Chapter 24

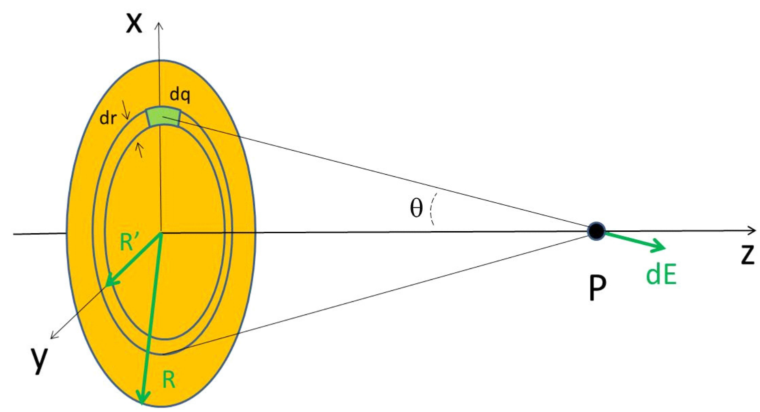
Motion of Charged Particles in Electric Fields

Fundamental Concepts

- The capacitor
- Field of an ideal Capacitor
- Motion of particles in a constant electric field

24.1 Sheet of Charge

Last lecture we found a procedure for calculating the electric field from a distribution of charge. We did one dimensional examples last time. Let's try a two dimensional distribution of charge, a uniform flat sheet of charge. We will assume that the sheet is infinitely large (so we don't have to deal with what happens at the edges). Let's call the surface charge density $\eta = Q/A$ where Q is the total charge and A is the total area. Of course, we can't calculate this surface charge density directly from the totals, because they are infinite. But we could take a square meter of area and find the amount of charge in that small area. The ratio should be the same for any area so long as η is uniform. We will find the electric field to the right of the sheet at point P.



Once again we start with

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int\frac{dq}{r^2}\hat{\mathbf{r}}$$

We need to find dq, an expression for r, and to get rid of $\hat{r}$ by taking components.

Since the disk is uniformly charged, then, knowing the surface charge density

$$\eta = \frac{Q}{A}$$

we can find the total amount of charge for an area

$$Q = \eta A$$

and we can use this to find an expression for a small amount of charge, dq

$$dq = \eta dA$$

But what area, dA, should we use? Notice the green patch in the figure that is marked dA. Think for a moment about arc length

$$s = R\phi$$

This little area is about $ds = Rd\phi$ long, and about dR wide. If we let dA be small enough, this is exact. So

$$dA = Rd\phi dR$$

Then our dq is just η times this

$$dq = \eta Rd\phi dR$$

From geometry we identify that r, the distance from our little charge dq to the point P where we want to know the field, is

$$r = \sqrt{z^2 + R^2}$$

And, due to symmetry we expect only the z-component of the field to survive. So to get rid of $\hat{r}$ we take the z-component by multiplying (dot product) by $\hat{k}$. There will be an angle, θ, between $\hat{r}$ and $\hat{k}$. So we expect the result of the dot product to be that we multiply by the cosine of θ

$$\cos\theta = \frac{z}{\sqrt{z^2 + R^2}}$$

We want to put all this into our basic equation. This time the radius R changes, so let's call it R' so we recognize that it is a variable over which we must integrate, then

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{r^2}\hat{\mathbf{r}}$$

becomes

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_o}\int \frac{\eta R' d\phi dR'}{(z^2 + R'^2)}\hat{\mathbf{r}}\cdot\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_o}\int \frac{\eta R' d\phi dR'}{(z^2 + R'^2)}\cos\theta \end{aligned}$$

and we will integrate from $R' = 0$ to $R' = R$.

$$E_z = \frac{1}{4\pi\epsilon_o}\int_0^{2\pi}\int_0^R \frac{z\eta R' d\phi dR'}{(z^2 + R'^2)^{\frac{3}{2}}}$$

Performing the integration over $d\phi$ just gives us a factor of 2π, leaving us with

$$E_z = \frac{z\eta\pi}{4\pi\epsilon_o}\int_0^R \frac{2R' dR'}{(z^2 + R'^2)^{\frac{3}{2}}}$$

where, for convenience, we have left the 2 inside the integral (it will be useful later).

