PH223: Physics for Chemists and Mechanical Engineers

R. Todd Lines
Brigham Young University-Idaho

Preface

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This document contains my lecture notes for a new, experimental course. The goal of the course is to teach the introductory physics of waves, optics, and electricity and magnetism for mechanical engineering students.

Forward

Acknowledgments

BYU-I *R. Todd Lines*.

Contents

1 Where We Start

Fundamental Concepts

What is this class?

This class is designed to teach the physics of wave motion, electricity and magnetism, and optics. We have three major goals. One is to teach the physics that is not covered by Statics, Dynamics, and the Engineering Electronics Course. This physics can affect the mechanical systems you will design, build, or test, so knowing this physics is a very good thing. The second objective is to teach a different method of thinking about how things work. The third goal is to describe electrical and wave motion enough that the quantum nature of atoms and molecules make sense as our chemists take physical chemistry.

In engineering, the design parameters are often the goal. In physics, the physical relationship is the goal. For design engineers, both views are useful and important. The design is no good if the underlying principles preclude it from working!

As an example, I once worked on an optics project with a strong mechanical component. The system had scanning mechanisms that were fantastic mechanical devices. It was part of an aircraft and integrated into the aircraft system. But the optical system required two lasers that were separated in wavelength by only a few nanometers. The chief engineer knew how to build all the systems, but did not understand the physics that required the close wavelength spacing. He judged that the difficulty in building the device at that wavelength spacing outweighed any benefit, and he changed the specs to give two wavelengths that were fifty nanometers apart. Fifty nanometers is a pretty small tolerance. Surely it would be good enough! The resulting product did not work. For two years he tried to fine tune the scanners, and servos to make it work. After ten million dollars and two years, he finally moved the wavelengths closer. The cost of the change was an extra $100,000 dollars, about 1/100 of the cost of the mistake. The system worked, but since this was a race to market, the time lost and the reputation lost on the faulty product destroyed the viability of the business. It is a bad day when you and your friends lose your jobs because you made a fundamental physics mistake!

Physics courses stress how we know what we know. They support the discipline called *system engineering*, which deals with the design of new and innovative products. As a more positive example, the National Weather service often releases requests for proposed weather sensing equipment. Their request might say something like the following:

> Measure the moisture of the soil globally from an altitude of $800\,\mathrm{km}$ with an accuracy of 5%. The suggested instrument is a passive microwave radiometer.

The job of a system engineer is to determine what type of instrument to build. What is the underlying principle that it will use to do its job? What signal processing will it need? What mechanical and electrical systems will support this? This must all be determined before the bearings and slip-rings, and structures can be designed and built.

The radiometer design that came out of this project is flying today (or one very like it based on the original design) and is a major part of the predictive models that tell us what the weather will be in a few days.

Because this type of reasoning is our goal, we will not only do typical homework problems, but we will also work on our conceptual understanding.

I will also emphasize a problem solving method that I used with my engineering team in industry. It is a structured approach to finding a solution that emphasizes understanding as well as providing a numeric answer for a particular design. When you are part of an innovative design team, you will have to repeat a calculation over and over again each time some other part of the sign changes. If you have produced a symbolic solution, a numerical model, or at least a curve, you are ready for any changes in specifications. But if you have just “found the answer” you will have to find that answer again every time the overall design specs change. This approach is too slow, and, at least in my team, would have you finding a new job because our design efforts were always done against exacting schedules and budgets. By thinking in a structured method, with an eye toward symbolic answers or relationships rather than end numbers, you will learn to be a more valuable engineer. The process we will use is the same approach I used to teach my new engineers in the defense industry. It has been proven useful over and over for decades.

This same problem solving process is useful in chemistry, particularly as you study physical chemistry.

So let's get started. To understand waves, we need to get the waves moving. You studied Oscillation in Dynamics or PH121. Oscillating systems are often the disturbance that starts a wave. We will begin with a review of oscillation.

Simple Harmonic Motion

You are, no doubt, an expert in simple harmonic motion (SHM) after your PH121 or Dynamics class. But this will get us warmed up for the semester. In class we will use our clickers and go through a few questions. We will usually use the clicker system to answer a few questions to test your understanding of the reading material. This allows me to not waste time on things you already know, and to help me find the ones you don't. Most lectures will consist of me asking you if you have questions, and then if you don't, I will ask you "clicker questions." Where there is reason to believe you don't understand (with a normal cutoff of 80% of the class answering correctly being our definition of "understanding"), I will use the material from these written lectures to teach the concepts. So we won't always go through all the ideas and skills demonstrated in these written lectures. If you feel you would have liked more explanation on something but we did not cover that concept in class because most people were "getting it," you can come and see me in my office.

SHM

Let's consider a mass attached to a spring resting on a frictionless surface[1]. This mass-spring system can oscillate.

In the position shown the spring is neither pushing nor pulling on the mass. We will call this position the *equilibrium position* for the mass.

Question 223.1.1

Question 223.1.2

QQuestion 223.1.3

Question 223.1.4

Question 223.1.5

Question 223.1.6

Question 15.3.8

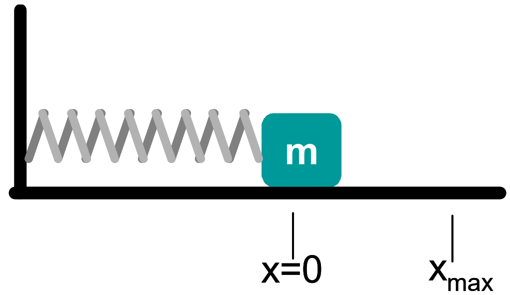
Question 15.3.9.2

Question 15.3.9.3

Question 15.3.9.5

Question 15.4

[1] Yes, I know there are no actual frictionless surfaces, but we are starting out at freshman level physics, so we will make the math simple enough that a freshman could do it by making simplifying assumptions. In this case, that the surface if frictionless.



Definition 1.1 *Equilibrium Position: The position of the mass when the spring is neither stretched nor compressed.*

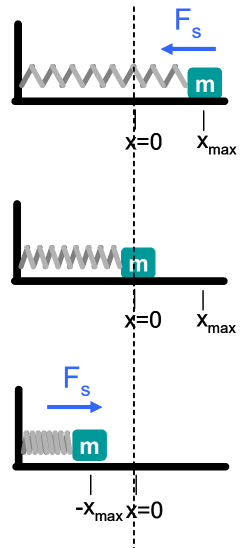
Hooke's Law

A law in physics is a mathematical expression of a mental model of how the universe works. Long ago it was noticed that the pull of a spring grew in strength as the spring was pulled out of equilibrium. The mathematical expression of this is

$$F_s = -kx \tag{1.1}$$

The force, F_s is directly proportional to the displacement from equilibrium, x. Since a man named Hooke wrote this down, it is called Hooke's law.

