Exercise – 4.1

1. Evaluate each of the following using identities:

(i) $\left(2x-\frac{1}{x}\right)^2$

(ii) $(2x+y)(2x-y)$

(iii) $\left(a^2b-ab^2\right)^2$

(iv) $(a-0\cdot1)(a+0\cdot1)$

(v) $\left[1\cdot5x^2-0\cdot3y^2\right]\left[1\cdot5x^2+0\cdot3y^2\right]$

Sol:

(i) We have,

$$\left(2x-\frac{1}{x}\right)^2=(2x)^2+\left(\frac{1}{x}\right)^2-2\cdot2x\cdot\frac{1}{x}$$

$$\left(2x-\frac{1}{x}\right)^2=4x^2+\frac{1}{x^2}-4 \qquad \left[\because(a-b)^2=a^2+b^2-2ab \text{ where } a=2x \text{ and } b=\frac{1}{x}\right]$$

$$\therefore\left(2x-\frac{1}{x}\right)^2=4x^2+\frac{1}{x^2}-4.$$

(ii) We have

$$(2x+y)(2x-y)$$

$$=(2x)^2-(y)^2 \qquad \left[\because(a+b)(a-b)=a^2-b^2\right]\text{where } a=2x \text{ and } b=y.$$

$$=4x^2-y^2$$

$$(2x+y)(2x-y)=4x^2-y^2$$

(iii) We have

$$\left(a^2b-ab^2\right)^2$$

$$=\left(a^2b\right)^2+\left(ab^2\right)^2-2\times a^2b\times ab^2 \qquad \left[\because(a-b)^2=a^2+b^2-2ab\right]$$

$$=a^4b^2+b^4a^2-2a^3b^3 \qquad \text{where} = a^2b \text{ and } b=ab^2$$

$$\therefore\left(a^2b-b^2a\right)^2=a^4b^2+b^4a^2-2a^3b^3$$

(iv) We have

$$(a-0\cdot1)(a+0\cdot1)=a^2-(0\cdot1)^2 \qquad \left[\because(a-b)(a+b)=a^2-b^2\right]$$

$$=a^2-0\cdot01 \qquad [a=a;b=0\cdot1]$$

$$(a-0\cdot1)(a+0\cdot1)=a^2-0\cdot01$$

(v) We have

$\left[1\cdot 5x^2-0\cdot 3y^2\right]\left[1\cdot 5x^2+0\cdot 3y^2\right]$

$=\left[1\cdot 5x^2\right]^2-\left[0\cdot 3y^2\right]^2$ $\left[\because (a+b)(a-b)=a^2-b^2\right]$

$=2\cdot 25x^4-0\cdot 09y^4$ $\left[\because a=1\cdot 5x^2 \text{ and } b=0\cdot 3y^2\right]$

$\left[1\cdot 5x^2-0\cdot 3y^2\right]\left[1\cdot 5x^2+0\cdot 3y^2\right]=2\cdot 25x^4-0\cdot 09y^4.$

2. Evaluate each of the following using identities:

(i) $(399)^2$

(ii) $(0\cdot 98)^2$

(iii) 991×1009

(iv) 117×83

Sol:

(i) We have

$(399)^2=(400-1)^2$

$=(400)^2+(1)^2-2(400)(1)$ $\left[\because (a-b)^2=a^2+b^2-2ab\right]$

$=1,60,000+1-8,000$ $[\because a=400\ \&\ b=1]$

$=159201$

$(399)^2=159201$

(ii) We have

$(0\cdot 98)^2=[1-0\cdot 02]^2$

$=(1)^2+(0\cdot 02)^2-2\times 1\times 0\cdot 02$

$=1+0\cdot 0004-0\cdot 04$ $[\because a=1; b=0\cdot 02]$

$=1\cdot 0004-0\cdot 04$

$=0\cdot 9604$

$\therefore (0\cdot 98)^2=0\cdot 9604.$

(iii) We have

991×1009

$=(1000-9)(1000+9)$

$=(1000)^2-(9)^2$ $\left[\because (a-b)(a+b)=a^2-b^2\right]$

$=1000000-81$ $[\because a=1000; b=9]$

$=999919$

$991\times 1009=999919$

(iv) We have

117×83

$=(100+17)(100-17)$

$=(100)^2-(17)^2$ $\quad\left[\because (a+b)(a-b)=a^2-b^2\right]$

$=10000-289$ $\quad[\because a=100; b=17]$

$=9711$

$117\times83=9711$

3. Simplify each of the following:

(i) $175\times175+2\times175\times25+25\times25$

(ii) $322\times322-2\times322\times22+22\times22$

(iii) $0\cdot76\times0\cdot76+2\times0\cdot76\times0\cdot24+0\cdot24\times0\cdot24$

