Physics Lectures

PH 121 Fundamentals of Physics I

Version 0.4000

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Preface

Preface Head

This set of notes is intended to be an aid to the student. There are likely errors and mistakes, so use the text, but don't expect perfection. If there are things that are confusing, please talk to me or ask questions in class.

Acknowledgments

I would like to thank the many students who suffered through classes giving feedback to make these lectures better.

BYU-I University *R. Todd Lines*.

Contents

1 What is Physics

This is sort of an introduction. All the regularly structured chapters will start in a particular way. I will give you the main topic in the title, and then give what I think are the four most significant concepts in the chapter. That will give you a warning about what is important in what you will read. You will read a chapter before each lecture and the list of important topics will act as a study guide both for the reading and for the lecture. You should test your reading after you read by reviewing this list of important topics and making sure you can explain each of them to someone else. Find a class mate or roommate, or spouse or minister or dog or whatever you have, and explain these concepts to them. By teaching, you will learn more completely. But for this first chapter, you likely not yet have your book nor will you have read in advance. So I structured this chapter differently. We will talk about this together, then you will have this printed text to refer to later. Think of this chapter as a set of instructions for the rest of the course.

Most of you will have taken some science classes in high school. Before we start learning how to solve detailed problems in physics, we should pause to discuss why this science called physics is different from what you already know. Let's start with the scientific method–something that you do know–and show how it is applied differently in this new science.

The scientific method

Most of us in this class are not science majors. So we should talk for a moment about what science is, and in particular, what physics is. Let's start with, the scientific method.

The scientific method

In your high school Earth science or biology class (or even a high school physics class) you probably learned the scientific method. It is usually presented as shown in figure

(1.1) on the left hand side (green bubbles).

We remember that first you must have an idea about how some part of the universe works. After studying the idea, noting what others have said on the subject we form a mental model that incorporates this new idea. This idea is formally called a *hypothesis*. To be science, we will have to test this hypothesis through experiment. That means you have to think of a consequence of the new idea of how the universe works, and try it out. You have to see if the universe really works according to your idea. If the experiment can be interpreted as a positive result, then we declare that we have learned something through the scientific method. We call our hypothesis that is now tested a *theory*. We publish our result so the rest of the scientific world can learn of our results. The details of doing this are taught in our program in PH150, our first lab class. But we won't do all the steps in this class.

So what will we do?

We will practice just part of the scientific method. The part where we use math and physics we know to solve physics problems. If you are a physicist that has a good hypothesis to test, you need to use the physics we know and math to form a prediction that can be tested in an experiment. The physics majors in our class need this experience. For example, Einsten's theory of relativity predicted that light rays would be bent by the gravity of large objects. Einstein calculated how much light would be bent by the Sun's gravity. This took understanding of the physical theory and a bunch of math. Eddington measured the light deflection, and found it in agreement with Einstein's calculations. This is the stuff physicists do.

But it turns out that we can use the same combination of physics and math to predict how many things will operate if we build them. And this is really useful for engineering! In this class we will just deal with this little bit of the scientific method because it is so useful for engineering. If you are an engineer, just realize that there is much more to science than what we will do.

If you are a chemist or other type of scientist, your goal is to get ready to understand the electrical and quantum mecanical processes in chemical bonding, etc. It is even a thinner slice of the scientific method for you chemical scientists.

For both engineers and scientists, you should know that, although this is a good statement of the scientific method, it is really not the end of the story in physics (or other sciences, for that matter). The story continues with the experiment that you did

Figure 1.1.Scientific_Method

being repeated by other researchers.

An odd but important belief that rests at the heart of the scientific method is that we cannot prove something to be true through scientific induction[1]. We can only prove things to be false! Thus every time you hear a news reporter say that "scientists have proven..." you should be sceptical! After you have tested your hypothesis, and declared it to be a theory, the theory must go through a rigorous set of attempts to prove it false. We call this process falsification. Many researchers over some time test the predictive power and consistency of the theory with new results from their experiments and the results of previous theories.

If these researchers agree with your conclusions, the idea you tested may become an accepted *model*. I have used the word "model" to describe a theory of limited applicability or a step on the way to becoming a general theory. Not all scientists use this term. Sometimes scientists use "model" to describe their mental image of how things work. I will use the word "model" as a limited theory, but I may slip occasionally and call my mental picture a "model."

If the theory is able to predict correct behavior beyond the experiment you used to test your hypothesis, and it's results are all positive, then the theory may become accepted as a general theory. In earlier times this would have been called a scientific law. Now the use of the words "scientific law" usually means a well established empirical relationship. This means a well tested equation which concisely embodies the ideas contained in our theory. Notice that if there are any negative results, it is back to the drawing board! The whole scientific process must start over.

Also notice that no theory is beyond doubt! New experiments may bring new results that will cause new hypothesis and new theories to be formed that will take the place of the old, established theories.

This makes the scientific method look a little complicated, but very ordered. In reality, the steps are not always done in order. Worse, the different branches of science disagree on the method of doing science a little. For example, in biology and psychology predictive experiments are harder to do! There is less emphasis on prediction in these sciences. But the green bubble steps are usually agreed on in introductory descriptions of science.

There are also differences in what the steps of the scientific process mean in different

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[1] The process of inferring a general law or principle from the observation of particular instances (OED)

branches of science. Most are the same, but in physics the hypothesis is generally reduced to a *mathematical equation* that predicts an outcome for the experiment. In physics concepts are transformed into equations that give predicted results. Even when the idea becomes accepted theory, with an accepted scientific law like

$$E = mc^2$$

Einstein's famous equation, you can see that complex ideas (the ideal that energy and mass are equivalent) is expressed as an equation that can be used to test the idea. In physics we talk of Einstein's theory, but physicists talk more about solutions to Einstein's equations–that is the real test of an idea in physics because solutions to the equations are predictions of what will actually happen. And if the predictions don't come true, we have falsified the theory!.

In our class, you won't be required to invent new ideas about how the universe works. We will be learning about the collection of ideas that others have already tested for the motion of single objects. But in our homework and tests, we will use this difference between physics and other branches of science. We will always try to find an equation as a solution. This will be our standard way to do physics. This might seem a little scary at first, but we will get used to it quickly. So in this class we will use two parts of the scientific method in each problem. We will make a mathematical expression of our hypothesis. We will call this a symbolic answer. It is the embodiment of our scientific theories that we will study in each chapter or lecture. The numerical answer is a numerical prediction from our hypothesis. When the problem is graded, you will know if your if your prediction is good!

A good theory is able to predict things that will happen. It must not predict anything that does not happen. When a theory's predictions are wrong, it must be changed or even abandoned.

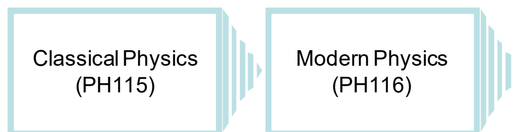
Notice that this expression of the scientific method does give theories that *explain* many things. But explaining is not enough to be accepted in physics. A theory must be able to predict in order to become named a scientific law or a general theory.

Identify several scientific laws, theories

Scope of Physics

Scope is a word which here means a limitation in topics (not a mouthwash). There must be some limitation in topics, or physics would be synonymous with biology or chemistry (or both). In practical usage, physics is *not* the study of living things and is generally *not* the study of how atoms form compounds. So what is physics?

It is the study of how things move. This may seem very restrictive, but it really is not. Physics incorporates everything from the motion of a ball at a play ground to the motion of atoms in a star to the motion of ions through a cell lining.



Our approach to physics will divide the world of physics into two main parts. The first part is called "classical" physics. It contains the theories that explain and predict motion of large objects at fairly slow speeds. It takes us three courses to cover all of classical physics, PH121, PH123, and PH220. The second part is called "modern" physics. It contains the theories that explain and predict motion of small objects or very fast objects. We start studying modern physics in PH279. In our PH121 class we will start with the part of classical physics called "mechanics." Mechanics deals with the motion of every-day objects like cars, or balls, or people. We will start with a one to a few objects. We will let those few objects travel in straight lines at first, but soon we will allow the objects to follow more complex paths or even to rotate.

But how does physics relate to biology, or chemistry, or any other branch of science? Purists think that everything is physics. That is, if you knew all the particles that make up something, and knew how they moved, you could apply the laws of physics, and predict the behavior of any object. When it comes to chemistry, this is true. We can explain what we see in chemistry by applying what we know from physics to the atoms and particles of mater. We can explain much of biology this way as well. We would like to explain psychology with basic physical laws, but this has yet to be done. I believe, in part, this is because there are actions by our spirits that physics does not yet recognize that affect the science of psychology. Maybe in your life this will be shown. But for now this is just speculation. What we will say for the purposes of PH121 is that physics is the basic underlying rules behind the other sciences, but at some point the mathematical book-keeping to use the basic laws becomes too difficult, so we tend to study compounds or whole animals rather than keep track of all the motions of their individual atoms.

Course Structure

This class is designed into several parts:

1. How we describe motion
2. Motion in one dimension
3. Motion in more than one dimension
4. Forces and motion
5. How objects interact
6. Impulse and momentum
7. Energy
8. Rotation
9. Gravitation

These topics will let you know everything you need to know to understand how things move and to solve motion problems.

By the time we are done, you can explain how bungee jumping, skydiving, scuba diving, and other fun things work. Let's take some examples

Four Ball Toy Experiment

Rotating Wheel Experiment

Just a note on testing: all tests are cumulative, in that the material builds. If you learn one chapter, and then figure you can forget all you learned because we are on the next chapter, you will be less pleased with your performance.

There is less emphasis on memorizing in this class. In Biology, and even in Chemistry much of what you need to know is memorized. In Physics, there are an infinite number of physics questions I could ask you. The only hope of getting any one of them right is to understand the concepts behind how things move (that is what we will learn!) and to have a systematic approach to solving the problems. So that is just what we will do. And I hope we will have fun along the way!

2 What is Motion?

We have said that we will study motion in this class. But what is motion?

The answer will take the rest of this class, most of PH 123, PH220, and PH 279. But we will have to start somewhere. Let's make a provisional definition of motion, one that we can refine as we become more knowledgeable.

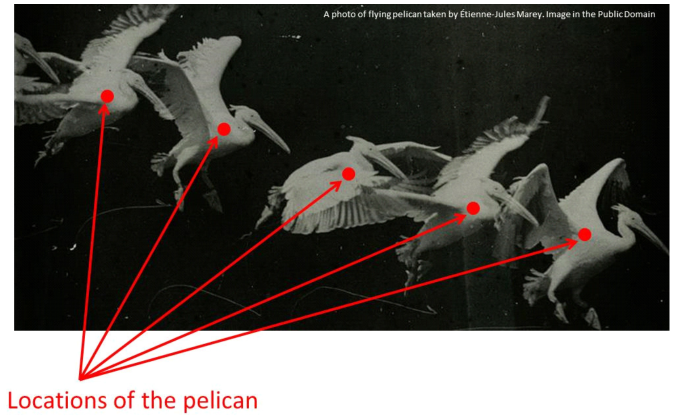
Definition 2.1 *Motion is the change of an object's position in an amount of time*

How do we depict motion?

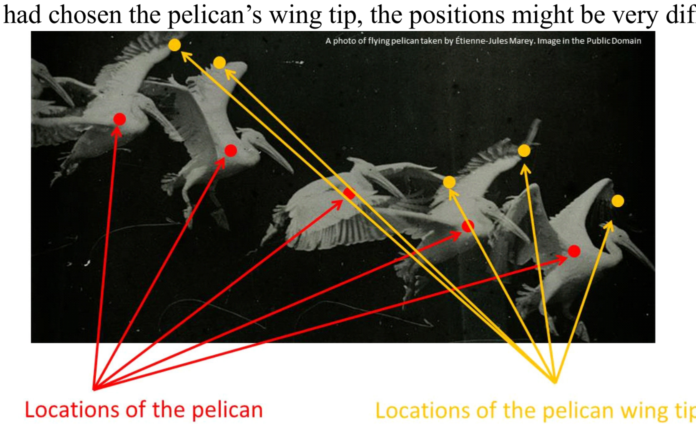
An early photographer named Étienne-Jules Marey developed a way of showing how an object's position changed in time. He made several photographs of the object on the same piece of photographic film, each exposure being a set time later than the previous. Here is his photograph of a pelican.

Photograph of flying pelican taken by Étienne-Jules Marey c. 1992. (Image in the Public Domain)

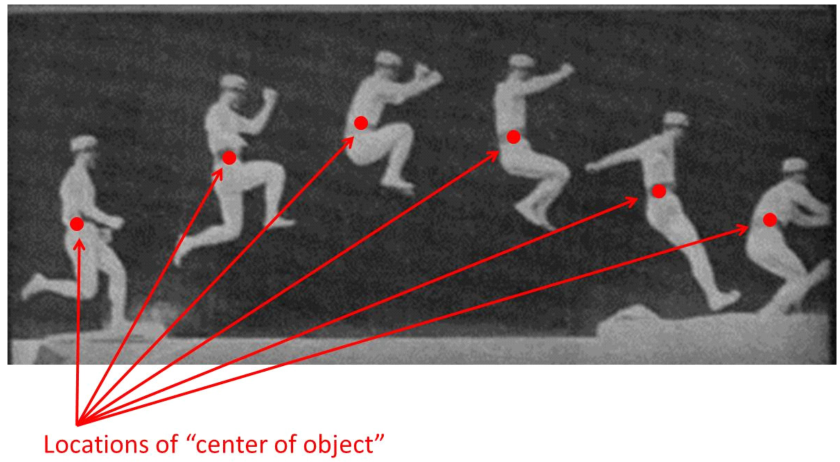
In this case, the object is the pelican, and we see that the pelican has changed position in the time between exposures. But this photograph also shows the problems with our simple definition. So far, we have considered the pelican as the object. So we could define a single position as the location of the pelican. In the next figure, a red dot represents the "location of the pelican" and we see that the position of this dot changes in time.



But if we had chosen the pelican's wing tip, the positions might be very different.

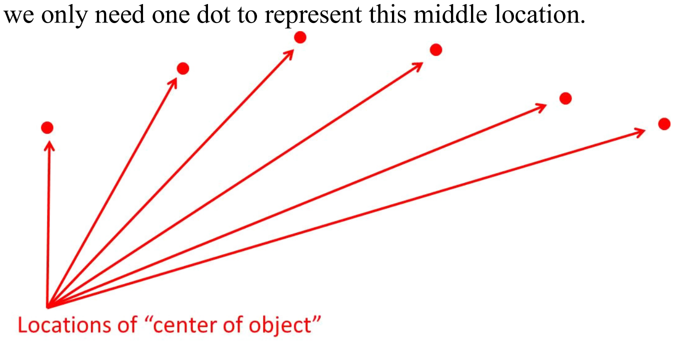


So when we say "an object" is in motion, just what do we mean? In the next figure, a man is jumping. But once he has left the ground, the middle of the man will follow the same path a ball would follow if it had the same mass and was launched at the same speed.



Man in White Jumping, Étienne-Jules Marey. (Image in the Public Domain)

The man could wave his arms or do a summersault in the air. But his middle would follow the same path. Then, we could start our study of motion by studying how the middle of things move. This is a good representation of the motion of the object as a whole. And we only need one dot to represent this middle location.



Of course this is an approximation. We could think of a case where the man was holding a ball and threw the ball part way through his jump. Then the "middle" of the man-ball system might move in a different way than the man. But that is a complication we will take on later. For now, we will deal with the motion of objects that can be described as begin one whole object that won't come apart during the motion we observe.

The photos we have used are tricky to take, so we will switch to diagrams that we can draw, but that represent the same thing. An object is shown at several positions with each new drawing representing the position of the object a set time Δt later. Here is such a diagram

The object is a ball. The motion starts from the left. Note that the position changes between the first two drawings of the ball. So the ball is moving. But also notice that the position change is less between the second and third drawings of the ball. We can tell from this that the ball is slowing down.

Let's give some names to what we have learned so far:

Definition 2.2 *Particle model: Using a "middle" point on an object to represent the location of the object*

Definition 2.3 *Motion diagram: A diagram that shows drawings of an object where each drawing shows the object a set time later.*

How do we describe when and where something happened?

Motion is the change of an object's position in an amount of time. Then we are going to have to have a way to express the time at which something happened and we are going to have to be able to express the position where it happened. Let's see how we do this in physics.

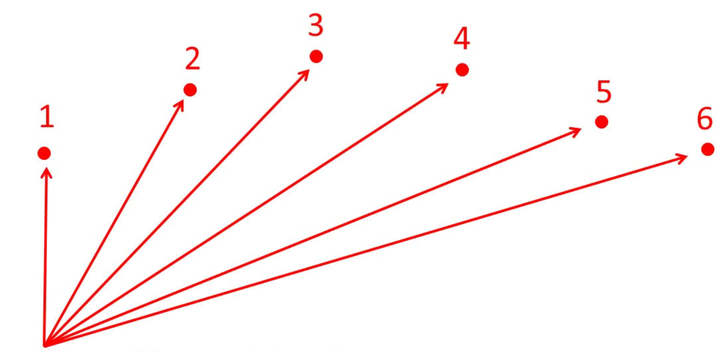
Duration: Experimental time

Let's consider measuring time in an experiment. Ideally, we would start timing our experiment right when the experiment begins. So we start with an initial time of $t_i = 0$. We could, say, start a stop watch right when the experiment starts, and stop the stop watch when we are done. I have an ap that does this on my phone. Then the reading on the watch is the duration of the experiment. But this ideal situation is not always possible. For one thing, you may be working in a team, and your lab partner may be slow (or fast) in starting the stop watch. But even more importantly, we may need to consider *parts* of an experiment. Think about our motion diagrams. We could consider the entire motion of an object as our experiment. Then each new position in our motion diagram represents part of the entire motion. And there is a beginning time and ending time for each part of the motion. We need a way to express both the starting and stopping times for parts of experiments and the duration of the part.

We can use a math symbol to represent our set time between two drawings in a motion diagram. That symbol is Δt. The first part is the Greek letter Δ pronounced "delta." We will use this symbol to mean a difference between two times.

$$\Delta t_{fi} = t_f - t_i$$

where t_f is the final time we are considering (for either the whole experiment or for the part of the experiment on which we are concentrating) and t_i is the initial time. For example, we could go back to our jumping man. We could number each of the positions where the middle of the man shows up in the photo.



Then the entire duration of the whole experiment would be

$$\Delta t_{total} = t_6 - t_1$$

and the duration of the first part of the experiment, the time between when the man was at position 1 and when he was at position 2, is given by

$$\Delta t_{\text{part 1}} = t_2 - t_1$$

When we were making our motion diagrams we said we would make series of pictures with each picture "being a set time later than the previous." We can use our new notation to see how to write our "set time." By "set time" we mean that Δt is not changing as we go from one part of the experiment to another. So for the first part of the diagram we can write the duration as just

$$\Delta t = t_2 - t_1$$

and since the duration of each part of the experiment is the same we could write

$$\begin{aligned} \Delta t &= t_3 - t_2 \\ \Delta t &= t_4 - t_3 \\ &\vdots \end{aligned}$$

and so forth.

This is not really too strange. If I ask you how long your physics class lasts, you would tell me "an hour." This is correct. It really does not matter if class starts at 9:00AM or 3:15PM. The class is still an hour. We could say that the duration of the class is a Δt so that

$$\Delta t_{class} = \text{10:00am} - \text{ 9:00am}$$

or

$$\Delta t_{class} = \text{4:15am} - \text{ 3:15am}$$

The duration is the same. And that hour is only part of your day. Some books call duration "*elapsed time*."

We will use the symbol Δ to mean a difference between two quantities, like time, often in this class.

Note that writing duration this way solves a lot of our experimental problems. If your lab partner starts the stop watch too late, then you just record the beginning and ending times and you can still find the duration. Beginning times can even be negative! Think of a rocket launch. The NASA official always starts the countdown by saying "$t - 10\,\mathrm{s}$." The time is measured from the launch time. If the launch time is $t = 0$, then before the launch time is negative time. But our Δt notation for duration handles this just fine. Suppose you try this. Suppose you are going out on a date. You tell yourself that $t = 0$ is the start of the date. But you have to get ready for the date and you start getting ready an hour before the date starts. Then the beginning of the date experience is at $t_i = -1\,\mathrm{h}$. Further suppose you go for a movie and ice cream and arrive home three hours after the start of the date. Then you could write

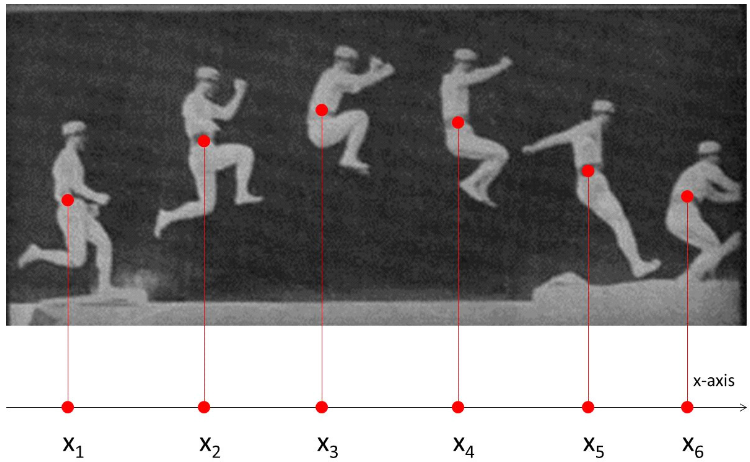
$$t_f = 3\,\mathrm{h}$$

and the duration of your date experience would be

$$\begin{aligned} \Delta t &= t_f - t_i \\ &= 3\,\mathrm{h} - (-1\,\mathrm{h}) \\ &= 4\,\mathrm{h} \end{aligned}$$

Position and Displacement

Now that we have a way to write the time involved with an experiment, let's find a way to describe how the object's position changes. Happily, we can use nearly the same notation!



We can mark the position of the man at each part of his jump. In each segment of the jump he will arrive at a different position. You can see this as red dots plotted on an axis under the picture of the man. Each of the positions (red dots) is labeled with a position label ($x_1, x_2, x_{3,}$ etc.). Then we can write how far he traveled in the horizontal direction as

$$\Delta x = x_f - x_i$$

We can write a displacement for the entire jump as

$$\Delta x = x_6 - x_1$$

we are using the Δ in the same way we did for time to mean a difference between two quantities, this time two positions.

Our jump picture has images of the man at equal time intervals. But the distance the man travels in each time interval is not necessarily the same. We could write the displacements for each part of the jump as

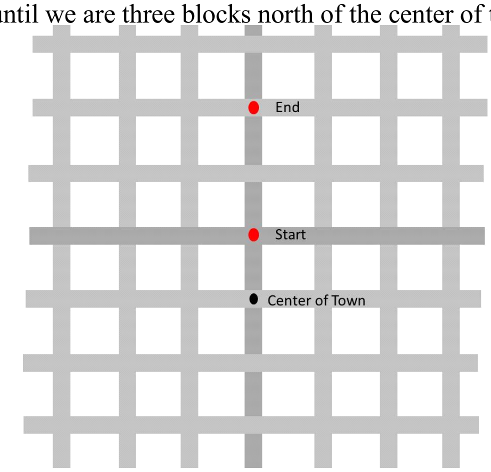
$$\begin{aligned} \Delta x_{21} &= x_2 - x_1 \\ \Delta x_{32} &= x_3 - x_2 \\ \Delta x_{43} &= x_4 - x_3 \\ &\vdots \end{aligned}$$

In each case, there are two subscripts. For example Δx_{21} has the subscripts “2” and “1.” It may look like a single subscript of “21,” but it’s not usual that we will mark up positions above $9.$ and then we use commas. For example

$$\Delta x_{43,42} = x_{43} - x_{42}$$

So we would interpret Δx_{21} as the displacement between positions x_1 and x_2.

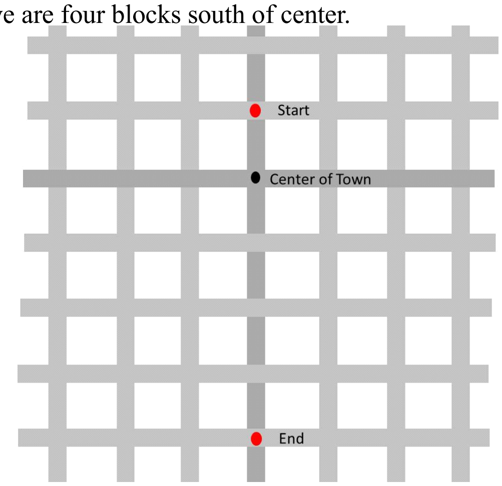
Note that displacement is a little different than distance. Displacements can be negative. If I am going to the right, my displacement is positive, but if I am going to the left my displacement is negative. For example, suppose we pick an origin for motion to be the center of Rexburg. And suppose we start our experiment a block north of the center of town and we walk until we are three blocks north of the center of town.



Then our displacement is

$$\begin{aligned} \Delta x &= 3\text{blocks} - 1\text{block} \\ &= 2\text{blocks} \end{aligned}$$

We could also go south. So suppose we start again one block north of the town center, but we walk until we are four blocks south of center.

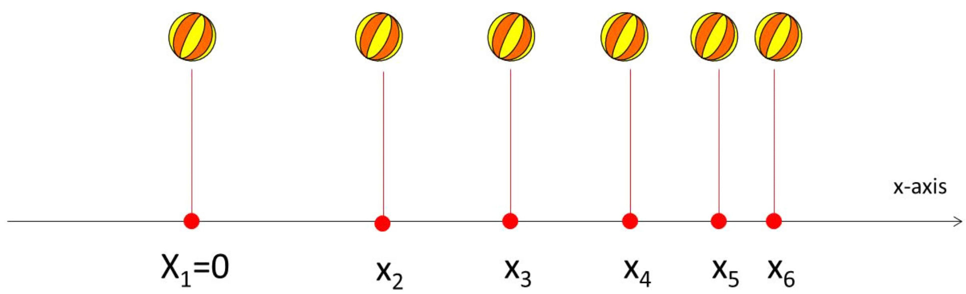


The our displacement is

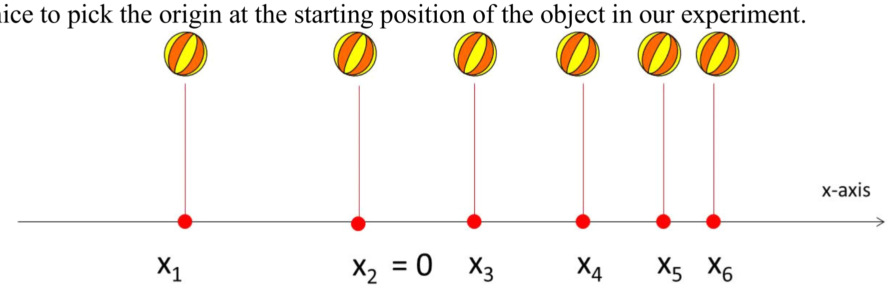
$$\begin{aligned}\Delta x &= -4\text{blocks} - 1\text{block} \\ &= -5\text{blocks}\end{aligned}$$

In effect, displacement gives not just the distance we traveled, but also tells us the direction we went. In our walking example, negative means “going South” and positive means “going north.” Directions will be very important in our study of motion, so we will prefer to use displacement rather than just distance often in this course.

In the case of the man jumping the Δx values do look like they are all equal. But in the ball diagram we can see that this is not the case.

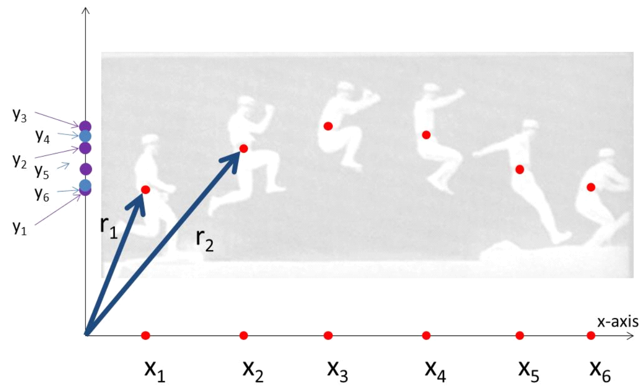


Note that in this figure we picked an *origin* as the starting point for measuring position. We label that $x = 0$ (in our case $x_1 = 0$, the first position of the ball). And we measure the other positions from this origin. We could pick the origin any where, but often it is nice to pick the origin at the starting position of the object in our experiment.



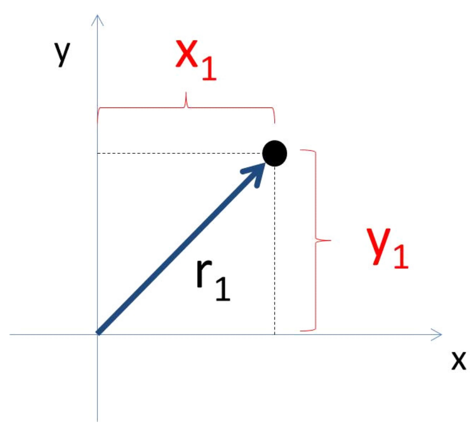
Two dimensional displacement

So far our displacement is only been in the x-direction, but our man's jump was not just horizontal motion, nor was it just vertical motion. It is really both at once. We should find a way to describe a motion that is part horizontal and part vertical. To do this let's step back and look at our jump again.

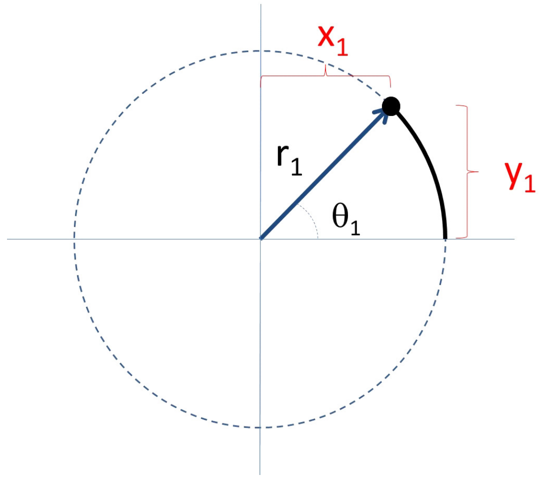


Notice now we have both x postions and y positions. We could write this with two quantities, one an x-part and one a y-part. Then the first red dot position could be written as (x_1, y_1) and the second as (x_2, y_2), etc.

Alternately, we could represent each position of our jumping man with new quantity. This new quantity must somehow have a vertical and a horizontal part to it.



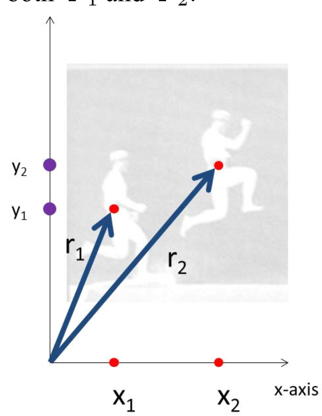
Let's take the point we labeled r_1 in the last figure and look at it carefully. We can see that r_1 must be made from x_1 and y_1 in some way.



If we think about it for a moment, this reminds us of polar coordinates. The distance r_1 at the angle θ_1 is related to the distances x_1 and y_1. So it really is quite natural to describe our position of point 1 by the line from the origin, r_1.

Notice in the figure we have drawn this new quantity like an arrow. And notice that this new quantity has not only a length, r_1, but also a direction, θ_1. We will call this new type of quantity a *vector*.

Specifically, the vector $\overrightarrow{\mathbf{r}}_1$ is a *position vector*. It gives the location of one of our points (x_1, y_1). Now let's consider both $\overrightarrow{\mathbf{r}}_1$ and $\overrightarrow{\mathbf{r}}_2$.



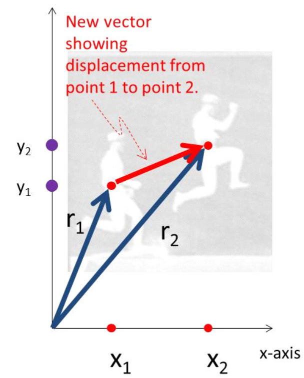
If these vectors represent our first and second positions, then it should be true that in some way

$$\Delta\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{r}}_1$$

must represent the displacement from point 1 to point 2. But what does it mean to subtract and add vectors?

2.1 Vector Addition (and Subtraction)

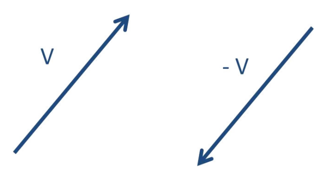
In our example of the jumping man, the displacement from point 1 to point 2 would be given by another arrow. After all, a displacement tells us how far we got and in what direction. The man jumping went a distance and in a particular direction. We use arrows to show how far and in what direction.



So this new arrow is also a vector, it has a length and a direction. So somehow we need to take $\overrightarrow{\mathbf{r}}_1$ and $\overrightarrow{\mathbf{r}}_2$, subtract them, and end up with the new vector.

Let's make a helpful definition: The negative of a vector is a vector of the same size going the opposite direction.

So if I have a vector, $\overrightarrow{V}$, as shown in the next figure



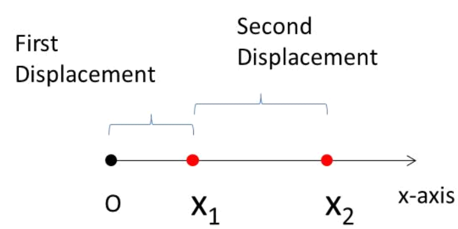
then $-\overrightarrow{V}$ will be the vector shown. Think, a negative sign gives us the opposite of a number (at least additively). The opposite of North is South, so with directions the negative of a direction should be "the other way."

The negative of $\overrightarrow{V}$ is just as long as $\overrightarrow{V}$, and goes the opposite direction. Then in our jumping man case we can see that

$$\begin{aligned}\Delta\overrightarrow{\mathbf{r}} &= \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{r}}_1 \\ &= \overrightarrow{\mathbf{r}}_2 + \left(-\overrightarrow{\mathbf{r}}_1\right)\end{aligned}$$

must mean to take $\overrightarrow{\mathbf{r}}_2$ and add to it a vector that has the length r_1 but goes the opposite direction of $\overrightarrow{\mathbf{r}}_1$. Let's draw $-\overrightarrow{\mathbf{r}}_1$ first

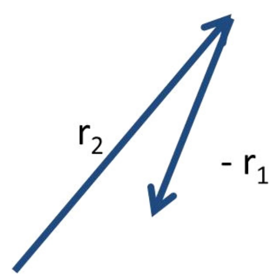
Now let's think about what it means to add displacements. In just the x-direction to add displacements we would go the first displacement, the go the second displacement starting at the end of the first displacement.



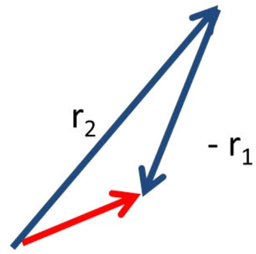
This works the same way for a set of displacements in the y-direction. So we might guess that this will work for our vector displacements. Our displacement is

$$\Delta\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_2 + \left(-\overrightarrow{\mathbf{r}}_1\right)$$

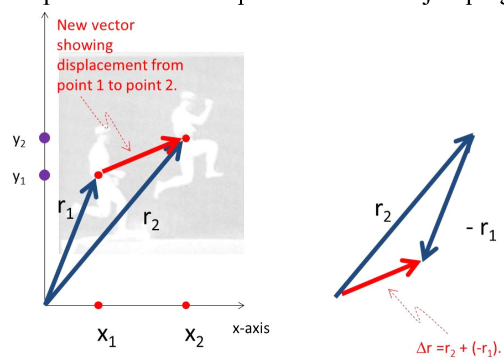
We will travel $\overrightarrow{\mathbf{r}}_2$ and then travel $-\overrightarrow{\mathbf{r}}_1$.



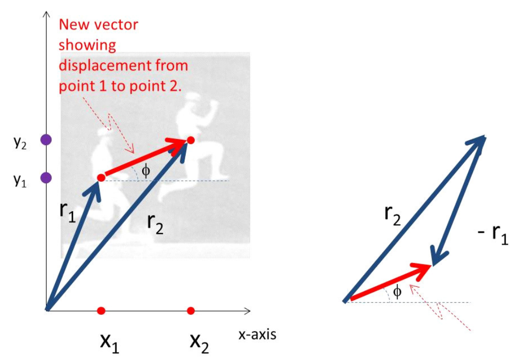
So after going the distance r_2 in the direction of r_2 we then turn into the direction of $-r_1$ and travel the distance r_1. The result is our red vector shown in the next figure



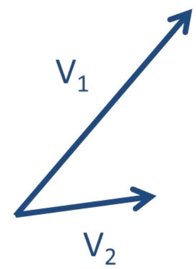
Notice that if we compare this to the displacement of our jumping man



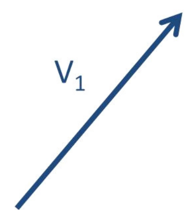
that the two red vectors are exactly the same length and that they point at exactly the same angle! This process seems to have worked!



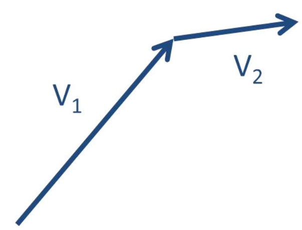
So we have a way to find the sum (and the difference) of two vectors. Let's summarize our steps. For a sum of two vectors, $\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2$



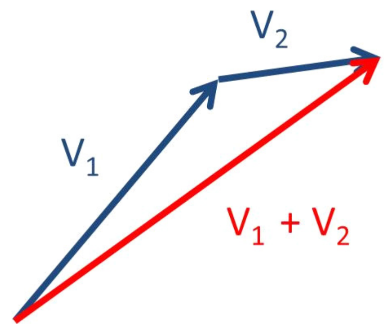
1. Draw $\vec{\mathbf{V}}_1$



2. Draw $\vec{\mathbf{V}}_2$ but start $\vec{\mathbf{V}}_2$ at the place $\vec{\mathbf{V}}_1$ stopped. We sometimes call this drawing the vectors "tip-to-tail." It's like two directions in a compass course for those of you who were Boy Scouts.



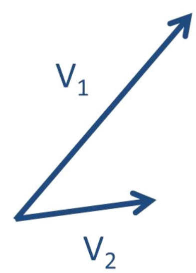
3. Finally draw a new vector from the tail of $\vec{\mathbf{V}}_1$ to the tip of $\vec{\mathbf{V}}_2$.



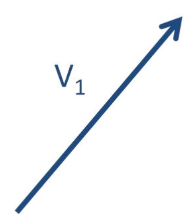
this new vector is $\overrightarrow{\mathbf{V}}_1 + \overrightarrow{\mathbf{V}}_2$.

Let's think about if this makes sense. If I tell you to walk in the $\overrightarrow{\mathbf{V}}_1$, a distance V_1 and then to stop and turn into the $\overrightarrow{\mathbf{V}}_2$ direction and walk a distance V_2. You would get to the same place as if you walked in the direction of the red arrow marked $\overrightarrow{\mathbf{V}}_1 + \overrightarrow{\mathbf{V}}_2$. So this does seem to be the sum of two vector displacements.

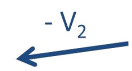
For subtracting two vectors, we just add one additional step. We have to reverse one of the vectors because we are adding a negative displacement For a difference of two vectors, $\overrightarrow{\mathbf{V}}_1 - \overrightarrow{\mathbf{V}}_2$



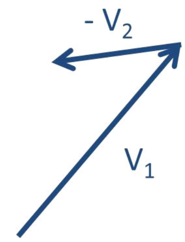
1. Draw $\overrightarrow{\mathbf{V}}_1$



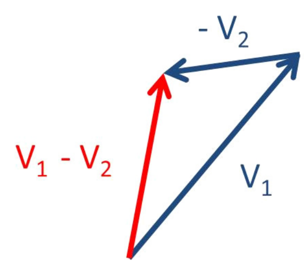
2. Draw $-\overrightarrow{\mathbf{V}}_2$ the inverse of $\overrightarrow{\mathbf{V}}_2$



3. Now move $-\overrightarrow{\mathbf{V}}_2$ so that it starts at the place $\overrightarrow{\mathbf{V}}_1$ stopped. We are adding $-\overrightarrow{\mathbf{V}}_2$ to $\overrightarrow{\mathbf{V}}_1$



4. Finally draw a new vector from the tail of $\overrightarrow{\mathbf{V}}_1$ to the tip of $-\overrightarrow{\mathbf{V}}_2$.



this new vector is $\overrightarrow{\mathbf{V}}_1 - \overrightarrow{\mathbf{V}}_2$.

We will use vectors for the rest of this class, for much of PH 123 and all of PH220. If you are a physics major or a mechanical engineering major, you will use vectors for the rest of your career. So it is worth getting used vectors and how to use them.

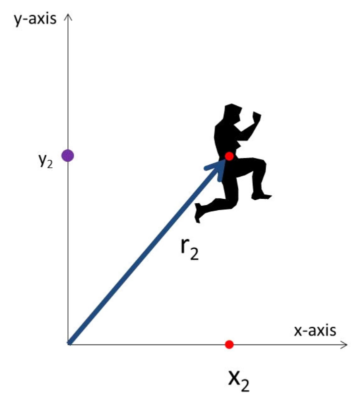
Now that we can describe how the time in an experiment changes, and we can describe how the position in an experiment changes, we can mathematically describe motion. In our next lecture, we will find that the velocity of an object is a combination of position (using displacement) and time

$$\overrightarrow{\mathbf{v}} = \frac{\Delta \overrightarrow{\mathbf{r}}}{\Delta t}$$

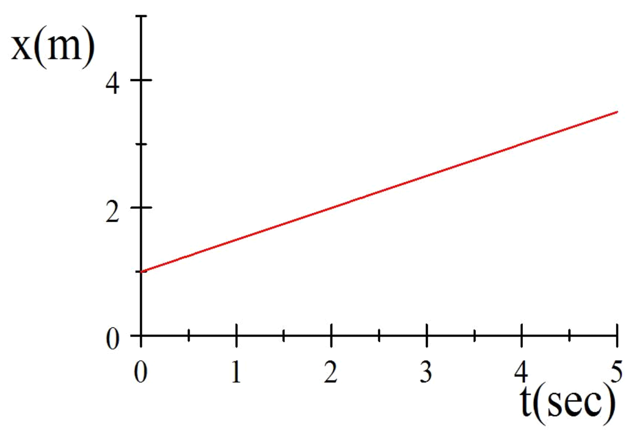
but this is not too much of a surprise. After all displacement is measured in miles sometimes, and time in hours, and we have been measuring velocities in miles per hour for years now. So we will be on familiar ground!

3.2 Position vs. time graphs

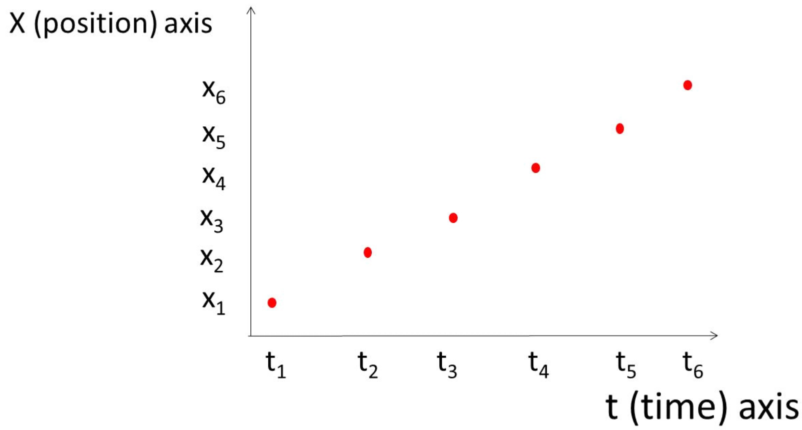
We have already used graphs in this class. But so far our graphs have given the position of an object, say, our jumping man.



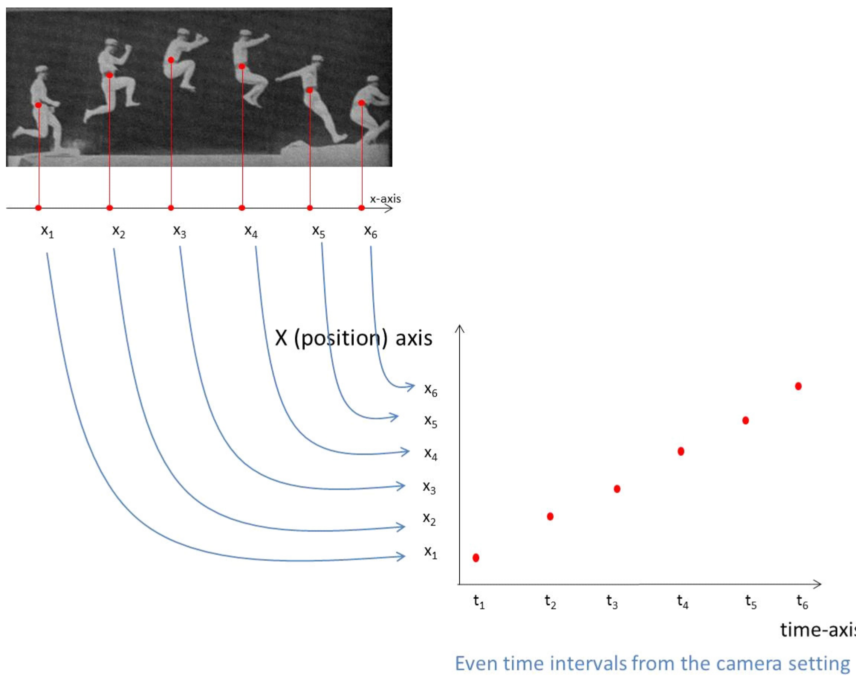
There is another kind of graph that is useful in describing motion. Since motion requires position and time, a graph of the position and time of the moving object helps us to know how the object is moving. We choose one of the axis of our graph to be time, usually the horizontal axis. Then the vertical axis will be position.



Here is an example for the jumping man.

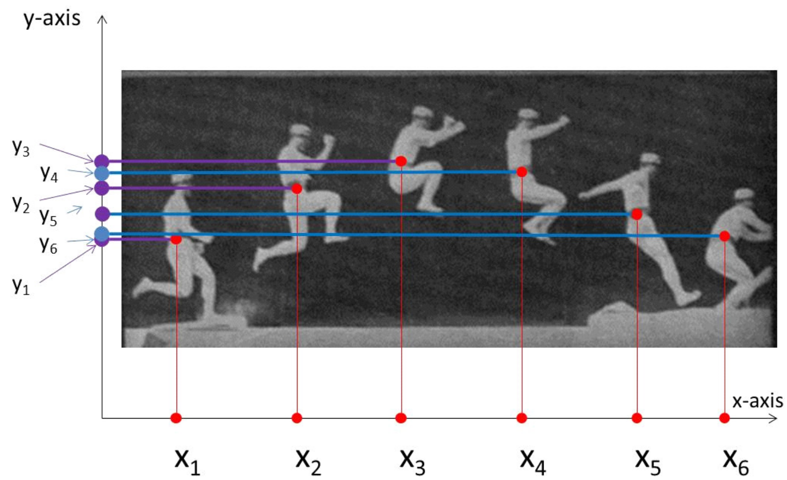


Let's take a moment and see how we formed this graph. The horizontal positions are placed on the vertical axis.

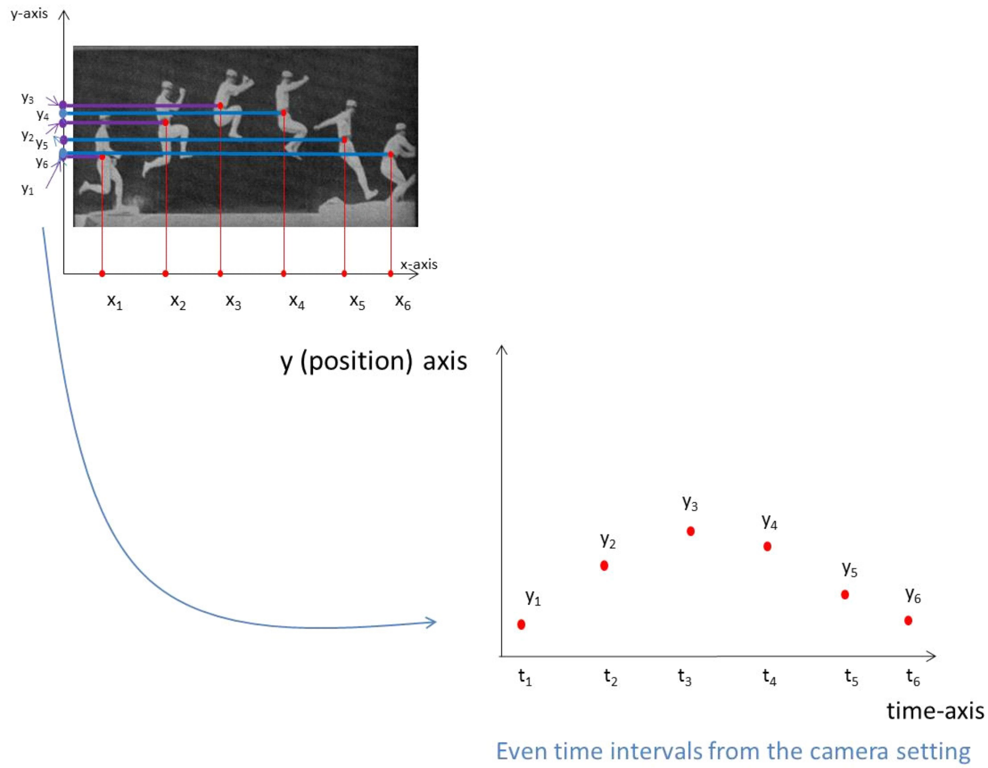


and we have assumed that the camera took pictures in even time increments. So each position is plotted a Δt away from the last along the t-axis. Since cameras usually operate in even time increments, but it is not required that the time steps be equal. But it is often convenient to use even time steps because it allows us to easily see changes in motion by noting the changes in position.

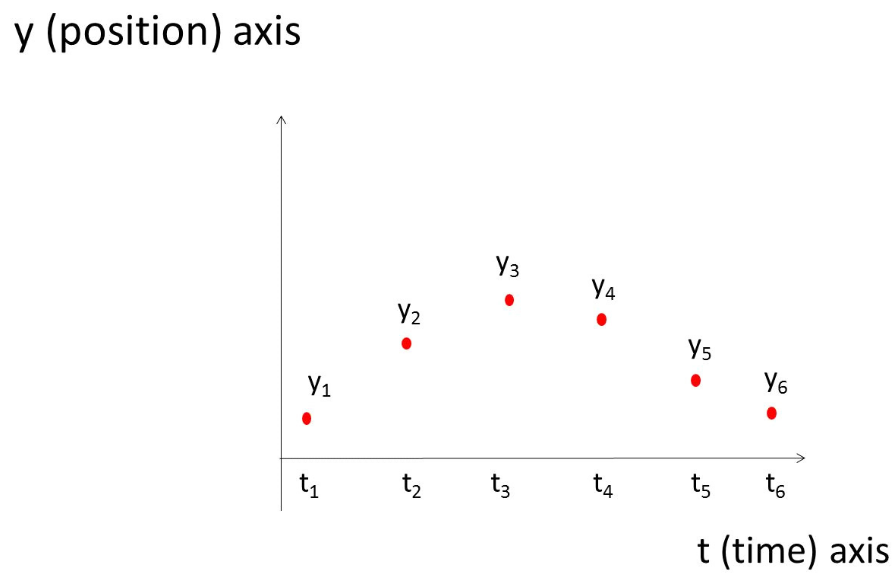
We will often use position vs. time graphs. Note that there are some difficulties with this type of graph. One is that we only plotted the x-position. But we know that we also have changes in the y-position for the jumping man. We could make another position vs. time graph for the jumping man for the horizontal (y) position. Immediately we realize that this will be more difficult because the y-positions don't fall in order on the y-axis. This is because the man goes up, and then comes back down.



But that is no problem. We can still make the graph, but this time we just have to be more careful about making the y-position axis.

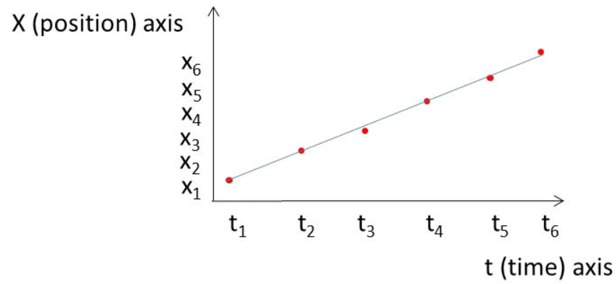


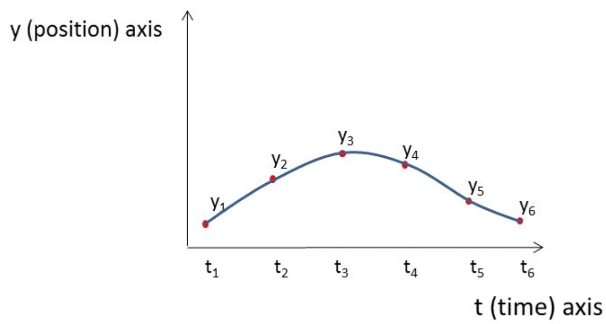
We are left with a graph that shows the vertical motion of the jumping man.



It is probably not to difficult that to see that we could combine these two graphs and plot r vs. time. But we will save that for another lecture.

It should also be clear that we have just plotted part of the jumping man's motion. We only have points for the positions where a photo was taken. But the man exists and is moving between the points. Sometimes we connect the dots with a line to show this missing motion



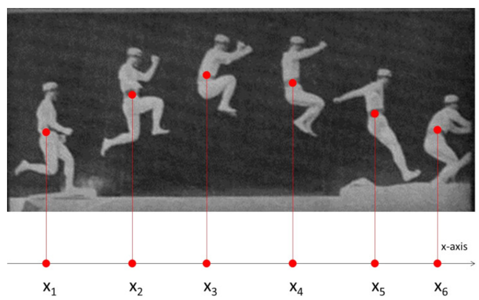


but it is important to realize that this is an estimate. We don’t have data (photographs) to show us for sure where the man was in between our known points.

Still, a position vs. time graph is a powerful visualization tool that helps us understand the motion of an object.

3 Velocity

Now that we have a way to describe the time and the displacement of an object, we are ready to describe how the object moves using mathematics. For example, how fast is our jumping man going?



If we know how far he has gone

$$\begin{aligned}\Delta x_{total} &= x_f - x_i \\ &= x_6 - x_1\end{aligned}$$

and how long it took for him to travel that displacement,

$$\begin{aligned}\Delta t_{total} &= t_f - t_i \\ &= t_6 - t_1\end{aligned}$$

can we find his speed?

The answer is of course *yes!* From everyday experience with motion, we know speed is how far we travel divided by how long it took to travel. So our speed should be something like

$$\text{speed} = \frac{\Delta x_{total}}{\Delta t_{total}}$$

We can see that the man travels about $1.5\,\mathrm{m}$ in each part of the jump. Then the total displacement is

$$\Delta x_{total} = 5\Delta x_{21} = 5 \times 1.5\,\mathrm{m} = 7.5\,\mathrm{m}$$

and suppose it took $1.07\,\mathrm{s}$ for the complete jump

$$\Delta t_{total} = 1.07\,\mathrm{s}$$

then the man's speed would be

$$v = \frac{\Delta x_{total}}{\Delta t_{total}} = \frac{7.5\,\mathrm{m}}{1.07\,\mathrm{s}} = 7.01\frac{\mathrm{m}}{\mathrm{s}}$$

Let's take another example. Say you go From Rexburg to Idaho Falls (IF). The displacement would be

$$\Delta x = 35.0\,\mathrm{mi}$$

and let's say it takes you a half hour to get to IF. Then the time would be

$$\Delta t = \frac{1}{2}\,\mathrm{h}$$

The speed we travel to IF is the displacement divided by the duration

$$v = \frac{\Delta x}{\Delta t} = \frac{35.0\,\mathrm{mi}}{\frac{1}{2}\,\mathrm{h}} = 70.0\frac{\mathrm{mi}}{\mathrm{h}}$$

This is a general formula. Speed is always how far we go divided by how long it took to get there.

Of course we don't want to use English units in this class, so we would say that

$$\Delta x = 56.33\,\mathrm{km}$$

and

$$\Delta t = \frac{1}{2}\,\mathrm{h} = 1800.0\,\mathrm{s}$$

so we would usually say

$$v = \frac{56.33\,\mathrm{km}}{1800.0\,\mathrm{s}} = 112.66\frac{\mathrm{km}}{\mathrm{h}} = 31.3\frac{\mathrm{m}}{\mathrm{s}}$$

In each example, notice that our equation has a displacement in it. And we found before that displacement can be negative. That means the speed could end up being negative. But negative speed does not seem to make much sense. What do we mean when we say speed is negative?

Average Velocity

Remember our example of walking through town. We found that if we started a block north of the center of town and we walk until we are three blocks north of the center of town. Then our displacement was

$$\begin{aligned} \Delta x &= \text{3blocks} - \text{1block} \\ &= \text{2blocks} \end{aligned}$$

If we started again one block north of the town center, but we walk until we are four blocks south of center. The our displacement would be

$$\begin{aligned}\Delta x &= -4\text{blocks} - 1\text{block}\\ &= -5\text{blocks}\end{aligned}$$

One displacement is positive and one is negative. What makes the difference? *Direction!* In our example, walking south gave negative displacement and going north gave positive displacement. So our speed quantity that we got for the jumping man is really more than speed. Using the displacement to find the speed gives us the speed we travel *and* the direction we travel. We need a new name for our speed term so we know it has both how fast we travel and which direction we travel. We will call this *velocity*.

Note that in physics speed and velocity are similar, but not the same. We will continue to call how fast we are going the speed. It won't be positive or negative. It is just a value that says how fast we go. But velocity will mean both the speed and the direction. So properly we should say that

$$v = \frac{\Delta x}{\Delta t}$$

is a velocity, not just a speed. But

$$|v| = \left|\frac{\Delta x}{\Delta t}\right|$$

is just a speed.

The absolute value signs are awkward, so lets make a new symbol that tells us we have not just a value for the speed, but also a direction. Let's put an arrow over any term that has direction in it.

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

then $\overrightarrow{\mathbf{v}}$ means velocity.

Without the arrow, v, means just the speed. Note that v won't ever be negative.

As an example, suppose it takes us 20 minutes to travel our 5 blocks. We say our walking velocity is

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t} = \frac{5\text{blocks}}{20\,\text{min}} = -0.24\frac{\text{blocks}}{\text{min}}$$

or about a quarter of a block a minute. Our speed would be

$$v = 0.24\frac{\text{blocks}}{\text{min}}$$

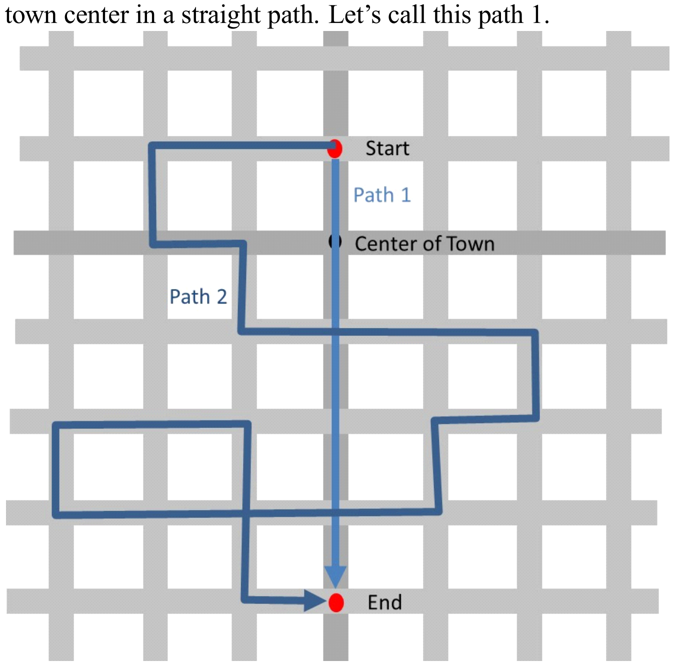
and our direction is given by the negative sign, where we have collectively agreed that South is the negative direction for our experiment.

Of course we know a name for a quantity that has both an amount and a direction. This is like our position vectors which had a length and a direction. Only now the length is a speed, and that is kind of strange. That length or speed part is the "amount" part of our quantity. Let's call the amount-part of a vector the *magnitude* of the vector and the direction part we will just keep calling "direction." So for a velocity vector the magnitude of the vector is the speed and the direction of the vector is which way the object is going. For position the magnitude of the position vector is how far away from the origin your position is, and the direction is the angle measured from the x-axis. Both position vectors and velocities are vectors because they have magnitudes and directions. But you can see that what they represent is quite different!

Before we go on, we should think about speed one more time. We have defined it as a displacement over a duration. But common use of the word defines speed as distance over duration.

$$v = \frac{d}{\Delta t}$$

Are these definitions different? Of course, they could be very different. For example, we have considered walking from one block north of the town center to four blocks south of the town center in a straight path. Let's call this path 1.



But what if we took path 2 as shown in the figure? And suppose we arrived in the same time for both paths. Clearly in some way we would have to be going faster if we take path 2 and arrive in the same time as we did in taking path 1. The distance traveled in path 2 is much larger. So to go farther in the same time we have to go faster along path 2. But our physics definition of speed marks our progress, not our total distance. So it won't show a difference between the speeds of the two paths. This is because our definition of speed (so far) gives an *average speed*. And on average, the speeds are the same for the two paths. Note that on path 2 sometimes we are going backwards. That cancels out some of our progress! So on average we are traveling the same speed. We will have to watch out for these two different definitions of speed as we go. But for now realize that

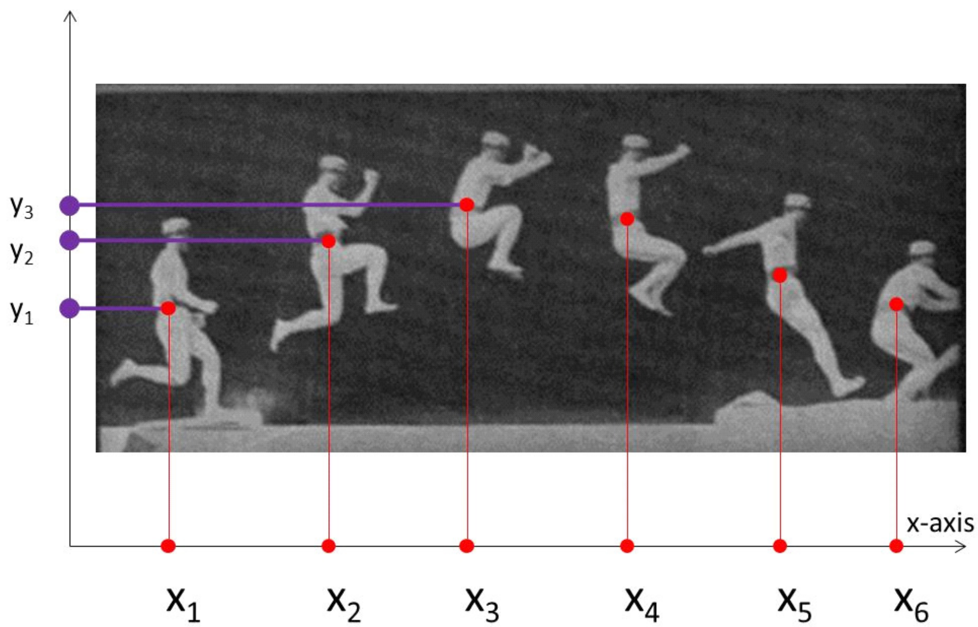
$$v = \frac{\Delta x}{\Delta t}$$

gives and average speed.

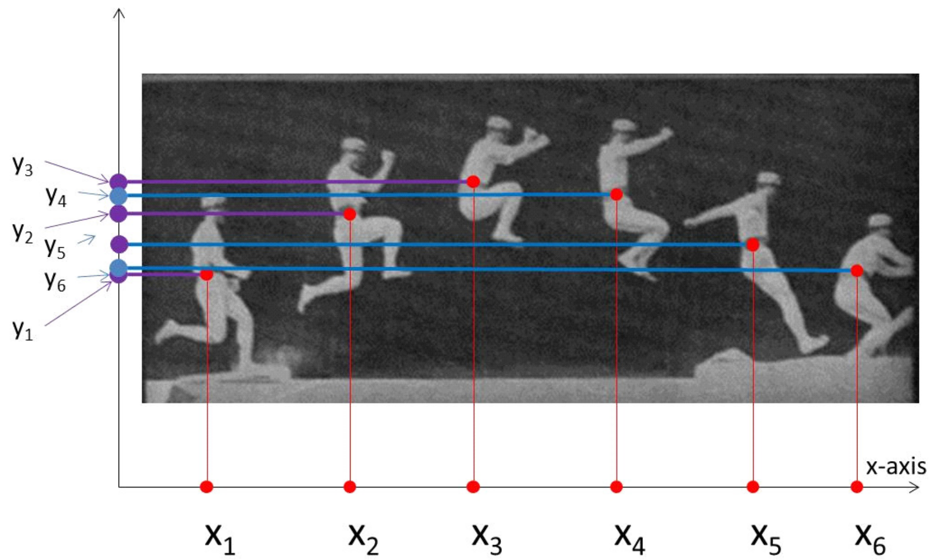
Vertical motion and equations

You might have objected to our analysis so far. It is true that our man travels horizontally, but he also travels vertically. How do we express displacement that is upward?

In the next figure we have marked the first three positions of the jumping man.



We can't use the variable x to describe how high up he went, because we have already used that variable for the horizontal displacement. So we picked another variable, y. And now we do just the same thing we did with horizontal displacement. We label the location of the center of the man at the first position as y_1 and the position of the center of the man at the second position as y_2 and so on until we have labeled each of the vertical positions of the center of the man.



Now we can define a vertical displacement

$$\Delta y = y_f - y_i$$

so that the total vertical displacement would be

$$\Delta y_{total} = y_6 - y_1$$

and we could find the displacement for each part of the man's jump

$$\begin{aligned} \Delta y_{21} &= y_2 - y_1 \\ \Delta y_{32} &= y_3 - y_2 \\ &\vdots \end{aligned}$$

We could even define a vertical velocity

$$\overrightarrow{\mathbf{v}}_y = \frac{\overrightarrow{\Delta \mathbf{y}}}{\Delta t}$$

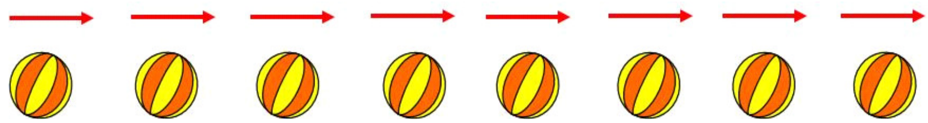
that would tell us how fast the man is going vertically and which direction he is going (up or down).

Of course, if we now label the vertical motion with the variable we chose for displacement in the vertical direction, we could also label the horizontal motion with the variable we chose for horizontal motion

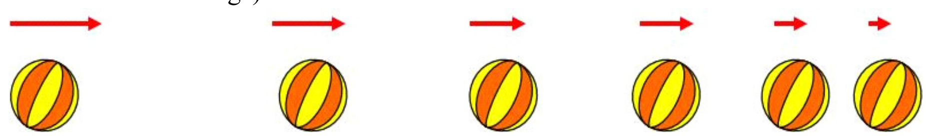
$$\overrightarrow{\mathbf{v}}_x = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t}$$

and this is just what we normally do.

It is convenient to add velocity vectors to our motion diagrams. Then we can know for sure which direction our object is going (we can even dispense with the numbers!). Here we have a ball traveling at a constant velocity (both speed and direction are constant!).



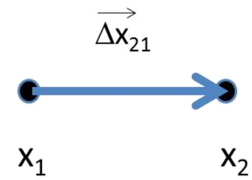
and here we have a ball changing speed so it is changing velocity (even though the direction does not change)



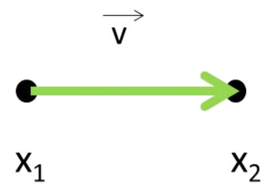
Often you will see motion diagrams where we have represented the object as just a dot (particle model) with the velocity vector attached to the dot.

It is important to realize that the vector is the average velocity of the object as it goes from x_1 to x_2, so it is might not be the velocity right at x_1 or at x_2. This is like driving to IF with an average velocity of $31.3\frac{\mathrm{m}}{\mathrm{s}}$. Your speedometer right when you start and right when you end your trip might not be exactly $31.3\frac{\mathrm{m}}{\mathrm{s}}$. In fact, you will find that your speed changes a little along the way due to traffic conditions. What we have calculated is only an average. So it might be better to think of the average velocity vector as being somewhere in between x_1 and $x_{2.}$ Soon we will calculate instantaneous velocity vectors, the kind of measurement made by your speedometer (plus a direction). Then the value for the velocity will be the speed and direction exactly at a point, say, x_1.

It is also good to realize that the artist that drew the last figure was being a little lazy. Here is a displacement vector Δx_{21}.



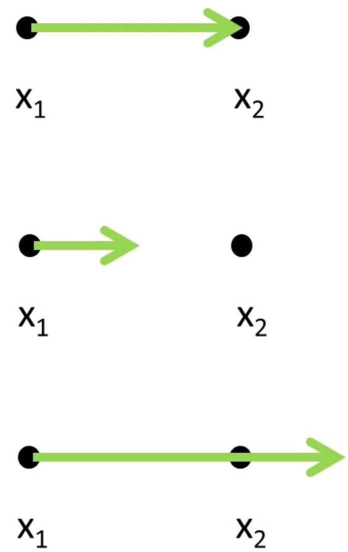
The magnitude of Δx_{21} should be exactly the distance from x_1 to $x_{2.}$ So the arrow is drawn correctly. It's length is just the distance from x_2 to x_2. But in the previous figure the artist drew the velocity vector as filling the space between x_1 and x_2.



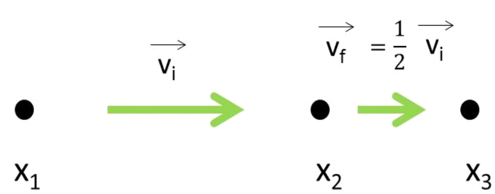
Should the velocity vector be the same length as the displacement vector? The answer is likely "no." Consider that

$$v = \frac{\Delta x}{\Delta t}$$

Unless we are very lucky, Δt would not likely be exactly $1\,\mathrm{s}$. So the numerical value for the speed won't be the same as the numerical value for the displacement. Unless we know the time, Δt, we don't even know how how long to make the vector $\overrightarrow{v}$! So we could draw the velocity vector in any of the following ways



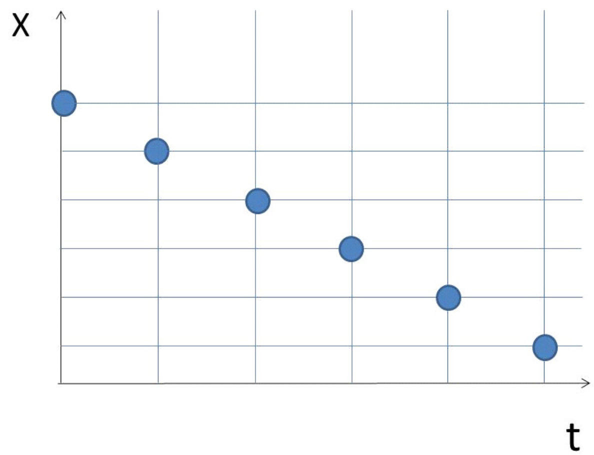
I think it is confusing to make the velocity vector fill the space between x_1 and x_2, so I will try to not draw velocity vectors that way. Velocity is a different thing that displacement. So it is hard to compare their "lengths." So I recommend not trying to do so. If you don't know the time, Δt, but need to draw a velocity vector on your motion diagram (and we often do), just pick a length for $\overrightarrow{v}$. Make any changes to the velocity proportional to the original length you chose, so you can see how the velocity changes. For example, once you have picked a length for a velocity, if the velocity is slower by half later in your diagram, choose a velocity vector for the new point that is half the size of the original.



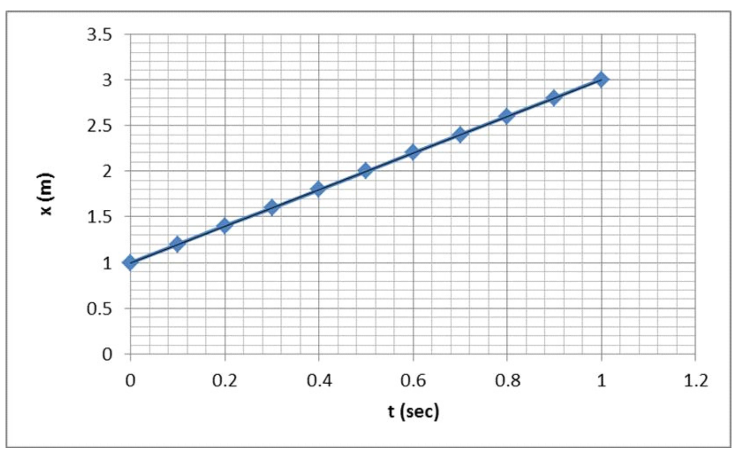
Uniform velocity

Uniform motion is a special case of motion that is useful, because we often have uniform motion. Think of driving your car. For long periods of time you may travel at

$70\,\mathrm{mi/\,h}$ along the freeway. That is uniform motion. Here is a position vs. time diagram for uniform motion and a motion diagram for the same uniform motion.



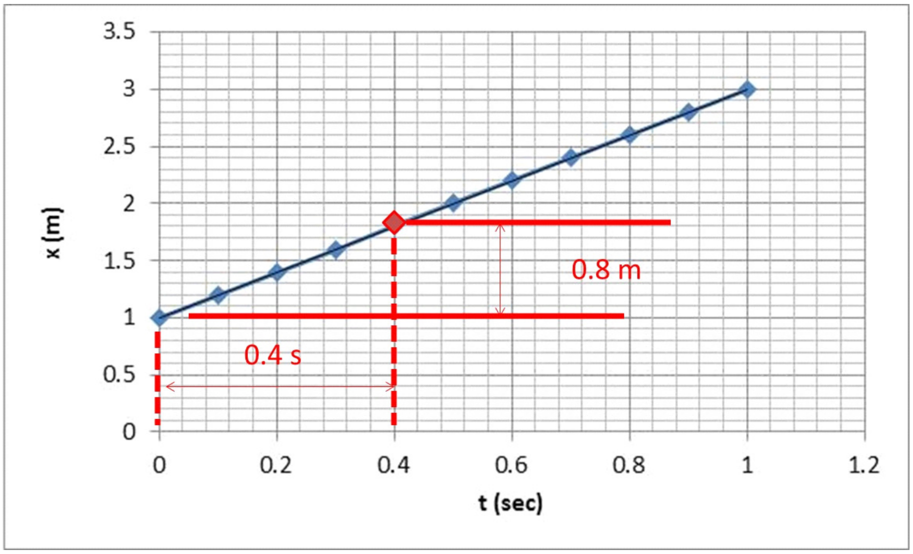
Let's take a specific position vs. time graph for a specific motion as an example. The following position vs. time graph describes a constant motion. What is the speed of the object?



The speed would be

$$v = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

where, because the motion is uniform, we can choose any two points for the final and initial points. Let's try x_0 and x_5 as our two points. Then we can see that



$$\begin{aligned} v &= \frac{x_5 - x_0}{t_5 - t_0} \\ &= \frac{0.8\,\mathrm{m}}{0.4\,\mathrm{s}} \\ &= 2\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Notice that this is just the "rise" over the "run" or the *speed is just the slope of the line of a position vs. time graph!*

Suppose that you know your speed, v, and your starting location, x_0 and you know how long you want to drive, could you find how far you will go? Of course! this just takes a little math. We just rearrange our equation to solve for x_f.

$$v = \frac{x_f - x_i}{t_f - t_i}$$
$$v\left(t_f - t_i\right) = x_f - x_i$$
$$x_i + v\left(t_f - t_i\right) = x_f$$

so

$$x_f = x_i + v\Delta t$$

Let's try such a problem. You wish to drive for half an hour, that is

$$\Delta t = \frac{1}{2}\,\mathrm{h} = 1800.0\,\mathrm{s}$$

and you wish to drive at

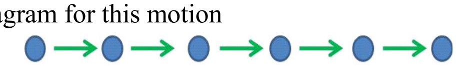
$$v = 70\,\mathrm{mi/\,h} = 31.\,293\frac{\mathrm{m}}{\mathrm{s}}$$

and let's define our starting location as

$$x_0 = 0$$

how far will we go?

We should draw a diagram for this motion



and realize that it is uniform motion, then we can use the equation for x_f that we just found

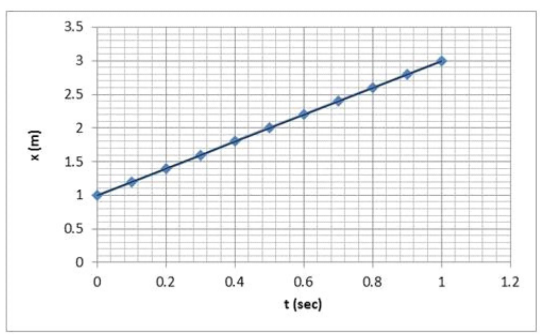
$$x_f = x_i + v\Delta t$$

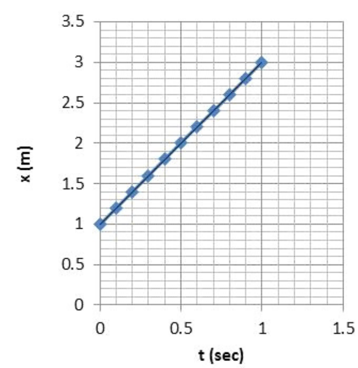
and fill in the pieces

$$\begin{aligned} x_f &= 0 + \left(31.\,293\frac{\mathrm{m}}{\mathrm{s}}\right)(1800.0\,\mathrm{s}) \\ &= 56327.\,\mathrm{m} \\ &= 56.327\,\mathrm{km} \\ &= 35.\,00\,\mathrm{mi} \end{aligned}$$

If you are taking this class in Rexburg, you might recognize this as a trip to Idaho Falls.

Let's look at two position vs. time graphs for uniform motion.





We know that the slope of the line in a position vs. time graph tells us the speed, but look at these two graphs carefully! Notice that they graph the same motion. One graph just has it's time graph compressed (kind of squished smaller) than the other. So although the second graph's slope looks steeper, this is only because of how we graphed the motion. If we had the same axes on both graphs, then they would look exactly the same. This means that we will have to be careful in how we interpret graphs of motion.

We need to actually calculate the rise over the run or the numeric slope instead of just looking at the picture.

Armed with understanding how to interpret graphs for uniform motion, we can take on a new quantity.

New Math

Likely your FDMAT 112 class is still teaching you how to take limits. And that is fine because we are going to use one today. Within the next three weeks, however, your calculus class will teach you...calculus! and calculus was invented by Newton to describe motion. So calculus is the mathematics of motion. We need to know how to do a little calculus today. I am going to let your math professor teach you the "why" of this new math. I am only going to teach you a little bit of "how" to do the new math.

Our new math is called a *derivative*. and for polynomial terms there is a procedure for taking a derivative.

If we have a function like

$$u = at^n$$

where a is a constant, then the derivative of this function is written as

$$\frac{du}{dt} = ant^{n-1}$$

where du/dy is the complicated symbol for the derivative of the function u, and the new function that results from taking the derivative of the function, u, is the constant a, times n, times t to the $n-1$ power.

Suppose we have a polynomial with two terms so a function w is given by

$$w = at^n + bt^m$$

the derivative of the sum of two functions is just the sum of the derivative of the two functions. We can treat at^n as a function (it's just our old function u) and we can treat bt^m as another function. Then we use our derivative rule twice and add the results

$$\frac{dw}{dt} = ant^{n-1} + bmt^{m-1}$$

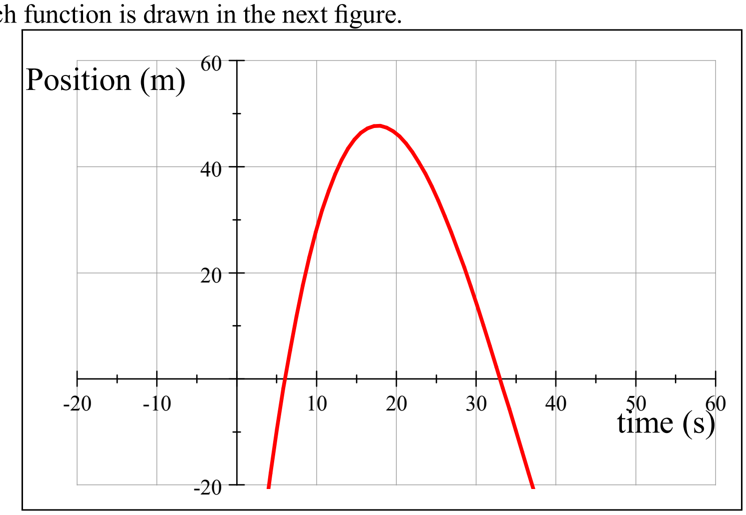
Notice that to use a derivative, we need a function. We will use this new math shortly.

Instantaneous Velocity

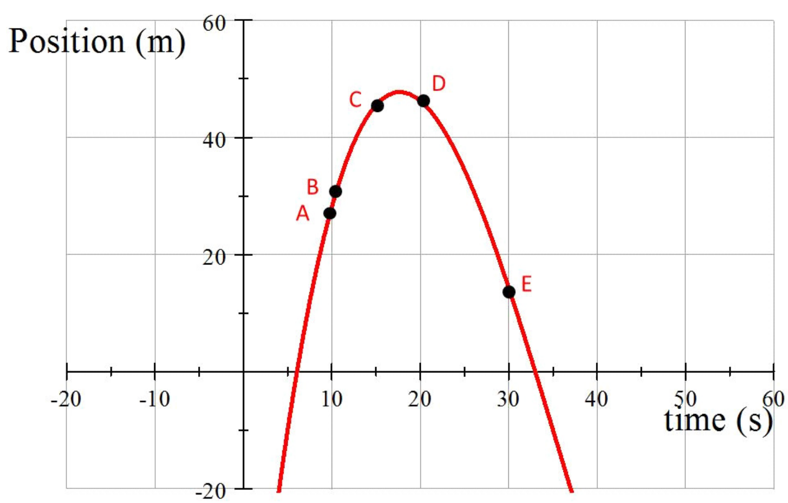
You may have thought as we have been talking that average velocity and average speed are nice, but generally when you think of speed, you are looking at a speedometer. The speedometer does not seem to be measuring average speed. It gives the speed you are traveling right now, an *instantaneous speed.* If we add in the direction we are going, we may also talk about an *instantaneous velocity*

For a constant velocity, the instantaneous velocity and the average velocity must be the same (no mater how small Δt, if v is constant, then it is constant). But what happens if we have a changing velocity so the velocity is a function of t?

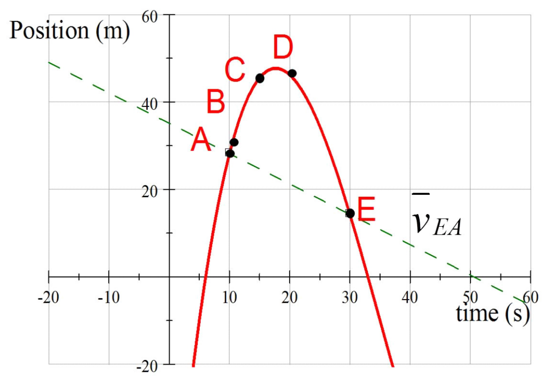
One such function is drawn in the next figure.



Let's start our experiment when the object is at position A as shown in the next figure.

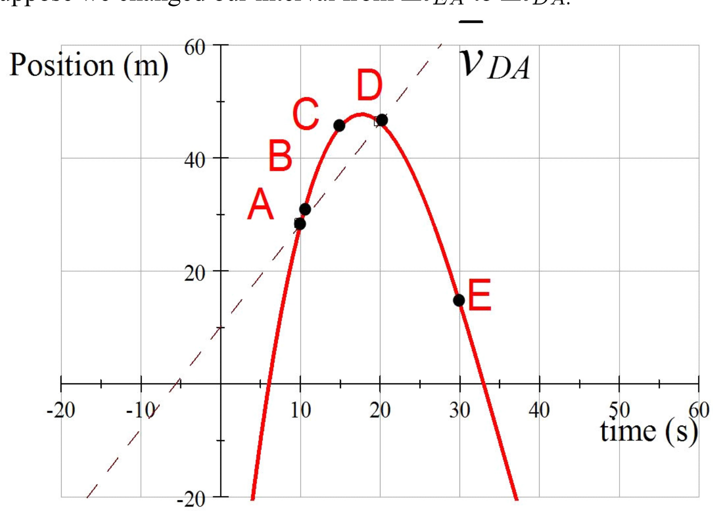


The position A we will call our initial position at our initial time, and position E as our final position and time, we have the average velocity given by the slope of the dashed line in the next figure that makes a chord-like cut across the function in the top part of the graph.



For this case the average speed v_{EA} measured over the time Δt_{EA} is really different that the instantaneous speed. In fact, between C and D the slope of our actual curve is zero. And the slope of the line through points A and E is really not zero. We can interpret that as meaning that the object stopped with zero speed between C and D! The time that the object stopped will affect the average speed. Think of going to Idaho Falls. You may travel at $70\,\mathrm{mi/\,h}$ on the freeway, but your average speed would be less if you stopped for lunch in Rigby.

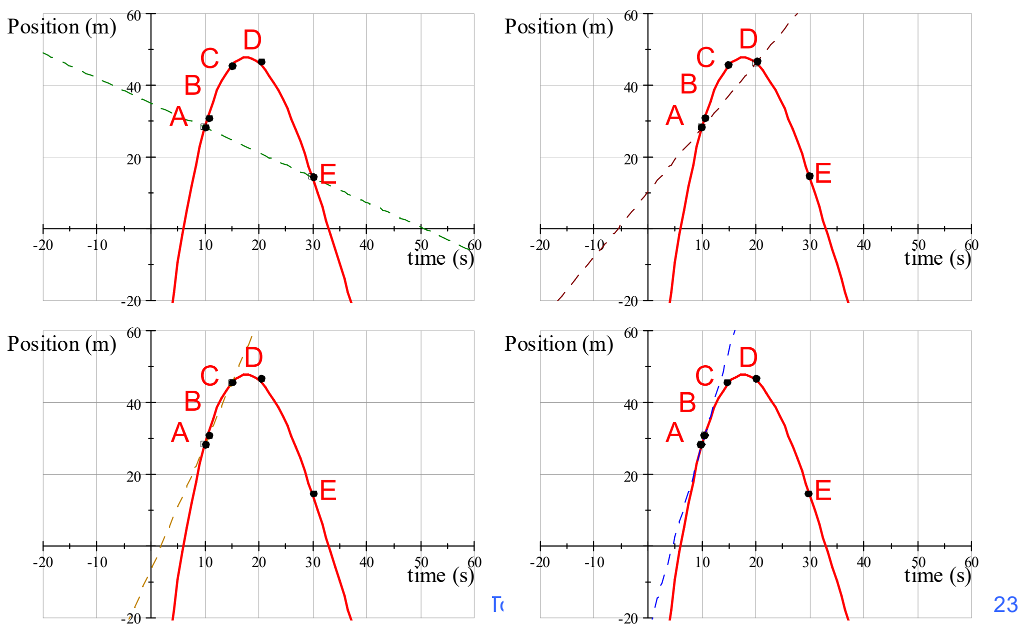
But suppose we changed our interval from Δt_{EA} to Δt_{DA}.



Notice that the slope of the dashed line is closer to the slope of the actual curve near point A. And we believe from our uniform motion analysis that the slope of a curve in a position vs. time graph is the speed!

So by making Δt smaller, we got closer to what we have reason to believe is the instantaneous speed.

We can keep getting closer to A, The results of trying this are in the bottom two parts of the next figure. We see that if we take smaller and smaller time intervals ($\Delta t \to 0$), we see the dashed line becomes a tangent!



And now you will recognize what you have been studying in FDMAT 112! We are really taking a limit. We are taking the limit of the average velocity as $\Delta t \to 0$.

