path length along our chosen path from A to B. For our capacitor, this was just the distance from one side to the other, but as we take the next step in finding the potential for a point charge we need to be more general. We should really write this as

$$\Delta V = -\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\Delta s}$$

because we really want the component of E in the Δs direction. In the capacitor $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\Delta s}$ were in the same direction so we just had $\cos(0) = 1$. But now we need to allow for $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\Delta s}$ not being entirely in the same direction. So far our E field is still constant, but now it doesn't have to be in exactly the same direction as the displacement.

And for the point charge, our field E changes, so technically this value for ΔV is not correct. But if we take vary small paths, $\Delta\overrightarrow{\mathbf{s}}$, then the field will be nearly constant over the small distances. It will change from one $\Delta\overrightarrow{\mathbf{s}}$ to another $\Delta\overrightarrow{\mathbf{s}}$ But for each $\Delta\overrightarrow{\mathbf{s}}$ the field is essentially constant. Then we can add up the contribution of each small distance, $\Delta\overrightarrow{\mathbf{s}}_i$ to deal with the entire path from A to B for our point charge geometry.



That is, we take a small amount of path difference $\Delta\overrightarrow{\mathbf{s}}_i$ and add up the contribution, $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\Delta s_i}$ from this small path. Then we can repeat this for the next $\Delta\overrightarrow{\mathbf{s}}_{i+1}$ and the next, until we have the contribution of each pice of the path. We can call the contribution from one piece.

$$\Delta V_i = -\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\Delta s_i}$$

The total potential difference would be

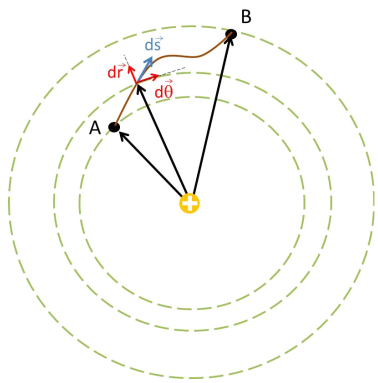
$$\Delta V = -\sum_i \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\Delta s_i}$$

In the limit that the Δs_i become very small this becomes an integral

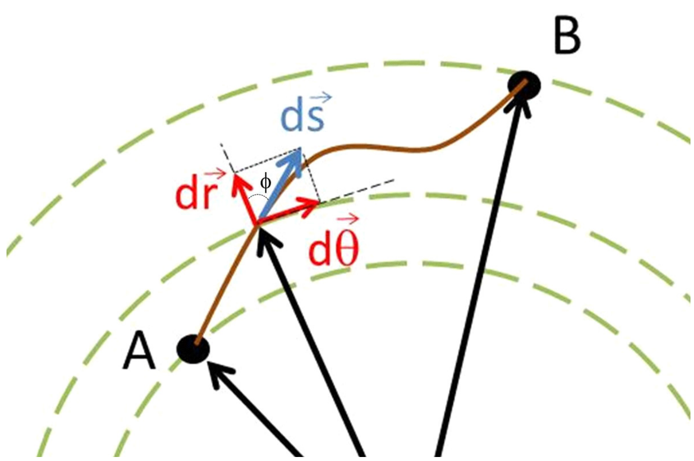
$$\Delta V = -\int_A^B \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} \tag{12.1}$$

and this is what we can use to find the electric potential from a point charge!

Let's find the electric potential between two points A and B where A and B could be any two points.



Here is an expansion of the region about A and B.



Let's divide up our $d\overrightarrow{\mathbf{s}}$ into components in the radial and azimuthal directions

$$d\overrightarrow{\mathbf{s}} = \left(dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}}\right)$$

from trigonometry we can see that

$$\cos\phi = \frac{dr}{ds}$$

and

$$\sin\phi = \frac{d\theta}{ds}$$

so

$$\begin{aligned} dr &= ds\cos\phi \\ d\theta &= ds\sin\phi \end{aligned}$$

and we can write

$$d\overrightarrow{\mathbf{s}} = \left(ds\cos\phi\hat{\mathbf{r}} + rds\sin\phi\hat{\boldsymbol{\theta}}\right)$$