(iv) $\dfrac{7\cdot83+7\cdot83-1\cdot17\times1\cdot17}{6\cdot66}$

Sol:

(i) We have

$175\times175+2\times175\times25+25\times25=(175)^2+2(175)(25)+(25)^2$

$=(175+25)^2$ $\quad\left[\because a^2+b^2+2ab=(a+b)^2\right]$

$=(200)^2=40000$ $\quad[\text{here } a=175 \text{ and } b=25]$

$\therefore 175\times175+2\times175\times25+25\times25=40000$

(ii) We have

$322\times322-2\times322\times22+22\times22$

$=(322-22)^2$ $\quad\left[\because (a-b)^2=a^2-2ab+b^2\right]$

$=(300)^2$

$=90000$

$\therefore 322\times322-2\times322\times22+22\times22=90000$

(iii) We have

$0\cdot76\times0\cdot76+2\times0\cdot76\times0\cdot24+0\cdot24\times0\cdot24$

$=[0\cdot76+0\cdot24]^2$ $\quad\left[\because a^2+b^2+2ab=(a+b)^2\right]$

$=[1\cdot00]^2$

$=1$

$\therefore 0\cdot76\times0\cdot76+2\times0\cdot76\times0\cdot24+0\cdot24\times0\cdot24=1$

(iv) We have

$$\frac{7\cdot83+7\cdot83-1\cdot17\times1\cdot17}{6\cdot66}$$

$$=\frac{(7\cdot83+1\cdot17)(7\cdot83-1\cdot17)}{6\cdot66} \qquad \left[\because (a^2-b^2)=(a+b)(a-b)\right]$$

$$=\frac{(9\cdot00)(6\cdot66)}{(6\cdot66)}$$

$$=9$$

$$\therefore \frac{7\cdot83\times7\cdot83-1\cdot17\times1\cdot17}{6\cdot66}=9$$

4. If $x+\frac{1}{x}=11$, find the value of $x^2+\frac{1}{x^2}$

Sol:

We have $x+\frac{1}{x}=11$

Now, $\left(x+\frac{1}{x}\right)^2=x^2+\left(\frac{1}{x}\right)^2+2\times x\times\frac{1}{x}$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2$$

$$\Rightarrow (11)^2=x^2+\frac{1}{x^2}+2 \qquad \left[\because x+\frac{1}{x}=11\right]$$

$$\Rightarrow 121=x^2+\frac{1}{x^2}+2$$

$$\Rightarrow x^2+\frac{1}{x^2}=119.$$

5. If $x-\frac{1}{x}=-1$, find the value of $x^2+\frac{1}{x^2}$

Sol:

We have

$$\left(x-\frac{1}{x}\right)^2=x^2+\left(\frac{1}{x}\right)^2-2\times x\times\frac{1}{x}$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}-2$$

$$\Rightarrow (-1)^2=x^2+\frac{1}{x^2}-2 \qquad \left[\because x-\frac{1}{x}=-1\right]$$

$$\Rightarrow 2+1=x^2+\frac{1}{x^2}$$

$$\therefore x^2+\frac{1}{x^2}=3.$$

6. If $x+\frac{1}{x}=\sqrt{5}$, find the values of $x^2+\frac{1}{x^2}$ and $x^4+\frac{1}{x^4}$

Sol:

We have

$$\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2\times x\times\frac{1}{x}$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2$$

$$\Rightarrow\left(\sqrt{5}\right)^2=x^2+\frac{1}{x^2}+2 \qquad \left[\because x+\frac{1}{x}=\sqrt{5}\right]$$

$$\Rightarrow 5=x^2+\frac{1}{x^2}+2$$

$$\Rightarrow x^2+\frac{1}{x^2}=3 \qquad \text{........}(1)$$

Now, $\left(x^2+\frac{1}{x^2}\right)^2=x^4+\frac{1}{x^4}+2\times x^2\times\frac{1}{x^2}$

$$\Rightarrow\left(x^2+\frac{1}{x^2}\right)^2=x^4+\frac{1}{x^4}+2$$

$$\Rightarrow 9=x^4+\frac{1}{x^4}+2 \qquad \left[\because x^2+\frac{1}{x^2}=3\right]$$

$$\Rightarrow x^4+\frac{1}{x^4}=7$$

Hence, $x^2+\frac{1}{x^2}=3; x^4+\frac{1}{x^4}=7.$

7. If $x^2+\frac{1}{x^2}=66$, find the value of $x-\frac{1}{x}$

Sol:

We have

$$\left(x-\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}-2\times x\times\frac{1}{x}$$

$$\left(x-\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}-2$$

$$\Rightarrow\left(x-\frac{1}{x}\right)^2=66-2 \qquad \left[\because x^2+\frac{1}{x^2}=66\right]$$

$$\Rightarrow\left(x-\frac{1}{x}\right)^2=64$$

$$\Rightarrow\left(x-\frac{1}{x}\right)^2=(\pm 8)^2$$

$$\Rightarrow x-\frac{1}{x}=\pm 8$$

8. If $x^2+\frac{1}{x^2}=79$, find the value of $x+\frac{1}{x}$

Sol:

We have

$$\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2\times x\times\frac{1}{x}$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=\left(x^2+\frac{1}{x^2}\right)+2$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=79+2$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=81$$

$$\Rightarrow\left(x+\frac{1}{x}\right)^2=(\pm 9)^2$$

$$\Rightarrow x+\frac{1}{x}=\pm 9.$$

9. If $9x^2+25y^2=181$ and xy = -6, find the value of 3x + 5y

Sol:

We have,

$$(3x+5y)^2=(3x)^2+(5y)^2+2\times 3x\times 5y$$

$$\Rightarrow(3x+5y)^2=9x^2+25y^2+30xy$$

$$=181+30(-6) \qquad \left[\because 9x^2+25y^2=181 \text{ and } xy=-6\right]$$

$$=181-180$$

$$\Rightarrow(3x+5y)^2=1$$

$$\Rightarrow(3x+5y)^2=(\mp 1)^2$$

$$\Rightarrow 3x+5y=\pm 1$$

10. If 2x + 3y = 8 and xy = 2, find the value of $4x^2 + 9y^2$

Sol:

We have

$(2x+3y)^2 = (2x)^2 + (3y)^2 + 2(2x)(3y)$

$\Rightarrow (2x+3y)^2 = 4x^2 - 9y^2 + 12xy$

$\Rightarrow (8)^2 = 4x^2 + 9y^2 + 24$ $\quad [\because 2x+3y=8, xy=24]$

$\Rightarrow 64 - 24 = 4x^2 + 9y^2$

$\Rightarrow 4x^2 + 9y^2 = 40.$

11. If 3x $-7y$ = 10 and xy = -1, find the value of $9x^2 + 49y^2$

Sol:

We have,

$(3x-7y)^2 = (3x)^2 + (-7y)^2 - 2(3x)(7y)$

$= 9x^2 + 49y^2 - 42xy$

$\Rightarrow [10]^2 = 9x^2 + 49y^2 - 42xy$ $\quad [\because 3x - 7y = 10]$

$\Rightarrow 100 = 9x^2 + 49y^2 - 42[-1]$ $\quad [\because xy = -1]$

$\Rightarrow 100 = 9x^2 + 49y^2 + 42$

$\Rightarrow 100 - 42 = 9x^2 + 49y^2$

$\Rightarrow 9x^2 + 49y^2 = 58.$

12. Simplify each of the following products:

(i) $\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

(ii) $\left(m + \frac{n}{7}\right)^3\left(m - \frac{n}{7}\right)$

(iii) $\left(\frac{x}{2} - \frac{2}{5}\right)\left(\frac{2}{5} - \frac{x}{2}\right) - x^2 + 2x$

(iv) $(x^2 + x - 2)(x^2 - x + 2)$

(v) $(x^3 - 3x^2 - x)(x^2 - 3x + 1)$

(vi) $[2x^4 - 4x^2 + 1][2x^4 - 4x^2 - 1]$

Sol:

(i) $\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

$$\Rightarrow\left[\left(\frac{1}{2}a\right)^2-(3b)^2\right]\left[\frac{1}{4}a^2+9b^2\right] \qquad \left[\because (a+b)(a-b)=a^2-b^2\right]$$

$$=\left[\left(\frac{1}{4}a^2\right)-9b^2\right]\left[\frac{1}{4}a^2+9b^2\right] \qquad \left[\because (ab)^2=a^2b^2\right]$$

$$=\left[\frac{1}{4}a^2\right]^2-\left[9b^2\right]^2 \qquad \left[\because (a-b)(a+b)=a^2-b^2\right]$$

$$=\frac{1}{16}a^4-81b^4$$

$$\therefore\left(\frac{1}{2}a-3b\right)\left(\frac{1}{2}a+3b\right)\left(\frac{1}{4}a^2+9b^2\right)=\frac{1}{16}a^4-81b^4$$

(ii) We have

$$\left(m+\frac{n}{7}\right)^3\left(m-\frac{n}{7}\right)$$

$$=\left(m+\frac{n}{7}\right)\left(m+\frac{n}{7}\right)\left(m+\frac{n}{7}\right)\left(m-\frac{n}{7}\right)$$

$$=\left(m+\frac{n}{7}\right)^2\left((m)^2-\left(\frac{n}{7}\right)^2\right) \qquad \left[\because (a+b)(a+b)=(a+b)^2 \ \& \ (a+b)(a-b)=a^2-b^2\right]$$

$$=\left(m+\frac{n}{7}\right)^2\left[m^2-\frac{n^2}{49}\right]$$

$$\therefore\left(m+\frac{n}{7}\right)^3\left(m-\frac{n}{7}\right)=\left(m+\frac{n}{7}\right)^2\left[m^2-\frac{n^2}{49}\right]$$

(iii) We have

$$\left(\frac{x}{2}-\frac{2}{5}\right)\left(\frac{2}{5}-\frac{x}{2}\right)-x^2+2x$$

$$\Rightarrow-\left(\frac{2}{5}-\frac{x}{2}\right)\left(\frac{2}{5}-\frac{x}{2}\right)-x^2+2x$$

$$\Rightarrow-\left(\frac{2}{5}-\frac{x}{2}\right)^2-x^2+2x \qquad \left[\because (a-b)(a-b)=(a-b)^2\right]$$

$$\Rightarrow-\left[\left(\frac{2}{5}\right)^2+\left(\frac{x}{2}\right)^2-2\left(\frac{2}{5}\right)\left(\frac{x}{2}\right)\right]-x^2+2x$$

$$\Rightarrow-\left[\frac{4}{25}+\frac{x^2}{4}-\frac{2x}{5}\right]-x^2+2x$$

$$\Rightarrow-\frac{x^2}{4}+\frac{2x}{5}-x^2+2x-\frac{4}{25}\Rightarrow-\frac{x^2}{4}-x^2+\frac{2x}{5}+2x-\frac{4}{25}$$

$$\Rightarrow -\frac{5x^2}{4}+\frac{2x}{5}+2x-\frac{4}{25}$$

$$\Rightarrow -\frac{5x^2}{4}+\frac{2x+10x}{5}-\frac{4}{25}$$

$$\Rightarrow \frac{-5x^2}{4}+\frac{12x}{5}-\frac{4}{25}$$

$$\therefore \left(\frac{x}{2}-\frac{2}{5}\right)\left(\frac{2}{5}-\frac{x}{2}\right)-x^2+2x$$

$$=\frac{-5x^2}{4}+\frac{12x}{5}-\frac{4}{25}$$

(iv) We have,

$$\left(x^2+x-2\right)\left(x^2-x+2\right)$$

$$\left[(x)^2+(x-2)\right]\left[x^2-(x-2)\right]$$

$$\Rightarrow \left[x^2\right]^2-(x-2)^2 \qquad \left[(a-b)(a+b)=a^2-b^2\right]$$

$$\Rightarrow x^4-\left(x^2+4-4x\right) \qquad \left[\because (a-b)^2=a^2+b^2-2ab\right]$$

$$\Rightarrow x^4-x^2-4+4x$$

$$\Rightarrow x^4-x^2+4x-4$$

$$\therefore \left(x^2+x-2\right)\left(x^2-x+2\right)=x^4-x^2+4x-4$$

(v) We have,

$$\left(x^3-3x^2-x\right)\left(x^2-3x+1\right)$$

$$\Rightarrow x\left(x^2-3x-1\right)\left(x^2-3x+1\right)$$

$$\Rightarrow x\left[\left[x^2-3x\right]^2-[1]^2\right] \qquad \left[\because (a-b)(a+b)=a^2-b^2\right]$$