Definition 3.1 *Instantaneous velocity v is the limit of the average velocity as the time interval Δt becomes very small.*

Algebraically we can write this as

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.1}$$

where the symbol

$$\lim_{\Delta t \to 0}$$

simply means that we let Δt become infinitesimally small. When

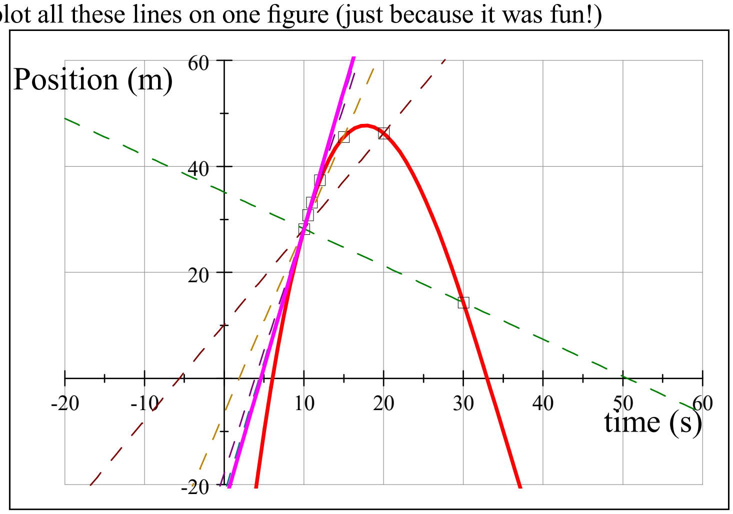
$$v_{ave} = \frac{\Delta x}{\Delta t}$$

has a Δt that is infinitesimally small we write it as

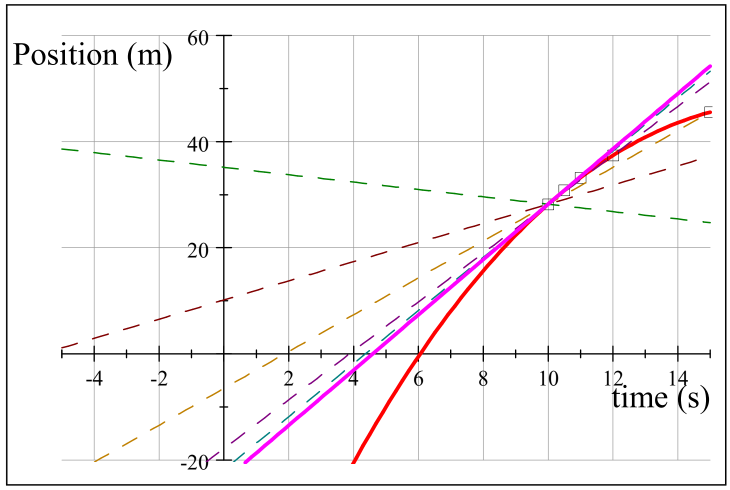
$$v(t) = \frac{dx}{dt}$$

to show that Δt is really small. The "d" tells us this delta is really really small. But notice that this is just the notation of our new derivative math! And this similarity in the notation is on purpose. This is great! The instantaneous velocity can be found using our new derivative math.

We can plot all these lines on one figure (just because it was fun!)



and even zoom in on the part right around our starting point. We see that the dashed lines do become more and more tangent.



This situation is general for any function (well, any function for which we can find the derivative). We can state that

> The instantaneous velocity is defined as the slope of the line tangent to the position vs. time curve at a given time.

Definition 3.2 *The instantaneous speed is the magnitude of the instantaneous velocity.*

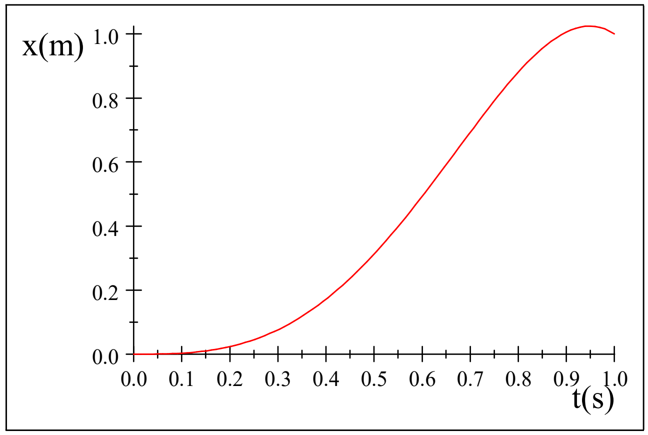
Remember that speed has no direction. This instantaneous speed is usually what we just call "speed."

Let's try an example:

Suppose the position vs. time has a functional form like

$$x\left(t\right)=-\left(2\frac{\mathrm{m}}{\mathrm{s}^{5}}\right)t^{5}+\left(3\frac{\mathrm{m}}{\mathrm{s}^{3}}\right)t^{3}$$

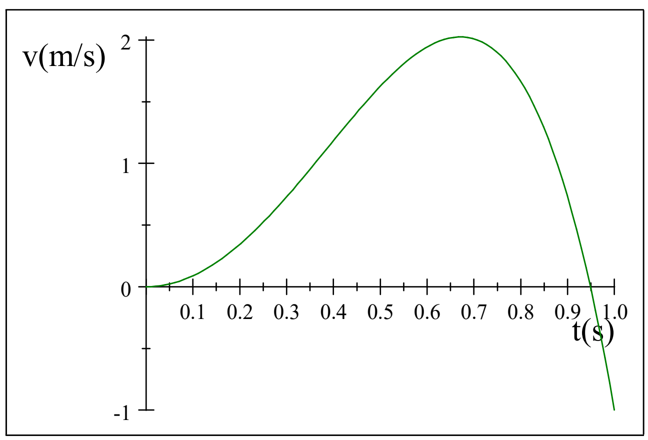
Recall that $x\left(t\right)$ means that x is a function of t, that is, x depends on t. It is not multiplication.



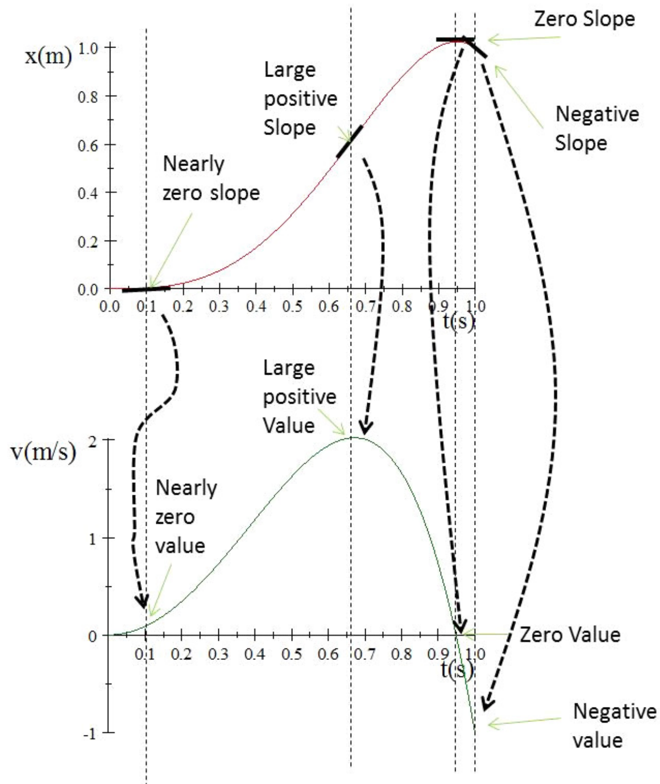
what would the velocity as a function of time be?

We can use our new math! the speed will be

$$\begin{aligned} v &= \frac{dx}{dt} = \frac{d}{dt}\left(-\left(2\frac{\mathrm{m}}{\mathrm{s}^5}\right)t^5 + \left(3\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3\right) \\ &= -\left(10\frac{\mathrm{m}}{\mathrm{s}^5}\right)t^4 + \left(9\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2 \end{aligned}$$



Let's look at these two graphs. The $v(t)$ graph should be the derivative, that is the slope, graph of $x(t)$. So if we find the slope at several places along the $x(t)$ graph, the value of that slope should correspond to the velocity on the $v(t)$ graph below.



So at about $0.1\,\mathrm{s}$ we see we have a small, nearly zero positive slope on the $x(t)$ graph. And in the $v(t)$ graph at $t = 0.1\,\mathrm{s}$ we have a small positive value for v. At about $0.65\,\mathrm{s}$ we have a large positive slope on the $x(t)$ graph, and at $0.65\,\mathrm{s}$ on the velocity vs. time graph we see a large positive value. At $0.95\,\mathrm{s}$, we have zero slope, and in the $v(t)$ graph we have a speed of zero at $0.95\,\mathrm{s}$. At $1\,\mathrm{s}$ we have a negative slope on the $x(t)$ graph, and at $1\,\mathrm{s}$ we have a negative value on the $v(t)$ graph. Since our one-dimensional graph of speed can have positive or negative values, we realize that this is really a velocity graph! So we are justified in writing our derivative equation as

$$\overrightarrow{\mathbf{v}} = \frac{d\overrightarrow{x}}{dt}$$

including direction.

Note that this new math is a powerful new tool to describe motion of a moving object. We will use the idea of a derivative of position as our velocity for the rest of this course, and if you are lucky and are majoring in physics, for many courses to come!

Speed, Average Speed, Instantaneous Speed

It's probably time to clarify our physics language a bit. Speed is how fast we are going. But we could define speed as how far we have gone (a distance) divided by how long it took (a duration).

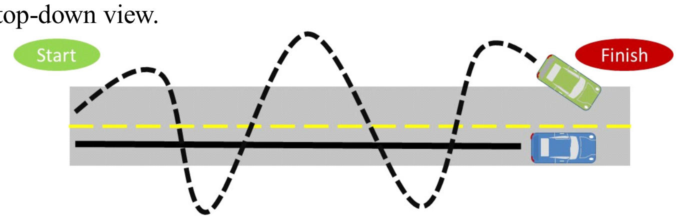
$$\text{speed} = \frac{\text{distance}}{\Delta t}$$

We could also define average speed and average velocity as

$$v = \frac{\Delta x}{\Delta t}$$

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta x}}{\overrightarrow{\Delta t}}$$

which tells us how far we have gotten toward some goal. Consider the car race shown below in a top-down view.



One car (the green one if you are seeing this in color) has clearly gone farther. Since the two cars are arriving at the finish at the same time, the green car must have gone faster since it traveled farther on it's zigzag path. To find out how fast the green car went we would indeed take

$$\text{speed} = \frac{\text{distance}}{\Delta t}$$

But in some since, the two cars started at the same place and ended at the same place, and they did this in the same time. Their average speeds must be the same!

$$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

We can see that both ideas, speed and average speed, are valuable. For this class, if a problem just asks for speed, we usually want distance traveled over time (duration). If the problem asks for average speed, we will report the magnitude of the displacement over duration.

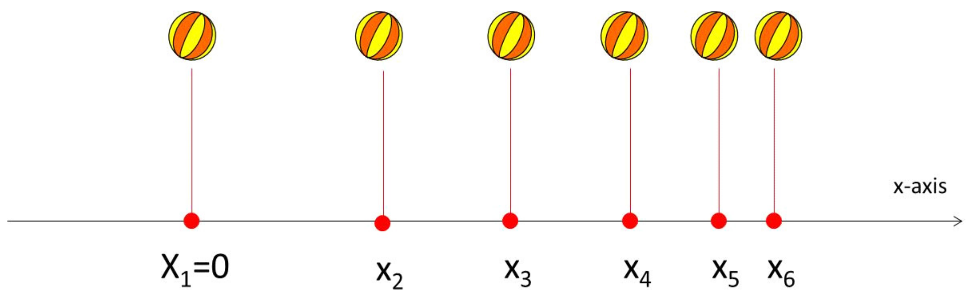
Of course each of our cars has a speedometer reading the instantaneous speed of the cars. This instantaneous speed might be very different than the average speed or the speed of the car for the entire trip. For example, the blue car might have slowed down to avoid hitting a deer (that happens in Idaho!). Then he would have speed up again. During some part of the blue car's path, the blue car instantaneous speed would have been zero! But just for the time the car was stopped for the deer.

We will have to keep these three ideas in our mind. Speed, average speed, and instantaneous speed (and of course, average velocity and instantaneous velocity as well).

4 Acceleration

In today's lecture we are going allow our velocity to change. We have a name for changing velocity. That name is *accelration*. From what we have done so far you might expect us to define an average acceleration and and instantaneous acceleration. And you would be right. Let's start with average acceleration.

Average Accelration



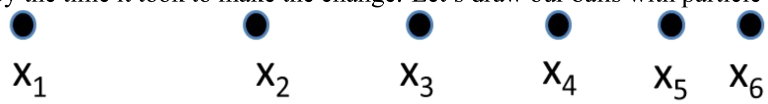
Let's look again at our moving ball case. Notice that at the first part of the motion the ball moves farther in Δt_{21} than it does in Δt_{65}. Since we draw motion diagrams with equal time increments, and the displacements are different, the ball's velocity must be changing. We need a way to express the change in velocity mathematically if we are going to be able to understand this motion. And our example of velocity can show us a way to express this. In finding velocity we found a change in displacement in an amount of time

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t} = \frac{x_f - x_i}{\Delta t}$$

Now we want a change in velocity in an amount of time

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta\mathbf{v}}}{\Delta t} = \frac{v_f - v_i}{\Delta t}$$

To show the change in the velocity we compare the velocity two points $v_f - v_i$ and divide by the time it took to make the change. Let's draw our balls with particle model.

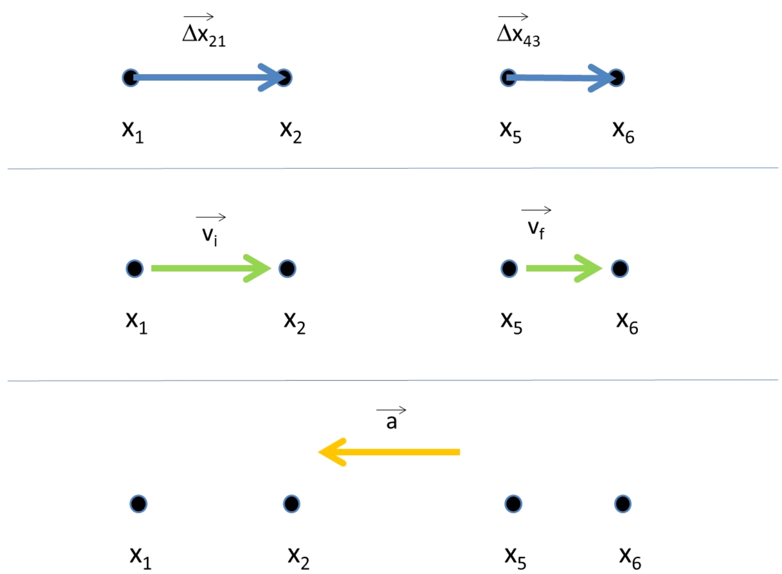


and pick just the end points.

Our process will be to find an average velocity v_{21} and average velocity v_{65}, and then use these to calculate the change in velocity

$$\Delta v = v_{65} - v_{21}$$

and then we divide by the amount of time that it took for the ball to go from about position x_1 to about position x_6. Here is a drawing of what we will do.



Our ball may have moved $3\,\mathrm{m}$ between points x_1 and x_2. Suppose $\Delta t_1 = 1\,\mathrm{s}$. Then between points x_1 and x_2 the average velocity would be

$$\overrightarrow{\mathbf{v}}_{12} = \frac{3\,\mathrm{m} - 0\,\mathrm{m}}{1\,\mathrm{s}} = 3\frac{\mathrm{m}}{\mathrm{s}}$$

but suppose by the time we get to points x_5 and x_6 the ball has moved from $x_5 = 9.75\,\mathrm{m}$

to $x_6 = 11\,\mathrm{m}$ so

$$\overrightarrow{\mathbf{v}}_{56} = \frac{11\,\mathrm{m} - 9.75\,\mathrm{m}}{1\,\mathrm{s}} = 1.25\frac{\mathrm{m}}{\mathrm{s}}$$

Note that the velocity $\overrightarrow{\mathbf{v}}_{12}$ happens at some time in between t_2 and t_1. We are not exactly sure when. Let's say it is close to half way in between the two times, so $t_{12} \approx 1.5\,\mathrm{s}$. Likewise $t_{56} \approx 5.5\,\mathrm{s}$

Then the change in the velocity is

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta\mathbf{v}}}{\Delta t} = \frac{1.25\frac{\mathrm{m}}{\mathrm{s}} - 3\frac{\mathrm{m}}{\mathrm{s}}}{5.5\,\mathrm{s} - 1.5\,\mathrm{s}} = -0.4375\frac{\mathrm{m}}{\mathrm{s}^2}$$

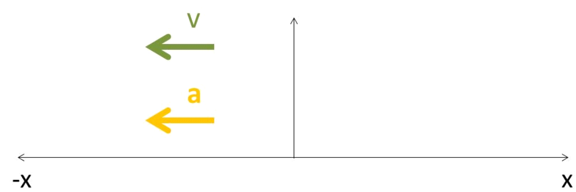
Notice that the units of our answer are $\mathrm{m/s^2}$ or $(\mathrm{m/s})/\mathrm{s}$. This is vaguely familiar. Think of how your car accelerates. We could go from 0 to $100\,\mathrm{km/h}$ in $3.6\,\mathrm{s}$. It is a change in our velocity in a given amount of time.

Also notice that the answer is negative! But what does this mean?

Like everything else, if the acceleration points to the left we call it negative and if it points to the right we call it positive. But really what is more important is that if the velocity and acceleration are in the same direction the mover is *speeding up*! likewise if the velocity and acceleration are in opposite directions the mover is *slowing down*. It matters whether the acceleration is in the same direction as the velocity or not.

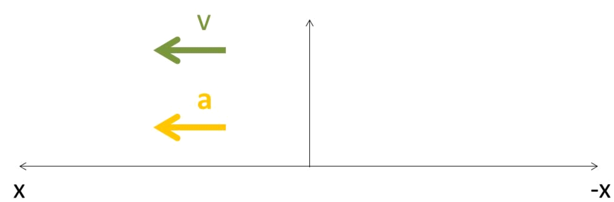
We can represent all this graphically by drawing the arrows. In the figure, we see a green arrow representing velocity and a yellow arrow representing acceleration. If they point the same way, the object is speeding up. If they point different directions, the object is slowing down.

There is a curious thing about acceleration in physics. We don't use the word "decelerate." We only use the word "accelerate." If we are slowing down, we are accelerating in the opposite direction we are going. It is important to notice that a negative acceleration is not necessarily slowing down. To see this, think about the following situation.



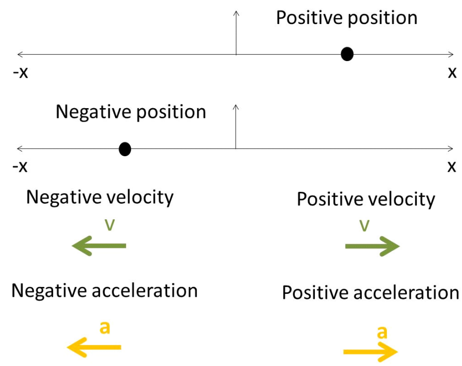
We have a velocity vector and an acceleration vector. We also have a coordinate system. The velocity would be negative simply because points in the negative direction (toward more negative numbers on the x-axis). The acceleration is also negative because it points in the negative direction. But if we had an object going in the negative x-direction an the acceleration was negative, the object would speed up. The acceleration and velocity are pointing the same way, so this is speeding up.

The coordinate system makes our velocity or acceleration negative or positive. We could define our coordinate system backwards so positive values were toward the left.

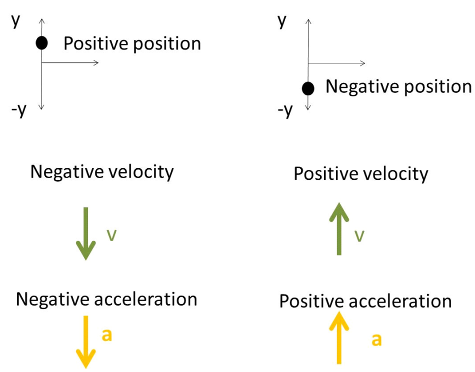


Then both velocity and acceleration would be positive, but still the object would be speeding up.

Since we are talking about coordinate systems, let's set up some standards for thinking about motion. We could define a coordinate system so that positive values are to the left, but usually we won't do that. Usually we will say that to the left in our coordinate system is negative and to the right is positive.



For the y-direction, we choose the upward direction to be positive and the downward direction to be negative.



Sometimes we will choose to break these standards, but for the most part these choices represent how we will define positive and negative directions in our study of motion.

Instantaneous Acceleration revisited

We started with displacement and duration and defined average velocity

$$\overrightarrow{v} = \frac{\overrightarrow{\Delta x}}{\Delta t}$$

and then we use the idea of a limit to find the instantaneous velocity

$$\lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta x}}{\Delta t} = \frac{\overrightarrow{dx}}{dt}$$

and we defined

$$\begin{aligned} \overrightarrow{v}_{\text{instantaneous}} &= \frac{\overrightarrow{dx}}{dt} \\ \overrightarrow{v}(t) &= \frac{\overrightarrow{dx}}{dt} \end{aligned}$$

and identified it as the velocity our object has at just one instant of time. Could we do the same for acceleration?

We defined average acceleration as

$$\overrightarrow{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

could we take

$$\lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{\overrightarrow{dv}}{dt} = a_{\text{instantaneous}}$$

so that

$$\overrightarrow{a}(t) = \frac{\overrightarrow{dv}}{dt}$$

What would this mean? If acceleration changes, then we must have a different acceeration at different points in time. The instantaneous acceleration would be the acceleration of our object at just one point in time.

We should paus to notice something cool. The velocity is the derivitive of the position, and the acceleration is the derivitive of the velocity. Our new math relates position, velocity, and acceleration in a cool easy way!

$$\begin{aligned} \overrightarrow{v}(t) &= \frac{\overrightarrow{dx}}{dt} \\ \overrightarrow{a}(t) &= \frac{\overrightarrow{dv}}{dt} \end{aligned}$$

we can use our new math to relate $x(t)$, $v(t)$, and $a(t)$.

Constant Acceleration

Physicists seem to like old words. The Greek work $\kappa\iota\nu\varepsilon\omega$ (kineo) means to move. And the ancient Greeks tough a lot about motion of thinks like rocks and arrows and spears. These things were all thrown at their enemies near the Earh's surface. And it turns out that near the Earth's surface we experience a nearly constant acceleration because of the Earth's gravity. So the study of motion with constant acceleration is often called *kinematics.* Since we live near the Earth's surface, we will start our mathematical study of motion with kinematics.

Let's think about driving a car. Hopefully the car is stopped when you first get in. We must get the car going. When we come to our destination, we must slow down. We need to change our velocity. We called a change in velocity acceleration. Recall that the average acceleration $\bar{a}$ during time interval Δt is the change in velocity Δv divided by Δt

Algebraically we may write

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta \overrightarrow{\mathbf{v}}}{\Delta t} = \frac{\overrightarrow{\mathbf{v}}_f - \overrightarrow{\mathbf{v}}_i}{t_f - t_i}$$

Since we are dealing with velocity, we would expect acceleration to be a vector quantity and we know that is true. In one dimension, we can indicate direction with a plus or minus sign. So we could write the equation above without the vector signs for one dimensional motion,

$$\mathbf{a}_{ave} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i}$$

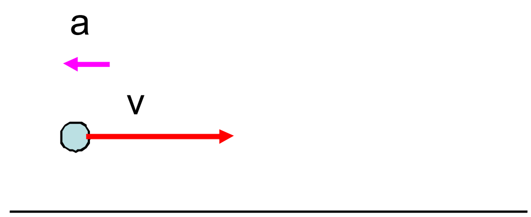
but we should remember that acceleration is a vector. Just for review, we should recall what the direction of acceleration means.

We have to consider the direction of *both* the acceleration and the velocity before we can determine the effect of acceleration on the motion of the object.

Let's start with a positive velocity and a positive acceleration.

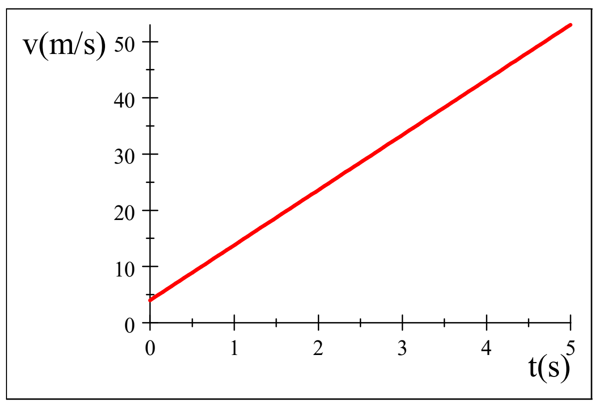
In this case, we have a positive initial velocity, and we would expect the velocity to get larger, so $v_f - v_i$ is positive.

Suppose we have a positive velocity and a negative acceleration.



This means that $v_f - v_i$ is negative ($t_f - t_i$ is never negative). For this to be true v_f must be smaller than v_i. The object is slowing down!

If we use average velocity as an example, we can guess that if the acceleration is constant, then the acceleration is the slope of the line in a *velocity* vs. time graph.



There are many physical phenomena that can be represented as a system with constant acceleration. Neglecting air resistance, all bodies attracted by gravitation act under uniform acceleration (we will find out why later!).

We can build our list of basic problem types by adding the special case of constant acceleration. Let's do that now.

What we will do is to do four problems, then we will save the results at the end and use these results over and over again when we have a problem to solve that involves constant acceleration. We will add four new equations to our list of basic equations! I suggest you label these equations "constant acceleration" or something like that, since we are going to assume constant acceleration when we form the equations, they will *only* work for constant acceleration cases. We are also adding a new problem type:

constant acceleration!

Lets start with our definition of average acceleration

$$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \tag{4.1}$$

since we are limiting ourselves to one dimension, we will use a $+$ or a $-$ sign to indicate direction. Also, since acceleration is constant, we have

$$a_{ave} = a \tag{4.2}$$

so

$$a = \frac{v_f - v_i}{t_f - t_i} \tag{4.3}$$

Remember, this is for our special case! What follows is **not true** in general! The equations we get only work if the acceleration does not change. We defined before $\Delta t = t_f - t_i$ so let's use it here

$$a = \frac{v_f - v_i}{\Delta t} \tag{4.4}$$

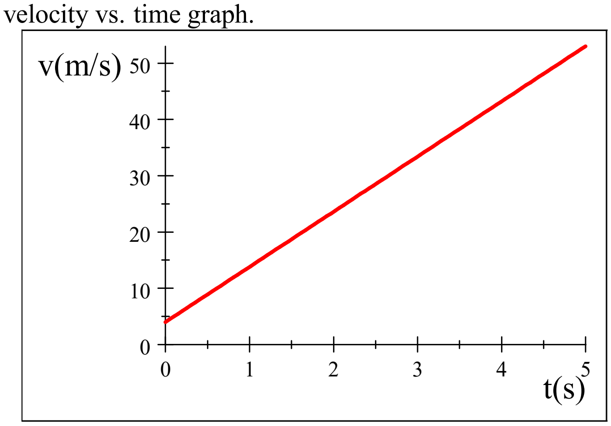
Rearranging gives

$$v_f = v_i + a\Delta t$$

This is the first of our new set of basic equations. And notice that is is the equaion in the form of a straight line just like we expect from our velocity vs. time graph for constant acceleration.

$$\begin{aligned} v_f &= a\Delta t + v_i \\ y &= mx + b \end{aligned}$$

For constant velocity problems, the velocity is a straight line with the acceleration as the slope on a velocity vs. time graph.



We have come up with the first in our new set of equation tools for our new constant acceleration problem type.

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{4.5}$$

If we have constant acceleration we can write the average velocity as

$$v_{ave} = \frac{v_f + v_i}{2} \qquad \text{Constant } a \tag{4.6}$$

We could use this specific equation for v_{ave} to write

$$v_{ave} = \frac{\Delta x}{\Delta t}$$

as

$$\frac{v_f + v_i}{2} = \frac{\Delta x}{\Delta t}$$

or even

$$\frac{\Delta t}{2}\left(v_f + v_i\right) = x_f - x_i$$

or even as

$$x_f = x_i + \frac{\Delta t}{2}\left(v_f + v_i\right)$$

I like this final form. We can interpret this as x_f consisting of the initial position plus how far our motion took us from x_i. This is the second of our new set of equations.

Now, how far will an object go that is experiencing constant acceleration in time Δt? This is harder to find, but could be useful (think if you are an ancient Greek trying to get your rock to land on your enemies). Let's write our displacement as

$$\Delta x = x_f - x_i \tag{4.7}$$

Let's take the equation for average velocity

$$v_{ave} = \frac{x_f - x_i}{t_f - t_i} \tag{4.8}$$

and write it as

$$v_{ave} = \frac{x_f - x_i}{\Delta t} \tag{4.9}$$

then

$$x_f - x_i = v_{ave}\Delta t \tag{4.10}$$

Using our equation for average velocity under constant a,

$$x_f - x_i = \left(\frac{v_f + v_i}{2}\right)\Delta t \tag{4.11}$$

or, by rearranging,

$$x = x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}v_f\Delta t \qquad \text{Constant } a \tag{4.12}$$

We can add this equation to our list of basic equations for constant acceleration problems. We could write this more compactly as

$$\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t \tag{4.13}$$

But we usually see this written another way. Let's take equation **??**

$$v_f = v_i + a\Delta t$$

and substitute it into equation 4.12.

$$\begin{aligned} x &= x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}\left(v_i + a\Delta t\right)\Delta t && (4.14) \\ &= x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}v_i\Delta t + \frac{1}{2}a\Delta t^2 && (4.15) \end{aligned}$$

$$x_f = x_i + v_i\Delta t + \frac{1}{2}at^2 \qquad \text{Constant } a \tag{4.16}$$

This one we definitely want in our new list of basic equations for constant acceleration problems. It tells us how far we get from our starting point as we move with constant acceleration. We shoudl interpret this equation as well. Think, the first term on the right is where the object started. The second term is how far it would have gotten if the motion was constant, and the third term is how much farther the object got because it is accelerating.

There is one more, let's go back to equation **??**

$$v_f = v_i + a\Delta t \tag{4.17}$$

and solve for Δt

$$\begin{aligned} v_f - v_i &= a\Delta t && (4.18) \\ \frac{v_f - v_i}{a} &= \Delta t && (4.19) \end{aligned}$$

now let's use this equation for Δt in equation 4.11

$$\begin{aligned} x_f - x_i &= \left(\frac{v_i + v_f}{2}\right)\Delta t && (4.20) \\ &= \left(\frac{v_i + v_f}{2}\right)\left(\frac{v_f - v_i}{a}\right) && (4.21) \\ &= \frac{1}{2}\left(v_f + v_i\right)\frac{1}{a}\left(v_f - v_i\right) && (4.22) \\ &= \frac{1}{2a}\left(v_f^2 - v_i^2\right) && (4.23) \end{aligned}$$

so

$$\Delta x = \frac{1}{2a}\left(v_f^2 - v_i^2\right) \tag{4.24}$$

or

$$2a\Delta x = -\left(v_f^2 - v_i^2\right) \tag{4.25}$$

Finally we can write this as

$$v_f^2 = v_i^2 + 2a\Delta x \qquad \text{Constant } a \tag{4.26}$$

This is the final equation of our new set. It is a good equation for problems where you are not given the time.

The Kinematic equations for constant acceleration

We have derived four main equations

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{4.27}$$

$$x_f = x_i + v_i t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a \tag{4.28}$$

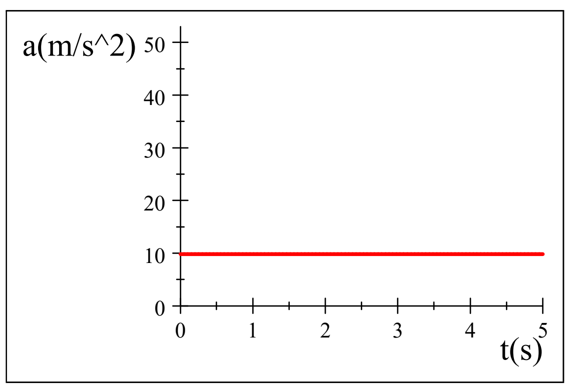
$$v_f^2 = v_i^2 + 2a(x_f - x_i) \qquad \text{Constant } a \tag{4.29}$$

$$(x_f - x_i) = \frac{1}{2}(v_f + v_i)\,\Delta t \qquad \text{Constant } a \tag{4.30}$$

The last one was used in deriving the second and third equation, so the four equations are not independent. The first three are the most useful. The first two are the most important for this chapter. The third is often convenient. The strategy for solving kinematic problems in this chapter should include (after restating the problem, drawing a diagram, and stating variables) selecting an equation from these three or four equations. This set of equations is so useful that it has a name. They are called the *kinematic equations*.

Graphs of kinematic equations

The graph of constant acceleration on and acceleration vs. time graph is not very exciting.



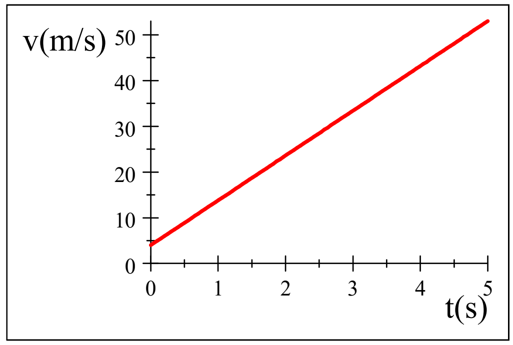
and knowing that

$$a = \frac{\Delta v}{\Delta t}$$

we can see that the acceleration will be the slope of the velocity vs. time graph. We can also see this from equation (4.27)

$$v_f = v_i + a\Delta t$$

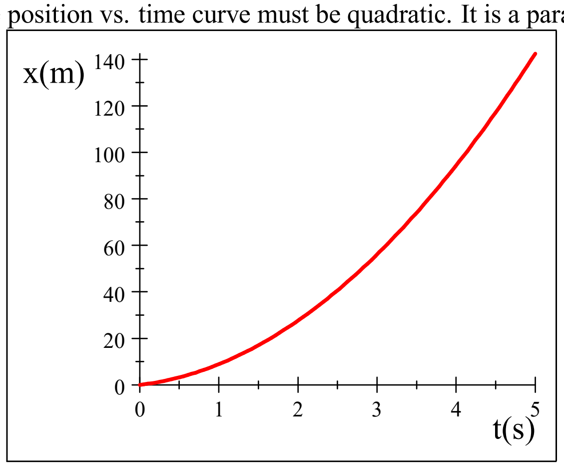
which is linear in t. The slope of a $v(t)$ vs. time graph will be the value from our a vs. time graph.



We know that the velocity is the slope of a position vs. time graph, and now our velocity is changing. But we can use one of our kinematic equations (4.28)

$$x_f = x_i + v_i t + \frac{1}{2}a\Delta t^2$$

to realize that the position vs. time curve must be quadratic. It is a parabola!



Now we know how to draw figures for constant acceleration problems, and we have constant acceleration equations. Let's try an example!

You build a new electric car. It has an acceleration of $0.5\,\mathrm{m/s^2}$. You start on your way to class with an initial velocity of $v_i = 0\,\mathrm{m/s}$. After $\Delta t = 3\,\mathrm{s}$, how fast are you going?

Lets list what we know:

$$a = 0.5\,\mathrm{m/\,s^2}$$
$$v_i = 0\,\mathrm{m/\,s}$$
$$\Delta t = 3\,\mathrm{s}$$

The acceleration is constant, so this is a kinimatic type problem, and we know the kinematic equations!

$$v_f = v_i + a\Delta t \qquad \text{Constant } a$$
$$x_f = x_i + v_i t + \frac{1}{2} a\Delta t^2 \qquad \text{Constant } a$$
$$v_f^2 = v_i^2 + 2a(x_f - x_i) \qquad \text{Constant } a$$
$$(x_f - x_i) = \frac{1}{2}(v_f + v_i)\,\Delta t \qquad \text{Constant } a$$

But what equation do we use? We can mark the parts we know in each of the equations, say, by underline them

$$v_f = \underline{v_i} + \underline{a\Delta t}$$
$$x_f = x_i + \underline{v_i}t + \frac{1}{2}\underline{a\Delta t^2}$$
$$v_f^2 = \underline{v_i^2} + 2\underline{a}(x_f - x_i)$$
$$(x_f - x_i) = \frac{1}{2}\left(v_f + \underline{v_i}\right)\underline{\Delta t}$$

And remember we want v_f. At this point is is clear that we can use equation (4.27)

$$v_f = v_i + a\Delta t$$

$$\begin{aligned} v_f &= 0 + \left(0.5\,\mathrm{m/\,s^2}\right)(3\,\mathrm{s}) \\ &= 1.5\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

If we define your house to be the $x_i = 0$ position, how far from home are you after $\Delta t = 3\,\mathrm{s}$?

Now that we know x_i we can add it to our list

$$a = 0.5\,\mathrm{m/\,s^2}$$
$$v_i = 0\,\mathrm{m/\,s}$$
$$\Delta t = 3\,\mathrm{s}$$
$$x_i = 0$$

and mark it in our equations

$$v_f = \underline{v_i} + \underline{a\Delta t}$$
$$x_f = \underline{x_i} + \underline{v_i}t + \frac{1}{2}\underline{a\Delta t^2}$$
$$v_f^2 = \underline{v_i^2} + 2\underline{a}(x_f - \underline{x_i})$$
$$\Delta x = \frac{1}{2}\left(v_f + \underline{v_i}\right)\underline{\Delta t}$$

Again we can see that in the second equation we know all by x_f, so we use equation (4.28)

$$x_f = x_i + v_i t + \frac{1}{2} a \Delta t^2$$

$$\begin{aligned} x_f &= 0 + 0 + \frac{1}{2}\left(0.5\,\mathrm{m/\,s^2}\right)(3\,\mathrm{s})^2 \\ &= 2.25\,\mathrm{m} \end{aligned}$$

You might need to put a bigger motor on your electric car.

You probably know that the Earth pulls on us to keep us from flying away as it turns through the Solar system. This pull is called gravity. The pull causes a constant acceleration for things falling near the Earth's surface. And that is just the sort of situation our new equations can handle. We will take up this case in the next lecture.

5 Problem Solving in Physics

You may have taken some physics in high school. Or you may have watched cool physics videos on YouTube. Just watching physics happen is like a real-world magic show! But our study of physics will be for more than entertainment and will be more than an introduction. Engineers use physics to build real world things. Chemists and biologists use physics to understand how chemical and biological systems will operate. So our study of physis will emphasis preicting the behavior of objects (how they will move). To do this we will use a lot of math. Math is very powerful. It allows us to predict the future in a way! We want to apply the power of mathematics to finding the future motion of an object. It is time to get ready to do that.

We haven't really studied much physics yet. But we have had a few lectures on physics and you have spent a decade or two observing the world around us. So you intuitively know a few basic equations, like you intuitively know that speed is how far you go in how much time.

$$\text{speed} = \frac{\text{distance traveled}}{\text{how long it took to travel}}$$

Some things will be less intuitive. For example, last we learned about displacement.

$$\Delta x = x_f - x_i$$

It is like distance, but really it tells you how far you got from where you started. We will use basic equations to predict motion. In what follows for this lecture and the next lecture, I will outline a proven problem solving process, then explain the parts of the process, then give an example of following that process. I will try to do this with things you already know. But it all may feel a bit forign. Don't worry. We will practice this process for the rest of this course.

Parts of a problem:

Here are the steps of our problem solving process. You received a copy of this process as part of the Syllabus package on the first day and it is posted on I-Learn under *Course Description*.

Problem Solving Process

Process Step	Purpose	Value if Present	Value if Absent
1. Label the problem with chapter and problem number	This is essential if I am to figure out what problem to grade	0	0 to -20
2. **Restate the problem in your own words.** One line may do! List any assumptions you are making.	Most major mistakes come from misinterpreting the problem. This step asks you to slow down and determine what the problem really is asking	1	-1
3. **Draw a picture**, label items, define coordinate systems, etc.	Many mistakes happen because we do not have a clear picture of the problem. This step may save hours of grief. Also, many physics problems will have different symbolic answers because of the freedom to choose coordinate systems, etc. Drawing a diagram gives the reader the ability to understand your vision of the problem.	2	-2
4. **Define variables** used, Identify known and unknown quantities	Choose reasonable names for physical quantities, and let me know what they are. Don't forget to include units.	2	-2
5. **List basic equations** that apply to the problem	This step gives you a firm starting place.	2	-2
6. **Solve the problem algebraically** starting from the basic equations,	This is the heart of the solution. The symbolic answer tells you the relationships between physical quantities.	10	-10
7. **Determine numerical answer**	The specific numerical answer is not the point of doing the problem in this class, but is a great indicator that you have succeeded in understanding the physics.	1	-1
8. **Check units.** If you have not done the algebra on the units earlier, do it here.	Many mistakes are evident in a units analysis. It is a good habit to always check units.	1	-1
9. **Determine if the numerical answer is reasonable.** Indicate if you are comfortable with the result, if you have little experience with the result and can't tell if it is reasonable, or if it is not reasonable, but you don't know why (or else you would fix it).	From your understanding of the physics, state whether the answer is reasonable. For example, if you are calculating the mass of a ping pong ball, and get an answer that is many times the mass of the earth, you should note that there may be a problem even if you do not know where you went wrong.	1	-1 to -25
Total Possible		**20**	**-25**

Figure 5.2.

Restate the Problem

The first thing to do when working a homework problem (or a problem given to you in your job, or a problem to test with an experiment, or whatever problem you are assigned to solve), is to restate the problem. You don't get as many points if you solve a different problem than was asked! It is common, especially on tests, to misread the problem. So take some time and make sure you are answering what was asked. In industry, I used to email my boss a restatement of each assignment to make sure I understood what I had been assigned!

Identify the type of Problem

If you can look at the problem and see it as part of a class of problems, then you know which equations to try, and what techniques to attempt. So far all we have are speed. But soon we will have many other types.

Suppose you are asked to find the displacement of an object that starts at position $x_i = 5\,\mathrm{m}$ and ends at position $x_f = 12\,\mathrm{m}$. The equation

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

is a great resource if you are looking for velocity, but not so great if you trying to find the displacement. Identifying that the problem asks for displacement helps you realize that the equation

$$\Delta x = x_f - x_i$$

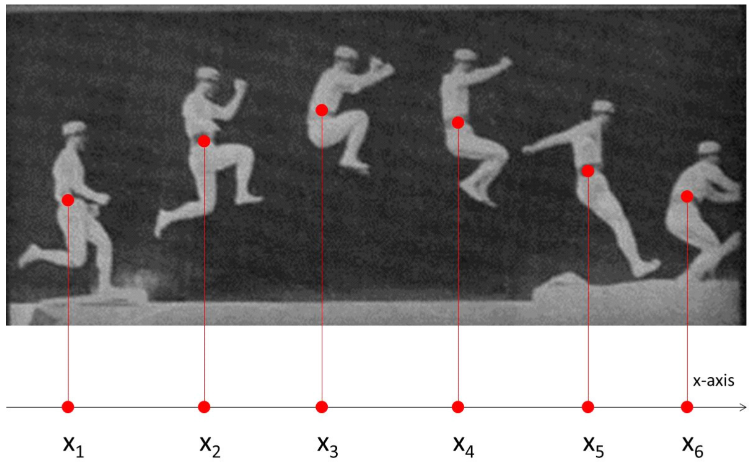
might help. Some problems might be a combination of more than one problem type. It is easier to solve problems if we view them as combinations of simple problems that we know how to solve. Identifying the problem type(s) for a homework or exam problem (or industry task) can save you hours of wondering.

Drawing the question

We will spend several lectures learning how to draw diagrams describing motion. We will spend more lectures learning how to draw diagrams for force problems, and equilibrium problems, and rotational problems. You would think it was an art class!

But seriously, learning to express the problem as a visualization is a large part of "doing physics," and the diagram is often the key to seeing how to solve the problem. It is tempting to skip this step. But you must convince yourself to learn how to make the diagrams and to use the diagrams in solving problems.

Defining Variables



Let's look at our jumping man again. It is a great help to realize what you know. Suppose that we know $x_i = 5\,\mathrm{m}$ and $x_f = 12\,\mathrm{m}$ for our man. And suppose we want to find the man's total displacement. It is much easier to see the known values if we call them out

$$\begin{aligned} x_i &= 5\,\mathrm{m} \\ x_f &= 12\,\mathrm{m} \end{aligned}$$

By placing these on your paper so you can see them, it is easier to find an answer. Seeing the positions like this it is obvious that all you need to do is subtract to get the answer. As the problems become more complicated, this will be an even more important step. It also lets the grader (or in your job, the reader of your report) know what the variable symbols mean. Not every field uses the same letters for the same quantities. And we reuse some letters! The letter T could be "tension" or "period of oscillation." By writing down what the letters mean, you avoid confusing yourself and others.

But what kinds of things are variables? Let's take some time to see what quantities we might define.

Objects

We have been talking about objects, like a person, a bird, or a ball. But what is an object? What is the universe made of?

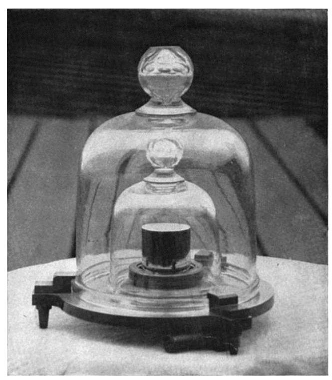
The startling answer is, we're not entirely sure! Oh, we know that the universe is made of stars and planets and galaxies and dust and many other things, but what are the things made of?

Answering that question is the job of particle physics, and the answer is still in the making. For our study of motion, we will assume there are fundamental things in the universe, and more complicated things are made of these fundamental things. Lets look at three quantities as our initial building blocks. *Mass, Length*, and *Time*.

Mass

When we think of an object, we usually think of something that has mass. Mass is an amount of matter. Exactly what matter is, still somewhat of a question (Job security for Physicists). Einstein equated mass and energy. The great experiments at the Conseil Européen pour la Recherche Nucléaire (CERN) are trying to understand exactly what mass is. You may have heard of the *Higgs boson*, a particle discovered at CERN that gives us a hint that our theory of what mass is might be right. But that is current on-going research. So for this class we shall take mass as just the amount of matter and trust our intuition on what that means. The standard unit of mass is the kilogram, abbreviated " kg." It is the mass of a standard piece of platinum alloy, again kept in France.

Question 1.1.3



US National Institute of Standards standard kilogram.

Note that mass and weight are very different quantities. you can see this if we use a bathroom scale. On earth the scale gives a reading that is proportional to the amount of matter in our bodies, but if we took it on a space craft in orbit, it would not measure any weight at all. Yet the amount of matter in our bodies has not changed!

Question 1.1.4

Length

Perhaps we should really say "space" here. We need to have some idea of how far away things are or how long or tall things are. In this class our view of space will be that it is a container in which things happen. When you study Einstein's Relativity we will change that a little, but for now space is a container, and length is a measure of how far away in this container something is.

In ancient Egypt, the standard of measuring length was when Pharaoh took his ceremonial reed and measured the length of the foundation for the temple (a little like our standard kilogram for mass). This might sound strange, but in essence this is what we all did until 1960. Prior to this, a meter, our unit of length, was defined as one ten millionth of the distance from the North Pole to the equator. Since this was not a very practical day-to-day measuring device, a standard "reed" (this time made of platinum) was kept in France, and meter sticks were made to match this standard. There are terrible problems with this! Each stick of a different materials changes length with temperature! So in 1983 the meter was defined as the distance light traves in vacuum during a time interval of

$$\frac{1}{299792458}\,\mathrm{s}$$

The abbreviation for meter is "m."

Time

You might object that time is not a thing. But we have already used time in our study of motion. It must be something! We should define it before we get to familiar with using time. But what is time? It turns out that time is hard to define. We usually use the idea that time is how long we wait.[2] That can be tricky to measure. Let's start with something simple. How much time will you spend in this class today? That is a time we can wait, about an hour. But it is harder to answer questions like "how long does it take for light to travel a foot." The answer is about a nanosecond. We cannot perceive of times this small. Likewise, we cannot wait for a million years (well, we could, but our vantage point might change after the first 70 years or so).

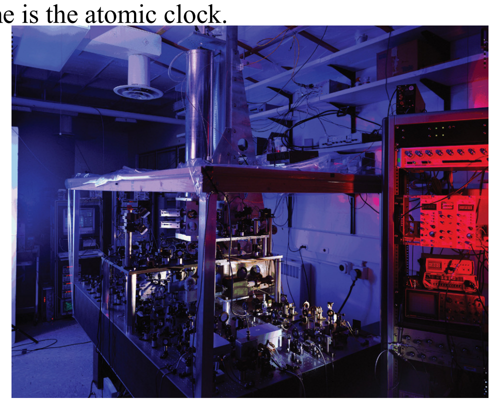
To measure time we use events that are *periodic*, that is, they occur at regular intervals. An early example is the pendulum of a clock. From a fundamental periodic phenomena, we can build up larger or smaller units of time.

The current unit is the *second*, abbreviated "s," which is given as 91926317000 times the period of oscillation of radiation from the cesium atom. But if you are lucky enough to go on in physics, you will find that this definition of time (like our simple definition

[2] Feynman, Richard, Robert Leighton, Matthew Sands, *The Feynman Lectures on Physics*, Vol. I, Addison-Sesley, Reading Massachusetts, 1963

of space) is not enough. But we can save that for a future class. For what we will study, the periodic motions in simple clocksor watchs are accurate enough to measure time in this class.

The standard for time is the atomic clock.



Atomic Clock

Other fundamental quantities

There are other fundamental quanties. The amount of electric charge, for example (sometimes expressed as part of electric current). And Temperature is another example. We will deal with electric charge and temperture and other fundamentals in the next few physics classes. For this class, time, space (distance), and mass will get us a long way toward understanding motion.

Derived quantities

So we have objects made of mass and space (length) and time to use in describing their motion.

Question 1.1.1

When we combine quantities we derive new quantities that are useful from the basic length, time, mass set. For example, we know speed is a combination of length and time.

$$\text{speed} = \frac{\text{distance traveld}}{\text{how long it took to travel}}$$

quantities like acceleration, force, momentum, etc. are derived quantities.

Dimensional Analysis

Analysis of the units in a measurement can be very useful. For example, if we take our first equation

$$\text{speed} = \frac{\text{distance traveld}}{\text{how long it took to travel}}$$

and write it in a more compact way

$$v = \frac{\Delta x}{\Delta t}$$

and look at the units, we find that Δx is a length in, say, meters. We find that Δt is a time in, say, seconds. Then when we calculate v we should have units of $\mathrm{m/s}$. If, instead, we have m^3 at the end of our calculation, something must have gone wrong! Sometimes it is useful to use generic units for our analysis. That is, any length is given a unit L and any time is given a unit T. So our equation gives

$$\frac{\Delta x}{\Delta t} \Rightarrow \frac{L}{T}$$

To see this lets take and acceleration (something we will study soon). It is given by

$$a = \frac{\Delta v}{\Delta t} \Rightarrow \frac{\frac{L}{T}}{T} = \frac{L}{T^2}$$

so from dimensional analysis, we expect that acceleration would be something like

$$a = c\frac{x}{t^2}$$

and we could deduce that

$$x = \frac{1}{c}at^2$$

note that I included a constant, c. Dimensional analysis cannot tell you the constants in an equation. We will see later that in this case $c = 2$ so

$$x = \frac{1}{2}at^2$$

We will find out that our dimensional analysis did not hit too far off the mark. This is part of the equation for position for an accelerating object.

Units

No value in physics is useful without a unit. For example, if I tell you to jump from a height of 100, it makes a difference whether it is $100\,\mathrm{cm}$ or $100\,\mathrm{m}$! Units tell us what standard was used to make the measurement so all who see the result can correctly interpret what it means.

System of Units

You will notice that we have only given metric units. We will use the *Système International* or *SI* units. There are, of course, other systems of units. We will try to ignore them in this class. Occasionally we may use feet for length and slugs (yes, slugs) for mass. we will usually use the following SI units for our basic quantities.

Quantity	Unit	Symbol
Mass	Kilogram	kg
Length	meter	m
Time	second	s

The SI system makes use of prefixes to modify the basic unit, like *centi*meter to mean $1/100$ of a meter. You should be familiar with the following prefixes.

Prefix	Symbol	Power		Prefix	Symbol	Power
nano-	n	10^{-9}		giga-	G	10^{9}
micro-	μ	10^{-6}		mega-	m	10^{6}
mili-	m	10^{-3}		kilo-	k	10^{3}
centi-	c	10^{-2}		deka-	da	10^{1}
deci-	d	10^{-1}				

We can already see that our unit for mass, the kilogram, must be 1000 grams. A centimeter must be $1/100^{th}$ of a meter. We obviously will need to be able to convert from centimeters to meters from time to time. We should be able to convert from any prefixed unit to any other prefixed unit. We need a strategy to do this.

Unit Conversions

Let's do a unit conversion that most of you do in your head. Let's convert hours to seconds. We know that

$$1\,\mathrm{h} = 60\,\mathrm{min}$$

and we know that

$$1\,\mathrm{min} = 60\,\mathrm{s}$$

Suppose we have 5 hours. How many seconds is this?

Most of us would say multiply by 3600, and that is right, but let's do it one step at a time so you can see the process. I want to multiply 5 by something, but I can't change the duration. At the end of our calculation, it still has to be a wait of $5\,\mathrm{h}$ even though we now give the value in seconds. For a person waiting 5 hours and a second one waiting $18\,000\,\mathrm{s}$ they must feel the same amount of time. So we need to adjust our 5 hours by something that does not change the wait.

I think you will agree that if I multiply by 1 nothing changes

$$5 \times 1 = 5$$

We can do this with units

$$5\,\mathrm{h} \times 1 = 5\,\mathrm{h}$$

Notice that the 1 can't have any units or this won't work. Now let's take our equation relating hours to minutes.

$$1\,\mathrm{h} = 60\,\mathrm{min}$$

and let me divide by $1\,\mathrm{h}$

$$\begin{aligned} \frac{1\,\mathrm{h}}{1\,\mathrm{h}} &= \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \\ 1 &= \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \end{aligned}$$

On the left hand side the h units cancel. So the 1 has no units just like we needed in our

$$5\,\mathrm{h} \times 1 = 5\,\mathrm{h}$$

equation. I can multiply by

$$1 = \frac{60\,\mathrm{min}}{1\,\mathrm{h}}$$

and all I am doing is multiplying by 1!

$$5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} = 5\,\mathrm{h}$$

This still must be true, but let's do the math

$$5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} = 300\,\mathrm{min}$$

This is how many minutes are in 5 hours. We can play the same trick with minutes to seconds

$$\begin{aligned} \frac{1\,\mathrm{min}}{1\,\mathrm{min}} &= \frac{60}{1\,\mathrm{min}}\,\mathrm{s} \\ 1 &= \frac{60\,\mathrm{s}}{1\,\mathrm{min}} \end{aligned}$$

so we can take our $300\,\mathrm{min}$ and find out how many seconds we have!

$$300\,\mathrm{min} \times \frac{60\,\mathrm{s}}{1\,\mathrm{min}} = 18000.0\,\mathrm{s}$$

Now we could do this all in one equation, using our strange way of writing 1 that converts from hours to minutes and our strange way of writing 1 that converts from minutes to seconds.

$$\begin{aligned} 5\,\mathrm{h} \times 1 \times 1 &= 5\,\mathrm{h} \\ 5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \frac{60\,\mathrm{s}}{1\,\mathrm{min}} &= 18000.0\,\mathrm{s} \end{aligned}$$

Notice that the units cancel like variables in algebra!

We will treat units like algebraic quantities that can be canceled. Let's do another example. We want to convert $10\,\mathrm{mi}$ (ten miles) to kilometers. We can look up that

$$1\,\mathrm{mi} = 1609\,\mathrm{m} \tag{5.1}$$

and we know that

$$1\,\mathrm{km} = 1000\,\mathrm{m} \tag{5.2}$$

Start with conversion from miles to meters. We recognize that with a small use of algebra

$$1 = \frac{1609\,\mathrm{m}}{1\,\mathrm{mi}} \tag{5.3}$$

then we can write

$$10\,\mathrm{mi} = 10\,\mathrm{mi}\frac{1609\,\mathrm{m}}{1\,\mathrm{mi}} = 16\,090\,\mathrm{m} \tag{5.4}$$

Then we also recognize that

$$1 = \frac{1000\,\mathrm{m}}{1\,\mathrm{km}} \tag{5.5}$$

or

$$1 = \frac{1\,\mathrm{km}}{1000\,\mathrm{m}} \tag{5.6}$$

then

$$16\,090\,\mathrm{m} = 16\,090\,\mathrm{m}\frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = 16.0900\,\mathrm{km} \tag{5.7}$$

so

$$10\,\mathrm{mi} = 16\,\mathrm{km} \tag{5.8}$$

We could do this all in one large, chained, conversion

$$10\,\mathrm{mi} = 10\,\mathrm{mi}\frac{1609\,\mathrm{m}}{1\,\mathrm{mi}}\frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = 16\,\mathrm{km} \tag{5.9}$$

If you think about it, to convert units, we have multiplied by 1 several times. So as you multiply to convert units, make sure your factors you multiply are equal to 1.

Uncertainty in measurements

In science, we must face the fact that no measurement is completely accurate. The reasons for uncertainty are limitations in our human sensory system or sensing apparatus. For example, if I measure a square of metal with a ruler. I am likely not able to tell the length to better than a tenth of a centimeter ($1\,\mathrm{mm}$). This is because of inaccuracies in the ruler and in my own ability to see the ruler clearly and consistently. So suppose I have a measurement of $16.3\,\mathrm{cm}$. I can really only tell you that the measurement is between $16.4\,\mathrm{cm}$ and $16.2\,\mathrm{cm}$. we could write this as $16.3 \pm 0.1\,\mathrm{cm}$. We will study uncertainty in measurements in some detail in PH150. But for PH121 we will need some provisional rules that let us make a guess on now good our answers are.

Significant figures

Scientists have devised a clever way to include the level of uncertainty in the statement of the measurement result. This is referred to as *significant figures* and it basically means to keep only the digits in a number that contain well known information. In the above example, we would say that in $16.3\,\mathrm{cm}$ that the 3 is the least significant digit. Now suppose we use the same ruler to measure the same object, but I tell you that the measurement is $16.3259357\,\mathrm{cm}$. If we know the measurement is only good to $\pm 0.1\,\mathrm{cm}$, what can we say about the digits 259357? We can say they are worthless! They are nonsense, so we cleverly leave them off! There are a series of rules to tell us which digits are significant. It is important to realize that zeros that just mark where the decimal place goes are not significant (e.g. in $0.00163\,\mathrm{cm}$ the three 0's are not significant, but in $1.400\,\mathrm{cm}$ the digits mean that the measurement is known to $\pm 0.001\,\mathrm{cm}$). We have to be careful when we use zeros!

We usually express numbers in scientific notation.

Propagation of uncertainty

Suppose we take two measurements, like measuring the sides of a rectangle.

$$\begin{aligned} l &= (2.3 \pm 0.1)\ \mathrm{cm} \\ w &= (4.5 \pm 0.1)\ \mathrm{cm} \end{aligned}$$

and we wish to find the area

$$A = l \times w$$

$$\begin{aligned} A &= 2.3\,\mathrm{cm} \times 4.5\,\mathrm{cm} \\ &= 10.35\,\mathrm{cm}^2 \end{aligned}$$

But we were a little uncertain about the length and width, wouldn't we also be uncertain about the area that we made from the uncertain length and the width? Of course there is some uncertainty in the area. Let's see how we could deal with this.

The length could have been as much as

$$l = 2.3\,\mathrm{cm} + 0.1\,\mathrm{cm} = 2.4\,\mathrm{cm}$$

and the width could be as much as

$$w = (4.5 + 0.1)\ \mathrm{cm} = 4.6\,\mathrm{cm}$$

So the area could be as much as

$$\begin{aligned} A_+ &= 2.4\,\mathrm{cm} \times 4.6\,\mathrm{cm} \\ &= 11.04\,\mathrm{cm}^2 \end{aligned}$$

But the length could be as little as

$$l = 2.3\,\mathrm{cm} - 0.1\,\mathrm{cm} = 2.2\,\mathrm{cm}$$

and the width could be as little as

$$w = (4.5 - 0.1)\ \mathrm{cm} = 4.4\,\mathrm{cm}$$

So the area could be as little as

$$\begin{aligned} A_- &= 2.2\,\mathrm{cm} \times 4.4\,\mathrm{cm} \\ &= 9.68\,\mathrm{cm}^2 \end{aligned}$$

We can see that these differ by about $\pm 1\,\mathrm{cm}^2$ total.

$$A_+ - A_- = 11.04\,\mathrm{cm}^2 - 9.68\,\mathrm{cm}^2 = 1.36\,\mathrm{cm}^2$$

Thus the tenths and hundredths places in our calculated area cannot be very certain. We drop these and write

$$A = 10 \pm 1\,\mathrm{cm}$$

Notice that our length and width had two digits,

$$\begin{aligned} l &= (2.3 \pm 0.1)\ \mathrm{cm} \\ w &= (4.5 \pm 0.1)\ \mathrm{cm} \end{aligned}$$

with the uncertainty in the second digit, and notice that our answer for the area has two digits, and the uncertainty is in the second digit!

In general:

> *In multiplying or dividing two quantities, the number of significant figures in the product or quotient is the same as the number of significant figures in the least accurate of the factors being combined.*

In our example, l and w both have two significant figures, so the result should be limited to two significant figures

For addition and subtraction the rules is:

> *The number of decimal places in the result should equal the smallest number of decimal places in any term in the sum or difference*.
> These two rules will help us determine how many digits to keep for most problems. You may know that there are several other rules, and thought we wont derive them like we did for multiplication, we will use them in our

problems. Here they are in a table

Significant Figure Rules
1. Non-zero digits are always significant
2. Embedded (i.e. captive) zero-digits are always significant
3. For the $number$ zero only the zero-digits after the decimal are significant
4. Leading zero-digits are not significant
5. Trailing zero digits:
If the number has a decimal, the trailing zero digits are significant
If the number does not have a decimal, the trailing zero digits are not significant
6. The final result of multiplication or division operation should have the same number of significant digits as the measured quantity with the least number of significant digits used in the calculation.
7. The final result of an addition or subtraction operation should have the same number digits to the right of the decimal as the measured quantity with the least number of decimal used in the calculation.
8. For a mixture of operations, work from left to right, do mathematical hierarchy of operations ($\times$ or $\div$, then $+$ or $-$).

This system of significant figures works, but there are better ways of dealing with uncertainty. In our PH150 class we teach standard error propagation. Engineers will get this in their classes as well. Chemists will get a different method in their quantitative analysis class. But for now sig figs will do.

Basic Equations

Equations are relationships in physics. The equation

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

tells us how the displacement and duration combine to form velocity. These equations are our way of expressing motion. They are the tools in our toolbox for solving problems. Once you have identified the type of problem you have, you can quickly write down a list of equations (tools) that you could use to solve that problem. You might write down more equations than you end up using for a particular problem. That's OK. You don't empty your tool box of all tools but the ones you think you might use when you start a fix-up job in your house! You should not do so when starting a problem. List your equations, then choose the ones that seem to work given your known values in your list of variables.

Solving with symbols

For years now, you have worked with numbers and answers that are numbers. And that is what the teacher was looking for. But in physics the equation is the important thing. It tells you how things relate to each other. Let's try an example. We will develop the equations for this example in the next few lectures, so don't panic if this seems mysterious!

`The initial speed of an object is` $3\,\mathrm{m/s}$ `and it's acceleration is` $2\,\mathrm{m/s^2}$ `in the same direction as the velocity. Find the speed half a second after the experiment start.`

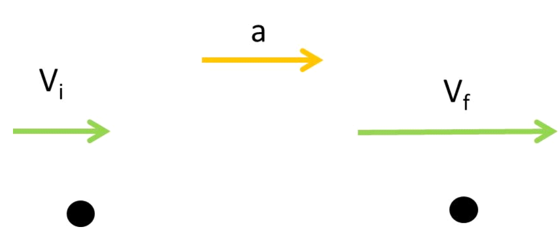
We want to start by restating the problem:

`Find the final speed knowing` a `and` v_i

Next identify the type of problem. I think it is an acceleration problem:

```
PT acceleration
```

Next we want to draw the picture



Our variables list is next:

```
VAR
```

$$\begin{aligned} v_i &= 3\frac{\mathrm{m}}{\mathrm{s}} \\ a &= 2\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 0.5\,\mathrm{s} \end{aligned}$$

and now basic equations:

`BE:`

$$\underline{\vec{\mathbf{a}}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\underline{\Delta t}}$$
$$\overrightarrow{\Delta \mathbf{v}} = \vec{\mathbf{v}}_f - \underline{\vec{\mathbf{v}}_i}$$
$$\vec{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$
$$\overrightarrow{\Delta \mathbf{x}} = \vec{\mathbf{x}}_f - \vec{\mathbf{x}}_i$$

Note that I underlined the known values from my list variables. The first two equations have v_f in them and they contain my known values, so it looks like they are the ones to use in my solution.

Solve Algebraically

Now we try to solve for the speed, but we do so symbolically. We already believe that the first two equations in our basic equation list will be helpful, so let's start with

$$\underline{\vec{\mathbf{a}}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\underline{\Delta t}}$$

and put in

$$\overrightarrow{\Delta \mathbf{v}} = \vec{\mathbf{v}}_f - \underline{\vec{\mathbf{v}}_i}$$

$$\underline{\vec{\mathbf{a}}} = \frac{\vec{\mathbf{v}}_f - \underline{\vec{\mathbf{v}}_i}}{\underline{\Delta t}}$$

At this point I recognize that I can solve for v_f and everything else is known. I could plug things in my calculator and have it solve for v_f using numbers, but we won't do that! We will continue with algebra

$$\vec{\mathbf{a}} \Delta t = \vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i$$

and

$$\vec{\mathbf{a}} \Delta t + \vec{\mathbf{v}}_i = \vec{\mathbf{v}}_f$$

or

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}} \Delta t$$

Since $\vec{\mathbf{a}}$ and $\vec{\mathbf{v}}_i$ are in the same direction, their magnitudes just add so

$$v_f = v_i + a\Delta t$$

This is the symbolic answer. It has the thing I want, v_f, an equals sign, and then symbols for what v_f is equal to.

Numeric answer

The numeric answer is easy. Just plug in numbers to your symbolic answer

$$v_f = v_i + a\Delta t$$

$$\begin{aligned} v_f &= 3\frac{\mathrm{m}}{\mathrm{s}} + \left(2\frac{\mathrm{m}}{\mathrm{s}^2}\right)(0.5\,\mathrm{s}) \\ &= 4.0\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Reasonableness check

If we had gotten $400000000000\,\mathrm{m/\,s}$ in our example, we would know something went wrong. Nothing can go this fast! It is good to check your answer and see if it seems to make sense. But how do we do that? One way is to do a quick estimate.

Estimates

Question 1.3

There are times when we simply do not have all the information we need, but we need a number for something anyway. There are also times when we wish to make a quick calculation (like checking on the reasonableness of a calculation). In such cases, we estimate. Many people in science are somewhat uncomfortable with estimates, because they are not "correct" (In business and politics, people may be a little too comfortable with estimates). But as a check on our exact calculations, estimates are very valuable!

Let's do a few examples together.

Example 1: How many sheets of paper fit between the Earth and Moon?

To do this calculation, we need to know how far away the Moon is and how thick a piece of paper is. I looked up the earth-moon distance and found

$$D_{EM} = 3.844\,7 \times 10^5\,\mathrm{km} \tag{5.10}$$

This might be a good number, but since I am building an estimate, I will round to the nearest power of 10.

$$D_{EM} \approx 4 \times 10^5\,\mathrm{km} \tag{5.11}$$

I got some paper and measured 28 pieces of paper, then took the measurement and

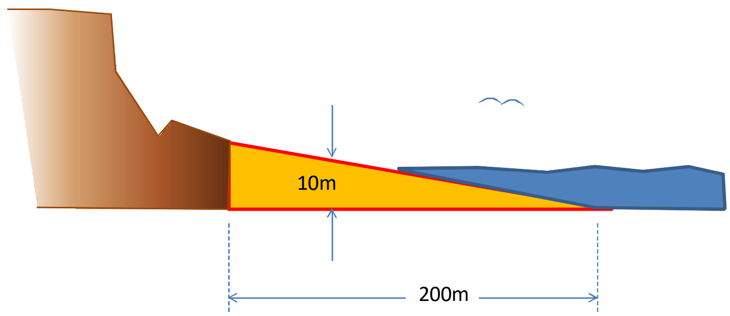


Figure 5.3.

devided by 28 to get the estimated thickness of just one piece.

$$\begin{aligned} t &= \frac{3\,\text{mm}}{28} = 1.071\,4 \times 10^{-4}\,\text{m} \qquad (5.12) \\ &\approx 1 \times 10^{-4}\,\text{m} \end{aligned}$$

So the number of pieces of paper would be

$$\begin{aligned} N &= \frac{D_{EM}}{t} = \frac{4 \times 10^{5}\,\text{km}}{1 \times 10^{-4}\,\text{m}} \frac{1000\,\text{m}}{\text{km}} \qquad (5.13) \\ &= 4\,000\,000\,000\,000 \\ &= 4.0 \times 10^{12} \\ &\approx 1 \times 10^{12} \qquad (5.14) \end{aligned}$$

Is this reasonable?

Example 2 How much sand is in the world's beaches?

We start by looking for a fundamental element of a beach, say, a grain of sand. We can calculate the total volume of the beaches, and divide by the volume of a grain of sand. This will tell us how many grains of sand there are in the world's beaches. If we can get an estimate of the mass of a grain of sand, then we can answer how much sand is in the world's beaches.

Let's estimate the grain of sand to have a mass of

$$m = 0.005\,\text{kg} \qquad (5.15)$$

Let's guess a volume of

$$V_s = 1.0 \times 10^{-15}\,\text{m}^3 \qquad (5.16)$$

We can estimate the length of the world's coastline to be

$$L = 40000000000\,\text{m} \qquad (5.17)$$

we need the volume of the coast line, lets say the beach is

$$w = 200\,\mathrm{m} \tag{5.18}$$

wide and

$$d = 10\,\mathrm{m} \tag{5.19}$$

deep.

Then the volume of the world's beaches would be

$$\begin{aligned} V &= Lwd = (40000000000\,\mathrm{m})\,(200\,\mathrm{m})\,(10\,\mathrm{m}) & (5.20)\\ &= 8.0\times 10^{13}\,\mathrm{m}^3 & (5.21)\end{aligned}$$

and the number of grains of sand would be

$$N = \frac{V}{V_s} = \frac{Lwd}{V_s} = 8.0\times 10^{28} \tag{5.22}$$

which gives us

$$\begin{aligned} M_{beach} &= Nm = \left(8.0\times 10^{28}\right)(0.005\,\mathrm{kg}) & (5.23)\\ &= 4.0\times 10^{26}\,\mathrm{kg} & (5.24)\end{aligned}$$

I looked up the mass of the Earth

$$M_E = 5.98\times 10^{24}\,\mathrm{kg} \tag{5.25}$$

Our answer doesn't seem too reasonable Where did we go wrong?

Consider silicon oxide. It has a density of

$$\rho = 2200\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{5.26}$$

If our estimate of

$$m = 0.005\,\mathrm{kg} \tag{5.27}$$

is good, then we should have used a volume of

$$\begin{aligned} V &= \frac{m}{\rho} = \frac{0.005\,\mathrm{kg}}{2200\frac{\mathrm{kg}}{\mathrm{m}^3}} & (5.28)\\ &= 2.2727\times 10^{-6}\,\mathrm{m}^3 & (5.29)\end{aligned}$$

for a grain of sound. So one problem is that our estimate of the volume of a grain of sand is very bad.

In general, you can be creative in making estimates, but you do have to be careful.

Units Check

We already have discussed units. But it is important to check your units in your final answer. Suppose we want to find the speed of something. The final units must be a length unit divided by a time unit. For speed this must be the case. If we had gotten a length unit divided by a time squared, then we would know something went wrong in our algebra. If the units don't work the answer is wrong. If the units are wrong, the anser can't be reasonable. So especially on a test (or in your real job) checking units is important!

Problem Solving Process

I have assembled all of the problem solving pieces that we have studied into a process that leads us to our solution. Here is the process we will use in a table:

Problem Solving Process

Process Step	Purpose	Value if Present	Value if Absent
1. Label the problem with chapter and problem number	This is essential if I am to figure out what problem to grade	0	0 to -20
2. **Restate the problem in your own words.** One line may do! List any assumptions you are making.	Most major mistakes come from misinterpreting the problem. This step asks you to slow down and determine what the problem really is asking	1	-1
3. **Draw a picture**, label items, define coordinate systems, etc.	Many mistakes happen because we do not have a clear picture of the problem. This step may save hours of grief. Also, many physics problems will have different symbolic answers because of the freedom to choose coordinate systems, etc. Drawing a diagram gives the reader the ability to understand your vision of the problem.	2	-2
4. **Define variables** used, Identify known and unknown quantities	Choose reasonable names for physical quantities, and let me know what they are. Don't forget to include units.	2	-2
5. **List basic equations** that apply to the problem	This step gives you a firm starting place.	2	-2
6. Solve the problem algebraically starting from the basic equations,	This is the heart of the solution. The symbolic answer tells you the relationships between physical quantities.	10	-10
7. Determine numerical answer	The specific numerical answer is not the point of doing the problem in this class, but is a great indicator that you have succeeded in understanding the physics.	1	-1
8. **Check units.** If you have not done the algebra on the units earlier, do it here.	Many mistakes are evident in a units analysis. It is a good habit to always check units.	1	-1
9. **Determine if the numerical answer is reasonable.** Indicate if you are comfortable with the result, if you have little experience with the result and can't tell if it is reasonable, or if it is not reasonable, but you don't know why (or else you would fix it).	From your understanding of the physics, state whether the answer is reasonable. For example, if you are calculating the mass of a ping pong ball, and get an answer that is many times the mass of the earth, you should note that there may be a problem even if you do not know where you went wrong.	1	-1 to -25
Total Possible		**20**	**-25**

Notice that I have assigned point values to each part. So if you leave out a part you will know how many points you will miss. Also notice that there are a lot of points for the symbolic answer and only a one for the numeric answer. Physics is about relationships that describe motion, the numbers just aren't that important. So it is not a winning strategy to only give me the numeric answer for a problem. you only get one out of twenty five points!

Also notice that if I ask for the mass of a ping pong ball, and you give me an answer that is three times the mass of Jupiter, I can take off more than one point for the very wrong numeric answer! Since you will be doing a reasonableness check, you won't have this problem. But what if you don't know if an answer is reasonable? Then say you don't know! You will act differently as a physicist, engineer, doctor, etc. if you admit in your calculations that you are not certain of the result. And this is very valuable! It can save your job! So if you are not sure, say so.

We will use this process for the rest of the semester, and this or a similar process for PH123, and PH220 (or PH223) and if you are a physicist for the rest of your career. This is also the process I used in engineering in industry. So it is worth practicing in our problems. It is also how I will grade the tests!

6 Motion near the Earth's Surface

You may have noticed that falling things speed up as they fall. You may even have noticed that heavy things like rocks tend to fall with about the same motion. It would be convenient if falling objects experienced constant acceleration. Then we could use our equation set for falling object problems.

At the end of the 16^{th} century Galileo Galilei tried an experiment to see how falling things fell. The legend says that Galileo dropped two balls with different masses off of the leaning tower of Pisa. Galileo predicted that the motion of the two balls would be the same. Galileo's experiment was successful in that he was able to show his model for motion was more correct than Aristotle's. But it was only an approximation to motion with constant acceleration. The reason for this is that on the Earth there is an atmosphere, and the air get's in the way. You may have heard of "air resistance." This resistance due to the air getting in the way slows the fall of the balls, but not by much.

By using smooth round balls, the air resistance was limited, so Galileo did not notice the problem.

But the experiment was repeated a few centuries later–on the Moon–where there is no air. In the figure below, the Apollo 15 astronauts dropped two objects, a hammer and a feather. The feather would have been strongly effected by air as it fell, but there is no air on the moon, so both the hammer and the feather experienced constant acceleration.

Applo 15 test of Galeleo's experiment on the moon. For a video, go to http://nssdc.gsfc.nasa.gov/planetary/lunar/apollo_15_feather_drop.html

We will give a name to such a situation, where objects fall with nothing, not even air, to get in their way. We will also use the same name for situations where the motion is so close to having no impediments that we would not know the difference, like Galileo's experiment. We will call such a motion *free fall.*

Free Fall

If we don't go too far above the surface, and if we pick an object for which we can neglect air resistance (like a smooth ball, and not like a feather), then we can use our constant acceleration equations for the falling object! Under these conditions, falling objects act like they are under the influence of constant acceleration.

It is a little counter intuitive, but the mass of the object does not matter in this type of problem! Think of dropping a large rock and a small rock off a bridge. They seem to fall at the same rate even though one rock is much less massive than the other.

All objects that are truly in free fall have the same acceleration. We will call this acceleration *free fall acceleration.*

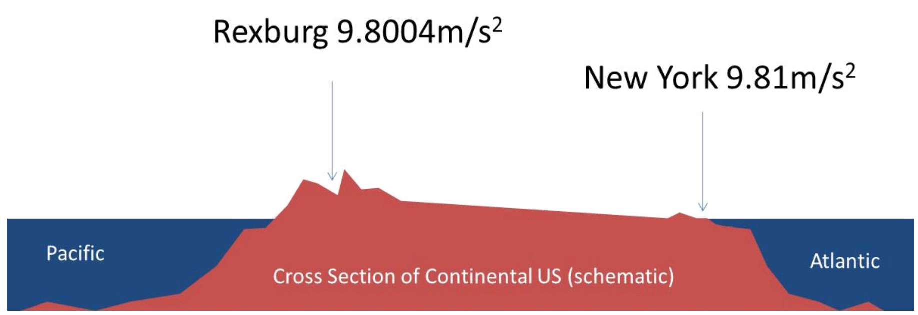
For the case of free fall near the surface of the Earth, we even give free fall acceleration a specific symbol, the letter "g." The value for this free fall acceleration near the Earths surface is about

$$g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \tag{6.1}$$

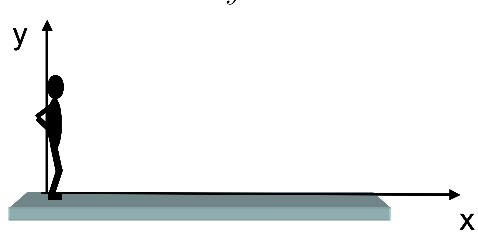
but for rough estimations

$$g \approx 10\frac{\mathrm{m}}{\mathrm{s}^2} \tag{6.2}$$

Because g really varies with height, it is closer to $9.81\,\mathrm{m/\,s^2}$ in New York and is about $9.8004\,\mathrm{m/\,s^2}$ in Rexburg. For our class $9.8\,\mathrm{m/\,s^2}$ is close enough most of the time.



Tradition guides us to choose an x-axis parallel to the ground. This leaves a chose between y and z for the axis perpendicular to the ground. Tradition again tells us to choose y. So heights are measured on the y axis[3].



Now you can define the axis any way you want, but if you do something different, you should warn people who might read your work (like our grader). With this choice of axis, "down" is in a negative y-direction. So our free fall acceleration near the Earth's surface is

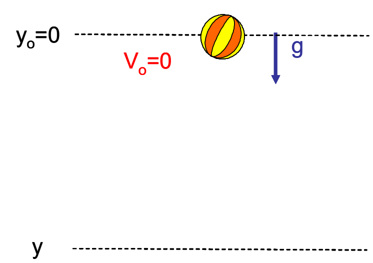
$$\overrightarrow{\mathbf{a}}_{ff} = -g = -\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right) \tag{6.3}$$

[3] This will change later in our course and in higher level physics classes.

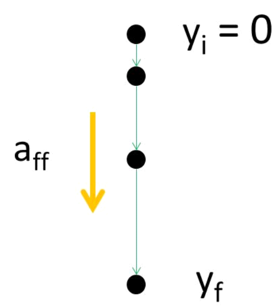
where g is the magnitude of the vector, and the minus sign is telling us that a_{ff} points down

Let's try a problem

An object is dropped from rest near the Earth's surface and we can say that free fall conditions apply. Determine the position of the object after $4.00\,\mathrm{s}$.



We can choose $y_i = 0.00$ as the point where the ball starts.



We could choose anywhere for $y = 0.00$, our origin, and the math would work out just fine. But for this problem I choose $y_i = 0.00$. We recognize that $v_i = 0.00$ (that is what is meant by the words "from rest").

So here is what we know

$$
\begin{aligned}
y_i &= 0.00\,\mathrm{m} \\
v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}} \\
\Delta t &= 4.00\,\mathrm{s} \\
g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2} \\
a_{ff} &= -g = -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)
\end{aligned}
$$

Since free-fall motion is constant acceleration motion, we can use our constant acceleration equation set:

$$v_f = v_i + a\Delta t \qquad \text{Constant } a$$
$$x_f = x_i + v_i\Delta t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a$$
$$v_f^2 = v_i^2 + 2a\Delta x \qquad \text{Constant } a$$
$$x_f = x_i + \frac{1}{2}\left(v_f + v_i\right)\underline{\Delta t} \qquad \text{Constant } a$$

but we can write them in terms of y because our motion is in the y-direction

$$v_f = v_i + a\Delta t$$
$$y_f = y_i + v_i\Delta t + \frac{1}{2}a\Delta t$$
$$v_f^2 = v_i^2 + 2a\left(y_f - y_i\right)$$
$$y_f = y_i + \frac{1}{2}\left(v_f + v_i\right)\Delta t$$

We know $v_{i,}$ Δt, and we know y_i and we know a, and we want y_f.

$$v_f = \underline{v_i} + \underline{a\Delta t}$$
$$y_f = \underline{y_i} + \underline{v_i}\underline{\Delta t} + \frac{1}{2}\underline{a\Delta t^2}$$
$$v_f^2 = \underline{v_i^2} + 2\underline{a}\left(y_f - \underline{y_i}\right)$$
$$y_f = \underline{y_i} + \frac{1}{2}\left(v_f + \underline{v_i}\right)\underline{\Delta t}$$

It looks like the second equation in our constant acceleration set will do for the final position

$$y_f = \underline{y_i} + \underline{v_i}\underline{\Delta t} + \frac{1}{2}\underline{a\Delta t^2} \tag{6.4}$$

only we need to change a to $-g$

$$y_f = \underline{y_i} + \underline{v_i}\underline{\Delta t} - \frac{1}{2}\underline{g\Delta t^2} \tag{6.5}$$

If a known value is zero, input it now. If it is not zero, it is better to wait to use it. We will use $y_i = 0$ and $v_i = 0$ now.

$$y_f = -\frac{1}{2}\underline{g\Delta t^2} \tag{6.6}$$

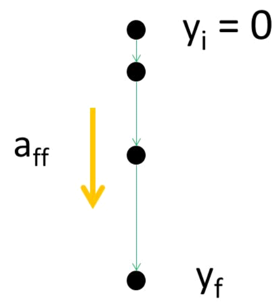
This is our symbolic solution! We can now put in the rest of our numbers

$$y = -\frac{1}{2}\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(4.00\,\mathrm{s}\right)^2 \tag{6.7}$$
$$= -78.4\,\mathrm{m} \tag{6.8}$$

Let's do another example:

What is the velocity of the object (same as in the last example) at $t = 4.0\,\mathrm{s}$?

Since everything is the same as in the previous example, we know that



$$
\begin{aligned}
y_i &= 0.00\,\mathrm{m} \\
v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}} \\
\Delta t &= 4.00\,\mathrm{s} \\
g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2} \\
a_{ff} &= -g = -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)
\end{aligned}
$$

and we can use the same equation set:

$$v_f = \underline{v_i} + \underline{a\Delta t} \qquad \text{Constant } a$$

$$y_f = \underline{y_i} + \underline{v_i\Delta t} + \frac{1}{2}\underline{a\Delta t^2} \qquad \text{Constant } a$$

$$v_f^2 = \underline{v_i^2} + 2\underline{a}\left(y_f - \underline{y_i}\right) \qquad \text{Constant } a$$

$$y_f = \underline{y_i} + \frac{1}{2}\left(v_f + \underline{v_i}\right)\underline{\Delta t} \qquad \text{Constant } a$$

We know v_i, a, and Δt, so we can use the first equation

$$v_f = \underline{v_i} + \underline{a\Delta t}$$

Let's put in $a = -g$

$$v_f = \underline{v_i} - \underline{g\Delta t}$$

We now use any zero terms

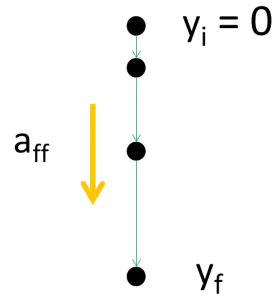
$$v_f = -g\Delta t$$

and we have a symbolic solution. We input numbers

$$
\begin{aligned}
v_f &= -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s}) \\
&= -39.2\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

Let's take a third example: An object is dropped from rest near the Earth's surface and we can say that free fall conditions apply. A short time later the object has dropped $80.0\,\mathrm{m}$. What is the velocity of the object at this point?

This is still free-fall



But this time we don't have a time

$$\begin{aligned} y_i &= 0.00\,\mathrm{m} \\ y_f &= -80.0\,\mathrm{m} \\ v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}} \\ g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2} \\ a_{ff} &= -g = -\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right) \end{aligned}$$

We can use the same set of equations

$$v_f = \underline{v_i} + \underline{a}\Delta t \qquad \text{Constant } a$$

$$y_f = \underline{y_i} + \underline{v_i}\Delta t + \frac{1}{2}\underline{a}\Delta t^2 \qquad \text{Constant } a$$

$$v_f^2 = \underline{v_i^2} + 2\underline{a}\left(\underline{y_f} - \underline{y_i}\right) \qquad \text{Constant } a$$

$$\underline{y_f} = \underline{y_i} + \frac{1}{2}\left(v_f + \underline{v_i}\right)\underline{\Delta t} \qquad \text{Constant } a$$

but since we know v_i, Δy, and a, and we want v_f, we should choose the third equation

$$v_f^2 = \underline{v_i^2} + 2\underline{a}\left(\underline{y_f} - \underline{y_i}\right)$$

and solve for v_f

$$v_f = \sqrt{\underline{v_i^2} + 2\underline{a}\left(\underline{y_f} - \underline{y_i}\right)}$$

and use our zeros

$$v_f = \sqrt{2\underline{a}\left(\underline{y_f} - \underline{y_i}\right)}$$

and recall that $a = a_{ff} = -g$

$$v_f = \sqrt{-2g\left(y_f - y_i\right)}$$

this is the algebraic answer. It might worry you that we have a negative sign inside a square root. Is this a problem? Won't we get an imaginary number?

But let's put in our values and we will see that this is not the case.

$$\begin{aligned} v_f &= \sqrt{-2\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-80.0\,\mathrm{m} - 0.00\,\mathrm{m})} \\ &= \pm 39.598\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

We have to remember that if we set $y = 0$ at y_i then our y_f is negative, so we have another negative sign that cancels the first! Note that we got two answers, because from a square root we don't know if the quantity that was squared was positive or negative, think

$$\begin{aligned} 2^2 &= 4 \\ (-2)^2 &= 4 \end{aligned}$$

but we can have only one velocity at a time for our object! We need to choose the answer that fits our situation. We know (from our picture) that the object is falling, so we will choose the negative value

$$v_f = -39.6\frac{\mathrm{m}}{\mathrm{s}}$$

Constant acceleration changes and Free-fall

Let's say that you have joined the BYU-I rocket team. You launch the rocket (and this time it does not blow up). The rocket moves upward with a constant acceleration $a = 30.0\,\mathrm{m/s}^2$ as long as the rocket fuel lasts, which is $\Delta t_1 = 4.00\,\mathrm{s}$. Once the fuel is spent, the rocket continues up for a while, then it begins to fall back to the ground (the parachute did not open. . . again). find

a) the velocity of the rocket and at the time when the fuel is spent

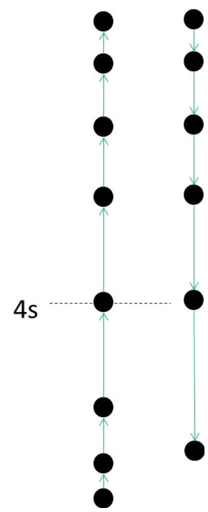
b) the position at the time when the fuel is spent

c) the maximum height of the rocket

We can only do problems for constant acceleration. But in this problem the acceleration changes! Fortunately, the change in acceleration is instantaneous, and the rest of the time we have constant acceleration, but we have *two* constant accelerations to deal with. One while the rocket is firing, and then free fall after the rocket fuel is spent. To deal with this we divide this problem into two phases of it's motion, one for each constant acceleration. In effect we make our one problem into two related problems. The first problem is the powered flight up. The second problem is after the fuel is spent and the rocket is in free fall.

Solution:

PT: constant acceleration problem *and* a free-fall problem. While the rocket engine is working it is constant acceleration, afterward, it is free-fall. We will divide the problem into two parts, one constant motion, the other free-fall.



