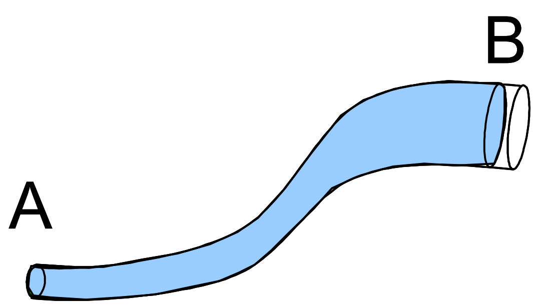
Chapter 7

Electric Flux

Flux might be a new word for us. But the idea is simple. We can think of fluid flow from something like a garden hose or a house plumbing pipe. The amount of fluid going through the pipe is called the *flow rate*. And think about water flow through a pipe. The pipe size might change, but that can't create new water in the pipe. There is the same mount of water flowing per second as there was in the smaller pipe. But with a bigger pipe, the water can spread out. So the velocity of the water goes down. The mass in the section of pipe is bigger because more water is in the pipe, but the water moves slower. Then the flow rate is the same. The same amount of water flows per unit time even if the pipe changes size. If we were studying fluid flow we would write this as

$$v_1 A_1 = v_2 A_2$$

which is called the equation of continuity



and we could do some great physics with this. But for us now, in our study of electrical forces, I just want us to think of this quantity, the flow rate, that tells us how much fluid is going through a cross section of the pipe per unit of time. It's a useful quantity that helps hydro-engineers and plumbers make sure the pressure in a pipe stays in safe regions.

We don't have a fluid flowing, but we do have an electric field. So far that field doesn't move, but it can penetrate a surface a little like a fluid flow. Could we use the idea of a field penetrating a surface to find the magnitude of the field – maybe even without a long integration?

Fundamental Concepts

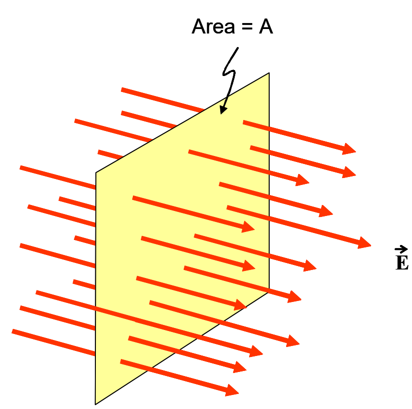
- Electric flux is the amount of electric field that penetrates an area.
- An area vector is a vector normal to the area surface with a magnitude equal to the area.
- For closed surfaces, flux going in is negative and flux going out is positive by convention.

7.0.2 The idea of electric flux

Let's call the amount of field that goes through a surface the *electric flux.* It's like a flow rate in a way, but this time we have an electric field instead of water and it just pokes through the area instead of flowing by it.

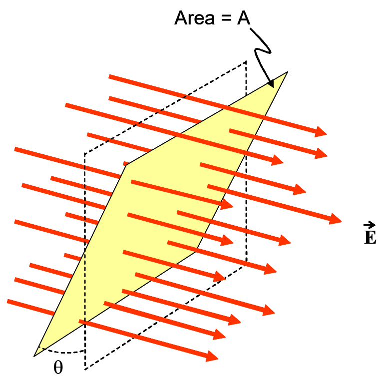
$$\Phi = EA$$

But still, this is like our flow rate in some ways. It is something multiplied by an area. In fact, it is how much of something goes through an area. It is the amount of electric field that passes through the area, A. Now the electric fields we have dealt with so far don't flow. They just stay put (we will let them change later in the course). So it is only *like* a flow rate. Our field doesn't pass by an area, it more penetrates the area. Like an arrow stuck through a target. But it is useful to think of this as "how much of something passes by an area," and the "something" is the electric field in this case. Let's consider a picture

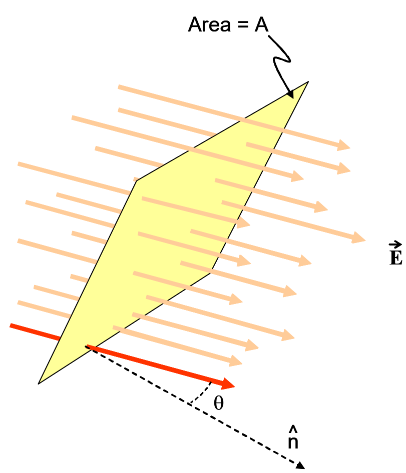


In this picture, we have a rectangular area, A, and the red arrows represent the field lines of the electric field. We can picture the quantity, Φ, as the number of field lines that pass through A. Remember that the number of field lines we draw is greater if the field strength is higher, so this quantity, Φ, tells us something about the strength of the field over the area.