We need to solve the integral over dR'. Looking up the integral in a table is a good way. But for this one a u-substitution is not too hard. Suppose we let

$$u = z^2 + R'^2$$

so

$$du = 2R' dR'$$

We will need to adjust the limits of integration, for $R' = 0$ we have

$$u = z^2$$

and for $R' = R$

$$u = z^2 + R^2$$

then our integral becomes

$$E_z = \frac{z\pi\eta}{4\pi\epsilon_o} \int_{z^2}^{z^2+R^2} \frac{du}{(u)^{\frac{3}{2}}}$$

We get

$$\begin{aligned}
E_z &= \frac{z\pi\eta}{4\pi\epsilon_o} \left[\frac{-2}{(u)^{\frac{1}{2}}} \right|_{z^2}^{z^2+R^2} \\
&= \frac{z\pi\eta}{4\pi\epsilon_o} \left(\frac{-2}{(z^2+R^2)^{\frac{1}{2}}} - \frac{-2}{(z^2)^{\frac{1}{2}}} \right) \\
&= \frac{-2z\pi\eta}{4\pi\epsilon_o} \left(\frac{1}{(z^2+R^2)^{\frac{1}{2}}} - \frac{1}{z} \right) \\
&= \frac{-2\pi\eta}{4\pi\epsilon_o} \left(\frac{z}{(z^2+R^2)^{\frac{1}{2}}} - 1 \right) \\
&= \frac{-2\pi\eta}{4\pi\epsilon_o} \left(\frac{1}{\frac{1}{z}(z^2+R^2)^{\frac{1}{2}}} - 1 \right)
\end{aligned}$$

The result is

$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o} \left(1 - \frac{1}{\left(1+\frac{R^2}{z^2}\right)^{\frac{1}{2}}} \right)$$

or

$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o} \left(1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}} \right)$$

This looks messy, but this is the answer.

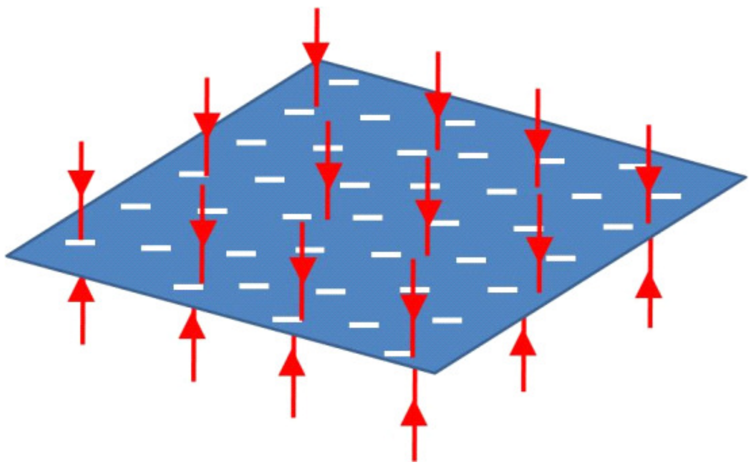
But wait, this is really a disk of charge with radius R. We wanted an infinite sheet of charge. So. suppose we let R get very big. Then

$$\begin{aligned}
E_{R\to\infty} &= \lim_{R\to\infty} \frac{2\pi\eta}{4\pi\epsilon_o} \left(1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}} \right) \\
&= \frac{2\pi\eta}{4\pi\epsilon_o} \\
&= \frac{\eta}{2\epsilon_o}
\end{aligned}$$

This is the field for our semi-infinite sheet of charge. We should take some time to figure out if this makes sense.

This sheet cuts the entire universe into to two parts. It is so big, that it is hard to say anything is very far away from it. So we can understand this answer, The field from such a sheet of charge is constant every where in all of space. No matter how far away we get, it will never look like a point charge, in fact, it never really looks any farther away!