Here is what we know from the problem statement

$$a_1 = 29.4\,\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$t_1 = 4.00\,\mathrm{s}$$
$$a_2 = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$y_{1i} = 0.00 \text{ (ground)}$$

Let's use y as position. We will choose $y = 0$ to be the ground. We will use v and a as velocity and acceleration, and g as the acceleration due to gravity. Our basic equations are

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{6.9}$$

$$y_f = y_i + v_i\Delta t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a \tag{6.10}$$

$$v_f^2 = v_i^2 + 2a\Delta y \qquad \text{Constant } a \tag{6.11}$$

$$y_f = y_i + \frac{1}{2}\left(v_f + v_i\right)\Delta t \qquad \text{Constant } a$$

Parts a) and b) Find the rocket's velocity and position after $4.00\,\mathrm{s}$ which is when the rocket fuel has just run out. These values are both from the first part of the problem where we have an acceleration a_1 (the rocket is firing). We can guess that $v_{1i} = 0$ (when the rocket starts, it is not moving) and $y_{1i} = 0$, and we are given $a_1 = 29.4\frac{\mathrm{m}}{\mathrm{s}^2}$ so

$a_1 = 29.4\frac{\mathrm{m}}{\mathrm{s}^2}$
$\Delta t_1 = 4.00\,\mathrm{s}$
$y_{1i} = 0.00$ (ground)
$v_{1i} = 0$

We choose from our kinematic equations. Let's write them for part 1, and mark them for what we know. The subscript 1 means this is the first position value we will find for part 1 of the motion.

$$v_{1f} = \underline{v_{1i}} + \underline{a_1 \Delta t_1} \qquad \text{Constant } a \tag{6.12}$$

$$v_{1f} = \underline{y_{1i}} + \underline{v_{1i}\Delta t_1} + \frac{1}{2}\underline{a_1}\underline{\Delta t_1^2} \qquad \text{Constant } a \tag{6.13}$$

$$v_{1f}^2 = \underline{v_{1i}^2} + 2\underline{a_1}\left(y_{1f} - \underline{y_{1i}}\right) \qquad \text{Constant } a \tag{6.14}$$

$$y_f = \underline{y_{1i}} + \frac{1}{2}\left(v_{1f} + \underline{v_{1i}}\right)\underline{\Delta t_1} \qquad \text{Constant } a$$

For part b) position, we might use

$$v_{1f} = \underline{y_{1i}} + \underline{v_{1i}\Delta t_1} + \frac{1}{2}\underline{a_1}\underline{\Delta t_1^2} \tag{6.15}$$

Using our zeros gives

$$y_{1f} = \frac{1}{2}\underline{a_1}\underline{\Delta t_1^2} \tag{6.16}$$

Now we can use numbers

$$y_{1f} = \frac{1}{2}\left(29.4\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s})^2 \tag{6.17}$$

$$= 235.2\,\mathrm{m} \tag{6.18}$$

$$= 235.\ \mathrm{m} \tag{6.19}$$

Now let's find the velocity at this altitude (when the rocket engine just runs out of fuel). We choose the equation

$$v_{1f} = \underline{v_{1i}} + \underline{a_1 \Delta t_1} \tag{6.20}$$

and recognize that v_{1i} is still zero, so

$$v_{1f} = \underline{a_1 \Delta t_1} \tag{6.21}$$

Now we use numbers

$$v_{1f} = \left(29.4\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s}) \tag{6.22}$$

$$= 117.6\frac{\mathrm{m}}{\mathrm{s}} \tag{6.23}$$

$$= 118\frac{\mathrm{m}}{\mathrm{s}} \tag{6.24}$$

Part c) Find the maximum Height. After the rocket fuel is spent the rocket keeps going up. But now the acceleration is due to just gravity. So for part 2 $\overrightarrow{a_2} = -g$. But this part 2 starts where part 1 stoped. We will use the ending values from part 1 that we found in questions a) and b) to be our starting values for part c). We will go back to all the digits we had before we rounded for sig-figs. That is because we keep all the digits from a calculation and only round at the end. At the end of part a) we did round because we reported v_{1f} and y_{1f}, but now we are continuing to calculate, so let's keep all the digits we have until the end once more. Here is where we start:

$$v_{1f} = v_{2i} = 117.6\frac{\mathrm{m}}{\mathrm{s}}$$
$$y_{1f} = y_{2i} = 235.2\,\mathrm{m}$$
$$a_2 = -9.80\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$t_{2i} = t_{1f} = 4.00\,\mathrm{s}$$

We know the initial speed, position, and acceleration. Since the rocket engine stopped working the acceleration is now $-g$. We are in free fall (but we are going up!). We know that at the highest point in the flight the velocity must be zero, so we can write another known value

$$v_{2f} = 0$$

We again need our kinematic equations, marked with what we know, and a subscript "2" to distinguish this set from the new part 2 set from part 1. We want y_{2f}.

$$\underline{v_{2f}} = \underline{v_{2i}} + \underline{a_2}\Delta t_2 \qquad \text{Constant } a \tag{6.25}$$

$$y_{2f} = \underline{y_{2i}} + \underline{v_{2i}}\Delta t_2 + \frac{1}{2}\underline{a_2}\Delta t_2^2 \qquad \text{Constant } a \tag{6.26}$$

$$\underline{v_{2f}}^2 = \underline{v_{2i}}^2 + 2\underline{a_2}\left(y_{2f} - \underline{y_{2i}}\right) \qquad \text{Constant } a \tag{6.27}$$

$$y_{2f} = \underline{y_{2i}} + \frac{1}{2}\left(v_{1f} + \underline{v_{2i}}\right)\underline{\Delta t_2} \qquad \text{Constant } a$$

It looks like we could use the second equation

$$y_{2f} = \underline{y_{2i}} + \underline{v_{2i}}\Delta t_2 + \frac{1}{2}\underline{a_2}\Delta t_2^2$$

but we don't know what Δt_2 is. But we can solve for Δt_2 from the first equation. Notice how we did this. After marking our equations with what we know, we looked for a way to solve for what we want, y_{2f}. But we could not do it with any one equation on it's own. So we looked at the other equations to find the missing parts in our equation for y_{2f}. Let's get the time Δt_2 first,

$$\underline{v_{2f}} = \underline{v_{2i}} + \underline{a_2}\Delta t_2$$

using our zeros

$$0 = \underline{v_{2i}} + \underline{a_2}\Delta t_2$$

and $-g$

$$\underline{v_{2i}} = g\Delta t_2$$

so

$$\Delta t_2 = \frac{\underline{v_{2i}}}{g}$$
$$\Delta t_2 = \frac{\underline{v_{1f}}}{g}$$

We don't need to know the value of $\Delta t_{2,}$ but let's calculate it anyway.

$$\begin{aligned} \Delta t_2 &= \frac{120.0\frac{\mathrm{m}}{\mathrm{s}}}{9.80\frac{\mathrm{m}}{\mathrm{s}^2}} && (6.28) \\ &= 12.245\,\mathrm{s} && (6.29) \end{aligned}$$

so the rocket goes up for $12.2\,\mathrm{s}$ after the fuel runs out!

Now that we know Δt_2 we can use the second of our equations to find y_{2f}

$$\begin{aligned} y_{\max} &= y_{2f} = y_{2i} + v_{2i}\Delta t_2 + \frac{1}{2}a_2\Delta t_2^2 && (6.30) \\ &= y_{1f} + v_{1f}\Delta t_2 - \frac{1}{2}g\Delta t_2^2 && (6.31) \\ &= y_{1f} + v_{1f}\left(\frac{\underline{v_{1f}}}{g}\right) - \frac{1}{2}g\left(\frac{\underline{v_{1f}}}{g}\right)^2 && (6.32) \\ &= y_{1f} + v_{1f}\left(\frac{\underline{v_{1f}}}{g}\right) - \frac{1}{2g}\left(\underline{v_{1f}}\right)^2 && (6.33) \end{aligned}$$

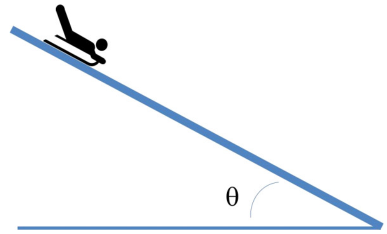
Notice that even though I calculated the value of Δt_2 I still put in the symbolic Δt_2 into my equation. Now we are ready for numbers

$$\begin{aligned} y_{\max} &= 240.0\,\mathrm{m} + \frac{\left(120.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{9.8\frac{\mathrm{m}}{\mathrm{s}^2}} - \frac{1}{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}\left(120.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2 && (6.34) \\ &= 974.69\,\mathrm{m} && (6.35) \end{aligned}$$

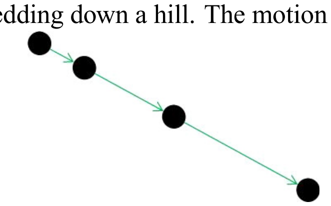
$$\begin{aligned} y_1 &= 240.0\,\mathrm{m} \\ v_1 &= 120.0\frac{\mathrm{m}}{\mathrm{s}} \\ t &= 12.245\,\mathrm{s} \end{aligned}$$

Motion on an inclined plane

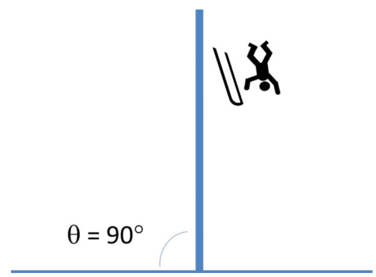
Let's take on an example that is almost free-fall. Suppose we have a person sledding down a (perfectly) frictionless surface



would this be free-fall? First, we should test for constant acceleration. So suppose we record video of a person sledding down a hill. The motion diagram looks like this



which we recognize as acceleration. But only if the slope angle θ were 90° would we have free-fall



And our intuition tells us that the guy who tried to sled off a vertical cliff will reach the bottom faster than the guy that slid down the slope. So we can guess that our slope has an acceleration that is less than the $a_{ff} = g = 9.8\,\mathrm{m/\,s^2}$. It is as though only part of the free-fall acceleration is able to pull the sled down the hill. The hill seems to be getting in the way!

Being physics students The obvious thing to do is to record video of sledders going down many different hills with different slopes to see if the acceleration is dependent on the steepness of the slope. You might guess that the steeper the slope the larger the acceleration.

Such an experiment has been done, and the result is

$$a_{slope} = g \sin \theta$$

From our trigonometry experience, we recall that $\sin \theta$ goes between 0 and 1 as the slope angle goes from 0° to 90°. Indeed, we seem to have only part of the free-fall acceleration. We have a fraction of the total possible falling acceleration with the fraction given by $\sin \theta$. But why $\sin \theta$? That is something we will take up next lecture.

Even Galileo actually used ramps to do his motion studies. By using a ramp (shown above) he could have constant acceleration, but a smaller constant acceleration than g. That made it easier to see the motion. Galileo did not have digital cameras to record the motion, instead he used bells along the ramp to tell where the balls were so he could make his motion diagrams.

7 Undoing Differentiation, and Two Dimensional Motion

So far we have found how far we go from

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

but solving for

$$\begin{aligned} \Delta x &= \bar{v}\Delta t \\ x_f &= x_i + \bar{v}\Delta t \end{aligned}$$

but this assumed constant motion. We made it better by allowing for constant accelertion

$$x_f = x_i + v_i \Delta t + \frac{1}{2} a \Delta t^2$$

What if our acceleration isn't constant? We don't have a way to deal with this. We need more new math!

More New Math - Integration

We now know how to take a derivative, but you were probably wondering if there is a way to undo the process of taking a derivative. We could call that an anti-derivative. Such a thing might be useful. If we consider that

$$v\left(t\right) = \frac{dx}{dt} \tag{7.1}$$

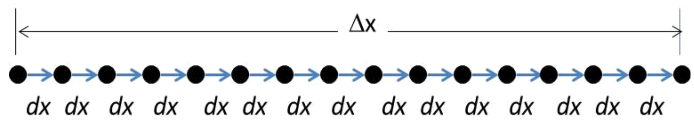
then undoing this process might be able to give us $x\left(t\right)$ if we know $v\left(t\right)$. Say, you record your speed as you travel from your speedometer, but you want to know how far you have gone.

There is such a process. Let's think about what the process might be. From equation (7.1) we can solve for the small displacement dx

$$dx = v(t)dt$$

This is a small amount of displacement dx in a small amount of time, dt. If I only want to go a very small distance, dx, then this in enough. The speed $v\left(t\right)$ won't change much at all over a small time dt, so dx is just how far I can go in dt. That is, with a small enough dt we can consider $v(t)$ to be constant no matter if we are accelerating or not. But if I want to go farther than and infintisimal dx, We need to do more.

Consider a larger displacement Δx. We could view this larger displacement as made up of a huge number of infintismal displacements, dx.



To find Δx we could just sum up all the little displacements to get the whole Δx.So we can write our big displacement like this

$$\Delta x = dx_1 + dx_2 + dx_3 + \cdots dx_N$$

where dx_N is the last of our small dx pieces. In math we can write this as

$$\Delta x = \sum_{i=1}^{N} dx_i$$

but since the dx units are so small, we give this a special, calculus notation for summation

$$\Delta x = \int_{x_1}^{x_N} dx$$

where the curly thing, $\int$, has replaced the Σ. The curly thing is a stylized "s" for sum. The Σ is a Greek "s" for sum. So they are doing the same operation, but $\int$ let's us know that we are working with infinitesimally small displacements, dx. We still go from the first dx to the N^{th} dx and that is shown by the "lower limit," x_1 and the "upper limit," x_N on the $\int$.

Recall that $\Delta x = x_f - x_i$ so

$$\begin{aligned} x_f &= x_i + \Delta x \\ &= x_i + \int_1^N dx \end{aligned}$$

but from what we said before each dx is given by

$$dx = v(t)\, dt$$

so we will substitute this in for dx

$$x_f = x_i + \int v(t)\, dt$$

But when we change from the counting over the number of dx elements to counting over time elements, dt, we also changed the limits. The first dx was traveled at t_i, the first time. The last was traveled at the final N^{th} time so the upper limit is t_f, so

$$x_f = x_i + \int_{t_i}^{t_f} vdt$$

This is our notation for taking the anti-derivative of $v(t)$. But we don't want to actually have to measure each $v(t)\,dt$ and sum them by hand. We need a procedure for finding the result of $\int_{t_i}^{t_f} vdt$. Once again I will let your calculus professor explain why this works. For this class I will just give a procedure. Let's take a function

$$u = at^n$$

then the anti-derivative of u between two times t_1 and t_2 is

$$\begin{aligned}\int_{t_1}^{t_2} udt &= \int_{t_1}^{t_2} at^n dt = \left.\frac{at^{n+1}}{n+1}\right|_{t_1}^{t_2} \\ &= \frac{at_2^{n+1}}{n+1} - \frac{at_1^{n+1}}{n+1}\end{aligned}$$

The strange bar $|_{t_1}^{t_2}$ just keeps track of our upper and lower limits. But notice that the way we use the upper and lower limits is to substitute *both* limits into the equation that is sitting just before the bar, and subtract the equation with the lower limit from the equation with the higher limit. This isn't so strange if you remember that we are adding up something like Δx which would have a x_f and an x_i and we would subtract the two.

Like with derivatives, antiderivatives of sums of functions are just the sums of the antiderivatives

$$\int_{t_1}^{t_2} (u + w)\,dt = \int_{t_1}^{t_2} udt + \int_{t_1}^{t_2} wdt$$

It is also important to notice that antiderivatives require us to know a function, just like derivatives do. In our case, we need to know the function $v(t)$, that is, the velocity as a function of time.

Let's try this on the case of constant motion. Then $v(t) = v_o$ where v_o is some constant speed. In this case,

$$x_f = x_i + \int_{t_1}^{t_2} v_o dt$$

doesn't seem to have a t in it. But remember $1 = t^0$, so we can write our x_f equation as

$$x_f = x_i + \int_1^N v_o t^0 dt$$

and use our formula for antiderivatives

$$\begin{aligned}
x_f &= x_i + \int_{t_1}^{t_2} v_o t^0 dt \\
&= x_i + \left.\frac{v_o t^{0+1}}{0+1}\right|_{t_1}^{t_2} \\
&= x_i + v_o t|_{t_1}^{t_2} \\
&= x_i + v_o t_2 - v t_1 \\
&= x_i + v_o \Delta t
\end{aligned}$$

which is not much of a surprise for constant motion. This is just

$$v_o = \frac{\Delta x}{\Delta t}$$

rearranged a bit!

Let's try another problem. Let's take a ball falling from y_i and let's start our experiment at $t_i = 0$. Then we would have $a = -g$. Let's find an expression for the final position y_f as a function of time. We can start with our new math

$$y_f = y_i + \int_0^{t_f} v(t)\, dt$$

and for free fall we know that

$$\begin{aligned}
v(t) &= v_i + a\Delta t \\
&= v_i - g(t - t_i)
\end{aligned}$$

and use our $t_i = 0$ up front

$$v(t) = v_i - gt$$

so our new math formula is

$$y_f = y_i + \int_0^{t_f} (v_i - gt)\, dt$$

We can split this into two pieces

$$y_f = y_i + \int_0^{t_f} v_i dt - \int_0^{t_f} gtdt$$

We have two antiderivitives. The first antiderivitive is just like the previous problem. Let's do it first.

$$y_f = y_i + \left.\frac{v_i t^{0+1}}{0+1}\right|_0^{t_f} - \int_0^{t_f} gtdt$$

$$y_f = y_i + v_I t|_0^{t_f} + \int_0^{t_f} atdt$$

$$y_f = y_i + v_i t_f - v(0) - \int_0^{t_f} gtdt$$

$$y_f = y_i + v_i t_f - \int_0^{t_f} gtdt$$

Now let's do the second antiderivitive

$$y_f = y_i + v_i t_f - \left.\frac{gt^{1+1}}{1+1}\right|_0^{t_f}$$
$$y_f = y_i + v_i t_f - \left.\frac{gt^2}{2}\right|_0^{t_f}$$
$$y_f = y_i + v_i t_f - \left(\frac{gt_f^2}{2} - \frac{g(0)}{2}\right)$$
$$y_f = y_i + v_i t_f - \frac{1}{2}gt_f^2$$

which looks amazingly like one of our kinematic equations for free fall.

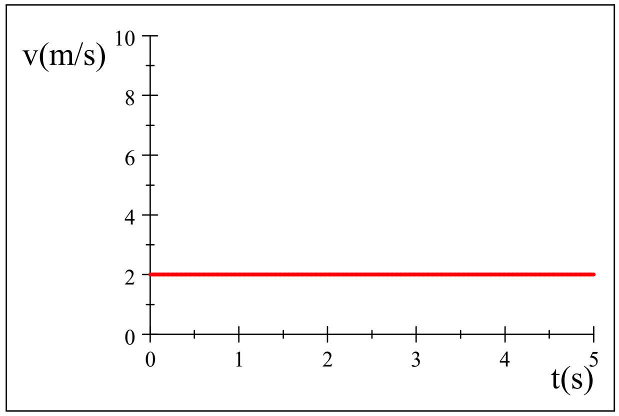
This seems like a lot of trouble to go through for constant motion or even constant acceleration problems. We got just what we would expect to get, and we already knew the equation. But not all motion is constant motion! And in many cases we have an $v(t)$ and $a(t)$ that really depend on t in complicated ways. In such a case, our antiderivative method may be the only way to get an answer for $x(t)$ knowing $v(t)$. Before we take on such a case, there are a few more things about antiderivatives we should know.

The first is that the official modern name for an antiderivative is *"integral"* and to find an antiderivative is called "taking an integral" or simply *"integrating."*

The second thing is a graphic interpretation for integrating. Let's graph the velocity vs. time for a constant motion case.

$$v = \frac{\Delta x}{\Delta t}$$

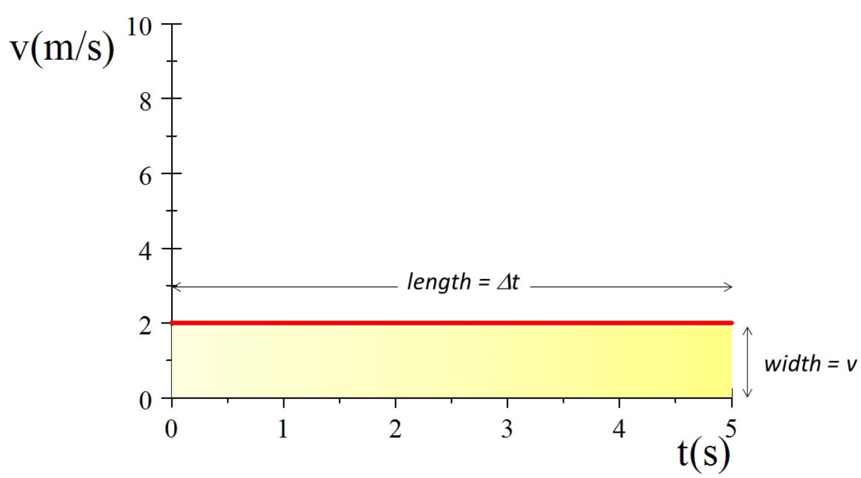
Let's say we have an object leaving the origin at $t_i = 0$ and traveling at $2\,\mathrm{m/s}$. The graph would look like this



Let's calculate the final position when $t_f = 5\,\mathrm{s}$. Our basic constant motion equation gives us

$$\begin{aligned} x_f &= x_i + v\Delta t \\ &= 0 + v\Delta t \\ &= v\Delta t \\ &= \left(2\frac{\mathrm{m}}{\mathrm{s}}\right)(5\,\mathrm{s} - 0\,\mathrm{s}) \\ &= 10\,\mathrm{m} \end{aligned}$$

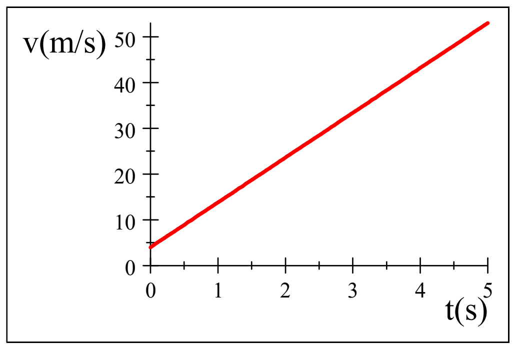
Looking at our graph we see that one axes is the time axes and the other is the velocity axis. We could define a kind of area for the surface of the graph, the part where we do the plotting, that would be length times width. But the length would be Δt and the width would be v. So the total "area" would be just $v\Delta t$.



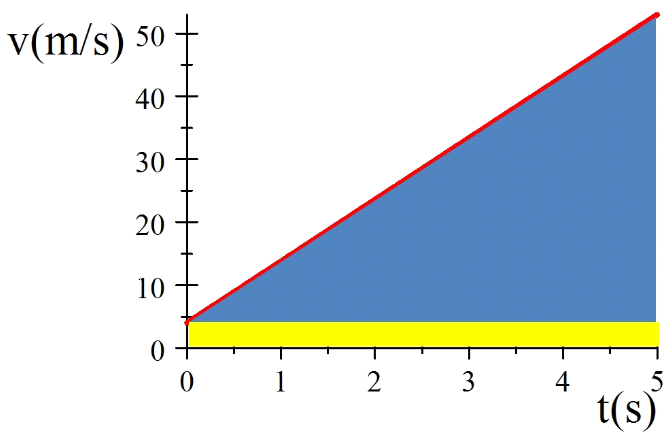
Notice that this "area" would be

$$\text{"area"} = \Delta t \times v = (5\,\mathrm{s} - 0\,\mathrm{s})\left(2\frac{\mathrm{m}}{\mathrm{s}}\right) = 10\,\mathrm{m}$$

It seems that the area under the red $v(t)$ line is equal to the final position, x_f! Of course real areas are constructed of lengths and widths that are both in the same units. So we don't mean that actual rectangle area as measured on the page with a ruler. What we mean is that if we take the value from the graph for the length, Δt, and multiply by the width, v, we get x_f. And this is what anti-derivatives or integrals do. they find the area under a curve. And this works for a $v(t)$ that is not constant as well. Consider the next graph.



We could find the "area under the curve" by splitting up the "area" into two pieces. A rectangle and a triangle.



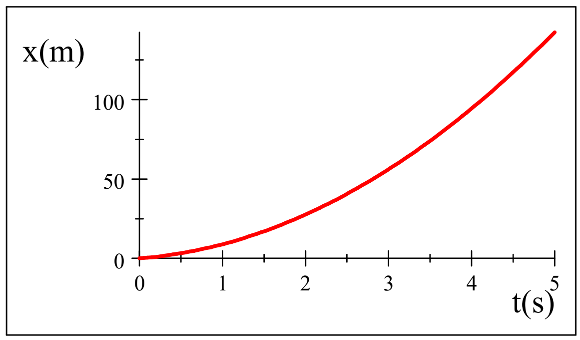
The yellow rectangle has a width of $4\,\mathrm{m/\,s}$ and a length of $5\,\mathrm{s}$. So the rectangle has an "area" of $20\,\mathrm{m}$. The triangle has a height of $53.0\frac{\mathrm{m}}{\mathrm{s}}$ and a base of $5\,\mathrm{s}$. So the area is

$$\begin{aligned} A_{triangle} &= \frac{1}{2}bh \\ &= \frac{1}{2}(5\,\mathrm{s})\left(53.0\frac{\mathrm{m}}{\mathrm{s}}\right) \\ &= 132.5\,\mathrm{m} \end{aligned}$$

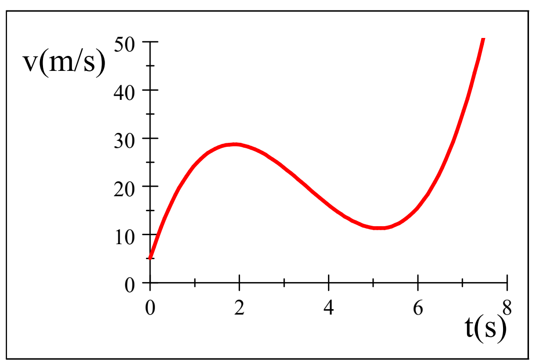
so the entire "area" is

$$\begin{aligned} x_f &= 20\,\mathrm{m} + 132.5\,\mathrm{m} \\ &= 152.5\,\mathrm{m} \end{aligned}$$

And, indeed, if we were to plot the position vs. time graph for this motion we would see that $x_f = 152.5\,\mathrm{m}$ is a reasonable answer.



Of course we could do much more complicated motions.



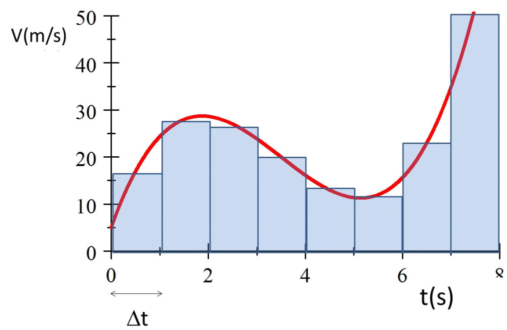
and with such a complicated motion we can see why our limiting process can turn a sum

$$\Delta x = \sum_{i=1}^{N} v\Delta t$$

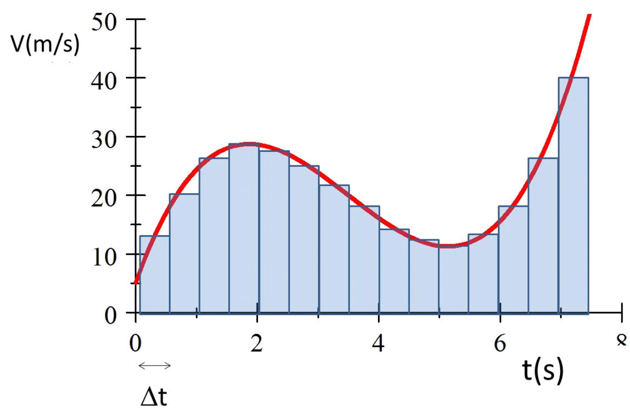
into an integral

$$\Delta x = \int_{t_i}^{t_f} v dt$$

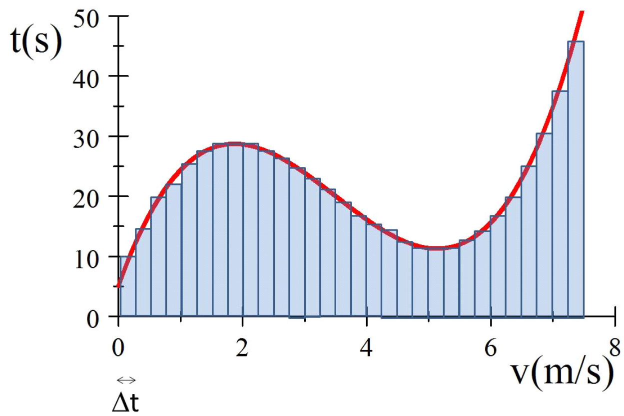
If Δt is large, our "areal under the curve" is only approximate because $v\Delta t$ is a box shape and although we could take many boxes and add them up (that is what the sum says to do) we miss pieces or get extra bits that are not under the curve



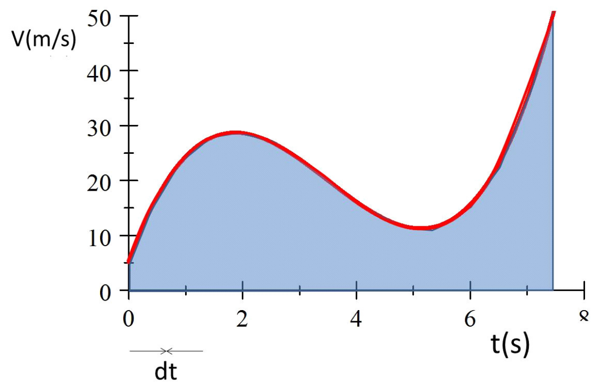
But if we let Δt get smaller our sum becomes a better approximation to the actual "area under the curve."



and each time we shrink Δt it gets better



until we finally get to an infinitesimal length for Δt, which we called dt.



and now the "area under the curve" is exact.

If I have a complicated velocity vs. time graph, then my object *must* be changing its speed or direction (or both). And we have a name for changing speed or direction. This is acceleration. Now that we have more powerful ways to think about changing velocity, it's time to reconsider acceleration. We will do that after practicing our new math a bit.

Examples of integration

Let's practice our integral procedure on some example functions. Suppose we have the function

$$v(t) = \left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right) t^2$$

What is the integral from $t_i = 0\,\mathrm{s}$ to $t_f = 3\,\mathrm{s}$?

Our procedure tells us to do the following:

$$\begin{aligned}\int_{t_i}^{t_f} udt &= \int_0^{3s}\left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2dt \\ &= \left.\frac{\left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3}{3}\right|_0^{3s} \\ &= \frac{5\frac{\mathrm{m}}{\mathrm{s}^3}\left(3\,\mathrm{s}\right)^3}{3}-\frac{5\frac{\mathrm{m}}{\mathrm{s}^3}\left(0\right)^3}{3} \\ &= 45.0\,\mathrm{m}\end{aligned}$$

Let's try another. Suppose we have the function

$$v\left(t\right)=5\frac{\mathrm{m}}{\mathrm{s}}$$

What is the integral from $t_i = 0\,\mathrm{s}$ to $t_f = 3\,\mathrm{s}$?

This function doesn't seem much like a function, but we remember that $1 = t^0$ so we could write our function as

$$v\left(t\right)=5\frac{\mathrm{m}}{\mathrm{s}}t^0$$

Then we just use our integration procedure:

$$\begin{aligned}\int_{t_i}^{t_f} udt &= \int_0^{3s}\left(5\frac{\mathrm{m}}{\mathrm{s}}\right)t^0dt \\ &= \left.\frac{\left(5\frac{\mathrm{m}}{\mathrm{s}}\right)t^1}{1}\right|_0^{3s} \\ &= \frac{5\frac{\mathrm{m}}{\mathrm{s}}\left(3\,\mathrm{s}\right)^1}{1}-\frac{5\frac{\mathrm{m}}{\mathrm{s}}\left(0\right)^1}{1} \\ &= 15\,\mathrm{m}\end{aligned}$$

We will have many more opportunities to practice this new math skill!

Velocity from acceleration

We can use our new math in another way! Recall that we started with constant motion,

$$v=\frac{\Delta x}{\Delta t}$$

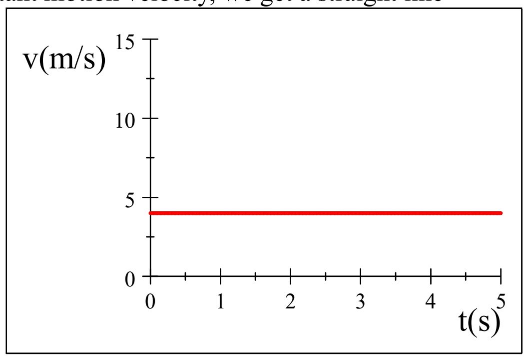
and we found that for constant motion

$$v=\frac{dx}{dt}$$

and we rewrote this to be

$$x_f = x_i + v\Delta t$$

If we plot the constant motion velocity, we get a straight line



and then we interpreted $v\Delta t$ as the area under a v vs. t graph

$$x_f = x_i + \text{area under a } v \text{ vs. } t \text{ graph}$$

and we found that if we added up many $v\Delta t$ segments with Δt very small so we called it dt, we could always find the area under a v vs. t graph using an integral

$$x_f = x_i + \int_{t_i}^{t_f} v dt$$

But earlier in our lecture we have been working with constant acceleration. And we recall that acceleration is

$$a = \frac{\Delta v}{\Delta t}$$

We could rewrite this as

$$v_f = v_i + a\Delta t$$

and we recognize this as one of our constant acceleration equation set of equations. If we make Δt so small that we could call it dt, we would have

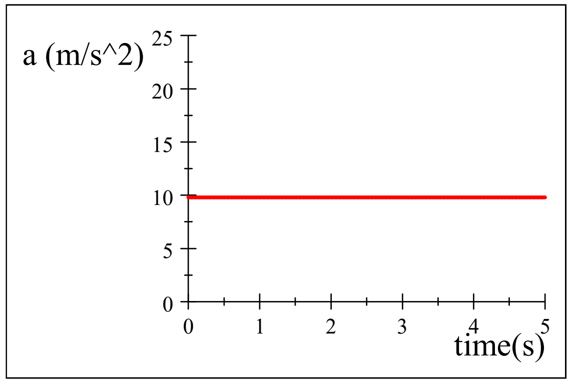
$$a(t) = \frac{dv}{dt}$$

and we have something grand! We recognize this derivative of the velocity as a function of time is the acceleration! Now suppose we know our acceleration. We cam use our new math to find our velocity from our acceleration! We could write our acceleration equation as

$$dv = a(t)\, dt$$

to find an infinatismal change in velocity dv in a infinatismal elapsed time dt. To find a bigger change Δv we would add up a whole lot of smaller chagnes dv.

Think, if we plot a constant acceleration, we get a straight line



It sure looks like we could interpret $a\Delta t$ as the area under a a vs. t graph

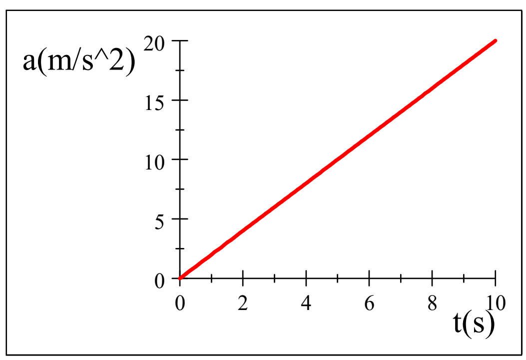
$$v_f = v_i + \text{area under a } v \text{ vs. } t \text{ graph}$$

So if we add up many $a\Delta t$ segments with Δt very small so we called it dt, we can find the area under a a vs. t graph using an integral

$$v_f = v_i + \int_{t_i}^{t_f} a dt$$

We have a way to find the velocity as a function of time knowing the acceleration! We get to use our integral process again! Let's try some problems using these new ideas.

Look at the graph below. If the initial velocity is zero, what would be the final velocity?



The “area” under the curve would be

$$\begin{aligned} A_{curve} &= \frac{1}{2}bh \\ &= \frac{1}{2}(10\,\mathrm{s})\left(20\frac{\mathrm{m}}{\mathrm{s}^2}\right) \\ &= 100\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Let’s try the same problem, but using the integral. The acceleration function is a straight line

$$a(t) = \left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right)t + 0$$

so

$$v_f = v_i + \int_{t_i}^{t_f} a dt$$

would be

$$\begin{aligned} v_f &= 0 + \int_0^{10\,\mathrm{s}} \left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right) t dt \\ &= \left.\frac{\left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2}{2}\right|_0^{10\,\mathrm{s}} \\ &= \left.\left(\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2\right|_0^{10\,\mathrm{s}} \\ &= \left(\frac{\mathrm{m}}{\mathrm{s}^3}\right)(10\,\mathrm{s})^2 - 0 \\ &= 100\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

What we have done is quite profound. We developed a process that will allow us to find the acceleration as a function of time given the velocity as a function of time, and the velocity as a function of time given the acceleration as a function of time. And we have a process to find the velocity as a function of time if we know the position and position as a function of time if we know the velocity.

position and velocity		velocity and acceleration	
$v = \frac{dx(t)}{dt}$	$x(t) = x_i + \int_{t_i}^{t_f} v(t)\,dt$	$a = \frac{dv(t)}{dt}$	$v(t) = v_i + \int_{t_i}^{t_f} a(t)\,dt$

These equations work for constant motion, constant acceleration, and all motion in general!

Two dimensional motion

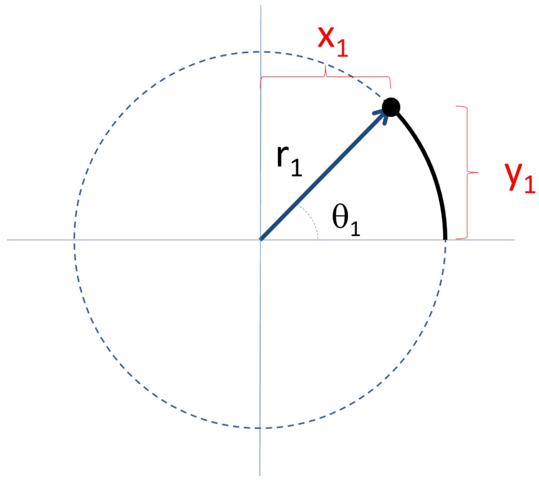
We have talked about motion in the x-direction and motion it the y-direction. So far we have our velocity as given by either

$$\overrightarrow{\mathbf{v}}_x = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

or

$$\overrightarrow{\mathbf{v}}_y = \frac{\overrightarrow{\Delta \mathbf{y}}}{\Delta t}$$

depending on which way our object is going. But what if our object isn't going right along the x-axis or the y-axis? What if it is going in some direction between these axes? We should realize that Δx and Δy are just parts of our vector displacement Δr.



So for this case, $\overrightarrow{\mathbf{v}}_x$ and $\overrightarrow{\mathbf{v}}_y$ must be parts of a vector velocity as well! We can write our velocity as

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{r}}}{\Delta t}$$

This works in any direction or combination of directions. But now our position is given by a vector. We have learned about vectors already. But there is more to know! Let's review what we know so far, then we will launch into new vector understanding.

We already know that some things in the universe have direction. And that direction is important. If you want to buy groceries in IF but end up in Ashton instead, things are not so good!

We will use the mathematical idea of a vector to describe these things that have direction. We draw a vector with an arrow. The arrow seems natural because it has both a direction that it points and a length. We can use the length of the arrow to show the *magnitude* of the vector. For example, a velocity is a vector quantity. We might travel at $26\,\mathrm{m/\,s}$ north. The $26\,\mathrm{m/\,s}$ is the magnitude of the vector. The "north" is the direction. We could draw an arrow to represent this.

Notice that we draw the arrow starting at the point (labeled P, for "position point"). This is always true for instantaneous vector values. We draw the vector starting at the point. The length of the arrow represents the $26\,\mathrm{m/\,s}$ The direction is given by the direction the arrow points. In this system for representing vectors, the magnitude *cannot be negative*. The magnitude is the amount of something, like speed. Negative speed does not make sense. We will only use positive values for magnitudes. But, you might say, I can write a velocity for a one-dimensional problem like $-26\,\mathrm{m/\,s}$. But remember the minus sign is the direction. In a one-dimensional problem, the minus sign means "to the left." So it is a direction. The $26\,\mathrm{m/\,s}$ is still a magnitude and it is still a positive value. When you put the $26\,\mathrm{m/\,s}$ together with the minus sign, then it is a vector that tells you that the object is traveling to the left at with a speed of $26\,\mathrm{m/\,s}$.

Our notion for the magnitude of a vector is a set of absolute value signs

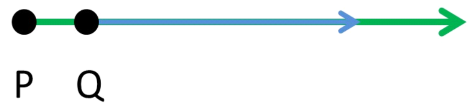
$$\text{magnitude of } \overrightarrow{\mathbf{v}} = \left|\overrightarrow{\mathbf{v}}\right|$$

but this is a lot to write, four individual characters required to make the symbol for a magnitude. So it is customary to write the magnitude of the vector with the same symbol as we use for the vector, but without the arrow.

$$\text{magnitude of } \overrightarrow{\mathbf{v}} = \left|\overrightarrow{\mathbf{v}}\right| = v$$

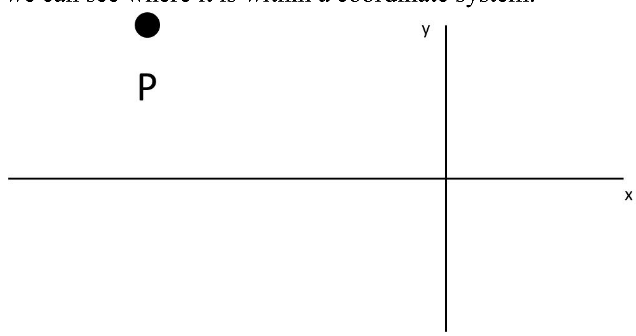
so $\overrightarrow{\mathbf{v}}$ is a vector and v is a magnitude. Now you see why v is the symbol for speed! Speed is the magnitude of a velocity. So naturally the symbol for speed is v.Note that in one dimensional problems being this careful with notation was not so important because we used the symbol for the vector as the magnitude and a plus or minus sign for the direction. But as we go to two and three dimensions, just using a plus or minus for direction won't work. So it pays to be careful and write $\overrightarrow{\mathbf{v}}$ for a vector and v for a magnitude.

It is *very important* to realize that both the magnitude and direction of an instantaneous vector quantity like velocity or acceleration represent values *at a single point, P.* The vector stretches to the right beyond the point, P, but the magnitude is the speed when the object was *at point P* and it does not tell us about any other point. For example, here is a drawing with the velocity show for a nearby point, Q drawn in blue so we can see it.

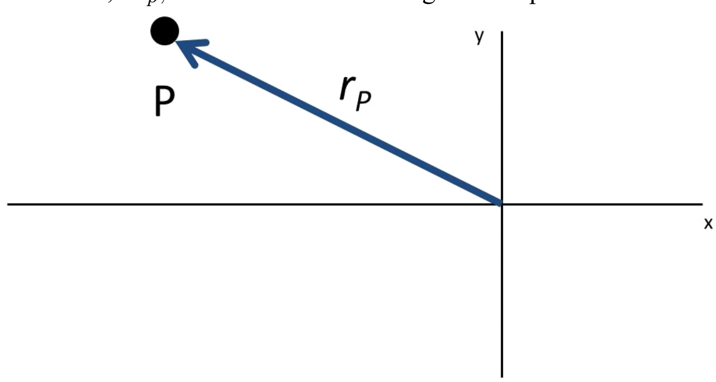


The green velocity vector from point P would completely cover the point Q and it's velocity vector! If we hadn't changed color, you might not know which arrow head went with which vector. Notice that the speed at Q is much less than it is at point P. The green vector $\overrightarrow{\mathbf{v}}_P$ does not represent the speed at Q. A vector will extend beyond the point, but it only gives us information about what happens at its single point.

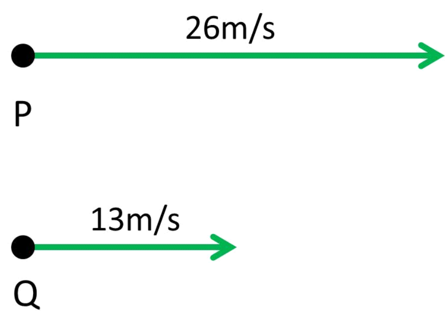
You might wonder how long to make the arrows for our drawings. For position vectors it is easy to know. A position vector must be as long as the displacement between a point and the origin. For example, here is our point P again, but this time with an origin drawn so we can see where it is within a coordinate system.



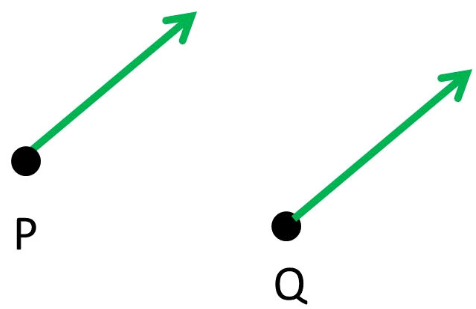
The position vector, $\overrightarrow{\mathbf{r}}_p$, must reach from the origin to the point P



This is different than the rule we just learned about velocities where the velocity vector would start at the point, P. But how long should the velocity vector be? Really we can choose any length we want for a velocity vector. Once we have chosen a length for a particular problem, we have to draw all other velocity vectors for that problem to the same scale. So, given our velocity at point P of $26\,\mathrm{m/s}$, a velocity of $13\,\mathrm{m}/s$ would have to be half as long.



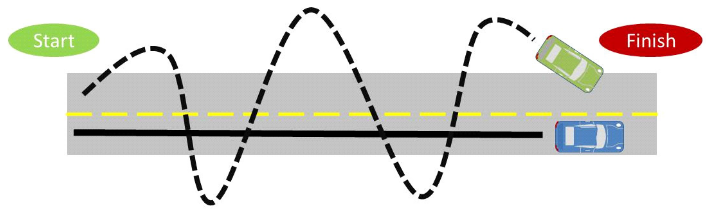
I is also important to realize that by “direction” we are talking about a direction like a compass direction. Both of the vectors in the next figure are pointing the same direction.

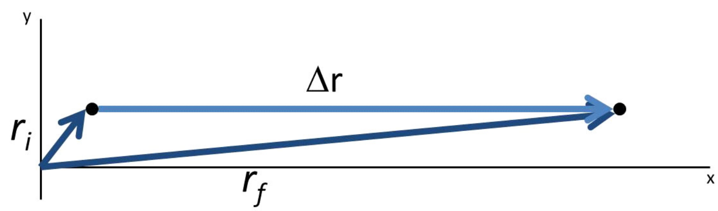


Notice that the object passing through P and the object passing through Q are going the same direction but will never meet. The will not even get closer together! Going the same compass direction is not “going to the same place.”

Of course there are quantities in physics that don’t have direction. Examples are mass and temperature. Mass does not have a direction at all. We don’t say “your mass is $75\,\mathrm{kg}$ due north.” That would make no sense. Temperature might seem to have a direction. We say it temperature is going up or going down. But this is not a compass direction. By this we only mean the temperature is rising or lowering, not that it is going north or south. Such quantities are called *scalar* quantities.

Let’s consider an example. Suppose we are again watching a race from above. One car travels the straight-line path from *Start* to *Finish.* Another travels the curved line. Both arrive after the same time interval Δt. Do both have the same displacement Δx? The answer is, yes.





We can see that vector displacement is equivalent to a straight line path, what we might colloquially call "as the crow flies." But vector displacement might not be the actual path taken by the object.

$$\Delta\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_f - \overrightarrow{\mathbf{r}_i}$$

only deals with the beginning and ending points.

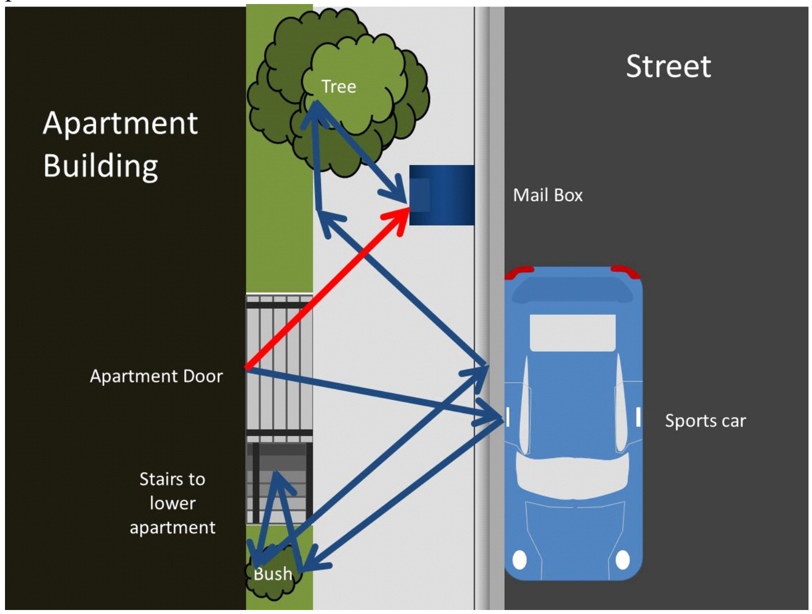
Vector Addition revisited

You have landed a nice job on campus grading papers for PH121 (after you get your "A" in this class). Suppose you agreed to work for $10.00 an hour. And most gradign jobs allow you to work about 10 hours per week.[4] Then in a month you would earn about $400. But when you get your pay check, will it be for $400 dollars? Of course you know the answer is "no." The $400 is called your "gross" pay. It is how much you earned. But you have to pay taxes, and the taxes, and insurance costs, and so forth are taken out of your paycheck before you ever see the money. You might get $230 dollars that actually comes to you. The $230 is called your "net" pay. I'm not sure where these words come from. I don't consider the larger amount to be "gross." I would like to have all that money to pay for food and rent, etc. But these are the words use by the financial people to describe our pay. And one of these words is useful in physics.

The word "net" is the part of the pay check you can do things with. The word "net" in physics is the part that actually mattered. Let's take our race car example again. All

[4] If you need more hours, you can grade for two classes.

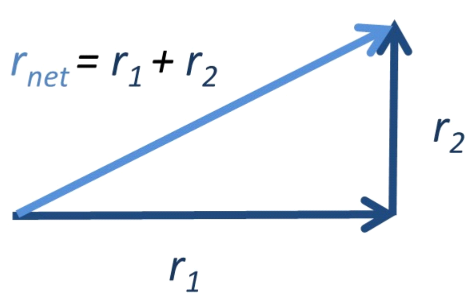
that mattered was how far the cars went toward the finish line. Not the actual distance traveled. Our displacement is a "net" value, because it is the part that actually mattered. Let's take another example. Suppose we send a child to a mail box to post a letter. Here is the path of the child in blue.



The path you wanted the child to travel is in red. The red arrow is the path that would get the job done. The part that actually mattered to you, the parent. It is the net displacement you wanted (get the letter to the box).

Notice that if we use vector addition to add up all the blue child displacements, we would end up with the red displacement! We will use the word "net" to mean "add vectors" in this class.

Our child going to the mailbox is fun to think about, but to learn the math of vector addition let's take on a simpler example.



and let's say that

$$\begin{aligned} r_1 &= 40\,\mathrm{m} \\ r_2 &= 30\,\mathrm{m} \end{aligned}$$

what would $\overrightarrow{r}_{net} = \overrightarrow{r}_1 + \overrightarrow{r}_2$ be? To answer this we need to find both the magnitude and the direction of $\overrightarrow{r}_{net}$. Let's start with the magnitude.

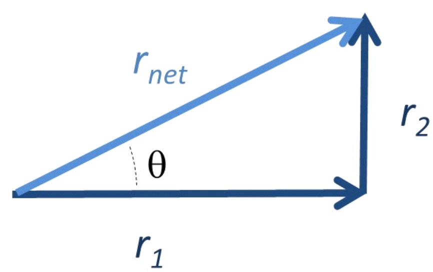
From our trigonometry experience we can see that finding the magnitude is an easy problem. Our position vectors form a right triangle. We use the Pythagorean theorem

$$r_{net} = \sqrt{r_1^2 + r_2^2}$$

to get

$$\begin{aligned} r_{net} &= \sqrt{(40\,\mathrm{m})^2 + (30\,\mathrm{m})^2} \\ &= 50\,\mathrm{m} \end{aligned}$$

Now we need to find the direction of $\overrightarrow{r}_{net}$. For our class, we will define the direction like a compass direction measured from the x-axis. We use Greek letters for angles, so our direction is the angle θ shown in the next figure.



Recalling our trig knowledge, we can see that

$$\tan\theta = \frac{\Delta r_2}{\Delta r_1}$$

so the direction θ would be

$$\theta = \tan^{-1}\left(\frac{r_2}{r_1}\right)$$

We can use this to solve for the direction in our example

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{30\,\mathrm{m}}{40\,\mathrm{m}}\right) \\ &= 0.643\,5\,\mathrm{rad} \\ &= 36.870\,^\circ \\ &= 37\,^\circ \end{aligned}$$

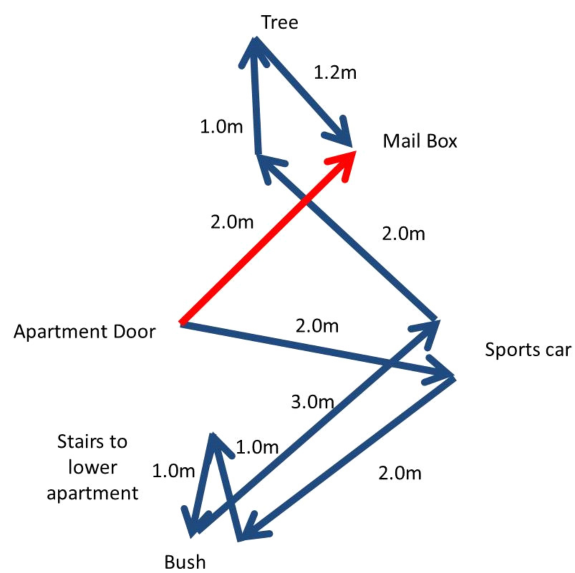
And we have our solution,

$$\overrightarrow{r}_{net} = 50\,\mathrm{m} \text{ at } 37\,^{\circ}$$

The answer **needs both parts**, magnitude and direction.

Of course, this problem would have been harder if our displacement $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ did not form a right angle. For such a problem, we could use the law of sines or law of cosines from trigonometry (*but we won't*, we are working our way toward a better way to do this!).

Let's go back to our child mailing a letter. Suppose the displacements involved are as shown in the next figure



What is the magnitude of the velocity of the child if the trip took $10\,\mathrm{min}$? Velocity uses net displacement. It only cares about what got the job done. So only our starting and ending positions matter. Then

$$\overrightarrow{r}_{net} = \overrightarrow{r}_{mailbox} - \overrightarrow{r}_{door}$$

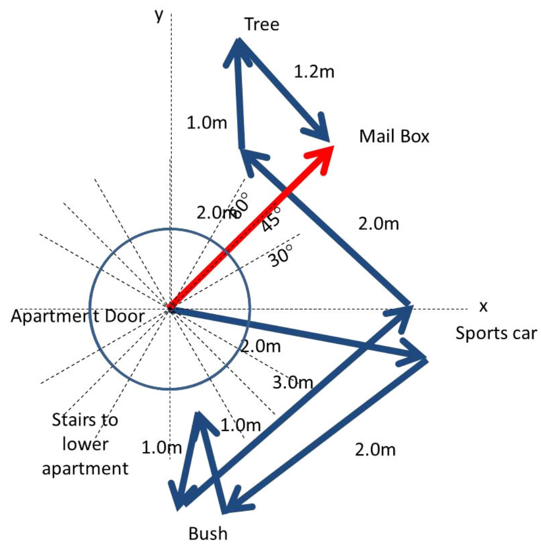
and according to the figure, $\Delta r_{net} = 2.0\,\mathrm{m}$. The duration Δt is $10\,\mathrm{min} = 600.0\,\mathrm{s}$. So the magnitude of the velocity is

$$v = \frac{2.0\,\mathrm{m}}{600.0\,\mathrm{s}} = 3.3 \times 10^{-3}\,\frac{\mathrm{m}}{\mathrm{s}}$$

which is small, but considering all the stops the child actually made, it is reasonable.

Direction would be harder, needing us to actually add up all the vector displacements.

Or we could use a compass or protractor to measure the direction. Let's do the latter.



It looks like $\overrightarrow{r}_{net}$ has a direction of about $45\,^{\circ}$.

It would be nice if there were a better way to find the magnitude and direction of the net displacement without resorting to quite so much trigonometry or giving up and doing a measurement. And our right triangle gives us an idea of what that better way might be.

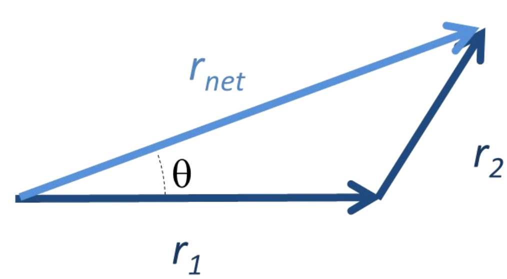
8 Components of vectors

We have been using vectors for some time now. And we know how to add and subtract vectors graphically. In a previous problem we were able to use math to find the magnitude and direction of the net displacement, but that was a special case because the displacements $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ formed a right angle. But in our mailbox problem, this was not the case. It would have been nice if our child had only made turns at right angles. But it is not realistic to require that of all moving objects (especially a child).

It would be nice if there were a way to turn a hard problem like the child's displacement into an easier problem like the right angle displacements.

And there is....

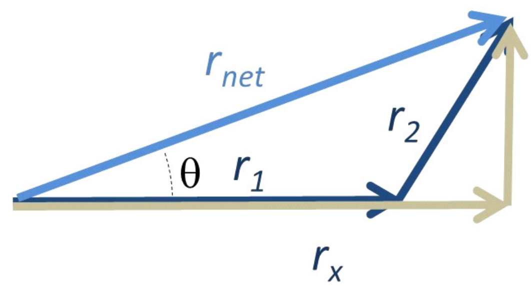
We will work our way up to making our displacement problems all easier using a clever trick, one we learned about with our sledder a few lectures ago. It will be easier to see how this works if we have a problem to work on, so here is one that is simpler than our child's displacement.



We wish to find

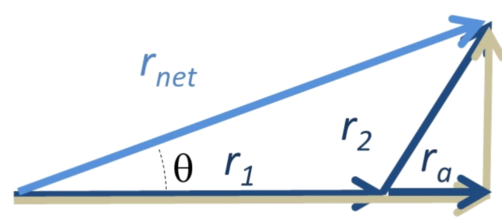
$$\overrightarrow{r}_{net} = \overrightarrow{r}_1 + \overrightarrow{r}_2$$

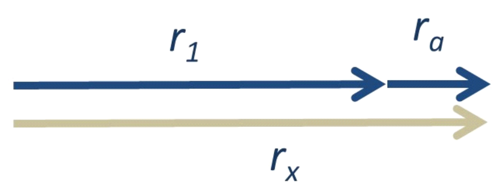
where $\overrightarrow{r}_1 = 40\,\mathrm{m}$ and now $\overrightarrow{r}_2 = 35\,\mathrm{m}$. The angle between $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ must now be larger than $90\,°$. And our intuition tells us that $\overrightarrow{r}_{net}$ must be larger in this case than it was in our previous problem. But we know that $\overrightarrow{r}_{net}$ could come from many different paths, and $\overrightarrow{r}_{net}$ only depends on the stopping and starting points. So suppose that we had two new vectors $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$ that also add up to $\overrightarrow{r}_{net}$. And suppose $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$ form a right angle. Then if we knew $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$ we could use the Pythagorean theorem and tangent function just like in our previous problem.



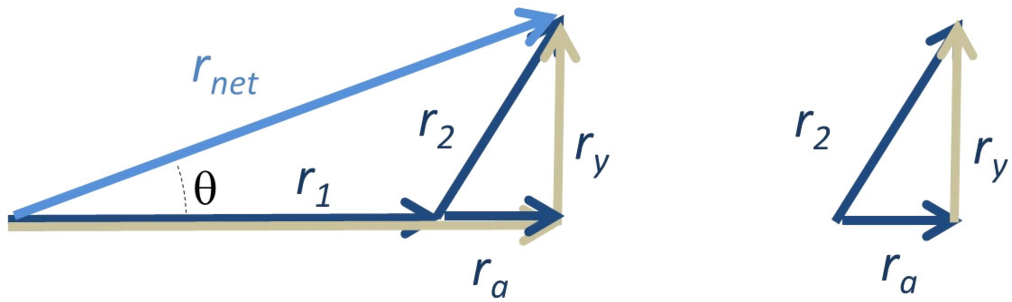
The only problem is that we don't know the length of $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$. But notice that $\overrightarrow{r}_x = \overrightarrow{r}_1 +$ a little bit more. Let's call the little bit more $\overrightarrow{r}_a$ so then

$$\overrightarrow{r}_x = \overrightarrow{r}_1 + \overrightarrow{r}_a$$





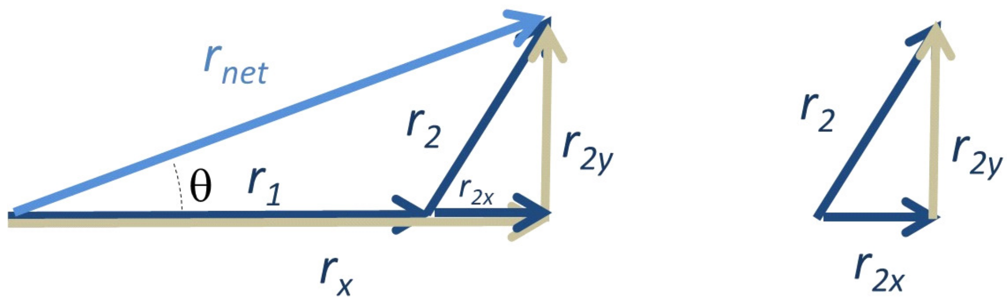
And further notice that there is a relationship between $\overrightarrow{r}_2$, $\overrightarrow{r}_y$, and $\overrightarrow{r}_a$



they form a right triangle. Then we can say that

$$\overrightarrow{r}_2 = \overrightarrow{r}_a + \overrightarrow{r}_y$$

We have broken $\overrightarrow{r}_2$ into two parts $\overrightarrow{r}_a$, and $\overrightarrow{r}_y$ that form a right angle. Since we can view $\overrightarrow{r}_a$, and $\overrightarrow{r}_y$ like parts of $\overrightarrow{r}_2$, let's relabel them



We will call $\overrightarrow{r}_a = \overrightarrow{r}_{2x}$ so we know it is part of $\overrightarrow{r}_2$. The x is appropriate because $\overrightarrow{r}_{2x}$ is along the x-axis. We will also call $\overrightarrow{r}_y = \overrightarrow{r}_{2y}$ because it is part of $\overrightarrow{r}_2$ but it is in the y-direction.

Now we can rewrite our solution for $\overrightarrow{r}_{net}$. Before we had

$$\overrightarrow{r}_{net} = \overrightarrow{r}_1 + \overrightarrow{r}_2$$

but now we can substitute $\overrightarrow{r}_{2x} + \overrightarrow{r}_{2y}$ in for $\overrightarrow{r}_2$

$$\overrightarrow{r}_{net} = \overrightarrow{r}_1 + \overrightarrow{r}_{2x} + \overrightarrow{r}_{2y}$$

We could even separate this into two parts, a part along the x-axis and a part along the y-axis.

$$\overrightarrow{r}_{net} = (\overrightarrow{r}_1 + \overrightarrow{r}_{2x}) + (\overrightarrow{r}_{2y})$$

and we could name the parts

$$\overrightarrow{r}_{net_x} = \overrightarrow{r}_1 + \overrightarrow{r}_{2x}$$
$$\overrightarrow{r}_{net_y} = \overrightarrow{r}_{2y}$$

and if you have been followng carefully you will recognize that

$$\begin{aligned}\vec{r}_{net_x} &= \vec{r}_x \\ \vec{r}_{net_y} &= \vec{r}_y\end{aligned}$$

where the "net" just tells us we are considering how far we actually got in the x and y-directions. So we have found a new way to write our two vectors that make the problem $\vec{r}_{net} = \vec{r}_1 + \vec{r}_2$ easier by turning it into a right triangle problem!. Now all we have to do is use the Pythagorean theorem to find r_{net}

$$\begin{aligned}r_{net} &= \sqrt{r_{net_x}^2 + r_{net_y}^2} \\ &= \sqrt{(r_1 + r_{2x})^2 + r_{net_y}^2}\end{aligned}$$

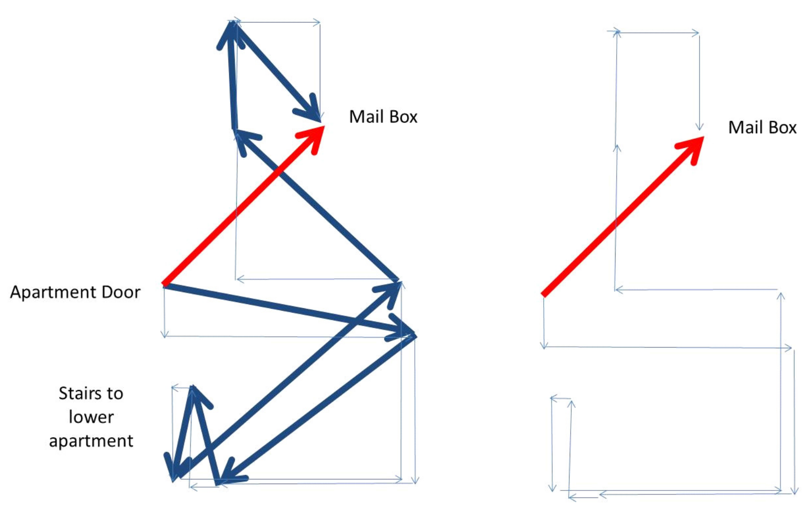
and the direction is given by

$$\theta = \tan^{-1}\left(\frac{r_{net_y}}{r_{net_x}}\right) = \tan^{-1}\left(\frac{r_{net_y}}{r_1 + r_{2x}}\right) == \tan^{-1}\left(\frac{r_{2_y}}{r_1 + r_{2x}}\right)$$

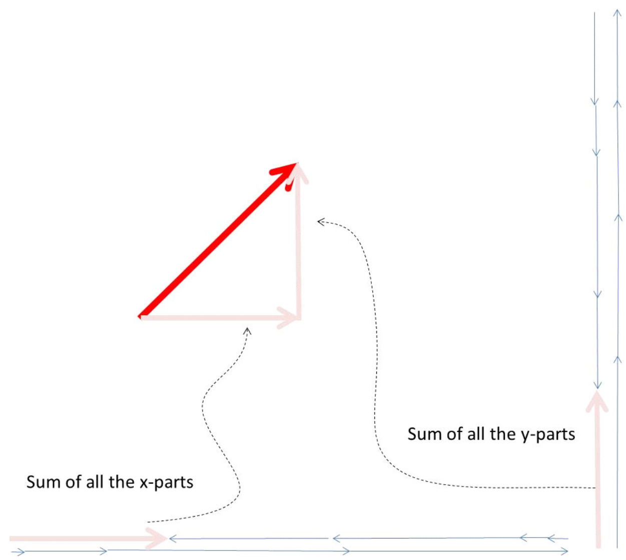
No matter how complicated the situation, we can separate every vector that is not along an axis into parts that are along an axis, and then all we have to do is to add up all the parts that are along each axis to find the net displacement.

Let's take on the child displacement case again!

None of the child's displacements are along an axis, so we need to find x-parts and y-parts for every vector.



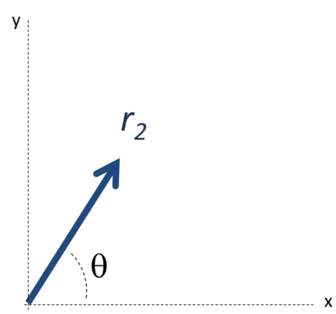
Once we have the parts, we add up all the x-parts separately, treating them like vectors so they go head to tail.



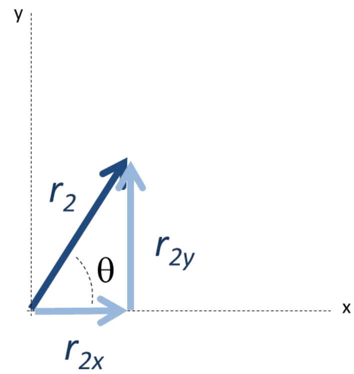
We also add up all the y-parts separately, placing them head to tail. The two resulting vectors are the x-part and the y-part of the net displacement (honest, they really worked!).

But you might have noticed that there is a part missing. We know how to find the x-part and the y-part of vectors graphically, just make right triangles. But we need to know how to mathematically find actual magnitudes for these x and y-parts. Fortunately, we know a little trigonometry! Since all our new vector-part triangles are right triangles, we know how to find the side lengths!

Suppose again that I have the following vector.



where I know from measurement, if nothing else, that $\theta = 60\,^{\circ} = 1.047\,2\,\mathrm{rad}$. Then if I make a right triangle with x and y-parts of the vector



and using trig we know that

$$\cos\theta = \frac{r_{2x}}{r_2}$$

so

$$r_{2x} = r_2 \cos\theta$$

To find the x-part, all we have to do is multiply the magnitude of the vector, $\vec{r}_2$ by the cosine of the angle it makes with the x-axis! So in our case, the x-part is just

$$\begin{aligned} r_{2x} &= 35\,\mathrm{m}\cos 60\,^{\circ} \\ &= 17.5\,\mathrm{m} \end{aligned}$$

Similarly, to find the y-part notice that

$$\sin\theta = \frac{r_{2y}}{r_2}$$

so that

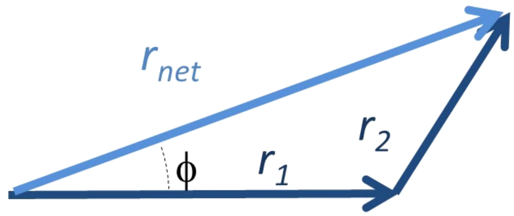
$$r_{2y} = r_2 \sin\theta$$

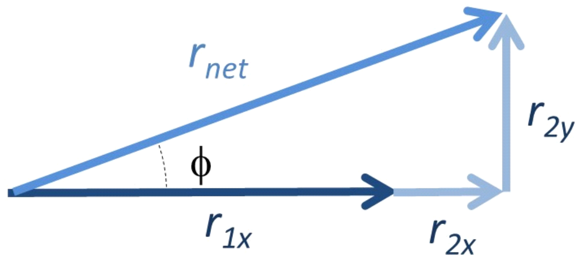
and so all we have to do to find the y-part is to multiply the magnitude of the vector, $\vec{r}_2$ by the sine of the angle it makes with the x-axis. In our case this gives

$$\begin{aligned} r_{2y} &= 35\,\mathrm{m}\sin 60\,^{\circ} \\ &= 30.311\,\mathrm{m} \end{aligned}$$

Armed with this we can do the problem we suggested so long ago.

Suppose we have two displacements $r_1 = 40\,\mathrm{m}$ along the x-axis and $r_2 = 35\,\mathrm{m}$ at an angle $\theta = 60\,^{\circ}$. What is the net displacement.





Our procedure is to divide up $\vec{r}_1$ and $\vec{r}_2$ into x and y-parts, add up the x and y-parts, and then use these to find the magnitude of $\vec{r}_{net}$.

We first find the x-parts of both vectors. Let's start with $\vec{r}_1$

$$\begin{aligned} \Delta r_{1x} &= \Delta r_1 \cos(0) \\ &= \Delta r_1 \\ &= 40\,\mathrm{m} \end{aligned}$$

this is because $\vec{r}_1$ lies right on the x-axis, so the angle from the x-axis is zero. which makes sense since $\vec{r}_1$ is along the x-axis. Now let's do the x-part of $\vec{r}_2$

$$\begin{aligned} r_{2x} &= r_2 \cos(\theta) \\ &= 35\,\mathrm{m} \cos(60\,^\circ) \\ &= 17.5\,\mathrm{m} \end{aligned}$$

We did the x-parts. Now let's find the y-parts. For $\vec{r}_1$

$$\begin{aligned} r_{1x} &= r_1 \sin(0) \\ &= 0 \end{aligned}$$

which makes sense if $\vec{r}_1$ is all along the x-axis. It shouldn't have a y-part. And the

y-part of $\overrightarrow{r}_2$ is

$$
\begin{aligned}
r_{2y} &= r_2 \sin(\theta) \\
&= 35\,\mathrm{m} \sin(60\,^\circ) \\
&= 30.311\,\mathrm{m}
\end{aligned}
$$

To summarize:

$$
\begin{aligned}
r_{1x} &= 40\,\mathrm{m} \\
r_{2x} &= 17.5\,\mathrm{m} \\
r_{1x} &= 0\,\mathrm{m} \\
r_{2y} &= 30.311\,\mathrm{m}
\end{aligned}
$$

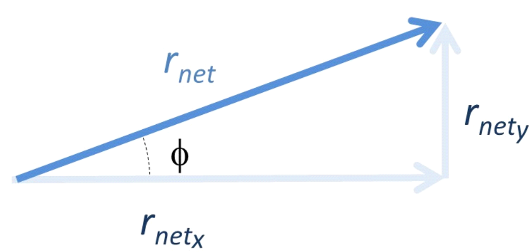
Next we add up all the x-parts

$$
\begin{aligned}
r_{net_x} &= r_{1x} + r_{2x} \\
&= 40\,\mathrm{m} + 17.5\,\mathrm{m} \\
&= 57.5\,\mathrm{m}
\end{aligned}
$$

and we add up all the y-parts

$$
\begin{aligned}
r_{net_y} &= r_{1y} + r_{2y} \\
&= 0 + 30.311\,\mathrm{m} \\
&= 30.311\,\mathrm{m}
\end{aligned}
$$

and now we have the sides for our final right triangle



and we can find r_{net} using the Pythagorean theorem

$$
\begin{aligned}
r_{net} &= \sqrt{\left(r_{net_x}\right)^2 + \left(r_{net_y}\right)^2} \\
&= \sqrt{(57.5\,\mathrm{m})^2 + (30.311\,\mathrm{m})^2} \\
&= 65.0\,\mathrm{m}
\end{aligned}
$$

But we are not done! r_{net} is a vector so we need a direction. Since we used the Greek letter θ already, I called the $\overrightarrow{r}_{net}$ direction ϕ. So, using our trig knowledge

$$\tan\phi = \frac{r_{net_y}}{r_{net_x}}$$

so

$$\begin{aligned}\phi &= \tan^{-1}\left(\frac{r_{net_y}}{r_{net_x}}\right) \\ &= \tan^{-1}\left(\frac{30.311\,\mathrm{m}}{57.5\,\mathrm{m}}\right) \\ &= 0.485\,13\,\mathrm{rad} \\ &= 27.796\,^\circ \\ &= 27\,^\circ\end{aligned}$$

What we have done is deeply profound. We now have an easy way to find the summation of any number of vectors. The steps are as follows:

1. We convert every vector in the sum into x and y-parts using
$$\begin{aligned}v_x &= v\cos\theta \\ v_y &= v\sin\theta\end{aligned}$$
where θ is usually different for every vector.

2. Then when we have the x and y-parts, we sum up each set (x or y) separately to find v_{net_x} and v_{net_y}.

3. We then use the Pythagorean theorem
$$v_{net} = \sqrt{v_{net_x}^2 + v_{net_y}^2}$$
and the inverse tangent
$$\phi = \tan^{-1}\left(\frac{v_{net_y}}{v_{net_x}}\right)$$
to convert the sums, v_{net_x} and v_{net_y} into a magnitude and direction for the net vector.

What we have done is to turn a two-dimensional problem into two easy one-dimensional problems, and then recombine the two one-dimensional results at the end to make it a two-dimensional problem again. We break our hard two-dimensional problem into parts that we know how to do!

We will do this over and over again in this class and in PH123 and in PH 220 and forever if you are a physics major or an engineer!

The name "x-part" is not very fancy. So let's give the x and y-parts of vectors a new name. We call the parts of vectors *components* of the vectors. So the x-part is called the

x-component of the vector and the y-part is called the y-component of the vector.

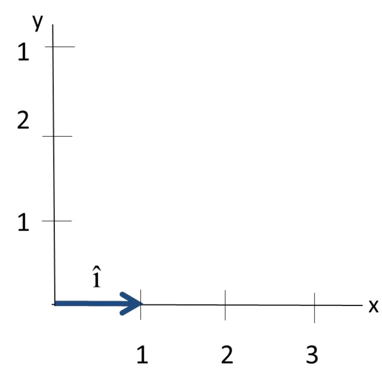
We are now all set to do motion problems in two dimensions.

Vectors in component form

After last lecture, you might have wondered if it wouldn't be just as well to give the components of the net vector and dispense with the magnitude and direction. And indeed, this is a perfectly fine way to express a vector. But we need a little bit of notation to help with this.

Unit vectors

To help with expressing a vector as components, consider the following vector.

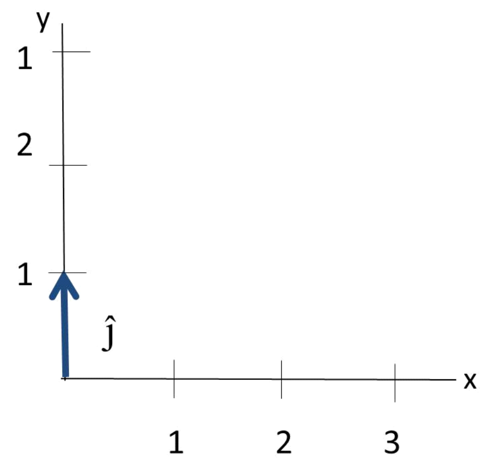


It has a magnitude of exactly 1 with no units. We give it a name, $\hat{\imath}$, pronounced "eye hat." It might sound like a restaurant name, but it is a useful little vector. Using this new vector we could write the x-component of a vector as

$$\overrightarrow{r}_x = r_x \hat{\imath}$$

The first part, r_x is the magnitude of the vector. The second part, our $\hat{\imath}$, has a direction, and it is along the x-axis. It also has a magnitude of 1, but multiplying anything by 1 does not change a value. So the product $r_x\hat{\imath}$ has a magnitude of r_x and a direction along the x-axis. By looking at $r_x\hat{\imath}$ you know the magnitude and direction of the component.

Likewise there is another useful vector, $\hat{\jmath}$



that also has a magnitude of 1 with no units, but it points in the y-direction. So we could write the y-component of a vector as

$$\overrightarrow{r}_y = r_y\hat{\jmath}$$

Although this makes sense, the beauty of $\hat{\imath}$ and $\hat{\jmath}$ may not yet be apparent. Let's write out $\overrightarrow{r}$ as the sum of $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$

$$\overrightarrow{r} = \overrightarrow{r}_x + \overrightarrow{r}_y$$

but we have new expressions for $\overrightarrow{r}_x$ and $\overrightarrow{r}_y$ in terms of $\hat{\imath}$ and $\hat{\jmath}$, so let's substitute them in

$$\overrightarrow{r} = r_x\hat{\imath} + r_y\hat{\jmath}$$

This completely defines a vector. And it is nice because you can clearly see that a 2-dimensional vector can be thought of as a sum of two 1-dimensional vectors. But it still has both the magnitude

$$r_{net} = \sqrt{\left(r_{net_x}\right)^2 + \left(r_{net_y}\right)^2}$$

and the direction

$$\phi = \tan^{-1}\left(\frac{r_y}{r_x}\right)$$

contained within it. Sometimes it is much easier to express a vector as a sum of it's components.

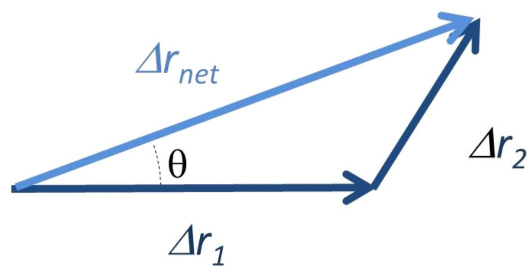
Let's take last time's example and write it in component form.

We had two vectors,

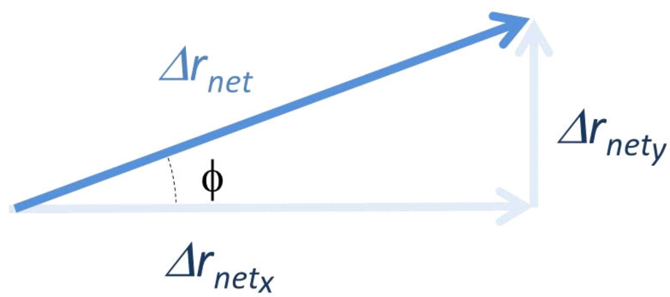
$$\overrightarrow{r}_1 = 40\,\mathrm{m}\angle 0^\circ$$

where $\angle$ is the symbol for "at the angle," and

$$\overrightarrow{r}_1 = 35\,\mathrm{m}\angle 60^\circ$$



and we turned this into two easier vectors $\overrightarrow{r}_{net_x}$ and $\overrightarrow{r}_{net_y}$



and found the magnitude and direction as

$$\overrightarrow{r}_{net} = 65.0\,\mathrm{m}\angle 27^\circ$$

We got this by finding the x and y-components of $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$

$$\begin{aligned} r_{1x} &= r_1 \cos(0) \\ &= r_1 \\ &= 40\,\mathrm{m} \\ r_{1y} &= r_1 \sin(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \Delta r_{2x} &= \Delta r_2 \cos(\theta) \\ &= 35\,\mathrm{m}\cos(60^\circ) \\ &= 17.5\,\mathrm{m} \end{aligned}$$

and finally

$$\begin{aligned} r_{2y} &= r_2 \sin(\theta) \\ &= 35\,\mathrm{m} \sin(60\,^\circ) \\ &= 30.311\,\mathrm{m} \end{aligned}$$

Then we added up all the x-parts

$$\begin{aligned} r_{net_x} &= r_{1x} + r_{2x} \\ &= 40\,\mathrm{m} + 17.5\,\mathrm{m} \\ &= 57.5\,\mathrm{m} \end{aligned}$$

and we added up all the y-parts

$$\begin{aligned} r_{net_y} &= r_{1y} + r_{2y} \\ &= 0 + 30.311\,\mathrm{m} \\ &= 30.311\,\mathrm{m} \end{aligned}$$

But now we see that the result could be written as

$$\begin{aligned} \overrightarrow{r}_{net} &= (r_{1x} + r_{2x})\,\hat{\imath} + (r_{1y} + r_{2y})\,\hat{\jmath} \\ &= 57.5\,\mathrm{m}\hat{\imath} + 30.311\,\mathrm{m}\hat{\jmath} \end{aligned}$$

Last lecture we found the magnitude and the direction for $\overrightarrow{r}_{net}$, we could have just left $\overrightarrow{r}_{net}$ in this new *component form* using unit vectors.

Notice that what we have done is very profound. We have written our original vectors in component form, then added up all the x-components and the y-components separately, and reported the sums as the components of the net vector. We have effectively turned a two-dimensional problem into two one-dimensional problems. This is huge, because we know how to do one dimensional problems already! By using components, we can turn a new, complicated kind of problem into two easy problems that we know how to do. We will often do this in our class. So it might be good to do another example.

Let's add the following two displacements

$$\begin{aligned} \overrightarrow{r}_1 &= 25\,\mathrm{m} \text{ at } 10\,^\circ \\ \overrightarrow{r}_2 &= 30\,\mathrm{m} \text{ at } 20\,^\circ \end{aligned}$$

To do this problem we need to take components of both vectors, but now we know how to do this:

$$\begin{aligned}
r_{1x} &= 25\,\mathrm{m}\cos\left(10\,^{\circ}\right) \\
&= 24.62\,\mathrm{m} \\
r_{1y} &= 25\,\mathrm{m}\sin\left(10\,^{\circ}\right) \\
&= 4.341\,2\,\mathrm{m} \\
r_{2x} &= 30\,\mathrm{m}\cos\left(20\,^{\circ}\right) \\
&= 28.191\,\mathrm{m} \\
r_{2y} &= 30\,\mathrm{m}\sin\left(20\,^{\circ}\right) \\
&= 10.261\,\mathrm{m}
\end{aligned}$$

and to find the net vector, we add up all the x-components

$$r_{net_x} = 24.62\,\mathrm{m} + 28.191\,\mathrm{m} = 52.811\,\mathrm{m}$$

and we add up all the y-components

$$r_{net_y} = 4.341\,2\,\mathrm{m} + 10.261\,\mathrm{m} = 14.602\,\mathrm{m}$$

Then we can write the net displacement as vector in component form

$$\overrightarrow{r}_{net} = 52.811\,\mathrm{m}\hat{\imath} + 14.602\,\mathrm{m}\hat{\jmath}$$

Notice that $\overrightarrow{r}_{net}$ has a vector sign. It *is* a vector, not just a magnitude, because we have expressed $\overrightarrow{r}_{net}$ in terms of a sum of terms with unit vectors. So it must be a vector. Of course we could find the magnitude and direction as well

$$\begin{aligned}
r_{net} &= \sqrt{r_{net_x}^2 + r_{net_y^2}} \\
&= \sqrt{\left(52.811\,\mathrm{m}\right)^2 + \left(14.602\,\mathrm{m}\right)^2} \\
&= 54.793\,\mathrm{m} \\
&= 54.8\,\mathrm{m}
\end{aligned}$$

and

$$\begin{aligned}
\phi &= \tan^{-1}\left(\frac{r_{net_y}}{r_{net_x}}\right) \\
&= \tan^{-1}\left(\frac{14.602}{52.811}\right) \\
&= 0.269\,76\,\mathrm{rad} \\
&= 15.456\,^{\circ} \\
&= 15.5\,^{\circ}
\end{aligned}$$

So we could write $\overrightarrow{r}_{net}$ as

$$\overrightarrow{r}_{net} = 54.8\,\mathrm{m}\angle 15.5\,^{\circ}$$

but the component form is just as good. Often we prefer the magnitude and direction form of the vector, because it is easy for us humans to interpret. But both forms are equally valid,

9 Two-Dimensional Acceleration 4.2

We have learned how things move in one dimension. We called the study of motion with constant acceleration Kinematics. So we know the Kinematics of one-dimensional motion. But if you have played futbol[5] you know that we can have motion in more than one dimension. We also have studied vectors. With the powerful mathematical notion of vectors, it is time to see how we can study motion in more than one dimension. We will start with two dimensions, but the extension to three or more dimensions is trivial[6] once we understand two dimensions.

Acceleration in Two Dimensions

We learned in one-dimensional kinematics that acceleration was very important for our understanding of motion. It is reasonable to assume that it will be just as important for our understanding of two-dimensional motion. So let's start by reviewing what we know about acceleration. Our equation for acceleration is

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

but now that we know more about vectors, we should ask, is acceleration really a vector quantity?

Acceleration has a magnitude, we know that. it matters whether the acceleration is in the same direction as the velocity or not.

[5] Americans call this soccer, but if you have played American football, motion in more than one dimension is just as important.

[6] To physicists, "trivial" means that once you have slogged through the hard work you are doing, you should be able to see how to do a little more without it being so much work to understand the little bit more.

So acceleration *does* have a direction . Acceleration must be a vector. We can write our equation for average acceleration as

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

But this acceleration vector seems different than a position vector. To get a feeling for what an acceleration vector means, let's practice our vector principles in one-dimension with acceleration. Suppose we have an object moving and the object's velocity is given by the motion diagram.

Let's say that the initial speed is $2\,\mathrm{m/\,s}$ and the final speed is $6\,\mathrm{m/\,s}$ and let's say that one second has transpired between our initial and final states, $\Delta t = 1\,\mathrm{s}$. What is the acceleration? We would take

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{6\frac{\mathrm{m}}{\mathrm{s}} - 2\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}} = 4\frac{\mathrm{m}}{\mathrm{s}^2}$$

and since the answer is positive we would say that the acceleration is to the right.

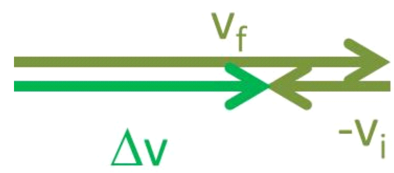
But we can do this graphically too. Recall that

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

and we know that

$$\Delta\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}_f} - \overrightarrow{\mathbf{v}_i}$$

and we even know how to do this graphically. We take $\overrightarrow{\mathbf{v}_i}$ and we turn it around to make it $-\overrightarrow{\mathbf{v}_i}$ and we add the result,$-\overrightarrow{\mathbf{v}_i}$, to $\overrightarrow{\mathbf{v}_f}$ by placing the tail of $-\overrightarrow{\mathbf{v}_i}$ on the tip of $\overrightarrow{\mathbf{v}_f}$. Then we draw a vector from the tail of $\overrightarrow{\mathbf{v}_f}$ to the tip of $-\overrightarrow{\mathbf{v}_i}$. It might look a little like this



We could measure out the lengths so that we have v_f has a length of 6 and v_i with a length of 2. The we could measure $\Delta\overrightarrow{\mathbf{v}}$ to have a length of 4. But that is a lot of work to do all the measuring. And since we know the magnitude of $\overrightarrow{\mathbf{v}}_f$ and $\overrightarrow{\mathbf{v}}_i$, it is easier to do the math than to draw the picture and measure. We can say that

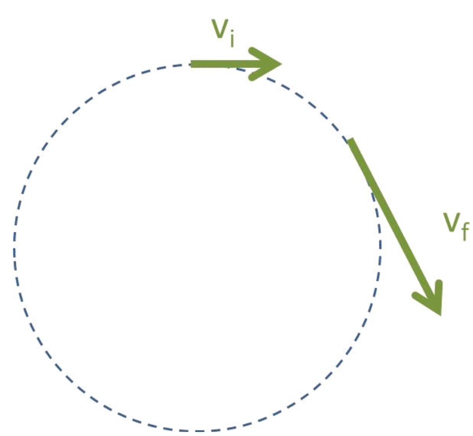
$$\overrightarrow{\mathbf{a}}_{ave} = \frac{4\,\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}} = 4\frac{\mathrm{m}}{\mathrm{s}^2}$$

and none of this is surprising. This is really just what we have been doing all along. But we have to give a direction as well as a magnitude. And we can use our new unit vectors to provide the direction.

$$\overrightarrow{\mathbf{a}}_{ave} = 4\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath}$$

The drawing does remind us of the need for a direction.

You may think that all we have done is complicate things with our new notation, but now let's do a problem in two dimensions. Let's consider an object that is moving in a circle. We can use the same magnitudes for v_i and v_f as before.

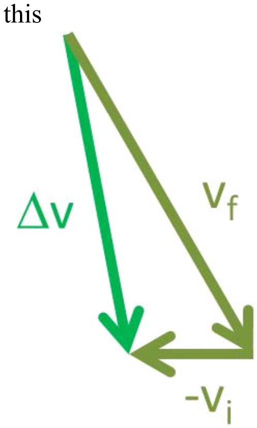


$v_i = 2\,\mathrm{m/\,s}$ and $v_f = 6\,\mathrm{m/\,s}$ with the same $\Delta t = 1\,\mathrm{s}$. But now the v_f has an angle of $\theta_f = -55\,^\circ$ so the vectors are pointing in different directions in two dimensions.