But, what if the area, A, is not perpendicular to the field?



We define an angle, θ (our favorite greek letter, but we could of course use β or α, or ζ or whatever) that is the angle between the field direction and the area. A more mathematical way to do this is to define a vector that is perpendicular to (normal to) the surface $\hat{\mathbf{n}}$. Then we can use this vector and one of the field lines to define θ. It will be the angle between $\hat{\mathbf{n}}$ and the field lines.



Of course either way gives the same θ.

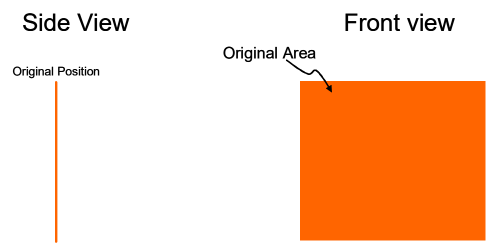
Now our definition of Φ can be made to work. We want the number of field lines passing through A, but of course, now there are fewer lines passing through the area because it is tilted. We can find Φ using θ as

$$\Phi = EA\cos\theta \tag{7.1}$$

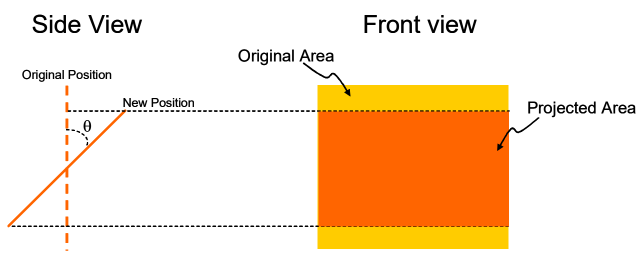
but let's consider what

$$A\cos\theta$$

means. We can start with our original area.



If we tip the area, it looks smaller



The smaller area is called the *projected area*. We can see that by tipping our area, we get fewer field lines that penetrate that area.

Really the number of field lines is just proportional to E, so we won't ever really count field lines. But this is a good mental picture for what flux means. Really we will calculate

$$\Phi = EA\cos\theta$$

The $\cos\theta$ with two magnitudes (field strength and area) multiplying it should remind you of something. It looks like the result of a vector dot product. If E and A were both vectors. Then we could write the flux as

$$\Phi = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} \tag{7.2}$$

where, we can define a vector that has A as it's magnitude and is in the right direction to make

$$\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} = EA\cos\theta$$

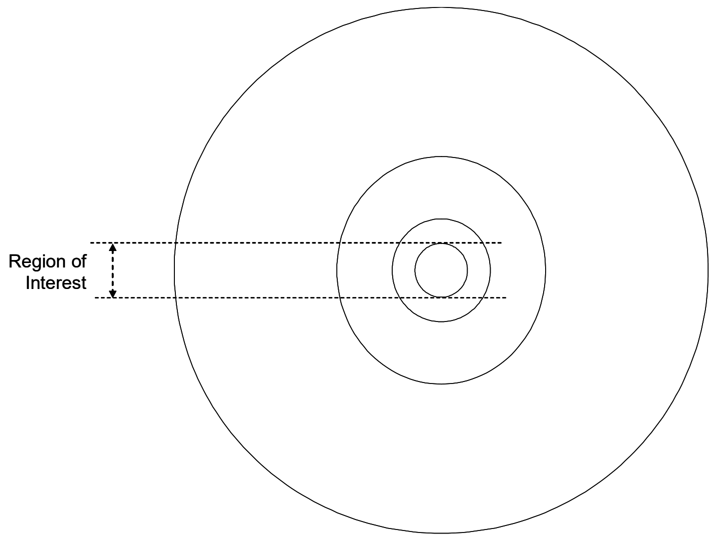
We define the *area vector*

$$\overrightarrow{\mathbf{A}} = \hat{\mathbf{n}}A \tag{7.3}$$

Notice that for an open surface (one that does not form a closed surface with a empty space inside) we have to choose which side $\hat{\mathbf{n}}$ will point from. We can choose either side. But once we have made the choice, we have to stick with it for the entire problem we are solving.