Note we did just one side of the sheet, there is a matching field on the other side. So this sheet of charge fills all of space with a constant field.



Of course this is not physically possible to build, but we will see that if we look at a large sheet of charge, like the plate of a capacitor, that near the center, the field approaches this limit, because the sides of the sheet are far away.

Let's go back and consider the disk of charge.

$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o}\left(1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}}\right)$$

Suppose we look at this distribution from vary far away for a finite disk. We expect that it should look like a point charge with total charge Q. Let's show that this is true. When z gets very large R/z is very small.

$$E_{z\gg R} = \frac{2\pi\eta}{4\pi\epsilon_o}\left(1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}}\right)$$

Let's look at just the part

$$\left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}}$$

This is of the form $(1+x)^n$ where x is a small number. We can use the binomial expansion

$$(1+x)^n \approx 1 + nx \qquad x \ll 1$$

to write this as

$$\left(1+\frac{R^2}{z^2}\right)^{-\frac{1}{2}} \approx 1-\frac{1}{2}\frac{R^2}{z^2}$$

so in the limit that z is large we have

$$\begin{aligned}
E_{z\gg R} &= \frac{2\pi\eta}{4\pi\epsilon_o}\left(1-1+\frac{1}{2}\frac{R^2}{z^2}\right) \\
&= \frac{1}{4\pi\epsilon_o}\frac{\pi\frac{Q}{\pi R^2}R^2}{z^2} \\
&= \frac{1}{4\pi\epsilon_o}\frac{Q}{z^2}
\end{aligned}$$

Which looks like a point charge as we expected. We have just a small, disk of charge very far away. That is looks like a point charge with total charge Q.

24.1.1 Spheres, shells, and other geometries.

I won't do the problem for the field of a charged sphere or spherical shell. We could, but we will save them for a new technique for finding fields from configurations of charge that we will learn soon. This new technique will attempt to make the integration much easier.

24.2 Constant electric fields

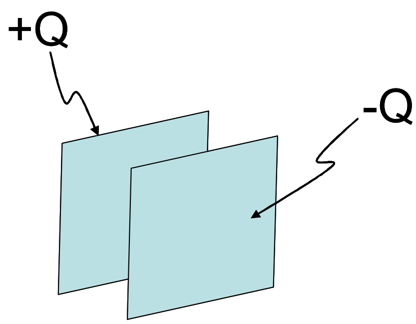
Let's try to use what we know about electric fields to predict the motion of charged particles that are placed in electric fields. We will start with the simplest case, a charged particle moving in a constant electric field. Before we take on such a case, we should think about how we could produce a constant electric field.

We know that a semi-infinite sheet of charge produces a constant electric field. But we realize that a semi-infinite object is hard to build and hard to manage. But if the size of the sheet of charge is very large compared to the charge size, using our solution for a semi-infinite case might not be too bad if we are away from the edges of the real sheet.

We want to study just such a device. In fact we will use two finite sheets of charge.

24.2.1 Capacitors

From what we know about charge and conductors, we can charge a large metal plate by touching it to something that is charged, like a rubber rod, or a glass rod that has been rubbed with the right material.



If we have two large metal plates and touch one with a rubber rod and one with a glass rod, we get two oppositely charged sheets of charge.

What would the field look like for this oppositely charged set of plates? Here is one of our thread-in-oil pictures of just such a situation. We are looking at the plates edge-on. Near the center, the field is close to constant. Near the