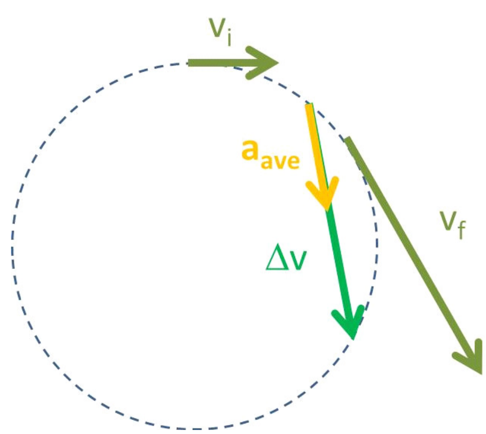
Still, we will do just the same thing we did before. Our acceleration will still be given by

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

where $\Delta\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}_f - \overrightarrow{\mathbf{v}}_i$. but now we need to add $\overrightarrow{\mathbf{v}}_f$ and $-\overrightarrow{\mathbf{v}}_i$ as vectors. To do this we take $\overrightarrow{\mathbf{v}}_i$ and we turn it around to make it $-\overrightarrow{\mathbf{v}}_i$ and then add the result to $\overrightarrow{\mathbf{v}}_f$ by placing the tail of $-\overrightarrow{\mathbf{v}}_i$ on the tip of $\overrightarrow{\mathbf{v}}_f$. Then we draw a vector from the tail of $\overrightarrow{\mathbf{v}}_f$ to the tip of $-\overrightarrow{\mathbf{v}}_i$. It might look a little like this

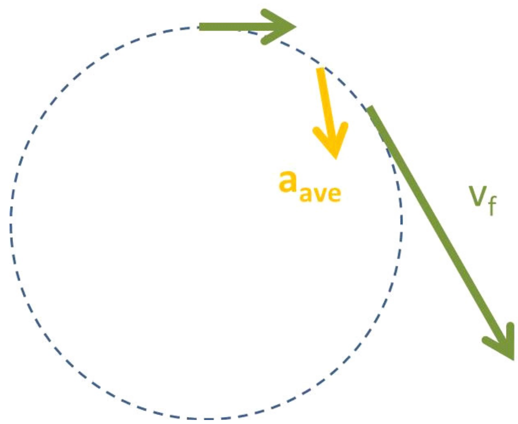


Notice that I recopied my vectors. I did not do the vector addition on the original diagram. This makes it much easier to do the vector addition. Once we know the length and direction of $\Delta\overrightarrow{\mathbf{v}}$, we divide by Δt and we have our acceleration. Note that Δt does not have a direction. The direction of our acceleration must come from $\Delta\overrightarrow{\mathbf{v}}$. If we draw our $\Delta\overrightarrow{\mathbf{v}}$ and our $\overrightarrow{\mathbf{a}}_{ave}$ on our original diagram, it is easy to see they point the same direction.



Usually we just place $\overrightarrow{\mathbf{a}}_{ave}$ on our diagram in between v_i and v_f along the trajectory.

That is because it is an average value.



But what do we do about the magnitude of the vector $\overrightarrow{\mathbf{a}}_{ave}$? In one dimension I could just add $v_f - v_i$ and divide by Δt. But this won't work here. Look at the length Δv compared to the length $v_{f.}$ They are nearly the same. This is really different than when they were all in the same dimension.

Working in components

The answer to our dilemma from the last section is to do our math in components of the vectors. Think, back to our one-dimensional problem. All we had to do was just add up the numbers to get Δv (remember one was negative, so we added a negative to subtract). Wouldn't it be great if we could reduce our difficult two-dimensional problem to two one-dimensional problems? Then we would know how to do the problem, and it would be easy!

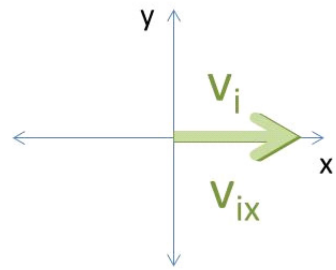
And that is just what we are going to do. suppose we have

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta \overrightarrow{\mathbf{v}}}{\Delta t}$$

and

$$\Delta \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}_f - \overrightarrow{\mathbf{v}}_i$$

as before. But we could make vector components for $\overrightarrow{\mathbf{v}}_f$ and $\overrightarrow{\mathbf{v}}_i$. Our vector $\overrightarrow{\mathbf{v}}_i$ is shown below.



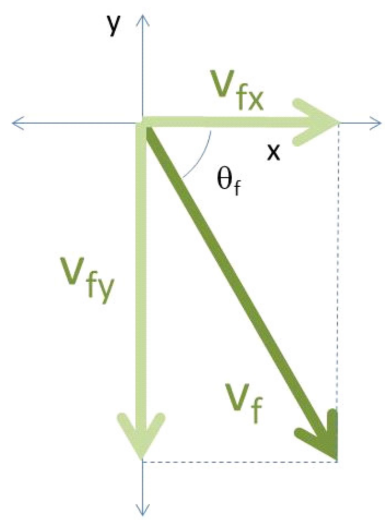
We can use our basic equations for taking components of vectors to find the components of $\overrightarrow{\mathbf{v}_i}$

$$\begin{aligned} v_{ix} &= v_i \cos\theta_i \\ v_{iy} &= v_i \sin\theta_i \end{aligned}$$

but $\theta_i = 0$ (see the diagram) because $\overrightarrow{\mathbf{v}_i}$ is all in the x-direction so

$$\begin{aligned} v_{ix} &= v_i \cos(0) = 2\frac{\mathrm{m}}{\mathrm{s}} \\ v_{iy} &= v_i \sin(0) = 0 \end{aligned}$$

Now let's do the same process for $\overrightarrow{\mathbf{v}_f}$

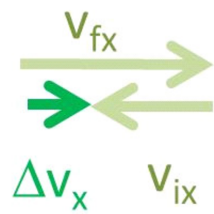


$$\begin{aligned} v_{fx} &= v_f \cos\theta_f \\ v_{fy} &= v_f \sin\theta_f \end{aligned}$$

and this one is harder because θ_f is not zero. Suppose $\theta_f = -55\,^{\circ}$ then

$$\begin{aligned} v_{fx} &= v_f \cos\theta_f = 6\frac{\mathrm{m}}{\mathrm{s}}\cos\left(-55\,^{\circ}\right) = 3.441\,5\frac{\mathrm{m}}{\mathrm{s}} \\ v_{fy} &= v_f \sin\theta_f = 6\frac{\mathrm{m}}{\mathrm{s}}\sin\left(-55\,^{\circ}\right) = -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

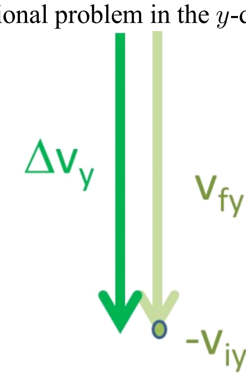
and now we can complete our two one-dimensional problems. First for the x-direction we now have the one-dimensional problem that looks like this:



and this is just the type of one-dimensional problem we did before! We know how to do it.

$$\begin{aligned} \Delta v_x &= v_{fx} - v_{ix} \\ &= 3.441\,5\frac{\mathrm{m}}{\mathrm{s}} - 2\frac{\mathrm{m}}{\mathrm{s}} \\ &= 1.441\,5\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

But we also have a one-dimensional problem in the y-direction. It looks like this:



It might be verticle, but it is just like our x-direction problem. We just take the difference between v_{fy} and v_{fi}

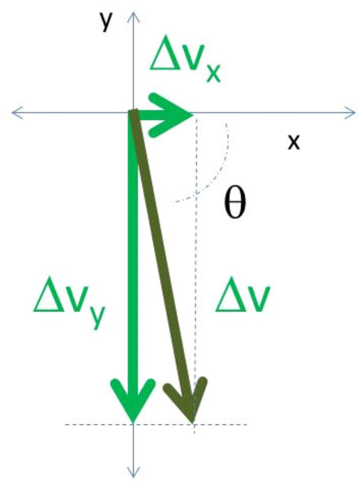
$$\begin{aligned} \Delta v_y &= v_{fy} - v_{iy} \\ &= -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} - 0\frac{\mathrm{m}}{\mathrm{s}} \\ &= -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

And now we have Δv_x and Δv_y! But really, we wanted $\Delta\overrightarrow{\mathbf{v}}$. We need to combine our x and y-component solutions into a solution for the two-dimensional problem. And we know there are two ways we could write it. We could give components and unit vectors, or magnitude and direction. Let's do both. The first is easy

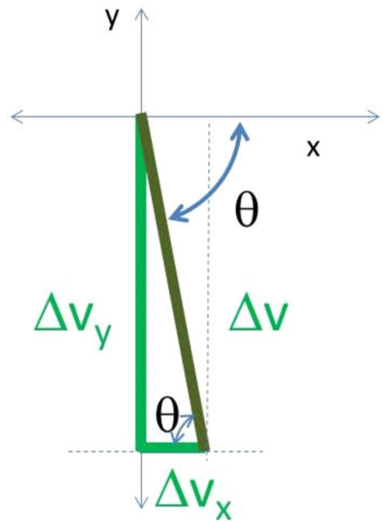
$$\Delta\overrightarrow{\mathbf{v}}=1.4415\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}-4.9149\frac{\mathrm{m}}{\mathrm{s}}\hat{\jmath}$$

And this is our answer! We have made a two-dimensional motion problem into two one-dimensional problems, and then combined the results of those two one-dimensional problems to form the answer of the two-dimensional problem.

But, of course, there is a second way to do the combination of our two one-dimensional results. We can find the magnitude and direction of $\Delta\overrightarrow{\mathbf{v}}$. To do this, we need to think a bit. The vector $\Delta\overrightarrow{\mathbf{v}}$ makes an angle, $-\theta$, with respect to the x-axis.



Note that $\Delta\mathbf{v}$, $\Delta v_{x,}$ and Δv_y form the sides of a right triangle



so to find Δv, we can use the Pythagorean theorem

$$\Delta v = \sqrt{\Delta v_x^2 + \Delta v_y^2}$$

and since it is a right triangle, we could find the direction of Δv using a tangent function

$$\tan\theta = \frac{\Delta v_y}{\Delta v_x}$$

so that

$$\theta = \tan^{-1}\left(\frac{\Delta v_y}{\Delta v_x}\right)$$

Let's try these for our case,

$$\begin{aligned} \Delta v &= \sqrt{\left(1.4415\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(4.9149\frac{\mathrm{m}}{\mathrm{s}}\right)^2} \\ &= 5.1219\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-4.9149\frac{\mathrm{m}}{\mathrm{s}}}{1.4415\frac{\mathrm{m}}{\mathrm{s}}}\right) \\ &= -1.2855\,\mathrm{rad} \\ &= -73.654\,^{\circ} \end{aligned}$$

so we could report like this our vector $\Delta\overrightarrow{\mathbf{v}}$ like this

$$\Delta\overrightarrow{\mathbf{v}} = 5.1\frac{\mathrm{m}}{\mathrm{s}}\angle-73.7\,^{\circ}$$

Then our acceleration is given by

$$\begin{aligned}\overrightarrow{a} &= \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t} \\ &= \frac{5.1\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}}\angle -73.7\,^{\circ} \\ &= 5.1\frac{\mathrm{m}}{\mathrm{s}^2}\angle -73.7\,^{\circ}\end{aligned}$$

Which is our answer. This was a little harder, but often conveys the meaning of the result better to other humans. But just the same, the magnitude and direction form of the solution is equivalent and is some times convinient

$$\overrightarrow{a} = 1.4\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath} - 4.9\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\jmath}$$

and either representation would work.

Again let me mention that we have done something profound. We have split a two-dimensional problem into two one-dimensional problems. That made it so we could solve the problem with what we already knew how to do. We did this by using components of the vectors and treating the x-components as one one-dimensional problem and the y-components as another one dimensional problem. Then at the very end we combined the results of our x-problem and our y-problem into the final vector. Doing this is the heart and soul of two-dimensional kinematics.

Two-Dimensional Kinematics

Armed with this technique, let's study the motion of things in two-dimensions in more detail. Let's start with displacement in two-dimensions

$$\Delta\overrightarrow{\mathbf{r}} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$$

Notice that our displacement, itself, can be split into components.

And recall that average velocity is just

$$\begin{aligned}\overrightarrow{\mathbf{v}}_{ave} &= \frac{\Delta\overrightarrow{\mathbf{r}}}{\Delta t} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath}}{\Delta t} \\ &= \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath}\end{aligned}$$

but we can recognize $\Delta x/\Delta t$ as v_x and $\Delta y/\Delta t$ ad v_y so

$$\overrightarrow{\mathbf{v}}_{ave} = v_x\hat{\imath} + v_y\hat{\jmath}$$

and we also know that

$$\begin{aligned}\overrightarrow{\mathbf{a}}_{ave} &= \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t} = \frac{\Delta v_x\hat{\imath}+\Delta v_y\hat{\jmath}}{\Delta t} \\ &= \frac{\Delta v_x\hat{\imath}}{\Delta t}+\frac{\Delta v_y\hat{\jmath}}{\Delta t} \\ &= a_x\hat{\imath}+a_y\hat{\jmath}\end{aligned}$$

We have proven mathematically that our technique of seeing a harder, two dimensional problem as a combination of easier one-dimensional problems really does work. It works for displacement, velocity, and acceleration. All the motion can be described in terms of components.

We remember that we had a set of equations for one-dimensional motion under constant acceleration.

$$\begin{aligned}\Delta x &= v_i\Delta t+\frac{1}{2}a\Delta t^2 \\ v_f &= v_i+a\Delta t \\ v_f^2 &= v_i^2+2a\Delta x \\ x_f &= x_i+\left(\frac{v_f+v_i}{2}\right)\Delta t\end{aligned}$$

But suppose we now have acceleration in two dimensions. No problem! If our acceleration is constant, then both a_x and a_y will be constant. And we can simply split our two-dimensional constant acceleration problem into two problems, one for each dimension. But we will need twice as many equations, one whole set for each one-dimensional problem part.

$$\begin{array}{ll}\Delta x = v_{ix}\Delta t+\frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t+\frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix}+a_x\Delta t & v_{fy} = v_{iy}+a_y\Delta t \\ v_{fx}^2 = v_{ix}^2+2a_x\Delta x & v_{fy}^2 = v_{iy}^2+2a_y\Delta y \\ x_f = x_i+\left(\frac{v_{fx}+v_{ix}}{2}\right)\Delta t & y_f = y_i+\left(\frac{v_{fy}+v_{iy}}{2}\right)\Delta t\end{array}$$

You are probably wondering if all this still works for instantaneous velocities and accelerations. And the answer is yes. We simply take the limit as $\Delta t \to 0$ and

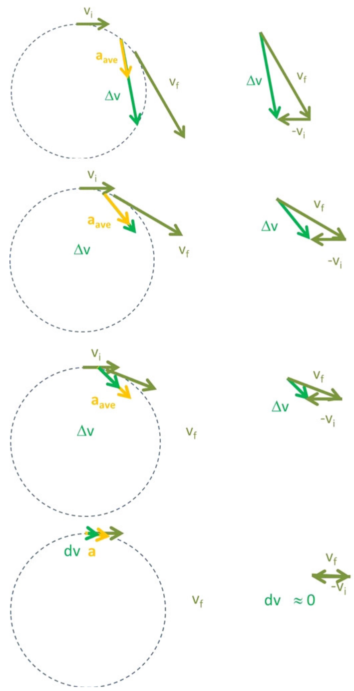
$$\overrightarrow{\mathbf{v}} = \lim_{\Delta t\to 0}\overrightarrow{\mathbf{v}}_{ave} = \frac{d\overrightarrow{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\imath}+\frac{dy}{dt}\hat{\jmath}$$

and

$$\begin{aligned}\overrightarrow{\mathbf{a}} &= \lim_{\Delta t\to 0}\overrightarrow{\mathbf{a}}_{ave} = \frac{d\overrightarrow{\mathbf{v}}}{dt} = \frac{dv_x}{dt}\hat{\imath}+\frac{dv_y}{dt}\hat{\jmath} \\ &= a_x\hat{\imath}+a_y\hat{\jmath}\end{aligned}$$

This brings up an interesting question, what happens to our diagram for motion as $\Delta t \to 0$? In the next figure, we shorten Δt from one frame to the next. In the last frame, $\Delta t \to 0$. So Δt gets smaller and smaller as we go down the page in the figure.

Notice that Δv also gets smaller and smaller until in the last figure $\Delta v \rightarrow 0$. But notice that the acceleration is not zero. This is what we expect from the math we did above. The dt may be small but so is $d\overrightarrow{\mathbf{v}}$ so the ratio of the two is not zero.

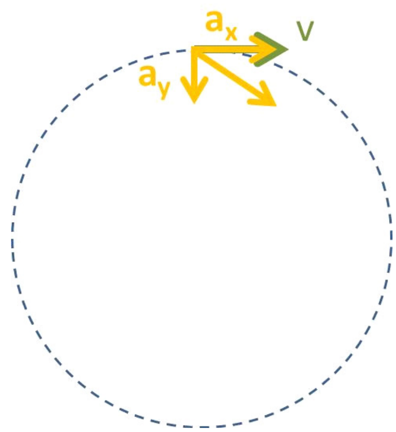


But now we have an instantaneous a vector that is right on top of the v_i vector. That is how we will draw the instantaneous value of the acceleration. Also notice that as we move along the trajectory (the path the object follows), the velocity vectors are always tangent to the trajectory path. This is also just what we expect from the math.

$$\overrightarrow{\mathbf{v}} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$

and dx/dt is the slope of the trajectory in the x direction and dy/dt is the slope of the line in the y direction. Combining them should give the slope along the trajectory, and that slope will be tangent to the actual path.

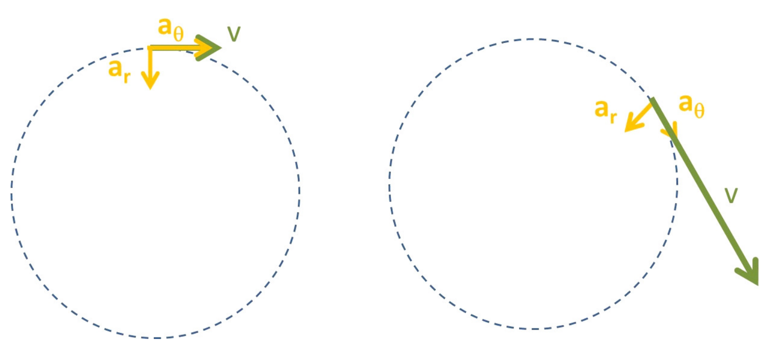
This gives us an idea. We know we can express our acceleration a in terms of a_x and a_y,



but for our circular problem, we realize that as we go around the circle, the velocity will stay tangent to the circle. Think of the component of the acceleration a_x in the picture. This component is making the object speed up because it is in the same direction as the speed. But what is the other component, a_y doing? Think that acceleration is a change in velocity

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta \overrightarrow{\mathbf{v}}}{\Delta t}$$

so acceleration can change speed *or direction.* The x component is changing speed, but the y-component is making the object turn or change direction. It might be convenient to keep our acceleration in terms of the part that makes the object speed up and the part that makes the object turn. The part that is parallel to the velocity is the speed-up-part. The part perpendicular to the velocity is the turn-part. We could define a component of the acceleration that is tangent to the trajectory. That will be the speed-up-part. And we could define a component perpendicular to the tangent. That would be the turning part.



This way we always know what is doing the speeding up and what is doing the turning. Notice that the turning-part always points to the center of the turn. A line from the center to the object would be along the same line. And we call a line from the center of a circle to the edge of the circle a “radius” so it is tradition to call the turning part of the acceleration the "radial acceleration" and the speeding-up part the "tangential acceleration." These names fit our "speeding-up" and "turning" acceleration parts even when the object has moved to another part of the circle. A math savvy student would immediately recognize these terms (radial and tangential) from polar coordinates. We will often use polar coordinates for motion in a circle.

Two dimensional examples

Let's try a problem with two-dimensional motion. Suppose we have Duke Mudwalker's Q-wing fighter taking off. The Q-wing must go up and forward at the same time. Suppose the Q-wing has $a_x = 3\,\mathrm{m/\,s^2}$ and $a_y = 6\,\mathrm{m/\,s^2}$. What is the speed of the Q-wing after $\Delta t = 2\,\mathrm{s}$?

This is a two-dimensional motion problem with constant acceleration. We need to split our problem into two one-dimensional problems. In the x-direction we have

$$\begin{aligned} v_{ix} &= 0 \\ a_x &= 3\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 2\,\mathrm{s} \end{aligned}$$

and in the y-direction

$$\begin{aligned} v_{iy} &= 0 \\ a_y &= 6\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 2\,\mathrm{s} \end{aligned}$$

Notice that the Δt values must be the same! We need two sets of equations

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \\ x_f = x_i + \left(\frac{v_{fx}+v_{ix}}{2}\right)\Delta t & y_f = y_i + \left(\frac{v_{fy}+v_{iy}}{2}\right)\Delta t \end{array}$$

Let's do the x part first. If we underline the parts we know

$$\begin{aligned} \Delta x &= \underline{v_{ix}}\underline{\Delta t} + \tfrac{1}{2}\underline{a_x}\underline{\Delta t}^2 \\ v_{fx} &= \underline{v_{ix}} + \underline{a_x}\underline{\Delta t} \\ v_{fx}^2 &= \underline{v_{ix}^2} + 2\underline{a_x}\Delta x \\ x_f &= x_i + \left(\frac{v_{fx}+\underline{v_{ix}}}{2}\right)\Delta t \end{aligned}$$

We can see that the second equation in the x set will give us the final x speed

$$v_{fx} = v_{ix} + a_x\Delta t$$

$$\begin{aligned} v_{fx} &= a_x \Delta t = \left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s}) \\ &= 6.0\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Now let's do the y part.

$$\begin{aligned} \Delta y &= \underline{v_{iy}}\underline{\Delta t} + \tfrac{1}{2}\underline{a_y}\underline{\Delta t}^2 \\ v_{fy} &= \underline{v_{iy}} + \underline{a_y}\underline{\Delta t} \\ v_{fy}^2 &= \underline{v_{iy}^2} + 2\underline{a_y}\Delta y \\ y_f &= y_i + \left(\tfrac{v_{fy}+\underline{v_{iy}}}{2}\right)\underline{\Delta t} \end{aligned}$$

Again the second equation in the set will work

$$v_{fy} = v_{iy} + a_y \Delta t$$
$$v_{fy} = 0 + a_y \Delta t = \left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s}) = 12\frac{\mathrm{m}}{\mathrm{s}}$$

then our the magnitude of the final velocity will be

$$\begin{aligned} v_f &= \sqrt{\left(6.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(12\frac{\mathrm{m}}{\mathrm{s}}\right)^2} \\ &= 13.416\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and the direction will be

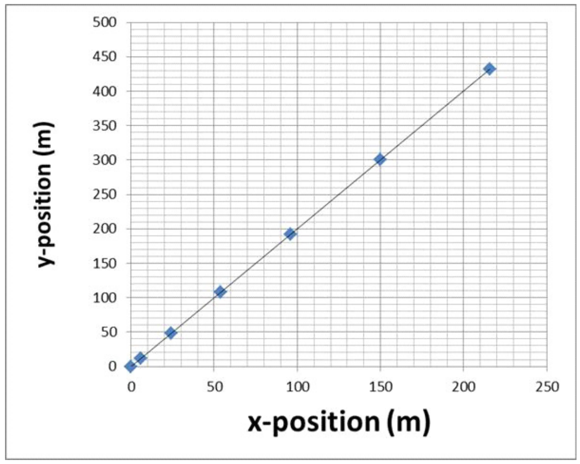
$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{12\frac{\mathrm{m}}{\mathrm{s}}}{6.0\frac{\mathrm{m}}{\mathrm{s}}}\right) \\ &= 1.1071\,\mathrm{rad} \\ &= 63.432\,^{\circ} \end{aligned}$$

We followed our pattern for solving a two-dimensional motion problem:

1. Split the two-dimensional problem into two one-dimensional problems by taking components of the vectors using the general form
$$\begin{aligned} V_x &= V\cos\theta \\ V_y &= V\sin\theta \end{aligned}$$
where θ is measured from the positive x-axis.

2. Solve the two one-dimensional problems separately

3. Combine the results of the one-dimensional problems together using
$$V = \sqrt{V_x^2 + V_y^2}$$
and
$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$
for the vector V that you are solving for, whether velocity, acceleration, or even displacement. If we use the first equation in our sets for both x and y we can get the position vs. time. A plot of the x and y positions for each time gives us a trajectory plot. A plot of the Q-wing trajectory looks like this.



Is it reasonable that the Q-wing goes in a straight line?

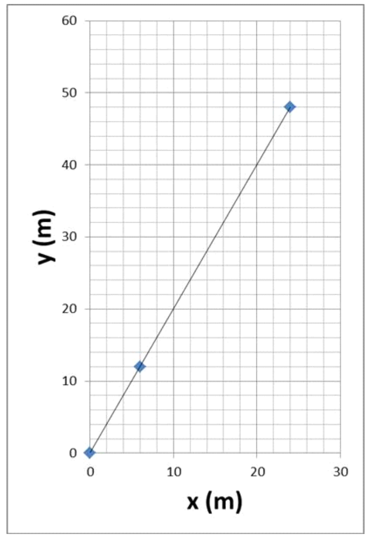
We can check this by doing a few calculations. After $\Delta t = 2\,\mathrm{s}$

$$\begin{aligned}\Delta x &= (0)\,(2\,\mathrm{s}) + \frac{1}{2}\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s})^2 \\ &= 6.0\,\mathrm{m} \\ \Delta y &= (0)\,(2\,\mathrm{s}) + \frac{1}{2}\left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s})^2 \\ &= 12.0\,\mathrm{m}\end{aligned}$$

and after $\Delta t = 4\,\mathrm{s}$

$$\begin{aligned}\Delta x &= (0)\,(4\,\mathrm{s}) + \frac{1}{2}\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4\,\mathrm{s})^2 \\ &= 24.0\,\mathrm{m} \\ \Delta y &= (0)\,(4\,\mathrm{s}) + \frac{1}{2}\left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4\,\mathrm{s})^2 \\ &= 48.0\,\mathrm{m}\end{aligned}$$

and we can plot these

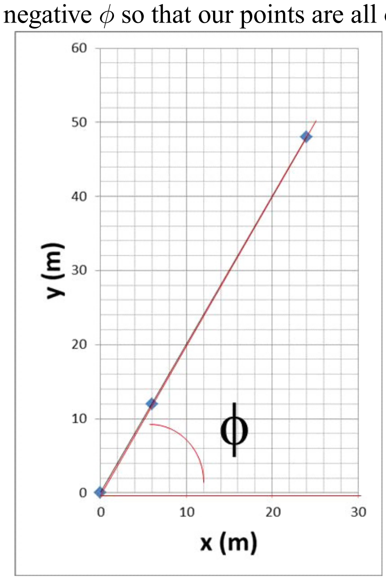


to see that we are right. It is linear.

A moment's thought will tell us that this has to be right. Suppose we rotated our axes by an angle

$$\begin{aligned}\phi &= \tan^{-1}\left(\frac{6\,\frac{\mathrm{m}}{\mathrm{s}^2}}{3\,\frac{\mathrm{m}}{\mathrm{s}^2}}\right)\\ &= 1.1071\,\mathrm{rad}\\ &= 63.432\,^{\circ}\end{aligned}$$

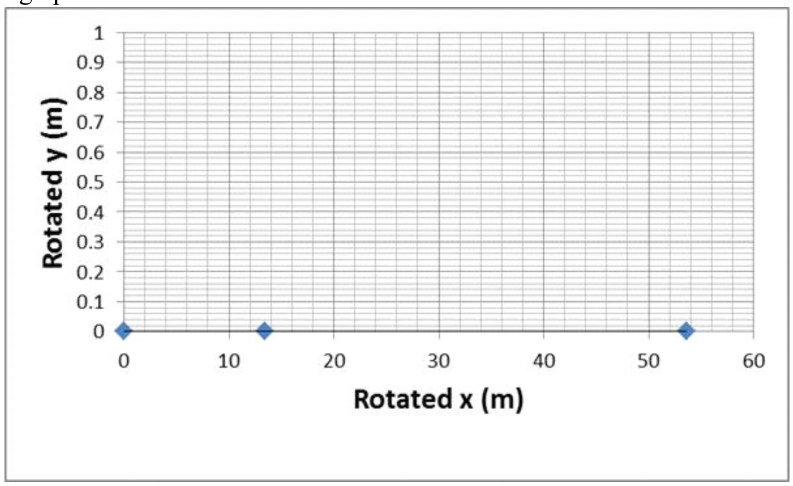
We'll, say we rotate by a negative ϕ so that our points are all on the x axes.



Then our acceleration would be a constant

$$\begin{aligned} a &= \sqrt{\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2 + \left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2} \\ &= 6.7082\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

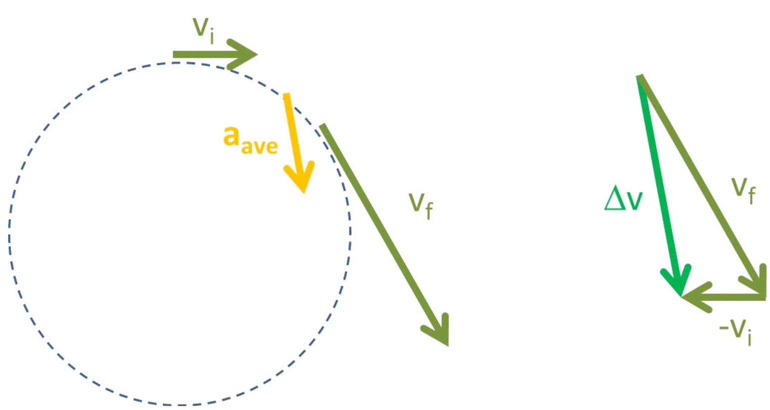
and our graph would look like this.



so indeed, we should have a straight line.

But two dimensional problems aren't always so easy. What if the Q-wing was already moving? or if it only experienced acceleration in only one direction?

Let's try a more difficult problem. Suppose we have a car moving on a circular track. When we first observe the car it is going $20\,\mathrm{m/s}$ at $0°$. 7 seconds later, the car is going $53\,\mathrm{m/s}$ at $-50°$. What is the average acceleration of the car?



This is a two dimensional problem. I don't know if the acceleration is constant or not. All I have is a before and after picture. And we want an average acceleration, so I will call this an average motion problem.

Our basic equations would be

$$\overrightarrow{v}_{ave} = \frac{\Delta \overrightarrow{r}}{\Delta t}$$
$$\overrightarrow{a}_{ave} = \frac{\Delta \overrightarrow{v}}{\Delta t}$$

but we have to split these into x and y-parts. So let's write these as

$$\begin{aligned} v_{ave_x} &= \frac{\Delta x}{\Delta t} \\ v_{avey} &= \frac{\Delta y}{\Delta t} \\ a_{ave_x} &= \frac{\Delta v_x}{\Delta t} \\ a_{ave_y} &= \frac{\Delta v_y}{\Delta t} \end{aligned}$$

We have split our two-dimensional equations into two one-dimensional equations.

We have several known values from the problem statement.

$$\begin{aligned} v_i &= 20\frac{\mathrm{m}}{\mathrm{s}} \\ \theta_i &= 0^\circ \\ v_f &= 53\frac{\mathrm{m}}{\mathrm{s}} \\ \theta_f &= -50^\circ \\ \Delta t &= 7\,\mathrm{s} \end{aligned}$$

but we need to split these initial values into x and y-parts. To do this we need to include our vector components equation set.

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \end{aligned}$$
$$v = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

We use these equations to turn our initial and final velocity vectors into initial and final velocity x and y-parts.

$$\begin{aligned} v_{ix} &= v_i\cos\theta_i \\ v_{iy} &= v_i\sin\theta_i \end{aligned}$$

$$
\begin{aligned}
v_{fx} &= v_f \cos\theta_f \\
v_{fy} &= v_f \sin\theta_f
\end{aligned}
$$

Now we can attempt to solve the problem. We are going to want a magnitude and a direction or at least the component form for the average acceleration. So our plan should be to find the x and the y parts of the acceleration, and then combine them for the total acceleration.

$$\overrightarrow{a}_{ave} = a_{ave_x}\hat{\imath} + a_{ave_y}\hat{\jmath}$$

so we need to solve two-one dimensional problems, one for a_{ave_x} and one for a_{ave_y}. Let's start with a_{ave_x}.

$$
\begin{aligned}
a_{ave_x} &= \frac{\Delta v_x}{\Delta t} \\
&= \frac{v_{fx} - v_{ix}}{\Delta t} \\
&= \frac{\underline{\underline{v_f}} \cos\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \cos\underline{\underline{\theta_i}}}{\underline{\underline{\Delta t}}}
\end{aligned}
$$

and it looks like we know all the parts, so we have solved for a_{ave_x}. Now for a_{ave_y}

$$
\begin{aligned}
a_{ave_y} &= \frac{\Delta v_y}{\Delta t} \\
&= \frac{v_{fy} - v_{iy}}{\Delta t} \\
&= \frac{\underline{\underline{v_f}} \sin\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \sin\underline{\underline{\theta_i}}}{\underline{\underline{\Delta t}}}
\end{aligned}
$$

so we could report symbolically

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}} \cos\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \cos\underline{\underline{\theta_i}}}{\underline{\underline{\Delta t}}}\hat{\imath} + \frac{\underline{\underline{v_f}} \sin\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \sin\underline{\underline{\theta_i}}}{\underline{\underline{\Delta t}}}\hat{\jmath}$$

We have some zeros. So let's use them

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}} \cos\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \cos(0\,^\circ)}{\underline{\underline{\Delta t}}}\hat{\imath} + \frac{\underline{\underline{v_f}} \sin\underline{\underline{\theta_f}} - \underline{\underline{v_i}} \sin(0\,^\circ)}{\underline{\underline{\Delta t}}}\hat{\jmath}$$

and we know that $\cos(0\,^\circ) = 1$ and $\sin(0\,^\circ) = 0$ so we can write our solution as

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}} \cos\underline{\underline{\theta_f}} - \underline{\underline{v_i}}}{\underline{\underline{\Delta t}}}\hat{\imath} + \frac{\underline{\underline{v_f}} \sin\underline{\underline{\theta_f}}}{\underline{\underline{\Delta t}}}\hat{\jmath}$$

This might be a better symbolic answer. Then putting in the rest of the numbers gives

$$\overrightarrow{a}_{ave} = \frac{53\frac{\mathrm{m}}{\mathrm{s}} \cos(-50\,^\circ) - 20\frac{\mathrm{m}}{\mathrm{s}}}{7\,\mathrm{s}}\hat{\imath} + \frac{53\frac{\mathrm{m}}{\mathrm{s}} \sin(-50\,^\circ)}{7\,\mathrm{s}}\hat{\jmath}$$

or

$$\overrightarrow{a}_{ave} = 2.0\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath} - 5.8\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\jmath}$$

and this seems to make sense. From looking at the $\Delta\overrightarrow{v}$ direction we can see that $\overrightarrow{a}_{ave}$ should point to the right a little and down a little more.

Of course we could report this in magnitude and direction form as well

$$\begin{aligned} a_{ave} &= \sqrt{\left(2.0\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2 + \left(-5.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2} \\ &= 6.1351\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

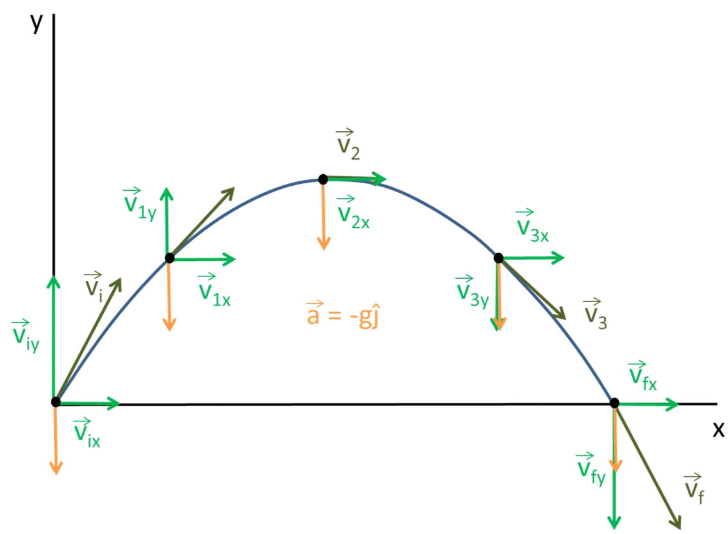
and

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{-5.8\frac{\mathrm{m}}{\mathrm{s}^2}}{2.0\frac{\mathrm{m}}{\mathrm{s}^2}}\right) \\ &= -1.2387\,\mathrm{rad} \\ & \quad -70.972\,^{\circ} \end{aligned}$$

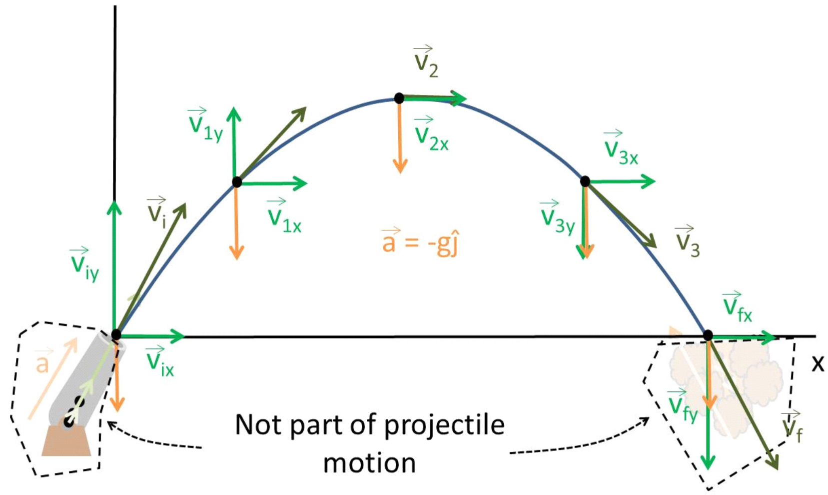
which seem reasonable.

10 Projectile Motion

Consider two people playing catch. One person throws the ball toward the other person. The ball rises in the air as it is thrown. As it travels, it reaches a maximum height, and then drops down into the glove of the catcher.



Let's look at this in some detail. First off, you may wonder, how did this motion start? Well of course the person throwing the ball must accelerate the ball. In the next figure, we have a cannon shooting a ball as our staring point.



While the ball is in the cannon, there is an acceleration acting on the ball. Then there is the acceleration due to gravity while the ball is in flight. And then there is a different acceleration as the ball crashes into the ground. Our kinematic equations can only deal with one constant acceleration at a time. Like in our rocket problem we did, we will need to split this complicated problem into three simpler problems, one for each time segment when the acceleration is constant. We would have part 1 when the ball is in the cannon. Part 2 would be when the ball is in the sky. And part 3 would be when the ball is burrowing into the ground. For today, I want do only one of these parts today, part 2 where the ball is in flight. We will consider the ball's motion only after it is already moving and out of the cannon. We will leave parts 1 and 2 for a future lecture.

This special case of part 2 when the ball is in flight should seem familiar. Part 2 is free-fall with an acceleration of $\overrightarrow{a} = -g\hat{\jmath}$.

But notice this is a *two-dimensional free fall problem*! So of course we would try to turn it into two one-dimensional problems.

First look at the horizontal motion. The figure above shows the velocity of the ball broken into components. Carefully look at the velocity component in the x direction. Notice the magnitude (size) of the vector $\overrightarrow{\mathbf{v}}_x$. It does not change. Is that a surprise?

Well, not really. To see where velocity will change we look for acceleration. We know there is an free-fall acceleration, $\overrightarrow{a} = -g\hat{\jmath}$. The $\hat{\jmath}$ tells us that this acceleration is all in the y-direction. We could say that

$$a_y = -g$$

and that is all the acceleration we have for our part of the motion that we are studying.

So we know that

$$a_x = 0$$

Since $a_x = \Delta v_x/\Delta t$ so if $a_x = 0$ then $\Delta v_x = 0$. The x-component of the velocity cannot change.

Let's give this special case of free-fall with $a_y = -g$ and $a_x = 0$ its own problem type name. We will call it *projectile motion*. Because projectile motion is a two dimensional free fall type problem, we will need two sets of kinematic equations in our projectile motion equation set, one for x and one for y.

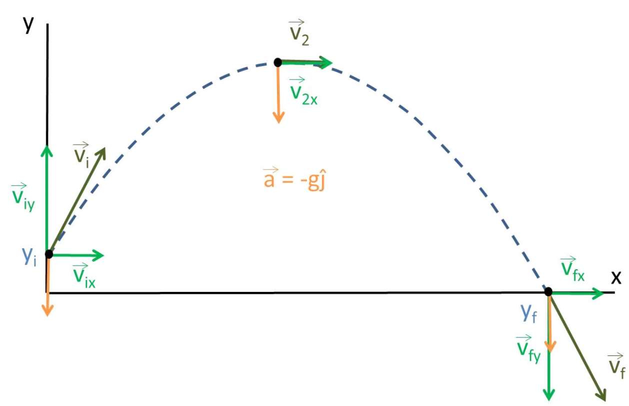
Ball Drop Demonstration

Let's try an example:

Suppose we have a small spring cannon that launches a metal ball with an initial speed of $7.00\,\mathrm{m/s}$ at an initial angle of $60.0\,^\circ$ starting from $0.270\,\mathrm{m}$ off the ground. What will the final speed be just before the ball hits the ground?

This is a PT=Projectile-motion problem. The ball is in free-fall with $a_y = -g$ and $a_x = 0$.

For our picture, let's draw the motion of the ball and mark the initial and final positions, velocities, velocity components, and accelerations.



For our variables we know

$$\begin{aligned}
x_i &= 0 \\
y_i &= 0.270\,\mathrm{m} \\
y_f &= 0 \\
v_i &= 7.00\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_i &= 60^\circ \\
a_y &= -g \\
a_x &= 0 \\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}$$

For basic equations we will need two sets of kinematic equations, one for the x-part and one for the y-part.

where the angle θ_o is the angle from the horizontal, the angle at which the ball is thrown. We call θ_o the projection angle.

Then we have two sets of our motion equations

$$\begin{array}{ll}
x_f = x_i + v_{ix}\Delta t + \frac{1}{2}a_x \Delta t^2 & y_f - y_i = v_{iy}\Delta t + \frac{1}{2}a_y \Delta t^2 \\
v_{fx} = v_{ix} + a_x \Delta t & v_{fy} = v_{iy} + a_y \Delta t \\
v_{fx}^2 = v_{ix}^2 + 2a_x\left(x_f - x_i\right) & v_{fy}^2 = v_{iy}^2 + 2a_y\left(y_f - y_i\right) \\
x_f = x_i + \frac{v_{fx}+v_{ix}}{2}\Delta t & y_f = y_i + \frac{v_{fy}+v_{iy}}{2}\Delta t
\end{array}$$

We also may need

$$\begin{aligned}
\Delta y &= y_f - y_i \\
\Delta x &= x_f - x_i
\end{aligned}$$

and we will need to split our problem into x and y-parts so we need the component of vectors set of equations

$$\begin{aligned}
v_x &= v\cos\theta \\
v_y &= v\sin\theta
\end{aligned}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Notice that projectile motion problems may take three of our previously collected sets of equations!

Now we are ready to try to solve for the final velocity.

The first part of our solution is to combine what we know with our equation set. So let's copy our equations and underline what we know.

$$\begin{aligned}\Delta y &= \underline{y_f} - \underline{y_i} \\ \Delta x &= \underline{x_f} - x_i\end{aligned}$$

$$\begin{array}{ll} x_f = \underline{x_i} + v_{ix}\Delta t + \frac{1}{2}\underline{a_x}\Delta t^2 & \underline{y_f} = \underline{y_i} + v_{iy}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = v_{ix} + \underline{a_x}\Delta t & v_{fy} = v_{iy} + \underline{a_y}\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2\underline{a_x}\left(x_f - \underline{x_i}\right) & v_{fy}^2 = v_{iy}^2 + 2\underline{a_y}\left(\underline{y_f} - \underline{y_i}\right) \\ x_f = \underline{x_i} + \frac{v_{fx}+v_{ix}}{2}\Delta t & \underline{y_f} = \underline{y_i} + \frac{v_{fx}+v_{ix}}{2}\Delta t \end{array}$$

We don't have enough information to solve for the final velocity yet. But we realize that several of the quantities in the kinematic set can be found with the component of vectors set. We can find the components of the initial velocity, for example

$$\begin{aligned}v_{ix} &= \underline{v_i}\cos\underline{\theta_i} \\ v_{iy} &= \underline{v_i}\sin\underline{\theta_i}\end{aligned}$$

so we can mark v_{ix} and v_{iy} as something we can know. Let's mark these and use our zeros.

$$\begin{array}{ll} x_f = 0 + v_{ix}\Delta t + 0 & 0 = \underline{y_i} + \underbrace{v_{iy}}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underbrace{v_{ix}} + 0 & v_{fy} = \underbrace{v_{iy}} + \underline{a_y}\Delta t \\ v_{fx}^2 = \underbrace{v_{ix}^2} + 0 & v_{fy}^2 = v_{iy}^2 + 2\underline{a_y}\left(0 - \underline{y_i}\right) \\ x_f = 0 + \dfrac{v_{fx} + \underbrace{v_{ix}}}{2}\Delta t & 0 = \underline{y_i} + \dfrac{v_{fx} + \underbrace{v_{iy}}}{2}\Delta t \end{array}$$

Now let's look at our set of equations. To find $\overrightarrow{v}_f$ we need to at least know v_{fx} and v_{fy} in order to write $\overrightarrow{v}_f$ in component form. So we need to search our set of equations and see if we know enough parts that we can solve for v_{ix} and v_{iy}.

It turns out that v_{ix} is fairy easy. From the second equation in our x-set we have

$$v_{fx} = \underbrace{v_{ix}} = \underline{v_i}\cos\underline{\theta_i}$$

It looks like from our third equation in the y-set we can find v_{fy}

$$v_{fy}^2 = \underbrace{v_{iy}^2} + 2\underline{a_y}\underbrace{\Delta y}$$

we will need to fill in the pieces from the vector components set and the displacement set

$$v_{fy}^2 = \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(0 - \underline{y_i}\right)$$

and solve for v_{fy}

$$v_{fy} = \pm\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(g\right)\left(-\underline{y_i}\right)}$$

where we will choose the the negative sign by looking at our picture to know that v_{fy}

must be negative.

$$v_{fy} = -\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(g\right)\left(\underline{y_i}\right)}$$

So our $\overrightarrow{v}_f$ can be written as

$$\overrightarrow{v}_f = \left(\underline{v_i}\cos\underline{\theta_i}\right)\hat{\imath} - \left(\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}}\right)\hat{\jmath}$$

but of course we could use

$$\begin{aligned}
v_f &= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}}\right)^2} \\
&= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}}
\end{aligned}$$

$$\theta_f = \tan^{-1}\left(\frac{-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}}}{\left(\underline{v_i}\cos\underline{\theta_i}\right)}\right)$$

then

$$\begin{aligned}
\overrightarrow{v}_f &= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}} \\
&\qquad \angle\tan^{-1}\left(\frac{-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2g\underline{y_i}}}{\left(\underline{v_i}\cos\underline{\theta_i}\right)}\right)
\end{aligned}$$

And we know every part of these equations except what we are solving for. Let's put in some numbers now

$$\begin{aligned}
\overrightarrow{v}_f &= \left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60^\circ\right)\right)\hat{\imath} - \left(\sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60^\circ\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)}\right)\hat{\jmath} \\
&= 3.5\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - 6.4840\frac{\mathrm{m}}{\mathrm{s}}\jmath \\
&= 3.50\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - 6.48\frac{\mathrm{m}}{\mathrm{s}}\jmath
\end{aligned}$$

or

$$\begin{aligned}
\overrightarrow{v}_f &= \sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60^\circ\right)\right)^2 + \left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60^\circ\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)} \\
&\qquad \angle\tan^{-1}\left(\frac{-\sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60^\circ\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)}}{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60^\circ\right)\right)}\right) \\
&= 7.3683\frac{\mathrm{m}}{\mathrm{s}}\angle-1.0758\,\mathrm{rad} \\
&= 7.3683\frac{\mathrm{m}}{\mathrm{s}}\angle-61.639^\circ \\
\overrightarrow{v}_f &= 7.37\frac{\mathrm{m}}{\mathrm{s}}\angle-61.6^\circ
\end{aligned}$$

This says that our ball is going a little faster at the end than it was at the beginning. Since it fell a little from the beginning to the end, this makes sense. It also makes sense that the angle is negative, and $-61\,^{\circ}$ seems reasonable looking at the picture.

You might think this was a lot of work! and it is! So why would human kind want to go through all this? From safety from falling rocks, to hunting food, to cannon balls in war, to moon launches, this process has been very useful! Perhaps even more useful is learning how to structure a solution so you can do a long complicated problem where you can't really know the answer intuitively when you start the problem. That kind of reasoning is useful in every technical field, medicine included! And it is in this kind of problem that our problem solving process actually saves us time.

Let's extend our example to ask what the total displacement in the x-direction would be. We can add to our known values from the work we have done.

$$
\begin{aligned}
x_i &= 0 \\
y_i &= 0.270\,\mathrm{m} \\
y_f &= 0 \\
v_i &= 7.00\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_i &= 60\,^{\circ} \\
a_y &= -g \\
a_x &= 0 \\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \\
v_{xf} &= 3.50\frac{\mathrm{m}}{\mathrm{s}} \\
v_{yf} &= -6.480\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

We need an updated set of equations as well.

$$
\begin{aligned}
\underbrace{v_{ix}} &= \underline{v_i}\cos\underline{\theta_i} \\
\underbrace{v_{iy}} &= \underline{v_i}\sin\underline{\theta_i}
\end{aligned}
$$

so we can mark v_{ix} and v_{iy} as something we can know. Let's mark these and use our zeros.

$$
\begin{aligned}
\Delta y &= 0 - \underline{y_i} \\
\Delta x &= \underline{x_f} - 0
\end{aligned}
$$

$$\begin{aligned}
\Delta x &= \underbrace{v_{ix}}\Delta t + 0 & \underbrace{\Delta y} &= \underbrace{v_{iy}}\Delta t + \tfrac{1}{2}\underline{a_y}\Delta t^2 \\
\underbrace{v_{fx}} &= \underbrace{v_{ix}} + 0 & \underbrace{v_{fy}} &= \underbrace{v_{iy}} + \underline{a_y}\Delta t \\
\underbrace{v_{fx}^2} &= \underbrace{v_{ix}^2} + 0 & \underbrace{v_{fy}^2} &= \underbrace{v_{iy}^2} + 2\underline{a_y}\,\underbrace{\Delta y} \\
x_f &= 0 + \frac{\underbrace{v_{fx}} + \underbrace{v_{ix}}}{2}\Delta t & 0 &= \underline{y_i} + \frac{\underbrace{v_{fy}} + \underbrace{v_{iy}}}{2}\Delta t
\end{aligned}$$

We can see that if we knew Δt then the first equation of our x-set would work. But we don't know Δt. But we can find Δt from the second equation in our y-set!

$$\underbrace{v_{fy}} = \underbrace{v_{iy}} + \underline{a_y}\Delta t$$

so

$$\underbrace{v_{fy}} - \underbrace{v_{iy}} = \underline{a_y}\Delta t$$

$$\frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}} = \Delta t$$

or

$$\Delta t = \frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}}$$

then from the x-set

$$x_f = \underbrace{v_{ix}}\left(\frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}}\right)$$

$$x_f = \underline{v_i}\cos\underline{\theta_i}\left(\frac{\underbrace{v_{fy}} - \underline{v_i}\sin\underline{\theta_i}}{\underline{a_y}}\right)$$

Putting in values gives

$$\begin{aligned}
x_f &= \left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60^\circ\right)\left(\frac{-6.480\frac{\mathrm{m}}{\mathrm{s}} - \left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60^\circ\right)}{-9.8\frac{\mathrm{m}}{\mathrm{s}^2}}\right) \\
&= 4.4793\,\mathrm{m}
\end{aligned}$$

We call this the "range" of the projectile motion. If you are in the army, or just out hunting, you might want to know this.

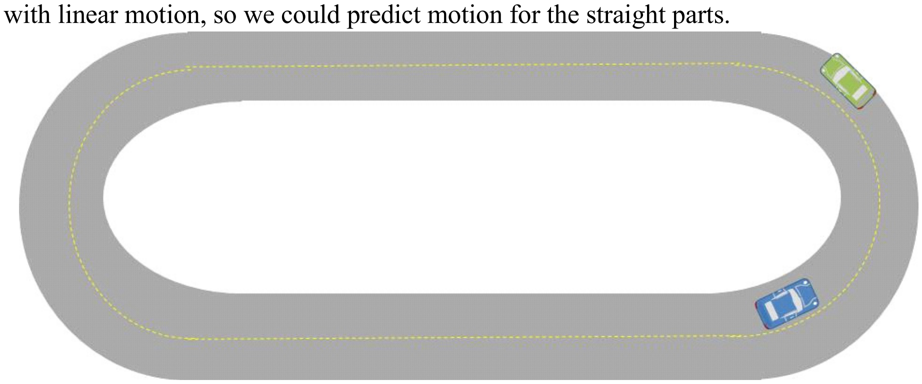
Our problems have become longer, now that we have two-dimensions. But not really harder if we take a systematic approach. We will continue with the topic of two-dimensional motion in our next lecture.

11 Beginning of Circular Motion

We have delt with and linear motion and projectiles that experience curved motion. But there is a special case of curved motion that is very important to physicists (and Engineers, etc.) This is the case of an object that moves in a perfect circle. This could be a satellite moving in a circular orbit, or it could be a geer moving in a circle in an engine. A circular path could be a part of a more complex motion (like a race track in the example below). This comes up so often in real applications, let's take a look at it now.

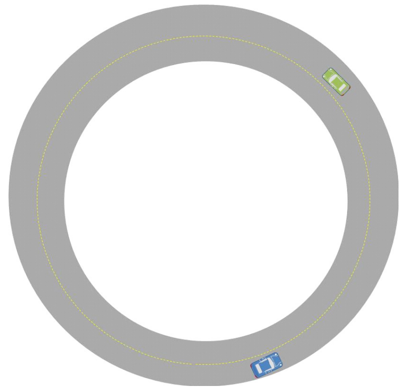
Uniform Circular Motion

Think of a race track. Many tracks consist of straight parts, and we know how to deal with linear motion, so we could predict motion for the straight parts.



We have also dealt a little with the curved parts. And the whole track could be considered to be a combination of straigtht parts and curved parts. Once again a complicated problem can be seen as a combination of simpler parts. But to understand this complicated problem, we need to know more about the curved parts. Lets consider what would happen if we have all curved parts.

Let's consider a completely curved track as a special case. A completely curved track would be a circle.



This would probably not be a fun track to drive on. But if we can find a way to express motion on a circle, then we could use this circular motion technique and our linear motion technique combined to deal with the entire track.

Let's start our analysis by considering uniform motion in a circle, that is, moving in a circle but not speeding up or slowing down.

This is a lot like a constant motion problem. We would expect to be able to use an equation like

$$v = \frac{dx}{dt}$$

but there is a problem, We only go purely in the x-direction for a very short time as we go around the circle. We need a new way to say how far we have gone as we travel around the circle. We could use our vector components, and that would work. But there is another way we could express our motion for this special case of circular motion. From Geometry, way back in junior high school, we know something about measuring how far we go around a circle. We all know this, but let's review it just to refresh our memories.

Lets consider how far we would travel in distance if we went all the way around the circle. That would be the entire circumference of the circle. So our distance would be

$$C = 2\pi r$$

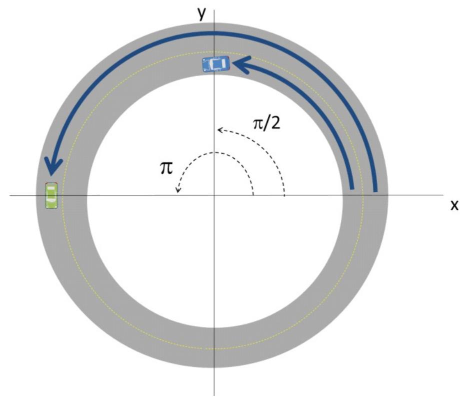
where r is the radius of the circle and where C is for circumference. But suppose we just traveled half way around the circle. The distance would be

$$\frac{C}{2} = \pi r$$

and notice that we swiped out an angle of $180\,^{\circ}$ (see next figure, green car). How about a quarter way around the circle?

$$\frac{C}{4} = \frac{\pi}{2}r$$

Notice that we swiped out an angle of $90\,^\circ$ (seen next figure, blue car).



Now notice that in radians $90\,^\circ$ is $\pi/2$ and $180\,^\circ$ is π. Then how far we travel around the circle seems to be given by

$$\begin{aligned} s &= \text{(angle in radius) (radius)} \\ &= \phi r \end{aligned}$$

The distance traveled around the circle we give the letter, s from di(s)splacement. And it is called the *arclength* because in geometry the distance we travel in a circle is given this name. We will just use the names right from geometry.

We also have a name for how long it takes to go around the whole circle. That is called a *period*, and it is given the symbol, T. This makes some sense, because a period is a *time* for going around the whole circle. We can finally describe the speed of the car as it travels the circular path

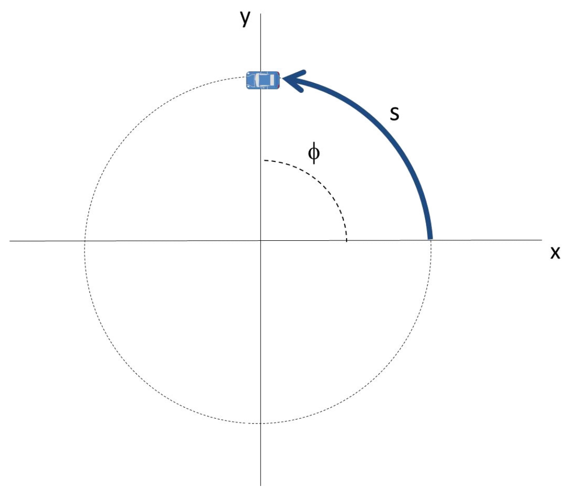
$$v = \frac{C}{T} = \frac{2\pi r}{T}$$

or we could say

$$v = \frac{\Delta s}{\Delta t}$$

if we don't go all the way around the circle.

We remember from geometry that all the way around the circle gives $365\,^\circ$ or $2\pi\,\mathrm{rad}$. And notice that the arclength is proportional to the angle, and the radius. The radius is not changing, so if we know the angle part of the displacement, and we know the car must be on the circular track, then the angle is enough to tell us where the car is.



This really only works for circular motion. But for circular motion knowing the angle is usually enough to know the position. Of course we need to know how big the track is, but if you have a car on a race track that us usually known (or it is at least easy to measure). We could write Δs as

$$\Delta s = r\Delta\phi$$

since the r is not changing. Then

$$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{r\Delta\phi}{\Delta t}$$

and we could even write

$$\frac{v_{ave}}{r} = \frac{\Delta\phi}{\Delta t}$$

as a sort of scaled speed. This tells us how much the angle changed in an amount of time. It is a lot like a velocity, but it has units of rad/s. It tells us how fast the angle of the car changes. Since for circular motion, this is equivalent to knowing how the position of the car changes, our new quantity is like a speed. Let's give it a name an a symbol. The name is *angular speed* and the symbol is a Greek letter ω. This is not a "w." It is an omega. But you may call it "w-looking thing" if that helps. Still, it is not a "w" and we should make the distinction because we will use "w" for something else in physics.

But ω does tell us how fast something spins around. It is how fast the angle changes.

$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$

notice that we can also write

$$\omega_{ave} = \frac{v_{ave}}{r}$$

from our definition. This relates how fast the object is going to how fast the angle is changing for circular motion.

If we take a limit, letting Δt get very small, then we will have an instantaneous angular speed

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \phi}{\Delta t} = \frac{d\phi}{dt}$$

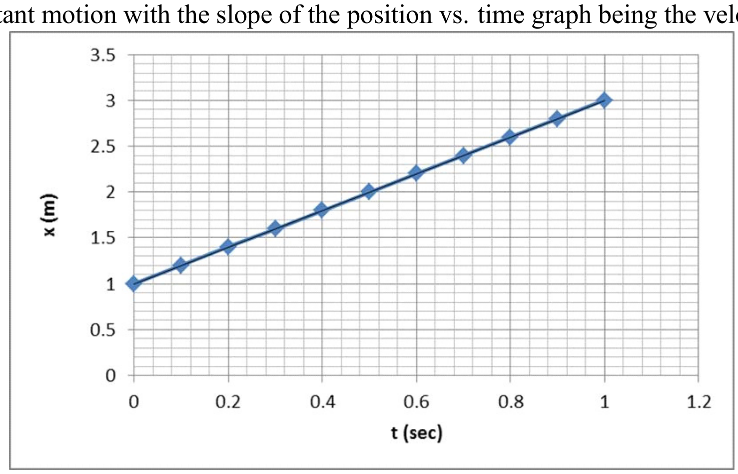
Notice that something wonderful has happened. The equation for angular speed looks just about the same as the equation for linear speed! It turns out we can make an entire set of equations for constant circular motion look very much like the equations for constant linear motion.

Linear	Circular
$\Delta r = r_f - r_i$	$\Delta \phi = \phi_f - \phi_i$
$\Delta t = t_f - t_i$	$\Delta t = t_f - t_i$
$v_{ave} = \frac{\Delta r}{\Delta t}$	$\omega_{ave} = \frac{\Delta \phi}{\Delta t}$
$v = \frac{dr}{dt}$	$\omega = \frac{d\phi}{dt}$

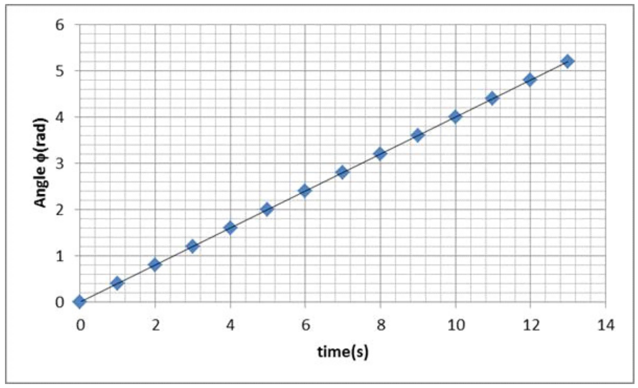
$$\omega = \frac{v}{r}$$

with an additional equation tying the two types of motion together (so long as the motion is in a circle).

This means we can use all the graphing techniques we learned for linear motion for our circular motion equations. For example, we used position vs. time graphs to show linear constant motion with the slope of the position vs. time graph being the velocity.



Now we could plot ϕ vs. time to get a plot like this



For linear constant motion we found that

$$x_f = x_i + v\Delta t$$

We can take our equation for angular speed

$$\omega = \frac{\Delta\phi}{\Delta t}$$

and write is using $\Delta\phi = \phi_f - \phi_i$ to get

$$\omega = \frac{\phi_f - \phi_i}{\Delta t}$$

or

$$\phi_f = \phi_i + \omega\Delta t$$

Let's try some problems. First, using the previous figure, what is ω?

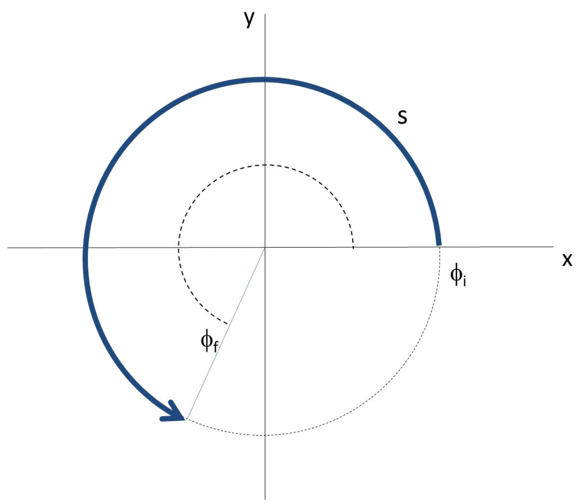
$$\omega = \frac{\phi_f - \phi_i}{t_f - t_i}$$

we recognize this as "rise over run" just like it was for linear constant motion. So the slope of the ϕ vs. time graph will be ω. We an see that at $t = 10\,\mathrm{s}$ we have $\phi = 4\,\mathrm{rad}$ and at $t = 0\,\mathrm{s}$ we have $\phi = 0\,\mathrm{rad}$, so

$$\begin{aligned} \omega &= \frac{4\,\mathrm{rad} - 0}{10\,\mathrm{s} - 0} = \\ &= 0.4\frac{\mathrm{rad}}{\mathrm{s}} \end{aligned}$$

For a second example, suppose a child is on a merry-go-round. The child hops on at $\phi_i = 0.0\,\mathrm{rad}$ and decides she does not like the ride, so she hops off at $\phi_f = 4.5\,\mathrm{rad}$. Suppose she was on the ride for $0.50\,\mathrm{s}$. What is the angular speed of the merry-go-round?

We can identify this as a constant angular speed problem



The final ϕ is about $258\,°$. So we draw a diagram showing where the child got on and off.

We know

$$\begin{aligned}\phi_i &= 0\,\text{rad}\\ \phi_f &= 4.5\,\text{rad}\\ \Delta t &= 0.50\,\text{s}\end{aligned}$$

and our basic equation is

$$\phi_f = \phi_i + \omega\Delta t$$

We can solve this for ω

$$\phi_f = \phi_i + \omega\Delta t$$

$$\begin{aligned}\phi_f - \phi_i &= \omega\Delta t\\ \frac{\phi_f - \phi_i}{\Delta t} &= \omega\end{aligned}$$

$$\omega = \frac{\phi_f - \phi_i}{\Delta t}$$

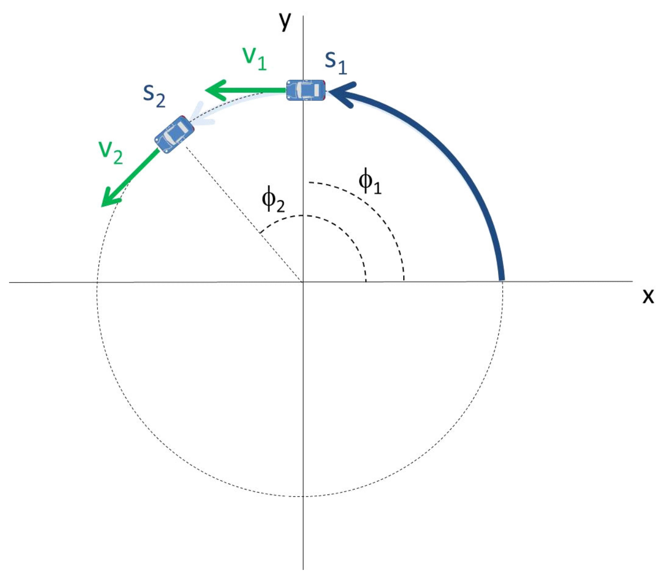
Using our numbers we get

$$\begin{aligned}\omega &= \frac{4.5\,\text{rad} - 0\,\text{rad}}{0.5\,\text{s}}\\ &= \frac{9.0}{\text{s}}\,\text{rad}\end{aligned}$$

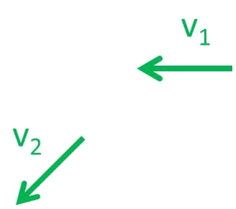
A quirk of history can make rotational problems tricky. sometimes people will talk in terms of "rotations per second" or, worse yet, "cycles per second." A rotation is just $2\pi\,\text{rad}$, and so is a cycle. They both mean, "go all the way around." So if you are given a value in rotations per second or cycles per second, just convert to radians per second.

Tangential and Radial Motion

Suppose once again we have a car going on a circular track at a constant speed.

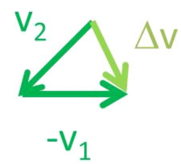


We know the car is accelerating, because it is turning. But it is not speeding up or slowing down. Notice that the velocity vectors always point along a tangent line to the circle. We will say they have a *tangential direction*. Let's find the direction of the acceleration. To do this we recopy our velocity vectors



and find

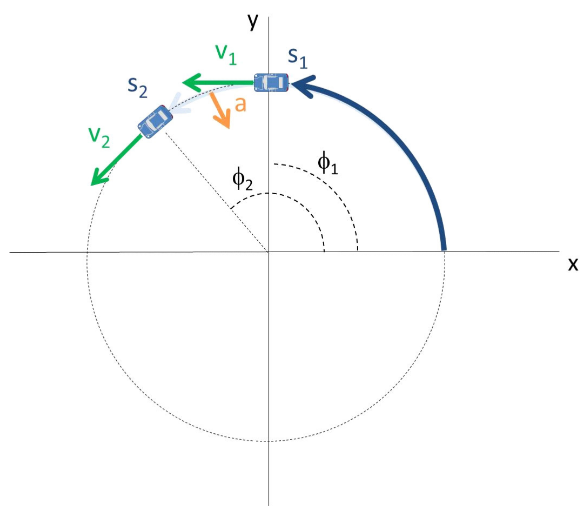
$$\Delta \overrightarrow{v} = \overrightarrow{v}_2 - \overrightarrow{v}_1$$



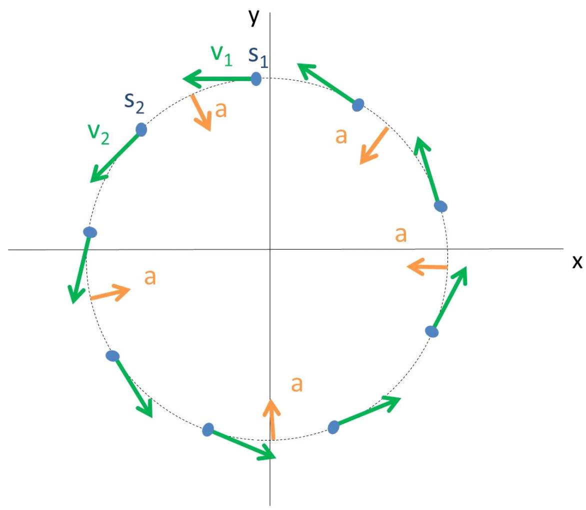
and divide $\Delta\overrightarrow{v}$ by Δt

$$a_{ave} = \frac{\Delta\overrightarrow{v}}{\Delta t}$$

There is nothing new in all this. We have found accelerations before. But now let's put our $\overrightarrow{a}$ back on our original diagram



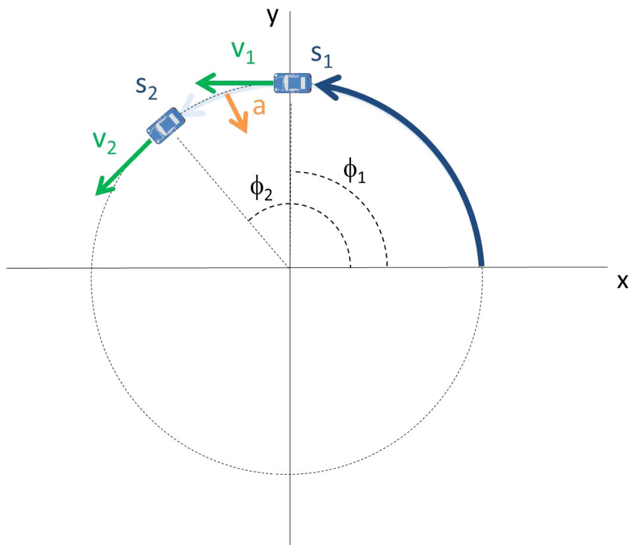
We have placed in between the two points where v_1 and v_2 are marked, because we have found an average acceleration. Note which way the acceleration points. It points into the center of the turn. Let's look at a few more places around the circular track.



Centripital Acceleration

Notice that no matter where we are on the track, the acceleration for constant circular motion always points to the center of the turn. We called the velocity "tangential" because it's direction was always along a tangent line. Notice that the acceleration is always along a radial line. So we could call it "radial" and sometimes we will. But the acceleration is not just radial, along a radial line. It specifically points to the *center* of the circular motion. There is a word that means "points to the center" and we will use this word to describe acceleration for uniform circular motion. That word is *centripetal.* like the word "tangential" just described the direction the velocity pointed, the word "centripetal" just describes the direction the acceleration points. This is not some new kind of acceleration. You could think of this like a person gaining a new title, say, "king." It is the same old person, but "king" does describe the direction their life is going!

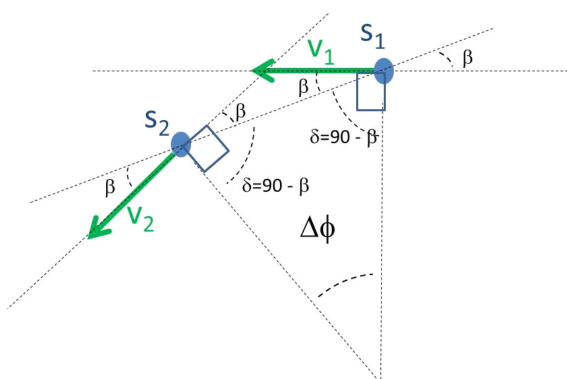
Let's go back to our traveling car going in a circle



Notice that the car sweeps out an angular displacement

$$\Delta\phi = \phi_2 - \phi_1$$

as it goes from s_1 to s_2. We can use this to find a very useful expression for a centripetal acceleration. But it will take some geometry. Here is a diagram of our motion with some extra lines added in.



Recall that the interior angles of a triangle must sum to $180\,^\circ$. So

$$\Delta\phi + \delta + \delta = 180\,^\circ$$

Also notice that the velocities are along tangent lines, and that the tangent lines and radial lines meet at right angles. Then

$$\beta + \delta = 90\,^\circ$$

and notice that $180\,^\circ = 90\,^\circ + 90\,^\circ$ (you probably already knew that!) so

$$180\,^\circ = 2\beta + 2\delta$$

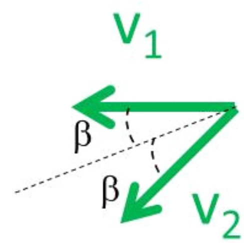
then

$$\Delta\phi + \delta + \delta = 2\beta + 2\delta$$

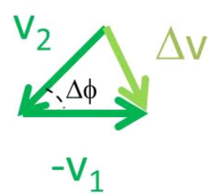
or

$$\Delta\phi = 2\beta$$

Not notice that the angle between v_1 and v_2 is exactly $2\beta = \Delta\phi$



Then in our triangle that defines $\Delta\overrightarrow{v}$ the angle between $-\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ will be $\Delta\phi$.



and these vectors form an isosceles triangle. If $\Delta\phi$ is not too big, then the tangen of $\Delta\phi$ is nearly equal to $\Delta\phi$, itself. And if $\Delta\phi$ is small then the angle between $-v_1$ and Δv is nearly $90\,^\circ$. Then we could write

$$\tan\Delta\phi \approx \frac{\Delta v}{v_1} \approx \Delta\phi$$

Then, the length the vector Δv is nearly equal to the arclength

$$\Delta v \approx v_1 \Delta\phi$$

If we let $\Delta\phi$ get very very small so it becomes $d\phi$, then we can write

$$dv = v d\phi \tag{11.1}$$

where I dropped the subscript, because the length of v_1 is the same as the length of v_2 so we can write $v_1 = v_2 \equiv v$. And we know the speed

$$v_1 = v_2 = v = \frac{ds}{dt}$$

and

$$v = \frac{ds}{dt} = \frac{r d\phi}{dt}$$

It is not obvious that this will give us anything good, but let's solve for dt

$$dt = \frac{rd\phi}{v}$$

Now let's substitute our new expression for dt for uniform circular motion into our equation for acceleration

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

but we will need to let $\Delta t \rightarrow dt$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{\frac{rd\phi}{v}} \\ &= \frac{v}{r}\frac{dv}{d\phi} \end{aligned}$$

and use equation (11.1) for dv again

$$\begin{aligned} a &= \frac{v}{r}\frac{vd\phi}{d\phi} \\ &= \frac{v^2}{r} \end{aligned}$$

This seems like a lot of work to go through for a simple equation

$$a_c = \frac{v^2}{r}$$

but it is a very important equation. It tells us that for constant circular motion the acceleration is equal to the speed squared divided by the radius. If we know the size of the circle, and how fast we are going, we can know the magnitude of the acceleration. Of course the direction of the acceleration is toward the center of the circle.

It is important to remember that we are studying circular motion at a constant speed–we are not speeding up or slowing down (yet). Our centripetal acceleration is just making our object change direction.

Lets try a problem. Suppose we are in our race car and we know from the speedometer that we are going a constant $26\,\mathrm{m/s}$ as we go around a circular track with a radius of $100\,\mathrm{m}$ (about the length of a football field). What is our acceleration?

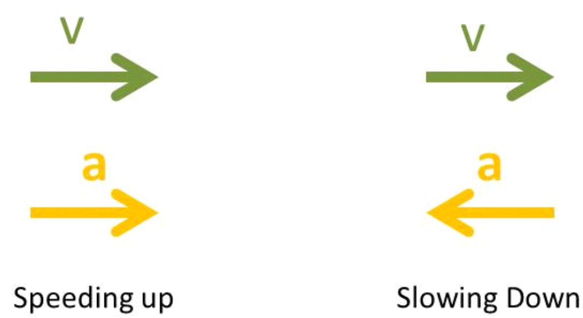
We are not speeding up or slowing down, so this must be a turning or centripetal acceleration problem. We can use our new equation

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{(26\,\mathrm{m/s})^2}{100\,\mathrm{m}} \\ &= 6.76\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

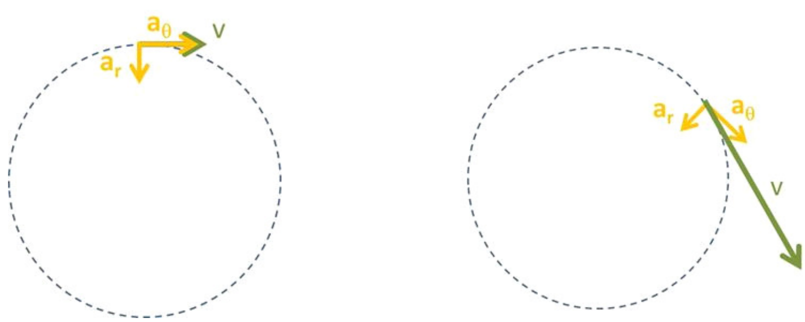
Constant Tangential Acceleration

We have dealt with the centripetal acceleration, that is, the part that points toward the center of a circle. This acceleration makes the object turn. It is in the radial inward direction. But we know there can be more to acceleration. We could speed up or slow down.

Remember that speeding up or slowing down comes from acceleration that is in the same direction or opposite direction of the motion.



Our centripetal acceleration can never make the object speed up or slow down. But we could have a tangential acceleration as well as a centripetal acceleration! The velocity as we go around the circle is always tangential.



so the tangential acceleration will make the object speed up or slow down. Looking at the figure is should be clear that we could add the centripetal and tangential accelerations to get a total acceleration.

$$a = \sqrt{a_r^2 + a_\theta^2}$$

or we could write this as $a_r = a_c$ for centripetal and $a_\theta = a_t$ for tangential.

$$a = \sqrt{a_c^2 + a_t^2}$$

Kinematics around the circle

Our arclength, s, is just a distance, and we can define a change in arclength

$$\Delta s = s_f - s_i$$

and we have defined the speed as we go around the circle as

$$v_t = \frac{ds}{dt}$$

By now you are probably guessing that I could define a whole set of kinematic equations for the arclength traveled around a circle.

$$s_f = s_i + v_{ti}\Delta t + \frac{1}{2}a_t\Delta t^2$$
$$v_{tf} = v_{ti} + a_t\Delta t$$

$$\begin{aligned} v_{tf}^2 &= v_{ti}^2 + 2a_t\left(s_f - s_i\right) \\ s_f &= s_i + \frac{v_{tf} + v_{ti}}{2}\Delta t \end{aligned}$$

but notice that we had to be careful. Only the tangential acceleration, a_t, will make the object speed up or slow down as it goes around the circle, so only the tangential acceleration can appear in our equations for arclength kinematics.

Let's try a problem using these equations.

suppose you get in your race car and the initial speed is $v_{ti} = 0$ but in $2.00\,\mathrm{s}$ you are going around a circular track with speed $v_{tf} = 25\,\mathrm{m/s}$. What is your tangential acceleration?

We know

$$\begin{aligned} v_{ti} &= 0 \\ v_{tf} &= 25\,\mathrm{m/s} \\ \Delta t &= 2.00\,\mathrm{s} \end{aligned}$$

and our basic equations are

$$s_f = s_i + v_{ti}\Delta t + \frac{1}{2}a_t\Delta t^2$$
$$v_{tf} = v_{ti} + a_t\Delta t$$

$$v_{tf}^2 = v_{ti}^2 + 2a_t\Delta s$$
$$s_f = s_i + \frac{v_{tf} + v_{ti}}{2}\Delta t$$

If we underline what we know and use our zeros we have

$$\underline{v_{tf}} = a_t \underline{\Delta t}$$

$$s_f = s_i + 0 + \frac{1}{2} a_t \underline{\Delta t}^2$$

$$\underline{v_{tf}^2} = 0 + 2 a_t \Delta s$$
$$s_f = s_i + \frac{\underline{v_{tf}} + 0}{2} \underline{\Delta t}$$

It looks like the first of these equations will work.

$$\underline{v_{tf}} = a_t \underline{\Delta t}$$
$$\frac{\underline{v_{tf}}}{\underline{\Delta t}} = a_t$$
$$\begin{aligned} a_t &= \frac{25\,\mathrm{m/s}}{2.00\,\mathrm{s}} \\ &= 12.5 \frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

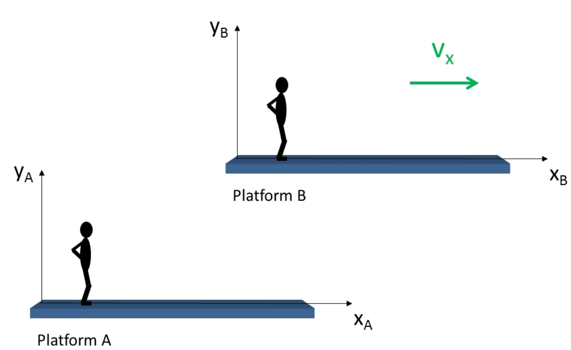
Kinematics has allowed us to solve linear, two dimensional, rotational, and arclength motion problems! We have gone a long way with the ideals of displacement, velocity, and constant acceleration.

12 Relative Motion

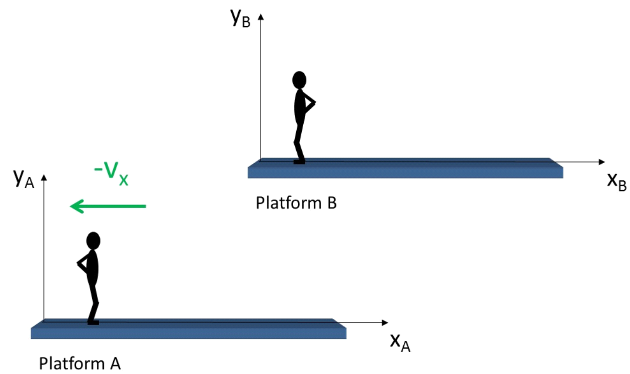
We are well into our study of two dimensional motion, but there is a complication that we have partially ignored. We have considered what would happen to our description of motion if we move the origin of our coordinate system. But what if we have a whole coordinate system that moves?

Relative Velocity

Let's consider a, somewhat far-fetched situation. Suppose we have two alien beings traveling on platforms, far from anything else in space. We can call the platforms A and B. Further suppose that each platform has it's own coordinate system attached to the platform. And suppose one of the platforms is moving with speed v_x to the right. We will give a name to these separate coordinate systems. We will call them *reference frames.* So, we can call the coordinate system of platform A reference frame A and the coordinate system of platform B reference frame B.



Alien A sees himself as stationary and sees alien B traveling with velocity v_x. But suppose only aliens A and B exist on their reference frame platforms and there are no other objects in the universe to give perspective. Alien B would see himself as stationary, and would see alien A traveling with velocity $-v_x$.



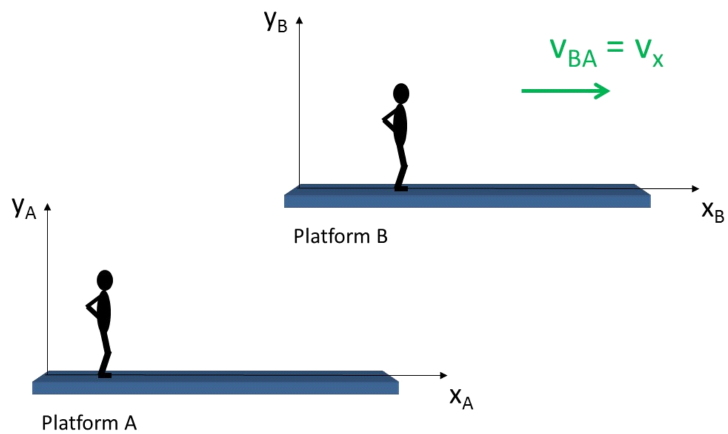
You might ask, who is right, A or B? If there are only these two aliens and their platforms, is there any way to tell which is "really moving?" It might surprise you to find out that the answer is - "no!" Recall that there is no universal zero point that is the center of everything in the universe. There is also no universal point that we know to be not moving. In fact, everything we can see in the universe, stars, planets, galaxies, etc. all seem to be moving closer to each other or away from each other, including our planet, Sun, and solar system. We believe we are moving around the galaxy, and that our galaxy is moving too. So we have no place that we can find that will work as a point that is not moving. since that is the case, it is the *relative* speed v_x that we must consider. That is all we can be absolutely sure of. Alien A really sees B moving and Alien B really sees A moving. And each viewpoint is valid.

We have all tried this experiment in real life, only, in our experiment there were other objects around us. Consider driving South on I-20. There are two lanes of traffic on the Southbound side of the freeway. As you drive, consider the cars next to you, also going south. You are all going south at $70\,\mathrm{mi/\,h}$ (carefully obeying speed laws). But think of the relative speed. Since you are all going about the same speed, the relative speed between your car and the next is very small. That is why you can wave to the person in the next car and consider if they are someone you would like to meet of if they are likely to be a maniac and force you off the road.

Now consider the cars going north. They whizz past seeming to go terribly fast. What is their relative velocity? If we take a coordinate system that is fixed to our car. The velocity of our car seems to be zero in that coordinate system. But then we see the northbound cars going past us at $70\,\mathrm{mi/\,h} + 70\,\mathrm{mi/\,h} = 140\,\mathrm{mi/\,h}$. The velocity of the two cars add. That is why you want to avoid a head-on collision with the Northbound traffic at nearly all costs!

But is a moving frame of reference useful–a coordinate system that moves with your car? Sure, if you want to hand a drink to a passenger, you don't want to have to consider where that passenger is with respect to Idaho Falls. You only need to know where the passenger is with respect to your position within the car. A more dramatic example might be finding your dining area on a cruse ship (I always wanted to go on a cruse). You don't want to have to consider your motion toward the Bahamas. You just need to know where the dinning area is with respect to your cabin.

Let's adopt a way to express our viewpoint of who is moving relative to whom.



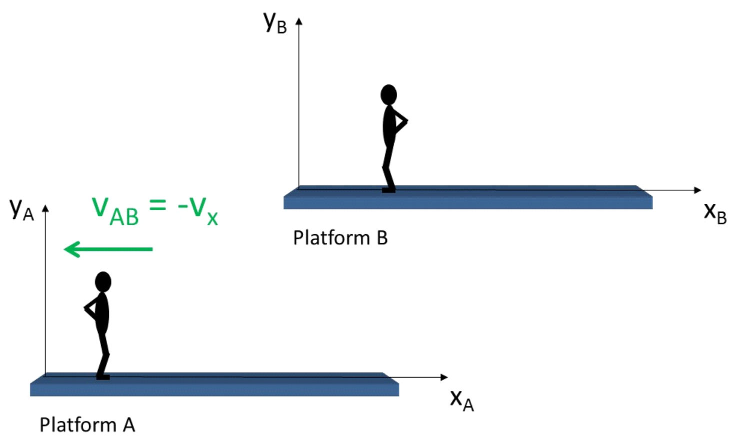
Returning to the aliens, suppose we again join space guy A and observe space guy B. The speed of B as observed from A's reference frame we will call v_{BA}. Note that in our case $v_{BA} = v_x$ in the positive x_A-direction.

Notice the order of the subscripts. First we write a letter that tells us which object is moving, in the last example, space buy B. Then we tell what view point (reference frame) we are using to observe the motion. In the last example, it was reference frame A.

If we jumped to the other platform and observed the motion of A from the perspective of B's platform we would see A moving with speed

$$v_{AB} = -v_x$$

The first subscript is always the moving object or mover, the second subscript tells the point of view from which the motion is observed.

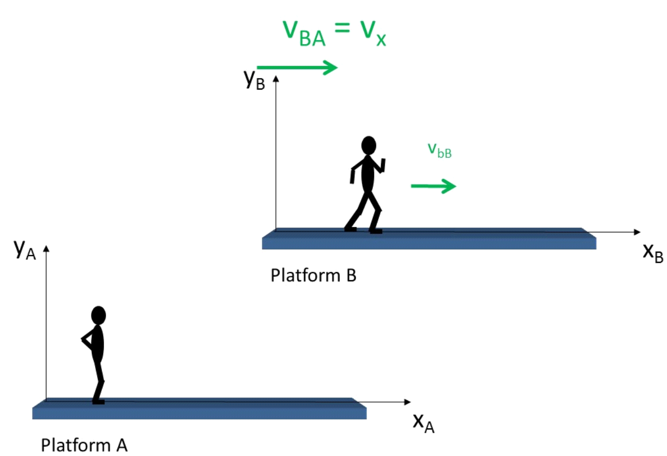


In this notation system, it will always be true that

$$v_{BA} = -v_{AB}$$

since the relative motion is always seen as opposite in direction when we switch view points.