7.0.3 Flux and Curved Areas

Suppose the area we have is not flat? Then what? Well let's recall that if we take a sphere the surface will be curved. But if we take a bigger sphere, and look at the same amount of area on that sphere, it looks less curved.



This becomes more apparent if we remove the rest of the circle or sphere to take away the visual cures our eyes and minds use to say something is curved

Suppose we take a curved surface but we just look at a very small part of that surface. This would be very like magnifying our circle. We would see an increasingly flat surface piece compared to our increased scale of our image.

This gives us the idea that for an element of area, ΔA we could find an element of flux $\Delta\Phi$ for this small part of the whole curved surface. Essentially ΔA is flat (or we would just take a smaller ΔA).

$$\Delta\Phi = \overrightarrow{\mathbf{E}} \cdot \Delta\overrightarrow{\mathbf{A}} \tag{7.4}$$

This is just a small piece of the total flux through the curved surface, the total flux through our whole curved surface is

$$\Phi_E \approx \sum \Delta\Phi \tag{7.5}$$

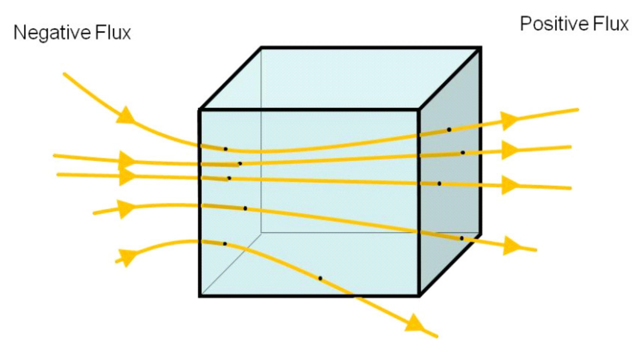
Of course, to make this exact, we will take the limit as $\Delta A \rightarrow 0$ resulting in an integral. We find the flux through a curved surface to be

$$\Phi_E = \lim_{\Delta A \rightarrow 0} \sum_i \overrightarrow{\mathbf{E}} \cdot \Delta \overrightarrow{\mathbf{A}}_i = \int_{surface} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} \tag{7.6}$$

Notice that this is a *surface integral.* It may be that you have not done surface integrals for some time, but we will practice this in the upcoming lectures.

7.0.4 Closed surfaces

Suppose we build a box with our areas.



Then we would have some lines going in and some going out. By convention we will call the flux formed by the ones going in negative and the flux formed by the ones going out positive. From these questions we see that if there is no charge inside of the box, the net flux must be zero. We could take any size or shape of closed surface and this would be true! But if we do have charge inside of the box we expect there to be a net flux. If it is a negative net charge, it will be an negative flux and if it is a positive net charge it will be a positive net flux. Next lecture we will formalize this as a new law of physics, but for now we need to remember from M215 or M113 how to write an integration over a closed surface. We use a special integral sign with a circle

$$\Phi_E = \oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} \tag{7.7}$$

You will also see this written as

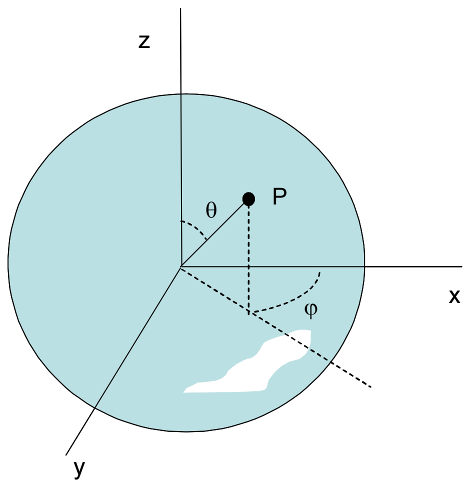
$$\Phi_E = \oint E_n dA \tag{7.8}$$

where E_n is the component of the field perpendicular (normal) to the surface at the point area increment dA.