$$\Rightarrow x\left[\left(x^2\right)^2+(-3x)^2-2(+3x)x^2\right]-1\Big]$$

$$\Rightarrow x\left[x^4+9x^2-6x^3-1\right]$$

$$\Rightarrow x^5-6x^4+9x^3-x$$

$$\therefore \left(x^3-3x^2-2\right)\left(x^2-3x+1\right)=x^5-6x^4+9x^3-x.$$

(vi) We have

$$\left[2x^4-4x^2+1\right]\left[2x^4-4x^2-1\right]$$

$$\Rightarrow \left[\left(2x^4-4x^2\right)^2-(1)^2\right] \qquad \left[\because (a+b)(a-b)=a^2-b^2\right]$$

$$\Rightarrow \left[\left(2x^4\right)^2+\left(4x^2\right)^2-2\left(2x^4\right)\left(4x^2\right)-1\right]$$

$\Rightarrow 4x^8 + 16^4 - 16x^6 - 1 \qquad \left[\because (a-b)^2 = a^2 + b^2 - 2ab\right]$

$\Rightarrow 4x^8 - 16x^6 + 16x^4 - 1$

$\therefore \left[2x^4 - 4x^2 + 1\right]\left[2x^4 - 4x^2 - 1\right] = 4x^8 - 16x^6 + 16x^4 - 1.$

13. Prove that $a^2 + b^2 + c^2 - ab - bc - ca$ is always non-negative for all values of a, b and c

Sol:

We have

$a^2 + b^2 + c^2 - ab - bc - ca$

$= \frac{2}{2}\left[a^2 + b^2 + c^2 - ab - bc - ca\right]$ [Multiply and divide by ‘2’]

$= \frac{1}{2}\left[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca\right]$

$= \frac{1}{2}\left[a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ca\right]$

$= \frac{1}{2}\left[\left(a^2 + b^2 - 2ab\right) + \left(a^2 + c^2 - 2ac\right) + \left(b^2 + c^2 - 2bc\right)\right]$

$= \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \qquad \left[\because (a-b)^2 = a^2 + b^2 - 2ab\right]$

$= \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq 0$

$\therefore a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

Hence, $a^2 + b^2 + c^2 - ab - bc - ca$ is always non negative for all values of a, b and c.

Exercise – 4.2

1. Write the following in the expanded form:

(i) $(a+2b+c)^2$

(ii) $(2a-3b-c)^2$

(iii) $(-3x+y+z)^2$

(iv) $(m+2n-5p)^2$

(v) $(2+x-2y)^2$

(vi) $(a^2+b^2+c^2)^2$

(vii) $(ab+bc+ca)^2$

(viii) $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2$

(ix) $\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)^2$

(x) $(x+2y+4z)^2$

(xi) $(2x-y+z)^2$

(xii) $(-2x+3y+2z)^2$

Sol:

(i) We have, $(a+2b+c)^2$

$= a^2+(2b)^2+(c)^2+2(a)(2b)+2ac+(2b)2c$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$\therefore (a+2b+c)^2 = a^2+4b^2+c^2+4ab+2ac+4bc.$

(ii) We have

$(2a-3b-c)^2 = \left[(2a)+(-3b)+(-c)\right]^2$

$=(2a)^2+(-3b)^2+(-c)^2+2(2a)(-3b)+2(-3b)(-c)+2(2a)(-c)$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$=4a^2+9b^2+c^2-12ab+6bc-4ac$

$\therefore (2a-3b-c)^2 = 4a^2+9b^2+c^2-12ab+6bc-4ca.$

(iii) $(-3x+y+z)^2 = [(-3x)+y+z]^2$

$= (-3x)^2 + y^2 + z^2 + 2(-3x)y + 2yz + 2(-3x)z$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= 9x^2 + y^2 + z^2 - 6xy + 2yz - 6xz$

$\therefore (-3x+y+z)^2 = 9x^2 + y^2 + z^2 - 6xy + 2xy - 6xz$

(iv) We have,

$(m+2n-5p)^2 = m^2 + (2n)^2 + (-5p)^2 + 2(m)(2n) + 2(2n)(-5p) + 2(m)(-5p)$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= m^2 + 4n^2 + 25p^2 + 4mn - 20np - 10pm$