Let's try a somewhat harder problem. Let's consider our spacemen again. and let's observe space guy b in frame B from the viewpoint of A. But let's have space guy b in frame B walk along his platform with a velocity $\vec{v}_{bB}$ relative to his own platform. The b stands for space guy b that is in frame B. How fast would the space guy in frame A see guy b go?



It's not too hard to see that guy in frame A would see guy b go

$$v_{bA} = v_{bB} + v_{BA}$$

We would add the platform speed and the walking speed to get how fast guy a in frame A sees guy b going.

Of course we could consider the spaceman in frame A (let's give him the subscript, a) walking on his platform with speed v_{aA}. From the perspective of platform B we would see spaceman a's speed as

$$v_{aB} = v_{aA} + v_{AB}$$

This gives a pair of equations for relative motion

$$\begin{aligned} \overrightarrow{\mathbf{v}}_{bA} &= \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BA} \\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}_{aA} + \overrightarrow{\mathbf{V}}_{AB} \end{aligned}$$

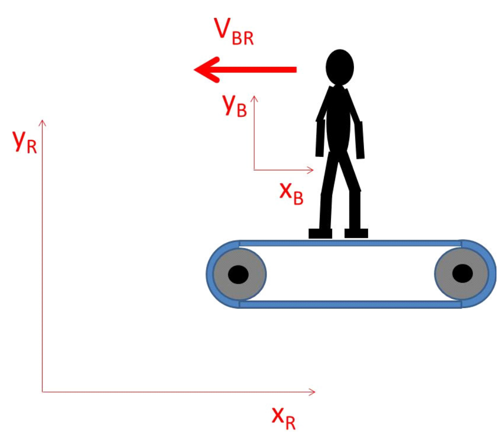
or we could write this as

$$\begin{aligned} \overrightarrow{\mathbf{v}}_{bA} &= \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BA} \\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}_{aA} - \overrightarrow{\mathbf{V}}_{BA} \end{aligned} \tag{12.1}$$

This equation set is often called the *Galilean relativity transformation* equation set because they were discovered by Galileo and because they "transform" our velocity from one perspective to another.

It might be tempting to try to do relative motion problems without the subscripts, but don't do it! The subscripts are really important for keeping the motions straight.

Let's do an example:



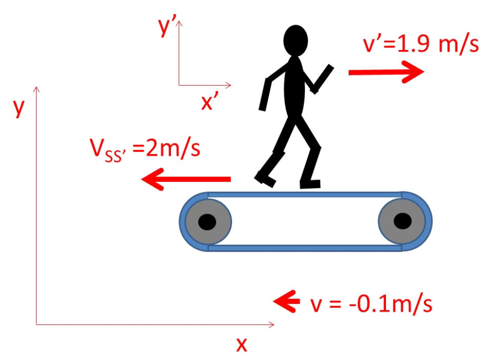
Suppose we have a boy (b) in the gym running on a treadmill. Let's call the room frame R and the top of the treadmill belt frame B (for belt). Then our transformation equations would be

$$\begin{aligned}\vec{\mathbf{v}}_{bR} &= \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BR} \\ \vec{\mathbf{v}}_{aB} &= \vec{\mathbf{v}}'_{aR} - \vec{\mathbf{V}}_{BR}\end{aligned}$$

The treadmill track belt has a relative velocity $\vec{\mathbf{V}}_{BR} = -2\frac{\text{m}}{\text{s}}\hat{\imath}_R$ with respect to the room. A person standing on the treadmill in frame B could see his/her self as not moving, and the rest of the room as moving the opposite direction.

The notation $\vec{\mathbf{V}}_{BR}$ means the velocity of the reference frame B with respect to frame R or in our case the speed of the treadmill belt top with respect to the room $\vec{\mathbf{V}}_{BR} = -2\frac{\text{m}}{\text{s}}\hat{\imath}_R$.

Now suppose the person is running at speed $\vec{\mathbf{v}}_{bB} = 1.9\frac{\text{m}}{\text{s}}\hat{\imath}_B$ on the treadmill in the treadmill frame B.



What is his speed with respect to the room? It seems obvious that we take the two speeds and add them. This is the first of our Galilean transformation equations

$$\vec{\mathbf{v}}_{bR} = \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BR}$$
$$\vec{\mathbf{v}}_{bR} = 1.9\frac{\text{m}}{\text{s}}\hat{\imath}_B - 2\frac{\text{m}}{\text{s}}\hat{\imath}_R = -0.1\frac{\text{m}}{\text{s}}\hat{\imath}$$

since the $\hat{\imath}_B$ and $\hat{\imath}_R$ directions are the same (look at the figures). Everyone in the room frame sees the person moving backward toward the end of the belt. The person is going to fall off the end of the treadmill unless he/she picks up the pace!

Likewise, if we want to know how fast the person is walking with respect to the treadmill belt frame, we would use the second of our Galilean transformation equations.

$$\vec{\mathbf{v}}_{bB} = \vec{\mathbf{v}}_{bR} - \vec{\mathbf{V}}_{BR}$$

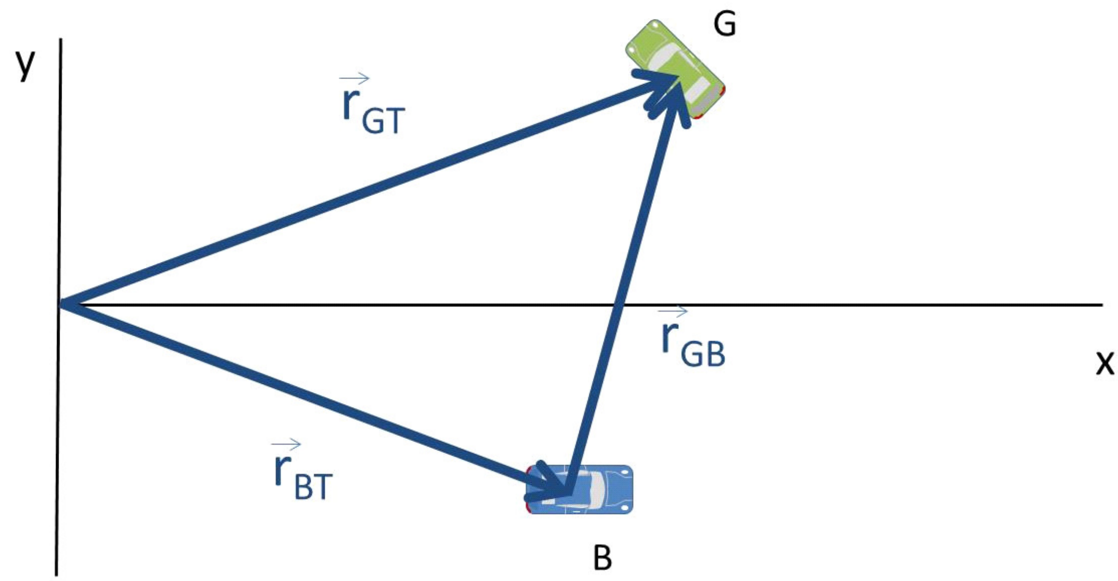
We take the room speed $\vec{\mathbf{v}}_{bR} = -0.1\frac{\text{m}}{\text{s}}\hat{\imath}$ and subtract from it the treadmill/room relative speed $\vec{V}_{BR} = -2\frac{\text{m}}{\text{s}}\hat{\imath}$ to obtain

$$\vec{\mathbf{v}}_{bB} = -0.1\frac{\text{m}}{\text{s}}\hat{\imath}_R - \left(-2\frac{\text{m}}{\text{s}}\hat{\imath}_R\right) = 1.9\frac{\text{m}}{\text{s}}\hat{\imath}_B$$

Using the Galilean transformation equations in one-dimension is tricky, but not too bad if we keep our subscripts straight. But now we can do two-dimensional problems. And our transformations are vector equations that work just fine for two-dimensional relative motion. Let's give this a try.

Relative velocity in two dimensions

The situation is even more complicated in two dimensions. Let's consider two cars. We will label them car G (for green) and car B (for blue). We will label the origin T (indicating that this is coordinate system is fixed with the track) and assume we have a third person at the origin observing our cars.



Then $\vec{r}_{GT}$ is the position of car G as measured by the person standing on the track, T. Likewise, $\vec{r}_{BT}$ is the position of car B as measured by the person on the track, T. And $\vec{r}_{GB}$ is the position of car G as measured by the person in car B. The first subscript tells us which object is being measured, and the second tells us what viewpoint is being used. Looking at the figure, if we know $\vec{r}_{BT}$ and $\vec{r}_{GT}$ as position vectors. How do we find $\vec{r}_{GB}$?

$$\vec{r}_{GB} = \vec{r}_{GT} + \vec{r}_{TB}$$

We just use our subscript notation, and relative motion problems become much simpler.

Suppose we want to know the instantaneous velocity of car G with respect to car B. We know

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt}$$

so let's try

$$\overrightarrow{v}_{GB} = \frac{d\vec{r}_{GB}}{dt}$$

this would be

$$\overrightarrow{v}_{GB} = \frac{d\vec{r}_{GB}}{dt} = \frac{d\vec{r}_{GT}}{dt} + \frac{d\vec{r}_{TB}}{dt}$$

or

$$\vec{v}_{GB} = \vec{v}_{GT} + \vec{v}_{TB}$$

This is just the same as one of our transformation equations, the first one (plus a little algebra)

$$\overrightarrow{\mathbf{v}}_{bR} = \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BR}$$

only with G for b because the moving object now is the green car, and with T in place of R because we are on a track and not in a room. The subscript B now means "car B" instead of "treadmill belt."

$$\overrightarrow{\mathbf{v}}_{GB} = \overrightarrow{\mathbf{v}}_{GT} + \overrightarrow{\mathbf{V}}_{TB}$$

Let's summarize what we have learned about solving relative motion problems.

1. Label each object involved with a letter. Letters that remind you of the object are better

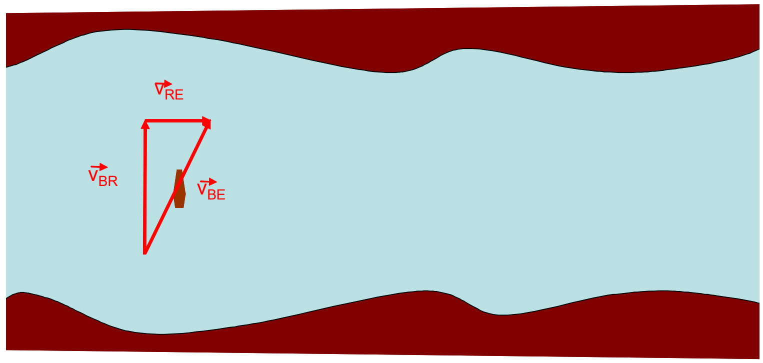
2. Look through the problem for phrases like "the velocity of object A relative to object B." If the velocity of one object is not specifically stated as being relative to another object, is usually the velocity with respect to the Earth, or a room or track that is attached to the Earth.

3. Take the velocities you have identified and arrange them into our transformation equations. If you don't have the velocities, you may need to look at position vectors and take a derivative.

4. Solve for the unknown x and y components of the velocity. Remember that we are still making our two-dimensional problem into two one-dimensional problems!

Example

Problem Statement: You have herd the fishing is great in Idaho and so you rent a boat and take your HFE group out fishing on the river. Your heading is due north and your speed with respect to the water is $v_{BR} = 10.0\,\mathrm{km/\,h}$. The river goes by to the east with a speed with respect to the shore is $v_{RE} = 5.00\,\mathrm{km/\,h}$. How fast would a forest ranger see the boat if the ranger is standing on the shore?



Let's choose the y axis to be positive in the northern direction. The x axis will be positive in the eastern direction.

Variables:

$\vec{v}_{BR} = 10.0\,\mathrm{km/\,h}\hat{\jmath}$	Velocity of the boat with respect to the river
$\vec{v}_{RE} = 5.00\,\mathrm{km/\,h}\hat{\imath}$	Velocity of the river with respect to the shore (Earth)
$\vec{v}_{BE}$	The velocity of the boat with respect to the shore

Basic Equations:

We will use our transformation equations

$$\begin{aligned} \vec{\mathbf{v}}_{bA} &= \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BA} \\ \vec{\mathbf{v}}_{aB} &= \vec{\mathbf{v}}_{aA} - \vec{\mathbf{V}}_{BA} \end{aligned}$$

but we will also need our vector recombination set

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ \theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right) \end{aligned}$$

Symbolic Solution:

Using the picture we can write the transformation equation with our vectors:

$$\vec{v}_{BE} = \vec{v}_{BR} + \vec{v}_{RE}$$

this is the first in our transformation set.

Now this is two-deminsional problem, so we will make it into two one-dimensional problems by taking components of the vectors.

$$v_{BEx} = v_{BRx} + v_{REx}$$

and

$$v_{BEy} = v_{BRy} + v_{REy}$$

so

$$\vec{v}_{BE} = (v_{BRx} + v_{REx})\,\hat{\imath} + (v_{BRy} + v_{REy})\,\hat{\jmath}$$

this is the vector, but we want the magnitude and direction. Let's take our direction to be ϕ, the angle between the direction we wanted (north) and the direction we actually go, then

$$\begin{aligned} v_{BE} &= \sqrt{v_{BEx}^2 + v_{BEy}^2} \\ \phi &= \tan^{-1}\left(\frac{v_{BEx}}{v_{BEy}}\right) \end{aligned}$$

Numeric Solution:

We can now plug in the numbers, noting that v_{RE} was conveniently all in the x-direction and v_{BE} was conveniently all in the y-direction.

$$\begin{aligned} \vec{v}_{BE} &= \left(0 + 5.00\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\imath} + \left(10.0\frac{\mathrm{km}}{\mathrm{h}} + 0\right)\hat{\jmath} \\ &= \left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\imath} + \left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\jmath} \\ v_{BE} &= \sqrt{\left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)^2 + \left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)^2} = 11.18\frac{\mathrm{km}}{\mathrm{h}} \\ \phi &= \tan^{-1}\left(\frac{\left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)}{\left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)}\right) = 26.565\,^{\circ} \end{aligned}$$

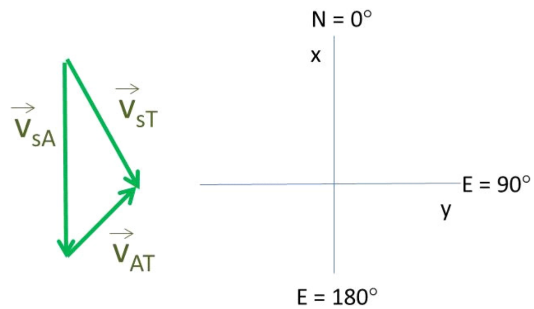
That wasn't too bad, in fact it seamed pretty easy. Let's do another example

On a distant planet, Duke Mudwalker is traveling due south at a speed of $60.0\,\mathrm{km/\,h}$ in his land speeder when a dust storm hits with winds from the southwest at a speed of $10.0\,\mathrm{km/\,h}$. What is the velocity of the land speeder with respect to the planet surface? Is Duke going to get where he wants to go?

ANSWER:

PT: Relative motion

Drawing:



VAR:

$$
\begin{aligned}
v_{sA} &= 60\,\mathrm{km/\,h} = 16.667\frac{\mathrm{m}}{\mathrm{s}} \\
v_{AT} &= 10\,\mathrm{km/\,h} = 2.7778\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_{sE} &= 180\,^{\circ} \\
\theta_{sE} &= 45\,^{\circ}
\end{aligned}
$$

BE

$$
\begin{aligned}
\vec{\mathbf{v}}_{bA} &= \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BA} \\
\vec{\mathbf{v}}_{bB} &= \vec{\mathbf{v}}_{bA} - \vec{\mathbf{V}}_{BA} \\
v_x &= v\cos\theta \\
v_y &= v\sin\theta
\end{aligned}
$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

We have to modify our basic equations for our situation. Let's say Duke lives on a planet called Tatoo. So we can let Tatoo's surface be one reference frame, with label T. Let's let the moving air mass be the other reference frame, labeled A. Then we know the speed of the speeder (s) with respect to the air (A) and the speed of the air (A) with respect to Tatoo (T) . We want

$$
\begin{aligned}
\vec{\mathbf{v}}_{sT} &= \vec{\mathbf{v}}_{sA} + \vec{\mathbf{V}}_{AT} \\
\vec{\mathbf{v}}_{sA} &= \vec{\mathbf{v}}_{sT} - \vec{\mathbf{V}}_{AT}
\end{aligned}
$$

and we see we want the first of the equations

$$\vec{\mathbf{v}}_{sT} = \vec{\mathbf{v}}_{sA} + \vec{\mathbf{V}}_{AT}$$

It would be easier to do this in components. So

$$\begin{aligned}
v_{sAx} &= v_{sA}\cos\theta_{sA} \\
v_{sAy} &= v_{sA}\sin\theta_{sA} \\
v_{ATx} &= v_{AT}\cos\theta_{AT} \\
v_{ATy} &= v_{sA}\sin\theta_{sA}
\end{aligned}$$

then

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_{sT} &= \overrightarrow{\mathbf{v}}_{sA} + \overrightarrow{\mathbf{V}}_{AT} \\
&= \left(v_{sA}\cos\theta_{sA} + v_{AT}\cos\theta_{AT}\right)\hat{\imath} \\
&\quad + \left(v_{sA}\sin\theta_{sA} + v_{sA}\sin\theta_{sA}\right)\hat{\jmath}
\end{aligned}$$

or

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_{sT} &= \left(\left(16.667\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(180^\circ\right) + \left(2.7778\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(45^\circ\right)\right)\hat{\imath} \\
&\quad + \left(\left(16.667\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(180^\circ\right) + \left(2.7778\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(45^\circ\right)\right)\hat{\jmath} \\
&= -14.703\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} + 1.9642\frac{\mathrm{m}}{\mathrm{s}}\frac{\mathrm{m}}{\mathrm{s}}\hat{\jmath}
\end{aligned}$$

and the magnitude would be

$$\begin{aligned}
v_{sT} &= \sqrt{\left(1.9642\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(14.703\frac{\mathrm{m}}{\mathrm{s}}\right)^2} \\
&= 14.834\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

with a direction of

$$\begin{aligned}
\phi &= \tan^{-1}\left(\frac{1.9642\frac{\mathrm{m}}{\mathrm{s}}}{-14.703\frac{\mathrm{m}}{\mathrm{s}}}\right) \\
&= -0.13281\,\mathrm{rad} \\
&= -7.6095^\circ
\end{aligned}$$

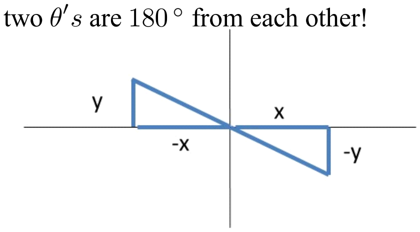
but this can't be. This is north west, and we can see from the figure we need south east. It needs to be more than 90°. What went wrong? Recall that tangent is the y part over the x part. But if one of these parts is negative, $\tan\theta$ can't tell which one.

$$\tan\theta = \frac{-y}{x}$$

looks just like

$$\tan\theta = \frac{y}{-x}$$

but the direction of the two $\theta's$ are 180° from each other!



And our calculator gave us the wrong direction. We need to add $180\,^{\circ}$.

$$\begin{aligned}\phi &= 180\,^{\circ} + -7.609\,5\,^{\circ} \\ &= 172.39\,^{\circ}\end{aligned}$$

where I had to provide my own minus sign from the drawing. Our units check and this seems reasonable. And Duke better turn into the wind a little or he is not going to get into town for those power converters.

Often we can arrange our problem to avoid haveing to deal with relative motion. But sometimes we can't. And we must always keep relative motion in mind.

13 Changing Motion: Newton's First Law

We have talked at length about displacement, velocity, and acceleration. But so far we have not discussed how we get motion started. When you get in your car, you know the engine is involved in changing your speed from $0\,\mathrm{m/s}$ to $26\,\mathrm{m/s}$. So there must be an acceleration. But how does this work? How do we start motion? And then, when you get to the parking lot, you want to go from your $26\,\mathrm{m/s}$ back to $0\,\mathrm{m/s}$. How do we change motion once we have it?

Really we have discussed this. We have had spring cannons, and people throwing balls. From the examples we have had, we know the answer. To change motion, we give something a push! The spring in the spring cannon pushes the ball. The person's hand pushes the ball. The tires of the car push against the road, and the road pushes back, making the car go forward.

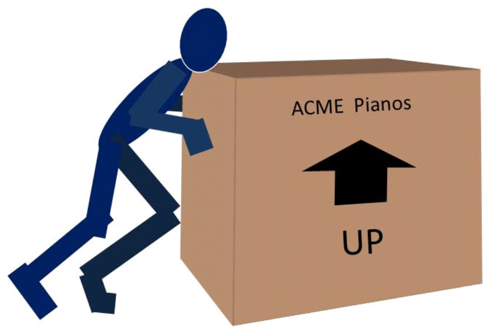
Of course, we could also pull something by tying a rope on it and tugging. This may not sound very scientific, changing motion by pushing or pulling. So let's give a new name to a push or a pull. Let's call a push or a pull a *force.*

To change a motion, there needs to be two objects involved. One that is doing the moving, and the other creating the force. In this course, we will call the moving object the *mover*. The other object makes the force that causes the change in the mover's motion. We will call the object that makes the force the *environmental object*. This environmental object is changing the environment in which the mover moves.

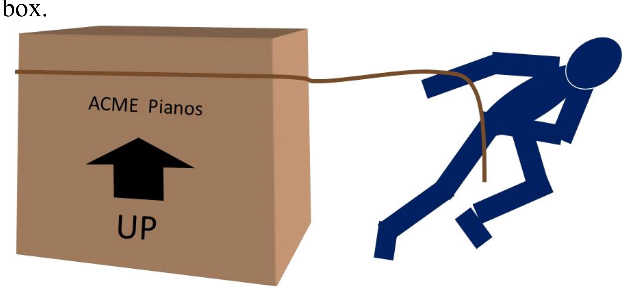
That last sentence was a whopper! What we mean is that the second object is doing the pushing or the pulling. It is something in the near vicinity of the mover (the mover's environment) that pushes or pulls on the mover. The volume of space around the mover we will call the *mover environment*. This is why we will call the object in the region around the mover that does the pushing and pulling the "environmental object" so we know it is not the object who's movement we are studying.

Types of forces

Some pushes and pulls are obvious. If I push on a piano, you see me with my hands on the piano, and watch it change it's motion. You are justified in concluding that the contact between my hands and the piano have somehow caused the motion of the piano to change. The piano is the mover in this example, and I am the environmental object making the force.



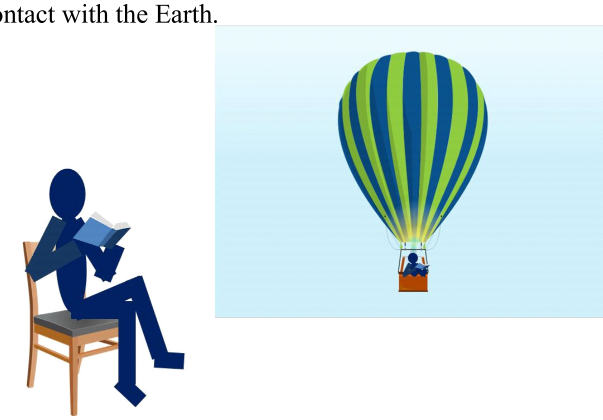
Likewise, if I tie a rope to the piano, and pull, you would see me, the rope, the piano, and the change in motion and conclude with justification that my pull from the rope was responsible for the change in motion. The piano box is still the mover, and now the rope is the environmental object pulling on the box. You might object to this analysis. Isn't it the person pulling that is making the force? of course you are right, ropes don't pull on objects unless there is another object pulling on them. But since the box is the mover, I am only concerned about what is creating the force on the box, not what is creating the force on the rope. So it is correct to say that the rope is the environmental object for the box.



In both of these cases, physical contact was necessary for the mover (the piano box) to change motion. We will call this type of force that requires physical contact a *contact force*.

But there are other ways to change motion. Suppose I give you a magnet and have you walk over to a workbench covered with nuts and bolts. You can predict that the magnet will change the motion of the nuts and bolts even before it makes contact with them. The nut (see the figure) is the mover and the magnet is the environmental object making the force due to magnetism. The magnet doesn't have to touch the nut to move it, so we will call this kind of force a *non-contact force.*

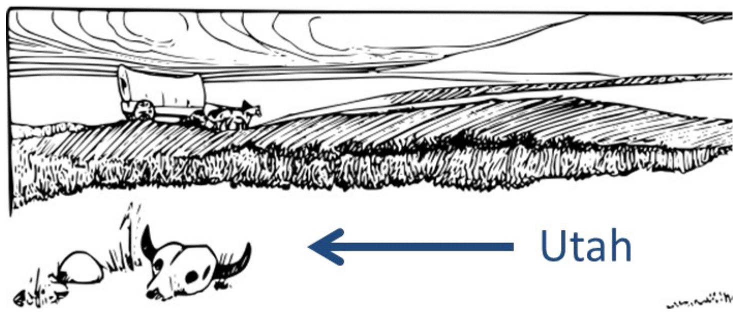
As you sit in your chair reading your physics reading assignment, another non-contact force is acting on you. It is the force of gravity. The Earth is the environmental object making the force due to gravity that is pulling you down. Right now you are probably in contact with the Earth, but the force of gravity would be pulling on you even if you were not in contact with the Earth.



The Earth's gravitational force is a non-contact force.

Forces are vectors

Suppose we hitch a team of oxen to a wagon. The wagon is our mover, and the oxen are the environmental objects pushing on the wagon tongue to change the wagon's motion. But does it matter which way the oxen pull?

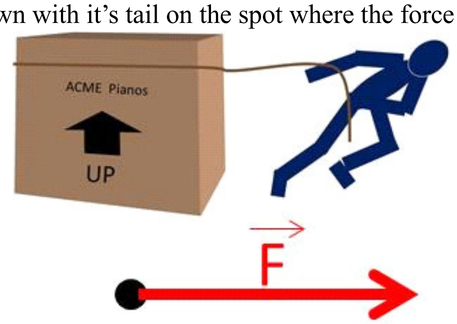


Of course it does! Forces must have direction, and we know what to call a quantity that has a magnitude and a direction. Forces are vectors.

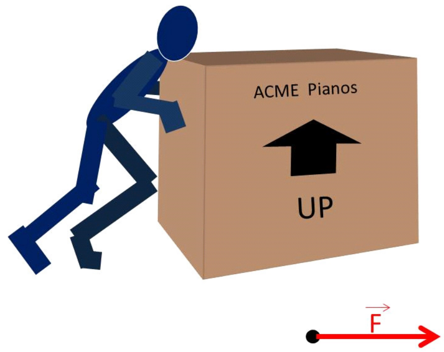
The magnitude of the force is how hard it pushes or pulls, and the direction is which way it pushes or pulls. In our wagon example, we want the force to pull to the west $270\,°$ on a compass. If they pull at $90\,°$, they get to Boston instead of Salt Lake City. That would make a difference!

Let's review vectors for forces.

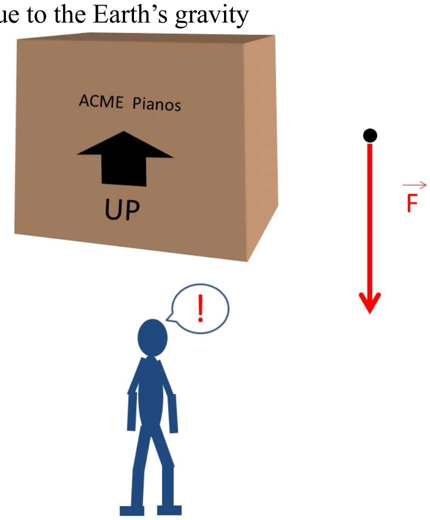
A force vector is drawn with it's tail on the spot where the force is applied.



We still use the particle model. So we draw the whole piano box as just a dot. Then we draw the force vector with the length proportional to how hard the rope is pulling on the box, and make the arrow point the way the box is begin pulled. Here is a vector representation of the box being pushed

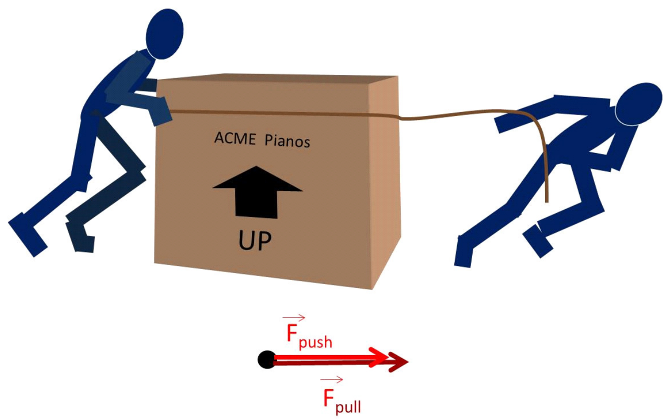


and here is the force due to the Earth's gravity

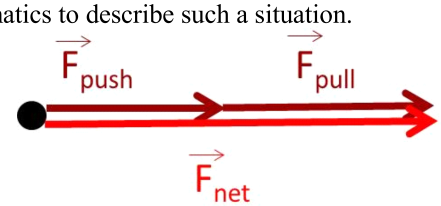


In each case the object is drawn as a particle, just a dot, and the vector shows how strong our push or pull is, and in what direction the push or pull is pointing.

Suppose we have more than one push or pull



You know from experience that two pushes or a push and a pull working together are more likely to change the motion of an object. We can use our wonderful vector notation and mathematics to describe such a situation.



Vectors add by placing the tail of the second vector on the tip of the first vector, then drawing a vector from the tail of the first to the tip of the second. This is the sum or *net* vector.

$$\overrightarrow{F}_{net} = \overrightarrow{F}_{push} + \overrightarrow{F}_{pull}$$

If the piano box was very heavy, it might take a whole group of people to push it. To keep our equations small, we could use summation notation. We give each person a number, so we can tell which person caused which force. Then the net force is just

$$\overrightarrow{F}_{net} = \overrightarrow{F}_1 + \overrightarrow{F}_2 + \overrightarrow{F}_3 + \overrightarrow{F}_4 + \overrightarrow{F}_5 + \overrightarrow{F}_6$$

which we can write as

$$\overrightarrow{F}_{net} = \sum_{n=1}^{6} \overrightarrow{F}_n$$

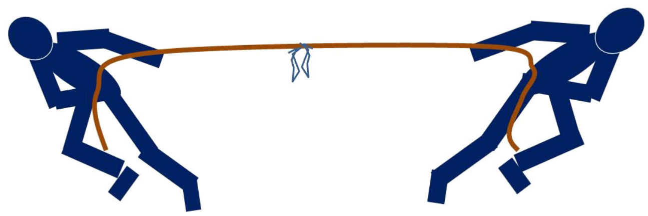
This looks like a smaller equation, but it really has just the same meaning as the equation before it. And note that the summation is a vector summation, That means for two-dimensional problems we really want to take components of the force vectors and separate our problem into x and y-parts

$$F_{net_x} = \sum_{n=1}^{6} F_{xn}$$

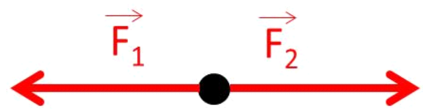
$$F_{net_y} = \sum_{n=1}^{6} F_{yn}$$

Let's try a problem.

Suppose we have two people pulling on a rope, but they pull opposite directions. They each pull with a force of $5\,\mathrm{N}$. What is the net force.



We first draw a vector diagram for the forces.



We can see that all the forces are in the x-direction and that they point opposite directions as expected.

We can write

$$\begin{aligned} F_{net_x} &= \sum_{n=1}^{6} F_{xn} \\ &= F_{1x} + F_{2x} \\ &= F_1 \cos\theta_1 + F_2 \cos\theta_2 \end{aligned}$$

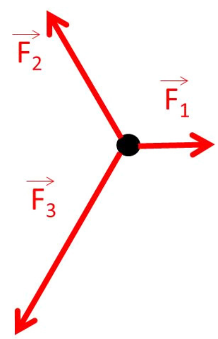
and we can see that $\theta_1 = 180\,^\circ$ and $\theta_2 = 0\,^\circ$. We know all the pieces so we can put in values

$$\begin{aligned} F_{net_x} &= (5\,\mathrm{N})\cos(180\,^\circ) + (5\,\mathrm{N})\cos(0\,^\circ) \\ &= -5\,\mathrm{N} + 5\,\mathrm{N} \\ &= 0 \end{aligned}$$

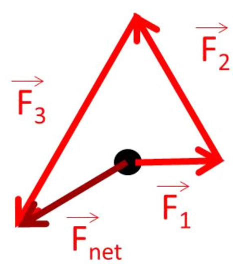
Notice how much direction mattered in this case! Also notice that we have introduced a unit for force, the *Newton*, abbreviated N.

Let's try another problem. Suppose we have three forces applied to an object, all spaced $120\,^\circ$ from each other. Force 1 has a magnitude of $10\,\mathrm{N}$ and is at $0\,^\circ$. Force 2 has a magnitude of $20\,\mathrm{N}$ and is at $120\,^\circ$. Force 3 has a magnitude of $30\,\mathrm{N}$ and is at $240\,^\circ$. What is the net force?

We will need a vector diagram for the forces



And we could use graphical vector addition to see what $\overrightarrow{F}_{net}$ will look like



and we recognize that from the problem statement we know

$$
\begin{aligned}
F_1 &= 10\,\mathrm{N} \\
F_2 &= 20\,\mathrm{N} \\
F_3 &= 30\,\mathrm{N} \\
\theta_1 &= 0\,^\circ \\
\theta_2 &= 120\,^\circ \\
\theta_3 &= 240\,^\circ
\end{aligned}
$$

But let's use our vector component set of equations to solve this:

$$
\begin{aligned}
v_x &= v\cos\theta \\
v_y &= v\sin\theta
\end{aligned}
$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Separate the two-dimensional problem into two one-dimensional problems using our force component summation equations

$$F_{net_x} = \sum_{n=1}^{6} F_{xn}$$

$$F_{net_y} = \sum_{n=1}^{6} F_{yn}$$

then

$$F_{net_x} = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3$$
$$F_{net_y} = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3$$
$$F_{net_x} = \sqrt{F_{net_x}^2 + F_{net_y}^2}$$
$$\theta = \tan^{-1}\left(\frac{F_{net_y}}{F_{net_x}}\right)$$

$$\begin{aligned} F_{net_x} &= (10\,\mathrm{N})\cos(0\,^\circ) + (20\,\mathrm{N})\cos(120\,^\circ) + (30\,\mathrm{N})\cos(240\,^\circ) \\ &= -15.0\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} F_{nety} &= (10\,\mathrm{N})\sin(0\,^\circ) + (20\,\mathrm{N})\sin(120\,^\circ) + (30\,\mathrm{N})\sin(240\,^\circ) \\ &= -8.660\,3\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} F_{net_x} &= \sqrt{(-15.0\,\mathrm{N})^2 + (-8.660\,3\,\mathrm{N})^2} \\ &= 17.321\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-8.660\,3\,\mathrm{N}}{-15.0\,\mathrm{N}}\right) \\ &= 0.523\,6\,\mathrm{rad} \\ &= 30.0\,^\circ \end{aligned}$$

Looking at our diagram we realize that this angle can't be right. We should be in the third quadrant because both F_{net_x} and F_{net_y} are negative. Our inverse tangent gave the angle with respect to the $-x$-axis. We usually want to report an angle with respect to the $+x$-axis. We need to add $180\,^\circ$ to our result to get the angle with respect to the positive x-axis.

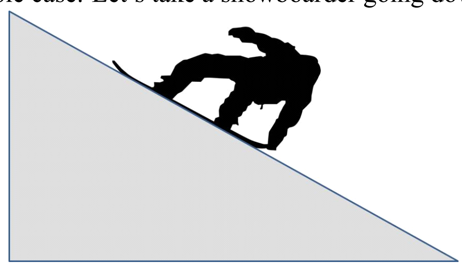
$$\begin{aligned} \theta &= 180\,^\circ + 30\,^\circ \\ &= 210\,^\circ \end{aligned}$$

This seems reasonable

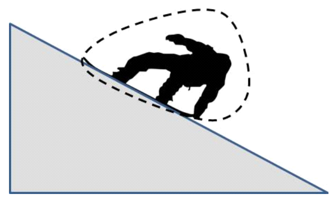
Force Diagrams, Mass, and Acceleration

You can guess that with a new topic, we will need to learn new techniques for restating our problems as drawings. A large part of the task of drawing the situation will be to identify what is the moving object, and what is the environment. You may not believe that this could be difficult. But it often is.

Let's start with a simple case. Let's take a snowboarder going down a slope.

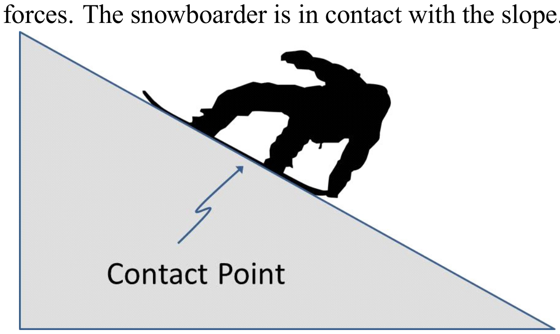


The snowboarder would be the moving object. If you draw a picture of the snowboarder situation as part of your drawing, it might be good to circle the snowboarder to indicate that it is the mover.



We should identify where forces may be acting on our mover object (the snowboarder). We can guess that gravity is pulling down on the snowboarder. Gravity is a non-contact force. And it pulls on just about everything that is near the Earth's surface. The Earth must be an environmental object acting on the snowboarder. The gravitational force will pull the snowboarder in the direction of the Earth's center.

It is unlikely that other non-contact forces will be acting on the snowboarder (the snowboarder is unlikely to have a magnetic suit or an overall electric charge). So let's consider contact forces. The snowboarder is in contact with the slope.

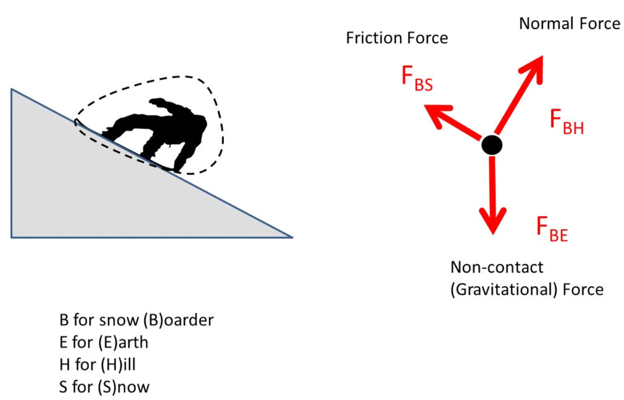


It is likely, then, that the slope is an environmental object acting on the snowboarder.

To be sure, we need to identify the mechanism for a slope force acting on the snowboarder. We can recognize that the hill atoms will be compressed by the weight of the snowboarder. The hill atoms will resist being compressed. They will push back on the snowboard, creating a normal force.

It is also true that there is a friction force due to the rough snow. You probably know that you need to wax your snowboard to reduce this friction. But some friction will still act on the snowboard. The friction force always acts in a direction opposite the direction the mover is going. So the friction force would act up the slope.

We put all this information about the forces acting on our mover object (the snowbaorder) into a diagram. Using the particle model, we draw a dot that represents the snowboarder.

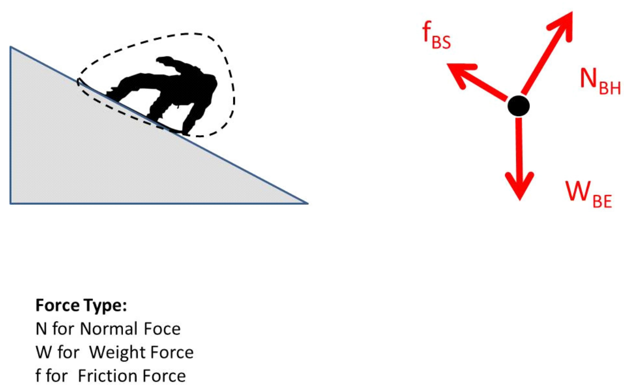


Arrows represent the force vectors. Their tails are on the dot, indicating that the forces act at the location of the mover. Notice that the length of the arrows are different. The force vector lengths give the relative magnitude of the forces. Also notice that we have drawn the vectors in the direction the forces act.

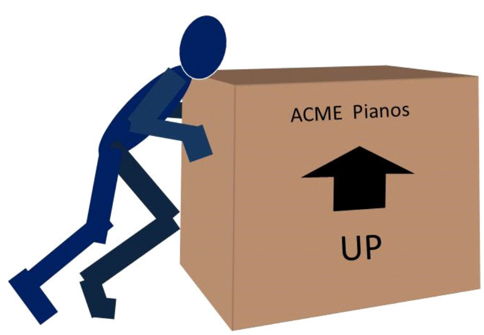
This particle model diagram with force vectors is called a *free body diagram*. In the old days, the word "body" was used instead of the word "object." So maybe we should call it a "mover object diagram." The diagram is for one object only! It just contains forces acting on the mover object. If we have two objects that can move, we need two free body diagrams, one for each mover. Our goal is always to be able to study the motion of an individual object. If I have more than one individual object, I split the problem into parts, one part for each mover object.

Notice that we have again used a set of subscripts. For each force we have identified the two objects involved in the force. The first subscript is for the mover object. In this case it is the snow(B)oarder. The second is the environmental object. In the case of gravity, it is the Earth that is pulling on the snowboarder, so we have used and "E" for "Earth" as the subscript for the gravity force.

We can further clarify our diagram by labeling different force types with different letters. After all, we know they are forces so the F is not telling us anything we don't already know! We could use a "N" to label a normal force, for example. A small "f" is traditional for friction forces. And for gravitational forces, we could use a "W" for "weight force." Sometimes the letters F_g are use for a force due to gravity, but the subscript "g" gets in the way of our mover-environment subscripts. So for now we will prefer "W." Here is how it might look.

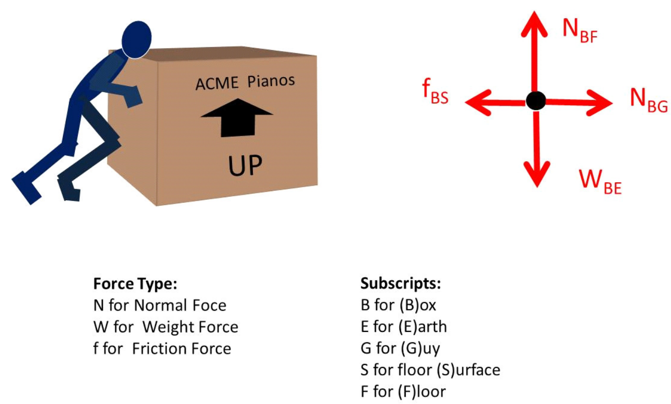


Let's try another example. Consider again a guy pushing a box. What forces act on the box?



Let's draw a free body diagram.

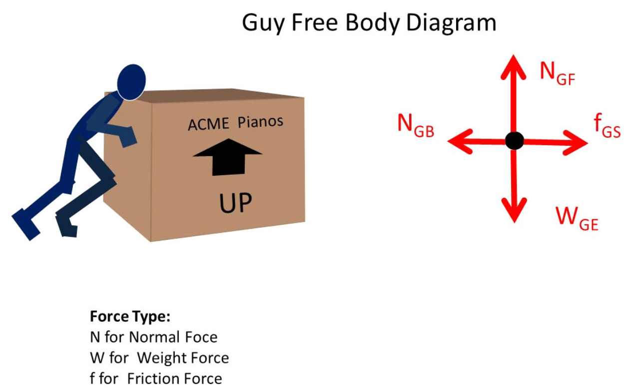
Box Free Body Diagram



Notice that each force is identified by it's letter symbol. The guy's atoms press against the box, so the guy's push is due to the guy's compressed atoms. It is a normal force! We recognize the normal force due to the floor holding up the box, the friction force, and the gravitational weight force.

The subscripts all start with "B" for "box." The box is our mover. The rest of the subscripts tell us what environmental objects act on the box. We can see that the Earth is pulling on the box as expected. We know the guy is pushing on the box. We know the floor is pushing up on the box. We could use the "F" for floor in both the normal force and the friction force. But I have chosen to indicate that the forces are fundamentally different by saying the floor rough surface is the source for the friction force, using an "S" for this aspect of the floor, and leaving the floor's compression strength to be indicated by the "F."

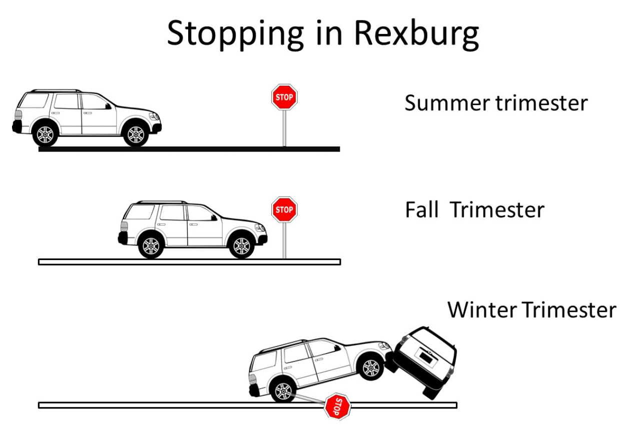
Now let's draw the diagram for the guy! We have two moving objects. We chose the box for our lasts diagram, but we could draw a diagram for the guy as well.



Note that for the guy’s diagram each subscript set begins with a “G” for “guy.” The guy pushes on the box, but the squashed atoms of the box push back. We see that in the normal force N_{GB}. The guy has a weight, W_{GE}, and is held up by a normal force due to the floor, N_{GF}. If the floor were perfectly smooth, the guy would not be able to move himself or the box. So there must be a friction force. The guy pushes his feet against the floor, and the rough floor pushes back. This friction force is why the guy moves forward. We label it f_{GS}.

Newton’s First Law

Let’s start with a BYU-I Rexburg area public service announcement. Suppose you are driving in your car and you come to a stop sign. You should stop, of course. Let’s start our experiment in summer. There is no problem, you press on the break peddle and the car stops



This is because the tires stop spinning. That is what your breaks do, stop the tires. And since there is a frictional force due to the road on the tires of the car, then stopping the tires from spinning leads to the car stopping.

During the fall trimester, however, there is some snow and it get’s cold. The friction between your tires and the road is reduced (the little roughness teeth are full of ice). So it takes longer to stop. During winter trimester, the city resurfaces the ice to smooth it out, making the ice less rough. This makes the force due to friction smaller. Now, with little to no force acting on your car, it is nearly impossible to stop!

This situation is an example of a profound understanding of how our universe works. What we found for an object (the car) is that unless there is a force acting on the car (in our case, friction) the motion of the object doesn’t change. You might hear this expresses like this:

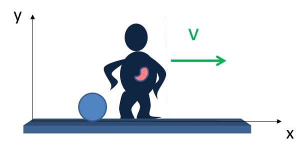
> An object that is in motion stays in motion unless a force is acting on it.

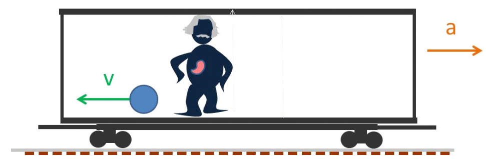
In our car case. when the ice removes the force due to friction, there is no force acting on the car to make the car stop. So the car does not stop. I prefer to express this by saying that it takes a force to change motion, something that we already know.

Newton wrote down this profound viewpoint with some flair, so he gets credit for this thought. We call it *Newton’s first law.* But this law only tells us that without forces, motion does not change. We need the idea called *Newton’s second law* to explain the origin of all the motion we experience.

Reference Frames and Forces

Let’s consider aliens on platforms again:





Notice that if we have a slightly larger space alien on his/her/its platform reference frame, he/she/it is free to believe that his/her/its reference frame has no motion. As we now know, that is because there is no net force acting on the ball, or the space alien's stomach, or anything else in the reference frame. We could write this as

$$\overrightarrow{F}_{net} = 0$$

But if we take a similar situation, say, a man in a train, but we make the train accelerate, we can see that the man will recognize that something is changing. The man's skeletal structure will move with the train, but his stomach, floating free of the skeletal structure, is not pushed by the train. It tries to stay at the same velocity (Newton's first law!). Likewise, the ball will try to stay in place. But since the man is viewing the ball and his stomach from the reference frame moving with the train car, he will "see" the ball move toward the back of the car and feel his stomach move backward relative to his skeleton. Both tell us that there is an acceleration, and therefore both tell us that there must be a net force.

$$\overrightarrow{F}_{net} \neq 0$$

Our idea of a reference frame needs a modification. We will call the alien's reference frame an *inertial reference frame*. This means that the reference frame, itself, is not accelerating. The train car we will call an *accelerated reference frame.* And we will leave accelerated reference frames for another physics class.[7] In this class we will only deal with intertial reference frames.

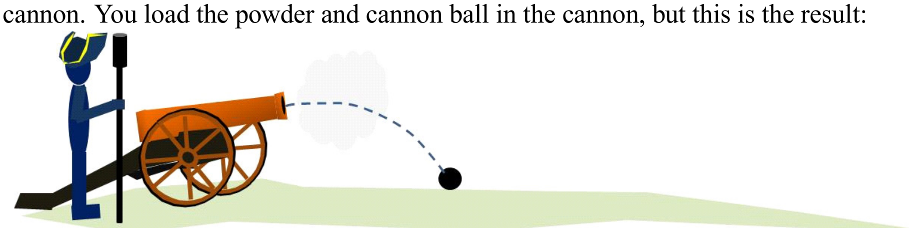
Now on to Newton's second law!

[7] Specifically, a physics class that deals with Einstein's Theory of General Relativity.

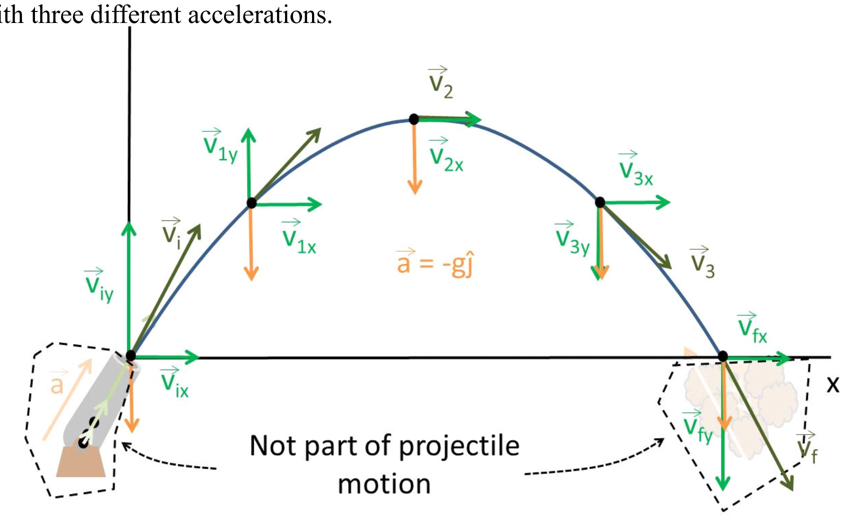
14 Newton's Second Law

We said last lecture that forces change motion. Changing motion is acceleration. So forces must cause acceleration. This should make intuitive sense. If I push on an object, it begins to move. my push accelerated the object. From experience you might guess that the harder we push, the larger the acceleration. Let's take an example to see that this is true.

Suppose you are a soldier in the French Revolution. You have been assigned to use a cannon. You load the powder and cannon ball in the cannon, but this is the result:

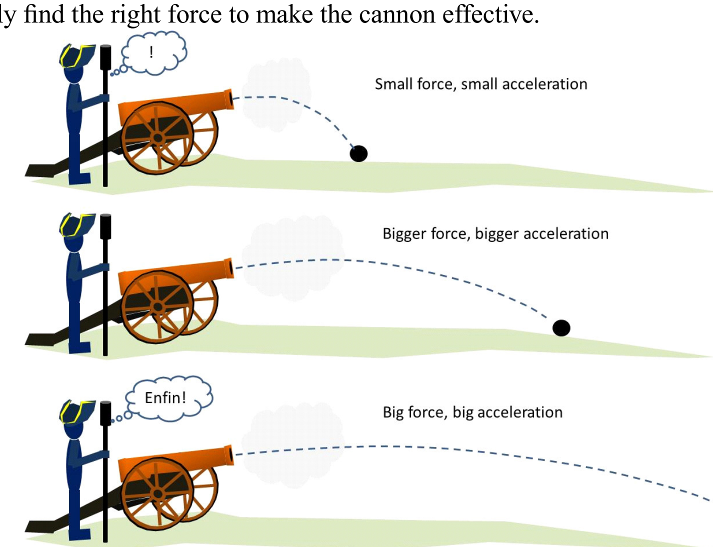


Last time we studied the flight of a cannon ball, we divided the problem into three parts with three different accelerations.



Back then, we only studied the second part of the motion, the projectile motion part. We are now experts in projectile motion, so let's use what we know about projectile motion to study the first part of the cannon problem, the part where the cannon ball is in the cannon and the exploding gunpowder is exerting a force on the ball.

Think about our projectile motion study. We know that the acceleration of the cannonball after it leaves the cannon is free-fall acceleration, $-g$. We know that how far the cannon ball goes depends on the initial speed of the cannon ball once it leaves the cannon. That initial speed depends on how much acceleration the ball has while it is inside the cannon. The power provides a force that causes this acceleration when the powder ignites. What would you think if you received the above result? Not enough powder! There was not enough force to accelerate the ball enough make a good initial velocity for the free-fall part of the cannon problem. How would we fix this? Add more powder! That is, we will make a bigger force. A bigger force makes a bigger acceleration, providing a larger initial velocity for the free fall part. By trial and error you would eventually find the right force to make the cannon effective.

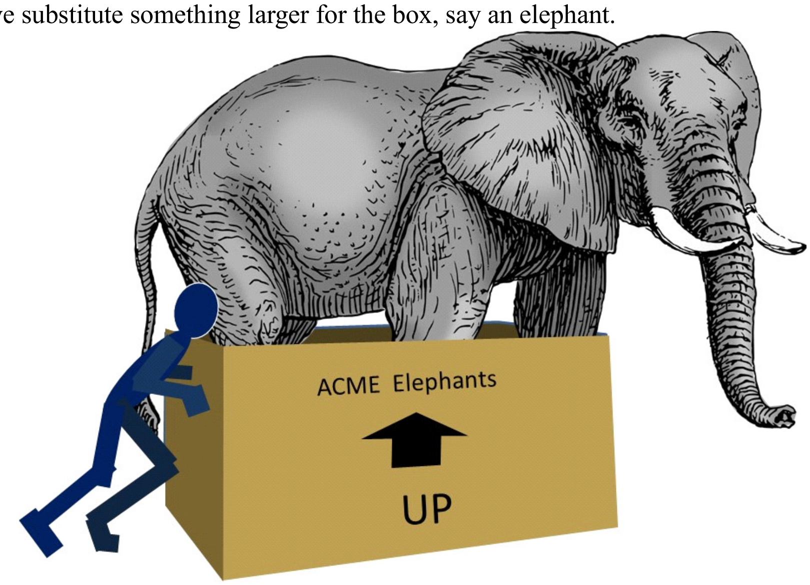


Let's summarize what we have said so far. It looks like the acceleration is proportional to the force that creates it.

$$a \propto F$$

The harder we push the more acceleration we get. But there must be more to this. Suppose we take our guy pushing the box again.

but we substitute something larger for the box, say an elephant.



You might guess that the elephant won't accelerate as much as the box (unless you get it angry). The amount of material in the object seems to matter in how effective a force is in causing acceleration. We could say this mathematically by writing

$$a \propto \frac{1}{m}$$

that is, we get less acceleration when we have a more massive object. We can combine these two effects into one equation

$$a = \frac{F}{m}$$

which tells us that our object will accelerate more if we push hard, or if we somehow reduce the mass of the object.

Sometimes you will see this written as

$$F = ma$$

which says that the harder we push the more acceleration we get, but that if the mass is larger the push has to be larger to get the same acceleration. Of course, forces and accelerations are vector quantities, but mass is scalar. So let's use vector notion for our equation.

$$\overrightarrow{F} = m\overrightarrow{a}$$

Now we can see that the force and acceleration must be in the same direction as well.

Units for force

We can see from our discussion above that the units of force must be the units of acceleration times the units of mass

$$\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} \tag{14.1}$$

We have given a name for this combination of units. It is called the *Newton*.[8] It is abbreviated N.

$$1\,\mathrm{N} = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} \tag{14.2}$$

The equation we just found

$$\overrightarrow{F} = m\overrightarrow{a}$$

has a name. It is called *Newton's Second Law.* But what is a scientific law? Recall that it is most often an equation that shows how our mental model of the universe works. The force laws are old enough that only one of them fits this definition. The second law can be stated as an equation, and it is the one we just found!

Newton's Second Law and Equilibrium

Our first of Newton's laws talks about an absence of forces, but we know there are forces. Newton's second law tells us how motion changes when forces act. Newton's second law states:

> *The acceleration of an object is directly proportional to the net force*

[8] After Sir Isaac Newton, who was an early researcher that studied forces.

acting on an object and is inversely proportional to the mass of the object.

Mathematically this is written as

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\mathbf{F}}_{net}}{m} \tag{14.3}$$

where we know the variable $\overrightarrow{\mathbf{F}}$ stands for force. Notice that both $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{F}}$ are vectors. Remember that because forces are vectors, they add like vectors. Then

$$\overrightarrow{\mathbf{F}}_{net} = \sum_{i=1}^{N} \overrightarrow{\mathbf{F}}_i$$

The symbol Σ means to sum or add up all the forces, so $\Sigma\overrightarrow{\mathbf{F}}$ is a way to write the net force. To find the net force, we add up all the forces as vectors. Also remember that forces are vectors, so we must use vector addition. Another way you may see this written is

$$\sum_{i=1}^{N} \overrightarrow{\mathbf{F}}_i = \overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}} \tag{14.4}$$

This Newton's second law applies to only one object at a time. The object who's mass is m is the object that is being pushed by $\overrightarrow{\mathbf{F}}_{net}$ and which has acceleration $\overrightarrow{\mathbf{a}}$. This means that *we will have to write out a separate Newton's second law equation for every moving object in our problems!*

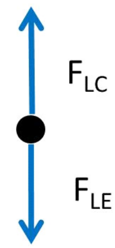
Suppose you are going to the temple and you observe the beautiful chandelier in one of the sealing rooms. The chandelier is not accelerating. This is a special case of equation (14.3)

$$0 = \frac{\overrightarrow{\mathbf{F}}_{net}}{m}$$

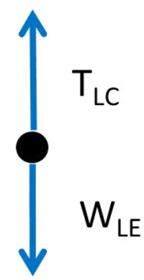
And it is an important special case. It says the net force is zero. It does not say that there are no forces acting on the object. But it says the forces balance, like two equally matched tug-o-war teams. So the object is not accelerating. We will call this situation *equilibrium.*

$$\overrightarrow{\mathbf{F}}_{net} = 0 \qquad \text{equilibrium}$$

The net force is the vector sum of all the forces. Let's find those forces and then sum them up to see if it makes sense that they would be zero. We start, of course, with a diagram



I used L for"light" to identify the object. I used C for "chain" to identify the environmental object for the upward force. I used E for "Earth" to identify the environmental object for the downward force (due to gravity). We should also note that the upward force is a tension force, and the downward force is a non-contact gravitational force. We could even modify our drawing to show this.



where the symbol "T" is used for tension forces and we know that "W" is used for weight forces (forces due to gravity).

We see that all our forces acting on the chandelier are in the y-direction, so for Newton's second law in the x-direction we have

$$F_{net_x} = 0$$

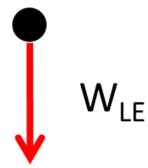
simply because none of the forces have x-components.

But

$$F_{net_y} = T_{LC} - W_{LE}$$

We should check, We only write out equations for one mover object at a time. So only forces with the first subscript, L, should be in our equation. Any force with a different first subscript would be a force acting on some other object, and could not be acting on our chandelier. Since we only have two objects (the chain and the light), it's not surprising that we got this right. But it might be more difficult if we had more objects (coming up soon). Notice that we might need a Newton's second law equation set for each free-body diagram we draw. That is, we could have a separate Newtons' second law equation set for every mover object in our problem.

We know about gravitational, weight, forces. Let's get a mathematical expression for such a force. Consider the case where the chandelier is not on a chain.[9] In that case there would only be one force on the chandelier (ignoring air drag). Then we would have just one force in our force diagram.



Then

$$F_{net_y} = -W_{LE}$$

and we know from Newton's second law that

$$F_{net_y} = ma_y$$

and since this would be a free fall situation we know

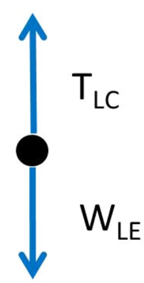
$$a_y = -g$$

thus

$$\begin{aligned} F_{net_y} &= -mg \\ F_{net_y} &= -W_{LE} \end{aligned}$$

So $W_{LE} = mg$ where m is the mass of the object, and g is the free-fall acceleration.

Let's go back to our chandelier hanging on a chain.



We could write our Newton's second law equation as

$$ma_y = 0 = T_{LC} - W_{LE}$$

but now from our free fall case we know that $W_{LE} = mg$ so we could write our net

[9] Not in the temple, maybe in a opera house in France, for example.

force in the y-direction as

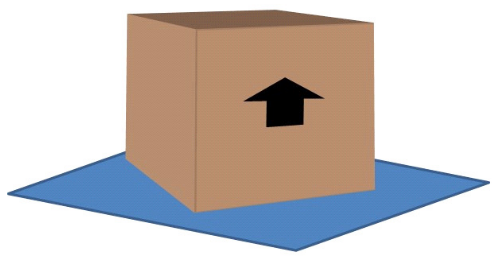
$$0 = T_{LC} - mg$$

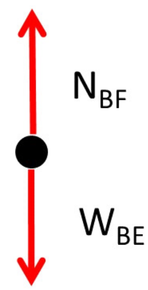
or just

$$T_{LC} = mg$$

which tells us that the chain tension must support the weight of the chandelier. This seems reasonable!

Let's try another force problem. Suppose you have been asked to help a neighbor move. The neighbor is a physics major and has conveniently marked each box with it's wight. You see a box marked $400\,\mathrm{N}$. What is the normal force that the floor must provide to support this box?





We would be surprised if the box accelerated on it's own, so we can identify this as an equilibrium problem. Since the box is not even moving (in our reference frame) we will give it another name, *static* (which means, "not moving"). Together, we call a problem where the net force is zero and the object is not moving *static equilibrium.* So this is a static equilibrium problem. The chandelier problem was also a static equilibrium problem. We know that

$$W_{BE} = 400\,\mathrm{N}$$

and we know that for static equilibrium we have no acceleration so $\overrightarrow{a} = 0$ and

$$\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{a} = 0$$

so we can see that

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

The net force is the sum of all the forces. So for the x-direction we could write

$$F_{net_x} = \Sigma_i F_{xi}$$

but for this example, this reduces to

$$0 = 0$$

because there are no x-components of the forces. In the y-direction

$$\begin{aligned} F_{net_y} &= \Sigma_i F_{yi} \\ &= N_{BF_y} + W_{BE_y} \end{aligned}$$

so that

$$F_{net_y} = N_{BF_y} + W_{BE_y}$$

and

$$F_{net_y} = ma_y = 0$$

We have two expressions for F_{net_y}. Let's set them equal to each other.

$$0 = N_{BF_y} + W_{BE_y}$$

Now let's find the y-components so we can put them into this equation we got from Newton's second law.

$$\begin{aligned} W_{BE_y} &= W_{BE} \sin\left(270\,^{\circ}\right) = -W_{BE} \\ N_{BF_y} &= N_{BF} \sin(90\,^{\circ}) = N_{BF} \end{aligned}$$

so or Newton's second law equation becomes just

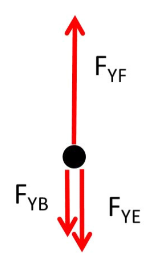
$$0 = N_{BF} - W_{BE}$$

and therefore

$$\begin{aligned} N_{BF} &= W_{BE} \\ &= 400\,\mathrm{N} \end{aligned}$$

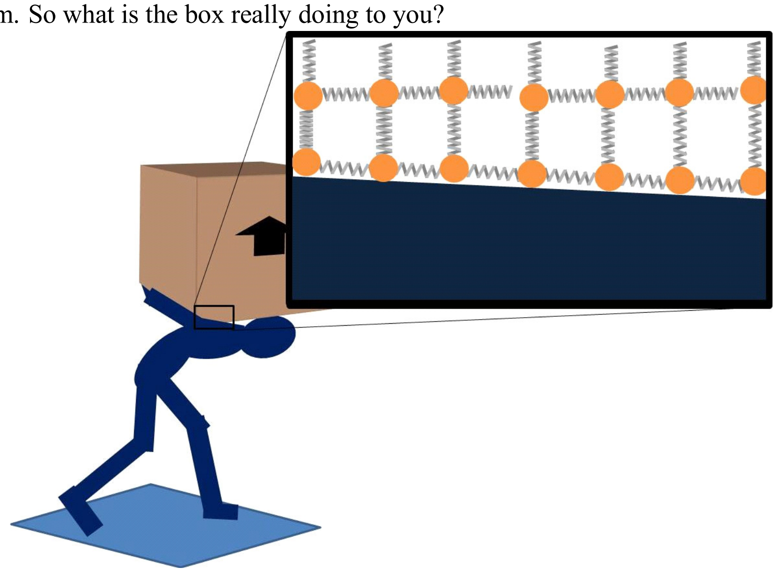
which says that the floor's normal force must be equal to the weight of the box. This seems to make sense.

But now you pick up the box. You know your own weight to be $667.2\,\mathrm{N}$. What is the normal force that the floor must supply to support you and the box?

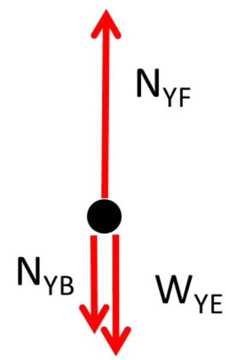


The picture shows our new situation. It seems reasonable that we have to have more forces than in the last case. After all now we have the box and you as objects. So at a minimum we must have your weight added into the situation. Since we are a little self-centered, let's consider you to be the mover (after all, you don't care about the box crashing through the floor nearly as much as you care about you crashing through the floor!). You would very much like to be in static equilibrium, not accelerating downward. We recognize that there will be a force due to the floor pushing you up. We also recognize that there is the Earth pulling you down. And we do need to account for the box pushing down on you. The Earth's pull is a weight, so we can write $F_{YE} = W_{YE}$ and the floor's push is a normal force, $F_{YF} = N_{YF}$. But what kind of force is the box's push?

It might be tempting to call this force the weight of the box. **But we must not do this!** The weight of the box is W_{BE} and this force is the force *on the box* due to *the Earth's gravity*. This is *not* a force on you! It is a force on the box. So it can't go on your force diagram. So what is the box really doing to you?



Let's look at the molecules that form the box. They have those spring-like molecular bonds. And as the Earth pulls the box down toward it, the molecules come into contact with your shoulder. The molecular bonds get squashed by your shoulder, and they push back. This is the force on you due to the box. And we recognize this type of force. This is a normal force! So $F_{YB} = N_{YB}$. We can complete our diagram.



We know

$$\begin{aligned} W_{BE} &= 400\,\mathrm{N} \\ W_{YE} &= 667.2\,\mathrm{N} \end{aligned}$$

and our basic equations are still

$$\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}}$$
$$\overrightarrow{\mathbf{F}}_{net} = \sum_i \overrightarrow{\mathbf{F}}_i$$

and for equalibrium

$$\overrightarrow{\mathbf{F}}_{net} = 0$$

or

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

We still have no x-part to this problem so let's just use our y-equation for $\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}}$

$$F_{net_y} = 0$$

then from the summation form of $\overrightarrow{\mathbf{F}}_{net}$

$$F_{net_y} = N_{YF} - N_{YB} - W_{YE}$$

and combining the two $\overrightarrow{\mathbf{F}}_{net}$ equations gives

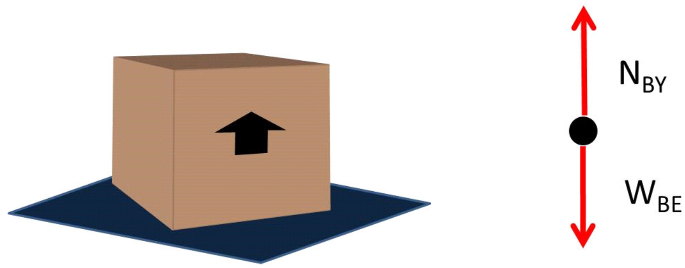
$$0 = N_{YF} - N_{YB} - W_{YE}$$

and we can solve for N_{YF}

$$N_{YF} = N_{YB} + W_{YE}$$

but we don't know N_{YB}! What can we do?

Let's consider our last problem. We found that the floor had to support the weight of the box for the box to not accelerate downward. So the normal force on the box had to be equal to the weight of the box.



This will be true for our case as well. Note! we are considering a second drawing for a second mover object – where N_{BY} is the normal force on the box due to the squashed atoms in your shoulder and W_{BE} is the force on the box due to the Earth's pull.

We should ask ourselves, which atoms get more squashed, your shoulder atoms or the box atoms? The answer is *neither!*. Think if we put the box on a large stack of Jello$^{®}$. The Jello atoms would not push back as hard on the box atoms as the box pushes on them, and the box would sink through the Jello. This does not happen with your shoulder! So we can reasonably assume that The box $N_{BY} = N_{YB}$. And we know from our analysis of the last problem that

$$N_{BY} = W_{BE}$$

so

$$N_{YB} = W_{BE}$$

You might feel a little cheated. Didn't I say that the force pushing down on you from the box was not the weight of the box? And that is right. But the force, a different force than the gravitational pull of the Earth on the box, a force that comes from little spring-like molecular bonds, *has the same magnitude as* of the weight of the box. Note, they are not the same force at all. They have different causes, one gravity and one compressed atomic bonds. But in this case they do have the same magnitude. So mathemtically now we can write

$$N_{YF} = W_{BE} + W_{YE}$$

$$\begin{aligned} N_{YF} &= 400\,\mathrm{N} + 667.2\,\mathrm{N} \\ &= 1067\,\mathrm{N} \end{aligned}$$

and we can see that the floor now supports both your wight and the box weight. This makes sense.

But we have learned something important along the way. We have to be very careful to correctly identify the source of each force. We may have to draw a free-body diagram for a second or third object to solve for the forces we want. If we incorrectly identify the source of the forces, we will eventually miss a force, and then the problem will be impossible until we correct our mistake. Let's take on a more difficult problem that will illustrate all this.

So far, we have only had forces in the y-direction, but you know we can handle two dimensional problems. Our same strategy that we used for kinematics motion problems works here. We will divide the two dimensional problem into two one-dimensional problems. We do this by finding components of the force vectors. Conveniently, the equation for net force can be resolved into components.

$$F_{netx} = ma_x \quad (14.5)$$

$$F_{nety} = ma_y \quad (14.6)$$

$$F_{netz} = ma_z \quad (14.7)$$

so we can turn two-dimensional force problems into two one-dimensional problems just like we did for projectile motion! Once again we turn a two or three-dimensional problem into two or three one-dimensional problems. We will take this on in the next lecture.

We also found that for normal foces the way the force is made wasn't so obvious. We will begin next time by looking at the origin of comon forces so we can understand how these forces will act.

15 Origins of Forces, Systems, and Newton's Third Law

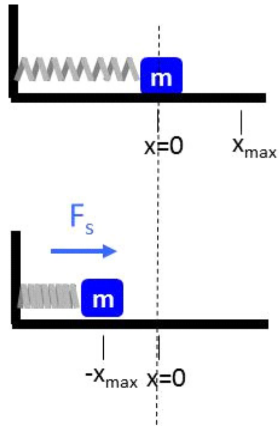
Let's start this lecture with a more in-depth look at where forces come from and how to draw force diagrams. One of our goals today is to determine what we can and can't include in our particle when using particle model. Once we have this figured out, we can tackle the last of the set Newton's force laws.

Origins of Forces

To practice solving problems with forces, we will need some environmental object acting on mover objects. So we need to know how environmental objects can act. We can't explain every type of force in this section, but we will explain a few forces, enough to start doing problems.

Spring Force

You may have had a toy dart gun as a child. The dart gun works because when you push in the dart, it compresses a spring. The compressed spring pushes on the dart–and that is a force!

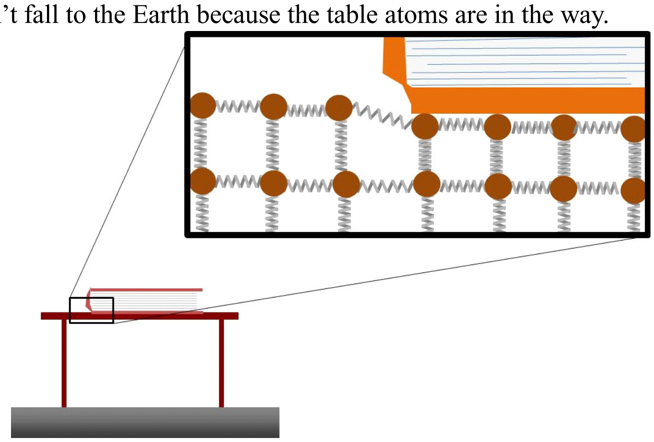


This is also the way our spring cannons work. Springs are combinations of tension like forces and normal like forces (both described below. So we won't go into the details of the spring forces yet.

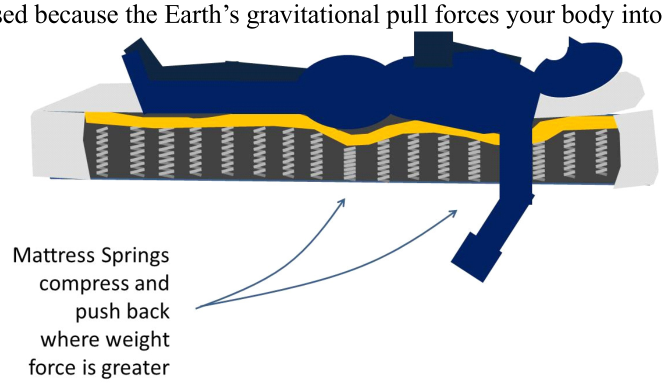
Pushing (Normal Forces)

We have already discussed direct contact and pushing and pulling. In PH220 we are going find that this direct contact is not so simple as we might think. It will involve forces due to the electric charges in the atoms that make up our objects. And those forces are non-contact forces! But that is getting way too far ahead of the physics story. We need just enough here to allow us to study forces.

Suppose we place a book on a table. The book has mass. You probably already know that the book will be pulled downward by the Earth's gravity, so the book has a weight. *Weight* is what we call the force due to gravity that pulls things toward the Earth. But the book can't fall to the Earth because the table atoms are in the way.



This is a little like sleeping on a mattress. The mattress is made of springs. The springs are compressed because the Earth's gravitational pull forces your body into the springs.



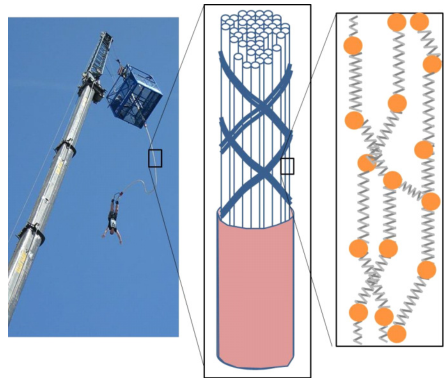
where more of your mass is concentrated, the springs are compressed more. Compressed springs push back with a spring force.

The molecular bonds in the table are a lot like little spring forces. The weight of the book compresses these bonds, and the spring-like bonds make the atoms push back. We call this a *normal force*. The old word "normal" used to mean "perpendicular." And this is where this force got it's name. The springs always push back perpendicular to the surface that is being squashed. The detailed physics of this kind of force will come in PH220.

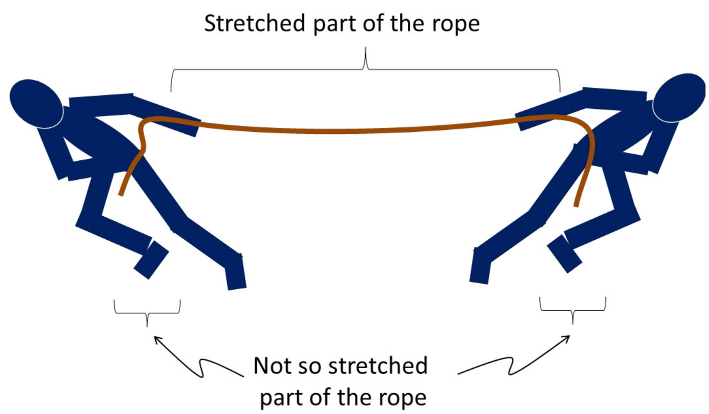
Tension

We have also discussed pulling on ropes. But how does a rope manage to get pulled without the fibers of the rope falling apart. To understand how it works, we need to jump ahead and think of the atoms that build up the rope fibers. The atoms are held together by bonding forces that are really due to the electric forces of the subatomic particles. These atomic bonds can stretch a little like the springs we were just discussing. The spring-like forces that hold the molecules are the reason the rope can pull on a box. The fibers stretch, like springs, pulling on the thing they are attached to. This force created by the stretched atomic bonds is called *tension*.

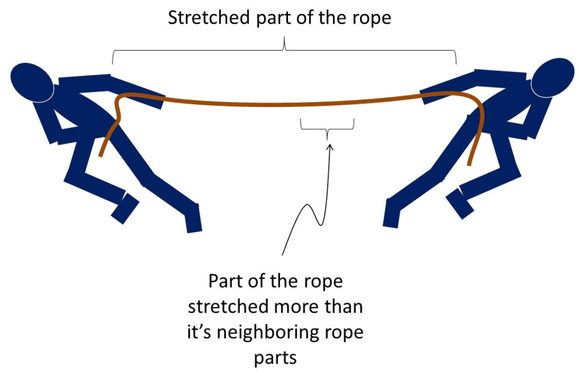
If the rope is a bungee rope, the fibers are very elastic.



But even if the rope is normal clothesline, the fibers can stretch. As the rope is stretched, the bonds resist the stretch, pulling back. This is tension force. Let's consider two guys pulling on a rope in a tug-o-war.



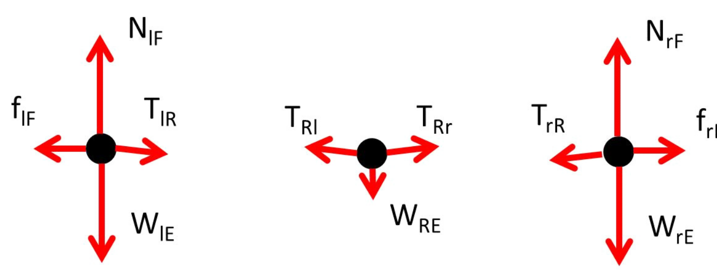
Notice that we will stretch atomic bonds for all the parts of the rope that are being used in the tug-o-war. Nearly the entire set of bonds in the whole rope are involved with creating the tension. If one region of the rope were to be stretched more than it's neighbors,



the stretchy spring-like bonds would quickly even out the stretches until the tension was mostly uniform throughout the rope.

Now Let's change our situation. Suppose that the guys are not actually pulling. They are just holding the rope.

Will there be a tension? Let's draw a free-body diagrams for each object involved. Then we can solve for the tension.



We have three objects, two guys and the rope. Notice that the rope pulls on the guys just as hard as the guys pull on the rope, well, almost. Let's write out Newton's second law for the rope

$$F_{net_{Rx}} = -T_{Rl}\cos\theta + T_{Rr}\cos\theta$$

and

$$F_{net_{Ry}} = +T_{Rl}\sin\theta + T_{Rr}\sin\theta - W_{RE}$$

Since we are in static equilibrium,

$$F_{net_x} = F_{net_y} = 0$$

so we can write

$$\begin{aligned} T_{Rl}\cos\theta &= T_{Rr}\cos\theta \\ W_{RE} &= T_{Rl}\sin\theta + T_{Rr}\sin\theta \end{aligned}$$

Let's look at the first of the set. This tells us that the right hand and left hand tensions are the same

$$T_{Rl} = T_{Rr} = T$$

which we had already surmised. But look at the y-equation

$$W_{RE} = 2T\sin\theta$$

so the tension for the guys just holding the rope is

$$T = \frac{W_{RE}}{2\sin\theta} = \frac{m_R g}{2\sin\theta}$$

If we have a slightly heavy rope like they use in a tug-o-war, we might find the length of rope having a mass of $2\,\mathrm{kg}$ and suppose we find the rope hands with a $5\,^{\circ}$ angle, then

$$\begin{aligned} T &= \frac{W_{RE}}{2\sin\theta} = \frac{(2\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{2\sin\left(5\,^{\circ}\right)} \\ &= 9.8\,\mathrm{N} \end{aligned}$$

We can see that the weight of the rope really does matter if the rope is massive enough. We get a little bit of tension because the rope has mass. So far we have been ignoring the effect of the rope's mass. We call doing this the *massless string approximation* and in this approximation the rope's weight does not affect its tension. We can see that if the rope is heaver the approximation will fail.

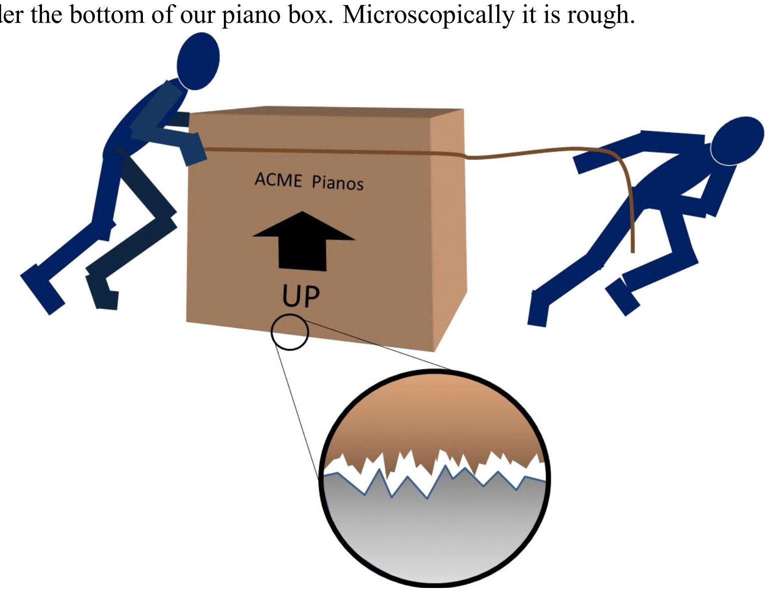
The heavy anchor chain on a large ship would really not be well approximated by a massless string!

USS John F. Kennedy Anchor Chain

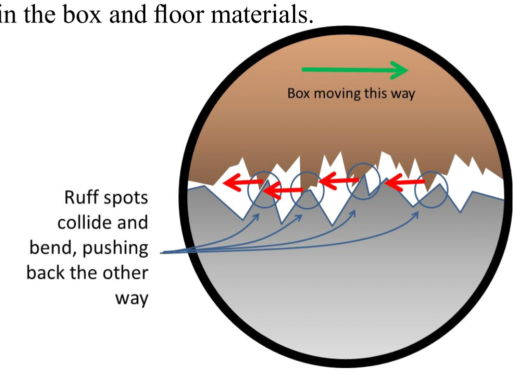
But so long as the weight of the rope or string is much less than the masses of the other objects in the system, then the massless string approximation can be used. In real situations, we will have to check to see if the massless string approximation is valid before solving the problem.

Friction Force

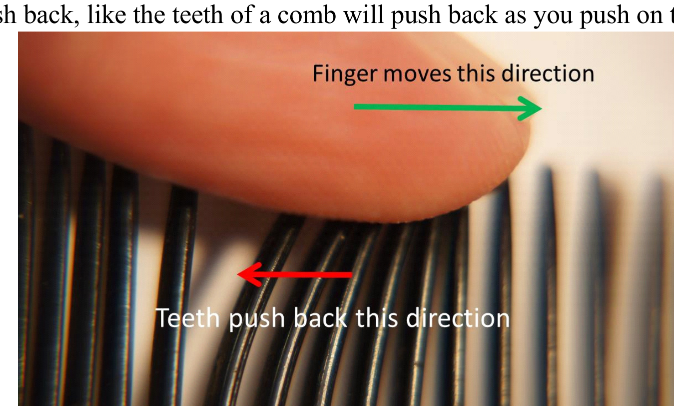
If you have been in Rexburg long, you have experienced surfaces with little friction. Friction is a push, and again it is due to the bonds in molecules that are being stretched. Consider the bottom of our piano box. Microscopically it is rough.



And the floor under the box is also microscopically rough. As the people push and pull the box the rough spots on the floor and the box catch against each other. As the box is pushed the rough spots collide and bend. This bending stretches the spring-like bonds between the atoms in the box and floor materials.



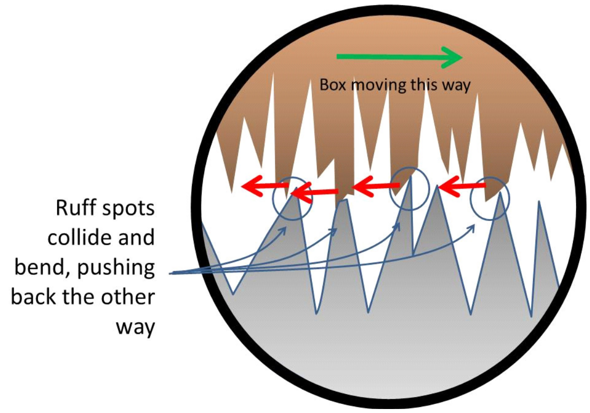
The bent and stretched molecules in the floor material push back against the box molecules. We used the analogy of pushing on the teeth of a comb. The roughness "teeth" push back, like the teeth of a comb will push back as you push on them.



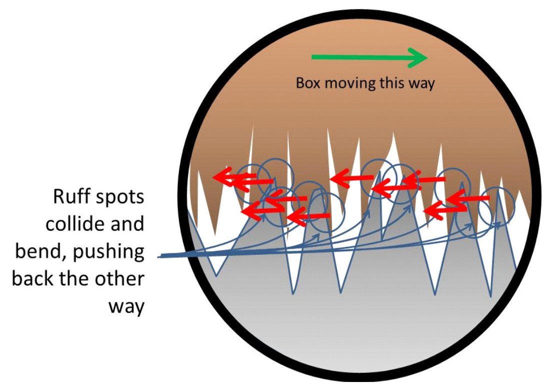
This backward push is what we call friction.

Details of Friction

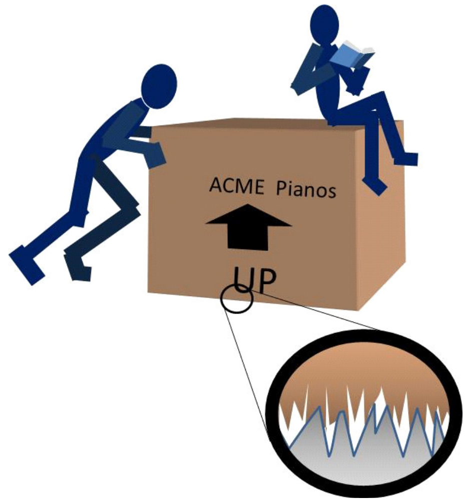
But let's take a different surface, but one with that still has large roughness (but not a rough as a comb).



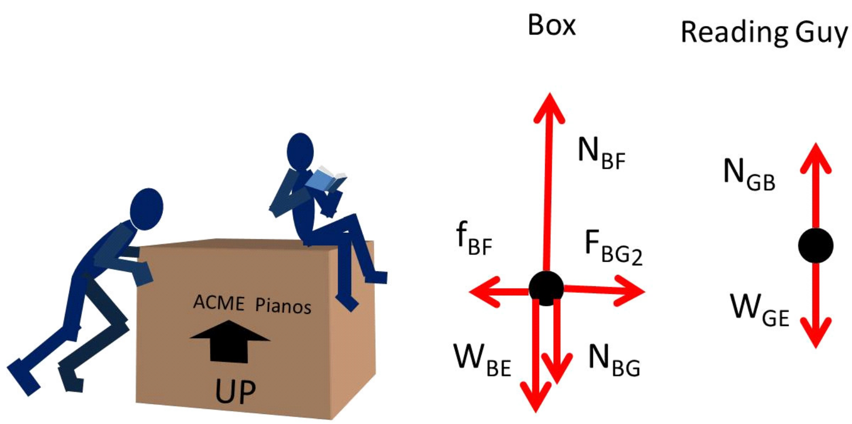
Only a few of the rough peaks hit each other. There would be friction. But we could make more friction if we made the roughness teeth collide more, making the surface area of the interaction larger and the amount of material who's bonds are stretched larger



But how can we make the rough "teeth" mesh more? One way is to push down on the object. Here are our guys moving a box. But one guy is pushing down on the box by sitting on it (with a normal force!).



The downward normal force due to the guy on the box will cause the box to accelerate downward until the force of the teeth pushing back up is matched with the force pushing down. Of course the force of the teeth pushing up must also match the force due to the box or the box would accelerate up or down. That is very unlikely to happen! Let's use Newton's second law to investigate this situation.