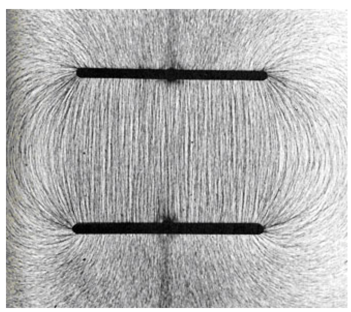
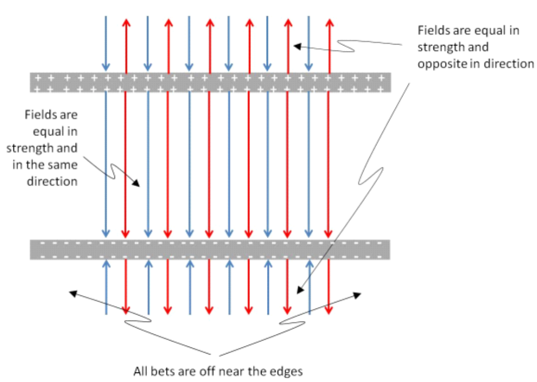


Figure 24.1: http://stargazers.gsfc.nasa.gov/images/geospace_images/electricity/charged_plates.jpg

sides it is not so much so. We are probably justified in saying the field in the middle is nearly constant. A look at the field lines shows us why.



Note that in between the plates, the electric field from the positive plate is downward. But so is the electric field from the negative plate. The two fields will add together. Outside the plates, the field from one plate is in the opposite direction from that of the other plate. The two fields will nearly cancel. If our device is made of semi-infinite sheets of charge, they will precisely cancel, because the field of a semi-infinite sheet of charge is uniform everywhere.

We call this configuration of two charged plates a *capacitor* and, as you might guess, this type of device proves to be more useful than just making nearly constant fields. It is a major component in electronic devices. Before we can build and iPad or a laptop, we will need to understand several different types of basic devices. This set of charged plates is our first.

Of course, for real capacitors, the fields outside cancel completely only near the center of the plates. Near the edges, the direction of the fields will change, and we get the sort of behavior that we see in figure 24.1 near the edges.

It is probably worth noting that outside the capacitor the field has a *magnitude of zero* (or nearly zero). It is not really correct to say that there is no field. In fact, there are two superimposed fields, or alternately, a field from each of the charges on each plates, all superimposed. The fields are there, but their magnitude is zero.

In the middle, then, we will have

$$\begin{aligned} E &= E_{+} + E_{-} \\ &\approx \frac{\eta}{2\epsilon_o} + \frac{\eta}{2\epsilon_o} \\ &= \frac{\eta}{\epsilon_o} \\ &= \frac{Q}{A\epsilon_o} \end{aligned}$$

24.3 Particle motion in a uniform field

Now that we have a way to form a uniform electric field, we can study charged particles moving in this field. Motion of particles in uniform fields is really something we are familiar with. It is very much like a ball in a uniform gravitational field. But we have the complication of having two different types of charge. The force on such a particle is given by

$$\overrightarrow{\mathbf{F}} = q_m \overrightarrow{\mathbf{E}}$$

But we can combine this with Newton's second law

$$\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}}$$

to find the particle's acceleration

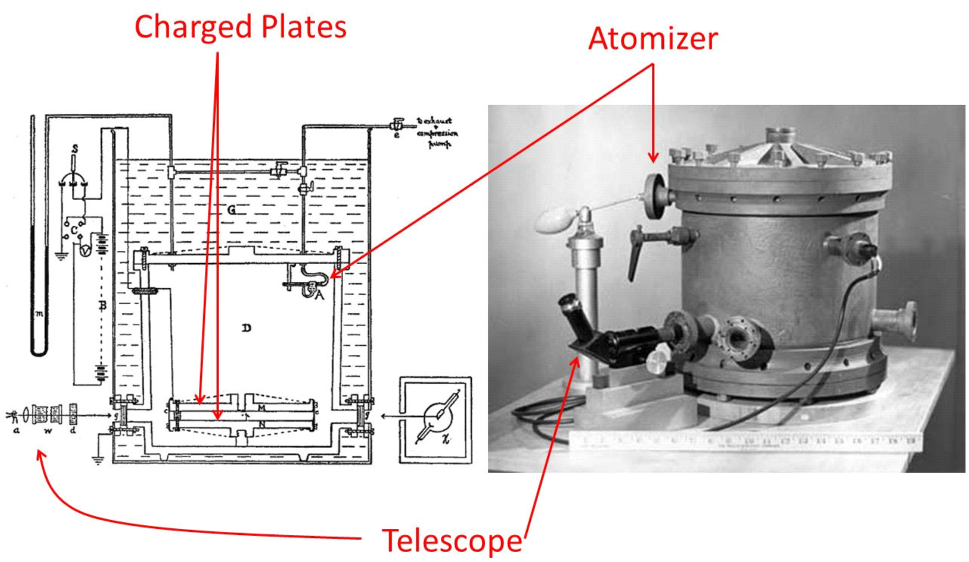
$$\overrightarrow{\mathbf{a}} = \frac{q_m \overrightarrow{\mathbf{E}}}{m}$$