The field due to the point charge is

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{12.2}$$

if we take

$$\begin{aligned} \overrightarrow{\mathbf{E}}\cdot d\overrightarrow{\mathbf{s}} &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}\cdot d\overrightarrow{\mathbf{s}} \\ &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}\cdot\left(ds\cos\phi\hat{\mathbf{r}} + ds\sin\phi\hat{\boldsymbol{\theta}}\right) \end{aligned}$$

we get only a radial contribution since $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = \mathbf{0}$. Then

$$\begin{aligned} \overrightarrow{\mathbf{E}}\cdot d\overrightarrow{\mathbf{s}} &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}\cdot ds\cos\phi\hat{\mathbf{r}} + 0 \\ &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}ds\cos\phi \end{aligned}$$

where ϕ is the angle between $d\overrightarrow{\mathbf{s}}$ and $\hat{\mathbf{r}}$ and where we recall that $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$. Recalling that

$$dr = ds\cos\phi$$

we can eliminate ϕ from our equation

$$\overrightarrow{\mathbf{E}}\cdot d\overrightarrow{\mathbf{s}} = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}dr$$

and we can integrate this!

$$\begin{aligned} \Delta V &= -\int_{r_A}^{r_B}\frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}dr \\ &= -\frac{Q}{4\pi\epsilon_o}\int_{r_A}^{r_B}\frac{1}{r^2}dr \\ &= \frac{q}{4\pi\epsilon_o}\frac{1}{r}\bigg|_{r_A}^{r_B} \end{aligned}$$

so

$$\begin{aligned}
\Delta V &= \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r_B}-\frac{1}{r_A}\right) \\
&= \frac{1}{4\pi\epsilon_o}\frac{q}{r_B}-\frac{1}{4\pi\epsilon_o}\frac{q}{r_A} \\
&= V_B - V_A
\end{aligned}$$

and we can recognize

$$\begin{aligned}
V_B &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r_B} \\
V_A &= \frac{1}{4\pi\epsilon_o}\frac{Q}{r_A}
\end{aligned}$$

which is what we expected. We can use this technique to get the potential from any electric field!

Note that the potential depends only on the radial distances from the point charge–not the path. We would expect this for conservative fields (where energy is conserved).

We know that, like potential energy, we may choose our zero point for the electric potential. For a point charge, we said we would take the $r_A = \infty$ point as $V = 0$.[1] So you will often see the potential for the point charge written as just

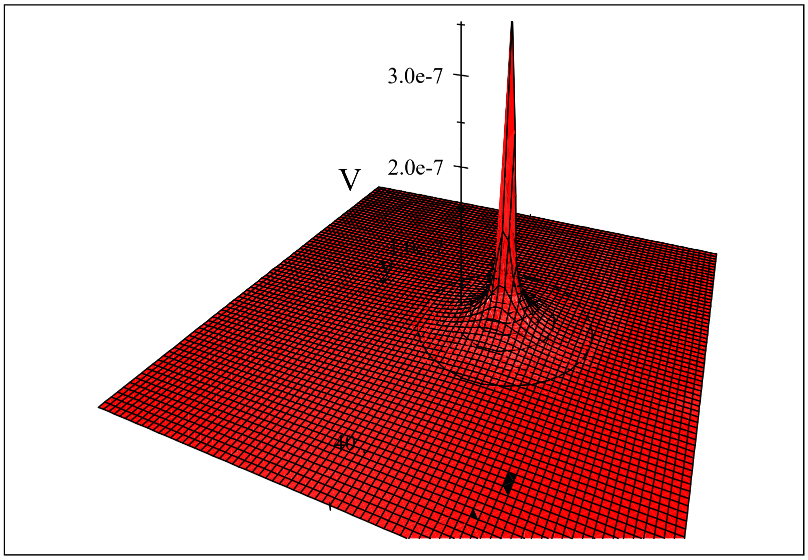
$$\Delta V = \frac{1}{4\pi\epsilon_o}\frac{Q}{r_B}$$

or simply as

$$V = \frac{1}{4\pi\epsilon_o}\frac{q}{r} \tag{12.3}$$

Here is a plot of this with $q = 2\times10^{-9}$ C and the charge placed right at $x = 10$ m.

[1] Remember this is because $U \rightarrow 0$ when $r \rightarrow \infty$.



It is probably a good idea to state that in common engineering practice we kind of do all this backwards. We usually say we will charge up something until it has a particular voltage. This is because we have batteries or power supplies that are charge delivery services. They can provide enough charge to make some object have the desired voltage. By "desired voltage" we always mean the voltage at a conductor surface in our apparatus.