Hooke's Law is, strictly speaking, not a law that is always obeyed. It is a good model for most springs as long as we don't stretch them too far. We will often use the word "law" to mean *an equation that gives a basic relationship.* In that sense, Hook's law is a law.



Lets write Hooke's Law using Newton's second Law

$$\Sigma F_x = ma_x$$

If we assume no friction, we have just

$$-kx = ma_x$$

We can write this as

$$a_x = -\frac{k}{m}x \tag{1.2}$$

This expression says the acceleration is directly proportional to the position, and opposite the direction of the displacement from equilibrium. We can see that the spring force tries to oppose the change in displacement. We call such a force a *restoring force*.

Definition 1.2 *Restoring force: A force that is always directed toward the equilibrium position*

This is a good definition of *simple harmonic motion.*

Mathematical Representation of Simple Harmonic Motion

Recall from your Dynamics or PH121 classes that acceleration is the second derivative of position

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Hook's Law tells us

$$\begin{aligned} F &= ma = -kx \\ m\frac{d^2x}{dt^2} &= -kx \end{aligned}$$

We have a new kind of equation. If you are taking this freshman physics class as a... well... freshman, you may not have seen this kind of equation before. It is called a differential equation. But really the chances are that you are a sophomore or junior (or even a senior) and have lot of experience with differential equations. The solution of this equation is a function or functions that will describe the motion of our mass-spring system as a function of time. We will need to know this function, so let's see how we can find it.

Start by defining a quantity ω as

$$\omega^2 = \frac{k}{m} \tag{1.3}$$

why define ω^2? Because experience has shown that it is useful to define ω this way! But you probably remember ω as having to do with rotational speed, and from trigonometry (trig) you may remember using ω to mean angular frequency

$$\omega = 2\pi f$$

so our definition of ω may hint that k/m will have something to do with the frequency of oscillation of the mass-spring system.

We can write our differential equation as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{1.4}$$

To solve this differential equation we need a function who's second derivative is the negative of itself. We know a few of these

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \qquad (1.5) \\ x(t) &= A\sin(\omega t + \phi_o) \end{aligned}$$

where A, ω, and ϕ_o are constants that we must find. Let's choose the cosine function and explicitly take its derivatives.

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ \frac{dx(t)}{dt} &= -\omega A \sin(\omega t + \phi_o) \\ \frac{d^2x(t)}{dt^2} &= -\omega^2 A \cos(\omega t + \phi_o) \end{aligned}$$

Let's substitute these expressions into our differential equation for the motion

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ -\omega^2 A\cos(\omega t + \phi_o) &= -\omega^2 A \cos(\omega t + \phi_o) \end{aligned}$$

As long as the constant ω^2 is our $\omega^2 = k/m$ we have a solution (now you know why we defined it as ω^2!). Since from trig we remember ω as the angular frequency.

$$\omega = 2\pi f$$

Thus

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f \tag{1.6}$$

The frequency of oscillation depends on the mass and the stiffness of the spring.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.7}$$

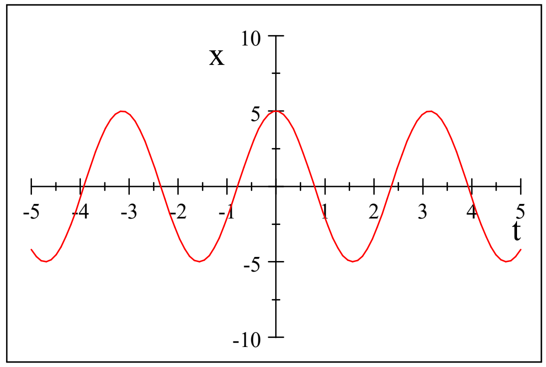
Let's see if this is reasonable. Imagine driving along in your student car (say, a 1972 Gremlin). You go over a bump, and the car oscillates. Your car is a mass, and your shock absorbers are springs. You have an oscillation. But suppose you load your car with everyone in your apartment[2]. Now as you hit the bump the car oscillates at a different frequency, a lower frequency. That is what our frequency equation tells us. Note also that if we changed to a different set of shocks, the k would change, and we would get a different frequency.

We still don't have a complete solution to our differential equation, because we don't know A and ϕ_o. From trigonometry, we recognize ϕ_o as the initial phase angle. We will call it the *phase constant* in this class. We will have to find this by knowing the initial conditions of the motion. We will do this in a minute.

A is the amplitude. We can find its value when the motion has reached its maximum displacement. Let's look at a specific case

$$A = 5$$
$$\phi_o = 0$$
$$\omega = 2$$

[2] If you are married, imagin taking two other couples with you in your car.



We can easily see that the amplitude A corresponds to the maximum displacement x_{max}.

Other useful quantities we can identify

We know from trigonometry that a cosine function has a period T.

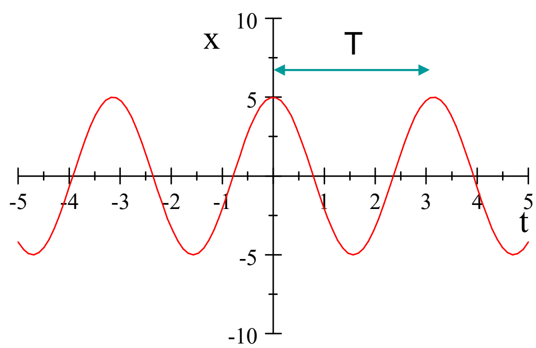


Figure 1.1.

The period is related to the frequency

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{1.8}$$

We can write the period and frequency in terms of our mass and spring constant

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.9}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.10}$$

Velocity and Acceleration

Since we know the derivatives of

$$x(t) = A\cos(\omega t + \phi_o) \tag{1.11}$$

we can identify the velocity of the mass and its acceleration

$$v(t) = \frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o)$$

Recall that $A = x_{\max}$

$$v(t) = \frac{dx(t)}{dt} = -\omega x_{\max} \sin(\omega t + \phi_o) \tag{1.12}$$

We identify

$$v_{\max} = \omega x_{\max} = x_{\max}\sqrt{\frac{k}{m}} \tag{1.13}$$

Likewise for the acceleration

$$\begin{aligned} a(t) &= \frac{dv(t)}{dt} && (1.14) \\ &= \frac{d}{dt}\left(-\omega x_{\max}\sin(\omega t + \phi_o)\right) && (1.15) \\ &= -\omega^2 x_{\max}\cos(\omega t + \phi_o) \end{aligned}$$