Note, this is NOT true in general. It is only true for constant electric fields.

24.3.1 Millikan

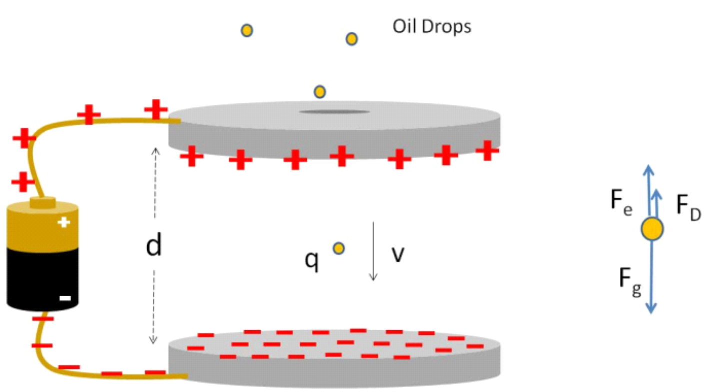
Let's try a problem. Perhaps you have wondered, "how do we know that charge comes in packets of the size of the electron charge?" This is a good story, and uses many of the laws we have learned.

Robert Millikan devised a clever device in the early 1900's. A picture of his device is given below.



Millikan's oil-drop apparatus: Diagram taken from orginal Millikan's paper, 1913, Image taken in 1906 (Both Images in the Public Domain)

Schematically we can draw the experiment like this.



Millikan made negatively charged oil drops with an atomizer (fine spray squirt bottle). The drops are introduced between two charged plates into what we know is essentially a constant electric field. A light shines off the oil drops, so you can see them through a telescope (not shown). We can determine the motion of the oil drops just like we did in PH121 or Dynamics. If the upper plate has the positive charge, then the electric field $\overrightarrow{\mathbf{E}}$ is downward. A free body diagram for a drop is shown in the figure to the left of the apparatus. We can write out Newton's second law for the drop (our mover charge).

$$\Sigma F_y = m_d a_y = -F_g \pm F_D + F_e$$

where F_D is a drag force because we have air resistance.

If the upper plate has the positive charge, then the electric field $\tilde{\mathbf{E}}$ is downward. So

$$\overrightarrow{\mathbf{F}}_e = -q_m \overrightarrow{\mathbf{E}}$$

The field points down, the charge is negative, so the force is upward (positive in our favorite coordinate system). We can write newtons's second law as

$$m_d a_y = -F_g \pm F_D + q_m E$$

If F_e is large enough, we can make the oil drop float up! Then the drag force is downward

$$m_d a_y = -mg - F_D + q_m E$$

and if we are very careful, we can balance these forces so we have the drop float upward at a small constant velocity.

$$0 = -mg - F_D + q_m E$$

The constant speed is really slow, hundredths of a centimeter per second. so we can watch the drop move with no problem (except for patience). Once he achieved a constant speed, by knowing the drop size and density Millikan could calculate the mass, and therefore the charge.

$$mg + F_D = q_m E$$

we see that

$$q_m = \frac{mg + F_D}{E}$$

Which is where our problem ends. But Millikan went farther. He had actual data, so he could compare charges on different droplets. He found that no matter what the value for q_m, it was a multiple of a value, $q_e = 1.602 \times 10^{-19}$ C. So

