# STRENGTH OF MATERIALS

## ■ CONCEPTS OF STRESS AND STRAIN

**STRESS :** When some external forces are applied to a body, then the body offers internal resistance to these forces. The magnitude of the internal resisting force is numerically equal to the applied forces. These internal resisting force per unit area is called "stress".

**STRAIN :** It is defined as change in length per unit length. The strain may be tensile or compressive depending upon whether the length increases (under tensile load) or decreases (under compressive load). It is a dimensionless quantity.

**EQUATIONS OF STATIC EQUILIBRIUM :**

Whenever a solid body under the action of various forces is in static equilibrium or in the state of rest then the algebraic sum of the components of these forces along any chosen three perpendicular axes reference are separately zero. In addition, the algebraic sum of the moments of these forces about any point is also zero. It is not rotating. Mathematically, these results are expressed in the form of following four equations :

$\sum X = 0$

$\sum Y = 0$

$\sum Z = 0$

$\sum M = 0$

where, $\sum X$, $\sum Y$ and $\sum Z$ represent the algebraic sum of the components of force along X, Y and Z direction respectively and $\sum M$ the algebraic sum of moments of these individuals forces about any point.

**UNITS OF STRESS AND STRAIN :**

**Stress :** The stress is defined as the internal force which is resisting to the applied force of deformation per unit area. Therefore, the stress is usually measured in **Newtons/Metre Square (N/m²).** It is also called Pascal **(Pa)**. Since Pascal is a very small unit, it is not uncommon to use Mega Pascal **(MPa)** or Giga Pascal **(GPa)** as unit of stress, where 1 MPa = $10^6$ Pa and 1 GPa = $10^9$ Pa.

**Strain :** It is defined as the change in length (or elongation in length) per unit length. Mathematically

it is defined as strain = $\dfrac{\text{Change in length } (l)}{\text{Unit length (L)}} = \dfrac{l}{L}$

which is unitless Quantity.

**ELASTIC LIMIT :** Considering a tension test of a speciman of round section being stretched in a **UTM**. The load recorded by the universal testing machine is the reaction of the test speciman. At room temperature, the load extension diagram in general , at strain rate less than $2 \times 10^{-3}$/sec will tend to follow the curve of the adjacent fig. I. Which is typical for an annealed metal like aluminium or copper.

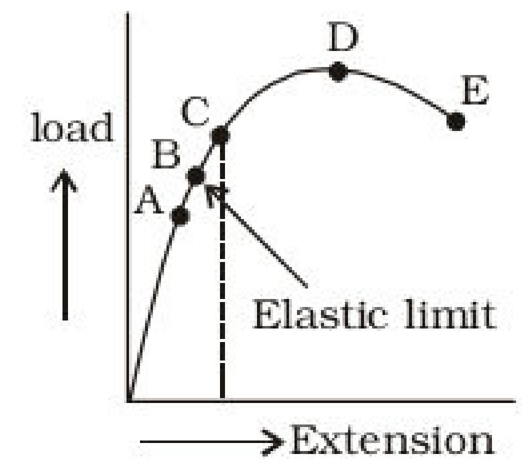



**Fig. I**

Initially, the relation between the load and extension is essentially linear–that is, portion OA of the curve, where A defines the limit of proportionality. On further straining the relation between load and extension linear–that is portion OA of the curve, where A defines the limit of proportionality. On further straining, the relation between the load and extension no longer remains linear but the material is still elastic. That is if the load is released the specimen will revert back to its original length. The maximum load which can be applied without causing permanent deformation is known as "Elastic Limit" represented by point B in the Fig-I attached. Usually the points A and B are very near to each other and the real difference can be achieved only by using highly sensitive measuring device. At the point B ends the elastic straining and the inelastic or plastic deformation ensues. Beyond the point B there is a general permanent extension of the test piece untill the load attains a maximum value, point D, corresponding to which the stress is called the "Ultimate tensile stress" $\sigma_{ult}$ (= $P_{max}/A_o$).

**TERMINOLOGIES:**

**Conventional or Engineering strain** : It is defined as the change in length per unit original length. By definition we have :

$$\in = \frac{l - l_0}{l_0} = \int_{l_0}^{l} \frac{dl}{l_0} \quad \frac{1}{l_0}\int_{l_0}^{l} dl.$$

where, $l$ = Changed or deformed length.

$l_0$ = Original length

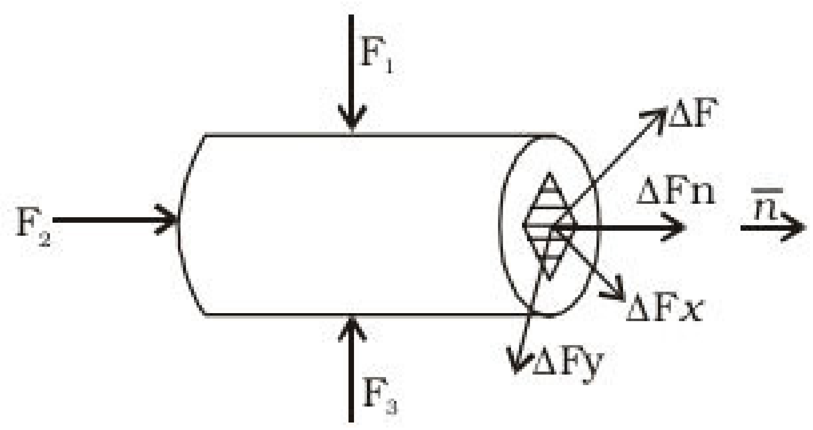
and $dl$ = Change in length.

**NATURAL STRAIN :** It is defined as change in length per unit instantaneous length. By definition we have, Natural strain

$$- = \int_0^l \frac{dl}{l} = ln\ \frac{l}{l_0} = l_0 n\ (1 + E)$$

**NORMAL STRAIN :** It is the strain produced under the action of direct or normal stress.

**SHEAR STRESS :** The force $\Delta F$ may be resolved into infinite number of components in the plane containing area $\Delta A$, because there are infinite number of directions in plane containing area $\Delta A$ which are perpendicular to the unit normal $\bar{n}$. However, in three dimensional co–ordinate system, we are left with only two directions $x$ and $y$ perpendicular to each other as shown in the diagram above. The shear stress is defined as :




**Defining stress at a point**

$$\tau_x = \text{limit}_{\Delta a \to 0} \frac{F_x}{A} = \frac{dF_x}{dA}$$

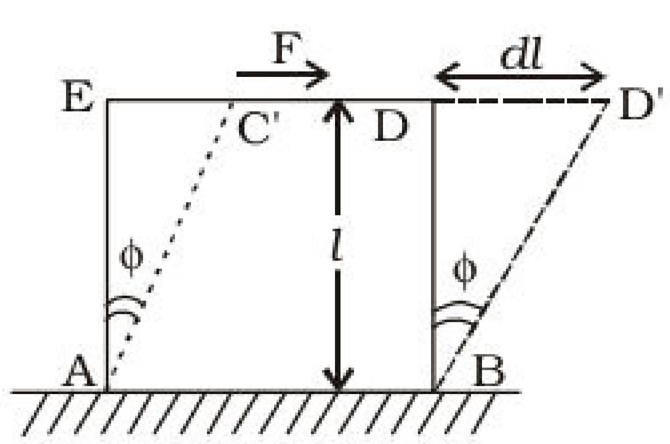
$$\tau_y = \text{limit}_{\Delta a \to 0} \frac{F_y}{A} = \frac{dF_y}{dA}$$