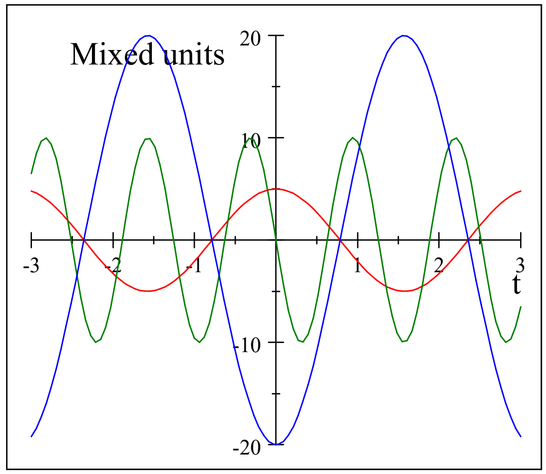
where we can identify

$$a_{\max} = \omega^2 x_{\max} = \frac{k}{m} x_{\max} \tag{1.16}$$

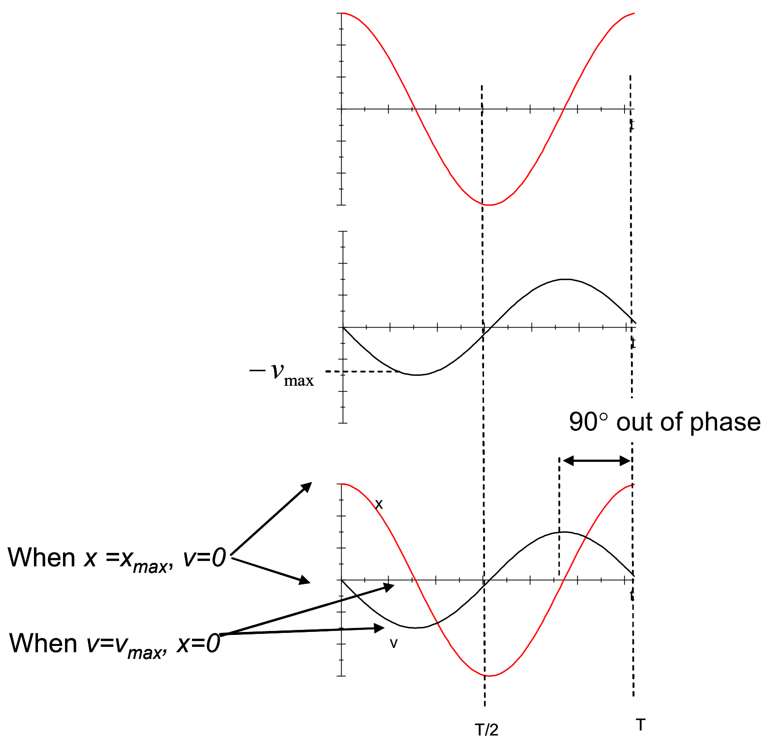
Comparison of position, velocity, acceleration

Don't do in class

Let's plot $x(t)$, $v(t)$, and $a(t)$ for a specific case



Red is the displacement, green is the velocity, and blue is the acceleration. Note that each has a different maximum amplitude. That is a bit confusing until we recognize that they each have different units. We have just plotted them on the same graph to make it easy to compare their phases. Note that they are not in phase!



The acceleration is 90° out of phase from the velocity.

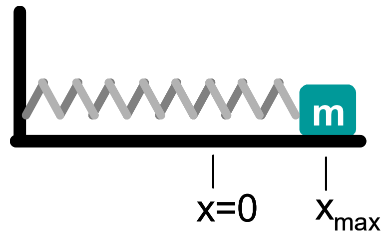
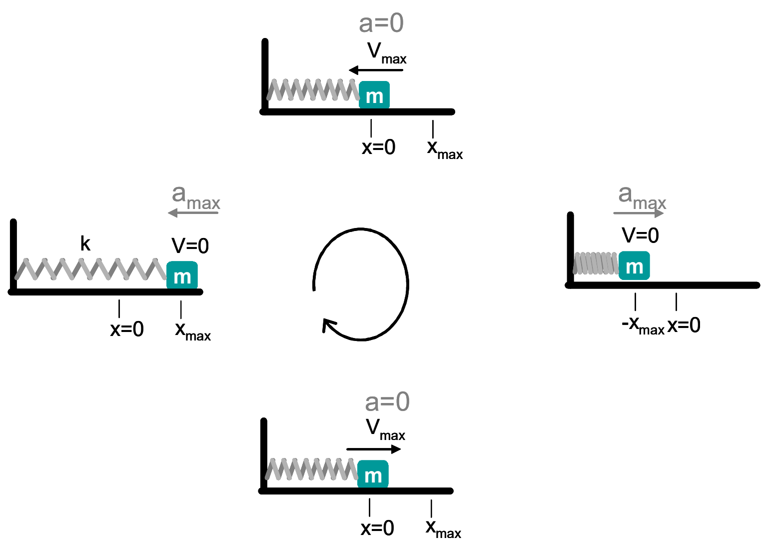


Figure 1.2.



An example of oscillation

We want to see how to find A, ω, and especially ϕ_o. These quantities will be important in our study of waves. So let's do a problem.

Let's take as our system a horizontal mass-spring system where the mass is on a frictionless surface.

Initial Conditions

Now let's find A and ϕ_o. To do this we need to know how we started the mass-spring motion. We call the information on how the system starts it's motion the *initial* *conditions.*

Suppose we start the motion by pulling the mass to $x = x_{\max}$ and releasing it at $t = 0$. These our our initial conditions. Let's see if we can find the phase. Our initial conditions require

$$\begin{aligned} x(0) &= x_{\max} \\ v(0) &= 0 \end{aligned} \tag{1.17}$$

Using our formula for $x(t)$ and $v(t)$ we have

$$\begin{aligned} x(0) &= x_{\max} = x_{\max}\cos(0+\phi_o) \\ v(0) &= 0 = -v_{\max}\sin(0+\phi_o) \end{aligned} \tag{1.18}$$

From the first equation we get

$$1 = \cos(\phi_o)$$

which is true if

$$\phi_o = 0, 2\pi, 4\pi, \cdots$$

from the second equation we have

$$0 = \sin\phi_o$$

which is true for

$$\phi_o = 0, \pi, 2\pi, \cdots$$

If we choose $\phi_o = 0$, these conditions are met. Of course we could choose $\phi_o = 2\pi$, or $\phi_o = 4\pi$, but we will follow the rule to take the smallest value for ϕ_o that meets the initial conditions.

A second example

Using the same equipment, let's start with

$$\begin{aligned} x(0) &= 0 \\ v(0) &= +v_i \end{aligned} \tag{1.19}$$

that is, the mass is moving, and we start watching just as it passes the equilibrium point.

$$\begin{aligned} x(0) &= 0 = x_{\max}\cos(0+\phi_o) \\ v(0) &= v_i = -v_{\max}\sin(0+\phi_o) \end{aligned} \tag{1.20}$$

from

$$0 = x_{\max}\cos(\phi_o)$$

(first equation above) we see that[3]

$$\phi_o = \pm\frac{\pi}{2}$$

but we don't know the sign. Using our initial velocity condition

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\pm\frac{\pi}{2}\right) \\ v_i &= -\omega x_{\max}\sin\left(\pm\frac{\pi}{2}\right) \end{aligned}$$

We defined the initial velocity as positive, and we insist on having positive amplitudes, so $x_{\max}$ is positive. Thus we need a minus sign from $\sin(\phi_o)$ to make v_i positive. This tells us to choose

$$\phi_o = -\frac{\pi}{2}$$

with a minus sign.