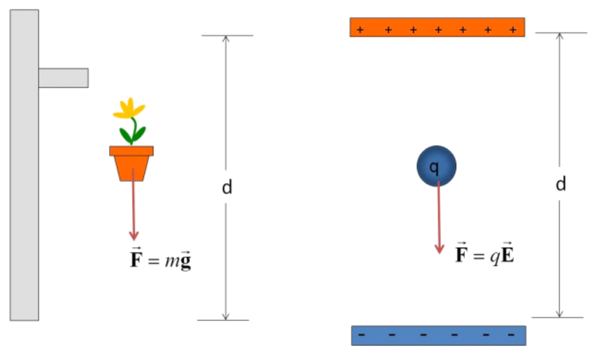
$$q_m = nq_e \qquad n = 0, \pm 1, \pm 2, \ldots$$

to within about 1%.[1]. So the smallest charge the drops could have added to them was $1 \times q_e$ and any other larger charge would be a larger multiple of q_e. The

[1] There is actually some controversy about this. Apparently Millikan and his students threw out much of their data, keeping only data on drops that behaved like they thought they should. They were lucky that this poor analysis technique did not lead to invalid results! (William Broad and Nicholas Wade, *Betrayers of the truth,* Simon and Schuster, 1983)

conclusion is that charge comes in units of q_e. We recognize q_e as the electron charge. You can't add half of an electron charge. This experiment showed that charge seems to only comes in whole units!

24.3.2 Free moving particles



We may recall that for an object falling in a gravitational field, say, near the Earth's surface, the acceleration, g, is nearly constant. If we have a charge moving in a constant electric field, we have a constant acceleration. From Newtons' second law,

$$ma = q_m E$$

we can see that this acceleration is

$$a = \frac{q_m E}{m}$$

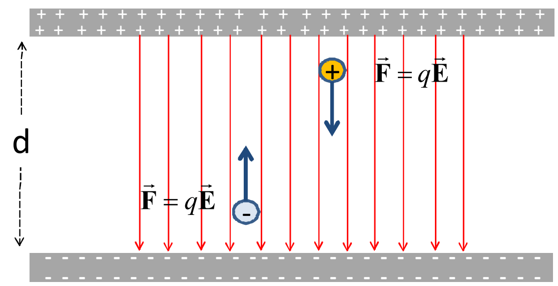
From our Dynamics or PH121 experience, we have a set of equations to handle problems that involve constant acceleration

$$\begin{aligned}
x_f &= x_i + v_{ix}\Delta t + \frac{1}{2}a_x \Delta t^2 \\
v_{fx} &= v_{ix} + a_x \Delta t \\
v_{fx}^2 &= v_{ix}^2 + 2a_x(x_f - x_i) \\
x_f &= x_i + \frac{v_{fx} + v_{ix}}{2}\Delta t
\end{aligned}$$

and

$$\begin{aligned}
y_f &= y_i + v_{iy}\Delta t + \frac{1}{2}a_y \Delta t^2 \\
v_{fy} &= v_{iy} + a_y \Delta t \\
v_{fy}^2 &= v_{iy}^2 + 2a_y(y_f - y_i) \\
y_f &= y_i + \frac{v_{fy} + v_{iy}}{2}\Delta t
\end{aligned}$$

These are know as the *kinematic equations*. You derived them if you took Dynamics (or derived them and then memorized them if you took PH121). Let's try a brief problem. Suppose we have a positive charge in a uniform electric field as shown.



Let $y = 0$ at the positive plate. How fast will the particle be going as it strikes the negative plate?