**SHEAR STRAIN :** It is the strain produced under the action of shear stresses. The shear strain is measured by the change in angle. As per attached diagram if $dl$ is the change in the length of face CD under the action of shear force F, then by definition, Shear strain $\gamma = \tan\phi$

For small strains,

$$\tan\phi = \phi, \text{ thus } \phi = \frac{dl}{l}$$

**GAUGE LENGTH :** It is the portion of the test specimen over which extension or deformation is measured.




**Shear Strain**

**Percentage Elongation :** It is the change in length per unit original length of the test specimen expressed as a percentage, i.e.,

Percentage elongation

$$= \frac{dl}{l} \times 100.$$

**Percentage reduction of area :** It is defined as the change in area per unit original area expressed as a percentage, i.e.,

$$\text{Percentage reduction area} = \left(\frac{A_0 - A}{A_0}\right) \times 100.$$

**Poisson's Ratio :** When a material is subjected to longitudinal deformation then the lateral dimensions also change. The ratio of lateral strain to longitudinal strain is a constant quantity called Poisson's Ratio and is designated by $\nu$ (called *nu*) or 1/m.

$$\therefore \nu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

**Superficial strain :** It is defined as change in area of cross–section per unit original area, i.e., superficial strain

$$\epsilon_s = \frac{A - A_0}{A_0} = \frac{dA}{A_0}$$

**Volumetric strain :** If a uniform stress is applied on all three faces of a body, then all the three dimensions of the body will change resulting in change in volume. Thus, volumetric strain,

$$\epsilon_v = \frac{V - V_0}{V_0} = \frac{dV}{V_0}$$

where, V = Final volume, and $V_o$ = original volume.

**HOOK'S LAW :** This law states that within elastic limits (proportional), strain is proportional to stress.

**MODULUS OF ELASTICITY :** Within elastic limits the ratio of normal stress to normal strain is a constant quantity and is defined as the young's modulus of elasticity, i.e., $E = — = \frac{Pl_0}{A_0 dl}$

**MODULUS OF RIGIDITY :** It is defined as the ratio of shearing stress to shearing strain.

i.e., G = —

**BULK MODULUS :** It is defined as the ratio of uniform stress intensity to volumetric strain, within the elastic limits and denoted by K. Thus,

$$K = \frac{}{E_v}$$

**PROOF STRESS :** It is the maximum stress which can be applied to a material without allowing the material to fail.

**FACTOR OF SAFETY :** Because of uncertainities of loading conditions, a factor of safety is introduced, defined as the ratio of maximum stress to allowable or working stress. The maximum stress is generally taken as the yield stress for ductile materials. This is also called the "Factor of ignorance."

$$\text{Factory of Saftey} = \frac{\text{Ultimate Strees}}{\text{Permissible Stress}}$$

**FREE BODY DIAGRAM :** The free body diagram of an element of a member in equilibrium is the diagram of only that member or element, as if made from the rest, with all the internal and external force.

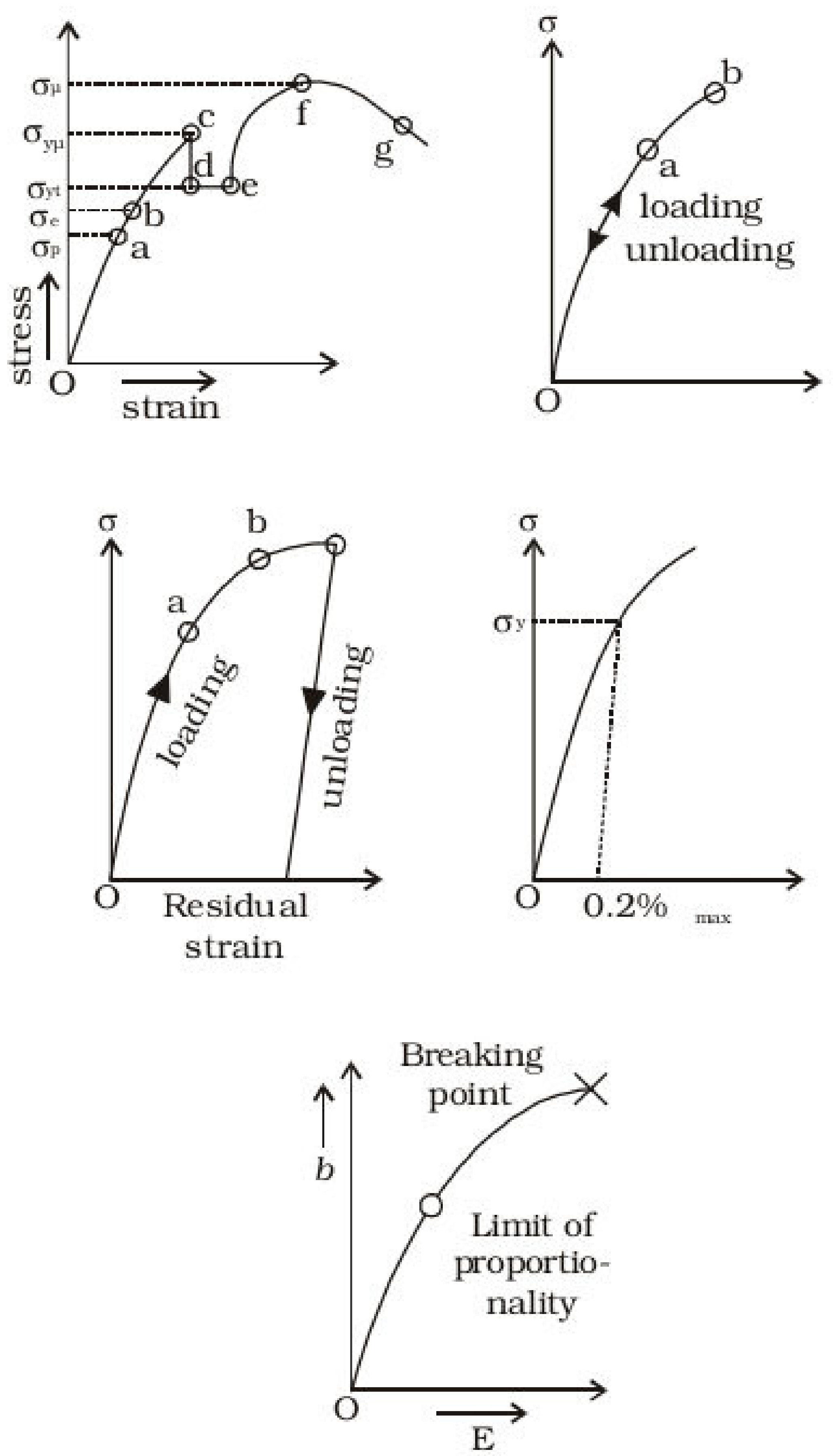



Figure ...I, II, III, IV V

**STRESS - STRAIN DIAGRAM :** The above diagrams fig I shows the stress- strain diagram for a ductile material like mild-steel. The curve starts from the origin O showing thereby that there is no initial stress or strain in the test specimen. Upto point '*a*' **Hook's law** is obeyed and stress is proportional to strain. Therefore, O*a* is a straight Line and point *a* is called the Limit of proportionality and the stress at a point is called the **proportionality Limit** stress, $\sigma_p$. The portion of the diagram between *ab* is not a straight Line but upto point *b*, the material remains elastic, i.e., on removal of the load, no permanent set is formed and the path is retraced. The point *b* is called the **elastic limit** point and the stress corresponding to that is called the elastic limit stress $\sigma_e$. In actual practice, the points *a* and *b* are so close to each other that it becomes difficult to differentiate between them.