Our solutions are

$$\begin{aligned} x(t) &= \frac{v_i}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right) \\ v(t) &= v_i\sin\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$

Remark 1.1 *Generally to have a complete solution to a differential equation, you must find all the constants (like A and ϕ_o) based on the initial conditions.*

A third example

So far we have concentrated on finding ϕ_o. Let's do a more complete example where we find ϕ_o, A, and ω.

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is $4.00\,\mathrm{cm}$, and the frequency is $1.50\,\mathrm{Hz}$.

a) show that the position of the particle is given by

$$x = (4.00\,\mathrm{cm})\sin(3.00\pi t)$$

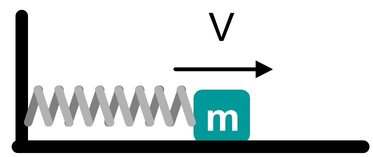
determine

b) the maximum speed and the earliest time $(t > 0)$ at which the particle has this speed,

[3] Really there are more possibilities, but we are taking the smallest value for ϕ_o as we discussed above.

c) the maximum acceleration and the earliest time $(t > 0)$ at which the particle has this acceleration, and

d) the total distance traveled between $t = 0$ and $t = 1.00\,\mathrm{s}$



Type of problem

We can recognize this as an oscillation problem. This leads us to a set of basic equations

Basic Equations

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

We should write down what we know and give a set of variables

Variables

t	time, initial time =0	$t_o = 0$
x	Position, Initial position =0	$x(0) = 0$
v		
a		
x_m	x amplitude	$x_m = 4.00\,\mathrm{cm}$
v_m	v amplitude	
a_m	a amplitude	
ω	angular frequency	
ϕ_o	phase	
f	frequency	$f = 1.50\,\mathrm{Hz}$

Now we are ready to start solving the problem. We do this with algebraic symbols first

Symbolic Solution

Part (a)

We can start by recognizing that we can find ω because we know the frequency. We just use the basic equation.

$$\omega = 2\pi f$$

We also know the amplitude $A = x_{\max}$ which is given. Knowing that

$$x(0) = 0 = A\cos(0 + \phi_o)$$

we can guess that

$$\phi_o = \pm\frac{\pi}{2}$$

Using

$$v(0) = -\omega x_{\max}\sin\left(0 \pm \frac{\pi}{2}\right)$$

again and demanding that amplitudes be positive values, and noting that at $t = 0$ the velocity is positive from the initial conditions:

$$\phi_o = -\frac{\pi}{2}$$

We also note from trigonometry that

$$x(t) = x_{\max}\cos\left(2\pi f t - \frac{\pi}{2}\right)$$

which is a perfectly good answer. However, if we remember our trig, we could write this using

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

Then we have

$$\begin{aligned} x(t) &= x_{\max}\cos\left(2\pi f t - \frac{\pi}{2}\right) \\ &= x_{\max}\sin(2\pi f t) \end{aligned}$$

Part (b)

We have a basic equation for $v_{\max}$

$$\begin{aligned} v_m &= \omega x_{\max} \\ &= 2\pi f x_{\max} \end{aligned}$$

to find when this happens, take

$$v(t) = v_{\max} = -\omega x_{\max} \sin\left(2\pi f t - \frac{\pi}{2}\right)$$

and recognize that $\sin(\theta) = 1$ is at a maximum when $\theta = \pi/2$ so the entire argument of the sine function must be $\pi/2$ when we are at the maximum displacement, so

$$\frac{\pi}{2} = \left(2\pi f t - \frac{\pi}{2}\right)$$

or

$$\pi = 2\pi f t$$

then the time is

$$\frac{1}{2f} = t$$

Part (c)

Like with the velocity we must use a basic formula, this time

$$a(t) = -\omega^2 A \cos(\omega t + \phi_o)$$

but recognize that the maximum is achieved when $\cos(\omega t + \phi_o) = 1$ or when $\omega t + \phi_o = 0$

$$\begin{aligned} t &= \frac{\phi_o}{\omega} \\ &= \frac{-\frac{\pi}{2}}{2\pi f} \\ &= \frac{-1}{4f} \end{aligned}$$

The formula for $a_{\max}$ is

$$\begin{aligned} a_{\max} &= -\omega^2 x_{\max} \\ &= -(2\pi f)^2 x_m \end{aligned}$$

Part (d)

We know the period is

$$T = \frac{1}{f}$$

We should find the number of periods in $t = 1.00\,\mathrm{s}$

$$N_{periods} = \frac{t}{T}$$

and find the distance traveled in one period, and multiply them together. In one period the distance traveled is

$$d = 4x_m$$

$$d_{tot} = d * \frac{t}{T} = 4fx_m t$$

Numerical Solutions

We found algebraic answers (or symbolic answers) to the parts of our problem above. We will always do this first. Then substitute in the numbers to find numeric answers.

Part (a)

$$\begin{aligned} x(t) &= x_{\max} \sin(2\pi f t) \\ &= (4.00\,\mathrm{cm}) \sin(3.00\pi t) \end{aligned}$$

Part (b)

$$\begin{aligned} v_m &= 2\pi\,(1.50\,\mathrm{Hz})\,(4.00\,\mathrm{cm}) \\ & \quad 0.377 \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

$$\frac{1}{2f} = t$$

$$\begin{aligned} \frac{1}{2\,(1.50\,\mathrm{Hz})} &= t \\ &= 0.333 \text{ s} \end{aligned}$$

Part (c)

$$\begin{aligned} t &= \frac{-1}{4f} \\ &= -0.166\,67\,\mathrm{s} \end{aligned}$$

$$\begin{aligned} a_{\max} &= (2\pi f)^2 x_m \\ &= (2\pi 1.5\,\mathrm{Hz})^2\,(4.00\,\mathrm{cm}) \\ &= 3.\,553\,1\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Part (d)

$$\begin{aligned} d_{tot} &= 4 f x_m t \\ &= 4 \times 4.00\,\mathrm{cm} * 1.50\,\mathrm{Hz} * 1.00\,\mathrm{s} \\ &= 0.24\,\mathrm{m} \end{aligned}$$

We should make sure the units check. We put in units along the way, so we can be confident that they do. But if you did not work along the way with units, check them now.