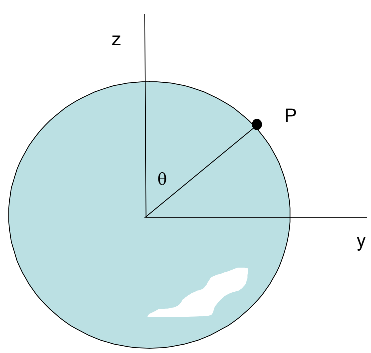
7.0.5 Flux example: a sphere

For each type of surface we choose, we need an area element to perform the integration. This is a lot like finding dq in our electric field integral. Let's take an example, a sphere.

We can start by finding the coordinates of a point, P, on the surface of the sphere.

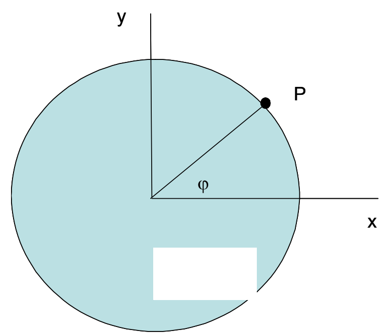


We define the coordinates in terms of two angles, θ and ϕ. Let's look at them one at a time. First θ



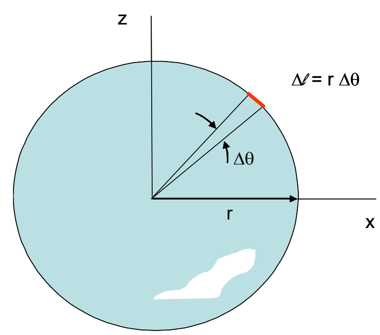
Side View

and now ϕ



Top View

Let's build an area by defining a sort of box shape on the surface by allowing a change in θ and ϕ ($\Delta\theta$ and $\Delta\phi$). First $\Delta\theta$,



Side View

The angle θ just defines a circle that passes through the "north pole" and "south pole" of our sphere. By changing θ we get a small bit of arc length. We remember that the length of an arc is

$$s_\theta = r\theta \tag{7.9}$$

where θ is in radians. So we expect that

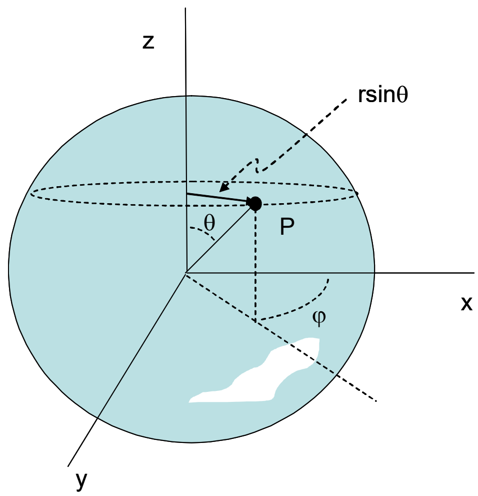
$$\Delta s_\theta = r\Delta\theta \tag{7.10}$$

We can check this by integrating

$$\int_0^{2\pi} r d\theta = r\int_0^{2\pi} d\theta = 2\pi r \tag{7.11}$$

Just as we expect, the integral of arc length around the whole circle is the circumference of the circle. Then Δs_θ is one side of our small box-like area, the box height.

Now let's look at ϕ

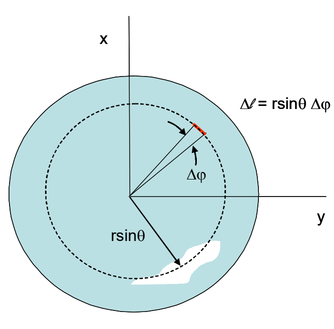


ϕ also forms a circle on the sphere, but it's size depends on θ. Near the north pole, the radius of the ϕ-circle is very small. At $\theta = 90^\circ$, the ϕ-circle is in the xy plane and has radius r. We can write the radius of the ϕ-circle as a projection over $90^\circ - \theta$ which gives us a radius of $r \sin\theta$. Then we use the arc length formula again to find