This is very like a problem we did last time. Notice that from the figure we can say that $N_{GB} = W_{GE}$ where the subscript G is for "guy" and the subscript B is for "box" and E is for "Earth."

$$F_{Gnet_y} = 0 = N_{GB} - W_{BE}$$

so

$$N_{GB} = W_{GE}$$

this means that the guy pushes down on the box with a force equal to his weight. We know this is true because we can see that the guy is not crushing his way through the box, and the box is not cutting it's way through the guy. Not braking the buy or the box is a constraint on our system.

$$N_{GB} = N_{BG}$$

Then from the box free-body diagram

$$F_{B_{net_y}} = 0 = N_{BF} - N_{BG} - W_{BE}$$

or

$$N_{BF} = N_{BG} + W_{BE}$$

but we know that $N_{BG} = N_{GB} = W_{GE}$ so

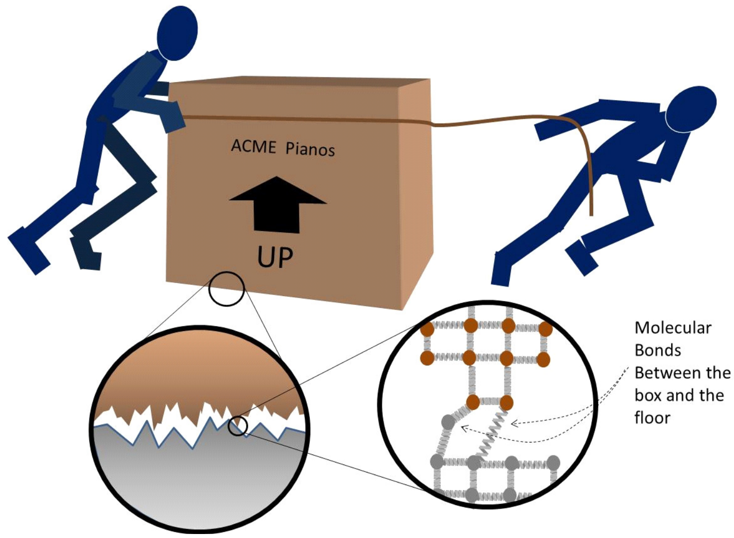
$$N_{BF} = W_{GE} + W_{BE}$$

This is not surprising. We know the floor must support the weight of both the guy and the box. But we can use what we have found to our advantage. The two things that are pushing the roughness teeth together are the wight of the box and the weight of the guy. Those two force magnitudes together are equal to the normal force pushing back up. The roughness teeth are sandwiched in between with forces above and below. They will be forced together. So since how much friction we should have depends on the two weights, and the normal force *is equal to* the two weights, we can say that if the normal force gets larger for an object, the friction gets larger too! We can say that

$$f_{BF} \propto N_{BF}$$

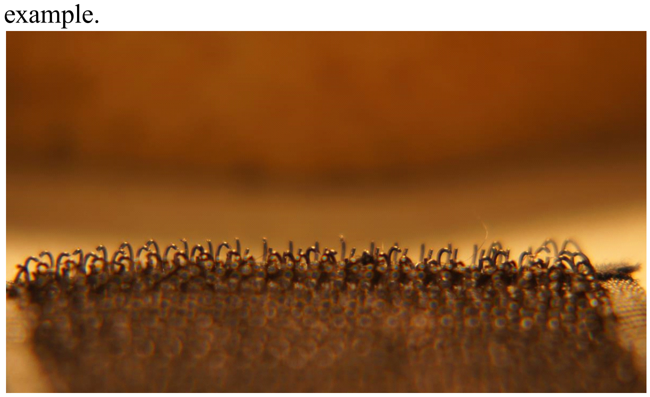
But let's be clear about what we mean. The weight of the box and the guy are doing the job of creating normal forces that are pushing the box down into the floor. The normal force is numerically equal to the sum of these weights. So it is fine to say that the force of friction on the box is numerically proportional to the normal force on the box. The downward force is due to the weight of the box and guy sort of like the guy pulling on a rope is the origin of the tension force in the rope. But it is the normal force that actually pushes the roughness teeth together to increase friction!

But we are still not done with our model for friction.



You may have been out on a hot day and noticed that your shoes would stick to the black top. Sticky materials actually form molecular bonds between themselves and other objects. Once these cohesive bonds are formed, they must be broken to make our object move. It's a little like adding a tension force into our frictional force.

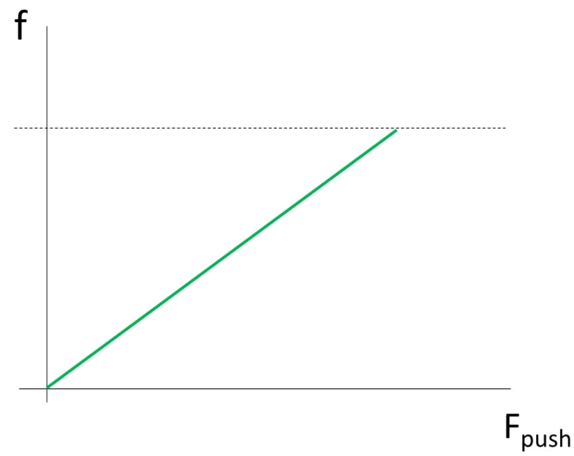
Some materials are more sticky than others, and some are more likely to form bonds than others. And some objects are designed to be rougher than others. Think of Velcro®, for example.



it is designed to be so rough that the little Velcro® hooks grab and hold on, creating tension forces as you move the Velcro® parts.

Let's envision our guys pushing the box again. If the box is at all sticky (and most things are, at least at the molecular level). The sticky bonds will stretch before they break. So by pushing on the box, we expect the backward push from the sticky teeth.

We could plot the situation as shown below.



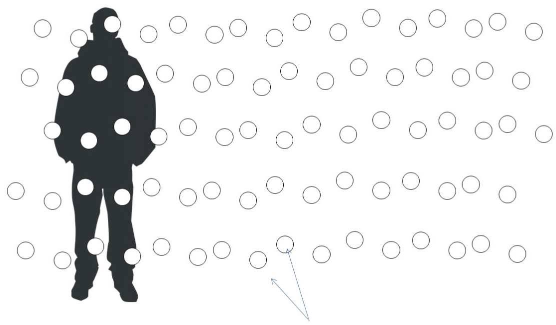
The harder we push, the more stretched the sticky bonds are, so the more friction we have. This model works until we break the sticky bonds. Then our friction force will become smaller in a hurry. More on this in the next lecture!

So far we know that our friction force is proportional to the normal force

$$f_{BF} \propto N_{BF}$$

Drag Force

In this class we really don't yet understand enough to see exactly how drag forces like air resistance work. We will set the groundwork for understanding the origins of drag force in PH123. But we can have some intuitive feel for an air resistance with some conceptual reasoning. Suppose we tied baseballs to strings and hung them all over our class room. Every time a student entered into the room the student would collide with the baseballs. Not only would this hurt, but it would slow the progress of the student. The baseballs would exert a force on the student with every collision. The collective resistance to the motion of the student due to all the baseballs is what we would call a drag force.



This, to have a drag type force we need our object to be colliding with something like air or water molecules.

Gravitational Force

We have discussed that the Earth's gravity tends to make things accelerate downward. This is because the Earth's gravity is a force! So far all of our forces have been contact forces where atoms are involved in making the force. But this gravitational force is fundamentally different. It is a non-contact force. It takes general relativity to truly understand gravity (even then there is some uncertainty!) so for now we will just state that the Earth and other objects with mass pull on other objects with mass. This is gravity.

We call the pull or force due to gravity *weight.*

Sometimes we refer to how much matter we have in our body as our weight. This is because the pull of gravity is proportional to our weight. But they are not the same thing. Weight is a force. How much matter we have is a mass.

Other force origins

There are other forces like thrust and air resistance that we won't use for a while. As we need these forces, we'll discuss their origins. But for now, we have plenty of forces to get started studying how motion is changed.

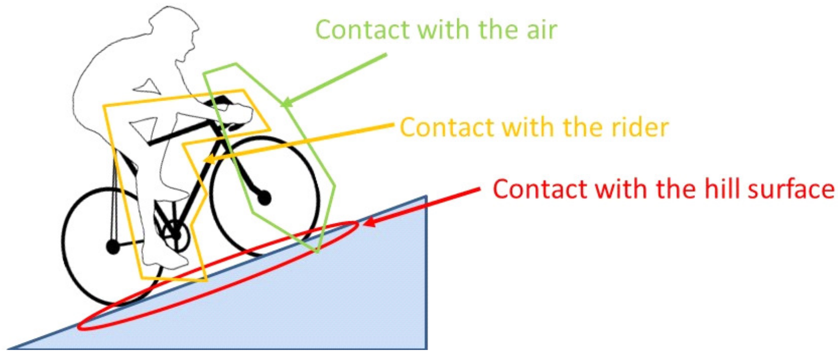
Strategy for Drawing Force (free-body) Diagrams

Let's think about how to make sure we draw our free-body or force diagrams correctly. What we have learned is that we must draw a separate diagram for each object that we will study. One object per diagram. And we have learned that we draw forces that act on that object, and only forces that act on that object. We have leaned to use subscripts to show what types of forces we have, and what object is causing the force. Our types of forces (so far) are given in the next table.

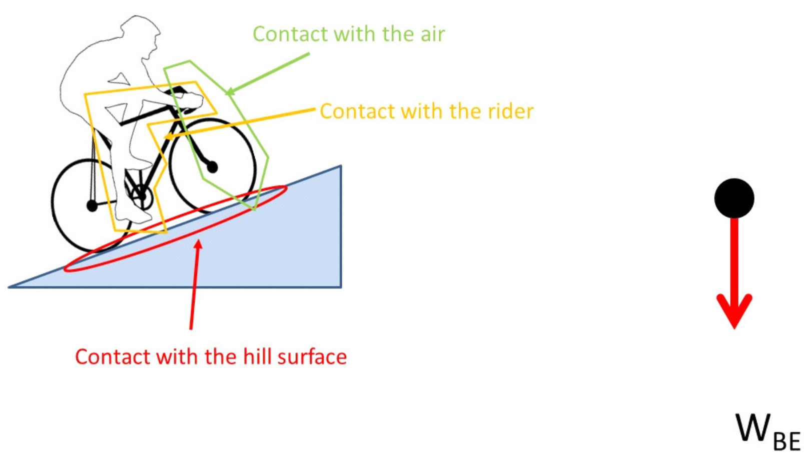
Force Type	Symbol	Causes (How this force is made)	Contact or Non-contact
Weight	W	Gravitational Attraction	Non-contact
Normal	N	Squashed or compressed atoms	Contact
friction	f	Molecular bonds or bending of roughness teeth	Contact
Tension	T	Stretched molecular bonds	Contact
Drag	D	Collisions with gas or liquid molecules	Contact

When we wish to make a force diagram we need to draw arrows for all the forces acting on the object, then go through our list to identify what type of force we have. It helps to look at where the object comes into contact with the surrounding environment. Only gravity (so far) is a non-contact force. So all forces that we know must come about by other objects making contact with our object. This also helps us identify the second subscript, the one that indicates what other object is causing the force. Let's try a complicated example, a bicycle going up a hill.

There are really five objects involved with this situation, but we only want the free-body diagram for the bicycle. We can start with the weight force. We know the bicycle has mass, so it has weight. And this is a great place to start because it takes care of our only non-contact force. The rest of the forces will have to be caused by contact with other objects. So we next look for where the bicycle is in contact with other objects.

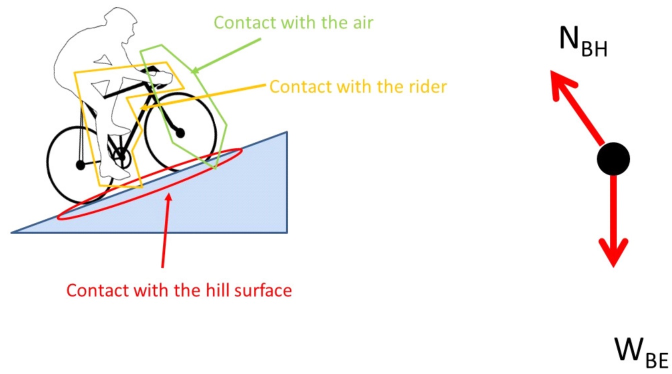


We can see that the bike tires are in contact with the hill. If we look at our table of possible forces we realize that this contact could cause two types of forces, a normal force (from compressing or squashing the hill's atoms, so they push back) and a friction force (because the tires push on the hill's surface roughness teeth, so the hill's roughness teeth push back). We can also see that the bike is in contact with the rider. The rider is being pulled down by the Earth's gravity, so his or her atoms are being pulled into the bike seat, the handle bars, and even the peddles. The rider's atoms don't like being squashed, so they push back. This creates a normal force. The bike is also in contact with the air, and as the bike moves it collides with the air molecules. This would make a drag force. We can place these forces on our diagram, but we have to be careful to think about what is causing the forces to get the directions right. Let's start with the weight force. It should be directed from the center of the bike to the center of the Earth. Usually this means straight down in our diagrams.

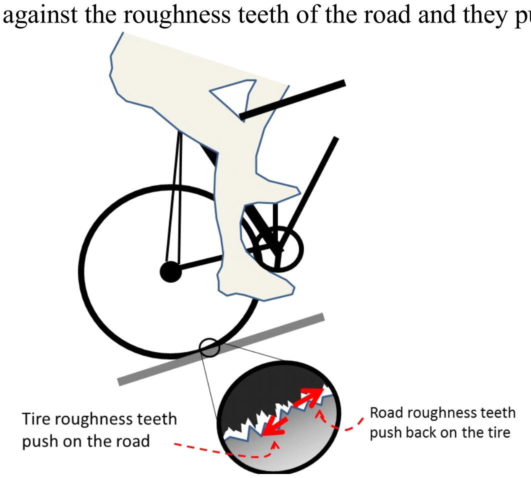


Notice that the first subscript is "B" for "bike." The second subscript is "E" for "Earth."

The normal force from the hill's atoms resisting being squashed will be perpendicular to the hill.



Now let's place the friction force due to the hill surface. To do this we need to realize that the tires push against the roughness teeth of the road and they push downhill!



That means the hill surface roughness teeth push uphill! We can add our friction force to the free-body diagram.Notice that the subscript is "S" for hill (S)urface roughness teeth. It is not just the hill, but the surface of the hill that matters for friction. After all, if the hill was perfectly smooth and not at all sticky, there would be no friction. But there would still be a normal force. It is the surface that matters for friction forces.

Let's take on the interaction between the bike and the rider next. The rider's atoms are resisting being squashed as the Earth pulls the rider into the bike. So there is another normal force due to the rider's atoms.

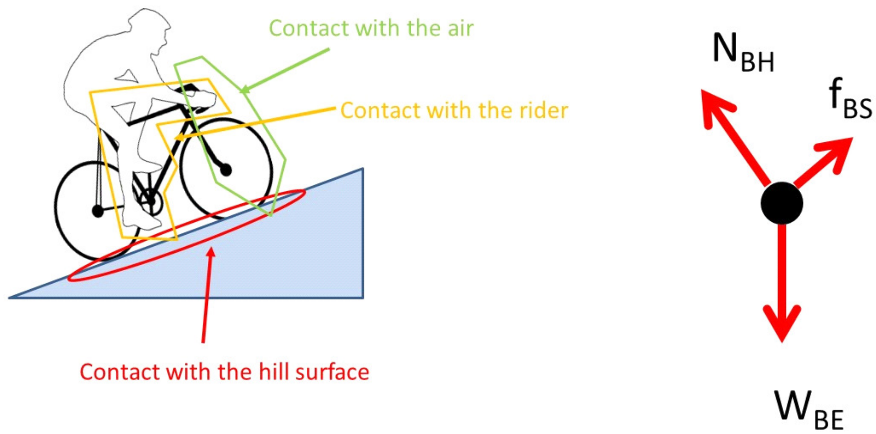
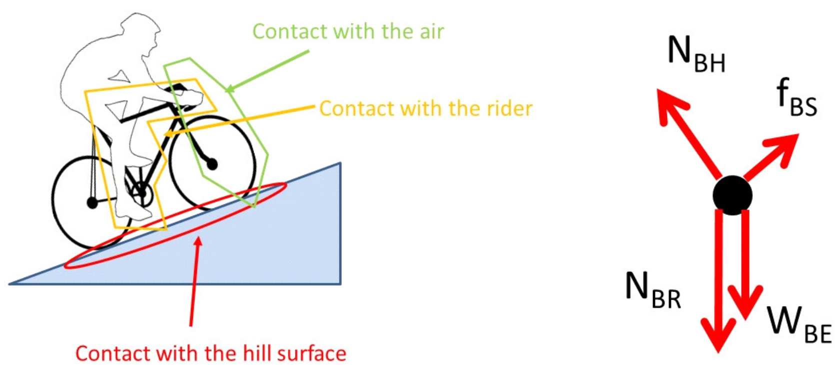
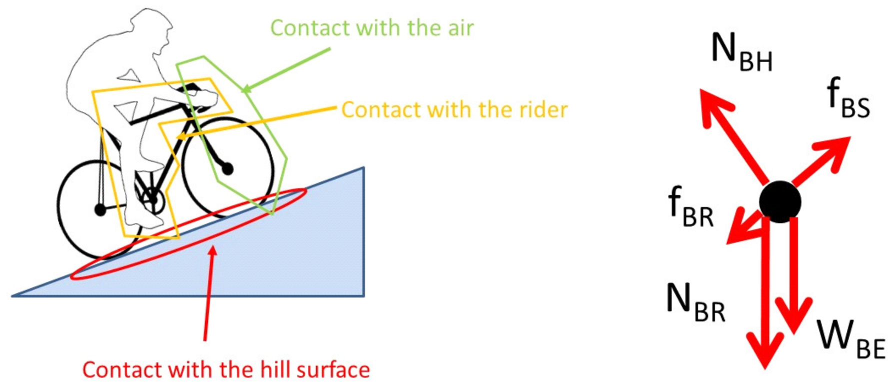


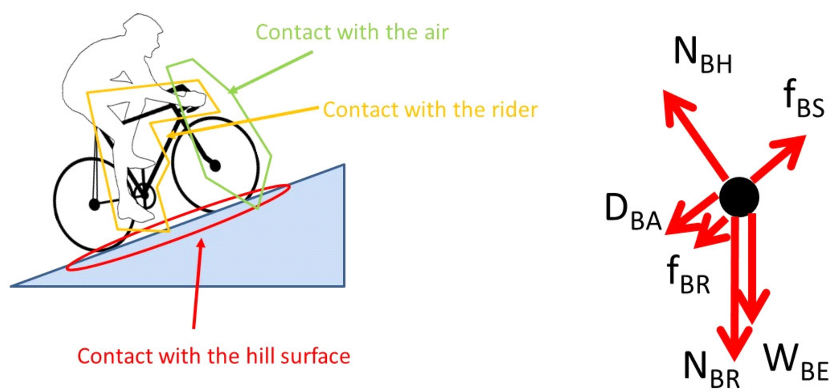
Figure 15.4.



The next force might not be obvious. But think, if we held the bike in place and the rider just sat on the bike. Wouldn't the rider slide off if the seat were frictionless? After all, the bike is on a hill, so the seat is tipped. There must be some friction between the rider and the bike. If we think that the rider would tend to slide down hill, the seat must be pushing the rider back up hill to keep the rider in place. But we want the force on the bike due to the rider. The rider must be pushing down hill on the bike seat.



We identified a drag force. From our study of drag forces, we know that the drag force must be opposite the direction the object is going. The bike is going uphill, so the drag force must be down hill.

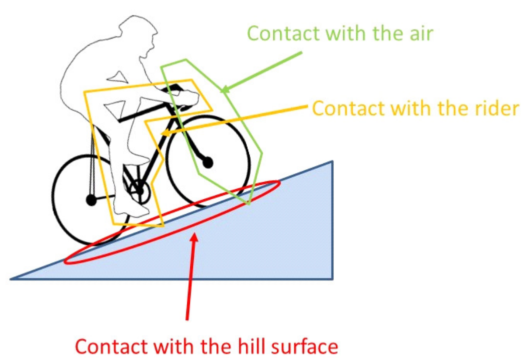


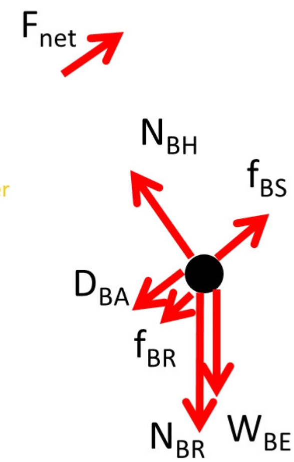
From our experience so far we can see that we need to get the forces right in our free-body diagram in order to find the motion of an object. So taking time to practice getting all the forces and labeling them so the mover object and the environmental object are obvious, and that the type of force is obvious help us to make sure we got the forces in the right directions. The diagrams may seem trivial, but really if you can't get the diagram right, you won't get the problem right. So it is worth taking time to carefully draw free-body diagrams. So let's review what we did

1. Identify the object who's diagram we will draw.

2. Identify any non-contact forces (for PH121 this is usually just a weight force).

3. Identify all environmental objects that make contact with our object
4. Identify what contact forces are caused by the environmental objects by using our table of forces and their causes.
5. Using the force causes from the table, determine the force direction and draw it on the diagram.
6. Label the force with the correct type symbol. Do this for each force.
7. Add the subscripts to indicate what object is being studied (in our case the bike) and what environmental object is making the force (e.g. the Earth for the weight force or the hill for one of the normal forces).
8. Don't put it on the diagram, but it probably a good idea to think of which direction the net force will be. If our cyclist is accelerating up the hill, then the net force will be uphill.





It is important to not put the net force as though it were acting on the dot representing the bike. Remember we will use our force diagram to form $\overrightarrow{F}_{net}$ which is the sum of all the forces, so we can't include $\overrightarrow{F}_{net}$ as part of that sum! If you want to mark $\overrightarrow{F}_{net}$ you can have it float near the free-body diagram, but don't make it part of the diagram.

Newton's Third Law

Newton' also realized that forces don't act alone in nature. They act in pairs. Let's start with the example of hammering a nail

Nail Demo

Notice how the hammer bounces! This means the motion of the hammer head has changed directions. We can see that there must be an acceleration of the hammer head. We now know that this acceleration is evidence of a force. We provided a force on the nail by hitting it. We now see that the nail provided a force back on the hammer!

We can now state Newton's third law:

> *If object 1 and object 2 interact, the force* $\overrightarrow{\mathbf{F}}_{12}$ *exerted on object object 1 by object 2 is equal to the force* $\overrightarrow{\mathbf{F}}_{21}$ *on object 2 by object 1.*

You ,may also hear this expressed as "for every action, there is an equal and opposite reaction," but this almost sounds like one force happens before the other. And that is not true. The forces happen at the same time.

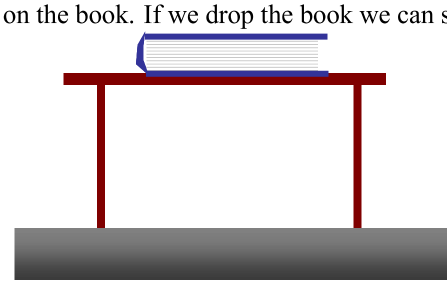
So in the case of our nail, we could write the force of the hammer striking the nail as $\overrightarrow{\mathbf{N}}_{NH}$ and the force of the nail on the hammer as $\overrightarrow{\mathbf{N}}_{HN}$ and state that

$$N_{NH} = N_{HN}$$

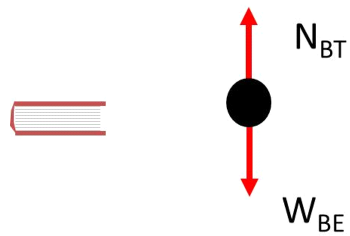
and that the direction of the two forces are opposite, that is

$$\overrightarrow{\mathbf{N}}_{NH} = -\overrightarrow{\mathbf{N}}_{HN}$$

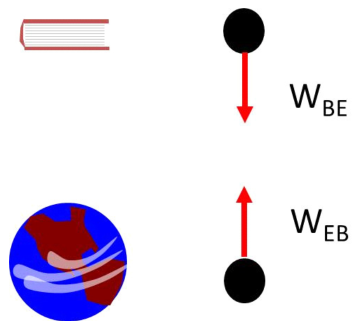
Let's take on a hard problem. Take a text book. It weights a lot. Now place it on your desk. Explain, now, why the book does not push the table to the floor. Certainly there is a force due to gravity on the book. If we drop the book we can see this.



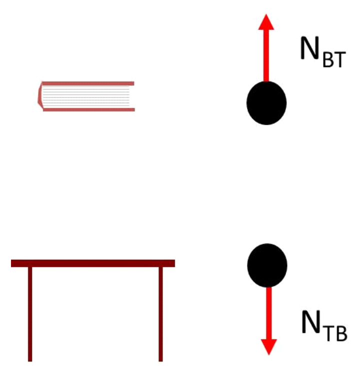
So if the book stays in one place when we put it on the table, there must be a force that opposes its motion so the net force is zero. We already know that we should call this force $\overrightarrow{\mathbf{N}}$ for "normal force" (meaning it is perpendicular to the surface). Let's look for force pairs.



The gravitational force is between the book and the Earth. So there is a force pair as shown below



The Earth is pulling on the book with a gravitational force. The book also has mass so it is pulling on the Earth with a gravitational force! Notice that in this force pair, the forces are the same type, both gravitational (weight) forces. Notice the forces in the force pair have the same subscripts, but in a different order. This is important! These two forces account for the gravitation, but not the normal force. There is another pair



The table pushes up on the book with force $\overrightarrow{\mathrm{N}}_{BT}$, and the book pushes down on the table with force $\overrightarrow{\mathrm{N}}_{TB}$. Notice again that in the force pair the forces are the same type (normal forces in this case). Notice again that the subscripts are the same, but in a different order.

Also notice that forces $\overrightarrow{\mathbf{N}}_{BT}$ and $\overrightarrow{\mathbf{N}}_{TB}$ both have the magnitude W_{BE}. And also notice that the forces acting on the book are $\overrightarrow{W}_{BE}$ and $\overrightarrow{\mathbf{N}}_{BT}$. The other forces, $\overrightarrow{\mathbf{N}}_{TB}$ and $\overrightarrow{W}_{EB}$ are are exerted *by* the book *on other objects* (the table and the Earth).

We can see that for the book $\overrightarrow{\mathbf{N}}_{BT}$ must equal $\overrightarrow{W}_{BE}$. The book is not even moving. So it clearly is not accelerating From Newton's *second* law we have

$$\Sigma F_{By} = ma_{By} = 0 = N_{BT} + W_{BE}$$

we know

$$W_{BE} = -mg$$

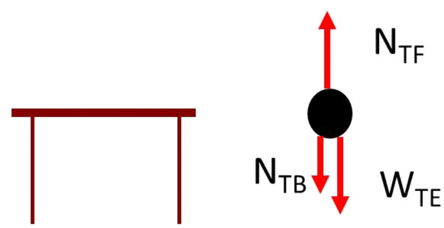
and have deduced that they are oppositely directed in the y direction so

$$0 = N_{BT} - mg$$

or

$$N_{BT} = mg$$

Now you may say that this seems incomplete (it did to me when I first learned this). We are left with the table having a net force of $\overrightarrow{\mathbf{N}}_{TB}$ acting on it. This should accelerate the table downward! But of course the table is sitting on the floor (which we will take to be part of the earth) so we really have another reaction pair that keeps the table in place. Here is the more complete table diagram.



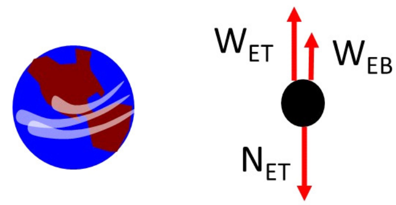
We can see that

$$\Sigma F_{Ty} = ma_{Ty} = 0 = N_{TF} - N_{TB} - W_{TE}$$

So that

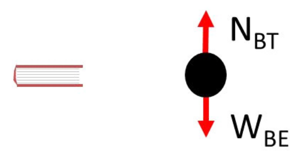
$$\begin{aligned} N_{TF} &= N_{TB} + W_{TE} \\ &= W_{BE} + W_{TE} \end{aligned}$$

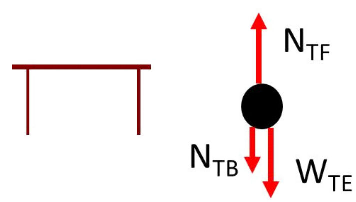
The normal force due to the floor must support the weight of the table and the book. This is no surprise. We could also draw a diagram for the Earth.

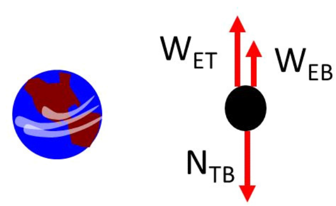


Notice that the table atoms must also resist the pull of the book and the table on the Earth!

But we were looking for force pairs. Notice that there are four force pairs W_{BE} and W_{EB}, W_{TE} and W_{ET}, N_{TF} and N_{FT}, and N_{BT} and N_{TB}. Notice that no two forces in a force pair are on the same diagram! This is important. Force pairs can never act on the same object. Also notice that each force in a force pair has the same subscripts but in reverse order.







also notice that each of the forces in a force pair are opposite in direction. These are the characteristics of a force pair:

1. Same type of force
2. On two different objects (same subscripts, but reversed)
3. Equal magnitudes
4. Opposite Direction

At this point we can start to recognize that Newton's third law can be viewed as a constraint in our problems. In our examples and homework we often found normal forces like N_{12} and N_{21}. We can see that all along these were just Newton's third law pairs.

In our next lecture, let's return to the idea of equilibrium and practice what we know about Newton's laws.

16 Friction, Systems, and Pulleys

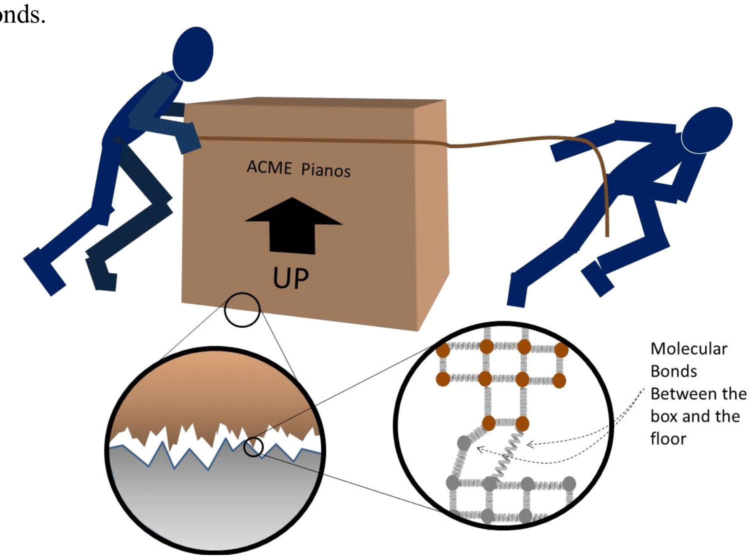
We started studying friction in our last lecture. Today we will finish our study of friction.

In what we did last lecture we treated each object separately. We will still do that in this lecture, but sometimes it is usuful to say our object is really a compound object that has several parts. We call such a compount object a *system.* We will study systems today,

And we will study a device that can change the direction of tension forces, the pulley.

Friction

From what we said last lecture, we know that friction is created by little stretchy molecular bonds.



And we know that our friction force is proportional to the normal force

$$f_{BF} \propto N_{BF}$$

but we would like to make this an equation with an equal sign. To do this, we can include a constant, μ_s that contains all the details of the surface of the substances (in our case, the box and floor substances) that are interacting. Are they very rough? Are they very sticky? etc.

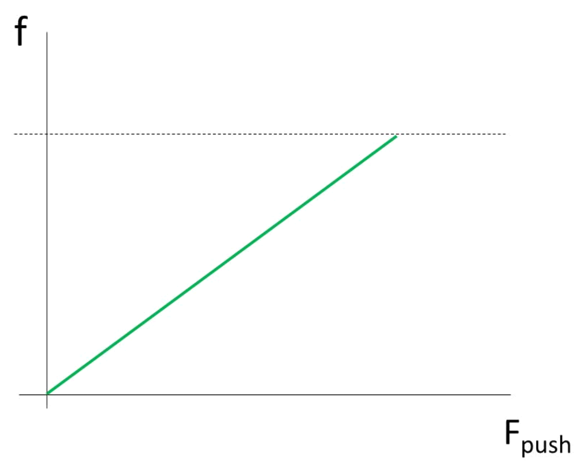
$$f_{BF} = \mu_s N_{BF}$$

The constant μ_s would be different for every material, and also different for every roughness of the specific material.

Static friction

This constant μ_s, is called the *coefficient of static friction*. And really must be different for nearly every item.

This equation for static friction is a little misleading still. Of course, if we push down harder on our box, the friction force can be larger. But look at our last graph (repeated below).



What if we don't push on the box. Suppose the box is just sitting on the floor. Then the little roughness teeth don't bend, and the molecular bonds don't stretch. In that case there is no frictional force even though we have a normal force! If we begin to push on the box, the teeth begin to stretch and push backward. But they won't push back as hard as they can.

If we push even harder on the box, the teeth push back harder, until we reach the point where the bonds start to break. That is the point where the equal sign works, $f_{BF} = \mu_s N_{BF}$. But if we push less on the box then

$$f_{BF} < \mu_s N_{BF}$$

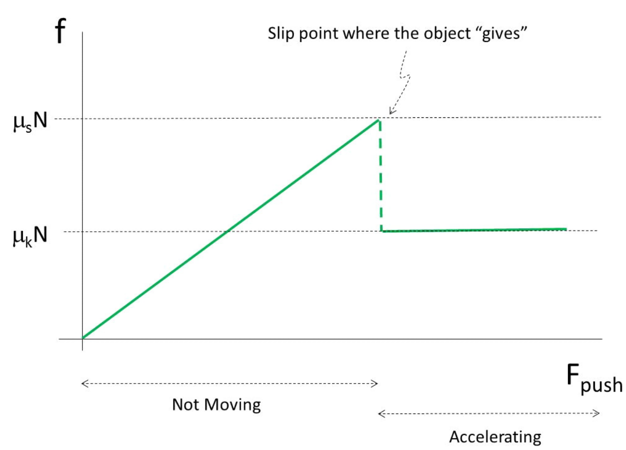
So the way we should write this equation is to say that

$$f_{BF} \leq \mu_s N_{BF} \tag{16.1}$$

meaning that the frictional force could be up to $\mu_s N_{BF}$ if we push hard on the object (in our case, the box), but could be less that $\mu_s N_{BF}$ if we push less hard on the box or even zero if we don't push at all.

Kinetic friction

Suppose we do push very hard on the box so hard that the bonds brake. Think about pushing a box. You push the box, and at first it does not move. The harder you push, the harder the floor roughness teeth push back. But then something seems to "give." That is the bonds breaking. Now the box scoots along the floor. So long as we keep the box moving, it can't form more bonds with the floor and it can't sink down, meshing the roughness teeth with the roughness of the floor. The breaking point, when the box begins to move is when $f_{BF} = \mu_s N_{BF}$. Beyond this point, the friction force has much less bonding and less teeth meshing, so the friction force is greatly reduced.



We can write a similar equation for the case when the box is finally moving

$$f_{BF} = \mu_k N_{BF}$$

It is still true that there is more friction if we push down harder on the box (making N_{BF} larger). But now we won't have much stickiness and will have less meshing of the roughness teeth. So our coefficient will be much less. To indicate that we have a different coefficient once the box is moving, let's give the new coefficient a new name. We call this the *coefficient of kinetic friction.*

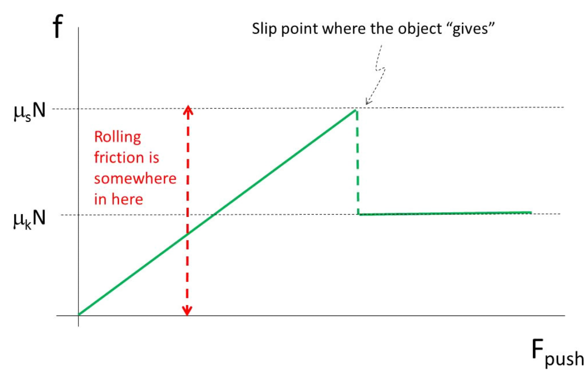
Here are some example coefficients of friction.

Material	μ_s	μ_k
Rubber on concrete	1.00	0.80
Steel on steel	0.74	0.57
Wood on wood	0.25 to 0.50	0.20
Waxed wood on snow	0 to 0.14	0.04 to 0.1
Ice on ice	0.10	0.03

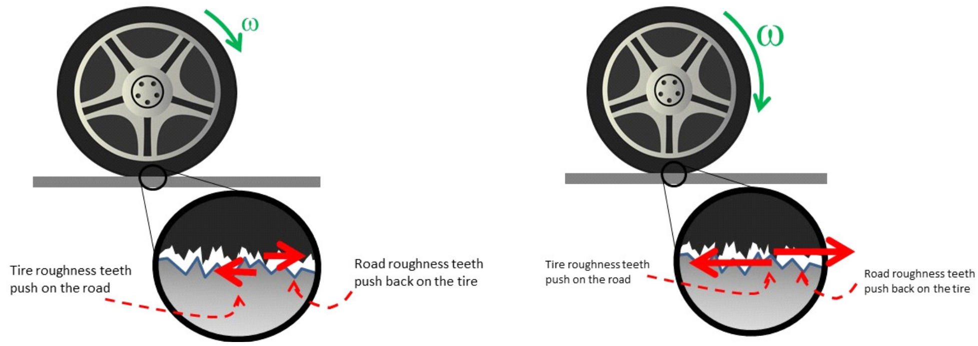
but recall that I could have wood that is more rough than other wood. That would change our table value for "Wood on wood" or "Waxed wood on snow." So these coefficients of friction are useful to get the general idea of how much frictional force we might get for a substance, but should be used with some caution.

Rolling friction

It is becoming popular to define another coefficient of friction, one for a rolling wheel or tire. But this is really just a special case of static friction. Think of a tire. You want your car tire to roll along without slipping. That means that the part of the tire that is on the ground is somewhere in the static friction area of our graph.



The rolling friction depends on how hard we work at spinning the tire (how big F_{push} is). Since the push force can change with how we rev our engine, so can the rotational friction force.



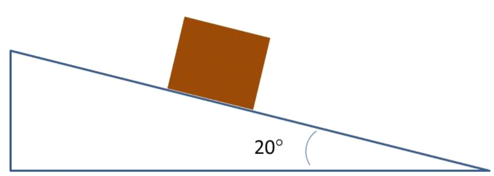
Some books define a coefficient of rolling friction, but we won't. We will realize that rolling friction is just a case of static friction and use

$$f_{BF} \leq \mu_s N_{BF}$$

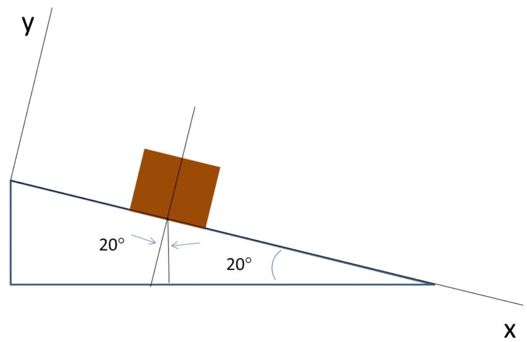
for rolling objects.

An example of friction

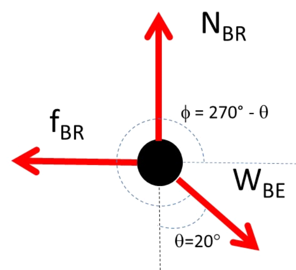
Suppose our moving guys have pushed the box to a ramp. They want to slide the box down the ramp. The ramp has an angle of $20\,^\circ$ and somehow you know that the coefficient of kinetic friction $\mu_k = 0.12$ for the box sliding on the ramp material. What is the acceleration of the box as it slides down the ramp?



It will be easier if we use a rotated coordinate system for this problem. Let's make the x-axis be along the ramp. Then the y-axis would be perpendicular to the ramp.



and we can draw a free-body diagram in this coordinate system



This is a friction problem, and since we want acceleration, it is also likely a Newton's second law problem.

We know

$$\begin{aligned}\theta &= 20\,^{\circ} \\ \phi &= 270\,^{\circ} + \theta \\ \mu_k &= 0.12\end{aligned}$$

and we know Newton's second law

$$\begin{aligned}F_{net_x} &= ma_x = \sum_i F_{xi} \\ F_{net_y} &= ma_y = \sum_i F_{yi}\end{aligned}$$

and we can use our new kinetic friction equation.

$$f_{BRk} = \mu_k N_{BR}$$

The box is not likely to fly off the ramp or to burrow into the ramp, so we can also identify

$$a_y = 0$$

then

$$F_{net_y} = 0 = N_{BR}\sin(90\,^\circ) + f_{BR}\sin(180\,^\circ) + W_{BE}\sin(\phi)$$

and the x-part would be

$$F_{net_x} = ma_x = N_{BR}\cos(90\,^\circ) + f_{BR}\cos(180\,^\circ) + W_{BE}\cos(\phi)$$

We have said that we should always use any value that is zero right away because it takes out whole terms. And we added to this using ± 1 values from sine and cosine functions.

$$F_{net_y} = 0 = N_{BR} + W_{BE}\sin(270\,^\circ + \theta)$$
$$ma_x = 0 - f_{BR} + W_{BE}\cos(\phi)$$

using our new friction equation, $f_{BRk} = \mu_k N_{BR}$, we can write this as

$$N_{BR} = -W_{BE}\sin(270\,^\circ + \theta)$$
$$ma_x = -\mu_k N_{BR} + W_{BE}\cos(\phi)$$

and substituting N_{BR} from the first equation into the second

$$ma_x = -\mu_k\left(-W_{BE}\sin(270\,^\circ + \theta)\right) + W_{BE}\cos(\phi)$$

finally, we know that $W_{BE} = mg$

$$ma_x = \mu_k\left(mg\sin(270\,^\circ + \theta)\right) + mg\cos(\phi)$$

the masses cancel, and we can take out a g

$$a_x = g\left(\mu_k\left(\sin(270\,^\circ + \theta)\right) + \cos(\phi)\right)$$

$$\begin{aligned} a_x &= \left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.12\left(\sin(270\,^\circ + 20\,^\circ)\right) + \cos(270\,^\circ + 20\,^\circ)\right) \\ &= 2.246\,7\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

We would not expect to have the full acceleration due to gravity because we have a slope and we have friction reducing the acceleration. So this seems reasonable.

Models vs. Laws

You might be saying to yourself at this point that friction seems a little less certain an idea than, say, Newton's law of gravity. And you would be right. We need to make a distinction between a way of thinking about how a complicated system works, and a physical law, meaning an equation that describes a physical relationship.

Our model for friction would be

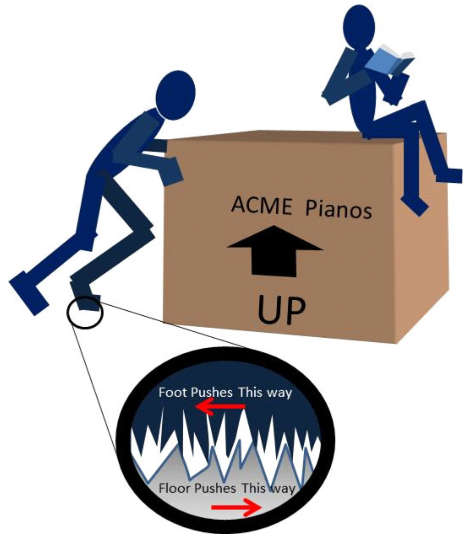
$$\begin{aligned} f_{BFs} &\leq \mu_s N_{BF} \\ f_{BF_k} &= \mu_k N_{BF} \end{aligned}$$

but there is a lot of leeway in our coefficients of friction. And you might further ask, does the amount of area on the object that is in contact with the floor matter? If I have two sets of Velcro®, but one set is much larger than the other, wouldn't it take more force to slide the two sides of the big set across each other than it would to slide the two sides of the smaller pieces across each other? For Velcro®, it does matter! But it turns out that for most surfaces (not like Velcro®) the area is not very important. But it could be for a designer surface that is intended to be extra rough. Our mental model works for most common surfaces. But it would not be to surprising to find that a measured coefficient of friction for, say, wood on wood, that might not match our table value if the wood had saw marks, or was varnished, etc. We might say that our equations for friction are a good mental model of how things work, but they are not a fundamental theory, describing the basic nature of the universe. The fundamental theory would be involved in the creation of the atomic bonds, etc. our friction model is like a summary of the effects of all the fundamental theories as they act on a particular object.

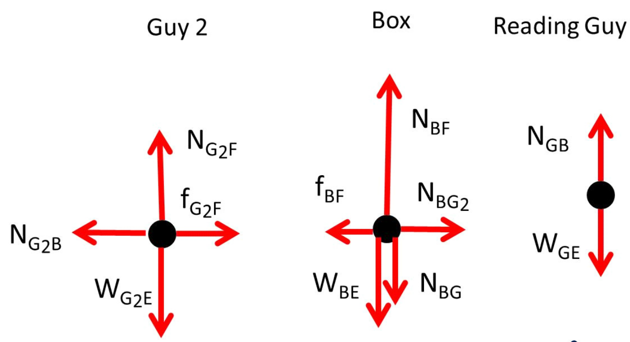
But all this doesn't seem much like a universal law. And that is right. We don't use the word "law" in physics to mean something that is always true. Instead, the word "law" means an equation that describes what our mental model says should happen. This might be like Newton's second law that seems to always be true, or like our friction equation that is sometimes true. They both could be considered laws in physics.

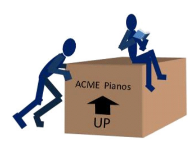
Propulsion

So far, we have worked rather hard to avoid the details of how we actually get something moving. We know we need a force. But we have avoided details by making surfaces frictionless. We know, for example, that if the floor of the apartment were frictionless, our moving guys would not be able to push on the box. Their feet would slide out from under them.



But with friction, the guy can push. However notice that the guy's foot pushes the opposite way he/she wants the box to go. The foot pushes backwards. That makes the little roughness teeth push *forwards!* This is not too much of a surprise. If we add the free body diagram for the guy that is pushig the box (labeled Guy2) it might look like this.



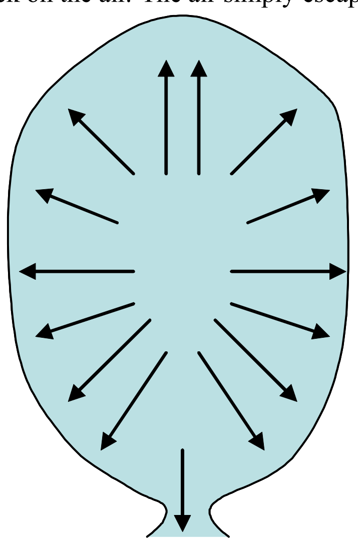


The roughness teeth are what keeps the person moving forward, so they must push forward.

Let's study a car wheel. Notice that as the tire turns, it will push against the road.

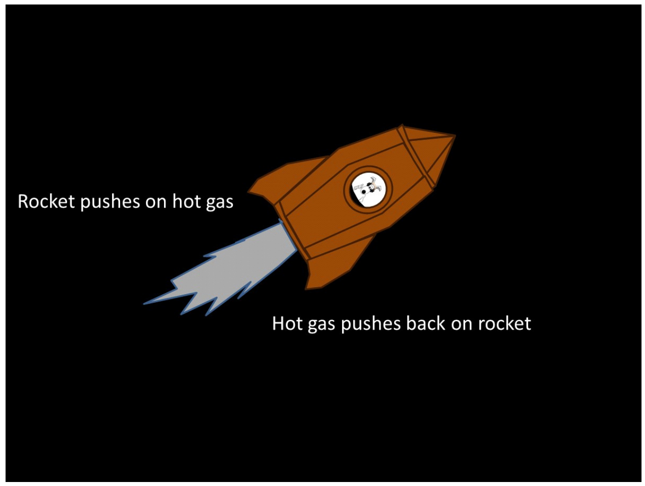
Again the roughness teeth in the road will push back on the tire. This is what pushes the car forward. Notice in propulsion problems so far there has always been a Newton's third law pair!

Let's take on a harder case. A balloon that has been released without being tied. The balloon moves around the room. What makes the balloon go? If you blow up the balloon, the extra air pressure in the balloon is distributed around the inside of the balloon. But if you open the stem and let if fly around, then where the stem is, there is no balloon wall to push back on the air. The air simply escapes.



Think of summing up all the forces acting on the surface of the balloon. The forces act to balance each other, except where the hole is. Thus, the force on the other side of the balloon is unbalanced. And the balloon flies around. The stretched balloon material pushes on the air, and the air pushes back.

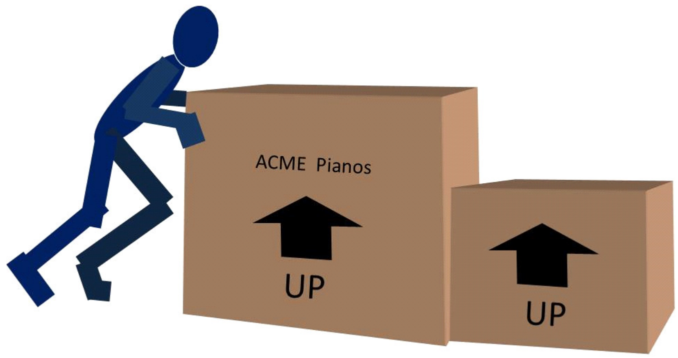
Rockets work this way, except that instead of inflating, they use a controlled explosion using the rocket fuel. The burning rocket fuel heats gas, making the gas expand.



The burning of the fuel effectively pushes the gas outward, out of the rocket. The gas resists this push, pushing back on the rocket. This push from the gas propels the rocket forward. To see how gas can push on something, think of the air pressure in your tires. The air pressure is literally pushing your car upward, keeping the rims off the ground. Air pressure is a subject for PH123. But hopefully you can see that air can provide a resistive force to propel something forward. For now, notice that Newton's third law pairs are the source of propulsion.

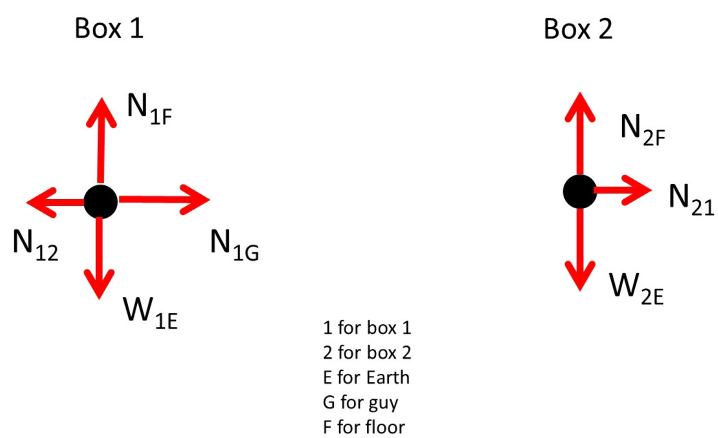
Systems

Suppose you are helping with a neighbor's move. There are two big boxes to move, and you decide you will push both boxes at once. One box has a weight of $50.0\,\mathrm{N}$ and the other has a weight of $40.0\,\mathrm{N}$. Further suppose that the floor is frictionless. Of course this is not a great assumption. If it were really true, you would not be able to walk on the floor. But let's suppose that you somehow can push on the boxes and that somehow the floor is frictionless.



What will be the acceleration of the smaller of the two boxes if you push with a force of $30.0\,\mathrm{N}$?

This is a Newton's second law problem, but a dynamic one. Newton's second law problems require us to draw free-body diagrams, so let's draw a diagram for each box.



We know

$$
\begin{aligned}
W_{1E} &= 50.\,\mathrm{N} \\
W_{2E} &= 40.\,\mathrm{N} \\
N_{1G} &= 30.\,\mathrm{N} \\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}
$$

And we will need our Newton's second law equations

$$\begin{aligned}\overrightarrow{\mathbf{a}} &= \frac{\overrightarrow{\mathbf{F}}_{net}}{m} \\ \overrightarrow{\mathbf{F}}_{net} &= m\overrightarrow{\mathbf{a}}\end{aligned}$$

$$W = mg$$

We need Newton's second law for both boxes. From our figure for box 1 we have

$$m_1 a_{1x} = N_{1G} - N_{12}$$

$$m_1 a_{1y} = N_{1F} - \underline{W_{1E}}$$

and we hope that $a_{1y} = 0$ because we don't expect the box to fly up or burrow into the ground.

$$0 = N_{1F} - \underline{W_{1E}}$$

$$N_{1F} = \underline{W_{1E}}$$

Now let's do box 2

$$m_2 a_{2x} = N_{21}$$

$$m_2 a_{2y} = N_{2F} - \underline{W_{2E}}$$

and again $a_{2y} = 0$

$$0 = N_{2F} - \underline{W_{2E}}$$

$$N_{2F} = \underline{W_{2E}}$$

Let's summarize what Newton's second law has taught us about this problem:

$$N_{1F} = \underline{W_{1E}}$$

$$N_{2F} = \underline{W_{2E}}$$

$$m_1 a_{1x} = \underline{N_{1G}} - N_{12}$$

$$m_2 a_{2x} = N_{21}$$

We have six things we don't know, and only two equations. It looks hopeless. But really we know more things. The boxes must accelerate together. If that were not true, one box would launch ahead of the other, or one would collapse as the other accelerated through it. Neither of these things are happening. So we can say

$$a_{1x} = a_{2x} = a$$

. This is what we call a "constraint." A constraint is a piece of information that comes from the physics of our situation. It is something we know from observing the physical setup of the problem. From Newton's third law we can pick up another constraint. N_{12} and N_{21} are Newton's third law pairs! They must be equal in magnitude.

$$N_{12} = N_{21} = N$$

This makes some sense. Unless one box is crushing the other, the forces between the two boxes must be equal This is another constraint. And each constraint added another equation!

We also know

$$\underline{W_{1E}} = m_1 \underline{g}$$
$$\underline{W_{2E}} = m_2 \underline{g}$$

This brings our equation count to six. We should be able to solve a set of six equations and six unknowns!

Using our constraints we could write our third equation from our Newton's second law set as

$$m_1 a = \underline{N_{1G}} - N$$

and the forth would be

$$m_2 a = N$$

let's solve this last equation for a

$$a = \frac{N}{m_2}$$

and substitute this into the previous

$$m_1 \frac{N}{m_2} = \underline{N_{1G}} - N$$

or

$$m_1 N = m_2 \underline{N_{1G}} - m_2 N$$

and some rearranging

$$m_1 N + m_2 N = m_2 \underline{N_{1G}}$$
$$N (m_1 + m_2) = m_2 \underline{N_{1G}}$$

gives

$$N = \frac{m_2 \underline{N_{1G}}}{(m_1 + m_2)}$$

We can substitute this into our equation for a

$$a = \frac{1}{m_2} \frac{m_2 \underline{N_{1G}}}{(m_1 + m_2)}$$

but don't yet know m_1 or m_2 But we have two equations relating m_1 and m_2 to the box weights

$$\underline{W_{1E}} = m_1 \underline{g}$$
$$\underline{W_{2E}} = m_2 \underline{g}$$

so we can find our masses

$$m_1 = \frac{\underline{W_{1E}}}{\underline{g}}$$
$$m_2 = \frac{\underline{W_{2E}}}{\underline{g}}$$

and substitute them into our equation for a

$$a=\frac{1}{\frac{W_{2E}}{g}}\frac{\frac{W_{2E}}{g}N_{1G}}{\left(\frac{W_{1E}}{g}+\frac{W_{2E}}{g}\right)}$$

Notice that some of the g terms cancel. Then we have just

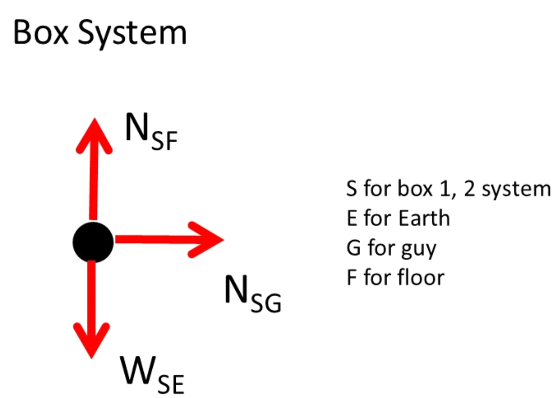
$$a=\frac{g}{W_{2E}}\frac{W_{2E}N_{1G}}{(W_{1E}+W_{2E})}$$

and we know all the pieces, so we can plug in numbers.

$$\begin{aligned} a &= \frac{9.8\frac{\mathrm{m}}{\mathrm{s}^2}}{40\,\mathrm{N}}\frac{(40\,\mathrm{N})(30\,\mathrm{N})}{(50\,\mathrm{N}+40\,\mathrm{N})} \\ &= 3.2667\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

This is a reasonable result and we can take a minute to rejoice that we found the answer.

But you might think that this was more work than we need to do. If it is really true that the accelerations are the same for both blocks, it should be that we can treat the two boxes together as though they were one object. We do this all the time. A car is one object, but it is made of many parts that all move together. We have used our particle model to find the motion of whole cars at once, so is should be true that we can treat both boxes together as one particle



The we can write out Newton's second law in the x and y-directions for the system consisting of box 1 and box 1

$$\begin{aligned} F_{net_x} &= m_S a_x \\ &= N_{SG} \\ F_{net_y} &= m_S a_y \\ &= N_{WF}-W_{WE} \end{aligned}$$

and

$$m_S = m_1 + m_2$$

then from the x-equation we have

$$m_S a_x = N_{SG}$$

and our acceleration is

$$a_x = \frac{N_{SG}}{m_1 + m_2}$$

We don't know the masses, but we have an expression for them from above, so

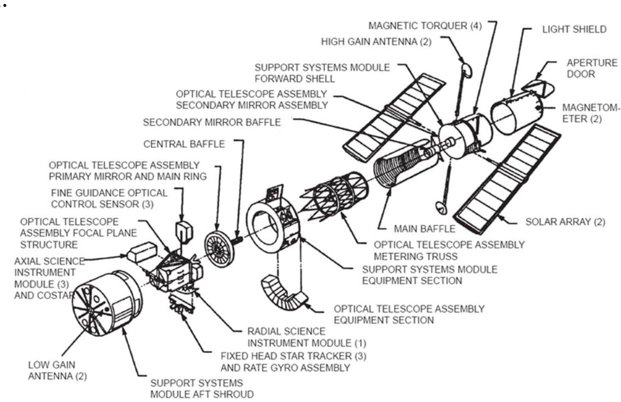
$$a_x = \frac{N_{1G}}{\left(\frac{W_{1E}}{g} + \frac{W_{2E}}{g}\right)}$$

$$a_x = \frac{gN_{1G}}{(W_{1E} + W_{2E})}$$

$$\begin{aligned} a_x &= \frac{\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(30\,\mathrm{N})}{(50\,\mathrm{N} + 40\,\mathrm{N})} \\ &= 3.266\,7\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

We got the same result, and it was so much easier!

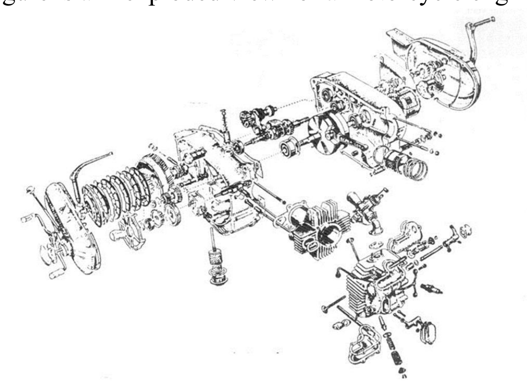
We can give a name to a group of objects that are somehow bound together so that they move together.



Hubble Telescope System

We call a group of objects that move together a *system*. Notice that in our first analysis of the two box system we found that there were forces between the two boxes. This is part of what makes a system a system. Something must keep the parts of the systems together. Bolts, welds, screws, and glue keep the Hubble Telescope together. But these welds, screws, bolts, and glue all really rely on molecular forces that can stretch, but hold together. For our boxes the forces that keep the boxes together are the normal forces N_{12} and N_{21}.

Notice that they figure prominently in our first solution, but don't show up at all in the second solution! These forces are not from outside the two-box system. Rather, they are *internal* forces. If we treat the whole system as a particle, we will never see these forces. The next figure is an "exploded view" of a motorcycle engine.



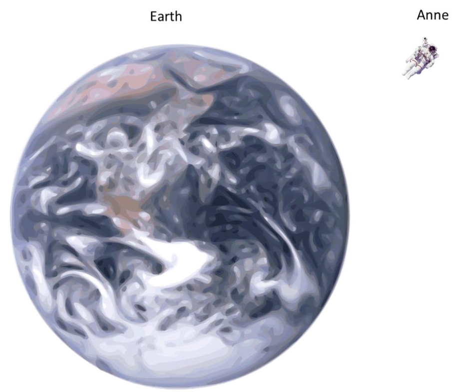
Motorcycle Engine System

As the motorcycle operates, there will be lots of internal forces among all these parts. But for most practical purposes, we will treat the whole motorcycle, engine parts and all, as one object. We call this one object the "motorcycle system."

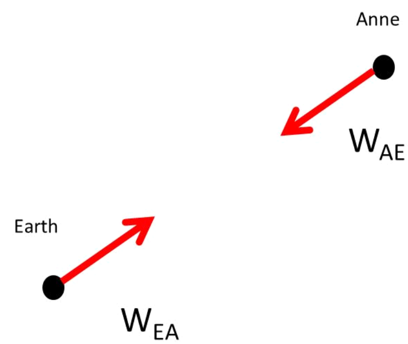
This is both a comfort, and a potential danger. Those internal forces really exist. And as we search for all the forces acting on an object, we will have to be sure that we don't include internal forces in our net force calculations. Internal forces won't accelerate an object. On the other hand, if you have ever had the misfortune of breaking a motor mount, you will recognize that making sure the internal forces are not too large is critical for a mechanical design!

Notice that internal forces come in matched sets, like normal forces N_{12} and N_{21}, for our two box example. In our subscript scheme, the internal forces have matched, but reversed subscripts. This is a dead giveaway that these forces are internal forces (and that they are Newton's third law pairs).

Almost any group of interacting objects could be considered a system. Suppose we take the Earth and you (or an astronaut) as a system.



Here are the free body diagrams for both the astronaut (or you) and the Earth.



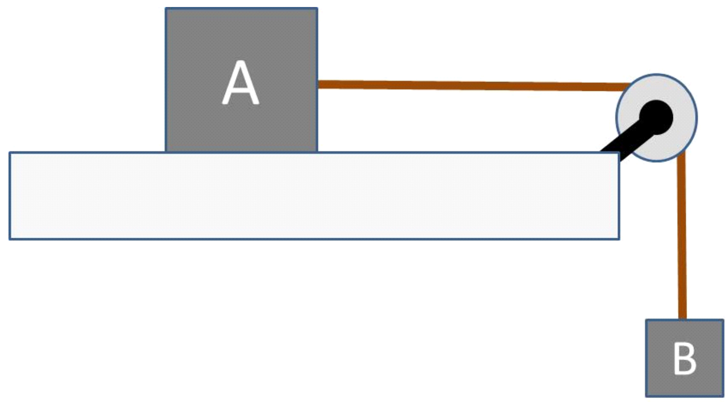
The internal forces will be W_{EA} and W_{AE}. Notice that they are the same type of force, both gravitational forces. Note the subscripts are the same, but reversed. You might also notice that their directions are opposite. Also notice that it took two free-body diagrams to draw both forces. That is because each of the forces is on a different object.

We call forces that match like this *force pairs*. All interactions between objects come as force pairs. But the parts of force pairs can never act on a single object. there must always be two objects for there to be a force pair.

Our normal forces between the boxes are a force pair, N_{12} and N_{21}. Notice that N_{12} and N_{21} are on different diagrams! Also notice that they are opposite in direction, but the same type of force (a normal force). It can be tricky to identify force pairs.

Pulleys

We have used ropes in our force situations already. But our ingenious ancestors invented devices that change the direction of a rope's tension. We need to be able to include such details in our force calculations. So let's take a closer look at the ropes, themselves, and lets take a look at this device that changes the direction of a tension, the pulley.



Look at figure. The first thing you will notice is that indeed the pulley changes the direction of the tension! This can be quite useful. But there is a cost.

Early pulley-like devices were just wooden blocks that the rope would slide over. They would have a lot of friction. That friction would reduce the tension in the rope. So, in our example above, Block A would not be pulled with as much force due to the tension created by block B because some of the force from block B was used just to make the rope move over the pulley. The engineering work on pulleys done over the years has been to reduce friction by making part of the pulley turn. By turning like a wheel. the rope and pulley don't slide against each other. The friction is greatly reduced because we changed the pully from having kinetic friction to having static friction

$$f_s \leq \mu_s N$$

and so long as the wheel moves with the rope the roughness teeth don't bend and we get

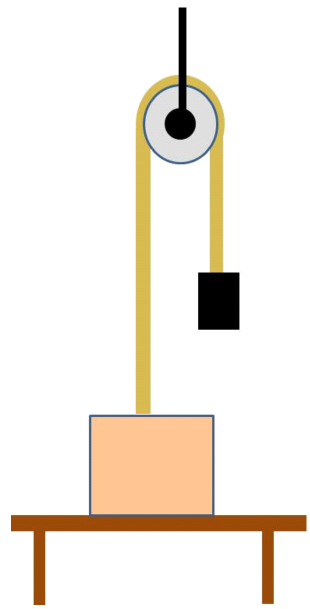
$$f_s \ll \mu_s N$$

Elevator pulley and cable.

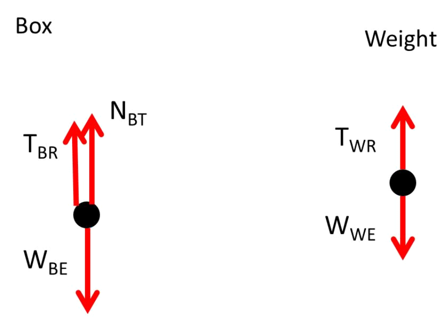
An ideal pulley would have absolutely no friction. This is an engineering feat yet to be accomplished. But many pulleys have very little friction. If the friction in the pulley axel and the static friction between the rope and the pulley wheel do not change the tension in the rope noticeably, we can ignore the friction. We call this the *frictionless pulley approximation.*

We will generally use both the massless string and the frictionless pulley approximation together in problems. Let's try one now.

A box with a weight of $100.0\,\mathrm{N}$ sits on a table. The table exerts a normal force on the box to keep it from smashing its way through the table material. But what would happen if you tied a rope to the box, and passed the rope through a pulley (that is perfectly frictionless), then attached a weight to the other end? What would the normal force be in this situation if the weight weighed $40\,\mathrm{N}$?



What type of problem is this? Well, we can see we have a rope and we have forces, so this is likely a Newton's second law problem. We will need a free body diagram for our objects.



Notice that we did not draw a diagram for the rope. If we take the massless string approximation, then the rope has no mass, so we can't exert a force on the rope! But the rope can exert a force on the box and weight (yes this is not realistic, but in many cases it is close enough).

VAR:

$$\begin{aligned} W_{BE} &= 100.0\,\mathrm{N} \\ W_{WE} &= 40\,\mathrm{N} \end{aligned}$$

BE

$$\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}} = \sum_i \overrightarrow{\mathbf{F}}_i$$
$$W = mg$$

We recognize that $\overrightarrow{a} = 0$ so our basic equation is

$$\overrightarrow{\mathbf{F}}_{net} = 0$$

or

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

With our frictionless pully approximation the pulley can't change the tension in the rope. Since we are assuming that the mass of the rope is negligible, and the pulley is frictionless. Then we can say

$$T_{BR} = T_{WR}$$

This is a constraint of the system. Now let's set up Newton's second for both objects. There are no x-direction forces, so we only need $F_{net_y} = 0$

For the box

$$0 = T_{BR} + N_{BT} - W_{BE}$$

and for the weight

$$0 = T_{WR} - W_{WE}$$

from the last equation

$$T_{WR} = W_{WE}$$

then

$$T_{BR} = W_{WE}$$

and

$$0 = W_{WE} + N_{BT} - W_{BE}$$

or

$$N_{BT} = W_{BE} - W_{WE}$$

and we can put in numbers

$$\begin{aligned} N_{BT} &= 100.0\,\mathrm{N} - 40\,\mathrm{N} \\ &= 60.0\,\mathrm{N} \end{aligned}$$

Indeed, our rope, pulley, and weight makes the table's N_{BT} less than it would have been. This is the idea of a counter weight that you see in theater lighting and scenery management that makes it easier to lift heavy equipment.

Of course this would have been harder (requiring some of our new math, an integral) if we did not use the massless string approximation and the frictionless pulley approximation.

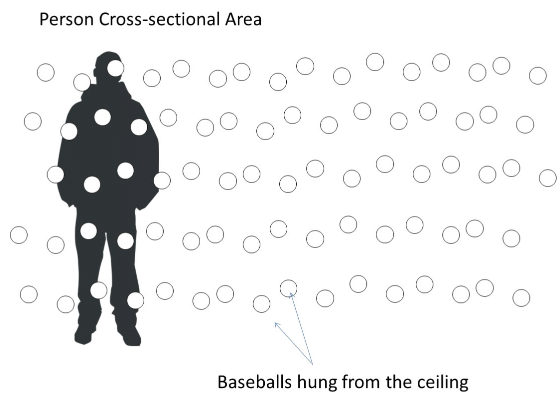
In our study of motion, we started with objects moving in straight lines, and then moved to objects moving in circles and even curved paths. In our study of forces we have studied objects moving in straight lines, you might expect that we will extend what we have learned to objects moving in circles and curved paths–and you would be right! We will start this next lecture.

17 Drag Force and Centripital Forces

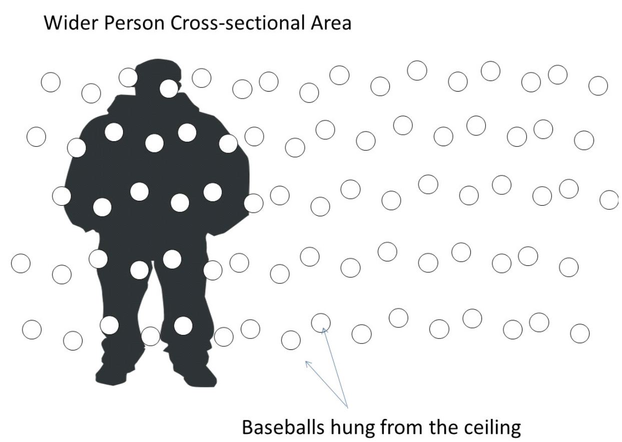
In this lecture, we will continue to look at forces and their origins. and we will practice using forces a bit. We are not going to learn much that is new in this lecture, but we are going to put pieces of physics that we already know together in a new way. We start with drag force, then go back to circular motion, and radial and tangential coordinates together to solve problems where objects experience forces as they travel in circular paths.

Drag Force

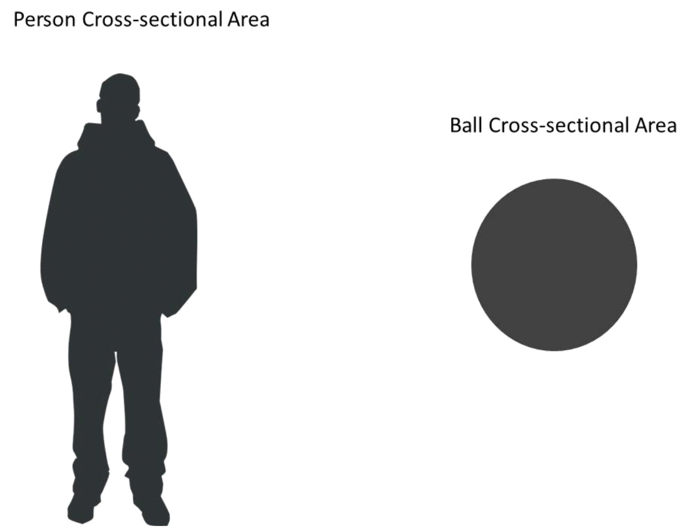
In this class we really don't yet understand enough to see exactly how drag forces like air resistance work. We will set the groundwork for understanding the origins of drag force in more detail in PH123. But we can have some intuitive feel for an air resistance with some conceptual reasoning. Consider again if we tied baseballs to strings and hung them all over our class room. Every time a student entered into the room the student would collide with the baseballs. Not only would this hurt, but it would slow the progress of the student. The baseballs would exert a force on the student with every collision. The collective resistance to the motion of the student due to all the baseballs is what we would call a drag force.



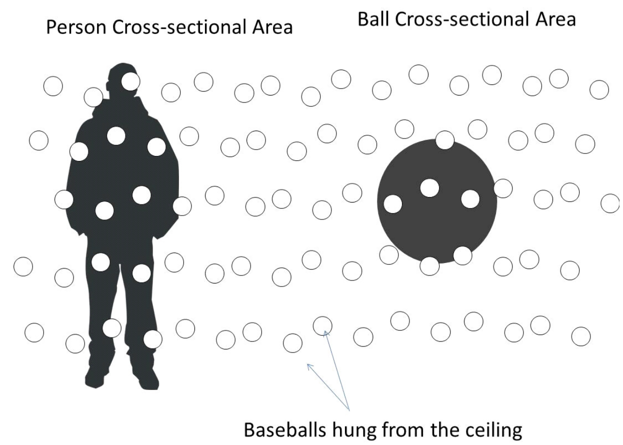
As the student tries to traverse the baseball laden room, we can see that the wider the student the more baseballs the student will strike.



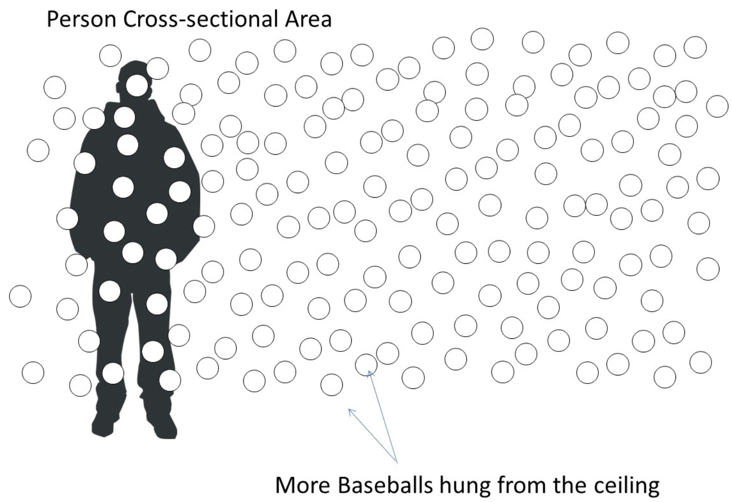
It doesn't matter much how long the person is, just how wide. A horse wouldn't not hit more baseballs due to it's long body, for example. We will call the area filled in from an outline of the student the cross sectional area.



Our drag force must be proportional to this cross-sectional area.



It must also be true that the number of collisions would be proportional to the density of baseballs in the room. The more tightly packed the balls, the more balls we will hit.



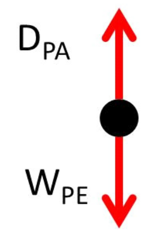
It's also true that the faster we try to run through the room, the more the baseballs will resist our motion. For one thing, the collisions would hurt more! That means the force due to the collision would be higher. This reasoning works for any shape that passes through the baseballs. We could try to push a large beach ball through the room, for example. The beach ball would be slowed down based on how big it's cross sectional area is, and how fast it is going.

We can give an equation for how strong the drag force will be (borrowed from PH123).

$$D = \frac{1}{2}C\rho A v^2$$

This is the magnitude of the drag force, and the direction is opposite the motion of the object. The value A is the cross-sectional area. The quantity ρ is the air density (think of this like a baseball density in our example). Of course v is the speed, and C is a a coefficient changes with the shape of the object. It's called the *drag coefficient* and it is smaller for pointed things like rockets and larger for blunt things like people

Suppose we have a person that wishes to parachute out of a plane. The person-parachute system will experience a drag force.



where the subscript P is for "person" and A is for "air" and E is for "Earth."

But notice something strange about drag forces. They increase as the speed increases. This makes sense, think of running vs. walking through or room with the baseballs. But for our falling person, it means that the drag force increases the faster the person falls. At some point the parachutist's drag force will be equal to his/her weight force. The parachutist will be in dynamic equilibrium!

$$F_{net} = D_{PA} - W_{PE} = 0$$

The person has stopped accelerating! which is the whole point of using a parachute. We can find out how fast the person will be going when he or she stops accelerating by using our borrowed formula for drag

$$D_{PA} - W_{PE} = 0$$

becomes

$$\frac{1}{2}C\rho A v^2 - mg = 0$$

or

$$\frac{1}{2}C\rho A v^2 = mg$$

so that we can write

$$v^2 = \frac{2mg}{C\rho A}$$

and finally

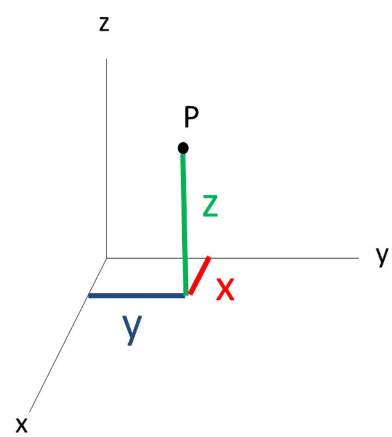
$$v = \sqrt{\frac{2mg}{C\rho A}}$$

We call this speed *terminal velocity*. Suppose we know that for a falling person on a parachute $C = 1.1$, $m = 89\,\mathrm{kg}$, (person plus parachute) the air density is $\rho = 1.3\frac{\mathrm{kg}}{\mathrm{m}^3}$ and suppose that for our person/parachute system our cross-sectional areal is about $A = \pi r^2$ with the radius about $5.5\,\mathrm{m}$ so $A = \pi\,(5.5\,\mathrm{m})^2 = 95.033\,\mathrm{m}^2$. Then

$$\begin{aligned} v &= \sqrt{\frac{2\,(89\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{(1.1)\left(1.3\frac{\mathrm{kg}}{\mathrm{m}^3}\right)(95.033\,\mathrm{m}^2)}} \\ &= 3.5828\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

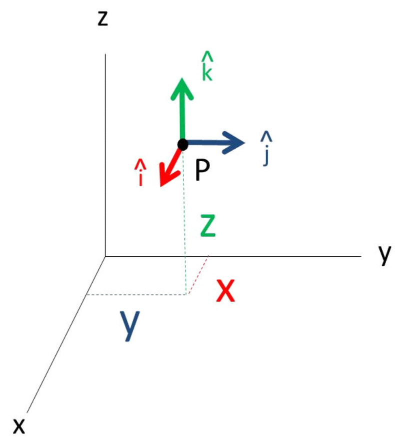
Centripital Forces

So far we have mostly used an Euclidean coordinate system (xyz) for forces.



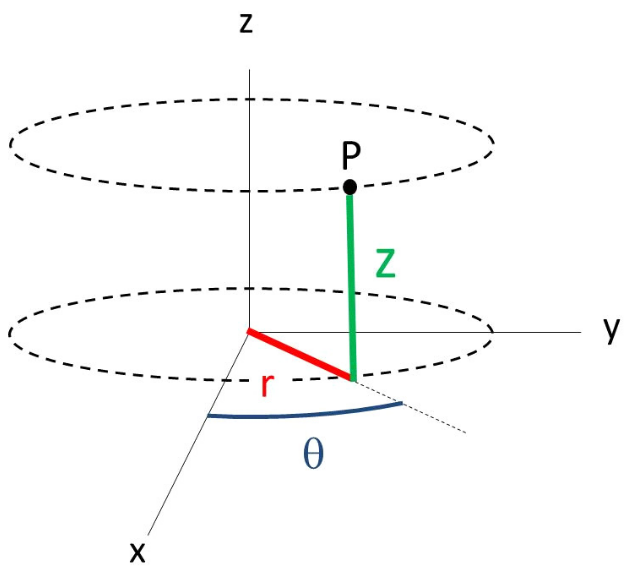
This coordinate systems has three axes, x, y, and z, and a point in space is given by how far away the point is from the xy , yz, and zx planes. A point is given by giving the three distances. We call the distance from the zy plane x. It is how far we have gone in the x-direction to get to our point. In Rexburg, it would be like saying how many blocks East we went. Likewise we call the distance from the xz plane y. It is how far we have gone in the y-direction. In Rexburg, it is like how many blocks North we went. And z is how far we are from the xy-plane. In Rexburg, it would be like how many floors up you went in a building.

We also defined three directions in this coordinate system.



These are our $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ unit vectors.

But really we have also used another coordinate system when we considered circular motion. Here it is



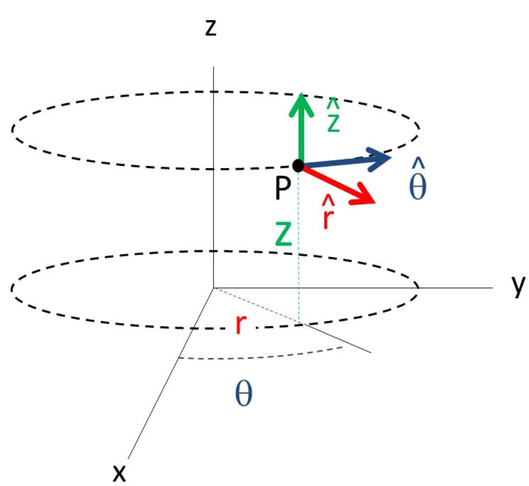
The figure shows the same point we considered in our Euclidean coordinate system. We still need three measurements in this coordinate system to describe where our point is in space. We will keep one of the measurements the same. We will call z the distance of our point from the xy plane. It is how far we went in the z-direction (how far up off the surface in Rexburg). But the other two measurements are different.

One of the new measurements is how far from the z-axis of the coordinate system we go to get to our point. But we won't follow in a block pattern like you would in a Western city like Rexburg. This is a direct "as the crow flies" measurement. We call this measurement r, because it is along a radius of the dotted circle shown. Notice that if I just tell you to go r meters, you could go in any direction. If the whole class received the same instruction, then you would all be lined up somewhere on the circle surrounding the starting point that has a radius, r.

The final measurement is an angle, measured from the x-axis. This will tell you which way to go from the origin. We call this angle measurement θ (or sometimes ϕ, or any Greek letter).

So if you were a large green super hero, you could start out at the city center, choose your direction by choosing θ, then smash through the buildings a distance r, and then climb up through what was left of the building a distance z to get to our point P. A point can be described by giving the measurements r, θ, and z.[10]

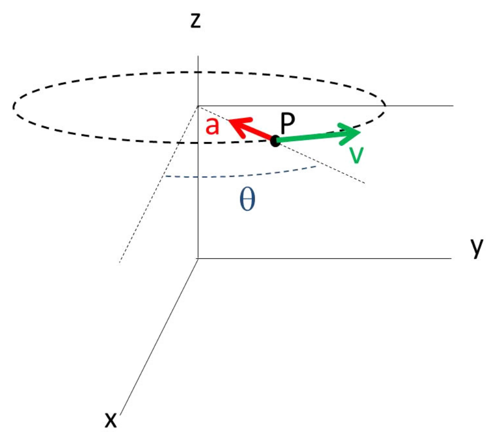
Of course you recognize this coordinate system as being made from polar coordinates, with a z-component added in. The proper name for this coordinate system is the cylindrical coordinate system. There are, of course, three unit vectors that describe directions in this coordinate system.



[10] But since there are no large green super heros, please do take the sidewalks through town and use the stairs to climb to floors in buildings.

Imagine you are in an airplane circling the airport. This coordinate system would be very natural. It tells you how far you are from the airport, what direction you are from the airport, and how high above the airport you are.

We called the $\hat{r}$ direction the radial direction because it is along the radius. The $\hat{\theta}$ direction is the tangential direction (also called azimuthal direction in older books). And we will keep calling the z-direction the $\hat{k}$-direction. Because of this, sometimes this coordinate system is called the rtz-coordinate system. Think of flying in the plane. You would have a turning (centripetal) acceleration keeping you going in a circle. That would be a radial acceleration because it points in the (negative) $\hat{r}$-direction.



The plane would also have a tangential velocity in the $\hat{\theta}$ direction.

Circular motion is familiar to us, and all this is really nothing new, with the small exception that we have added in the z-axis so we can have circular motion of flying things. We have a set of equations for uniform circular motion

$$\Delta\theta = \theta_f - \theta_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\theta}{\Delta t}$$
$$\omega = \frac{d\theta}{dt}$$
$$\omega = \frac{v_t}{r}$$
$$a_r = a_c = \frac{v_t^2}{r} = \omega^2 r$$

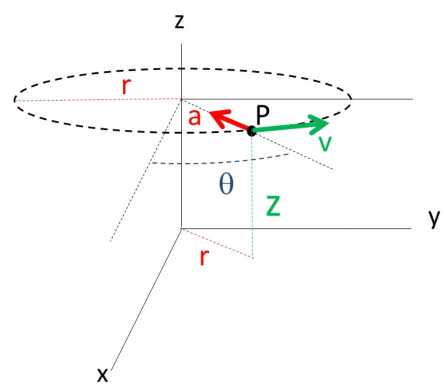
and if the motion is uniform circular motion we know

$$\begin{aligned} a_t &= 0 \\ a_z &= 0 \\ v_r &= 0 \\ v_z &= 0 \end{aligned}$$

Let's try a problem.

Suppose you decide to live in a particular apartment complex that gives obnoxious helicopter rides for signing up for an apartment (clear evidence that the rent is too high!).[11] The helicopter is flying $152.4\,\mathrm{m}$ above the ground (because any lower would be illegal) and is going around campus in a big circle (radius of $3218.7\,\mathrm{m}$) with a constant speed of $7.0\,\mathrm{m/s}$. What is the acceleration of the helicopter?

This is a uniform circular motion problem, but with our new twist that the motion is off the surface of the Earth.



We can see that the tangential and z-components of the acceleration are zero, and that the radial and z-components of the velocity are zero.

$$\begin{aligned} a_t &= 0 \\ a_z &= 0 \\ v_r &= 0 \\ v_z &= 0 \end{aligned}$$

[11] Yes, one semester this really happened.

all from our picture and knowing the helicopter is creating constant circular motion. We also know that

$$\begin{aligned} z &= 152.4\,\mathrm{m} \\ r &= 3218.7\,\mathrm{m} \\ v_t &= 7.0\,\mathrm{m/s} \end{aligned}$$

where we can see that the helicopter speed must be in the tangential direction (which direction around the circle we don't know, but if you were in the helicopter, it would be obvious).

The radial acceleration is related to the tangential speed of the helicopter. We can tell that this must be true by recognizing that a_r is pointing toward the center of the circular flight path. And remember, there is a word that means "points toward the center." Physicists use this word as a description of anything, force, acceleration, etc. that points toward the center of a circule. Then a_r can be given the title "centripital" and that means we can say

$$a_r = a_c = \frac{v_t^2}{r}$$

and we know all these parts

$$a_r = \frac{\underline{v_t}^2}{\underline{r}}$$

so we can find

$$\begin{aligned} a_r &= \frac{(7.0\,\mathrm{m/s})^2}{3218.7\,\mathrm{m}} \\ &= 1.5224 \times 10^{-2}\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

This is not a very large acceleration, the pilot probably does not want you to feel motion sickness (and therefore question the value of that high rent payment).

Circular motion and forces

We have reviewed circular motion and centripital acceleration. Now let's add in forces. We are force experts using Newton's second law.

$$\overrightarrow{a} = \frac{\overrightarrow{F}_{net_x}}{m}$$

But in the past we have written Newton's second law as

$$
\begin{aligned}
a_x &= \frac{F_{net_x}}{m} \\
a_y &= \frac{F_{net_y}}{m} \\
a_z &= \frac{F_{net_z}}{m}
\end{aligned}
$$

but now we need to write these as

$$
\begin{aligned}
a_r &= \frac{F_{net_r}}{m} \\
a_\theta &= \frac{F_{net_\theta}}{m} \\
a_z &= \frac{F_{net_z}}{m}
\end{aligned}
$$

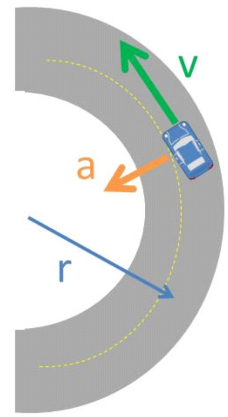
and for constant circular motion we can even write the first of our new Newton's second law set as

$$a_r = \frac{F_{net_r}}{m} = \frac{v_t^2}{r}$$

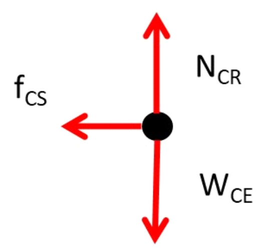
Notice that we are turning a three-dimensional problem into three one-dimensional problems. This is just what we always do with mulit-dimensional problems. But this time we have split the problem into r, θ, and z parts instead of x, y, and z parts.