We should also make sure our answers are reasonable. If the amplitude came out to be a billion miles, you might guess something went wrong. Always look over your answers to make sure they seem reasonable.

Energy of the Simple Harmonic Oscillator

Stop class here

If there is motion, there is energy. We can find the energy in a harmonic oscillator. Let's start with kinetic energy. Recall that

$$K = \frac{1}{2}mv^2$$

for our Simple Harmonic Oscillator (SHO) we have

$$\begin{aligned} K &= \frac{1}{2}m\left(-\omega x_{\max}\sin\left(\omega t + \phi_o\right)\right)^2 \\ &= \frac{1}{2}m\omega^2 x_{\max}^2 \sin^2\left(\omega t + \phi_o\right) \\ &= \frac{1}{2}m\frac{k}{m}x_{\max}^2 \sin^2\left(\omega t + \phi_o\right) \\ &= \frac{1}{2}k x_{\max}^2 \sin^2\left(\omega t + \phi_o\right) \end{aligned}$$

The potential energy due to a spring is given by (from your PH121 class or Statics/Dynamics)

$$U = \frac{1}{2}kx^2 \tag{1.21}$$

Again for our SHO we have

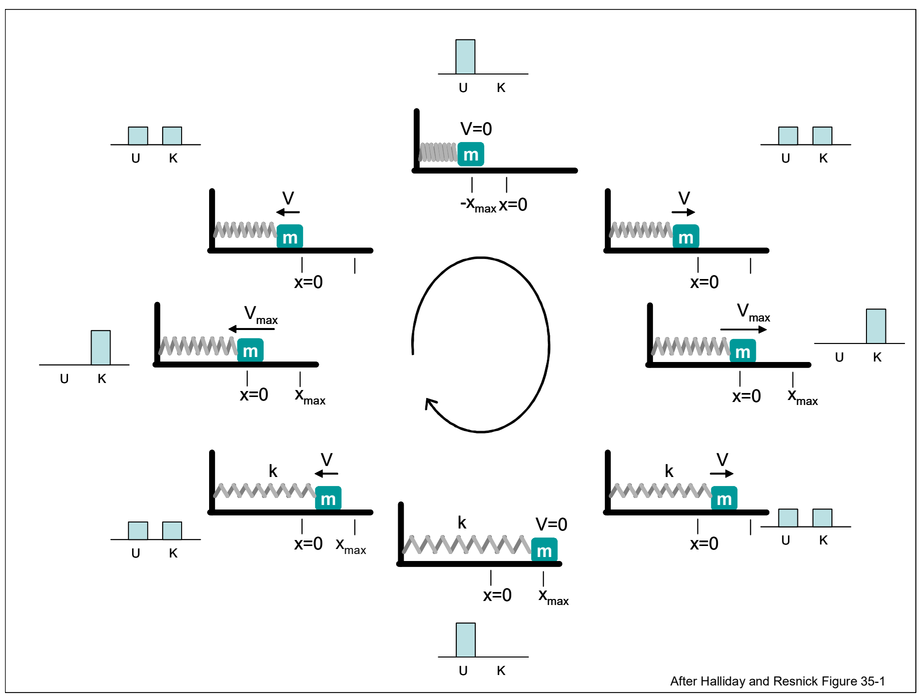
$$U = \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi_o) \tag{1.22}$$

The total energy is given by

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}kx_{\max}^2 \sin^2(\omega t + \phi_o) + \frac{1}{2}kx_{\max}^2 \cos^2(\omega t + \phi_o) \\ &= \frac{1}{2}kx_{\max}^2 \left(\sin^2(\omega t + \phi_o) + \cos^2(\omega t + \phi_o)\right) \\ &= \frac{1}{2}kx_{\max}^2 \end{aligned} \tag{1.23}$$

This is an astounding result! The amount of energy at any given time is equal to the amount of energy we started with. We are not changing how much energy we have. We call such a value that does not change a *constant of motion.*

Remark 1.2 *The total mechanical energy of a SHO is a constant of motion*



In the figure you can see that the kinetic and potential energies trade back and forth, but the total amount of energy does not change. Note that the kinetic and potential en-

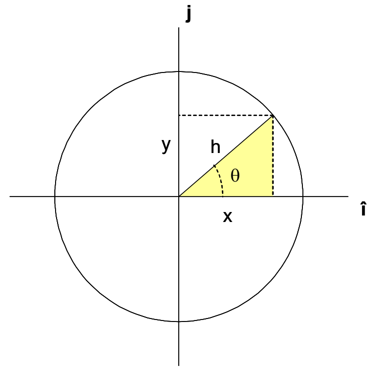
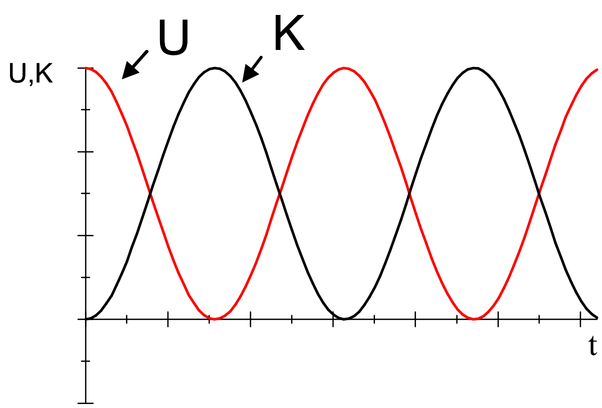


Figure 1.3.

ergy are out of phase with each other. If we plot them on the same scale (for the case $\phi_o = 0$) we have

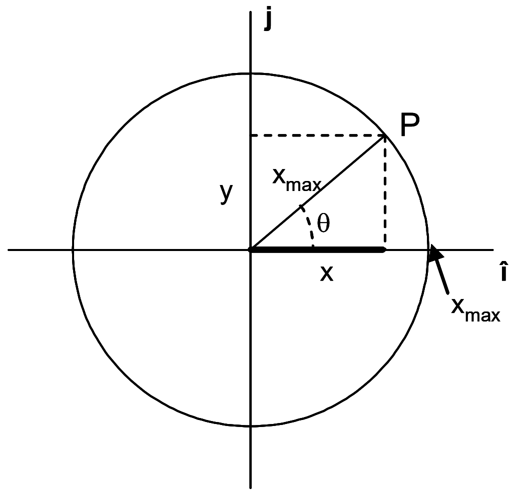


Circular Motion and SHM

That circular motion and SHM are related should not be a surprise once we found the solutions to the equations of motion were trig functions. Recall that the trig functions are defined on a unit circle

$$\tan\theta = \frac{x}{y} \tag{1.24}$$
$$\cos\theta = \frac{x}{h} \tag{1.25}$$
$$\sin\theta = \frac{y}{h} \tag{1.26}$$

Let's relate this to our equations of motion.