Early *electrodes* were spherical, so let's consider making a spherical conductor have a particular potential at it's surface. A sphere of charge with radius R would have

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{R}$$

at it's surface. We can guess this because Gauss' law tells us that the field of a charged sphere is the same as that of a point charge with the same Q. Then it takes

$$Q = 4\pi\epsilon_o RV$$

to get the voltage we want. The battery or power supply must provide this. If the power supply or battery has a large amperage (ability to supply charge) this happens quickly. But away from the electrode the potential falls off. We can find how it falls off by again using

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

but with charge

$$Q = 4\pi\epsilon_o RV_o$$

so that

$$V = \frac{1}{4\pi\epsilon_o}\frac{4\pi\epsilon_o RV_o}{r}$$

or

$$V = \frac{R}{r} V_o$$

where V_o is the voltage at the surface. We can see that as r increases, V decreases.

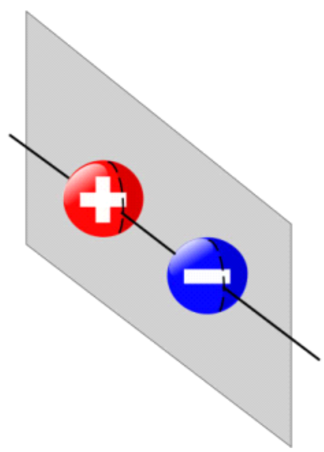
We will come back to using $\Delta V = -\int \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}}$ later in our course, but for now let's do a little more work with point charges.

12.1.1 Two point charges

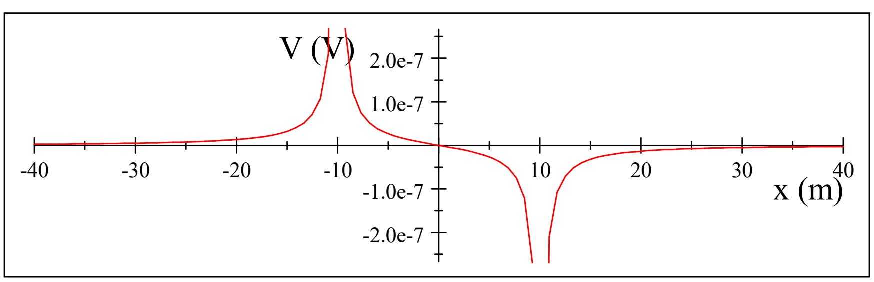
We can guess from our treatment of the potential energy of two point charges that the electric potential of two point charges is just the sum of the individual point charge potentials.

$$\begin{aligned} V &= V_1 + V_2 \\ &= \frac{1}{4\pi\epsilon_o}\frac{Q_1}{r_1} + \frac{1}{4\pi\epsilon_o}\frac{Q_2}{r_2} \\ &= \frac{1}{4\pi\epsilon_o}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right) \end{aligned}$$

It is instructive to look at the special case of two opposite charges (our dipole). We can plot the electric potential in a plane through the two charges.



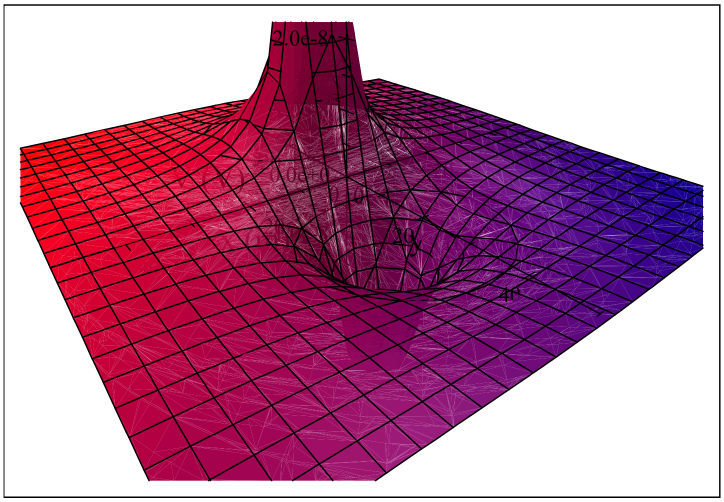
It would look like this



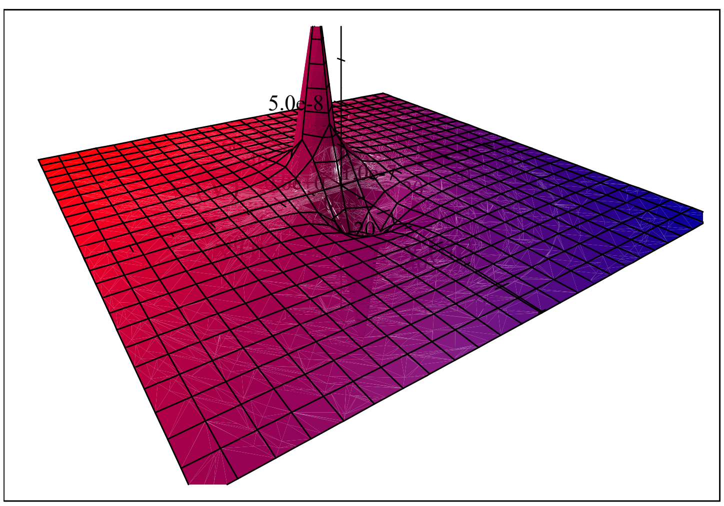
The charges $\left(Q = 2 \times 10^{-9}\,\mathrm{C}\right)$ were placed right at $x = \pm 10\,\mathrm{m}$. The potential