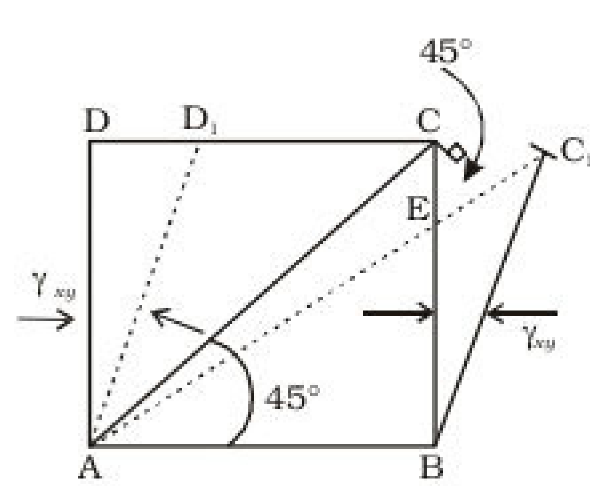
Beyond the point *b*, the material goes to plastic stage unit the **upper yield point 'c'** is reached. At this point, the cross-sectional area of the material starts decreasing and the stress decreases to a lower value to a point *d*, called the **lower yield point.** Corresponding to ponit *c*, the stress is known as upper yield stress $\sigma_{yu}$ and corresponding to point *d*, the stress is known as yield point stress, $\sigma_{yl}$. At point *d* the specimen elongates by a considerable amount without any increase in stress and upto point *e*. The portion *de* is called the yielding of the material at constant stress. From point *e* onwards, the strain hardening phenomenon becomes predominant and the strength of material increases thereby requiring more stress for deformation, until *f* is reached. Point *f* is called the ultimate point and the stress corresponding to this point is called the **ultimate stress $\sigma_u$.** It is the maximum stress to which the material can be subjected in a simple tensile Test. At point *f*, the necking of the material begins and the cross-sectional area starts decreasing at a rapid rate. Due to this local necking, the stress in the material goes on decreasing inspite of the fact that actual stress intensity goes on increasing. Ultimately the specimen breaks at point *g*, known as the breaking point and the corresponding stress is called the **nominal breaking stress** based upon the original area of cross-section at the neck. The initial portions of the diagram are shown in fig II and III.

Sometimes it is not possible to locate the yield point quite accurately in order to determine the yield strength of the material. For such materials, the yield point stress is defined at some particular value of the permanent set. It has been observed that if load is removed in the plastic range then the unloading path line is parallel to the straight portion of the strees-strain diagram in fig III. The commonly used value of permanent set for determining the value of yield strength for mild steel is 0.2 per cent of the maximum strain as shown in fig IV.

**BRITTLE MATERIALS :** The stress-strain diagram for brittle material like cast iron is shown in fig-V. There is very little elongation and reduction in area of the specimen for such materials. The yield point is not marked at all. The straight line portion of the diagram is also very small.

**RELATION BETWEEN ELASTIC CONSTANTS :**

Consider a cube of Unit size acted upon by simple shear stress $\tau_{xy}$ as shown in the adjacent diagram.



shear strain = $\gamma_{xy} = \frac{CC_1}{BC} = \frac{xy}{G}$

From C drop CE $\perp$ AC, so that AC = AE Elong ation of diagonal AC,

$$EC_1 = CC_1 \text{ Cos } 45^\circ = \frac{CC_1}{\sqrt{2}}$$

Now $AC = \sqrt{2}$ AB.

Strain in the diagonal = $\frac{EC_1}{AC} = \frac{CC_1}{\sqrt{2}}$ ; $\frac{1}{\sqrt{2}AB}$

$= \frac{1}{2}$ ; $\frac{CC_1}{AB} = \frac{1}{2} \cdot \frac{xy}{G}$, $[\because AB = BC]$

Thus the strain in the diagonal of a cube subjected to simple shear is half the amount of the shear strain. Now if the cube is subjected to simple shear, then the normal stress on the diagonal of the cube is equal to the shear stress, *i.e.*,

$$\sigma_n = \quad xy$$

$\therefore$ Strain in the diagonal = $\frac{1}{2} \cdot \frac{n}{G}$

where, $\sigma_n$ is the intensity of the equal and opposite direct stress on planes in the direction of the diagonals. If $\sigma_n$ acted alone in the direction AC, then the resultant strain would be $\sigma_n$ E, but there is an equal and opposite direct stress in the direction of the diagonal BD. The strain caused by it in the direction of AC.

$\frac{n}{E}$.

Total strain in the direction of the diagonal AC

$= \frac{n}{E} \quad \frac{n}{E}$

$= \frac{n}{E} + (1 + v)$

$\therefore \quad \frac{1}{2} \frac{n}{G} = \frac{n}{E} (1 + v)$

or, $G = \frac{E}{2(1 \quad )}$ .......(I)

Now, considering a cube under an equal compressive stress $\sigma_n$ action on all its faces, we have :

Bulk or Volume Strain : $\epsilon_v = \frac{n}{K}$

where, K = bulk Modulus

Total strain along the edge = $\frac{n}{E}$ $(1 - 2v)$

$\therefore \quad \epsilon_u = 3 \frac{n}{E} (1 - 2v)$

Hence, $\frac{n}{K} = \frac{3 \; n}{E} (1 - 2v)$

or, $E = 3 K (1 - 2v)$ ........(II)

From equation (I) and (II) above we get

$2 G (1 + v) = 3 K (1 - 2v)$

or, $2 G + 2 Gv = 3 K - 6 Kv$

or, $v(2 G + 2 G) = 3 K - 2 G$

$\therefore \quad v = \frac{(3K \quad 2G)}{(2G \quad 6K)}$ .........(III)

Now from equation (I) above we have : $v = \frac{E \quad 2G}{2G}$

or, $\frac{E \quad 2G}{2G} = \frac{3K \quad 2G}{2G \quad 6K}$

On solving we get $\frac{E}{2G} = 1 + \frac{3K \quad 2G}{2G \quad 6K}$

$= \frac{E}{2G} = \frac{9K}{2G \quad 6K} = \frac{9K}{2(G \quad 3K)}$

$\therefore \quad E = \frac{9KG}{G \quad 3K}$ .........(IV)

Also the following equation hold good :

E = 2G (1 + v) ........ derivation from equation (I) above and E = 3 K (1 – 2v)

**Example :** A M.S. rod $20_{mm}$ diameter is subjected to an axial pull of 50 kN. Determine the tensile stress induced in the road and the elongation if the unloaded length is 5m. E = 210 GN/m².

**Solution :** Given, *d* = 20 mm ; P = 50 kN; *l* = 5m.

Area of cross-section of the rod

$A = \frac{}{4} d^2$

$= \frac{}{4} (20)^2 \times 10^{-6}$

Stress, $\sigma = \frac{P}{A} = \frac{50 \quad 10^3}{314 \quad 10^{-6}}$

$= 159.155 \text{ MN/m}^2$

Elongation, $\delta = \frac{Pl}{AE} = \frac{50 \quad 10^3 \quad 5 \quad 10^3}{314 \quad 10^{6} \quad 210 \quad 10^9}$