$\therefore (m+n-5p)^2 = m^2 + 4n^2 + 25p^2 + 4mn - 20np - 10pm.$

(v) We have,

$(2+x-2y)^2 = [2+x+(-2y)]^2$

$= (2)^2 + x^2 + (-2y)^2 + 2(2)(x) + 2(x)(-2y) + 2(2)(-2y)$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= 4 + x^2 + 4y^2 + 4x - 4xy - 8y$

$\therefore (2+x-2y)^2 = 4 + x^2 + 4y^2 + 4x - 4xy - 8y$

(vi) We have

$(a^2+b^2+c^2)^2 = (a^2)^2 + (b^2)^2 + (c^2)^2 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2$

$\therefore (a^2+b^2+c^2)^2 = a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2$

(vii) We have

$(ab+bc+ca)^2 = (ab)^2 + (bc)^2 + (ca)^2 + 2(ab)(bc) + 2(bc)ca + 2(ab)(ca)$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2bc^2a + 2a^2bc$

$\therefore (ab+bc+ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2bc^2a + 2a^2bc$

(viii) We have

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2=\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+2\cdot\frac{x}{y}\cdot\frac{y}{z}+2\frac{y}{z}\cdot\frac{z}{x}+2\cdot\frac{z}{x}\cdot\frac{x}{y}$$

$$\left[\because (a+b+c)^2=a^2+b^2+2ab+c^2+2bc+2ca\right]$$

$$\therefore \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2=\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+2\frac{x}{z}+2\frac{y}{x}+2\frac{z}{y}$$

(ix) We have

$$\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)^2=\left(\frac{a}{bc}\right)^2+\left(\frac{b}{ca}\right)^2+\left(\frac{c}{ab}\right)^2+2\left(\frac{a}{bc}\right)\left(\frac{b}{ca}\right)+2\left(\frac{b}{ca}\right)\left(\frac{c}{ab}\right)+2\left(\frac{a}{bc}\right)\left(\frac{c}{ab}\right)$$

$$\left[\because (a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca\right]$$

$$\therefore \left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)^2=\frac{a^2}{b^2c^2}+\frac{b^2}{c^2a^2}+\frac{c^2}{a^2b^2}+\frac{2}{a^2}+\frac{2}{b^2}+\frac{2}{c^2}$$

(x) $(x+2y+4z)^2=x^2+(2y)^2+(4z)^2+2x(2y)+2(2y)(4z)+2x(4z)$

$=x^2+4y^2+16z^2+4xy+16yz+4xz.$

(xi) $(2x-y+z)^2=\left[(2x)+(-y)+z\right]^2$

$=(2x)^2+(-y)^2+(z)^2+2(2x)(-y)+2(-y)(z)+2(2x)(z)$

$=4x^2+y^2+z^2+4x(-y)-2yz+4xz$

$\therefore (2x-y+z)^2=4x^2+y^2+z^2-4xy-2yz+4xz$

(xii) $(-2x+3y+2z)^2=((-2x)+3y+2z)^2$

$=(-2x)^2+(3y)^2+(2z)^2+2(-2x)(3y)+2(3y)(2z)+2(2z)(-2x)$

$=4x^2+9y^2+4z^2-12xy+12yz-8xz$

2. Simplify:

(i) $(a+b+c)^2+(a-b+c)^2$

(ii) $(a+b+c)^2-(a-b+c)^2$

(iii) $(a+b+c)^2+(a-b+c)^2+(a+b-c)^2$

(iv) $(2x+p-c)^2-(2x-p+c)^2$

(v) $(x^2+y^2-z)^2-(x^2-y^2+z^2)^2$

Sol:

(i) We have

$$(a+b+c)^2+(a-b+c)^2$$
$$=\left(a^2+b^2+c^2+2ab+2bc+2ca\right)+\left(a^2+b^2+c^2-2ab-2bc+2ac\right)$$
$$\left[\because (x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx\right]$$
$$=2a^2+2b^2+2c^2+4ca$$
$$\therefore (a+b+c)^2+(a-b+c)^2=2a^2+2b^2+2c^2+4ca.$$