We use the acceleration

$$\begin{aligned} a_y &= \frac{q_m E}{m} \\ a_x &= 0 \end{aligned}$$

For this problem we don't have any x-motion, So we can limit ourselves to.

$$\begin{aligned} y_f &= y_i + v_{iy}\Delta t + \frac{1}{2}a_y \Delta t^2 \\ v_{fy} &= v_{iy} + a_y \Delta t \\ v_{fy}^2 &= v_{iy}^2 + 2a_y(y_f - y_i) \\ y_f &= y_i + \frac{v_{fy} + v_{iy}}{2}\Delta t \end{aligned}$$

We don't have the time of flight of the particle, but we can identify

$$\Delta y = y_f - y_i = d$$

The particle started from rest, so

$$v_{yi} = 0$$

Therefore it makes sense to use the third of the three equations, because we know everything that shows up in this equation but the final speed, and that is what we want to find.

$$\begin{aligned} v_{yf}^2 &= v_{yi}^2 + 2\left(\frac{q_m E}{m}\right)\Delta y \\ v_{yf}^2 &= 0 + 2\left(\frac{q_m E}{m}\right) d \\ v_{yf} &= \sqrt{\frac{2q_m E d}{m}} \end{aligned}$$

There is a complication, however. With gravity, we only have one kind of mass. But with charge we have two kinds of charge. Suppose we have a negative particle.

Of course the negative particle would not move if it was started from the positive side. It would be attracted to the positive plate. But suppose we start the negative particle from the negative plate. It would travel "up" to the positive plate. We defined the downward direction as the positive y-direction without really thinking about it. Now we realize that the upward direction must be opposite, so upward is the negative y-direction. Our negative particle will experience a displacement $\Delta y = -d$.

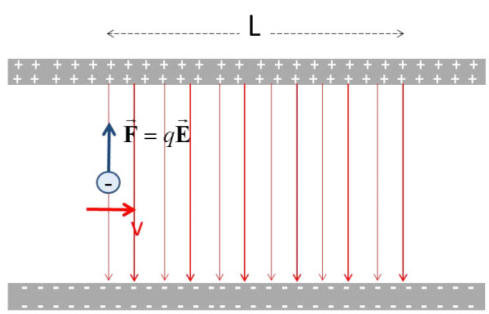
Then

$$\begin{aligned} v_{yf}^2 &= v_{yi}^2 + 2\left(\frac{-q_m E}{m}\right)\Delta y \\ v_{yf}^2 &= 0 + 2\left(\frac{-q_m E}{m}\right)(-d) \\ v_{yf} &= \sqrt{\frac{2q_m E d}{m}} \end{aligned}$$

we get the same speed, but this illustrates that we will have to be careful to watch our signs.

In this last problem we have had only an electric force, no gravitational force. This is important to notice. If there were also a gravitational force, we would need to use Newton's second law to add up the forces like we did with the Millikan problem.

Let's take another example. This time let's send in a negatively charged particle horizontally through the capacitor. The particle will move up due to the electric field force. How far up will it go as it travels across the center of the capacitor?



Let's define the starting position as

$$\begin{aligned} x_i &= 0 \\ y_i &= 0 \end{aligned}$$

We can identify that

$$\begin{aligned} v_{ix} &= v_0 \\ v_{iy} &= 0 \end{aligned}$$

And that

$$\begin{aligned} a_y &= \frac{q_m E}{m} \\ a_x &= 0 \end{aligned}$$

We can fill in these values in our kinematic equations

$$\begin{aligned} x_f &= 0 + v_{xi}t + \frac{1}{2}(0)\Delta t^2 \\ v_{fx} &= v_{ix} + (0)t \\ v_{xf}^2 &= v_{ix}^2 + 2(0)\Delta x \end{aligned}$$

and

$$\begin{aligned} y_f &= 0 + (0)\Delta t + \frac{1}{2}\left(\frac{q_m E}{m}\right)\Delta t^2 \\ v_{yf} &= (0) + \left(\frac{q_m E}{m}\right)\Delta t \\ v_{yf}^2 &= (0) + 2\left(\frac{q_m E}{m}\right)(y_f - 0) \end{aligned}$$

From the first set we see that $v_{fx} = v_{ix}$, that is, the x-direction velocity does not change. That makes sense because we have no force component in the x-direction.