$$s_\phi = (r \sin\theta)\, \phi \tag{7.12}$$

A change in this arch length will be

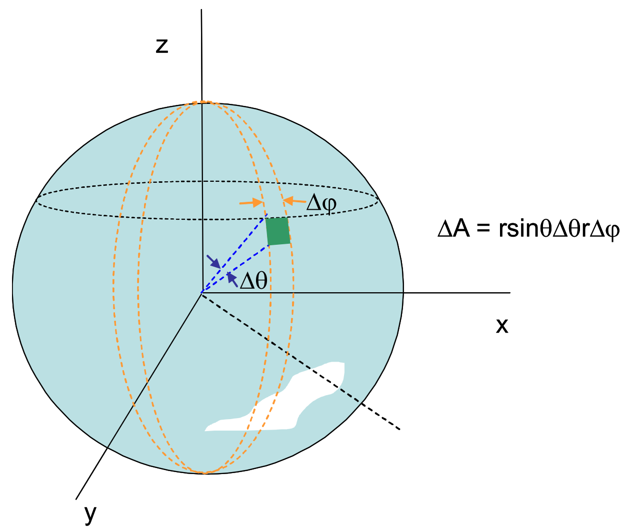
$$\Delta s_\phi = (r \sin\theta)\, \Delta\phi \tag{7.13}$$



Top View

This is the other side of our box, the box width.

Now let's combine them. We multiply $\Delta s_\theta \times \Delta s_\phi$ to obtain a roughly rectangular area.



$$\Delta A \approx \Delta s_\theta \times \Delta s_\phi = r\Delta\theta r \sin\theta \Delta\phi \tag{7.14}$$

which is the area of our small box. We have found an element of area on the surface of the sphere! Let's check our element of area by integration. After changing Δ to d and rearranging

$$dA = r^2 \sin\theta d\theta d\phi \tag{7.15}$$

then

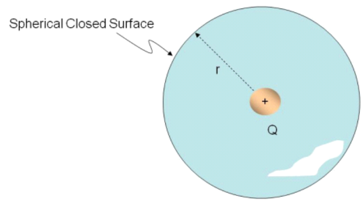
$$A = \int\int r^2 \sin\theta d\theta d\phi \tag{7.16}$$

we have to be careful not to over count area. Let's view this as first integrating around the circle of radius $r\sin\theta$ over the variable ϕ, then an integration of all these circles as θ changes from 0 to π

$$\begin{aligned} A &= \int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\phi d\theta \qquad (7.17) \\ &= r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= 2\pi r^2 \int_0^\pi \sin\theta d\theta \\ &= 4\pi r^2 \end{aligned}$$

as we expect.

We are now ready to do a simple flux problem.



Let's calculate the flux through a spherical surface if there is a point charge at the center of the sphere. The field of the point charge is

$$\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{Q_E}{r^2}\hat{\mathbf{r}}$$

then the flux through the surface is

$$\begin{aligned}\Phi_E &= \oint \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}} \\ &= \oint \frac{1}{4\pi\epsilon_o}\frac{Q_E}{r^2}\hat{\mathbf{r}} \cdot \mathbf{d}\tilde{\mathbf{A}}\end{aligned}$$

Because our surface is closed, let's let the direction of $\mathbf{d}\tilde{\mathbf{A}}$ be outward. We will always make $\mathbf{d}\tilde{\mathbf{A}}$ point out ward for closed surfaces. And look, the direction of $\hat{\mathbf{r}}$ is also outward. For this problem the direction of $\hat{\mathbf{r}}$ is always in the same direction as $\mathbf{d}\tilde{\mathbf{A}}$, so

$$\hat{\mathbf{r}} \cdot \mathbf{d}\tilde{\mathbf{A}} = (1)dA\cos(0) = dA$$

which gives us just

$$\Phi_E = \frac{Q_E}{4\pi\epsilon_o}\oint \frac{1}{r^2}dA$$

and we have just built an expression for dA for a spherical geometry!

$$\begin{aligned}\Phi_E &= \frac{Q_E}{4\pi\epsilon_o}\oint \frac{1}{r^2}r^2\sin\theta d\theta d\phi \\ &= \frac{Q_E}{4\pi\epsilon_o}\int_0^{\pi}\left(\int_0^{2\pi} d\phi\right)\sin\theta d\theta \\ &= \frac{Q_E}{4\pi\epsilon_o}4\pi \\ &= \frac{Q_E}{\epsilon_o}\end{aligned}$$

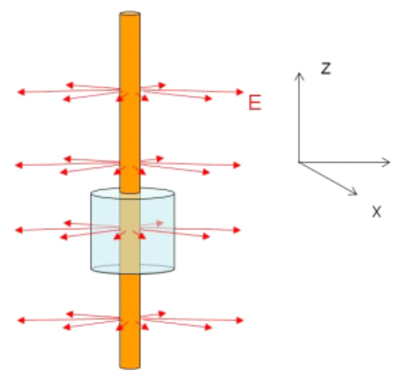
Some comments are in order. Our surfaces that we are using to calculate flux might be a real object. You might calculate the electric flux leaving a microwave oven, or a computer case to make sure you are in keeping emissions within FCC rules. But more likely the surface is purely imaginary–*just something we make up*.