(ii) We have

$$(a+b+c)^2-(a-b+c)^2$$
$$=\left[(a+b+c)^2\right]-[a-b+c]^2$$
$$=a^2+b^2+c^2+2ab+2bc+2ca-\left[a^2+b^2+c^2-2ab-2bc+2ca\right]$$
$$=a^2+b^2+c^2+2ab+2bc+2ca-a^2-b^2-c^2+2ab+2bc-2ca$$
$$=4ab+4bc$$
$$\therefore (a+b+c)^2-(a-b+c)^2=4ab+4bc$$

(iii) We have

$$(a+b+c)^2+(a-b+c)^2+(a+b-c)^2$$
$$=\left[a^2+b^2+c^2+2ab+2bc+2ca\right]+\left[a^2+b^2+c^2-2bc-2ab+2ca\right]$$
$$+\left[a^2+b^2+c^2-2ca-2bc+2ab\right]$$
$$\left[\because (x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx\right]$$
$$=3a^2+3b^2+3c^2+2ab+2bc+2ca-2bc-2ab+2ca-2ca-2bc+2ab$$
$$=3a^2+3b^2+3c^2+2ab-2bc+2ca$$
$$=3\left(a^2+b^2+c^2\right)+2(ab-bc+ca)$$
$$\therefore (a+b+c)^2+(a-b+c)^2+(a+b-c)^2=3\left(a^2+b^2+c^2\right)+2[ab-bc+ca]$$

(iv) We have

$$(2x+p-c)^2-(2x-p+c)^2$$
$$=\left[(2x)^2+(p)^2+(-c)^2+2(2x)(p)+2(p)(-c)+2(2x)(-c)\right]$$
$$-\left[(2x)^2+(-p)^2+c^2+2(2x)(-p)+2(2x)(c)+2(-p)c\right]$$
$$=\left[4x^2+p^2+c^2+4xp-2pc-4cx\right]-\left[4x^2+p^2+c^2-4xp-2pc+4cx\right]$$
$$=4x^2+p^2+c^2+4xp-2pc-4cx-4x^2-p^2-c^2+4xp+2pc-4cx$$

$= 8xp - 8xc$

$= 8x(p-c)$

$\therefore (2x+p-c)^2 - (2x-p+c)^2 = 8x(p-c)$

(v) We have

$(x^2+y^2-z)^2 - (x^2-y^2+z^2)^2$

$= \left[x^2+y^2+(-z)^2\right]^2 - \left[x^2+(-y^2)+(z^2)\right]^2$

$= \left[(x^2)^2+(y^2)^2+(-z^2)^2+2(x^2)(y^2)+2(y^2)(-z^2)+2(x^2)(-z^2)\right]$

$-\left[(x^2)^2+(-y^2)^2+(z^2)^2+2(x^2)(-y^2)+2(-y^2)z^2+2x^2z^2\right]$

$\left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca\right]$

$= x^4+y^4+z^4+2x^2y^2-2y^2z^2-2z^2x^2-x^4-y^4-z^4+2x^2y^2+2y^2z^2-2z^2x^2$

$= 4x^2y^2-4z^2x^2$

$\therefore (x^2+y^2-z^2)^2 - (x^2-y^2+z^2)^2 = 4x^2y^2-4z^2x^2$

3. If a + b + c = 0 and $a^2+b^2+c^2=16$, find the value of ab + bc + ca.

Sol:

We know that,

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$\Rightarrow (0)^2 = 16+2(ab+bc+ca)$ $\left[\because a+b+c = \text{and } a^2+b^2+c^2=16\right]$

$\Rightarrow 2(ab+bc+ca) = -16$

$\Rightarrow ab+bc+ca = -8$

4. If $a^2+b^2+c^2=16$ and ab + bc + ca = 10, find the value of a + b + c.

Sol:

We know that,

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$\Rightarrow (a+b+c)^2 = 16+2(10)$ $\left[\because a^2+b^2+c^2=16 \text{ and } ab+bc+ca=10\right]$

$\Rightarrow (a+b+c)^2 = 16+20$

$\Rightarrow (a+b+c) = \sqrt{36}$

$\Rightarrow a+b+c = \pm 6$

5. If a + b + c = 9 and ab + bc + ca = 23, find the value of $a^2 + b^2 + c^2$.

Sol:

We know that,

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$\Rightarrow (9)^2 = a^2+b^2+c^2+2(23)$$

$$\Rightarrow 81 = a^2+b^2+c^2+46 \qquad [\because a+b+c=9 \text{ and } (ab+bc+ca=23)]$$

$$\Rightarrow a^2+b^2+c^2 = 81-46$$

$$\Rightarrow a^2+b^2+c^2 = 35.$$

6. Find the value of $4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20zx$ when x = 4, y = 3 and z = 2.

Sol:

We have,

$$4x^2+y^2+25z^2+4xy-10yz-20zx$$

$$\Rightarrow (2x)^2+(y)^2+(-5z)^2+2(2x)(y)+2(y)(-5z)+2(-5z)(2x)$$

$$\Rightarrow (2x+y-5z)^2$$

$$\Rightarrow [2[4]+3-5(2)]^2 \qquad [\because x=4, y=3 \text{ and } z=2]$$

$$=[8+3-10]^2$$

$$=[1]^2$$

$$=1$$

$$\therefore 4x^2+y^2+25z^2+4xy-10yz-20zx=1.$$

7. Simplify each of the following expressions:

(i) $(x+y+z)^2+\left(x+\dfrac{y}{2}+\dfrac{2}{3}\right)^2-\left(\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}\right)^2$

(ii) $(x+y-2z)^2-x^2-y^2-3z^2+4xy$

(iii) $(x^2-x+1)^2-(x^2+x+1)^2$

Sol:

(i) We have,

$$(x+y+z)^2+\left(x+\frac{y}{2}+\frac{2}{3}\right)^2-\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{4}\right)^2$$

$$=\left[x^2+y^2+z^2+2xy+2yz+2zx\right]+\left[x^2+\frac{y^2}{4}+\frac{z^2}{9}+2x\cdot\frac{y}{2}+2\frac{zx}{3}+\frac{yz}{3}\right]$$

$$-\left[\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{10}+\frac{xy}{3}+\frac{xz}{4}+\frac{yz}{6}\right]$$

$$=x^2+y^2+z^2+x^2+\frac{y^2}{4}+\frac{z^2}{9}-\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{16}+2xy+2x\cdot\frac{y}{2}-\frac{xy}{3}+2yz+\frac{yz}{3}-\frac{yz}{6}+2zx+\frac{2zx}{3}-\frac{xz}{4}$$

$$=\frac{8x^2-x^2}{4}+\frac{36y^2+9y^2-4y^2}{36}+\frac{144z^2+16z^2-9z^2}{144}+\frac{6xy+3xy-xy}{3}+\frac{13yz}{6}+\frac{29xz}{12}$$

$$=\frac{7x^2}{4}+\frac{41y^2}{36}+\frac{151z^2}{144}+\frac{8xy}{3}+\frac{13yz}{6}+\frac{29zx}{12}$$

$$\therefore (x+y+z)^2+\left(x+\frac{y}{2}+\frac{z}{3}\right)^2-\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{4}\right)^2$$

$$=\frac{7x^2}{4}+\frac{41y^2}{36}+\frac{151Z^2}{144}+\frac{8xy}{3}+\frac{13yz}{6}+\frac{29zx}{12}$$

(ii) We have,

$$(x+y-2z)^2-x^2-y^2-3z^2+4xy$$

$$=\left[x^2+y^2+(-2z)^2+2xy+2(y)(-2z)+2x(-2z)\right]-x^2-y^2-3z^2+4xy$$

$$=x^2+y^2+4z^2+2xy-4yz-4xz-x^2-y^2-3z^2+4xy$$

$$=z^2+6xy-4yz-4zx$$

$$\therefore (x+y-2z)^2-x^2-y^2-3z^2+4xy=z^2+6xy-4yz-4zx$$

(iii) We have,

$$\left[x^2-x+1\right]^2-\left[x^2+x+1\right]^2$$

$$=\left[\left(x^2\right)^2+(-x)^2+1^2+2\left(x^2\right)(-x)+2(-x)(1)+2x^2(1)\right]$$

$$-\left[\left(x^2\right)^2+(x)^2+(1)^2+2x^2(x)+2(x)(1)+2\left(x^2\right)(1)\right]$$

$$=x^4+x^2+1-2x^3-2x+2x^2-x^2-x^4-1-2x^3-2x-2x^2$$

$$\left[\because (a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca\right]$$

$$=-4x^3-4x$$

$$=-4x\left[x^2+1\right]$$

$$\therefore \left[x^2-x+1\right]^2-\left[x^2+x+1\right]^2=-4x\left[x^2+1\right]$$