= 3.789 mm. **.......Ans.**

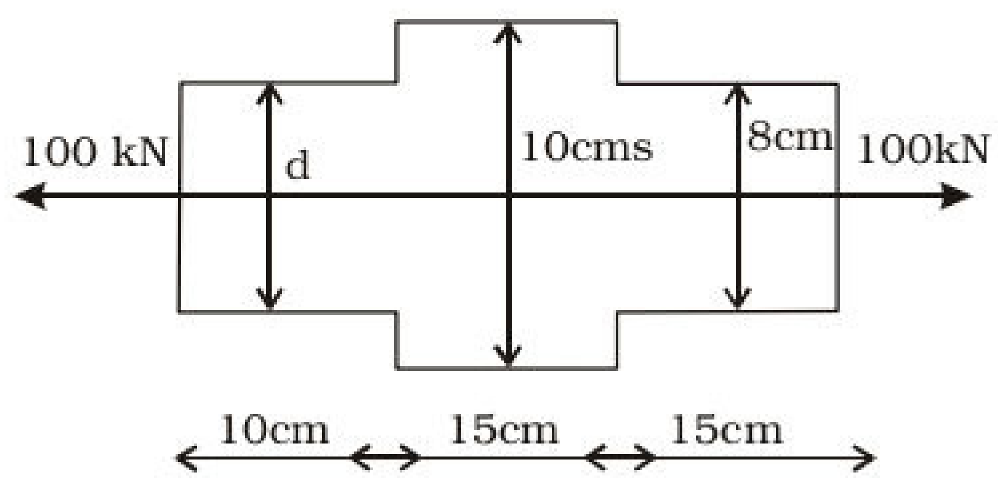
**Example :** Around bar as shown in the figure below is subjected to an axial tensile load of 100 kN. what must be the diameter '*d*' if the stress there is to be 100 MN/m² ? Find also the total elongation. E = 200 GPa.

**Solution :**

We know stress = $\frac{P}{( \; / 4)d^2}$

Putting the values we get

$$100 \times 10^6 = \frac{100 \quad 10^3}{(\quad /4)d^2}$$




$\therefore \quad d$ = Diameter = $\sqrt{\frac{4}{10^3}}$ = 0.03568 m = 35.68mm.

Total Elongation, $\Delta l = \frac{P}{E}\left[\frac{l_1}{A_1} \quad \frac{l_2}{A_2} \quad \frac{l_3}{A_3}\right]$

$$\Delta l = \frac{100 \quad 10^3}{200 \quad 10^9}\left[\frac{0.10}{10^3} \quad \frac{0.15}{\frac{}{4}\ (100)^2} \quad \frac{0.15}{\frac{}{4}\ (80)^2} \quad \frac{0.15}{\frac{}{4}\ (80)^2}\right] \times \frac{1}{10^{\ 6}}$$

$$\Delta l = \frac{10^{\ 4}}{2}[1 \quad 0.191 \quad 0.299]$$

$$\Delta l = \frac{1.490 \quad 10^{\ 4}}{2}$$

$= 0.0745$ mm.

**Example :** A piece of material 15 cms long by 2.5 cms square, is in compression under a load of 100 kN. If the modulus of elasticity of the material is 105 GPa and Poisson's Ratio is 0.25, Find the alteration in length if all lateral strain is prevented by the application of uniform lateral external pressure of suitable intensity.

**Soultion :** $\quad _x = \frac{P}{A} = \frac{-100 \quad 10^3}{2.5 \quad 2.5 \quad 10^2} = -160 \text{ MP}a$

Now $\in_x = \frac{1}{E}\left[\ _x - (\ _y \quad _z)\right]$

$$\in_y = \frac{1}{E}\left[\ _y - (\ _z \quad _x)\right]$$

$$\in_z = \frac{1}{E}\left[\ _z - (\ _x \quad _y)\right]$$

Since the strain in the $y$ and $z$ directions are prevented, therefore $\in_y = \in_z = 0$

$\therefore \sigma_y - \nu(\sigma_z + \sigma_x) = \sigma_z - \nu(\sigma_x + \sigma_y)$

or, $\sigma_y - \nu\sigma_z = \sigma_z - \nu\sigma_y$

$\sigma_y (1 + \nu) = \sigma_z (1 + \nu)$

$\therefore \quad \sigma_y = \sigma_z$

Now $\sigma_y - \nu(\sigma_y + \sigma_z) = o$

or, $(1 - \nu)\sigma_y - \nu\,\sigma_y = o$

or, $\sigma_y = \left[\frac{}{(1 - \quad)}\right]\sigma_x$

$$\therefore \quad \in_x = \frac{1}{E}\left[\ _x - 2 \quad _y\right]$$

$$\in_x = \frac{1}{E}\left\{\ _x - \left[\frac{2 \quad ^2}{1 -}\right]\ _x\right\}$$

$$\in_x = \frac{_x}{E}\left[1 - \frac{2 \quad ^2}{1 -}\right] \qquad \text{.....(1)}$$

Now putting the values from above in equation no (1) we get

$$\in_x = \frac{-160}{105 \quad 10^3}\left[1 - \frac{2 \quad 0.0625}{0.75}\right]$$

$$\in_x = \frac{-160}{105 \quad 10^3}\left[\frac{5}{6}\right]$$

$\in_x = -0.143 \times 10^{-2}$

Contraction in length $= l \times \in_x = 15 \times 0.143 \times 10^{-2}$ $= 2.145 \times 10^{-2} = 0.02145$ cm.

**PROOF RESILIENCE :**

The maximum strain energy stored in a body is known as proof resilience. The strain energy stored in the body will be maximum when the body is stressed upto elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained upto elastic limit.

**MUDULUS OF RESILIENCE :**

It is defined as the proof resilience of a material per unit volume. It is important property of a material Mathematically:

$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}$$

## BENDING MOMENTS AND SHEAR FORCE DIAGRAM

**BEAM :** Any member of a machine or structure whose one dimension is very large as compared to the other two dimensions and which can take lateral forces in the axial plane is called a beam. The lateral forces acting on the beam shall cause bending in it. Beams may be classified as simply supported beam, fixed beam, continuous beam, over-hanging beam and cantilever type, also inverted beam.

Otherwise also a member of a machine or structure under the action of lateral forces in the axial plane is called a beam. The effect of these forces will be to cause bending of the member. Depending on the end conditions there are different types of beams. A beam with both ends rigidly fixed is known as fixed or encastre beam. If one of the ends is rigidly fixed and the other is absolutely free it is known as cantilever beam. If both the supports at the ends are such that the beam is capable of taking any angular position at the end and not the lateral movement, the beam is called to be a freely or simply supported beam; in actual practice this requirement is met by providing a well lubricated hinge at one end and a support on rollers at the other.

If such a beam has ends extending beyond the supports, it is called an overhanging beam. A beam which is supported at more than two supports is called a continuous beam.

As beams are invariably horizontal, and external forces are weights, the beams will be taken horizontal and the external forces as vertical, although the same conclusions will hold good in other cases. Members of strutures and machines transmit axial loads in addition to lateral forces acting upon them. A concentrated or point load is one which is applied to the beam through a knife edge. A uniformly distributed load is that which is spread evenly over a certain length of the beam so that the load carried by a portion of the beam is proportional to the length of that portion. A distributed load will be called varying load when the rate of loading varies from one point to another.