$$V = \frac{1}{4\pi\epsilon_o}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right)$$

becomes large near $r_1 = R_o$ or $r_2 = R_o$ where R_o is the charge radius (which is very small, since these are point charges). Plotting the potential in two dimensions is also interesting. We see that near the positive charge we have a tall mountain-like potential and near the negative charge we have a deep well-like potential.



Notice the equipotential lines. The more red peak is the positive charge (hill), the more blue the negative charge (valley).A view from farther away looks like this



Of course the hill and the valley both approach an infinity at the point charge because of the $1/r$ dependence.

12.1.2 Lots of point charges

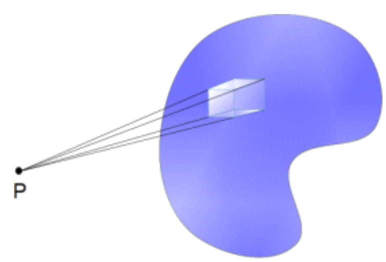
Suppose we have many point charges. What is the potential of the group? We just use superposition and add up the contribution of each point charge

$$V = \frac{1}{4\pi\epsilon_o}\sum_i \frac{Q_i}{r_i} \tag{12.4}$$

where r_i is the distance from the point charge Q_i to the point of interest (where we wish to know the potential). Note that this is easier than adding up the electric field contributions. Electric potentials are not vectors! They just add as scalars.

12.2 Potential of groups of charges

Suppose we have a continuous distribution of charge. Of course, this would be made of many, many point charges, but if we have so many point charges that the distance between the individual charges is negligible, we can treat them as one continuous thing. Suppose the total charge in the distribution is Q. If we know how the charge is distributed we can just interpret the distribution as a set of small amounts of charge dq acting like point charges all arranged into some shape.



Then for each charge dq we will have a small amount of potential

$$dV = \frac{1}{4\pi\epsilon_o}\frac{dq}{r} \tag{12.5}$$

and the total potential at some point will be the summation of all these small amounts of charge, weighted by r, the distance from the bit of charge (dq) to our point P.

$$V = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r} \tag{12.6}$$

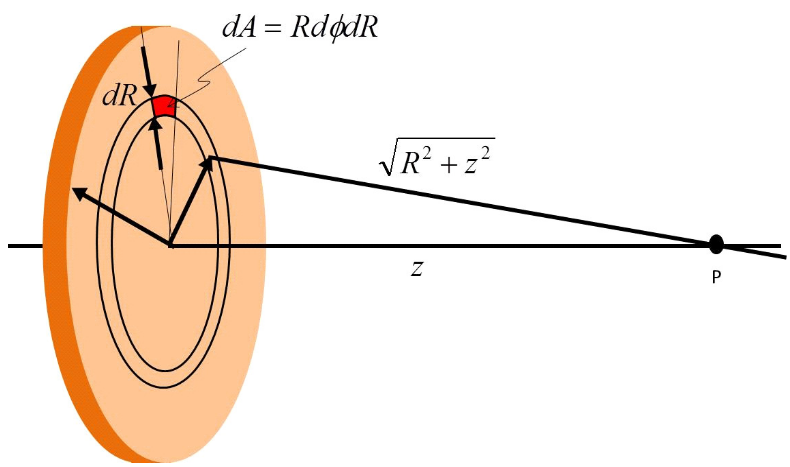
This looks a little like our integral for finding the electric field from a configuration of charge, but there is one large difference. There is no vector nature to this integral. So our procedure will have one less step

- Start with $V = \frac{1}{4\pi\epsilon_o}\int\frac{dq}{r}$
- find an expression for dq
- Use geometry to find an expression for r, the distance from the group of charges, dq, and the point P
- Solve the integral

Let's try one together

12.2.1 Electric potential due to a uniformly charged disk

We have found the field due to a charged disk. We can use our summation of the potential due to small packets of charge to find the electric potential of an entire charged disk.