Look at the projection x of the point P on the x axis. Lets follow this projection as P travels around the circle. We find it ranges from $-x_{\max}$ to $x_{\max}$. If we watch closely we find its velocity is zero at the extreme points and is a maximum in the middle. This projection is given as the $\cos$ of the vector from the origin to P. This model, indeed fits our SHO solution.

Now lets define a projection of P onto the y axis. Again we have SHM, but this time the projection is a $\sin$ function. Because

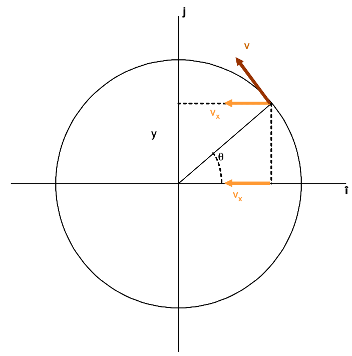
$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta) \tag{1.27}$$

we can see that this is just a SHO that is 90° out of phase.

Remark 1.3 *We see that uniform circular motion can be thought of as the combination of two SHOs, with a phase difference of* 90°.

The angular velocity is given by

$$\omega = \frac{v}{r} \tag{1.28}$$



A particle traveling on the x-axis in SHM will travel from $x_{\max}$ to $-x_{\max}$ and from $-x_{\max}$ to $x_{\max}$ (one complete period, T) while the particle traveling with P makes one complete revolution. Thus, the angular frequency ω of the SHO and the angular velocity of the particle at P are the same. (Now we know why we used the same symbol). The magnitude of the velocity is then

$$v = \omega r = \omega x_{\max} \tag{1.29}$$

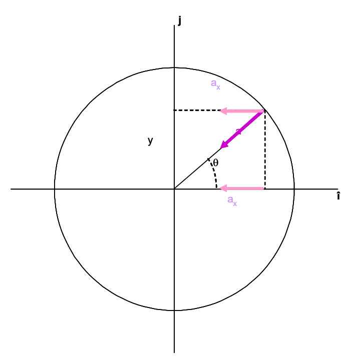
and the projection of this velocity onto the x-axis is

$$v_x = -\omega x_{\max} \sin\left(\omega t + \phi_o\right) \tag{1.30}$$

Just what we expected!

The angular acceleration of a particle at P is given by

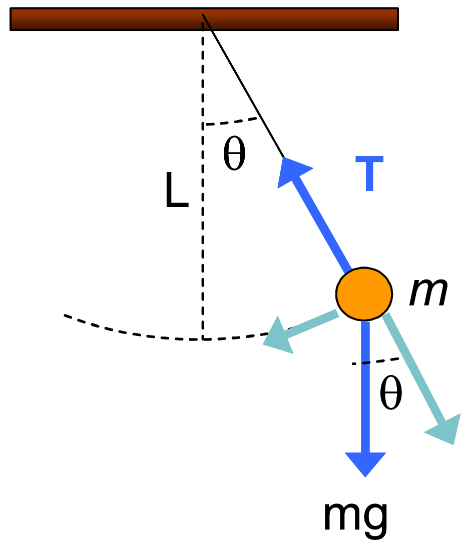
$$\frac{v^2}{r} = \frac{v^2}{x_{\max}} = \frac{\omega^2 x_{\max}^2}{x_{\max}} = \omega^2 x_{\max} \tag{1.31}$$



The direction of the acceleration is inward toward the origin. If we project this onto the x-axis we have

$$a_x = -\omega^2 x_{\max} \cos\left(\omega t + \phi_o\right) \tag{1.32}$$

The Pendulum



A simple pendulum is a mass on a string. The mass is called a "bob."

A simple pendulum exhibits periodic motion, but not exactly simple harmonic motion.

The forces on the bob, m, are $\mathbf{F}_g$, $\mathbf{T}$ the tension on the string. The tangential component of F_g is always directed toward $\theta = 0$. This is a restoring force!

Let's call the path the bob takes s. The path, s, is along an arc, then from Jr. High geometry[4], we can use the arc-length formula to describe s

$$s = L\theta \tag{10.01a}$$

and we can write an equation for the restoring force that brings the bob back to its equilibrium position as

$$\begin{aligned} F_t &= -mg\sin\theta \\ &= m\frac{d^2s}{dt^2} \\ &= mL\frac{d^2\theta}{dt^2} \end{aligned} \tag{1.33}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is a harder differential equation to solve. But suppose we are building a grandfather

[4] From Jr. High, but if you are like me you have forgotten it until now.

clock with our pendulum, and we won't let the pendulum swing very far. Then we can take θ as a very small angle, then

$$\sin(\theta) \approx \theta \tag{1.34}$$

In this approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

and we have a differential equation we recognize! Compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{1.35}$$

if

$$\omega^2 = \frac{g}{L} \tag{1.36}$$

we have all the same solutions for s that we found for x. Since ω changed, the frequency and period will now be in terms of g and L.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{1.37}$$

Remark 1.4 *the period and frequency for a pendulum with small angular displacements depend only on L and g!*

Damped Oscillations

Question 223.1.7

Question 223.1.8

Suppose we add in another force

$$\mathbf{F}_d = -b\mathbf{v} \tag{1.38}$$

This force is proportional to the velocity. This is typical of viscus fluids. So this is what we would get if we place our mass-spring system (or pendulum) in air or some other fluid. We call b the damping coefficient. Now our net force is

$$\Sigma F = -kx - bv_x = ma$$

We can write the acceleration and velocity as derivatives of the position

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is another differential equation. It is harder to guess its solution

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi_o) \tag{1.39}$$

but now our angular frequency, ω, is more complicated

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{1.40}$$

We have three cases:

1 1. *the retarding force is small:* ($bv_{\max} < kA$) *The system oscillates, but the amplitude is smaller as as time goes on. We call this "underdamped"*

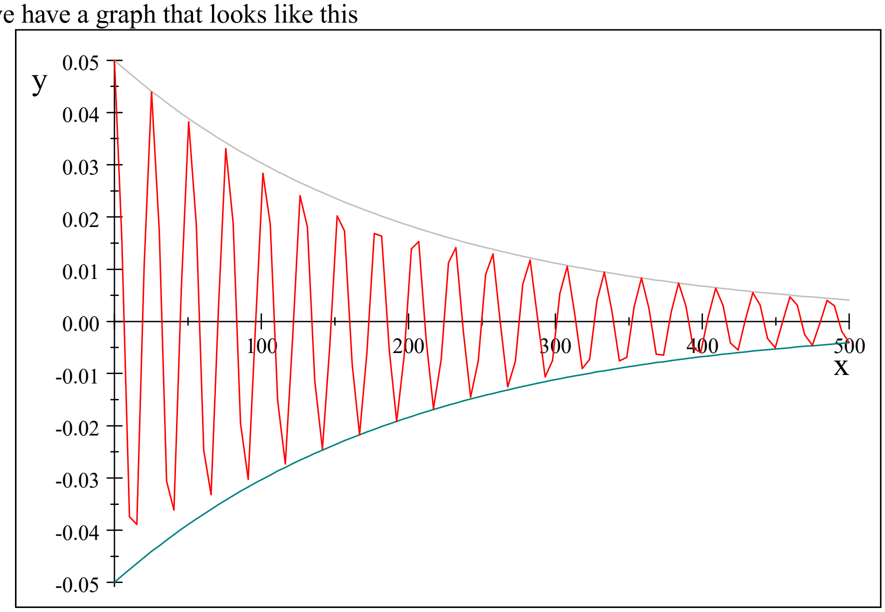
2. *the retarding force is large:* ($bv_{\max} > kA$)*The system does not oscillate. we call this "overdamped." We can also say that* $\frac{b}{2m} > \omega_o$ *(after we define* ω_o *below)*