Let's try another problem, one with forces involved this time:

A car travels at a constant speed of $13.411\,\frac{\mathrm{m}}{\mathrm{s}}$ on a level road and experiences a circular turn of radius $50.0\,\mathrm{m}$. What minimum coefficient of static friction, μ_s, between the tires and roadway will allow the car to make the circular turn without sliding? You might want to know something like this if you were designing the tire tread for race cars! A rougher tread will increase μ_s. But you don't want to make the tread too rough, or you have more friction than you need for the straight parts of the track. Here is a picture to illustrate what we want to find,



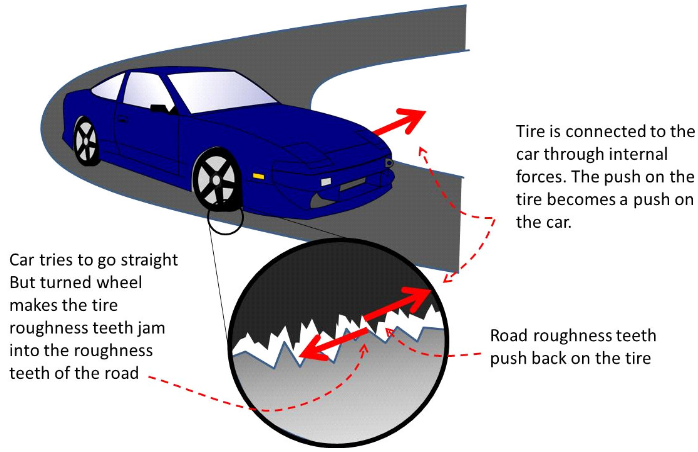
and since this is a force problem, here is the free body diagram for the car



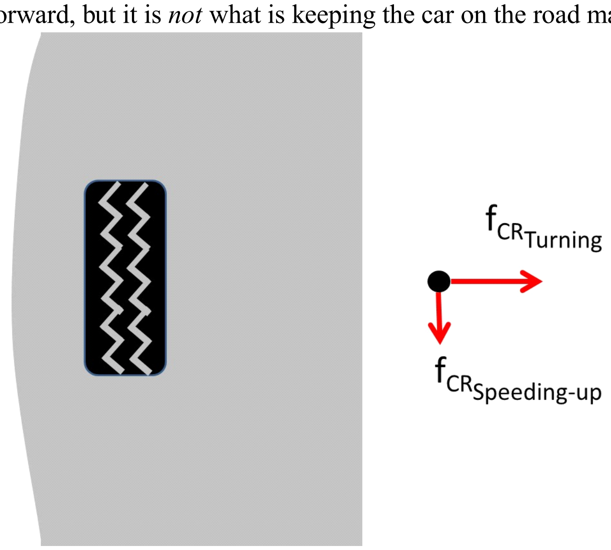
From the problem statement we know that

$$\begin{aligned} a_t &= 0 \\ a_z &= 0 \\ v_r &= 0 \\ v_z &= 0 \\ z &= 0 \\ v_t &= 13.411 \frac{\mathrm{m}}{\mathrm{s}} \\ &= 50.0\,\mathrm{m} \end{aligned}$$

and we can see that the friction force must be in the (negative) radial direction. That means that $\overrightarrow{a}_r$ must also be in the $-\hat{r}$ direction. This makes sense because as the car tires turn, the car wants to go straight (think of Newton's first law, and also think what would happen if the road were covered with smooth, frictionless ice), but the angled tire roughness teeth jam into the road roughness teeth and bend the road roughness teeth.



Of course, the road roughness teeth push back on the tire roughness teeth. So the tire is shoved toward the center of the circle. The tire is connected to the car through a series of internal forces, so a push on the tire becomes a push on the car. The car experiences a friction force toward the center of the circle. Note that this is static friction. We really never want the tires to slide sideways. There would also be some "rolling friction" as the tires roll forward, but it is *not* what is keeping the car on the road making a turn.



The "rolling" part of the friction makes the car speed up forward, not turn to stay in the circle. Usually you don't speed up on a turn, so while the car is turning we mostly have just $f_{CR_{Turning}}$. We can identify this static friction as a *centripetal force.* A centripetal force is the force that causes the centripetal acceleration.

Now that we have a free-body diagram, we can set up Newton's second law in our cylindrical coordinate system. Notice that the z-direciton is "up." So

$$\begin{aligned}\Sigma F_z &= 0 = N_{CR} - W_{CE} \\ \Sigma F_r &= -ma_r = -f_{CS_{Turning}} \\ \Sigma F_\theta &= ma_\theta = 0\end{aligned}$$

Note that the net force in the r-direction is inward, so we need the minus sign in $-ma_r$. We can drop the "Turning" subscript, because we understand it is the turning part of the friction that is in the $-\hat{r}$ direction. We will do so in what follows.

From the first (z) equation we find that

$$N_{CR} = W_{CE}$$

which is not a surprise. We know that

$$f_{CS} = \mu_s N_{CR}$$

so we can write

$$f_{CS} = \mu_s W_{CE}$$

and

$$-ma_r = -f_{CS} = -\mu_s W_{CE}$$

so we have for a

$$-a_r = -\frac{\mu_s W_{CE}}{m}$$

But now consider that the force, ma_r, is the force that is changing our car's direction. It is in the radial direction. This *is* our centripetal force, so we can identify $a_r = a_c$

$$-a_c = -\frac{\mu_s W_{CE}}{m}$$

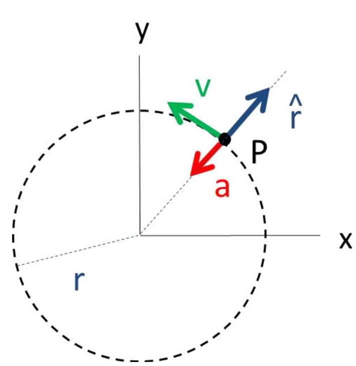
and we know that

$$a_c = \frac{v_t^2}{r}$$

so

$$-\frac{v_t^2}{r} = -\frac{\mu_s W_{CE}}{m}$$

Consider that our acceleration is radial, but it is in the $-\hat{r}$ direction toward the origin.



so in our cylindrical coordinate system our radial acceleration is negative.

Now we are down to some algebra

$$-\frac{v_t^2}{r} = -\frac{\mu_s m g}{m}$$

We can cancel the minus signs

$$\frac{v_t^2}{r} = \mu_s g$$

so

$$\mu_s = \frac{v_t^2}{rg}$$

If we plug in numbers,

$$\begin{aligned} \mu_s &= \frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{50\,\mathrm{m}\cdot 9.8\frac{\mathrm{m}}{\mathrm{s}^2}} \\ &= 0.367\,05 \end{aligned}$$

Consider for a moment what this means. Will the car slip? We can look up the value of μ_s for rubber on pavement, and note that it is close to 1, so on a clear road (no ice or water) the car will not slip.

Note that we now have a centripetal force as well as a centripetal acceleration. Once again, this is not a new kind of force, but just the same old force with a new title.

More angular acceleration

Suppose that we are traveling in a circle, but the speed is not constant, then we have two acceleration components, one component towards the center, and one tangential (along a tangent line)

$$a_t \tag{17.1}$$

This is a little weird, because often in the past we have taken x and y components, but *centripetal* and *tangential* components work just as well. And we have used a_t and a_c in circular problems before. To get back to a total acceleration vector we use,

$$a = \sqrt{a_t^2 + a_c^2} \tag{17.2}$$

and

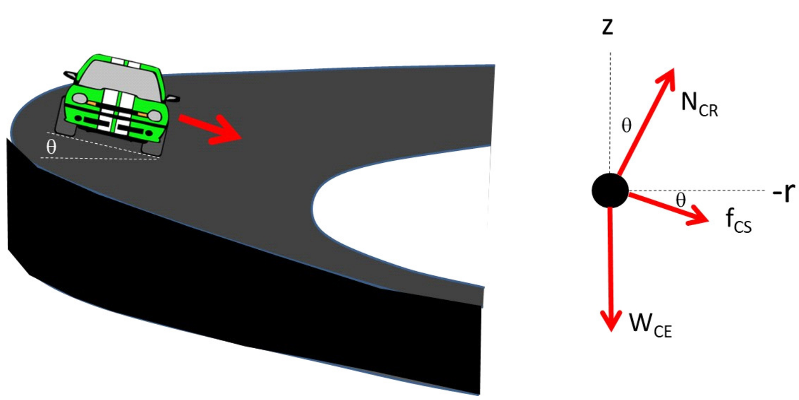
$$\theta = \tan^{-1}\left(\frac{a_t}{a_r}\right) \tag{17.3}$$

Banked roadways

Suppose we try our turning car problem again, but we play a trick to let cars travel around the curve faster. We bank the curve. This is done in real race tracks. Here is a NASCAR ® example.

Notice that the roadway is not flat. The roadway has a slope where there is a turn. So let's do our car problem again, but with a sloped roadway of, say, 20.0° slope angle. Will we need as much friction to keep the car on the road?

Once again we draw a free-body diagram.



and write out Newton's second law. We have forces in the z and r directions. So we will need Newton's second law in at least these two directions.

Let's do the z-axis first. From the figure we have the angle θ, but for N_{CR} that angle isn't with respect to the $-r$ axis. So we need to be careful with taking components. We could find the angle ϕ that is with respect to the $-r$ axis. Then our normal equations for components

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \end{aligned}$$

will work. And notice that $\phi = 90\,^\circ - \theta$. Now let's look at f_{CS}. We have two choices. We could use the angle measured from the $-r$ axis. That would be $\eta = 360\,^\circ - \theta$. Or we could just use $-\theta$. I will use the first choice, but either work. Here is Newton's second law for the z-direction.

$$\Sigma F_z = ma_z = N_{CR}\sin(90\,^\circ - \theta) + f_{CS}\sin(360\,^\circ - \theta) - W_{CE}$$

Now for the r-axis. The net force is in the $-\hat{r}$ direction and all the components will be in the $-\hat{r}$ as well.

$$\Sigma F_r = -ma_r = -N_{CR}\cos(90\,^\circ - \theta) - f_{CS}\cos(360\,^\circ - \theta)$$

Of course, all the negative signs cancel. But starting out with them keeps us honest in our thinking about which direction the forces go.

Using some of our beloved trig identities

$$\begin{aligned} \sin(90\,^\circ - \theta) &= \cos\theta \\ \cos(90\,^\circ - \theta) &= \sin\theta \\ \sin(360\,^\circ - \theta) &= -\sin\theta \\ \cos(360\,^\circ - \theta) &= \cos\theta \end{aligned}$$

we can write our Newton's second law as

$$\begin{aligned} \Sigma F_z &= ma_z = N_{CR}\cos(\theta) - f_{CS}\sin(\theta) - W_{CE} \\ \Sigma F_r &= -ma_r = -N_{CR}\sin(\theta) - f_{CS}\cos(\theta) \end{aligned}$$

and we can do the same thing we did in our last example, mostly...

It might be tempting to rotate our coordinate system, but we need to be careful. We know something about $a_r = a_c = v^2/r$. If we rotate our coordinates, we won't be in our cylindrical coordinates, and we will have to be careful when we calculate the centripetal acceleration. Let's not rotate coordinates this time.

We know that the car won't launch off the surface, and is not likely speeding up through the turn, so we can say that

$$a_z = 0$$

$$\begin{aligned}
a_t &= 0\\
v_r &= 0\\
v_z &= 0\\
z &= 0\\
v_t &= 13.411\frac{\mathrm{m}}{\mathrm{s}}\\
r &= 50.0\,\mathrm{m}\\
\theta &= 20.{}^\circ
\end{aligned}$$

Then our z-part of Newton's second law becomes

$$N_{CR}\cos(\theta) - f_{CS}\sin(\theta) - W_{CE} = 0$$

and we know

$$f_{CR} \leq \mu_s N_{CR}$$

and again we want to take the case were we find the minimum μ_s so we will set.

$$f_{CR} = \mu_s N_{CR}$$

Note that is was a choice. We picked the equal sign because we want the minimum μ_s so the car just stays on the road. That is the case where f_{CR} is just equal to $\mu_s N_{CR}$. So our Newton's second law pair now look like this:

$$N_{CR}\cos(\theta) - \mu_s N_{CR}\sin(\theta) - mg = 0$$
$$-ma_r = -N_{CR}\sin(\theta) - \mu_s N_{CR}\cos(\theta)$$

Using the first of these,

$$N_{CR}\left(\cos(\theta) - \mu_s \sin(\theta)\right) = mg$$

or

$$N_{CR} = \frac{mg}{\left(\cos(\theta) - \mu_s \sin(\theta)\right)}$$

then we can take our r-equation

$$-ma_r = -N_{CR}\left(\sin(\theta) - \mu_s\cos(\theta)\right)$$

and substitute in for N_{CR}

$$-ma_r = -\frac{mg}{\left(\cos(\theta) - \mu_s\sin(\theta)\right)}\left(\sin(\theta) + \mu_s\cos(\theta)\right)$$

and we know a_r is our centripetal acceleration so

$$a_r = a_c = \frac{v^2}{r}$$

then

$$-m\frac{v^2}{r} = -mg\frac{\left(\sin(\theta) + \mu_s\cos(\theta)\right)}{\left(\cos(\theta) - \mu_s\sin(\theta)\right)}$$

We know v, r, m, g, and θ, so messy as this is, we should be able to solve for μ_s

$$-m\frac{v^2}{r}\left(\cos(\theta)-\mu_s\sin(\theta)\right)=-mg\left(\sin(\theta)+\mu_s\cos(\theta)\right)$$
$$-m\frac{v^2}{r}\cos(\theta)+\mu_s m\frac{v^2}{r}\sin(\theta)=-mg\sin(\theta)-\mu_s mg\cos(\theta)$$
$$-m\frac{v^2}{r}\cos(\theta)+mg\sin(\theta)=-\mu_s mg\cos(\theta)-\mu_s m\frac{v^2}{r}\sin(\theta)$$
$$-m\frac{v^2}{r}\cos(\theta)+mg\sin(\theta)=\mu_s\left(-mg\cos(\theta)-m\frac{v^2}{r}\sin(\theta)\right)$$

$$\frac{-m\frac{v^2}{r}\cos(\theta)+mg\sin(\theta)}{-mg\cos(\theta)-m\frac{v^2}{r}\sin(\theta)}=\mu_s$$
$$\mu_s=\frac{-m\frac{v^2}{r}\cos(\theta)+mg\sin(\theta)}{-\left(m\frac{v^2}{r}\sin(\theta)+mg\cos(\theta)\right)}$$

So our answer is

$$\mu_s=\frac{-\frac{v^2}{r}\cos(\theta)+g\sin(\theta)}{-\left(\frac{v^2}{r}\sin(\theta)+g\cos(\theta)\right)}$$

We should check this. We should get the last answer if $\theta=0$

$$\mu_s=\frac{-\frac{v^2}{r}(1)+g(0)}{-\left(\frac{v^2}{r}(0)+g(1)\right)}$$

$$\frac{-\frac{v^2}{r}}{-(g)}=\mu_s$$
$$\frac{v^2}{(gr)}=\mu_s$$
$$\mu_s=\frac{v^2}{(gr)}$$
$$\begin{aligned}\mu_s &= \frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(50.0\,\mathrm{m})}\\ &= 0.36705\end{aligned}$$

which is the same answer we got for the un-banked road. This gives some confidence that we got our tilted problem right.

Let's put in our numbers for our tilted ramp.

$$\mu_s=\frac{-\frac{v^2}{r}\cos(\theta)+g\sin(\theta)}{-\left(\frac{v^2}{r}\sin(\theta)+g\cos(\theta)\right)}$$

so

$$\mu_s = \frac{-\frac{\left(13.411\,\frac{\text{m}}{\text{s}}\right)^2}{50.0\,\text{m}}\cos(20\,^\circ) + \left(9.8\,\frac{\text{m}}{\text{s}^2}\right)\sin(20\,^\circ)}{\left(-\frac{\left(13.411\,\frac{\text{m}}{\text{s}}\right)^2}{50.0\,\text{m}}\sin(20\,^\circ) - \left(9.8\,\frac{\text{m}}{\text{s}^2}\right)\cos(20\,^\circ)\right)} = 2.7176 \times 10^{-3}$$

This is *much* less than the un-titled road.

Let's see if we can tell why. For the un-tilted road, the normal force just had to support the car's weight

$$N_{CR} = W_{CE} = mg$$

so the frictional force was

$$f_{CR} = (0.36705)\,W_{CE}$$

but now look at our normal force with the banked road

$$\begin{aligned} N_{CR} &= \frac{W_{CE}}{(\cos(\theta) - \mu_s \sin(\theta))} \\ &= \frac{W_{CE}}{(\cos(20\,^\circ) - (2.7176 \times 10^{-3})\sin(20\,^\circ))} \\ &= 1.0652 W_{CE} \end{aligned}$$

The normal force has increased with the banking. That is because the car is pushing harder on the road due to the banked turn. The road is helping the car turn with the radial part of the normal force. Then the frictional force can be much less.

$$f_{CR} = \mu_s \frac{W_{CE}}{(\cos(\theta) - \mu_s \sin(\theta))}$$

Since the minimum μ_s is now tiny, this is the case. Let's put in numbers.

$$\begin{aligned} f_{CR} &= \left((2.7176 \times 10^{-3})\right) \frac{W_{CE}}{(\cos(20\,^\circ) - ((2.7176 \times 10^{-3}))\sin(20\,^\circ))} \\ &= 2.8949 \times 10^{-3} W_{CE} \end{aligned}$$

This is a reduction in the required friction of

$$\frac{f_{CR}}{f_{CR}} = \frac{2.8949 \times 10^{-3} W_{CE}}{(0.36705)\,W_{CE}} = 7.8869 \times 10^{-3}$$

or we now need only 0.78869% of the friction force that we needed without the banking. This allows NASCAR ® cars to travel much faster without slipping. In Rexburg, if the curves were banked might even be able to turn with snow and ice!

Although we really did not do anything radically new in this lecture, we did use physics pieces we knew in new ways. So let's review.

1. We called our rotational coordinate system a cylindrical coordinate system (or rtz coordinate system).

2. We used this coordinate system to solve constant motion problems, first with kinematics, and then using forces and Newton's second law.

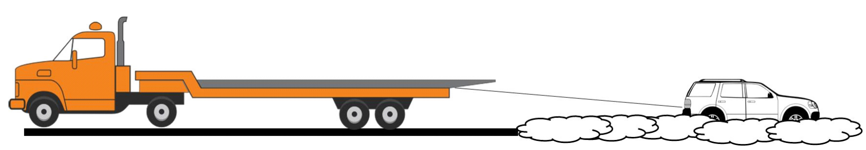
These combinations will allow us to solve more rotational problems, even with aircraft that fly off the surface. Of course, we could also let the car or airplane speed up or slow down. And that is coming. But armed with what we have done in this lecture, we are ready to take on orbital motion, loop-de-loop roller coasters, and a bunch of great problems that require both rotation and forces. And that is our next lecture.

Newton's Second Law and Kinematics

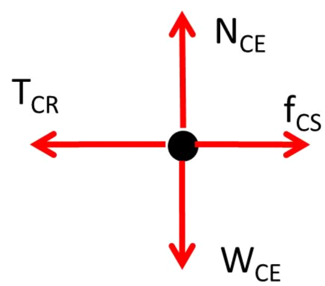
Of course, its not always true that forces sum to zero for all objects. But that's not bad. If we have an acceleration we just use our equations for constant acceleration problems. We might have to use our Newton's second law techniques to find the acceleration, however. Lets try this type of problem.

Suppose your car is off the road in the snow. Since the car is just sitting there, it has a velocity $v_i = 0$. But you would like it towed out of the snow, back on the road. So you wish to change the car's motion. You are a distance $\Delta x = 10.0\,\mathrm{m}$ from the road. The deep snow causes a constant friction force of $f_{CS} = 1000.0\,\mathrm{N}$. Your car has a mass of $1615.4\,\mathrm{kg}$ (weight of $W_{CE} = 15831\,\mathrm{N}$). The tow truck wench pulls with a force of $2000.0\,\mathrm{N}$. How long does it take to get the car back on the road?

We can see that this is an acceleration problem, we don't want the forces to balance. We may even suspect that this is a constant acceleration problem. But we don't know that constant acceleration. If we did we could us kinematics to solve the problem. But now that we have Newton's second law, we have a way to find the acceleration! Let's try it out.



Our free-body diagram for the car might look like this.



Notice that this is the free-body diagram for the car. I didn't draw the diagram for the truck (or the snow).

We know

$$\begin{aligned} f_{CS} &= 1000\,\mathrm{N} \\ T_{CR} &= 2000\,\mathrm{N} \\ m &= 1614.4\,\mathrm{kg} \\ W_{CE} &= 15831\,\mathrm{N} \\ \Delta x &= 10.0\,\mathrm{m} \end{aligned}$$

and we know that

$$\begin{aligned} F_{net_x} &= ma_x \\ F_{net_y} &= ma_y \end{aligned}$$

and if the acceleration is constant, we can use our kinematic set.

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + 0 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + 0 \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 0 \end{array}$$

We can realize that we know many more things based on a choice of coordinate system.

$$\begin{aligned} v_{ix} &= 0 \\ v_{iy} &= 0 \\ x_i &= 0 \\ y_i &= 0 \\ y_f &= 0 \end{aligned}$$

We should realize that we actually know $a_y = 0$ (or at least we hope it does!) but we don't know a_x.

$$\begin{array}{ll} \underline{\Delta x} = \underline{v_{ix}} + \frac{1}{2}a_x\Delta t^2 & \underline{\Delta y} = \underline{v_{iy}}\underline{\Delta t} + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + a_x\Delta t & \underline{v_{fy}} = \underline{v_{iy}} + \underline{a_y}\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 2a_x\underline{\Delta x} & \underline{v_{fy}^2} = \underline{v_{iy}^2} + 2\underline{a_y}\underline{\Delta y} \end{array}$$

and using all the zeros, we have

$$\begin{array}{ll} \underline{\Delta x} = 0 + \frac{1}{2} a_x \underline{\Delta t}^2 & \Delta y = v_{iy} \Delta t + 0 \\ v_{fx} = 0 + \underline{a_x} \Delta t & v_{fy} = v_{iy} + 0 \\ v_{fx}^2 = 0 + 2\underline{a_x} \Delta x & v_{fy}^2 = v_{iy}^2 + 0 \end{array}$$

It looks like our best bet is to use the first x-equation.

$$\underline{\Delta x} = \frac{1}{2} a_x \underline{\Delta t}^2$$

$$\frac{2\Delta x}{a_x} = \Delta t^2$$

$$\sqrt{\frac{2\Delta x}{a_x}} = \Delta t$$

But we don't know a_x. We need a strategy to find it. We know that the net force has our acceleration in it. Let's try Newton's second law.

Let's start with the forces to see this. Remember that the net force is the sum of the forces, so

$$\begin{aligned} F_{net_x} &= (T_{CR} - f_{CS}) \\ F_{net_y} &= (N_{CE} - W_{CE}) \end{aligned}$$

and Newton's law tells us that

$$\begin{aligned} F_{net_x} &= m a_x \\ F_{net_y} &= m a_y \end{aligned}$$

so putting these two together tells us

$$\begin{aligned} m a_y &= (N_{CE} - W_{CE}) \\ m a_x &= (T_{CR} - f_{CS}) \end{aligned}$$

We can guess from our experience that $N_{CE} = W_{CE}$ so $a_y = 0$. The car is not lifting off the surface or burrowing into the ground! but T_{CR} and f_{CS} are not the same.

$$\begin{aligned} a_x &= \frac{(T_{CR} - f_{CS})}{m} \\ a_x &= \frac{1}{1614.4\,\mathrm{kg}} (2000\,\mathrm{N} - 1000\,\mathrm{N}) \\ &= 0.619\,43 \frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

This *is* a constant acceleration. So our guess that we could use kinematics is justified! Let's finish it up. We know

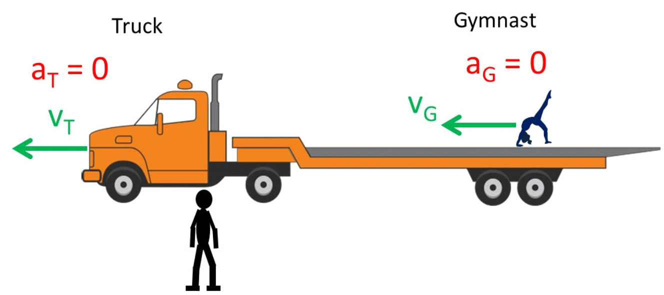
$$\Delta t = \sqrt{\frac{2\Delta x}{a_x}}$$

$$\begin{aligned} \Delta t &= \sqrt{\frac{2\,(10\,\mathrm{m})}{0.619\,43 \frac{\mathrm{m}}{\mathrm{s}^2}}} \\ &= 5.682\,2\,\mathrm{s} \end{aligned}$$

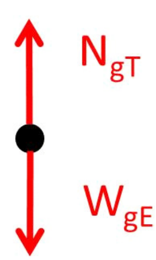
The problem was not too hard! It was a little longer because we had to use two different problem types to solve this single problem, but it really was not too bad.

Fictitious Forces

As we travel in circular paths, at a minimum we have a centripetal acceleration. This is a problem for us because we agreed to only use inertial reference frames for our calculations, and really we have done this so far. We do our measurements from a reference frame that is not accelerating, and study the acceleration of an object from this viewpoint. Suppose we consider, for an example, a gymnast on a flatbed truck in the Rexburg Parade.

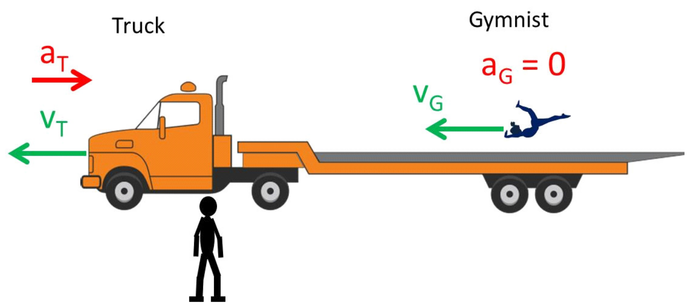


We know that as long as the truck travels with a constant speed, the truck bed is an inertial reference frame and all our physics works just fine. The gymnast is safe. The forces acting on the gymnast are all in the y-direction. She or he would be able to do his or her routine just as though they were in their "stationary" gym.

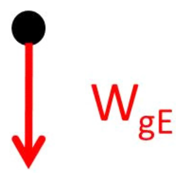


There could be static friction if the gymnast's feet are on the truck, but since there is no acceleration, the gymnast's feet won't push on the roughness teeth of the truck, so the truck roughness teeth won't push back. There would be no friction if the truck has a constant speed! We don't need a force to make the gymnast move with constant speed. Of course if there was a strong wind, then we would have some drag force. But let's assume this is negligible.

Now if the truck driver slammed on the breaks, the truck would quickly change it's motion. The gymnast would try to keep going at a constant speed (Newton's first law).



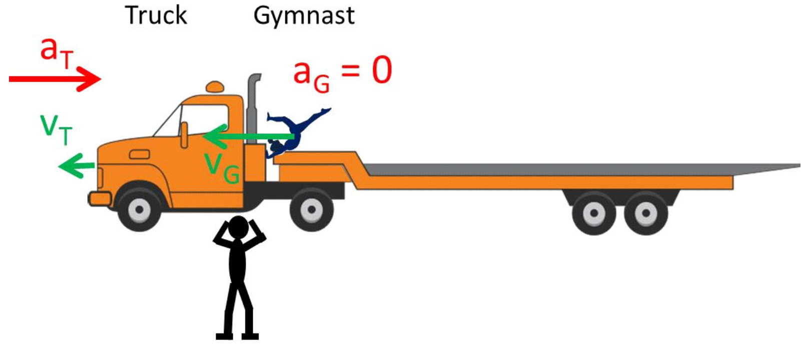
But notice that we took all our measurements of acceleration and velocity from the point of view of a person watching this happen from the street. The street is an inertial reference frame. Let's say that the gymnast has just jumped into the air as the truck puts on it's breaks. From the street viewpoint, the forces acting on the gymnast are still all in the y-direction.



and the gymnast just keeps going. She has no acceleration in the x-direction. So there is no force on her in the x-direction

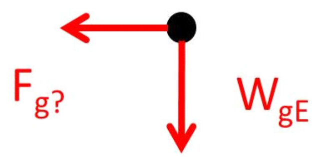
$$F_{net_x} = ma_x = m(0) = 0$$

But we can see that this is really not going to be good on the gymnast.



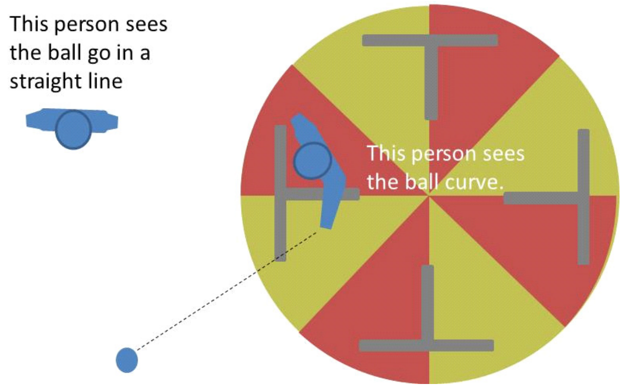
The truck has slowed, the gymnast has not, so the gymnast is going to run into the cab of the truck.

Let's consider what the gymnast would see (say, we attache a video camera on the gymnast and watch the video from her point of view). The gymnast sees herself jump, and then she sees herself thrown forward into the truck. She would see an x-direction force acting on her (in the $-\hat{\imath}$ direction).



But what would cause this force? Forces need two objects, the object that is being acted on (mover) and the environmental object creating the force. There is no environmental object creating a horizontal force on the gymnast. And really we realize that the problem is that the gymnast is using the accelerating truck as her reference system. It is an accelerated reference frame, and we won't allow that in PH121! But now we can see what an accelerated reference frame would do to our physics. It makes it look like there are forces there that are not real. We call these *fictitious forces*. They appear to be there only because we are using an accelerating point of view.

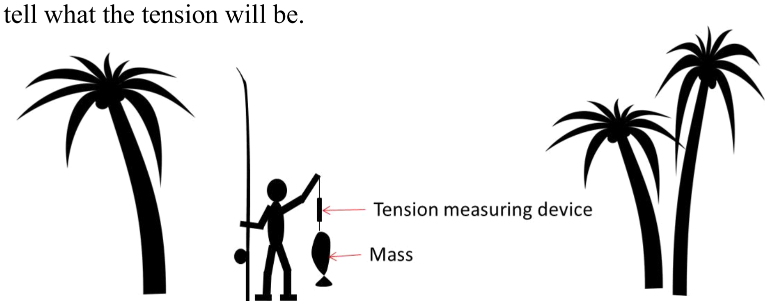
Let's look at another fictitious force case. Perhaps you played on a rotating platform as a kid. We call them merry-go-rounds. If you have a ball on a merry-go-round, and you throw the ball, to you it looks like the ball curves as it flies away from you. But for someone who is standing on the playground, not on the merry-go-round, the ball appears to go in a straight line. Which is correct? This is important, because if the ball curves, then the direction has changed. That would be acceleration. It takes a force to cause acceleration. So for you there appears to be a force, and for your partner on the playground there does not appear to be a force. Serious stuff!



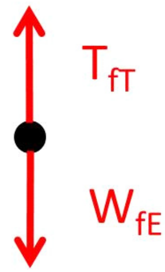
The apparent force on the ball is another fictitious force. The ball only appears to change direction because you are on a rotating platform. Really you have moved away from the ball, not the ball curved away from you. So the apparent force causing the ball to turn is not real. This particular fictitious force even has a name, it is called a *centrifugal force*. This is different than centripetal force which is a title for the force that makes things turn. To keep from confusing these two forces, remember that centri(f)ugal has an "f" for "fictitious" in it.

This centrifugal force comes up a lot because we all really live on a rotating platform, the Earth. It is common to see things appear to curve on a global scale that really don't curve. Let's take an example to see how important it might be to take into account the rotation of the Earth (and therefore acknowledge that we are on a rotating platform).

Suppose we are on the equator and have a mass on a rope with a tension measuring device to tell what the tension will be.



A free body diagram for the mass (a fish) might look like this



that the net force

$$F_{net_y} = m_f a$$

and the y-net force is equal to the sum of all the y-forces

$$F_{net_y} = T_{fT} - W_{fE}$$

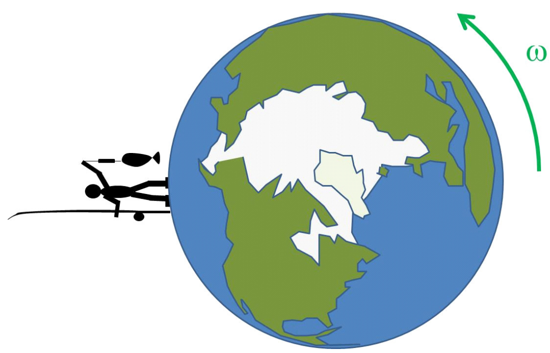
so in our flat-non-rotating Earth approximation we would say that the mass is not accelerating so

$$F_{net_y} = m_f a = 0$$
$$0 = T_{fT} - W_{fE}$$

so

$$T_{fT} = W_{fE}$$

But if we realize that the Earth is really turning,



we can see that the mass must have a radial acceleration,

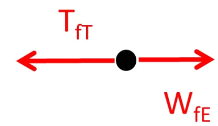
$$\overrightarrow{a}_r = a\left(-\hat{r}\right)$$

and this radial acceleration could be called a centripetal acceleration

$$a_r = a_c = \frac{v^2}{r} = \omega^2 r_E$$

where r_E is the radius of the Earth. So

$$F_{net_r} = -m_f a = -m_f\left(\omega^2 r_E\right)$$



and then

$$-m_f\left(\omega^2 r_E\right)=T_{fT}-W_{fE}$$

and we can see that the tension in the rope is

$$T_{fT}=-m_f\left(\omega^2 r_E\right)+W_{fE}$$

This is smaller than it would be if the Earth were flat and not rotating. And what is $m_f\left(\omega^2 r_E\right)$? It is a correction to our physics because we violated the no-accelerating (rotating) reference frame rule! But it *looks like* a force. This is the fictitious force called the centrifugal force. It's not real, it is a correction to our physics because we did not use a proper reference frame. But the meter in the rope really will read a little bit less because the Earth is rotating.

This gives us a way to test our flat earth assumption. If $m_f\left(\omega^2 r_E\right)$ is not negligible, then we need to do more work on our problem! Of course we did this at the equator. The value changes with latitude. But we get the idea. The Earth can be used as an inertial reference frame only if the correction factor is negligible. In later physics courses, we will take up this problem again!

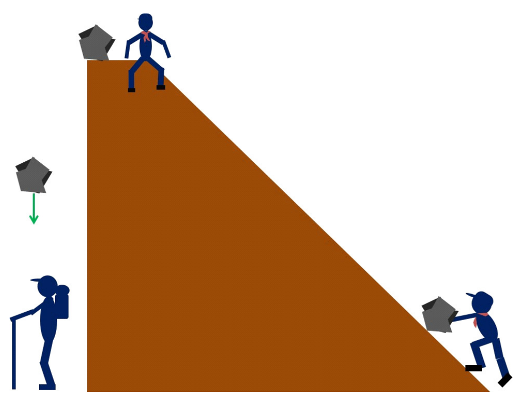
We will return to rotation later in our lectures. In our next lecture we will take on a new way of looking at motion. One that is really more fundamental than the forces and accelerations we have been studying. But we had to work a while so we could understand it. We will study *energy* next.

18 Energy

In our last lecture, we said we would study energy. We have an intuitive feel for what energy is. It is what makes something go. But what is it? You might be surprised to know that we don't have a very good answer to that question. Energy is involved in making motion happen. And we know how energy behaves. So we can define energy by describing how it acts. This is a little bit like describing an airline pilot by describing what an airline pilot does (wears a uniform, flies planes[12]). And that is a perfectly good way to define something. It will take several classes to complete this definition of energy (many, many classes for the physicists among us). But we will start with how energy causes motion to happen. We will start with a form of energy that is directly related to forces moving objects.

Work

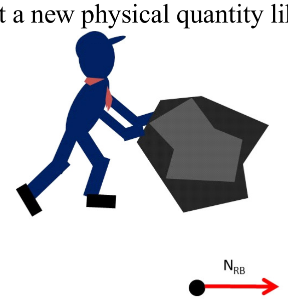
For the moving object let's take a rock, and for the environmental object let's take a deacon. The deacon is on a hike and sees the rock. You know what will happen, the rock will be moved to the edge of a cliff and pushed off the edge.[13] We will get motion from the push due to the deacon, and motion due to the pull of gravity. The rock will travel.



[12] some pilots give talks in Conference

[13] This is real. I have learned, in Idaho hike on the high ground!

In the next figure there is a force diagram showing just the horizontal forces of such a boy pushing on a (frictionless) rock. Only the horizontal force is shown, and for some strange reason this boy is pushing a rock on a frictionless surface (which makes our math easier as we learn about a new physical quantity like energy).



Work-energy equation

Suppose we push our rock in the s-direction. Before we used s to be arc length. But now the s-direction could be the x-direction, or the y-direction, or the r-direction or the θ-direction, or the z-direction or the direction along the axis of any coordinate system. We call s a generalized coordinate. So F_s would be the component of the force in the s-direction, whichever direction that actually is. We know that the force is

$$F_s = ma_s = m\frac{dv_s}{dt}$$

and we can use the chain rule of calculus to get an energy equation.[14] This is how it works:

$$m\frac{dv_s}{dt} = m\frac{dv_s}{ds}\frac{ds}{dt} = m\frac{dv_s}{ds}v_s$$

so we can write

$$F_s = mv_s\frac{dv_s}{ds}$$

and, let's multiply both sides by ds

$$F_s ds = mv_s\frac{dv_s}{ds}ds$$

It's not obvious that this did anything useful for us, but it did! Notice that the ds terms cancel so we have

$$F_s ds = mv_s dv_s$$

[14] If your calculus class hasn't gotten to the chain rule yet, just ignore this step and trust that the result is OK.

and we can now integrate both sides

$$\int_{s_i}^{s_f} F_s ds = \int_{v_i}^{v_f} m v_s dv_s$$

Let's start with the right hand side. We get

$$\int_{s_i}^{s_f} F_s ds = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

But what does it mean?

The left hand side is the result of the push on the rock. It is a form of energy. We will give it a name, one that the deacon's would like. It is called *work*.

$$w = \int_{s_i}^{s_f} F_s ds$$

It might surprise you to learn that in physics "work" is a type of energy. But it's not too strange if we think about it. It takes energy to do work, so it makes some sense to think of the energy from the deacon's lunch being converted into the energy of the deacon's work in moving the rock.

The right hand side must also be a form of energy. Since it has the speed of the rock in it, it must be the energy tied up in the movement of the rock. This matches our intuition. Think of the Sunbeam class when Sunday is the day after Halloween. The kids are full of energy, and they run around and bounce off the walls. Because this amount of energy is related to moving, we will use our greek word for motion to describe this energy. We call it *kinetic energy* and we give it the symbol K.

$$K = \frac{1}{2} m v^2$$

In fact, there is a beginning energy and an ending energy in the right hand side of our equation. So the right hand side is a change in kinetic energy

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \Delta K$$

Notice that this kinetic kind of energy can't be negative! At least as far as we know, mass can't be negative, and the speed in the equation is squared. So kinetic energy can only have positive values. Another thing to notice is that kinetic energy has no direction, it is a scalar. This is a wonderful thing! If we can solve problems using the idea of energy, we don't have to use vectors and their components.

Work-energy theorem

Suppose just for a second that the force F_s is constant, Then our work equation would look like this

$$w = F_s \int_{s_i}^{s_f} ds$$

and we know how to take this integral

$$\begin{aligned} w &= F_s \left(s \right|_{s_i}^{s_f} \\ &= F_s \left(s_f - s_i \right) \\ &= F_s \Delta s \end{aligned}$$

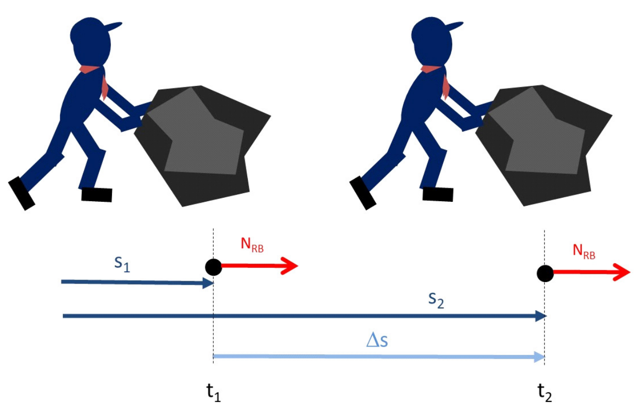
and we can see that once again physics is concerned with how much we accomplish. Work can be thought of as how effective a force has been. If the force moves the object a displacement Δs, then it has done work. If the force doesn't move the object, it might feel like work, but for physics it isn't.

Now let's go back to our integral form and take just the integrand and look at it carefully. We can see that it is a force times a small distance.

$$F_s ds$$

This makes sense. The small distance is how far the boy pushed the rock in a time dt. If the time is larger, Δt, and the force was constant, then we could write this as

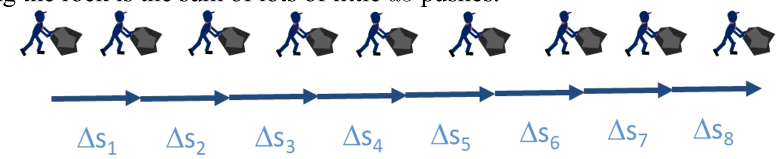
$$w = F_s \Delta s$$



The work from t_1 to t_2 would be the force F_s multiplied by how far the boy pushed the rock $\Delta s = s_2 - s_1$. We can see with our calculus experience that

$$dw = F_s ds$$

is a small amount of work, where we can assume that the distance ds is so small that the boy does not change how hard he pushes during the time dt. Then, the total work in moving the rock is the sum of lots of little ds pushes.



or

$$w = \sum dw_i = \sum F_{si} \Delta x_i$$

and of course we will let our Δs_i be infinitesimal ds's, so the sum is really an integral.

$$w = \int_{s_i}^{s_f} F_s ds$$

but the integral means, divide up the distance traveled into small segments of size ds. For each segment, the component of the force in the direction we are going can be assumed to be constant. Multiply the distance ds by the force, F_s at that ds point. This is the change in kinetic energy as the rock moves the distance ds. To get all of the change in kinetic energy, we add up the contribution for each ds segment of the path,

$$w = \int_{s_i}^{s_f} F_s ds$$

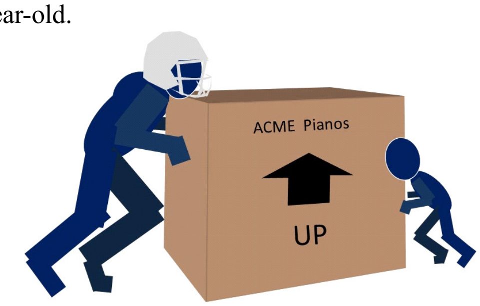
and this is the change in kinetic energy while the boy is pushing.

$$w = \Delta K$$

This equation is very important. But the idea is very simple. If you push on something hard enough that it moves, you have changed it's kinetic energy. The equation is so important that it has a name. It is called the *work energy theorem.*

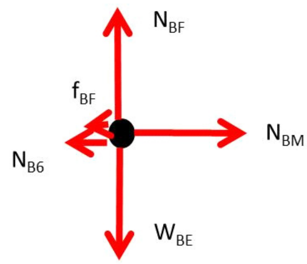
Negative work

We said that kinetic energy can't be negative, but how about work? Suppose we have two people exerting a force on a box. One is a professional American football player, The other is a six-year-old.



They push in opposite directions. How much work is being done?

To answer this, let's consider a free-body diagram.



where the subscript B is for the box and the subscript 6 is for the six-year-old and M is for the football playing (M)an. We can see that there will be a net force.

$$F_{net_s} = F_{net_x} = N_{BM} - N_{B6} - f_{BF}$$

where here we have explicitely chosen s to be in the x-direction. So the total work done on the box would be

$$\begin{aligned} w &= \int_{s_i}^{s_f} F_{net_s} ds \\ &= \int_{x_i}^{x_f} F_{net_x} dx \\ &= \int_{x_i}^{x_f} \left(N_{BM} - N_{B6} - f_{BF}\right) dx \\ &= \int_{x_i}^{x_f} \left(N_{BM}\right) dx + \int_{x_i}^{x_f} \left(-N_{B6}\right) dx + \int_{x_i}^{x_f} \left(-f_{BF}\right) dx \end{aligned}$$

But which way will the box go? I suspect that the football player will push the box *and* the six-year-old to the right. So Δx will be positive. We can identify the first term in total work equation as the the work done on the box by the man

$$\begin{aligned} w_{BM} &= \int_{x_i}^{x_f} N_{BM} dx \\ &= N_{BM} \Delta x \end{aligned}$$

The next term is the work done by the child

$$\begin{aligned} w_{B6} &= \int_{x_i}^{x_f} -N_{B6} dx \\ &= -N_{BM} \Delta x \end{aligned}$$

The child's work is negative! What can that mean? Well, for starters, the child's force can't be the force that is making the box move. In fact, the child's force is another obstacle that the man's force must overcome to make the box move. This means that the man must do enough work (push hard enough) to make the box go, *and* to overcome the backward push of the child. The work that actually makes the box move would be

$$\begin{aligned} w_{net} &= w_{BM} + w_{B6} \\ &= N_{BM}\Delta s - N_{BM}\Delta x \\ &= (N_{BM} - N_{BM})\,\Delta x \end{aligned}$$

But wait! we did not include friction. Does the frictional force do work? Well, yes, like our child is doing work. The frictional force is not moving the box forward, but it is making it harder for the man to move the box. The last term in our integral is the work due to friction. And now we know to expect that the work due to friction will also be negative.

$$\begin{aligned} w_{bf} &= \int_{x_i}^{x_f} -f_{Bf}dx \\ &= -f_{Bf}\Delta x \end{aligned}$$

So we really should include an amount of work w_{Bf} in our net work equation

$$\begin{aligned} w_{net} &= w_{BM} + w_{B6} + w_{Bf} \\ &= N_{BM}\Delta x - N_{BM}\Delta x - f_{Bf}\Delta x \\ &= (N_{BM} - N_{BM} - f_{Bf})\,\Delta x \\ &= F_{net_x}\Delta x \end{aligned}$$

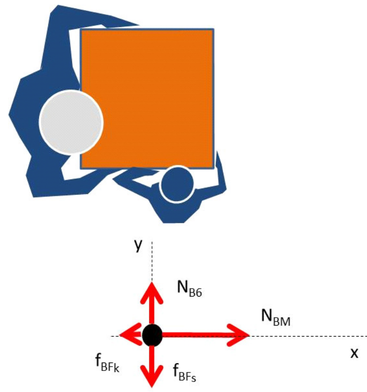
or we could write this using our generalized coordinate, s

$$w_{net} = F_{net_s}\Delta s$$

So it is the net force that makes our box move! That is not a surprise. The net force is what makes the box's acceleration, so the net force *should be* what gives the box some kinetic energy.

Work and forces perpendicular to the direction of travel

Let's consider another case. Suppose our man and child are still pushing on the box, but the child is tired of being pushed backwards. So now the six-year-old pushes on the side of the box (see next figure, it's a top down view).



Top down view of man and boy pushing on a box. Notice that z is up and x and y are horizontal.

Now the child pushes at a right angle to the man. But suppose the box still goes to the right, the same direction as it did before. What can we say about the work done by the man and the child? Let's take a coordinate system where up is the z-direction and to the right is the x-direction. Then the y-direction would be to the man's left or in our top-down figure it would be toward the top of the page. Then Δs is in the x-direction

$$\Delta s = \Delta x$$

and $\Delta y = 0$ because the box is not going in the y-direction. So the man's work is

$$\begin{aligned} w_{BM} &= \int_{x_i}^{x_f} N_{BM} dx \\ &= N_{BM} \Delta x \end{aligned}$$

just as before but the work done by the child would be

$$\begin{aligned} w_{B6} &= \int_{y_i}^{y_f} N_{B6} dy \\ &= N_{B6} \Delta y \\ &= 0 \end{aligned}$$

The child is pushing, but the child has not overcome the static friction force between the box and the floor in the y-direction. So the box has no y-displacement. The man has overcome the floor friction, so in the x-direction there is an amount of kinetic friction.

$$\begin{aligned} w_{bf_s} &= \int_{y_i}^{y_f} -f_{Bfs} dy \\ &= 0 \\ w_{bf} &= \int_{x_i}^{x_f} -f_{Bf} dx \\ &= -f_{Bf_k} \Delta x \end{aligned}$$

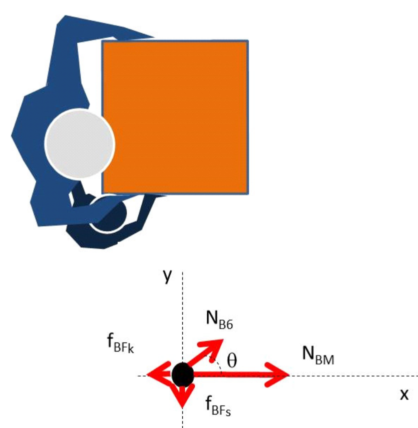
Notice that the man doesn't have to push as hard now, because he is no longer having to overcome the child's push.

$$\begin{aligned} w_{net} &= w_{BM} + w_{Bf_k} + w_{B6} + w_{Bfs} \\ &= N_{BM}\Delta x - f_{Bf}\Delta x + 0 + 0 \\ &= (N_{BM} - f_{Bf})\,\Delta x \\ &= F_{net_x}\Delta x \end{aligned}$$

The man only has to overcome friction to keep the box moving (and avoid tripping over the child).

Work and forces at an angle

But suppose our child get's smarter. Now the six-year-old pushes mostly in the same direction as the man. The child pushes with a force that is $30\,^\circ$ from the x-axis.



Then we can use Newton's second law to find the net force

$$\begin{aligned} F_{net_x} &= N_{BM} + N_{B6}\cos\theta - f_{BF_k} \\ F_{net_y} &= N_{B6}\sin\theta - f_{BF_s} \end{aligned}$$

and $F_{net_y} = 0$ still because the box is still going in the x-direction. because the box is not accelerating in the y-direciton. Notice that from the y-equation

$$N_{B6}\sin\theta = f_{BF_s}$$

so the static friction force in the y-direction is canceled with the y-component of the child's push. The y-components will do no work. But we can see that now part of the child's push is helping the man. The net work would be in the x-direction

$$\begin{aligned} w_{net} &= w_{BM} + w_{Bf_k} + w_{B6} + w_{Bfs} \\ &= N_{BM}\Delta x - f_{Bf}\Delta x + N_{B6}\cos\theta\Delta x + 0 \\ &= F_{net_x}\Delta x \end{aligned}$$

as expected.

And we have learned something important. If a force pushes at an angle to the direction the object is traveling, then only part of the force can be causing the motion. In our original work equation

$$w = \int_{s_i}^{s_f} F_s ds$$

we had only the s-component of the force. But now we recognize that

$$F_s = F\cos\theta_{Fs}$$

where θ_{Fs} is *the angle between* the $\overrightarrow{F}$ and $\overrightarrow{\Delta s}$ directions. This is a new concept, the angle between two vectors. Up till now we have always used the angle measured from the x-axis (or we have used our trigonometry experience to do equivalent math). But now we can't do this. We *have to* find the angle between the direction of the force is pushing and the direction the object is actually going so we can take a component of the force in the direction that the box is going.

Let's put the cosine in the basic work equation

$$w = \int_{s_i}^{s_f} F\cos\theta_{Fs} ds$$

What we are really doing is finding the component of the force in the direction that the box is going and multiplying by ds!

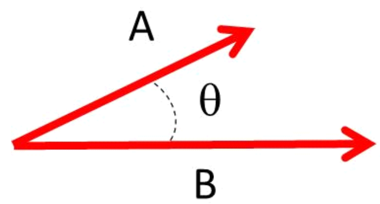
Once again, new math

We have a quantity

$$F\cos\theta_{Fs} ds$$

that represents something fundamentally new. We are taking a component of a vector in a direction that is not one of our axes directions. It would be great if we had a standard math notation to indicate we are doing this. And there is one!

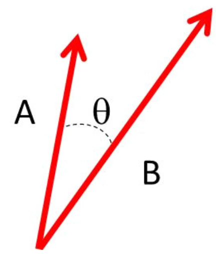
If we have a generic vector $\overrightarrow{A}$ and we want the component in the direction of another vector $\overrightarrow{B}$ as shown in the next figure,



Then the A_B component would be.

$$A_B = A\cos\theta_{AB}$$

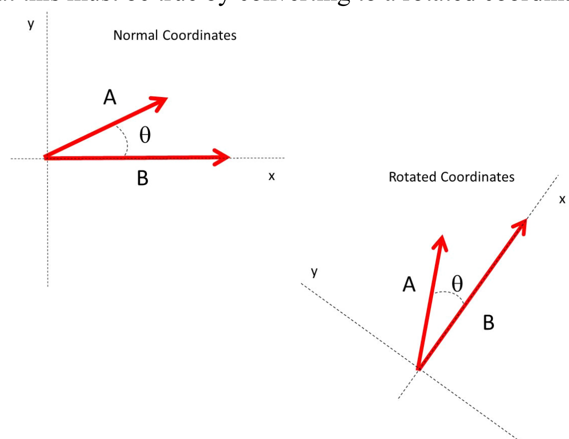
Since $\overrightarrow{B}$ is in the x-direction, this is no surprise. But suppose the vectors are as shown in the next figure:



Really the situation has not changed. The lengths are the same, the angle θ_{AB} is the same. So the result is still

$$A_B = A\cos\theta_{AB}$$

We could see that this must be true by converting to a rotated coordinate system.



These two situations look the same once you see the coordinates that support them.

Now let's consider

$$A\cos\theta B$$

This is

$$A\cos(\theta)\,B = A_B B$$

or the component of $\overrightarrow{A}$ in the $\overrightarrow{B}$ direction multiplied by B. This strange quantity has a name and a symbol. The symbol is

$$\overrightarrow{A}\cdot\overrightarrow{B} = A\cos(\theta)\,B$$

and it is called the *dot product* of $\overrightarrow{A}$ and $\overrightarrow{B}$. It turns out that (a phrase that here means that I am going to let your math professor prove all this) we can find the value of $\overrightarrow{A}\cdot\overrightarrow{B}$ if we know the components of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

$$\overrightarrow{A}\cdot\overrightarrow{B} = A_x B_x + A_y B_y$$

We can see that this might be true if we write $\overrightarrow{A}$ and $\overrightarrow{B}$ in terms of their components

$$\begin{aligned}\overrightarrow{A} &= A_x\hat{\imath} + A_y\hat{\jmath}\\ \overrightarrow{B} &= B_x\hat{\imath} + B_y\hat{\jmath}\end{aligned}$$

then

$$\begin{aligned}\overrightarrow{A}\cdot\overrightarrow{B} &= (A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath})\\ &= A_x\hat{\imath}\cdot B_x\hat{\imath} + A_x\hat{\imath}\cdot B_y\hat{\jmath} + A_y\hat{\jmath}\cdot B_x\hat{\imath} + A_y\hat{\jmath}\cdot B_y\hat{\jmath}\end{aligned}$$

now consider

$$\begin{aligned}\hat{\imath}\cdot\hat{\imath} &= (1)(1)\cos(0\,^\circ)\\ &= 1\end{aligned}$$

since $\hat{\imath}$ and $\hat{\imath}$ are in the same direction, and

$$\begin{aligned}\hat{\imath}\cdot\hat{\jmath} &= (1)(1)\cos(90\,^\circ)\\ &= 0\end{aligned}$$

So, using the zeros first

$$\begin{aligned}\overrightarrow{A}\cdot\overrightarrow{B} &= A_x\hat{\imath}\cdot B_x\hat{\imath} + 0 + A_y\hat{\jmath}\cdot B_x\hat{\imath} + 0\\ &= A_x B_x + A_y B_x\end{aligned}$$

All of this leads to

$$F\cos\theta_{Fs}ds = \overrightarrow{F}\cdot d\overrightarrow{s}$$

so we can write our work equation as

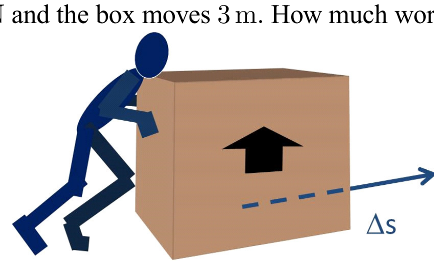
$$w = \int_{s_i}^{s_f}\overrightarrow{F}\cdot d\overrightarrow{s}$$

This just means that we take the component of the force in the s-direction and multiply by ds and integrate the result. But it is a handy, compact notation.

Example, Work from a Constant Force

Let's try a problem.

Suppose we let our guy push his box without the "help" of a child. He pushes with a constant force of $20\,\mathrm{N}$ and the box moves $3\,\mathrm{m}$. How much work did the guy do?

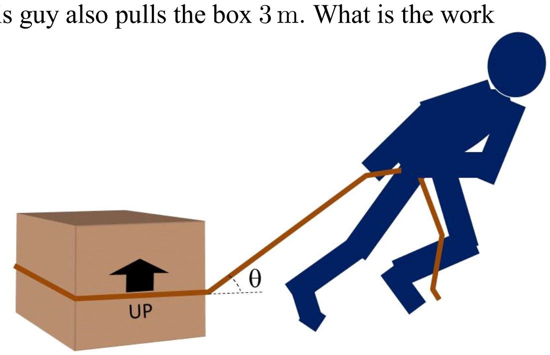


The push and the displacement are in the same direction so

$$\begin{aligned} w_{BG} &= \int_{s_i}^{s_f} \overrightarrow{F} \cdot d\overrightarrow{s} \\ &= \int_{s_i}^{s_f} N_{BM} ds \cos\theta \\ &= N_{BM} \cos\theta \int_{s_i}^{s_f} ds \\ &= N_{BM} \Delta s \cos\theta \end{aligned}$$

$$\begin{aligned} w_{BG} &= (20\,\mathrm{N})\,(3\,\mathrm{m})\cos\left(0\,^{\circ}\right) \\ &= 60.0\,\mathrm{J} \end{aligned}$$

Let's try another. Suppose another guy pulls a box with the same $20\,\mathrm{N}$ but now at an angle of $20\,^{\circ}$. This guy also pulls the box $3\,\mathrm{m}$. What is the work



$$\begin{aligned}
w_{BG} &= \int_{s_i}^{s_f} \overrightarrow{F} \cdot d\overrightarrow{s} \\
&= \int_{s_i}^{s_f} N_{BM} ds \cos\theta \\
&= N_{BM} \cos\theta \int_{s_i}^{s_f} ds \\
&= N_{BM} \Delta s \cos\theta
\end{aligned}$$

$$\begin{aligned}
w_{BG} &= (20\,\mathrm{N})\,(3\,\mathrm{m})\cos\left(20\,^{\circ}\right) \\
&= 56.382\,\mathrm{J}
\end{aligned}$$

But now that we have work, why do we want it? Recall the work energy theorum.

$$\begin{aligned}
w &= \Delta K \\
&= K_f - K_i \\
&= \frac{1}{2}mv_f^2 + \frac{1}{2}mv_i^2
\end{aligned}$$

So if our guy pulling the box in the last problem started with the box and rest, so $v_i = 0$, how fast would the box be going after $3\,\mathrm{m}$? Normally we would approach this kind of problem with kinematics. We would find the acceleration from the force, so this kind of problem would be both a kinaematics and a Newton's second law type problem. But now we can just use the work to find how fast the box is going.

$$w = \frac{1}{2}mv_f^2 + \frac{1}{2}mv_i^2$$

and using our zero for v_i

$$w = \frac{1}{2}mv_f^2 + 0$$
$$2w = mv_f^2$$
$$\frac{2w}{m} = v_f^2$$
$$v_f = \sqrt{\frac{2w}{m}}$$

Since we know the weight of the box, W_{BE}, and not m we can find $m = W_{BE}/g$ and put it in for m

$$v_f = \sqrt{\frac{2wg}{W_{BE}}}$$

or

$$\begin{aligned}
v_f &= \sqrt{\frac{2\,(56.382\,\mathrm{J})\,(9.8\,\mathrm{m/s^2})}{20\,\mathrm{N}}} \\
&= 7.4333\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

That is pretty fast! He should probably be more careful with the box!

For a third problem, let's look at the work done by gravity in making something fall. We can say that the force done by gravity is

$$W = mg$$

downward. So if we drop a ball the work done would be

$$\begin{aligned} w_{BE} &= \int \overrightarrow{F} \cdot \overrightarrow{ds} \\ &= \int (mg)\, ds \cos\theta_{Fx} \end{aligned}$$

but our θ_{FX} will be zero because the ball is going in the negative direction so dy downward and the force due to gravity is downward. So

$$w_{BE} = \int (mg)\, dy\, (1)$$

Note that I have used dy in place of ds because I want to say explicitly that the ball is moving in the downward direction.

$$w_{BE} = \int_{yi}^{y_f} mgdy$$

the mass and the acceleration due to gravity don't change, so we have

$$\begin{aligned} w_{BE} &= mg \int_{yi}^{y_f} y^0 dy \\ &= mg \left(\frac{y^1}{1} \right) \Big|_{y_i}^{y_f} \\ &= mg\,(y_f - y_i) \\ &= mg\Delta y \end{aligned}$$

So the pull of Earth's gravity does work in making balls fall.

Suppose we want the ball to go back up to it's starting point. We will have to push at least a hard as the Earth's gravity is pushing down just to keep it moving upward. So we must at least have

$$N_{push} = -W$$

Then the ball won't accelerate, but it will keep moving up at a constant speed. In that case we will do work

$$\begin{aligned} w_{By} &= \int_{yi}^{y_f} \overrightarrow{F} \cdot \overrightarrow{dy} \\ & \int_{yi}^{y_f} N_{push} dy \cos\theta_{Nx} \\ &= \int_{yi}^{y_f} (mg)\, dy\, (1) \\ &= mg\Delta y \end{aligned}$$

which is just as much work as the gravitational force did in making the ball fall. Of course we could have pushed harder up than gravity pushed down. In that case the ball would accelerate upward. But $mg\Delta y$ is the *minimum* work needed to put the ball back up where it started.

It is probably worth noting that while we are pushing the ball upward, gravity is trying to push the ball downward. The work done by gravity in this case is

$$\begin{aligned} w_{BE} &= \int \overrightarrow{F} \cdot \overrightarrow{ds} \\ &= \int (mg)\, ds \cos\theta_{Fx} \\ &= \int (mg)\, dy\, (-1) \end{aligned}$$

because the gravitational force is opposite the direction we are pushing. Then

$$w_{BE} = -mg\Delta y$$

as we push upward at a constant speed. The gravitational force is not causing the motion, so we expect to see negative work. This is a special situation. We could have pushed harder, and not kept the object moving at a constant rate upward. But this is a minimum work required to lift the object.

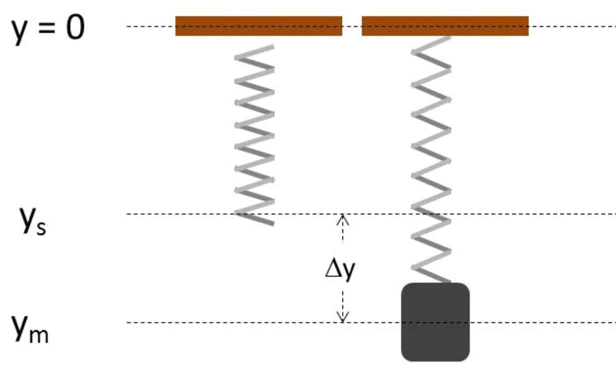
We have done some good work problems with constant forces. Of course, if the force is not constant, the problems might be a little more difficult. We will take on this more difficult situation in the next lecture.

19 Work Done By Changing Forces

In our last lecture, we had normal, tension, and gravitational forces do work. These forces didn't change as the object moved. But some forces are not at all constant. In this lecture we will study a very important example of a force that changes as an object moves. We will study an object connected to a spring.

Hook's "Law"

Think of a spring. If it is stretched, it will pull back against the stretching. If it is compressed, it pushes back against the compression. The more you pull, the harder it pulls back. Or the more you compress it the more it resists being compressed. This force depends on how stretched or compressed the spring has become. The spring likes to be at an equilibrium length and opposes any displacement from that length.



Sir Robert Hooke first came up with a mathematical expression for how the force of a spring works

$$F_s = -k\left(y - y_{\text{equilibrium}}\right) \tag{19.1}$$

where k is a constant that depends on the material and manufacturing method used in making the spring. It really tells us how stiff the spring is. Notice that the direction of the force is in the opposite the direction of the displacement. When this is true we call the force a *restoring force* because it tends to restore the object to it's equilibrium position. The quantity $y_{\text{equilibrium}}$ is where the end of the spring would be if it were not stretched. Of course, if we hang a mass on the end of the spring, the mass will pull the spring down past it's equilibrium position to a new position, y. Sometimes you will see our spring force equation written as

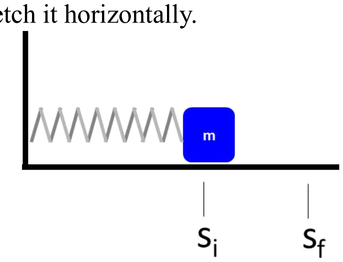
$$F_s = -k\Delta y$$

but we have to be careful to recognize that $\Delta y = y - y_{\text{equilibrium}}$ is a very special displacement. It is a displacement from a chosen starting point, the equilibrium point. We could place a small subscript on the "y" to replace the long word, "equilibrium."

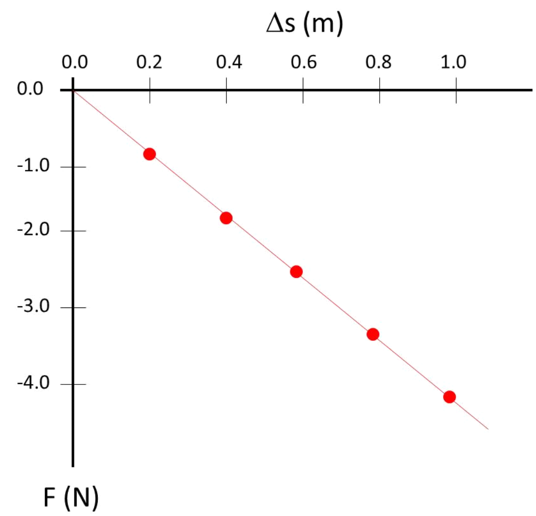
$$F_s = -k\left(y - y_o\right) \tag{19.2}$$

Anyone who has owned a slinky and has had younger siblings knows that you can overstretch a spring. Once overstretched, the spring will not regain its original shape. So with this "law" we have to be careful to use our formula in a region of x (or y) that does not damage the spring. This demonstrates that a scientific law is not something that is true all the time, but rather is a mathematical formula that describes the way we think the physical universe works in some way.

Let's take a spring and stretch it horizontally.

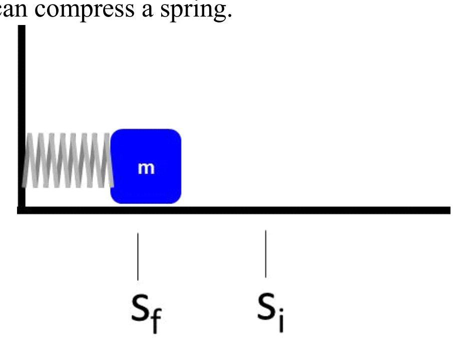


We can plot the effect of stretching the spring. The restoring force of the spring increases as we stretch the spring.



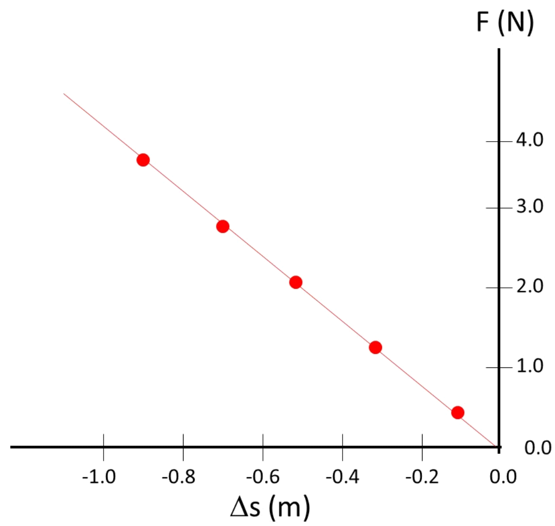
The more we stretch, the more restoring force we have.

It's also true that we can compress a spring.



If Δs is negative, then the restoring force will be positive.

$$F_s = -k\Delta s$$



Let's try a problem:

Suppose we suspend a spring of length $0.50\,\mathrm{m}$ and spring constant of 0.30 from a support. And further suppose that we suspend a $0.50\,\mathrm{kg}$ mass from this spring. How much does the spring stretch?

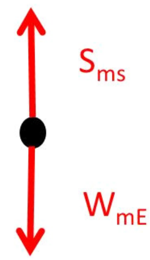
We'll call this a *spring force problem*, and the basic equation is Hooke's law plus Newton's second law

$$F_s = S = -k\left(y - y_o\right)$$

We know

$$\begin{aligned} L_s &= 0.05\,\mathrm{m} \\ k &= 28\frac{\mathrm{N}}{\mathrm{m}} \\ m &= 0.50\,\mathrm{kg} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^3} \end{aligned}$$

We will need a free body diagram



We will call the spring force $F_s = S$ for our free body diagrams. The subscript s is also for spring. When the mass stretches the spring it will come to rest.

$$F_{net_y} = S_{ms} - W_{mE} = 0$$

so

$$S_{ms} = W_{mE}$$

so

$$k\Delta s = mg$$

Notice that we did not put in the negative sign. We already took care of direction when we wrote Newton's second law. So we just put in magnitudes after we write out Newton's second law.

Then

$$\Delta s = \frac{mg}{k}$$

or

$$\begin{aligned} \Delta s &= \frac{(0.50\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{\left(28\frac{\mathrm{N}}{\mathrm{m}}\right)} \\ &= 0.175\ \mathrm{m} \end{aligned}$$

Work for a spring force

Recall our basic equation for work is

$$w = \int \overrightarrow{F} \cdot \overrightarrow{ds}$$

where we have used our new dot product notation.

But what if $\overrightarrow{F} = \overrightarrow{F}(s)$ that is, what if the force changes as we travel in the s-direction? Let's take an example, and then generalize what we find.

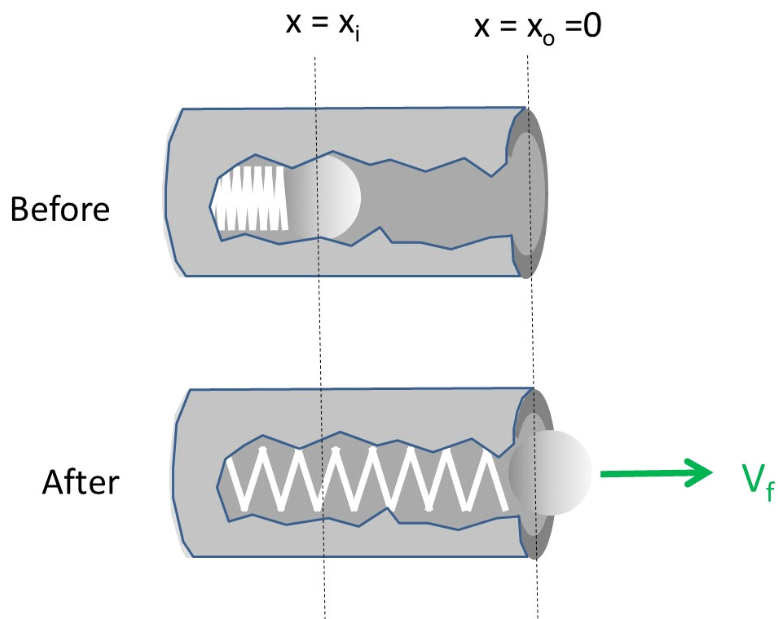
Suppose we have a spring, say, the one in our spring cannon. As the spring pushes on the ball, is the force constant? We know the answer is no, because for spring forces

$$S(s) = -k(s - s_o)$$

so as we let the spring become less compressed (we change s), the magnitude of the force changes. So in such a case, how do we find the work done by the force? We just place this force in our work equation

$$w = \int \overrightarrow{S}(s) \cdot \overrightarrow{ds}$$

or, if our spring cannon operates in the x-direction.



then the work done on the ball is

$$w_b = \int \overrightarrow{S}(x) \cdot \overrightarrow{dx}$$

or

$$w_b = \int |-k(x - x_o)| \cos\theta_{Sx} dx$$

where θ_{Sx} is the angle between the $\overrightarrow{S}$ and the $\overrightarrow{dx}$ directions. From the figure we can see that $\theta_{Sx} = 0$ so

$$w_b = \int |-k(x - x_o)| (1) dx$$

It would be convenient to set our $x = 0$ point right where the spring is at equilibrium. So

$$x_o = 0$$

We will do this, but we have to be careful and make sure $|-k(x - x_o)|$ stays a positive number. After all, it is a magnitude. The problem is that if we choose the end of the cannon as our $x = 0$ point, then x_i and all the other x values from x_i up to $x = 0$ are negative. k can't be negative, and $|-k(x - x_o)|$ must be positive. We can achieve this by using the absolute value signs, but they are clunky. We could just write

$$|-k(x - x_o)| = -k(x - x_o) \text{ for } x \leq 0$$

Then we can state that

$$x_o = 0$$

because we chose this as our zero point. Note that if $x > 0$ we have to make

$$|-k(x - x_o)| = +k(x - x_o) \text{ for } x > 0$$

to keep $|-k(x - x_o)|$ positive. This is tricky, make sure you understand why this works!

Using $|-k(x - x_o)| = -k(x - x_o)$ for $x \leq 0$ the work would be

$$\begin{aligned} w_b &= \int -k(x - 0)\,dx \\ &= \int -k(x)\,dx \\ &= -k \int x dx \end{aligned}$$

and we need limits for our integrals, making sure $x \leq 0$ in our limits. We start with the ball at $x = x_i$ and end with the ball and the end of the spring at $x = x_o = 0$ with the spring uncompressed. So

$$w_b = -k \int_{x_i}^{0} x dx$$

Our limits fit our condition on our equation. We can do this integral with our basic integral formula that we learned earlier

$$w_b = \left(-k\left(\frac{x^2}{2}\right)\right|_{x_i}^{0}$$

$$w_b = -k\left(\frac{0}{2}\right) - \left(-k\left(\frac{x_i^2}{2}\right)\right)$$

$$w_b = -\left(-k\left(\frac{x_i^2}{2}\right)\right)$$

$$w_b = \frac{1}{2}kx_i^2$$

This was not too hard! We can do the same for any force that changes with position.

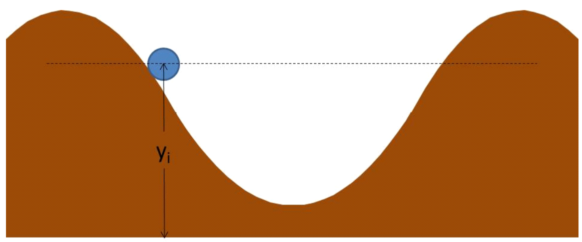
Work-energy missing piece, dissapation

@@@ This feels awkward. Maybe push this discussion back until after potential energy @@@

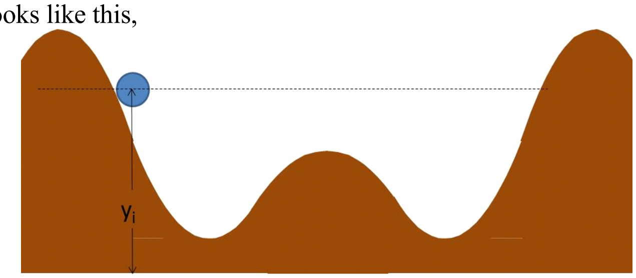
We started by saying our Deacons were pushing rocks on frictionless surfaces. But in a real situation, we know friction exists. Let's consider what happens to the energy of our motion if we do have friction.

Conservative and non-conservative forces

Let's imagine a test situation with a ball rolling down a hill/valley system. We could have a ball roll down a hill/valley system that looks like this,



or one that looks like this,



The gravitational pull due to the Earth will do work on the ball, and that work energy will become kinetic energy. The ball will pick up speed. As the ball reaches the valley floor, the work done by gravity will be negative, so the ball will slow down as it climbs the next hill. The place where the ball stops, and turns around are called *turning points.* The turning points would be just the same height for both balls. The actual path from the beginning y_i and the ending y_f doesn't matter at all.

This is because energy is being *conserved*. The Earth's gravitational force is the force causing the ball to move, that is, causing the gravitational work to be there. It is the environment that makes the y-position matter. So we will give this gravitational force a new title. We will add the name *conservative* to gravitation. Note this is not a new force, just a new name applied to an old force.

But the words "conservative force" apply to a class of forces that all act this way. Only the beginning and ending positions matter when calculating work for these forces. For conservative forces, energy is conserved. Electrical and magnetic forces are both this type of force. And it turns out that so are spring forces for ideal springs!

If we have only conservative forces, then the amount of energy we have for a systems won't change. This simple idea is amazingly powerful! In the case of our ball sliding down the hill, we start with an amount of work that it took to get the ball up the hill. From a previous problem, we can guess that this will be

$$w_{gi} = mgy_i$$

As the ball slides down the hill, the amount of work energy used to get the ball up the hill will be converted into kinetic energy. But as the ball slides up the hill, the kinetic energy will be converted back into work energy. We can predict that no energy will be lost from our ball-hill system. so

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \end{aligned}$$

If we stop our experiment at the bottom of the hill we will see that $w_{gf} = 0$, that is, it took no work to get the ball from the bottom of the hill to the bottom of the hill–the ball's displacement $\Delta y = 0$, so no work. If we start the ball from rest then $K_i = 0$. So

$$0 + w_{gi} = K_f + 0$$

and we find that the kinetic energy at the bottom of the hill, K_f, is equal to the work it took to get the ball up the hill. We converted all our work energy into kinetic energy.

Suppose we let the ball keep going. How high will it go up the other side of valley? Assuming we have only conservative forces, then

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \end{aligned}$$

but now the final situation is up the hill. Just as the ball reaches it's highest point, $v_f = 0$ (the ball will momentarily stop) and so $K_f = 0$. Then

$$0 + w_{gi} = 0 + w_{gf}$$

and we see that the work to get the ball up on the other side is equal to the initial work in placing the ball up the hill.

$$mgy_i = mgy_f$$

so the ball goes up just as high on the far hill as it was originally on the starting hill. Notice we did these problems very easily just knowing that we had conservative forces. When we can say that the total amount of energy does not change in a before and after picture, we say that *energy is conserved.* This is such a powerful idea, let's write it anther way

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \\ w_{gi} - w_{gf} &= K_f - K_i \\ -\Delta w_g &= \Delta K \end{aligned}$$

which says as we change the amount of work in a system, we change the kinetic energy of that system. Since work comes from a force, this says that if we push something enough to make it move, we will gain kinetic energy. We convert work into kinetic energy, and notice that the amount of energy lost from our change in work becomes the same energy gained in kinetic energy. This just a restatement of the ideal of conservation of energy. We also recognize it as a form of the *Work-Energy Theorem!*

But there is another class of forces that are *non-conservative forces*. And the archetype of this class is friction.

Consider our two hills again. The ball's path on the two-hump hill is shorter than the ball's path on the three-hump hill. Let's see if this matters for friction forces. The work done by a friction force would be

$$w_f = \int \overrightarrow{f} \cdot \overrightarrow{ds}$$

Notice that as the ball slides down the hill, f and ds are in opposite directions. So the angle θ in

$$w_f = \int f ds \cos\theta$$

will be $180\,^{\circ}$. The cosine of $180\,^{\circ}$ is -1 so

$$\begin{aligned} w_f &= -\int f ds \\ &= -f\Delta s \end{aligned}$$

and our friction force will give us negative work. This means it would take more work to make the ball go the same speed as it would have with no friction.

Let's look at this in our energy equation. For energy to be conserved, we have to add in the energy being taken out by friction. Notice in the next equation there is a work w_{ff}

which is the final amount of work taken out of the system by friction.

$$\begin{aligned}
E_i &= E_f \\
K_i + w_{gi} &= K_f + w_{gf} + w_{ff} \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff}
\end{aligned}$$

if we use our zeros, and we take final position of the ball at the bottom of the hill,

$$0 + mgy_i = \frac{1}{2}mv_f^2 + 0 + w_{ff}$$

so

$$\frac{1}{2}mv_f^2 = mgy_i - w_{ff}$$

so

$$v_f = 2\frac{mgy_i - w_{ff}}{m}$$

we can see that the ball won't go as fast if there is a friction term. Some of the original gravitational work energy is no longer available to make into the kinetic energy of the ball. And since $|w_f| = |f\Delta s|$

$$v_f = 2\frac{mgy_i - f\Delta s}{m}$$

the larger the Δs, the more original work energy becomes unavailable for making the kinetic energy of the ball.

Suppose the ball rolls all the way down the hill and up the other side. Since we had a lower speed at the bottom of the hill, the ball won't go up the other side as far. Let's show this in our math, where now the final case is up on the other side of the valley. At the turning point $v_f = 0$ we have

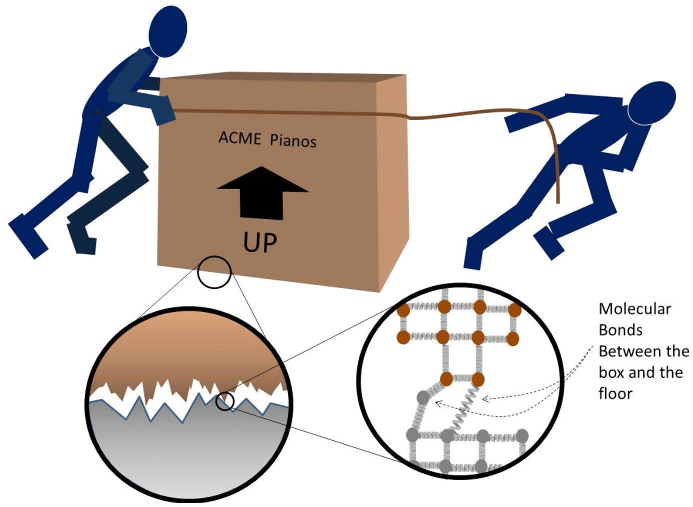
$$\begin{aligned}
E_i &= E_f \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\
0 + mgy_i &= 0 + mgy_f + w_{ff}
\end{aligned}$$

and let's solve for y_f, the final height of the ball

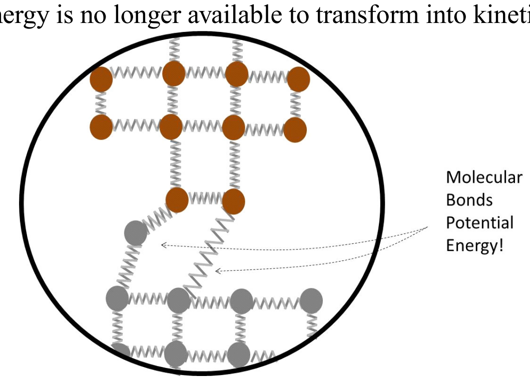
$$\begin{aligned}
mgy_f &= mgy_i - w_{ff} \\
y_f &= \frac{mgy_i - w_{ff}}{mg} \\
&= y_i - \frac{w_{ff}}{mg}
\end{aligned}$$

and as expected, the ball does not get as far up the other side. The friction force has taken energy away from both the kinetic energy and the potential energy. But where did it the energy go?

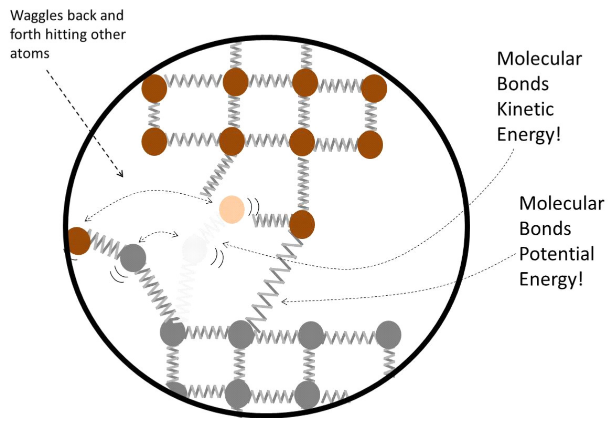
To understand this, let's go back to our model for friction



Consider our Elder's quorum moving team again. Recall that as the guys push the box, the roughness teeth are bent, and the molecular bonds are stretched. Those stretched, spring-like bonds will build up what we might call "bond potential energy." We will deal with this type of potential energy in detail in PH220, but for now we will just model the bonds like springs. The important point for now is that some of the energy has been removed from the gravitational potential energy and turned into bond potential energy. So this energy is no longer available to transform into kinetic energy of the ball.



When the bonds break, the atoms are free to move. The spring like bond forces convert the bond potential energy into kinetic energy of the atoms. As more roughness teeth come by the atoms are struck again and again.



They oscillate and strike other atoms. The whole solid is made of atoms and bonding forces. So as atoms are struck, the spring-like bonds are compressed and push back. After a short time, the whole collection of atoms in the solid are moving, compressing and uncompressing their bond forces. All this is kinetic and potential energy of the atoms in the solid floor martial (and of the box material). If you put your hand on the floor or the box some of the energy would be transferred to your hand atoms by collisions between the floor atoms and your hand atoms. So soon the atoms in your hand would have some kinetic energy transferred to them and would be stretching and compressing their bonds. What you would feel is that the floor got hot. We will call the energy lost to the motion of an object due to friction *thermal energy* because it triggers our body temperature sensors. We will study this in more detail in PH123, but for now the important point is that we have spread some of the energy all over the floor, the box, your hand, and eventually all over the room. This energy is no longer available to be transformed into kinetic energy of the box. We will say that energy that has been lost from our box system (so that it can't participate in creating motion of the box or gravitational potential energy of the box) has been *dissipated***.** The word "dissipated" means "lost from the system."

Let's be a little more specific. We call the combination of gravitational work energy, spring work energy, and kinetic energy *mechanical energy.*

$$E_{mech} = K + w_g + w_s$$

these energies are all conservative. We call the work energy due to friction *non-conservative energy* and forces like friction that dissipate energy are called *non-conservative forces.*

Let's write non-conservative work as

$$w_{nc}$$

Then from our ball sliding down the hill we can see that

$$\begin{aligned}
E_i &= E_f \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\
K_i + w_{gi} &= K_f + w_{gf} + w_{nc}
\end{aligned}$$

The left hand side is the initial mechanical energy (we don't have a spring, so $w_{si} = 0$). The right hand side is the final mechanical energy. So or we could write this as

$$E_{mech_i} = E_{mech_f} + w_{nc}$$

So the final mechanical energy must be less than the initial mechanical energy, and the difference is w_{nc}. We could write this as

$$\Delta E_{mech} = w_{nc}$$

20 Power and Stored Energy

Power

We now have a better understanding of energy and how we can use the concept of energy to solve problems. But there is more to our energy picture. The rate at which we use energy, say, the rate at which we perform work, is important. Suppose you are in the thriving metropolis next to the offices of the Daily Universe, and you see a super guy

Both you and Super Guy go to the top of the Daily Universe building. Super Guy does this in a single bound. You take the stairs. What is the difference in the work done by you and Super Guy assuming we can neglect the work done by frictional forces.

This is an energy problem so we set up our energy equation

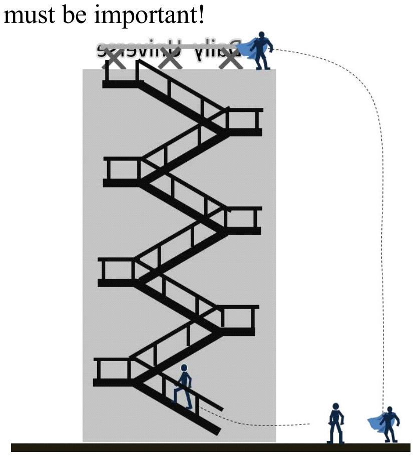
$$K_i + w_{gi} + w_{si} + w_{ei} + E_{th_i} = K_f + w_{gf} + w_{sf} + w_{ef} + E_{th_f}$$

where I have written the thermal energy due to non-concervative work as E_{th}. In this problem we don't have springs, or electric forces, so we can cancel all the spring and electric force terms

$$K_i + w_{gi} = K_f + w_{gf}$$

At the beginning you are standing on the ground stationary, and so is Super Guy. At the end you are standing stationary at the top of the Daily Universe building, and so is Super Guy. So $K_i = 0$ and $K_f = 0$. The only change in energy is that both you and Super Guy have done work against the gravitational field in order to get to the top of the building $w = mg\,(y_f - y_i)$. We can see that energy is not conserved for you or Super Guy, there must be an input of energy at the start that we missed. And that energy we missed is the chemical potential energy of the food you and Super Guy ate for breakfast that can be turned into the work you and Super Guy do to reach the top. But notice that since you and Super Guy have the same mass (just distributed differently) the two of you have done the same amount of work. But something must be different. Otherwise, we would all be Super Guy!

And you can see immediately what the difference is. Supper guy got to the top of the building in moments. It took you 15 minutes of hard running to get up the stairs. The time it takes to do work must be important!