Suppose we have a uniform charge density η on the disk, and a total charge Q, with a disk radius a. We wish to find the potential at some point P along the central axis.

To do this problem let's divide up the disk into small areas, dA each with a small amount of charge, dq. The area element is

$$dA = Rd\phi dR$$

so the charge element, dq, is

$$dq = \eta Rd\phi dR$$

For each dq we have a small part of the total potential. The variable r is the distance from our small group of charges that we called dq to the point P.

Then $r=\sqrt{R^2+z^2}$ and our integral becomes

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_o}\int\frac{dq}{r} \\ &= \frac{1}{4\pi\epsilon_o}\int\int\frac{\eta R d\phi dR}{\sqrt{R^2+z^2}} \end{aligned}$$

We will integrate this. We will integrate over r from 0 to a and ϕ from 0 to 2π which will account for all the charge on the disk, and therefore all the potential.

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_o}\int_0^{2\pi}\int_0^a\frac{\eta R d\phi dR}{\sqrt{R^2+z^2}} \\ &= \frac{\eta 2\pi}{4\pi\epsilon_o}\int_0^a\frac{R dR}{\sqrt{R^2+z^2}} \end{aligned} \tag{12.7}$$

At this point we can use an integral table or a symbolic math processor to solve the integral.

$$\begin{aligned} V &= \frac{\eta 2\pi}{4\pi\epsilon_o}\left.\sqrt{R^2+z^2}\right|_0^a \\ &= \frac{\eta 2\pi}{4\pi\epsilon_o}\sqrt{a^2+z^2}-\frac{\eta 2\pi}{4\pi\epsilon_o}z \end{aligned}$$

so

$$V=\frac{\eta}{2\epsilon_o}\left(\sqrt{a^2+z^2}-z\right) \tag{12.8}$$

This is the potential at point P.

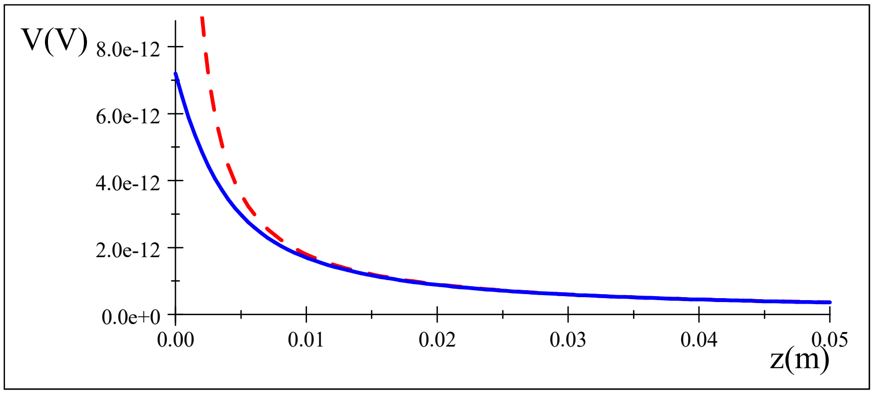
We compared our electric field solutions with the solution for a point charge. We can do the same for electric potentials. We can compare our solution to a point charge potential for an equal amount of charge. Far away from the disk, we expect the two potentials to look the same. The point charge equation is

$$V=\frac{Q}{4\pi\epsilon_o}\frac{1}{z}$$

Our disk gives

$$V=\frac{Q}{4\epsilon_o\pi}\frac{2}{a^2}\left(\sqrt{a^2+z^2}-z\right) \tag{12.9}$$

They don't look much alike! But plotting both yields



The dashed line is the point charge, the solid line is our disk with a radius of 0.05 m and a total charge of 2 C. This shows that far from the disk the potential is like a point charge, but close the two are quite different as we would expect. This is a reasonable result.

We will calculate the potential due to several continuous charge configurations.

But, you may ask, since we knew the field for the disk of charge, couldn't we have found the electric potential from our equation of the field? We will take up this question in the next two lectures.

Basic Equations

The electric potential of a point charge is given by

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

where the zero potential point is set at $r = \infty$.

Electric potentials simply add, so the potential for a collection of point charges is just

$$V = \frac{1}{4\pi\epsilon_o}\sum_i \frac{q_i}{r_i}$$

To find the potential due to a continuous distribution of charge we use the following procedure:

- Start with $V = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r}$
- find an expression for dq
- Use geometry to find an expression for r
- Solve the integral

Since electric fields and electric potentials are both representations of the environment created by the environmental charge, there must be a way to calculate the potential from the field and *vice versa*. It will take us two lectures to do both.