3. *The system is critically damped (see below)*

For the following values,

$$A = 5\,\mathrm{cm}$$
$$b = 0.005\,\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k = .5\,\frac{\mathrm{N}}{\mathrm{m}}$$
$$m = .5\,\mathrm{kg}$$

we have a graph that looks like this

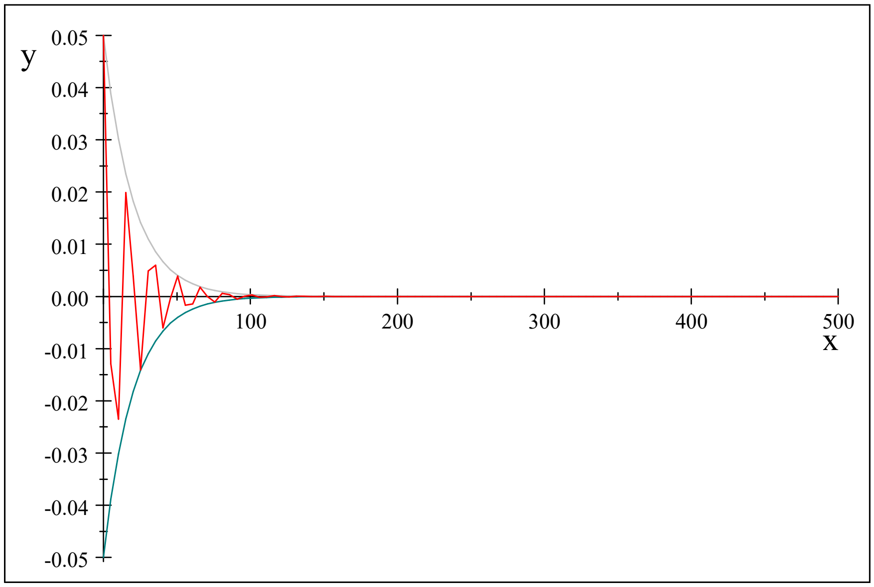


The gray lines are

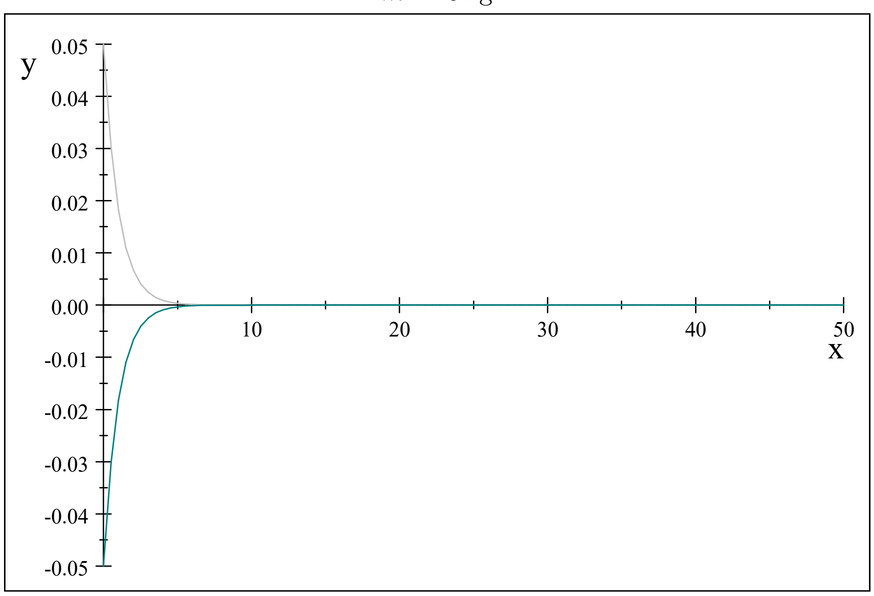
$$\pm A e^{-\frac{b}{2m} t} \tag{1.41}$$

They describe how the amplitude changes. We call this the *envelope* of the curve.

$$A = 5\,\mathrm{cm}$$
$$b = 0.05\,\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k = .5\,\frac{\mathrm{N}}{\mathrm{m}}$$
$$m = .5\,\mathrm{kg}$$



$$A = 5\,\mathrm{cm}$$
$$b = 0.5\,\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k = .5\,\frac{\mathrm{N}}{\mathrm{m}}$$
$$m = .5\,\mathrm{kg}$$



What happened?

When the damping force gets bigger, the oscillation eventually stops. Only the exponential decay is observed. This happens when

$$\frac{b}{2m} = \sqrt{\frac{k}{m}} \tag{1.42}$$

then

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0 \tag{1.43}$$

We call this situation we call critically damped. We are just on the edge of oscillation. We define

$$\omega_o = \sqrt{\frac{k}{m}} \tag{1.44}$$

as the *natural frequency* of the system. Then the value of b that gives us critically damped behavior is

$$b_c = 2m\omega_o \tag{1.45}$$

Remark 1.5 *When $\frac{b}{2\pi} \geq \omega_o$ the solution in equation (1.39) is not valid! If you are a mechanical engineer you will find out more about this situation in your advanced mechanics classes.*

Driven Oscillations and Resonance

Question 223.1.9

Question 223.1.10

Question 223.1.11

We found in the last section that if we added a force like

$$\mathbf{F}_d = -b\mathbf{v} \tag{1.46}$$

our oscillation died out. Suppose we want to keep it going? Let's apply a periodic force like

$$F(t) = F_o \sin(\omega_f t)$$

where ω_f is the angular frequency of this new driving force and where F_o is a constant.