Symmetry is going to be very important in doing problems with flux. So we will often make up very symmetrical surfaces to help us with our problems. In today's problem, the fact that $\hat{\mathbf{r}}$ and $d\mathbf{A}$ were in the same direction made the integral *much* easier.

Until next lecture, it may not seem beneficial to invent some strange symmetrical surface and then to calculate the flux through that surface. But it is, and it will have the effect of turning a long, difficult integral into a simple one, when we can pull it off.

7.0.6 Flux example: a long straight wire

Let's take another example. A long straight wire.



We remember that the field from a long straight wire is approximately

$$E = \frac{1}{4\pi\epsilon_o}\frac{2\left|\lambda\right|}{r}$$

We got this by integrating the field from small dq sized bits of charge and summing each field contribution to get the total field. Let's look at this equation. It is constant except for the factor of $1/r$. So if we pick a particular r value, the E is constant all around the wire for that value of r.

The symmetry of the field suggests an imaginary surface for measuring the flux, and in this case a cylinder matches the geometry well. Let's find the flux through an imaginary cylinder that is L tall and has a radius r and is concentric with the line of charge. Note that we are totally making up the cylindrical surface. There is not really any surface there at all.

The flux will be

$$\Phi_E = \oint \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}}$$

We can view this as three separate integrals

$$\Phi_E = \oint_{top} \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}} + \oint_{side} \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}} + \oint_{bottom} \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}}$$

since our cylinder has end caps (the top and bottom) and a curved side.

Let's consider the end caps first. For both the top and the bottom ends, $\tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}} = 0$ everywhere. No field goes thorough the ends. So there is no flux through the ends of the cylinder.

There is flux through the side of the cylinder. Note that the field is perpendicular to the side surface everywhere. Then $\tilde{\mathbf{E}}$ is outward, and we agreed before to make $\mathbf{d}\tilde{\mathbf{A}}$ outward for closed surfaces. Then $\tilde{\mathbf{E}}$ and $\mathbf{d}\tilde{\mathbf{A}}$ are once again in the same direction everywhere on the curved surface. So, $\tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}} = EdA$. We can write our flux as

$$\begin{aligned} \Phi_E &= \oint_{side} EdA \\ &= \oint \frac{1}{4\pi\epsilon_o} \frac{2\left|\lambda\right|}{r} dA \end{aligned}$$

Integrated over the side surface. But we will need an element of surface area dA for a cylinder side. Cylindrical coordinates seem logical so let's try

$$dA = rd\theta dz$$

then

$$\begin{aligned} \Phi_E &= \oint\oint \frac{1}{4\pi\epsilon_o} \frac{2\left|\lambda\right|}{r} rd\theta dz \\ &= \frac{2\left|\lambda\right|}{4\pi\epsilon_o} \int_0^L \int_0^{2\pi} d\theta dz \\ &= \frac{\left|\lambda\right|}{2\pi\epsilon_o} \left(2\pi L\right) \\ &= \frac{\left|\lambda\right|}{\epsilon_o} L \end{aligned}$$

So far we have, indeed, made integrals that look hard but are really easy to do. But note that this would be *much* harder if the wire were not at the center of the cylinder, or if in the previous example the charge had been off to one side of the sphere.

We would still like to remove such difficulties if we can. And often we can by choosing our imaginary surface so that the symmetry is there. But sometimes that is harder. or worse yet, we don't know exactly where the charges are in a complicated configuration of charge. We will take this on next lecture when we study a technique for finding the electric field invented by Gauss.

Basic Equations

The electric flux is defined as

$$\Phi_E = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} = EA\cos\theta$$

where the area vector is given by

$$\overrightarrow{\mathbf{A}} = \hat{\mathbf{n}}A$$

and for a curved area, we integrate

$$\Phi_E = \oint \tilde{\mathbf{E}} \cdot \mathbf{d}\tilde{\mathbf{A}}$$