**SHEARING FORCE :** The shearing force at any point along the loaded beam is the algebraic sum of all the vertical forces acting to one side of the point. Thus the beam shown in the figure enclosed. The shear force at cross-section $x$-$x$ as measured from left hand side is $F_1 = R_1 - W_1 - W_2$ which is equal to the shear force $F_2 = W_3 - R_2$

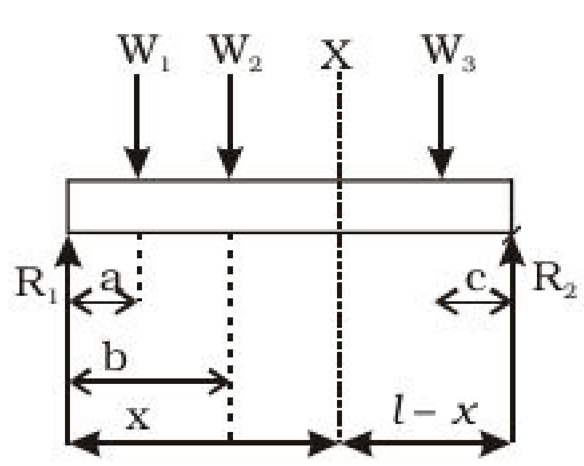



Fig.I

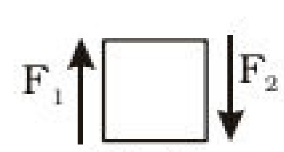



Fig.II

as measured from right hand side, i.e., $F_1 = F_2$, since the beam is in equilibrium. A thin slice of the beam at section $x$-$x$ subjected to these shear forces is shown in fig-II shear force is assumed to be positive if it produces a clockwise moment and negative if it produces an anticlockwise moment.

**BENDING MOMENT :** The bending Moment at any point along a loaded beam is the algebaric sum of the moments of all the vertical forces acting to one side of the point about the point. It fig-I, the clockwise moment at cross-section $x$-$x$ is :

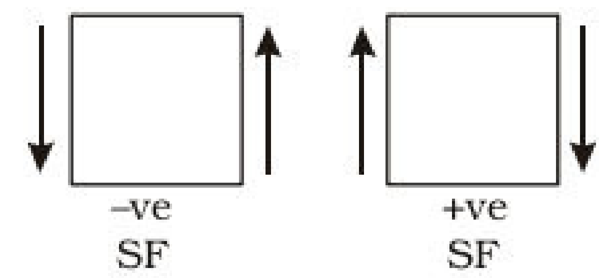
$$M_1 = R_1\, x - W_1\,(x-a) - W_2\,(x-b)$$

whereas anticlockwise moment is $M_2 = R_2\,(l-x) - W_3\,(l-x-c)$ for equilibrium of the beam $M_1 = M_2$.

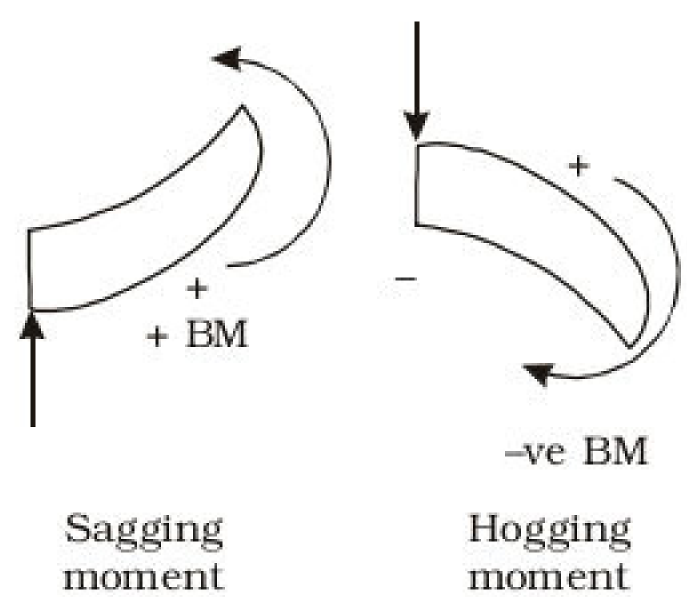
Clockwise moments are assumed to be positive and anticlockwise moments are assumed as negative due to all loads acting on the left of a section.

**BENDING MOMENT AND SHEAR FORCE DIAGRAM :**

In drawing the Bending Moment Diagrams (B.M.D.) and shearing force diagram (S.F.D.),




Sign Conventions



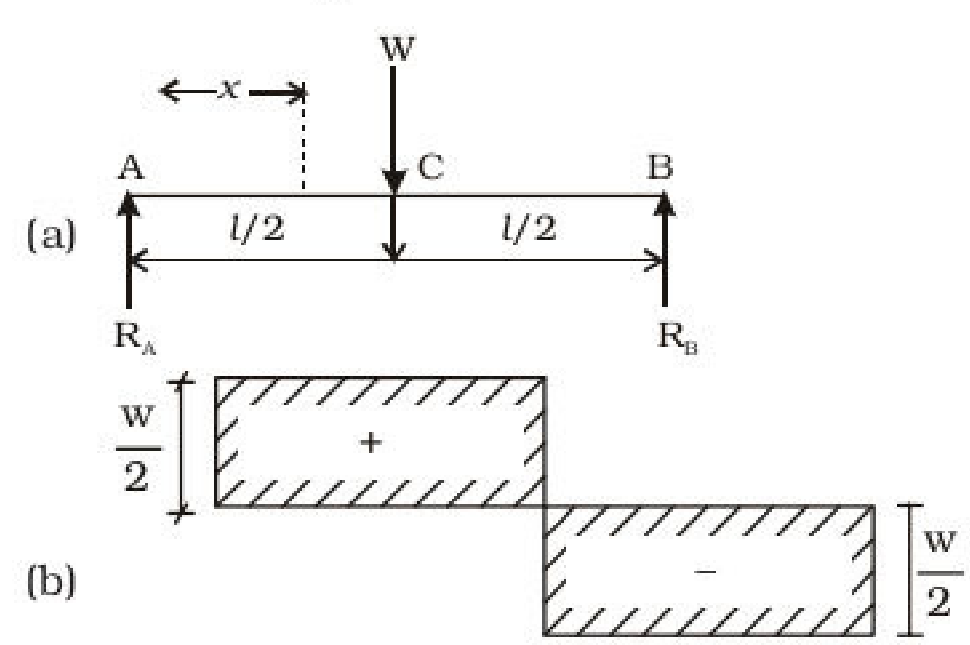

The following sign conventions are given in the enclosed figure are followed. The shear force is positive if the net resultant external force to the left of a section is upwards. Positive bending moment produces compression on the top fibres of the beam and negative bending moment produces tension on the top fibres of the beam.

**1. Simply supported Beam carrying a concentrated Load at mide-span.**

Taking moments we get

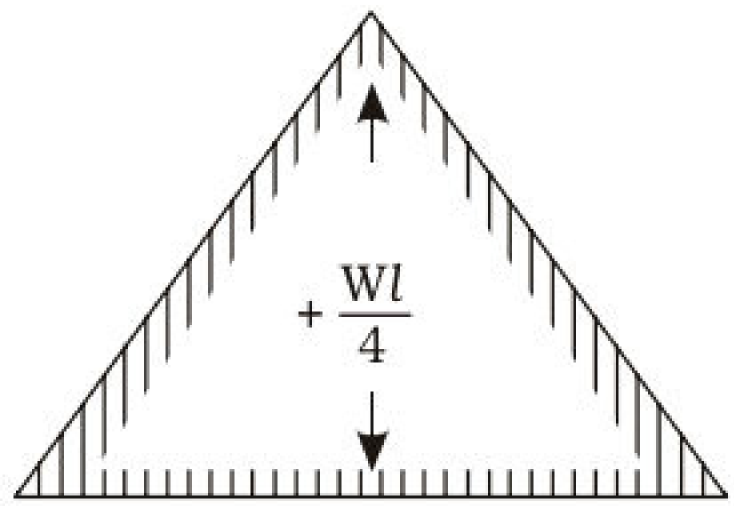
$$R_A \times l = W \times \frac{l}{2}$$

$$\therefore \quad R_A = R_B = \frac{W}{2}$$




Shearing Force Diagram

(c)




Bending moment Diagram

Taking a section at a distance $x$ from $R_1$ for values of $x$ lying between $R_A$ and W, the shear force is $\left(W - \frac{W}{2}\right) = \frac{W}{2}$, and is negative in nature. Hence, the S.F. diagram is shown in fig- (b).