$$\Sigma F = F_o \sin(\omega_f t) - kx - bv_x = ma$$

When this system starts out, the solutions is very messy. It is so messy that we will not give it in this class! But after a while, a steady-state is reached. In this state, the energy added by our driving force $F_o \sin(\omega_f t)$ is equal to the energy lost by the drag force, and we have

$$x(t) = A\cos(\omega_f t + \phi_o) \tag{1.47}$$

our old friend! BUT NOW

$$A=\frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2-\omega_o^2\right)^2+\left(\frac{b\omega_f}{m}\right)^2}} \tag{1.48}$$

and where

$$\omega_o=\sqrt{\frac{k}{m}} \tag{1.49}$$

as before. It is more convenient to drop the f subscripts

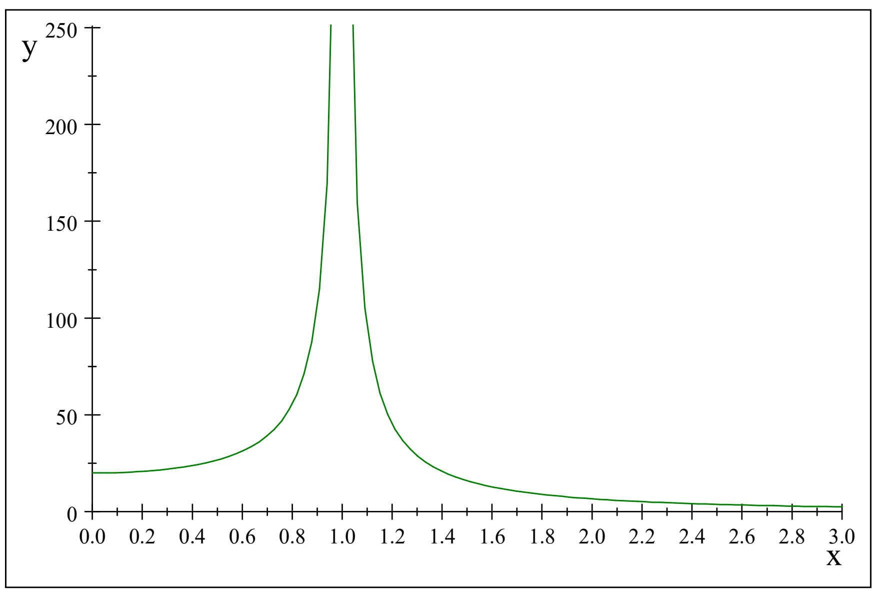
$$x\left(t\right)=A\cos\left(\omega t+\phi_o\right) \tag{1.50}$$

$$A=\frac{\frac{F_o}{m}}{\sqrt{\left(\omega^2-\omega_o^2\right)^2+\left(\frac{b\omega}{m}\right)^2}} \tag{1.51}$$

so now our solution looks more like our original SHM solution (except for the wild formula for A).

Lets look at A for some values of ω. I will pick some nice numbers for the other values.

$$F_o=2\,\mathrm{N}$$
$$b=0.5\frac{\mathrm{kg}}{\mathrm{s}}$$
$$k=0.5\frac{\mathrm{N}}{\mathrm{m}}$$
$$m=0.5\,\mathrm{kg}$$



now let's calculate ω

$$\begin{aligned}\omega_o &= \sqrt{\frac{0.5\frac{\mathrm{N}}{\mathrm{m}}}{0.5\,\mathrm{kg}}}\\ &= \frac{1.0}{\mathrm{s}}\end{aligned}$$

Notice that right at ω our solution gets very big. This is called *resonance*. To see why this happens, think of the velocity

$$\frac{dx(t)}{dt} = -\omega A \sin(\omega t + \phi_o) \tag{1.52}$$

note that our driving force is

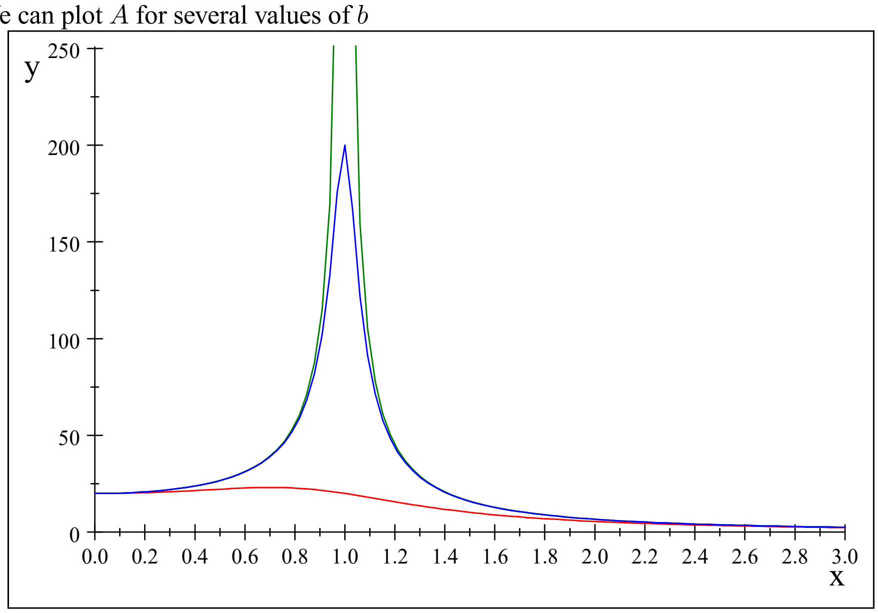
$$F(t) = F_o \sin(\omega t) \tag{1.53}$$

The rate at which work is done (power) is

$$\mathcal{P} = \frac{\mathbf{F} \cdot \Delta \mathbf{x}}{\Delta t} = \mathbf{F} \cdot \mathbf{v} \tag{1.54}$$

if F and v are in phase, the power will be at a maximum!

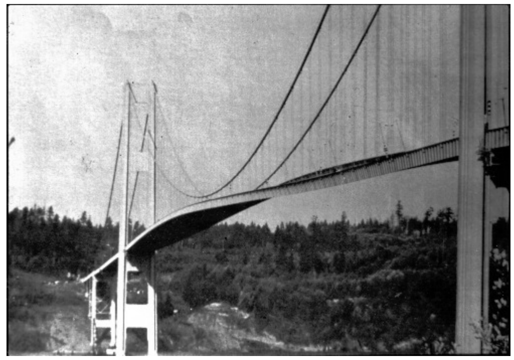
We can plot A for several values of b



Green: b=0.005kg/s; Blue: b=0.05kg/s; Red b=0.01 kg/s

As $b \to 0$ we see that our resonance peak gets larger. In real systems b can never be zero, but sometimes it can get small. As $b \to$large, the resonance dies down and our A gets small.

An example of this is well known to mechanical engineers. The next picture is of the Tacoma Narrows Bridge. As a steady wind blew across the bridge it formed turbulent wind gusts.



Tacoma Narrows Bridge (Image in the Public Domain)

The wind gusts formed a periodic driving force that allowed a driving harmonic oscillation to form. Since the bridge was resonant with the gust frequency, the amplitude grew until the bridge materials broke.