Let's define a new quantity that is the amount of work we do divided by the time it took to do the work.

$$\frac{w}{\Delta t}$$

and we need a name and a symbol. In respect of Super Guy, let's say that this new quantity is called *power*. Super Guy is more powerful because he can do the work of going from street level to the top of a building in a very short Δt. You can do the same work, via the stairs, but it will take a much longer Δt.

$$\frac{w}{\Delta t_{small}} > \frac{w}{\Delta t_{big}}$$

The symbol for power is a capital P. Notice that power is a rate at which work happens. If we take the Δt to be very small, so we can call it dt, then

$$P = \frac{dw}{dt}$$

and this definition works for any type of energy

$$P = \frac{dE_{any}}{dt}$$

Notice that the units for power would be $\mathrm{J/s}$. This combination of units has a name, the *watt.*

$$\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}}$$

Let's go back to our definition of power, to find an alternate equation for power.

$$P = \frac{w}{\Delta t}$$

which we could write as

$$P = \frac{w_f - 0}{\Delta t}$$

where 0 is the initial work before we start pushing or pulling. Then

$$P = \frac{dw}{dt}$$

as well. And we know that

$$w = \int \overrightarrow{F} \cdot \overrightarrow{ds}$$

so the integrand, $\overrightarrow{F} \cdot \overrightarrow{ds}$ must be a little bit of work

$$dw = \overrightarrow{F} \cdot \overrightarrow{ds}$$

Let's divide both sides of this equation by dt

$$\frac{dw}{dt} = \frac{\overrightarrow{F} \cdot \overrightarrow{ds}}{dt}$$

the left hand side is power

$$P = \frac{\overrightarrow{F} \cdot \overrightarrow{ds}}{dt}$$

The right hand side could be rewriting as

$$P = \frac{F ds \cos\theta_{Fs}}{dt}$$

or even as

$$P = F \cos\theta_{Fs} \frac{ds}{dt}$$

and we recognize ds/dt as the speed of the object we are pushing or pulling with our force F. So

$$P = F \cos\theta_{Fs} v$$

and now we know that F and v are vectors, and that θ_{Fs} is the same direction as θ_{Fv} because we are going in the s direction, so v and s are in the same direction. Then

$$P = \overrightarrow{F} \cdot \overrightarrow{v}$$

So if we know the force and the velocity, we can also find the power.

Let's try a problem.

Suppose that the Daily Universe Building roof is $45\,\mathrm{m}$ above the street below. And suppose that both you and Super Guy have a mass of $90\,\mathrm{kg}$. Super Guy gets to the roof in $1.2\,\mathrm{s}$. You get to the roof in $10.1\,\mathrm{min}$. What is Super Guy's power, and what is your power?

We know

$$\begin{aligned}
m &= 90\,\mathrm{kg}\\
\Delta y &= 45\,\mathrm{m}\\
\Delta t_y &= 10.1\,\mathrm{min} = 606.0\,\mathrm{s}\\
\Delta t_{SG} &= 1.2\,\mathrm{s}\\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}$$

Our basic equation for this would be

$$P = \frac{w}{\Delta t}$$

so

$$\begin{aligned}
(w_f - w_i) &= (mgy_f - mgy_i)\\
&= mgy_f
\end{aligned}$$

so

$$P = \frac{mg\Delta y}{\Delta t}$$

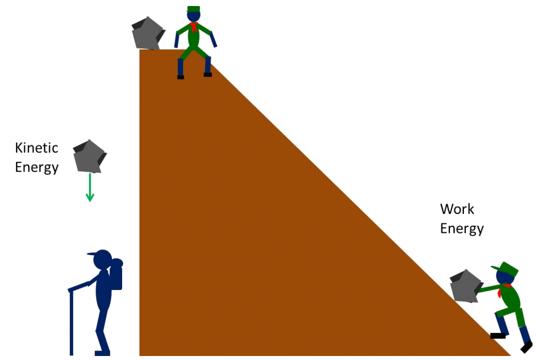
This is true for both you and Super Guy. Numerically, then

$$\begin{aligned}
P_{SG} &= \frac{(90\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(45\,\mathrm{m})}{1.2\,\mathrm{s}}\\
&= 33075\,\mathrm{W}\\
P_y &= \frac{(90\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(45\,\mathrm{m})}{606.0\,\mathrm{s}}\\
&= 65.495\,\mathrm{W}
\end{aligned}$$

and we see that Super Guy is much more powerful than you are, just as expected.

Potential Energy

So now let's review. Our deacons quorum is once again hiking with rocks. To get the rocks up to the cliff, our boys do work. That work creates some kinetic energy as the rocks move Once they are on the top of the hill, the gravitational force will do work on the rocks further creating kinetic energy. But as the boys reach the top of the hill, they usually momentarily stop before pushing the rock onto the unsuspecting hikers below[15]. Where did the energy go? There was lots of work done, but we no longer have kinetic energy.



So far we have learned about motion (kinematics) and forces (Newton's laws). We changed our view point to consider energy and we learned about work and kinetic energy. But we can't yet do a kinematics problem with our energy view point. In our deacons example, the deacons pushed rocks up a hill. We know we could use kinematics to find how the rocks will move, as they fall, but work and kinetic energy aren't enough.

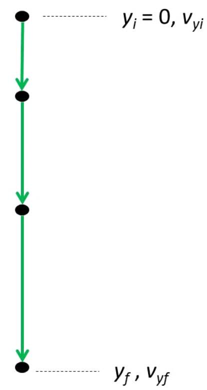
To find the missing piece of our energy picture, let's start with the kinematic equations.

$$
\begin{array}{ll}
x_f = x_i + v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & y_f = y_i + v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\
v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\
v_{fx}^2 = v_{ix}^2 + 2a_x\left(x_f - x_i\right) & v_{fy}^2 = v_{iy}^2 + 2a_y\left(y_f - y_i\right) \\
x_f = x_i + \frac{v_{fx}+v_{ix}}{2}\Delta t & y_f = y_i + \frac{v_{fy}+v_{iy}}{2}\Delta t
\end{array}
$$

and let's take an example problem. Let's take a falling object (a spy, a super hero, a pharmacist, or whatever).

For all of these objects our motion diagram would look like this

[15] Please don't really roll rocks down hills, this is dangerous. This is why I don't hike in valleys or at the bottom of cliffs!



When we were trying to find an easier way to do force problems in the last few lectures, we did the force problem and then generalized the result. Since each falling object seems to have the same motion diagram, it looks like we could do this again. Let's try to generalize our kinematic equations for falling objects. Let's use the third equation in the y-kinematics set

$$v_{fy}^2 = v_{iy}^2 + 2a_y\left(y_f - y_i\right)$$

In doing energy problems so far we have often ended up separating our equations into a "before" picture and an "after" picture. Think of the work energy theorum.

$$\begin{aligned} w &= \Delta K \\ &= K_f - K_i \end{aligned}$$

The kinetic part is obviously a"before" and "after" picture, but really so is the work. We integrate from an inital to a final location. So let's take all the initial values to the left side of the equation and all the final values to the right side. Then

$$v_{fy}^2 = v_{iy}^2 + 2a_y y_f - 2a_y y_i$$

becomes

$$v_{fy}^2 - 2a_y y_f = v_{iy}^2 - 2a_y y_i$$

or

$$v_{iy}^2 - 2a_y y_i = v_{fy}^2 - 2a_y y_f$$

This looks very useful. If we know the initial speed and position, we know the combination $v_{iy}^2 - 2a_y y_i$ won't change so we can predict what the combination $v_{fy}^2 - 2a_y y_f$ will be.

I would like to make some cosmetic changes to this equation. We won't change the fact that our quantity $\left(v_y^2 - 2a_y y\right)$ is not changing. I will divide both sides of the equation by $1/2$

$$\frac{1}{2}v_{iy}^2 - a_y y_i = \frac{1}{2}v_{fy}^2 - a_y y_f$$

and multiply through by the mass of the object.

$$\frac{1}{2}mv_{iy}^2 - ma_y y_i = \frac{1}{2}mv_{fy}^2 - ma_y y_f$$

These mathematical changes did not affect the equality at all. Finally, we know for our falling things that $a_y = -g$, so

$$\frac{1}{2}mv_{iy}^2 + mgy_i = \frac{1}{2}mv_{fy}^2 + mgy_f$$

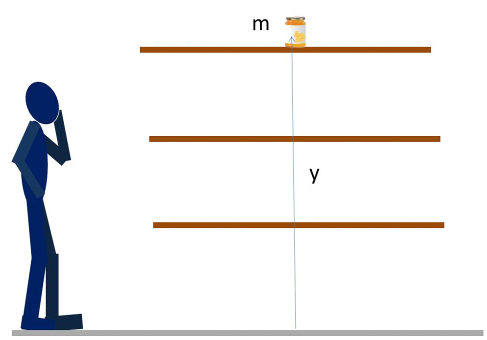
Then, the combined quantity $\frac{1}{2}mv^2 + mgy$ is not changing as the object moves. If a quantity doesn't change, in physics we say it is *conserved*. It is the same in our "before picture" and in our "after picture." But what is this quantity?

We recognize one part of this of our conserved quantity!

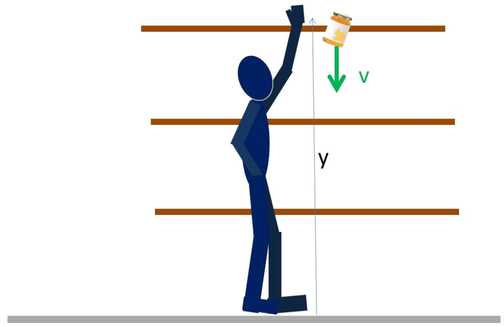
$$K = \frac{1}{2}mv^2$$

is kinetic energy. And we know all the terms in a sum must be the same kind of thing. So this combination must be *energy!* So I could use this to, say, measure the final y, and calculate the final speed without using our kinematic equations. This is wonderful enough all on it's own, but let's pause for a minute and ask, what is the mgy part?

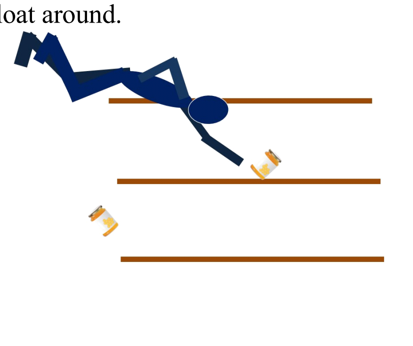
We recognize that this is the work we did in putting the object up high to begin with! This is the minimum work it took to place the object up high, like our deacons placing a rock on a hill. For another example, consider a jar on your pantry shelf.



The jar has mass, m, and is up a height y. There is no motion. But you did work at some time earlier to get the jar up on the high shelf. If you knock the jar off the shelf,



the jar will move. We could say that by doing the work of putting the jar high on the shelf we have created the potential to have motion. We have effectively stored our work for use later! We call this kind of stored energy *potential energy.* We give it the symbol U. Since it takes the Earth's gravity to create the environment where this potential energy is possible, we will say that this is *gravitational potential energy*. Think about this for a minute. If there were no gravitational pull, you, the things in your pantry, and the pantry itself would float around.



It is that our jar is up high with the Earth's gravity pulling on it that makes the jar have potential to move downward. Think that the acceleration due to gravity was in our kinematic equation that we started with. The gravitational pull is required to make gravitational potential energy. And to say we have gravitational potential energy we write a subscript, g, on our symbol for potential energy, U_g.

$$U_g = mgy$$

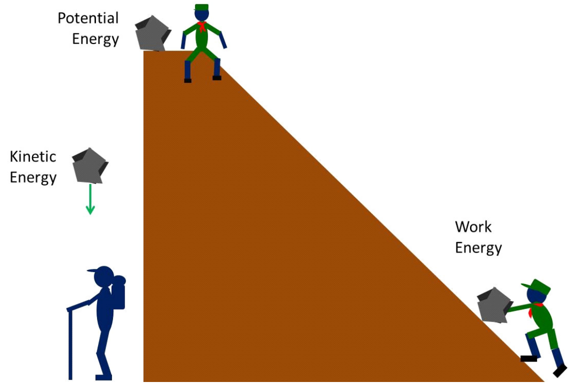
So our energy combination can be written as

$$\frac{1}{2}mv^2 + mgy = K + U_g$$

and our energy equation can be written as

$$\begin{aligned} \frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \\ K_i + U_i &= K_f + U_f \end{aligned}$$

This is the missing piece of our energy picture from our deacons.



We can see that as the deacons moved the rock up the hill, some of the work they did became kinetic energy but some of the work was converted into potential energy. That is why the rock can easily move once it is pushed over the edge. The work of the boys moving the rock up the hill against the force of gravity effectively stores part of their work as potential energy. That potential energy is converted into kinetic energy as the rock falls.

Notice that we have to be very careful when we talk about work and potential energy. The work done by the gravitational force when the pushing the rock up the hill is

$$w_g = \int mg\Delta y \cos\theta$$

but the rock is going up and the gravitational force is down so $\theta = 180\,°$. The work done by the gravitational force is negative. That makes sense because the Earth's gravitation is not making the motion, the deacon is. So we expect the losing force to make negative work.

$$w_g = -mgh$$

where

$$h = \Delta y = y_f - y_i = y_{top} - y_{bottom}$$

The gravitational potential energy change that we store when pushing the rock up the hill is given by

$$\Delta U_g = mg\Delta y = mgh$$

so we see that

$$\Delta U_g = -w_g$$

The change in potential energy is the negative of the work done by the Earth's gravitational force. Notice that the deacon's work in pushing the rock up the hill is

$$w_d = \int mg\Delta y \cos\theta$$

but this time the angle θ is 0 degrees because the rock is going in the direction of the push. So

$$w_d = +mg\Delta y = +mgh$$

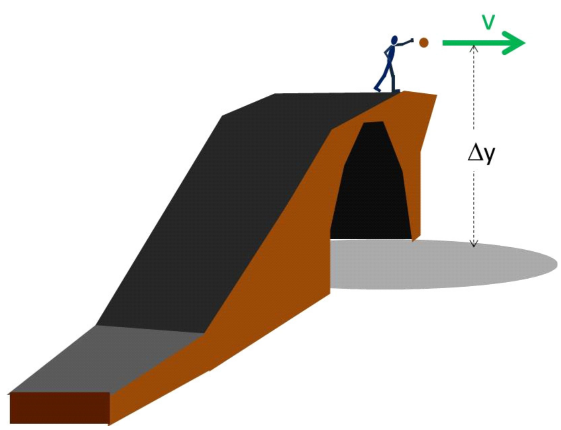
But in comparing gravitational potential energy to work, we choose to compare the gravitational potential energy to the gravitational work, not the boy's work, so we will use the minus sign.

$$\Delta U_g = -w_g$$

and this will be a general expression, so long as we always consider the same force (gravitational in this case) for both sides of the equation

$$\Delta U = -w$$

Let's try some problems using the idea of potential energy.



Suppose a deacon throws a $0.5\,\mathrm{kg}$ rock off a bridge. The boy throws the rock with an initial velocity of $\overrightarrow{v} = 10.0\,\mathrm{m/s}\hat{\imath}$. The bridge is $20.\,\mathrm{m}$ high. What is the final speed of the rock just before it hits the ground below the bridge assuming air resistance is negligible?

This certainly could be a kinematics problem, but let's do this with our conservation of energy equation. We know that

$$\begin{aligned}
\overrightarrow{v}_i &= 10\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} \\
y_i &= 20\,\mathrm{m} \\
y_f &= 0\,\mathrm{m} \\
m &= 0.5\,\mathrm{kg} \\
a_y &= -g \\
g &= -9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}$$

and our equation is

$$\begin{aligned}
K_i + U_i &= K_f - U_f \\
\frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f
\end{aligned}$$

To solve this, first let's notice that the mass of the rock in every term in our equation. So we can divide both sides by the mass of the rock and eliminate the mass.

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

and let's underline what we know

$$\frac{1}{2}\underline{v_i^2} + \underline{gy_i} = \frac{1}{2}v_f^2 + \underline{gy_f}$$

The only thing we don't know is v_{fy} which is what we want to find. We just need to do some algebra,

$$\begin{gathered}
\frac{1}{2}\underline{v_i^2} + \underline{gy_i} - \underline{gy_f} = \frac{1}{2}v_f^2 \\
\frac{1}{2}v_f^2 = \frac{1}{2}\underline{v_i^2} + \underline{gy_i} - \underline{gy_f} \\
\frac{1}{2}v_f^2 = \frac{1}{2}\underline{v_i^2} + \underline{g}\left(\underline{y_i} - \underline{y_f}\right) \\
v_f^2 = \underline{v_i^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right) \\
v_{fy}^2 = \underline{v_{iy}^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right) \\
v_f = \sqrt{\underline{v_i^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right)}
\end{gathered}$$

or

$$\begin{aligned}
v_f &= \sqrt{\left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(20\,\mathrm{m} - 0\,\mathrm{m})} \\
&= 22.181\frac{\mathrm{m}}{\mathrm{s}} \\
&= 22.\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

We could compare this to doing the problem using kinematics.

We would need a full two-dimensional set of kinematic equations

$$\begin{array}{ll} x_f = x_i + v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & y_f = y_i + v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\left(x_f - x_i\right) & v_{fy}^2 = v_{iy}^2 + 2a_y\left(y_f - y_i\right) \\ x_f = x_i + \frac{v_{fx}+v_{ix}}{2}\Delta t & y_f = y_i + \frac{v_{fy}+v_{iy}}{2}\Delta t \end{array}$$

and we would need to add into our knowns

$$\begin{aligned} v_{xi} &= 10\frac{\mathrm{m}}{\mathrm{s}} \\ v_{yi} &= 0 \\ y_i &= 20\,\mathrm{m} \\ x_i &= 0\,\mathrm{m} \\ y_f &= 0\,\mathrm{m} \\ m &= 0.5\,\mathrm{kg} \\ a_y &= g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \\ a_x &= 0 \end{aligned}$$

then marking what we know

$$\begin{array}{ll} x_f = \underline{x_i} + \underline{v_{ix}}\Delta t + \frac{1}{2}\underline{a_x}\Delta t^2 & \underline{y_f} = \underline{y_i} + \underline{v_{iy}}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + \underline{a_x}\Delta t & v_{fy} = \underline{v_{iy}} + a_y\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 2\underline{a_x}\left(x_f - \underline{x_i}\right) & v_{fy}^2 = \underline{v_{iy}^2} + 2a_y\left(\underline{y_f} - \underline{y_i}\right) \\ x_f = \underline{x_i} + \frac{v_{fx}+\underline{v_{ix}}}{2}\Delta t & \underline{y_f} = \underline{y_i} + \frac{v_{fy}+\underline{v_{iy}}}{2}\Delta t \end{array}$$

and using our zeros

$$\begin{array}{ll} x_f = 0 + \underline{v_{ix}}\Delta t + 0 & 0 = \underline{y_i} = 0 + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + 0 & v_{fy} = 0 + \underline{a_y}\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 0 & v_{fy}^2 = \underline{0} + 2\underline{a_y}\left(0 - \underline{y_i}\right) \\ x_f = 0 + \frac{v_{fx}+\underline{v_{ix}}}{2}\Delta t & \underline{y_f} = 0 + \frac{v_{fy}+\underline{0}}{2}\Delta t \end{array}$$

we could identify that the x-component of our rock velocity does not change,

$$v_{fx} = \underline{v_{ix}}$$

and from the third of the y-set we can find the final y-component of the velocity

$$\begin{aligned} v_{fy}^2 &= \underline{0} + 2\underline{a_y}\left(-\underline{y_i}\right) \\ v_{fy} &= \sqrt{2\underline{a_y}\left(-y_i\right)} \end{aligned}$$

and then we need to combine the components to get the magnitude of the final velocity

$$\begin{aligned}
v_f &= \sqrt{(v_x)^2 + (v_y)^2} \\
&= \sqrt{(\underline{v_{ix}})^2 + \left(\sqrt{2\underline{a_y}\,(-y_i)}\right)^2} \\
&= \sqrt{(\underline{v_{ix}})^2 + 2\underline{a_y}\,(-y_i)} \\
&= \sqrt{\left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(-9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-20\,\mathrm{m})} \\
&= 22.181\frac{\mathrm{m}}{\mathrm{s}} \\
&= 22.\,\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

with a direction of

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{-\sqrt{2\underline{a_y}\,(-y_i)}}{v_{ix}}\right) \\
&= \tan^{-1}\left(\frac{-\sqrt{2\left(-9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-20\,\mathrm{m})}}{10\frac{\mathrm{m}}{\mathrm{s}}}\right) \\
&= 1.1031\,\mathrm{rad} \\
&= -63.203\,^\circ
\end{aligned}$$

We got the same speed! But the energy method seemed easier. Did you notice that the easier energy method came with a cost? We did get the speed of the rock, and if that is all we wanted, there is no problem. But the energy method did not give us a direction, and it can't give direction. We have an easier way to get the speed, but at the cost of not knowing the direction.

Also notice, that like with conservation of momentum and Newton's laws, the kinematic equations are really buried down deep in the energy equation. They are still there, and for some problems (ones where we don't need to know direction) the energy approach works very well and may be much easier.

Let's do another problem. Suppose our boy scout throws another $0.5\,\mathrm{kg}$ rock, but this time he throws the rock at a $30\,^\circ$ angle. The initial speed is still $10\,\mathrm{m/\,s}$. If we do this with kinematics, we would have to start over and do the whole problem again. But let's try with energy.

Our initial kinetic energy is still

$$K_i = \frac{1}{2}mv_i^2$$

and our initial potential energy is still

$$U_i = mgy_i$$

and the final kinetic energy is still

$$K_f = \frac{1}{2}mv_f^2$$

and the final potential energy is still

$$U_f = mgy_f$$

nothing has changed! When we did the first problem we did all problems where the initial and final conditions are the same. We still don't know the direction, and the direction of v_f for the two problems is very different. But the final speed is the same. This is another great time savings if we only need to know the speed.

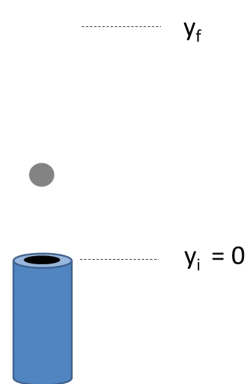
Zero point for U$_g$

You probably noticed that U_g depends on y

$$U_g = mgy$$

But we know about y. It is part of a coordinate system. And we know we can set the origin of the coordinate system anywhere we want. So does that mean that we can make U_g anything we want? To a point, this is true. Let's a problem. Let's shoot a ball out of our spring cannon. Let's try our problem shooting a ball straight up, but let's do it twice, once with the $y = 0$ point at the muzzle of the cannon, and once with the $y = 0$ point at the top of the ball's flight.

For the first case, we have the situation as shown in the next figure.



We can see that

$$\begin{aligned} y_i &= 0 \\ v_f &= 0 \end{aligned}$$

if we let v_f be right at the top of the ball's flight, and let's say that the top of the ball's flight is

$$y_f = 1.34\,\mathrm{m}$$

above the muzzle of the cannon.

We believe from what we have done in this lecture, that the quantity $K + U_g$ won't change, so

$$K_i + U_{gi} = K_f + U_{gf}$$

and we can write this out using our equations for kinetic energy and gravitational potential energy

$$\begin{aligned} K_i + U_{gi} &= K_f + U_{gf} \\ \frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \end{aligned}$$

The masses cancel

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

Now let's use our zeros, so

$$\frac{1}{2}v_i^2 + 0 = 0 + gy_f$$

then

$$v_i^2 = 2gy_f$$

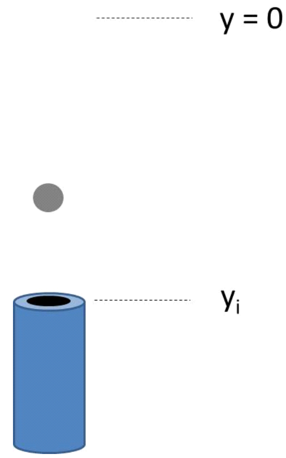
and finally

$$v_i = \sqrt{2gy_f}$$

or

$$\begin{aligned} v_i &= \sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(1.34\,\mathrm{m})} \\ &= 5.12\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Now let's do the problem again but with $y = 0$ at the top of the ball's flight. The situation is as shown in the next figure.



Now we have

$$
\begin{aligned}
y_f &= 0 \\
v_f &= 0
\end{aligned}
$$

and we realize that it must be true that

$$y_i = -1.34\,\mathrm{m}$$

Note the minus sign!

We still believe that the quantity $K + U_g$ won't change, so

$$
\begin{aligned}
K_i + U_{gi} &= K_f + U_{gf} \\
\frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f
\end{aligned}
$$

The masses cancel

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

Now let's use our zeros, but they are different in the second case

$$\frac{1}{2}v_i^2 + y_i = 0 + 0$$

then

$$v_i^2 = -2gy_i$$

and finally

$$v_i = \sqrt{-2gy_f}$$

and the initial velocity is then

$$
\begin{aligned}
v_i &= \sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-1.335\,\mathrm{m})} \\
&= 5.1153\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

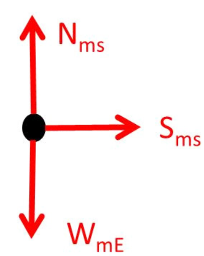
we got the same result.

This is amazing, we can pick our $y = 0$ point anywhere that is convenient, and the math still works. But WARNING, once you have picked a $y = 0$ point for a problem, you have to keep the same $y = 0$ point for the whole problem. You can't switch your origin of your coordinate system half way through the problem!

21 Using Conservation of Energy

Spring potential energy

Since a spring can exert a force, we can store up energy in the spring by stretching or compressing it. This energy will be a potential energy. Let's do what we did a few lectures ago to find the gravitational potential energy. We started with the net force. Let's take a horizontal spring resting on a (frictionless) surface. The free body diagram would be



We can write out Newton's second law

$$\begin{aligned} F_{net_y} &= ma_y = 0 \\ &= N_{ms} - W_{mE} \end{aligned}$$

which tells us that

$$N_{ms} = W_{mE}$$

and

$$\begin{aligned} F_{net_x} &= ma_x \\ &= S_{ms} \end{aligned}$$

so

$$ma_x = S_{ms}$$

We know

$$S_{ms} = -k\Delta x$$

so

$$ma_x = -k\Delta x$$

Again we write

$$a_x = \frac{dv_x}{dt}$$

so

$$m\frac{dv_x}{dt} = -k\Delta x$$

Let's write dv_x/dt in terms of dv_x/dx. We use the chain rule

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{dv_x}{dx}\frac{dx}{dt} \\ &= \frac{dv_x}{dx}v_x \end{aligned}$$

so

$$m\frac{dv_x}{dx}v_x = -k\left(x - x_0\right)$$

where x_o is the equilibrium position of the spring. Then

$$mv_x dv_x = -k\left(x - x_0\right)dx$$

and we integrate both sides

$$\int_{v_i}^{v_f} mv_x dv_x = \int_{x_i}^{x_f} -k\left(x - x_0\right)dx$$

The right hand side is

$$\int_{v_i}^{v_f} mv_x dv_x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

The left hand side is

$$\begin{aligned} \int_{x_i}^{x_f} -k\left(x - x_0\right)dx &= \int_{x_i}^{x_f} -kxdx + \int_{x_i}^{x_f} -k\left(-x_0\right)dx \\ &= \left(-k\left(-\frac{1}{2}x^2\right)\right|_{x_i}^{x_f} + \left(-kx_0x\right|_{x_i}^{x_f} \\ &= +\frac{1}{2}kx_f^2 - \left(+\frac{1}{2}kx_i^2\right) + \left(-kx_0x_f - \left(-kx_0x_i\right)\right) \\ &= +\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 - kx_0x_f + kx_0x_i \\ &= +\frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right) \end{aligned}$$

At this point let's play a trick we learned in high school. I want to add zero to this

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = +\frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right) + 0$$

this does not affect the value of the right hand side at all. But let's write zero as

$$0 = \frac{1}{2}kx_0^2 - \frac{1}{2}kx_0^2$$

then

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = \frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right) + \frac{1}{2}kx_0^2 - \frac{1}{2}kx_0^2$$

which we can rearrange as

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = \left(\frac{1}{2}kx_f^2 - kx_0x_f - \frac{1}{2}kx_0^2\right) - \left(\frac{1}{2}kx_i^2 - kx_0x_i - \frac{1}{2}kx_0^2\right)$$

and we can factor the quadratic terms

$$\begin{aligned}\int_{x_i}^{x_f} -k\left(x - x_0\right)dx &= \left(\frac{1}{2}k\left(x_f - x_o\right)\left(x_f - x_o\right)\right) - \left(\frac{1}{2}k\left(x_i - x_o\right)\left(x_i - x_o\right)\right) \\ &= \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2\end{aligned}$$

then substituting in our results for the right and left hand sides our energy equation is

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2$$

The right hand side we recognize as

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i$$

so the left hand side must be our spring energy change as we move the spring from an initial position x_i to a final position x_f

$$\frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2 = U_{sf} - U_{si}$$

Then grouping initial and final terms gives

$$\frac{1}{2}mv_i^2 - \frac{1}{2}k\left(x_i - x_o\right)^2 = \frac{1}{2}mv_f^2 - \frac{1}{2}k\left(x_f - x_o\right)^2$$

which is

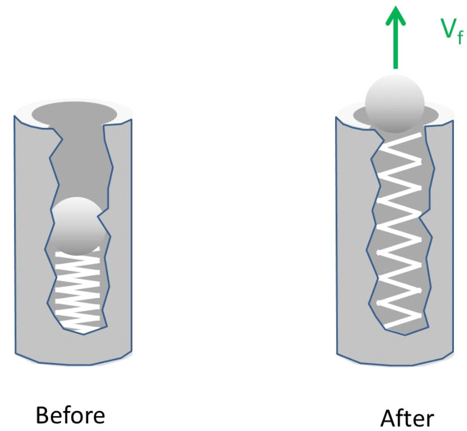
$$K_i + U_{si} = K_f + U_{sf}$$

We have identified that the potential energy of a spring is given by

$$U_s = \frac{1}{2}k\left(x - x_o\right)^2$$

which isn't too much of a surprise. The potential energy of the spring should be the work it took to compress or stretch the (ideal) spring.

Let's try a problem with this.



Our spring cannon gives a muzzle velocity of about $5.1\,\mathrm{m/\,s}$. We found this last time. Let's calculate the spring constant for our spring. The spring is compressed $0.05\,\mathrm{m}$ when the cannon is loaded. The initial speed of the ball is $0\,\mathrm{m/\,s}$. The final position of the spring is at it's equilibrium point. Our ball has a mass of $67.1\,\mathrm{g}$.

We know

$$\begin{aligned} v_i &= 0 \\ v_f &= 5.1\frac{\mathrm{m}}{\mathrm{s}} \\ y_o &= 0 \\ y_i &= -0.05\,\mathrm{m} \\ y_f &= y_o \\ m &= 67.1\,\mathrm{g} = 0.067\,1\,\mathrm{kg} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Our basic equations are

$$K_i + U_{gi} + U_{si} = K_f + U_{gf} + U_{sf}$$
$$K = \frac{1}{2}mv^2$$
$$U_s = \frac{1}{2}k\left(y - y_o\right)^2$$

We can write out our conservation of energy equation as

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}k\left(y_i - y_o\right)^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}k\left(x_f - x_o\right)^2$$

and use our zeros

$$0 + mgy_i + \frac{1}{2}k\left(y_i - 0\right)^2 = \frac{1}{2}mv_f^2 + 0 + \frac{1}{2}k\left(y_o - y_o\right)^2$$
$$mgy_i + \frac{1}{2}k\left(y_i\right)^2 = \frac{1}{2}mv_f^2 - 0$$

so

$$mv_f^2 = 2mgy_i + k\,(y_i)^2$$
$$\frac{mv_f^2 - 2mgy_i}{(y_i)^2} = +k$$

so

$$k = \frac{m\left(v_f^2 - 2gy_i\right)}{(y_i)^2}$$

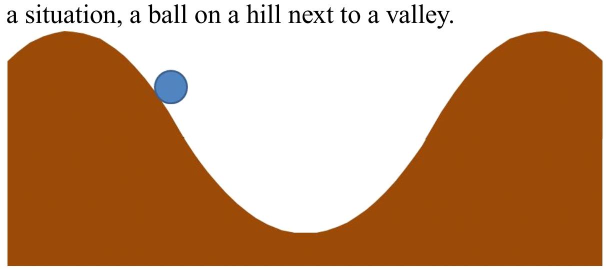
$$\begin{aligned} k &= \frac{(0.067\,1\,\mathrm{kg})\left(\left(5.1\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - 2\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-0.05\,\mathrm{m})\right)}{(-0.05\,\mathrm{m})^2} \\ &= 724.41\frac{\mathrm{N}}{\mathrm{m}} \end{aligned}$$

Adding in springs and spring forces and spring potential energy allows us to design wonderful things from dart guns to car shocks to vibration isolation systems we use on big laser systems (and on temples!).

Conservation of Energy and Energy Graphs

We spent a lot of time learning to interpret motion diagrams and position, velocity and acceleration vs. time graphs. Using these graphs and diagrams we could understand how something moved. We also drew diagrams for forces. And for energy, we need to learn to raw *before* and *after* pictures to find the initial and final speeds and positions. But energy is such a useful way to look at motion that there are some standard ways to show energy for a system and we will need to be able to use these ways to depict energy.

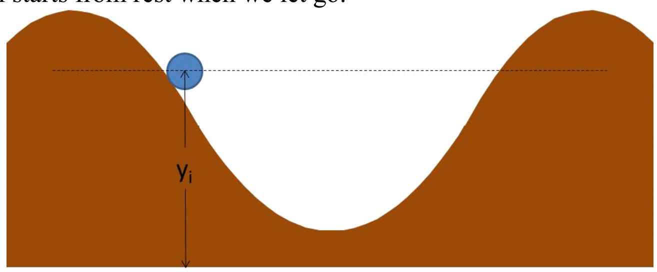
Let's start with a situation, a ball on a hill next to a valley.



We know a lot about this situation. The ball will have an initial energy

$$\begin{aligned} E_i &= K_i + U_i \\ &= \frac{1}{2}mv_i^2 + mgy_i \\ &= mgy_i \end{aligned}$$

since the ball starts from rest when we let go.



So the initial energy is all potential energy and that potential energy depends on the initial height of the ball. We know that if there is nothing to take energy away from the system, then energy is conserved

$$E_f = E_i$$

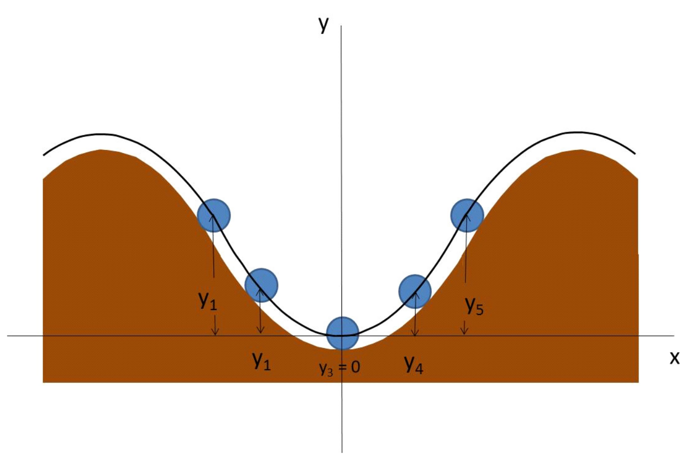
So at some time later we will have a final energy

$$E_f = \frac{1}{2}mv_f^2 + mgy_f$$

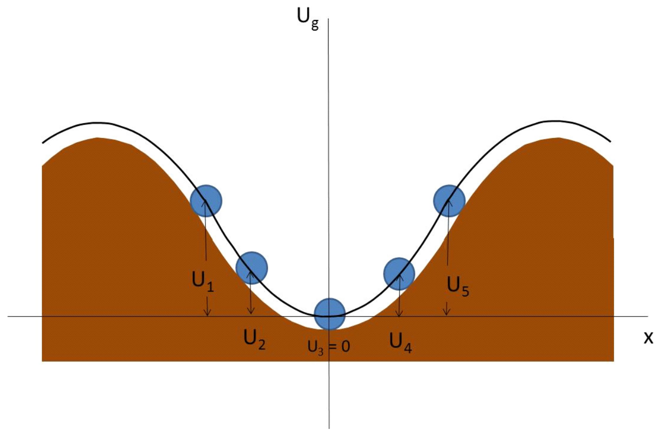
and this final energy must be equal to the initial energy

$$mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

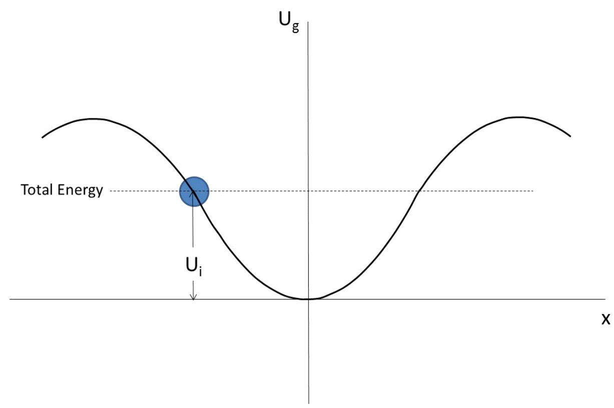
Let's think about how high up the ball will be as the ball travels down the hill and through the valley to the other side. The potential energy, U_g depends on the height of the ball. We can plot the ball height for each x-position along the ball path.



Since the gravitational potential depends on the y-position. a graph of the gravitational potential vs. x should look like the position (y vs. x) graph along the ball's path.



Of course, we could draw the graph without the hill



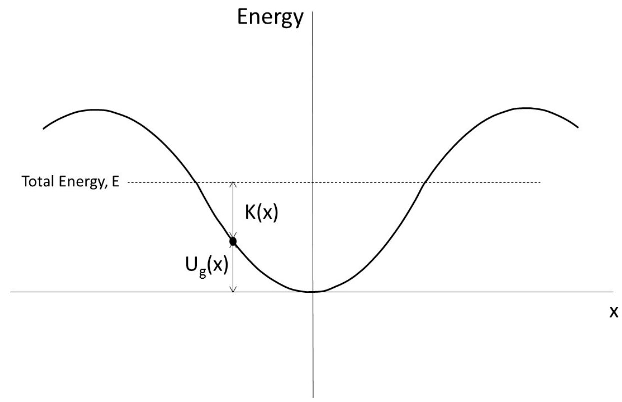
This type of graph is very commonly used to describe the energy of a system. Notice that if we plot $U_g(x)$ and we also plot the total energy, $E(x)$ then from the graph we can always find the kinetic energy too because

$$E(x) = K(x) + U_g(x)$$

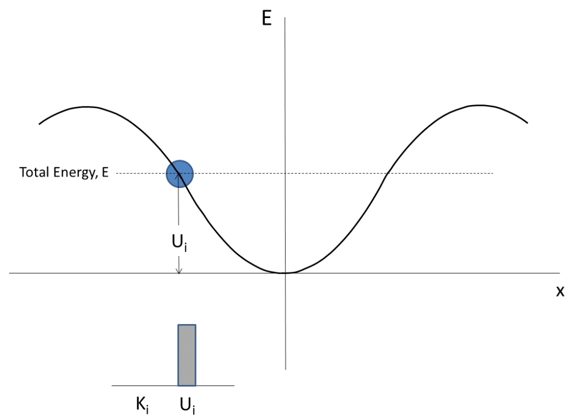
so

$$K(x) = E(x) - U_g(x)$$

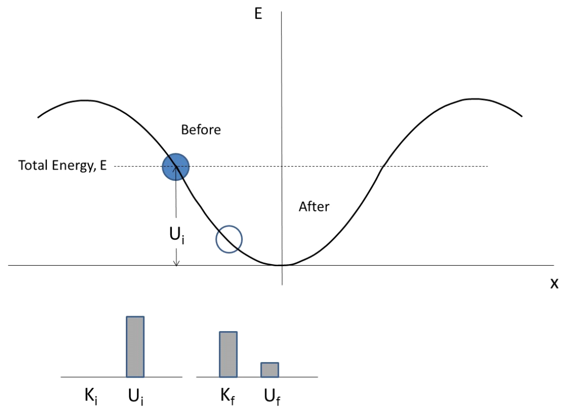
so one graph shows the entire energy situation for the system!



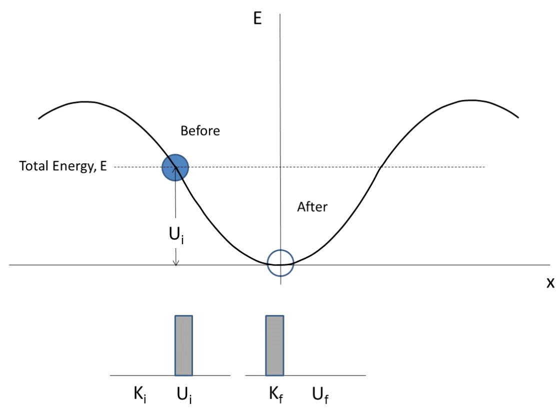
Since the kinetic and potential energies must add up to the total energy $E(x) = K(x) + U_g(x)$. We could plot the energy at a point in another way



We could use a bar chart to tell us how much of the energy is potential energy at a given x-position and how much is kinetic energy at that x-position. Notice in the figure, when $U_g(x) = E(x)$ then $K(x)$ is zero. And the bar chart shows that. After we let go of the ball, the amount of potential energy decreases and the amount of kinetic energy increases.



Right at the bottom, the ball will be moving quickly



We can see from the bar graph that all of the potential energy has been changed to kinetic energy. Remember that

$$K(x) = \frac{1}{2}mv(x)^2$$

so

$$v(x) = \sqrt{\frac{2K(x)}{m}}$$

and the speed will be fastest right at the bottom.

If the ball is allowed to keep going it will roll up the other side of the valley. We could predict that it will stop at the same height on the other side of the valley.

$$U_g(x_f) = E(x_f) - K(x_f)$$

so when $v(x_f) = 0$ then $K(x_f) = 0$ so

$$U_g(x_f) = E(x_f)$$

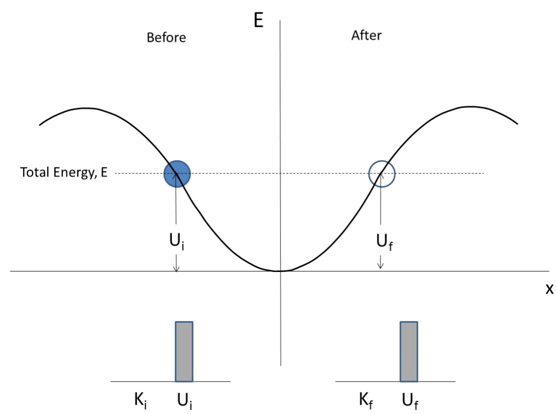
and we started with

$$E(x_f) = U_i = mgy_i$$

so

$$U_g(x_f) = mgy_i$$

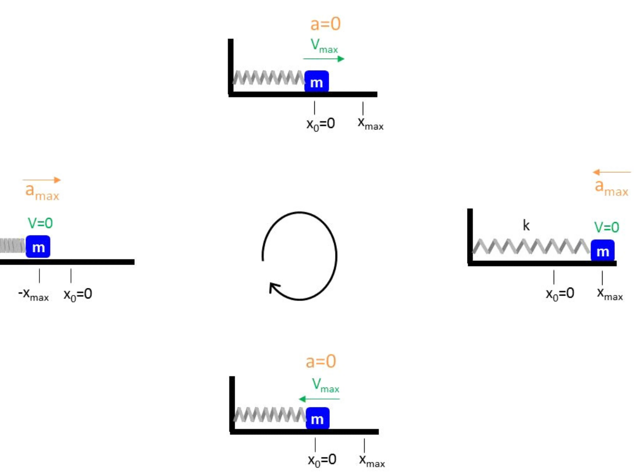
and indeed, the ball will stop (briefly) when it reaches the same height on the other side.



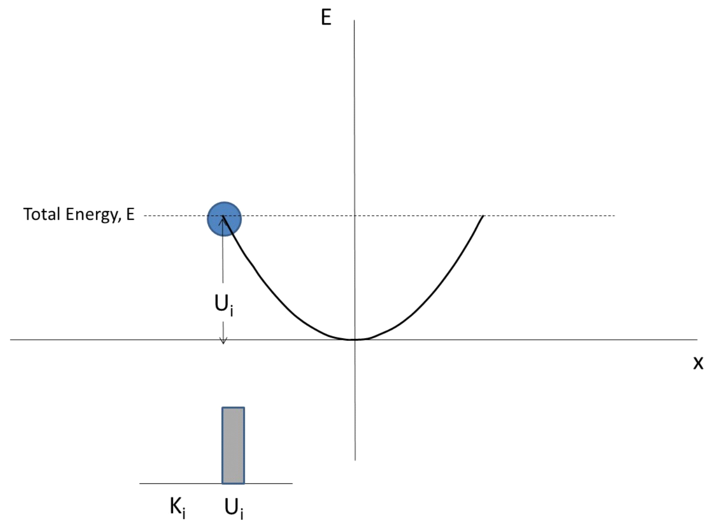
If we let the ball keep going, it will roll back down into the valley and go back and forth over and over again.

Spring potential and energy graphs

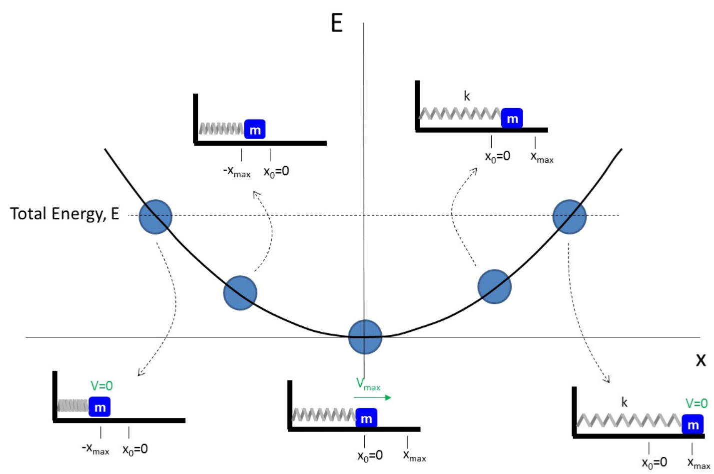
Let's try using our new graphs for spring potential energy. Consider what would happen if we compress the spring and let go.



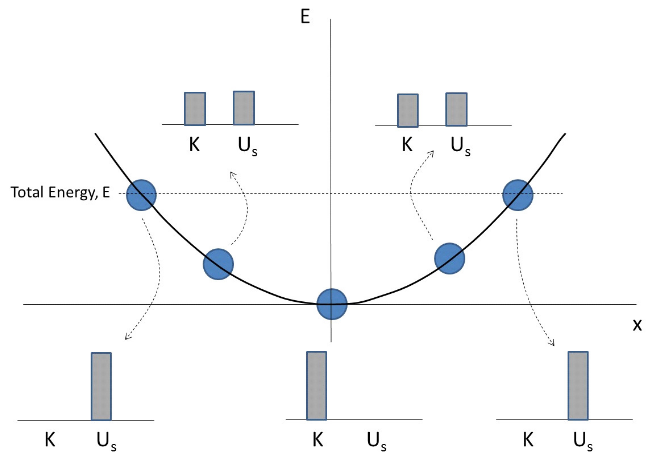
This is a little bit like letting the ball go on the hill by the valley



At first, we have all spring potential energy. But quickly this spring potential energy starts to change into kinetic energy.



The mass picks up speed, so it gains kinetic energy and loses potential energy. At the midpoint the mass has all kinetic energy and no potential energy. But it has momentum so it is hard to stop! So it will keep going and it will start stretching the spring, building up potential energy and slowing down so there will be less kinetic energy.

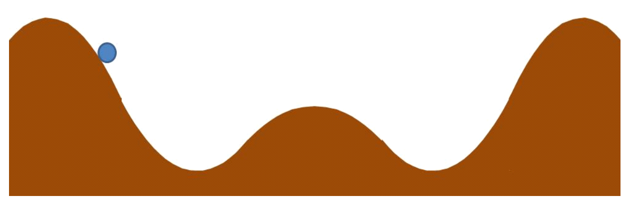


Notice how similar this set of graphs is to the gravitational potential energy set. Since the graphs look so much the same, we can use gravitational potential energy to help us gain intuition into other types of potential energy and how the objects that experience the potential energy will act.

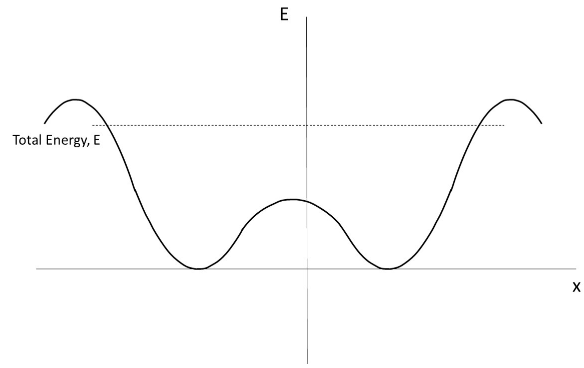
In the case of the spring, we can see that the mass will go back and forth, a little like the ball went back and forth between the two hills.

Equilibrium points

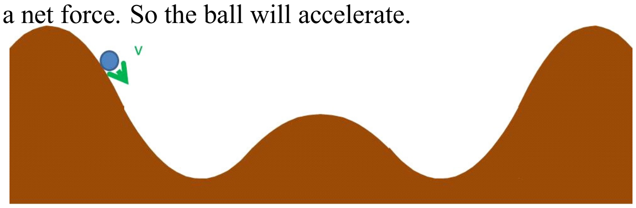
Suppose we have a more complicated hill and valley system.

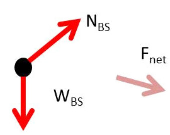


We recognize that our gravitational potential energy graph would have the same shape as the hill/valley system.

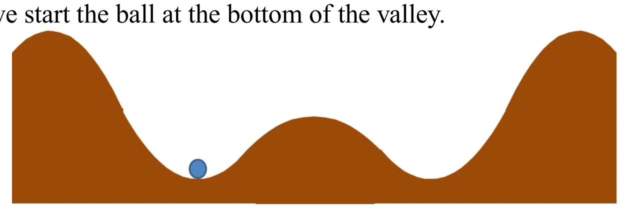


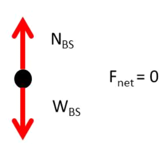
Let's relate our potential energy graphs to Newton's laws. At the point where the ball starts there is a net force. So the ball will accelerate.



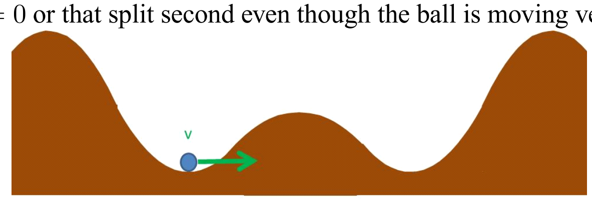


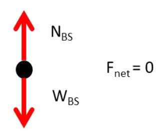
But suppose we start the ball at the bottom of the valley.



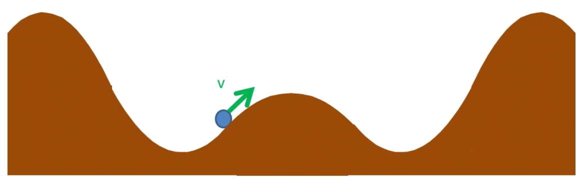


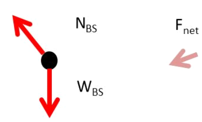
The condition $\overrightarrow{F}_{net} = 0$ is what we have called equilibrium. Notice that our force diagram would be the same if we started the ball on the hill and let it go down to the valley. The forces would be the same at the bottom of the valley, but now with the ball moving. Still the bottom of the valley is an equilibrium. The equation $\overrightarrow{F}_{net} = 0$ tells us that we are not accelerating, but it does not tell us if we are moving. Right at the bottom of the hill the ball is not speeding up or slowing down for a split second. So, indeed, $\overrightarrow{F}_{net} = 0$ or that split second even though the ball is moving very fast.



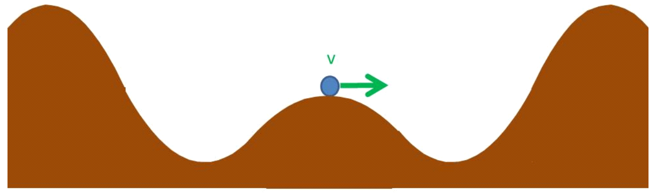


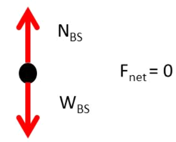
As the ball goes up the little hill, it will slow down, and a free body diagram can shown us why. There is a net force, so there is an acceleration in the opposite direction the ball is going.





Let’s join the ball again as it hits the top of the hill.





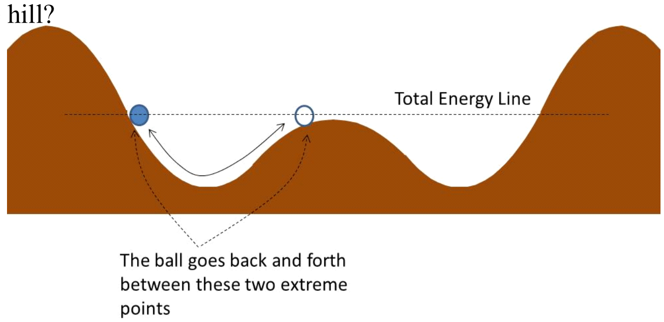
Once again the net force is zero. Of course the ball is moving. We can tell that the middle hill is lower than the side hills, so

$$K_i + U_i = K_{lh} + U_{lh}$$

where the subscripts "lh" are for "little hill." Then

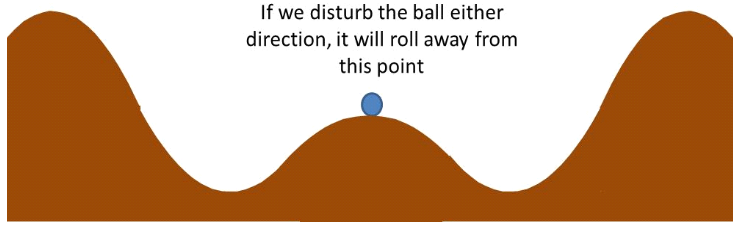
$$\begin{aligned} K_{lh} &= K_i + U_i - U_{lh} \\ &= 0 + mgy_i - mgy_f \\ \frac{1}{2}mv_{lh}^2 &= mg\left(y_i - y_f\right) \end{aligned}$$

so since the hills have different heights the speed of the ball won't be zero. We can see that this is another equilibrium point. But what would happen if we started the ball lower on the hill?



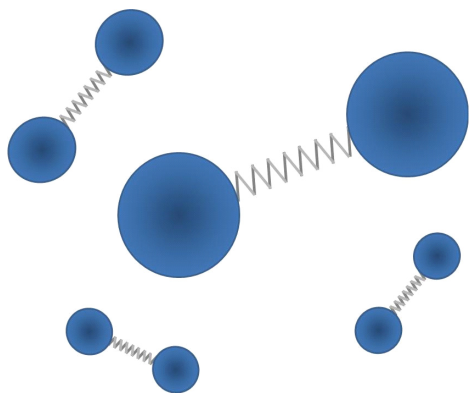
We already know the ball will only go up as high as it started. So if we start it lower than the top of the middle hill, the ball will roll back and forth between the left hill and the middle hill. If we allowed friction in our problem, eventually the ball would stop right at the bottom of the valley. If we gave the ball a small kick, again it would return to the bottom of the valley. This equilibrium position is an important one, we call it a *stable equilibrium* because an object with this particular potential energy won't move away from that point or will return to that point if disturbed.

Let's consider our other equilibrium. If we place the ball on the top of the middle hill, it is in equilibrium, but any disturbance either direction will result in a net force away from the top-of-the-hill equilibrium position.

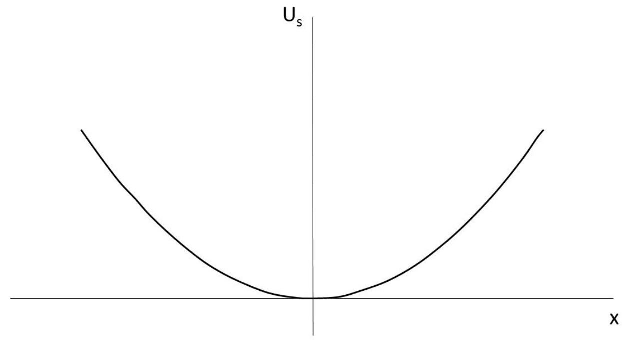


The ball would have to have someone go get it and put it back on the top of the hill. It won't return there of it's own accord. We call an equilibrium point where the object would leave the point permanently if disturbed an *unstable equilibrium.*

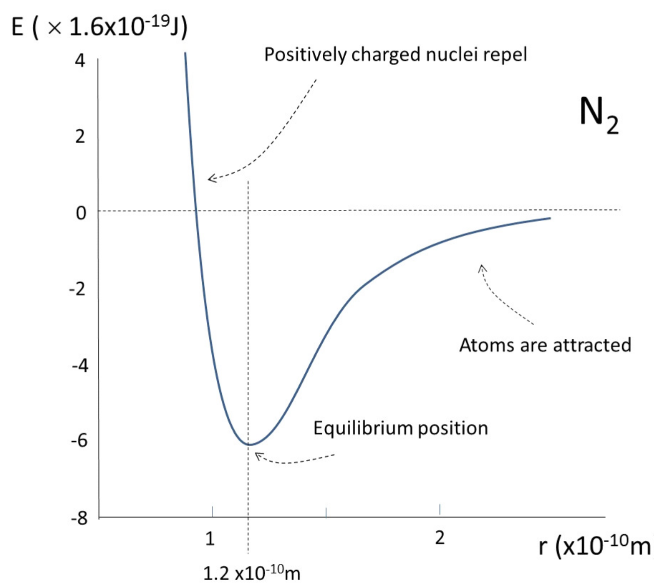
Let's use what we have learned to take on a real problem. Our atmosphere is mostly made of diatomic nitrogen or N_2. We have used spring forces as a model for molecular bonds. So let's picture the two nitrogen atoms as tied together by a spring.



As the spring is compressed, the spring pushes back. As the spring is stretched it pulls back. So we expect a potential energy that looks like our spring potential energy.

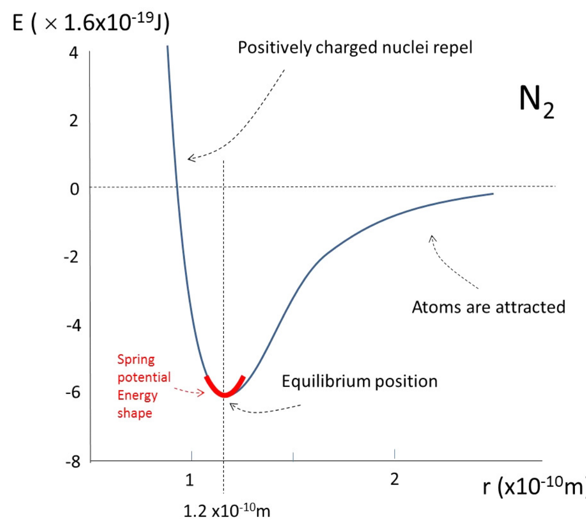


But there is a little more to our molecule. We know that if we pull the two atoms hard enough, we can pull the molecule apart. So as the distance between the atoms gets larger, the potential energy must show that we can get the atoms apart. If we push the atoms closer together the strong electrical repulsive force between the two nuclei will make it harder and harder to put get the atoms closer together. So the potential energy must show that it is hard to force the atoms together. Here is what the potential energy graph looks like for one possible diatomic nitrogen molecule.



Notice that the potential energy graph is not symmetric. The left hand side shows that the potential energy gets very large as we push the atoms together. We could mentally think of this as leaving one atom in place and pushing the other atom toward the first.

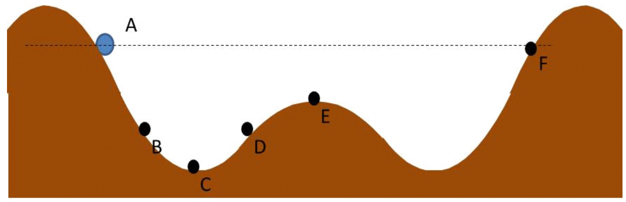
The moving atom is like our ball and the potential energy from the electrical force is like the hill. It makes a potential energy. So moving our mover atom toward the environmental atom is like moving the ball up a very steep hill. On the right hand side, we can see that we have a much less steep hill. This is part of the graph that tells us that we could break the molecular bond by pulling our mover atom farther away. It is going "up hill," but much less slowly. Now observe that in between the two sides there is a part of the graph that looks a lot like the spring potential energy. Here is our N_2 graph again but with a spring potential energy graph superimposed over the equilibrium point of our N_2 graph.



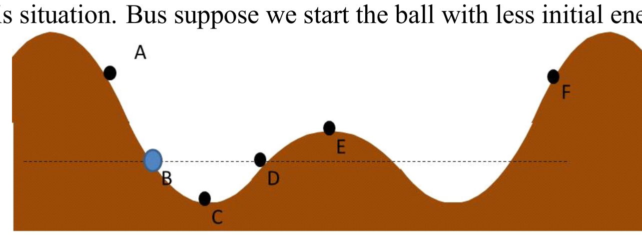
Notice that it fits pretty well. Then we could predict that if we pull on one of the atoms just a little, the bond would pull it back. It is like our ball near the stable equilibrium point. The atom would oscillate back and forth but never get very far from the equilibrium position. If we allow a friction-like force to remove the extra energy, then the atom would settle right back to the equilibrium position.

Turning points

Let's go back to our hill. If we start the ball from rest as shown, where will the ball stop on the far hill?



We now know that this will be at point F, when the energy is all potential energy again. But at point F, the force due to gravity and the normal force don't point the same direction, so we will have a net force. The ball will roll back down the hill. Since the ball stopped, and then went the other direction, we can say that the ball turned around at this point. So we call point F a *turning point*. When the ball reaches it's original position, it will also turn around, so the initial position and point F are both turning points for this situation. Bus suppose we start the ball with less initial energy.



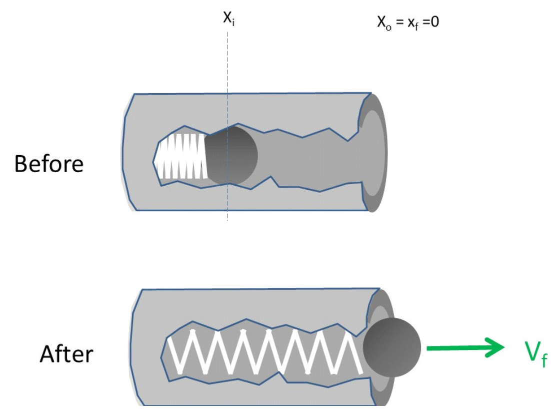
Now points B and D will be turning points.

22 Force and Potential Energy-work Relationships

Let's start this lecture with a review. Let's take our equation for spring potential energy

$$U_s = \frac{1}{2}k\left(s - s_o\right)^2$$

and consider a before and after case. Let's review our spring cannon case where we compress the spring and then let it go back to it's equilibrium position.



The initial potential energy of the spring in our case is

$$U_{si} = \frac{1}{2}k\left(x_i - x_o\right)^2$$

and

$$U_{sf} = \frac{1}{2}k\left(x_f - x_o\right)^2$$

but $x_f = x_o$ for this case, so

$$U_{sf} = \frac{1}{2}k\left(x_o - x_o\right)^2 = 0$$

and the change in spring potential energy is

$$\begin{aligned}\Delta U_s &= U_{sf} - U_{si} \\ &= 0 - \frac{1}{2}k\left(x_i - x_o\right)^2 \\ &= -\frac{1}{2}k\left(x_i - x_o\right)^2\end{aligned}$$

The spring has lost energy, so ΔU_s is negative. This is just the spring potential energy lost by the spring as it transfers its stored energy into kinetic energy of the ball. Now let's find the work done on the ball

$$\begin{aligned}w_{ball} &= \int \overrightarrow{S}\left(x\right) \cdot \overrightarrow{dx} \\ &= \int \left|-k\left(x - x_o\right)\right| \cos\left(\theta_{sx}\right) dx\end{aligned}$$

Let's use the fact that $x_f = x_o = 0$ to make the math easier

$$w_{ball} = \int \left|-k\left(x\right)\right| \cos\left(\theta_{sx}\right) dx$$

and let's think that $\left|-k\left(x\right)\right|$ must be positive. But our x values from x_i to x_f will be negative with our choice of origin. So

$$\left|-k\left(x\right)\right| = -kx$$

is a positive value for our range of x's. Then

$$w_{ball} = \int_{x_i}^{x_f=0} -k\left(x\right) \cos\left(\theta_{sx}\right) dx$$

and note that the ball is going to the right, and the spring is pushing to the right so $\theta_{sx} = 0$

$$\begin{aligned}w_{ball} &= \int_{x_i}^{x_f=0} -k\left(x\right) dx \\ &= -k\int_{x_i}^{x_f=0} x dx \\ &= -k\left.\frac{x^2}{2}\right|_{x_i}^{x_f=0} \\ &= -\frac{1}{2}kx_f^2 - \left(-\frac{1}{2}kx_i^2\right) \\ &= 0 + \frac{1}{2}kx_i^2\end{aligned}$$

But this is a positive amount of work! The spring lost energy and the ball received energy by work. Losing energy gets a minus sign, gaining energy gets a plus sign. And we can see that

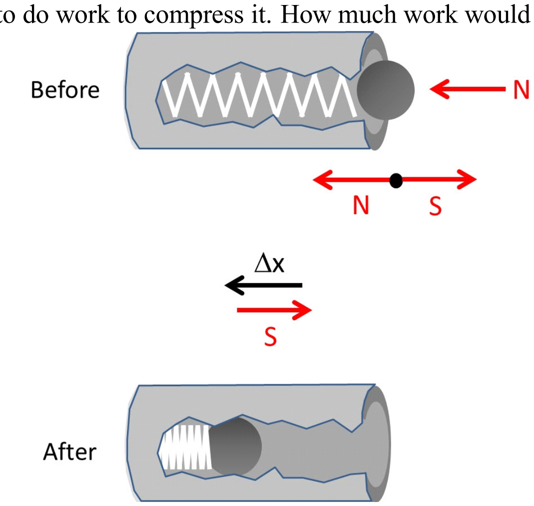
$$\Delta U_{spring} = -w_{ball}$$

It turns out that this is an important result. Let's write it as

$$\Delta U = -w$$

without subscripts so it is more general. But, you might fairly object, that leaving off the subscripts obscures what is changing energy and what has work done on it!

Another way to look at it is that to do the problem of causing ΔU to be stored in the spring. We have to do work to compress it. How much work would we have to do?



Really it is the normal force that will compress the spring. We don't know what the normal force magnitude is. But we know it must be at least as much at each moment as the spring force or the ball won't keep compressing the spring. We can use the spring force in our work equation because N and S must at least be equal. Again we would have

$$w_s = \int \overrightarrow{S}(x) \cdot \overrightarrow{dx}$$

where both S and dx are both negative, making the angle between them $\theta_{Sx} = 0$ again.

$$w_s = \int_{x_o}^{x_f} -k(x - x_o)(1)\,dx$$

$$w_s = \left(-k\left(\frac{x^2}{2}\right)\right|_{x_o}^{x_f} + (kx_o x dx|_{x_o}^{x_f}$$

$$w_s = -k\left(\frac{x_f^2}{2}\right) - \left(-k\left(\frac{x_0^2}{2}\right)\right) + kx_o x_f - kx_o x_0$$

$$w_s = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_o^2 + kx_o x_f - kx_o^2$$

$$w_s = -\frac{1}{2}kx_f^2 - \frac{1}{2}kx_o^2 + kx_o x_i$$

$$\begin{aligned} w_s &= -\frac{1}{2}k\left(x_f^2 - 2x_o x_f + x_o^2\right) \\ &= -\frac{1}{2}k\left(x_f - x_o\right)^2 \end{aligned}$$

But the change in potential energy as we compress the spring is

$$\begin{aligned} U_f - U_i &= \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_o - x_o\right)^2 \\ &= \frac{1}{2}k\left(x_f - x_o\right)^2 \end{aligned}$$

which again gives

$$w_s = -\left(U_f - U_i\right)$$

That is, the work we do to compress the spring is minus the energy stored in the spring. This is not so surprising. While we do work to compress the spring, the spring pushes back. The spring's push is not causing the motion, so it's work would be negative. Maybe it would be better to write this as

$$w_{\text{done on the spring}} = -\left(U_f - U_i\right)_{\text{stored in the spring}}$$

The big thing is to notice that the force that compressed the spring wasn't really the spring force. It is just equal to it. So in a sense we got the sign wrong on w because we used the wrong force. The spring force is the force that makes the stored energy possible, thought. So we might write this as

$$-w_{\text{by force that makes the potential energy possible}} = \left(U_f - U_i\right)_{\text{stored}}$$

It takes another object (you) exerting a force (you pushing) to make this energy storage happen.

Let's compare this to our previous example of launching the ball. The launch is using the stored potential energy to do this.

$$\Delta U_{spring} = -w_{ball}$$

Now the energy stored by an external force is able to be retrieved by doing work on another object. Both in the storage of potential energy and in the use of potential energy there were two objects involved. In this example, we have a spring in both cases, but in the storage case you pushed and in the retrieval case a ball was pushed.

To summarize To gain an amount of spring potential energy ΔU_s by doing work on the spring we need an amount of work

$$\Delta U_s = -w$$

but that work *must be caused by another force, not the spring force*. To use the potential energy stored in a spring, we let the spring act with a force *on some other object*. In both cases, the work done is the inverse of the change in potential energy.

$$\Delta U = -w$$

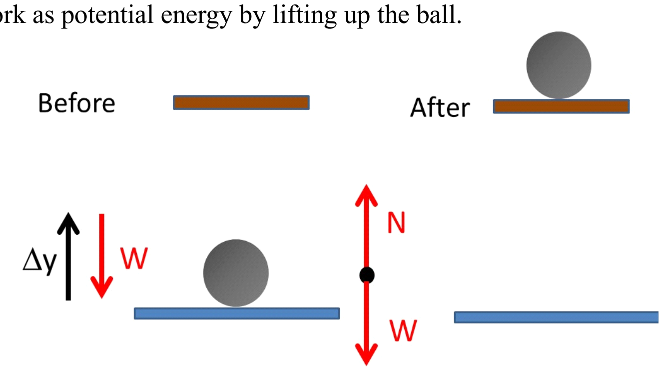
We can use this idea for gravitational potential energy as well, and any other type of mechanical potential energy.

We know how to find potential energy now if we know the force causing that potential energy to exist. Because we know that $w = -\Delta U$, so we can write

$$\begin{aligned} \Delta U &= -w \\ &= -\int \overrightarrow{F}(s) \cdot \overrightarrow{ds} \end{aligned}$$

So if I know the force on a mover object, and I know how far I move the mover object I can find the change in potential energy. Let's see how this works.

Let's take the case of a ball being lifted up on a shelf, and them falling off the shelf. We will store work as potential energy by lifting up the ball.



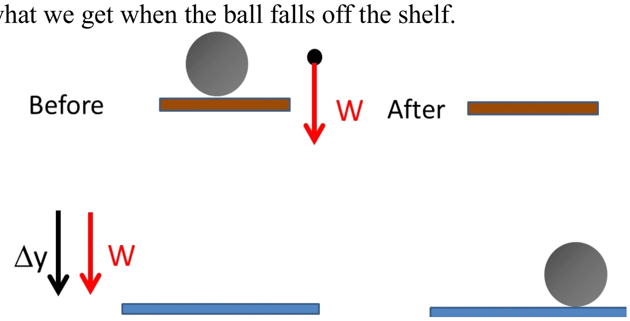
Again we don't know N, the normal force we are using to lift up the ball. But we know it must be at least equal to W to keep the ball moving up. If we make it bigger, we would expend more energy, but the extra energy would not be stored, only the energy for the case where $N = W$. So we can find the work done by letting $N = W$.

$$\begin{aligned} w &= \int_{s_i}^{s_f} \overrightarrow{F}(s) \cdot \overrightarrow{ds} \\ &= \int_{y_{bottom}}^{y_{top}} W \cos(\theta_{Wy})\, dy \\ &= \int_{y_{bottom}}^{y_{top}} mg(-1)\, dy \\ &= -mgy|_{ybottom}^{y_{top}} \\ &= -mgy_{top} - (-mgy_{bottom}) \\ &= -mgy_{top} + mgy_{bottom} \\ &= -\left(U_{g_{top}} - U_{g_{bottom}}\right) \\ &= -\Delta U_g \end{aligned}$$

so again

$$w = -\Delta U$$

but again the w came from using the force that makes it possible to have gravitational potential energy, not the force (N) that actually causes the potential energy to be stored. Let's look at what we get when the ball falls off the shelf.



We know the potential energy for a falling ball is due to the gravitational force, and that the gravitational potential is mgy.

$$\begin{aligned} \Delta U_g &= U_{g_f} - U_{g_i} \\ &= U_{g_{bottom}} - U_{g_{top}} \\ &= mgy_{bottom} - mgy_{top} \end{aligned}$$

but because $y_f = y_{bottom}$ is now smaller than $y_i = y_{top}$ we can see that ΔU must be negative.

Let's find the work done by the gravitational force in making the ball fall and compare.

$$W_{bE} = mg$$

and points downward and the ball falls in the $-y$ direction, so

$$ds = dy$$

and $(\theta_{Wy}) = 0$ and we know that w must be positive. Then

$$\begin{aligned} w &= \int \overrightarrow{F}(s) \cdot \overrightarrow{ds} \\ w &= \int \overrightarrow{W} \cdot \overrightarrow{ds} \\ &= \int_{y_{top}}^{y_{bottom}} W \cos(\theta_{Wy}) \, |dy| \end{aligned}$$

Remember that using a dot product $\overrightarrow{W} \cdot \overrightarrow{ds} = W \cos \theta_{Ww} ds$ the W and the ds are magnitudes. We need both the W and the dy to be positive numbers. We need to be careful because as the ball goes downward any $\Delta y = y_f - y_i$ will be negative. Then even a small dy will be negative. So we need to make it dy positive in the dot product because magnitudes are not negative. The absolute value signs do this for us, but they make the integral awkward. Since we know that dy is negative for this specific case, we could make it positive by supplying our own minus sign

$$w = \int_{y_{top}}^{y_{bottom}} W \cos(\theta_{Wy}) (-dy)$$

for this specific case. Now we are guaranteed to have w positive as we know it must be for this case. And now we can perform the integral

$$w = \int_{y_{top}}^{y_{bottom}} -mg(1) \, dy$$

The mass and the acceleration are not changing, so we can take them out of the integral

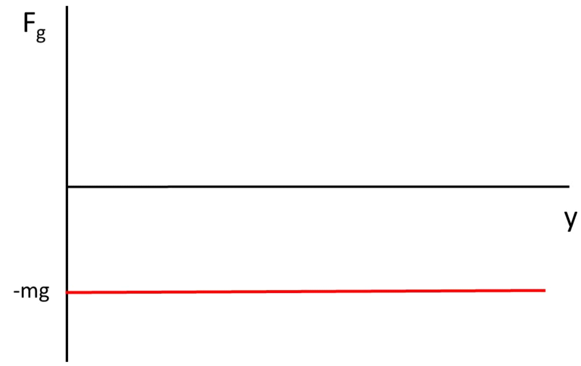
$$\begin{aligned} w &= -mg \int_{y_{top}}^{y_{bottom}} dy \\ &= -\left(mgy|_{y_{top}}^{y_{bottom}}\right. \\ &= -\left(mgy_{bottom} - mgy_{top}\right) \\ &= -\left(U_f - U_i\right) \\ &= -\Delta U \end{aligned}$$

And we can see that we did, indeed, find the potential energy change for our mass but our work is the opposite sign of the change in potential energy.

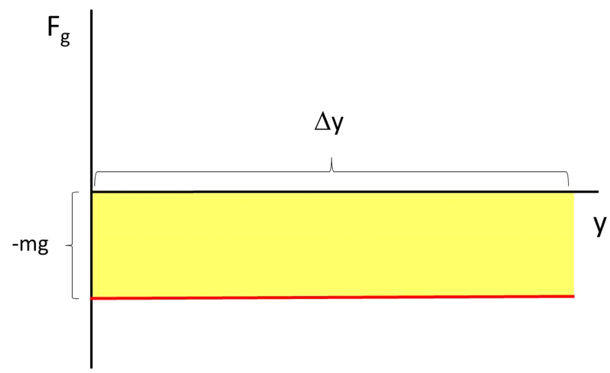
$$w = -\Delta U$$

Force from potential energy

Let's look at a graph of the falling ball situation. We learned earlier that an integral is a way to find the "area" under a curve. So let's graph force vs. y



The "area" would be



$$\text{"A"} = -mg \times \Delta y$$

which is just what we expect for our potential energy difference.

$$\Delta U = -mg\Delta y$$

It would also be good to find the force if we know the potential energy change. Let's go back to our relationship between force and the change in potential energy.

$$\Delta U = -\int \overrightarrow{F}(s) \cdot \overrightarrow{ds}$$

The integrand is a small amount of potential energy change. The integral adds up many small changes to get the entire potential energy change. So we could say that

$$dU = -\overrightarrow{F}(s) \cdot \overrightarrow{ds}$$

where dU is a small change in potential energy.

$$dU = -F ds \cos\theta_{Fs}$$

$$dU = -F \cos\theta_{Fs} ds$$

We can again write

$$F \cos(\theta_{Fs}) = F_s$$

so that

$$dU = -F_s ds$$

Then

$$F_s = -\frac{dU}{ds}$$

where all along we have used our generic component direction s. We could have used x, or y

$$F_x = -\frac{dU}{dx}$$

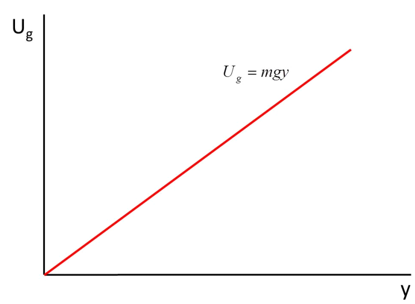
$$F_y = -\frac{dU}{dy}$$

We can try this again for our falling ball. If we know $U_g = mgy$ then

$$\begin{aligned} F_y &= -\frac{d}{dy}(mgy) \\ &= -mg \end{aligned}$$

which is just right!

We should study this graphically to understand it better. If we plot U_g as a function of y, we get a graph that looks like this



This is not too surprising. Consider the equation for a straight line

$$y = mx + b$$

and we see that with the y-axis now the U_g-axis and with the x-axis now the y-axis.

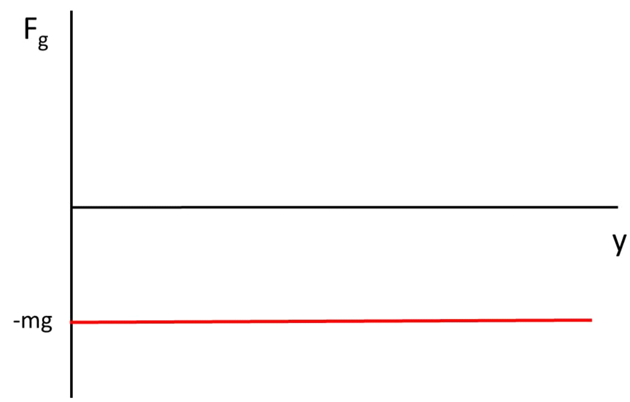
then

$$\begin{aligned} y &= mx + b \\ Ug &= mg\,(y) + 0 \end{aligned}$$

then we can see that if the slope of the curve is mg, we have a straight line that goes through the origin.

Now that we are more familiar with calculus, we easily recognize that dU/dt is the slope of the U vs. y graph.

$$\begin{aligned} F_y &= -\frac{dU}{dy} \\ &= -mg \end{aligned}$$



This was pretty easy for a constant force like the force due to gravity. But it would be more complicated for a spring force. Let's try it!

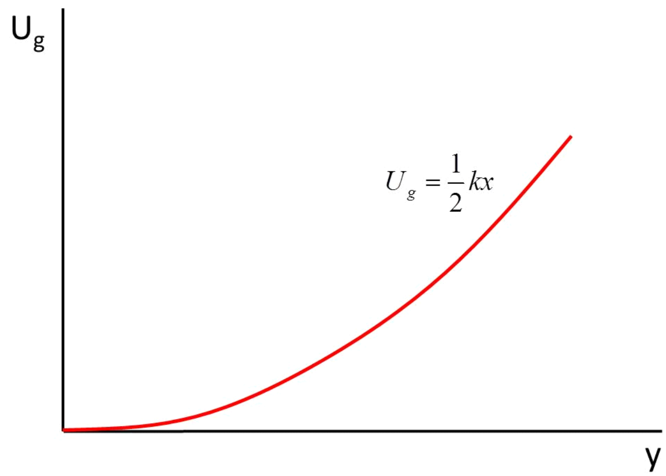
We know that

$$U_s = \frac{1}{2}k\,(x - x_o)^2$$

let's choose our origin so that $x_o = 0$ to make the math easy, then

$$U_s = \frac{1}{2}k\,(x)^2$$

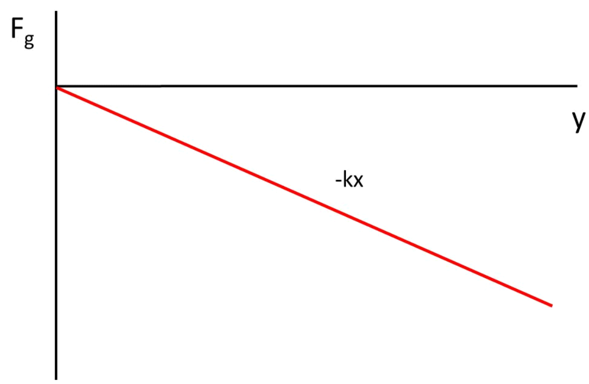
The graph looks like this



Then the force would be

$$
\begin{aligned}
F_x &= -\frac{dU_s}{dx} \\
&= -\frac{d}{dx}\left(\frac{1}{2}k\left(x\right)^2\right) \\
&= -\frac{1}{2}\left(2kx\right) \\
&= -kx
\end{aligned}
$$

The plot looks like this.



And we are have found the spring force! We will use these techniques again in PH123 and PH220. We have solved our problem of non-constant forces. Next lecture we will take on frictional forces.

Work-Energy Theorem Revisited

We already know the work energy theorem

$$w = \Delta K$$

but we want to include our idea of potential energy into this basic equation, and we have a missing piece to fill in.

Work-energy and potential energy

Now that we have the concept of thermal energy, we can begin to give some detail to our work-energy equation.

$$w = \Delta K$$

First let's split our work term into two terms, a term for conservative work and a term for non-conservative work.

$$w_c + w_{nc} = \Delta K$$

The non-conservative work takes energy out of the mechanical system and turns it into thermal energy. We observe this by noting that friction makes temperature go up. So w_{nc} will cause a change in thermal energy. Let's give thermal energy a symbol E_{th}, so the change in thermal energy is

$$\Delta E_{th} = -w_{nc}$$

The minus sign may seem mysterious, but remember that to get the same ΔK the conservative work would have to be larger to overcome the non-conservative work due to friction, so really the minus sign is real. Another way to look at this is that the energy due to friction work is lost to the mechanical system. If we have friction, we lose some energy so, like in our ball-hill/valley example, ΔK will be smaller. Since it is a loss, w_{nc} must be negative. Then

$$w_c - \Delta E_{th} = \Delta K$$

Let's also think about what forces are conservative. We have the gravitational force, and spring forces. And for these forces, we have learned that

$$\begin{aligned} \Delta U_g &= -w_g \\ \Delta U_s &= -w_s \end{aligned}$$

that is, the energy stored as potential energy is minus the work done to store the energy. We could split our w_c into specific types of work

$$w_c = w_g + w_s$$

then the work energy theorem would be

$$w_g + w_s - \Delta E_{th} = \Delta K$$

or, using potential energy

$$-\Delta U_g - \Delta U_s - \Delta E_{th} = \Delta K$$

Usually we prefer to not have negative signs, so let's take all the negative terms to the other side of the equation

$$0 = \Delta K + \Delta U_g + \Delta U_s + \Delta E_{th}$$

This is a very exciting equation (really it is!). It tells us that if we find the change in all these energy terms, these changes sum to zero. Another way to write this is

$$0 = K_f - K_i + U_{gf} - U_{gi} + U_{sf} - U_{si} + E_{th_f} - E_{th_i}$$

or

$$K_i + U_{gi} + U_{si} + E_{th_i} = K_f + U_{gf} + U_{sf} + E_{th_f}$$

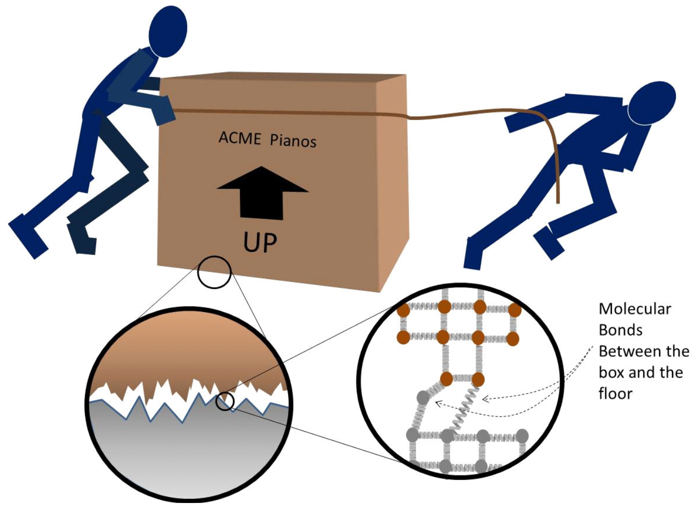
which is conservation of energy again! But this time we have included friction and similar dissipative terms. Note that we have included a term for all the possible types of energy we have studied. In PH220 we will find that there are more conservative forces (e.g. ones for electrical forces) which will also create potential energies. And we can just add these into our conservation of energy equation.

$$K_i + U_{gi} + U_{si} + U_{ei} + E_{th_i} = K_f + U_{gf} + U_{sf} + U_{ef} + E_{th_f}$$

This will become a general procedure, if you have a new kind of energy in your problem, just add it into our conservation of energy equation until all forms of energy for the problem are accounted for.

But is energy always conserved? Is this equation always true?

Let's go back to our man pushing a box.



Some of the thermal energy generated by friction increases the temperature of the box, but some of the energy increases the temperature of the floor. So if we include the floor in our system, energy is conserved, but if we only consider the box, the energy is not conserved. Energy is leaving the box.

In fact, if we just consider the box, the push and pull from the guy and rope are adding energy to the box system. So we have energy coming in and some energy coming out and we would have to account for both of these before we could say that energy is conserved for the box or not. Generally if we allow external forces to act on our system we would have to guess that energy will not be conserved for the system. After all, we are adding or subtracting energy from our system (the box). But if we consider a system that includes more of the environment (say, the box, the guys, the rope, and the floor), it would be more likely that for the larger system energy would be conserved. If a system has no external forces acting on it, we call it an *isolated system*. For isolated systems our conservation of energy equation always works.

But as long as we account for every external energy input and every external energy dissipation in our energy equation, we can use it.

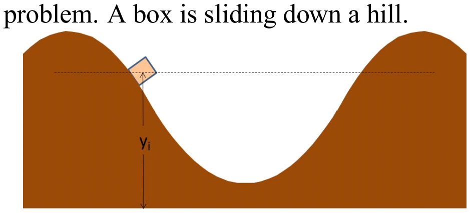
$$K_i + U_{gi} + U_{si} + U_{ei} + U_{push} + E_{th_i} = K_f + U_{gf} + U_{sf} + U_{ef} + E_{th_f}$$

You might think this distinction of external forces and systems is somewhat artificial, after all we are choosing our system and not nature, so, you might ask, if we take the ultimate system, the universe, is energy conserved? We believe that it is![16]

[16] Well, there is some question about universal conservation of energy. If the universe is expanding, there

In doing energy problems, then, we have to be sure to identify all the types of energy that our system has. We can use bar energy graphs and energy vs. position graphs to aid in this process. Then we write our energy equation to include each type. Then we use algebra to solve for the item we are looking for to finish the problem.

Let's try an example problem. A box is sliding down a hill.



The box starts from rest. The hill is $20.\,\mathrm{m}$ high. The box has a mass of $20.\,\mathrm{kg}$. The box is going $10.\,\mathrm{m/\,s}$ at the bottom of the valley. How much energy was lost due to friction?

This is a conservation of energy problem but with friction.

We know

$$\begin{aligned} y_i &= 20\,\mathrm{m} \\ y_f &= 2\,\mathrm{m} \\ v_i &= 0 \\ v_f &= 10.0\frac{\mathrm{m}}{\mathrm{s}} \\ m &= 20.\,\mathrm{kg} \end{aligned}$$

Our basic equation is

$$K_i + U_{gi} + U_{si} + E_{th_i} = K_f + U_{gf} + U_{sf} + E_{th_f}$$

but we don't have any springs, so we can cancel all the spring potential terms (and any other terms for forces we don't have).

$$K_i + U_{gi} + E_{th_i} = K_f + U_{gf} + E_{th_f}$$

with

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ U_g &= mgy \end{aligned}$$

and we know so

$$K_i + U_{gi} = K_f + U_{gf} + E_{th_f} - E_{th_i}$$

needs to be an input of energy to cause the expansion. We don't know where such energy would come from (what is outside the universe?). This is the problem called "dark energy" that you hear about in the press. But for normal systems, energy is conserved.

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f + \Delta E_{th}$$

Let's use our zero

$$0 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f + \Delta E_{th}$$

then

$$mgy_i - mgy_f - \frac{1}{2}mv_f^2 = \Delta E_{th}$$
$$mg\left(y_i - y_f\right) - \frac{1}{2}mv_f^2 = \Delta E_{th}$$
$$\Delta E_{th} = m\left(g\left(y_i - y_f\right) - \frac{1}{2}v_f^2\right)$$

$$\begin{aligned}\Delta E_{th} &= (20.\,\mathrm{kg})\left(\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(20\,\mathrm{m} - 2\,\mathrm{m}) - \frac{1}{2}\left(10.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2\right)\\ &= 2528.0\,\mathrm{J}\end{aligned}$$