Similarly, for values of $x$ lying between $R_A$ and W, the bending moment at any point is $\frac{W}{2}x$ and is positive in nature. This bending moment at $x_2 = \frac{l}{2}$ i.e., at the midspan and is given by

$$M_{max} = \frac{Wl}{4}$$

For Values of $x$ lying between W and $R_B$, the Bending Moment is $\frac{W}{2}\ x - W\left(x - \frac{l}{2}\right) = -\frac{Wx}{2} + \frac{Wl}{2}$

$$= \frac{W}{2}(l - x)$$

This bending moment is also positive. The Bending moment diagram is shown in fig-(c).

**Simply supported beam carrying a concentrated load not a Mid span,**

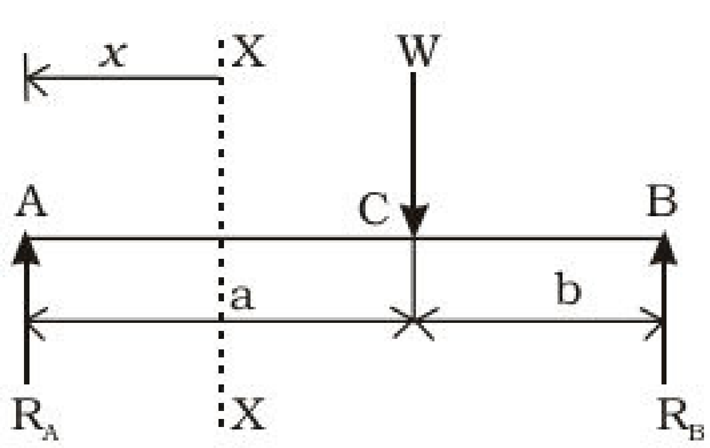
Consider a beam simply supported at the ends and carrying a concentrated load W at a distance a from left hand support as shown in fig (a). Taking the moments about $R_B$, we have :

$R_A\ (a + b) = Wb$

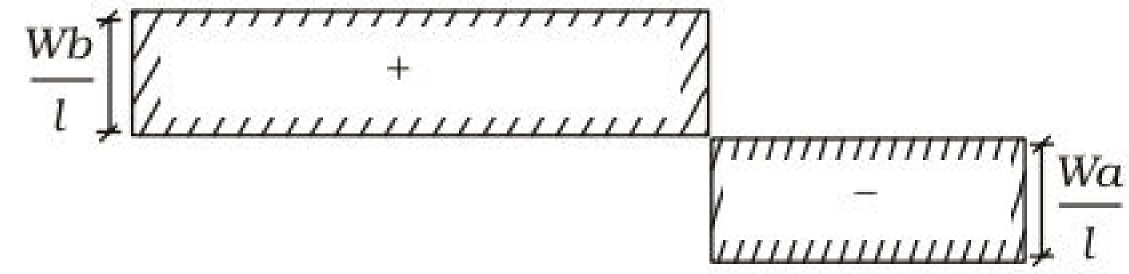
$$\therefore \quad R_A = \frac{Wb}{a \quad b}$$

$$\text{and } R_B = \frac{Wa}{a \quad b}$$

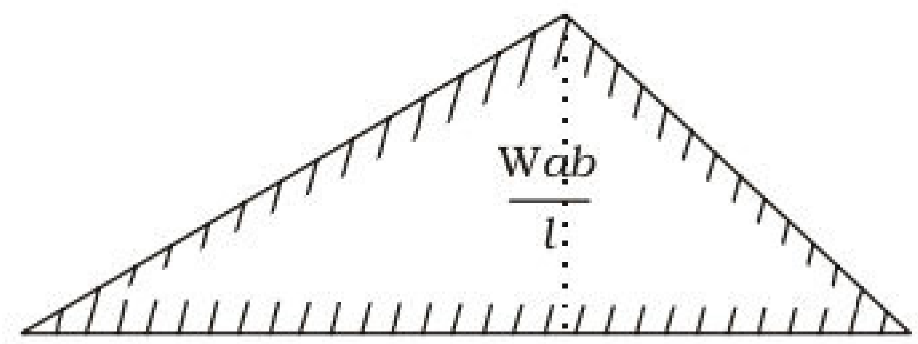
(a)




(b)




(c)




Take a section at a distance $x$ from $R_A$. When $x$ lies between $R_A$ and W, the shear force is $R_A$ and is positive in nature. When $x$ lies between W and $R_B$, the shear force is

$$W - R_A = \frac{Wa}{(a \quad b)} = \frac{Wa}{l}$$

and is negative in nature. Hence, the S.F. diagram is as shown in diagram (*b*)

Similarly for $x$ lying between $R_A$ and W, the bending moment is

$$M = R_A\, x$$

$$\text{or, } M = \frac{Wb}{(a \quad b)}\ x = \frac{Wbx}{l}$$

This is the equation of a straight line. When $x = a$, the bending moment is maximum and becomes,

$$M_{max} = \frac{Wab}{(a \quad b)} = \frac{Wab}{l}$$

This is a positive bending moment. For $x$ lying between W and $R_B$, the bending moment is

$$M = R_A\, x - W\,(x-a)$$

$$= \frac{Wb}{(a \quad b)} \cdot x - W\,(x-a) = Wa - \frac{Wa}{(a \quad b)} \cdot x$$

$$= Wa\left(1 - \frac{x}{l}\right)$$

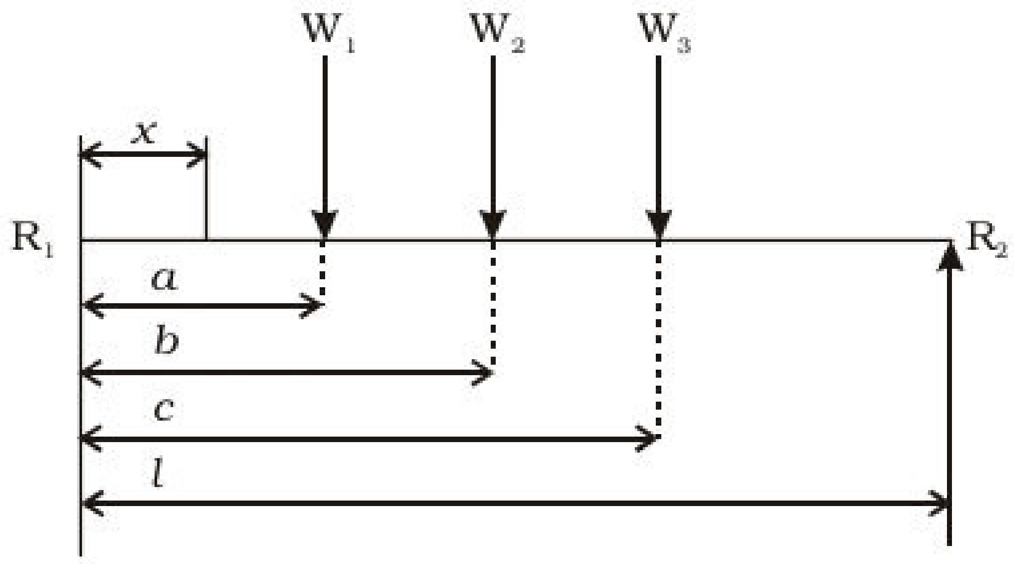
This also represents a straight line. When $x = a$, $M = \frac{Wab}{(a \quad b)}$ and when $x = a + b$, M = 0. This bending moment is also positive. Hence the B.M. diagram is also shown in fig - (c).