After t seconds we see that the charged particle has traveled a distance

$$x_f = v_{xi}t$$

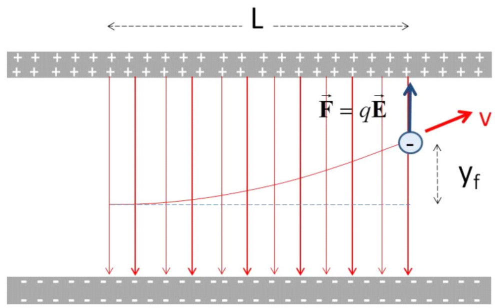
If we measure $x_f = L$ then we can see how long it took for the particle to travel through the capacitor

$$t = \frac{L}{v_{ix}}$$

Now let's look at the deflection. We can use the first equation of the y-set

$$\begin{aligned} y_f &= \frac{1}{2}\left(\frac{q_m E}{m}\right)t^2 \\ &= \frac{1}{2}\left(\frac{q_m E}{m}\right)\left(\frac{L}{v_{ix}}\right)^2 \end{aligned}$$

Let's see if this makes sense. If the electric field gets larger, the particle will deflect more.



This is right. The field causes the force, so more field gives more effect from the force. If we increase the charge, the deflection grows since the force depends on the charge of the moving particle. This also seems reasonable. If the mass increases, it is harder to move the particle, so it makes sense that a larger mass makes a smaller deflection. If the particle is in the field longer, the deflection will increase, so the dependence on L makes sense. Finally, if the initial speed is larger the particle spends less time in the field, so the deflection will be less.

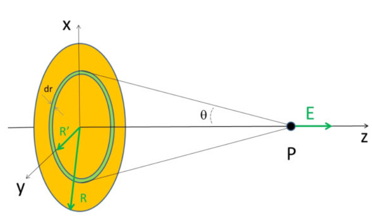
Of course all of this depends on the field being uniform. For a non uniform field the force is still

$$\overrightarrow{\mathbf{F}} = q_m \overrightarrow{\mathbf{E}}\,(x, y, z)$$

but now the field is a function of position. This makes for a more difficult problem. For now we will stick to constant fields. If we had to take on a non-uniform field, we would likely use a numerical technique.

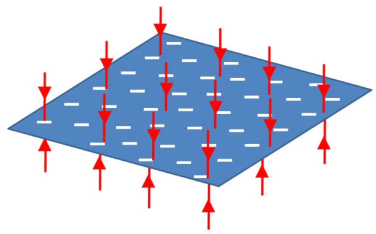
Basic Equations

The magnitude of the electric field due to a disk of charge along the disk's axis



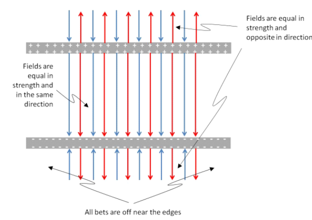
$$E_z = \frac{2\pi\eta}{4\pi\epsilon_o}\left(1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}}\right)$$

The magnitude of the electric field due to a semi-infinite sheet of charge



$$E = \frac{\eta}{2\epsilon_o}$$

The magnitude of the electric field inside an ideal capacitor



$$E = \frac{Q}{A\epsilon_o}$$

Motion of a charged particle in a constant electric field

$$\vec{\mathbf{a}} = \frac{q_m \vec{\mathbf{E}}}{m}$$

It's time to put the kinematic equations back on your equation sheet

$$\begin{array}{ll}
x_f = x_i + v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & y_f = y_i + v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\
v_{xf} = v_{xi} + a_x\Delta t & v_{yf} = v_{yi} + a_y\Delta t \\
v_{xf}^2 = v_{xi}^2 + 2a_x\left(x_f - x_i\right) & v_{yf}^2 = v_{yi}^2 + 2a_y\left(y_f - y_i\right) \\
x_f = x_i + \frac{v_{xf}+v_{xi}}{2}\Delta t & y_f = y_i + \frac{v_{yf}+v_{yi}}{2}\Delta t
\end{array}$$