**Simply supported Beam Carrying Many Concentrated Loads :**

Consider a beam simply supported at the ends and carrying many concentrated loads as shown in fig (a). Taking the moments about $R_2$, We get :

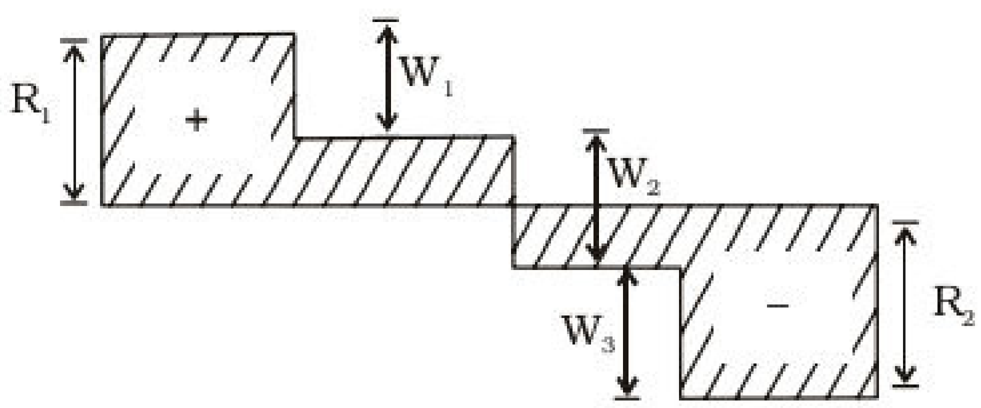
$R_1 \times l = W_1\ (l{-}a) + W_2\ (l - b) + W_3\,(l - c)$

$$\therefore\ R_1 = W_1\left(1-\frac{a}{l}\right) + W_2\left(1-\frac{b}{l}\right) + W_3\left(1-\frac{c}{l}\right)$$

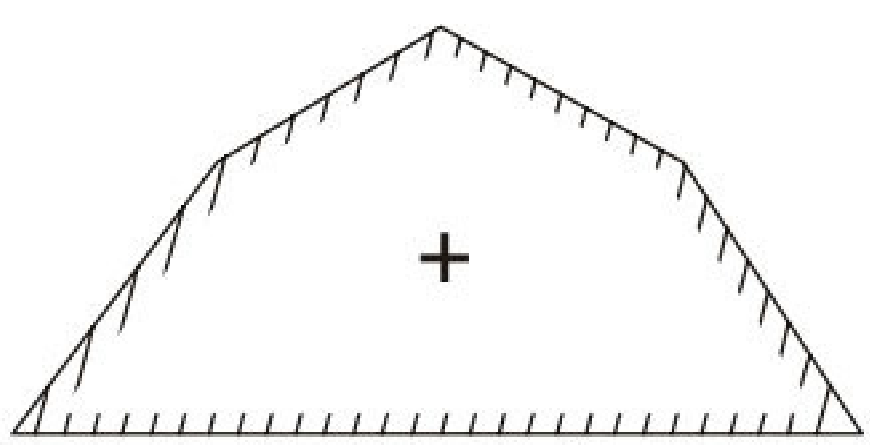



(a)

(b)




(c)




B.M. Diagram

$$\text{or, } R_1 = (W_1 + W_2 + W_3) - \frac{1}{l}\ (W_1\, a + W_2\, b + W_3\, c)$$

and $R_2 = (W_1 + W_2 + W_3) - R_1$

Let us consider a section at a distance $x$ from $R_1$. When $x$ lies between $R_1$ and $W_1$, the shear force is $R_1$ and is positive in nature. When $x$ lies between $W_1$ and $W_2$, the shear force is $(R_1 - W_1)$. Between $R_1$ and $W_1$ and $W_1$ and $W_2$, the shear force remains constant. Shear force between $W_1$ and $W_2$ will remain positive, if $W_1 < R_1$, when $x$ lies between $W_2$ and $W_3$, the shear force is $(R_1 - W_1 - W_2)$. We have assumed that $(W_1 + W_2) > R_1$ and hence the shear force becomes negative between $W_2$ and $W_3$. When $x$ lies between $W_3$ and $R_2$, the shear force is $R_1 - W_1 - W_2 - W_3 = R_2$ and is negative in nature. Hence, the shear force diagram is as shown in fig (b). Similarly for $x$ lying between $R_1$ and $W_1$, the bending moment is $M = R_1\, x$.

At $x = a$, $M = R_1 a$.

The variation of bending moment is linear in nature and it is positive, for $x$ lying between $W_1$ and $W_2$,

$M = R_1\, x - W_1(x - a)$

At $x = b$, the bending moment becomes,

$M = R_1\, b - W_1\, (b - a)$

and as per assumptions made, this is the maximum bending moment.

For $x$ lying between $W_2$ and $W_3$, the bending moment is

$M = R_1\, x - W_1\, (x - a) - W_2\, (x - b)$

At $x = c$, the bending moment becomes :

$M = R_1\, c - W_1\, (c - a) - W_2\, (c - b)$

And for $x$ lying between $W_3$ and $R_2$

$M = R_1\, x - W_1\, (x - a) - W_2\, (x - b) - W_3\, (x - c)$

$= (R_1 - W_1 - W_2 - W_3)\, x + (W_1 a + W_2 b + W_3 c)$

At $x = l$

$M = (R_1 - W_1 - W_2 - W_3)\, l + (W_1 a + W_2 b + W_3 c)$

$= (W_1 + W_2 + W_3)\, l - (W_1 a + W_2 b + W_3 c)$

$- (W_1 + W_2 + W_3)\, l + (W_1 a + W_2 b + W_3 c) = 0$

The B.M. diagram is shown in fig (c)

**Simply supported Beam carrying a Uniformly distributed load :**

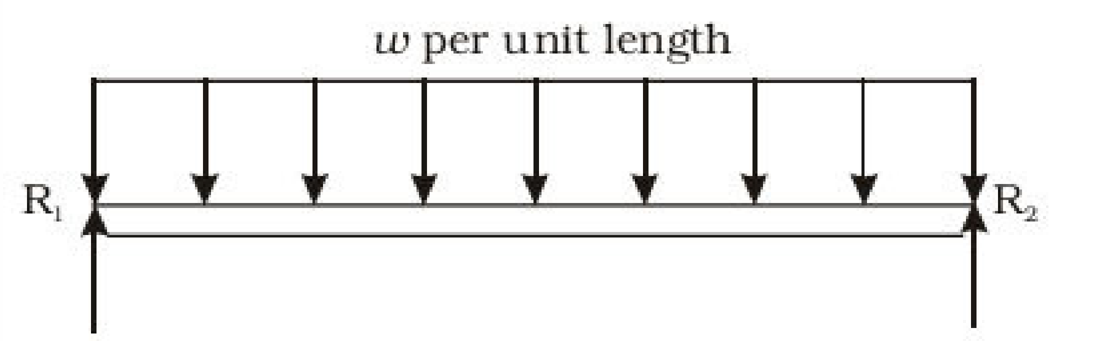
Consider a simply supported beam carrying uniformly distributed load of intensity $w$ per uint length over its whole span as shown in fig (a).

Taking moments about $R_2$, we have

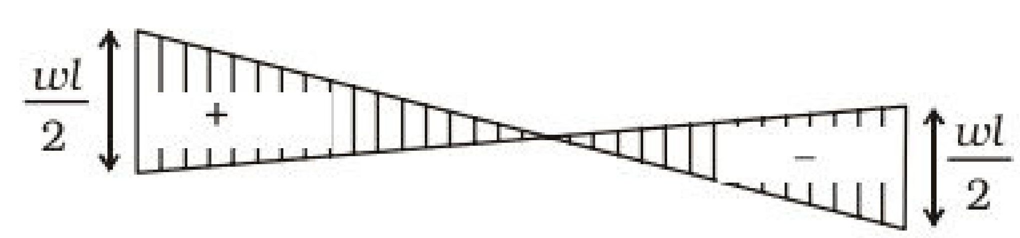
$$R_1 \times l = wl \times \frac{l}{2}$$

$$\therefore\ R_1 = \frac{wl}{2} \quad \text{and } R_2 = \frac{wl}{2}$$

(a)

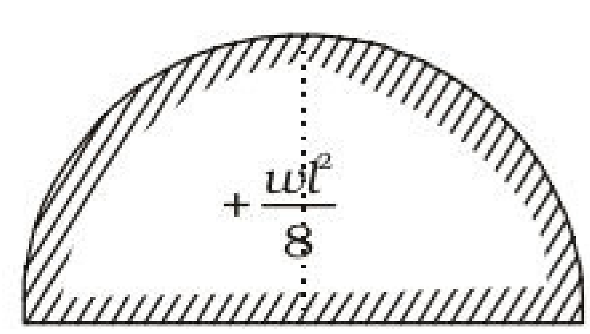



(b)




S.F. Diagram

(c)




B.M. Diagram

Consider a section at a distance $x$ from $R_1$. Shear force at this cross-section is

$$F = R_1 - wx = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right)$$

This represents the equation of a straight line. At $x = \frac{l}{2}$, the shear force is zero and is positive in nature upto this length. At $x = l$, the shear force $F = -\frac{wl}{2}$, i.e., it becomes negative. Hence, the shear force diagram is shown in fig – (b).

The bending moment at a section $x$ from $R_1$ is :

$$M = R_1 x - wx.\frac{x}{2}$$

$$= R_1\, x - \frac{wx^2}{2}$$

$$= \frac{wl}{2}\, x - \frac{wx^2}{2} \qquad \left[\because R_1 = \frac{wl}{2} \text{ from above}\right]$$

$$= \frac{w}{2}\,(lx - x^2)$$

This represents the equation of parabola. Hence the variation of bending moment is parabolic in nature.

At $x = 0$, $\quad M = 0$

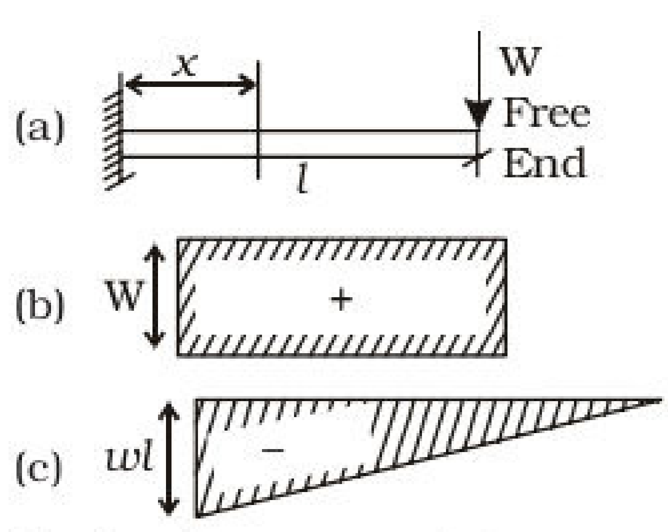
At $x = \frac{l}{2}$, $\quad M_{max} = \frac{wl^2}{8}$

and is the maximum bending moment.

At $x = l$, $M = 0$.

The bending moment remains positive on the whole span of the beam. Hence the bending moment diagram is as shown in figure (c).

**CANTILEVERS CARRYING A CONCENTRATED LOAD AT FREE END :** Consider a cantilever beam of span $l$ and carrying a concentrated load W at the free end as shown in figure (a). Consider a section at a distance '$x$' from the fixed end. Shear force at any section is equal to W and is positive in nature. Hence the S.F. diagram is as shown in figure (b).




The bending moment at any section is given by :

$M = W\,(l - x)$

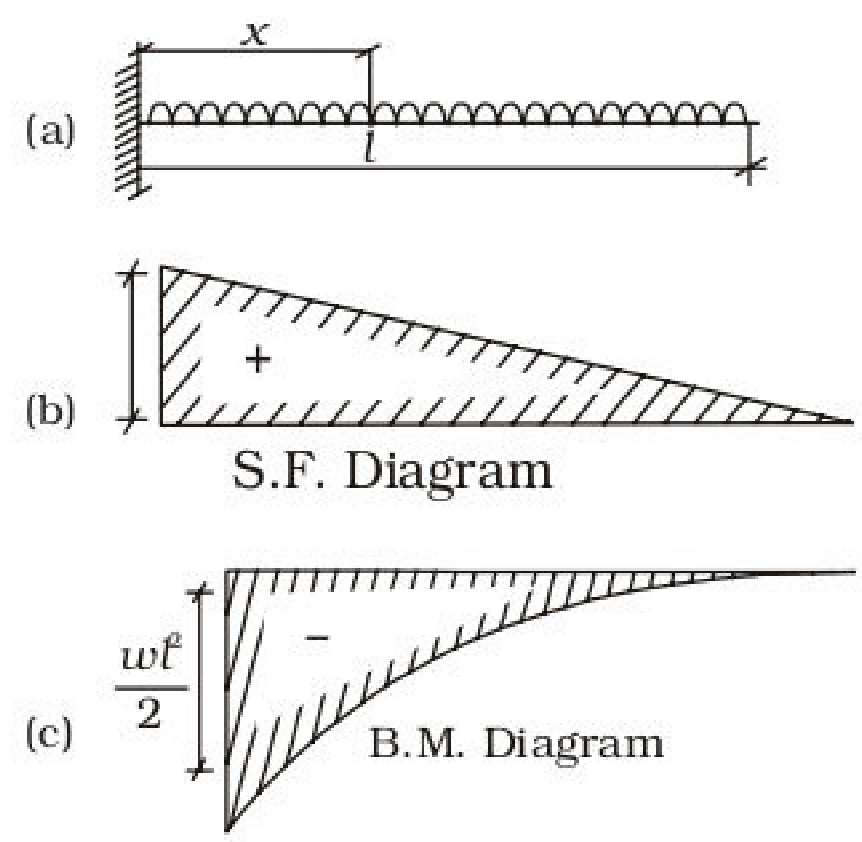
At $x = l$, $M = 0$

and at $x = 0$, $M = wl$

The bending moment is negative in nature and varies linearly as shown in figure (c).

**Cantilever carrying Uniformly distributed load**

Consider a cantilever beam of span $l$ and carrying a uniformly distributed load of intensity $w$ per unit length over its whole span as shown in fig (a). Consider a section at a distance $x$ from the fixed end. The shear force at this section is :




$F = w\,(l - x)$

At $x = l$, $F = 0$ and

at $x = 0$, $F = wl$.

This is a positive shear force and varies linearly. Hence, the S.F. diagram is as shown in fig (b).

The bending moment at any section is,

$$M = w\,(l - x).\ \frac{(l - x)}{2}$$

$$= \frac{w}{2}\,(l - x)^2$$

At $x = l$, $M = 0$

and at $x = 0$, $M = \frac{wl^2}{2}$

This is a negative bending moment and varies parabolically as shown in fig (c).

**CONTINUOUS BEAMS**

Continuous beam is a beam which is supported on more than two supports. Fig. shows such a beam, which is subjected to some external loading (here a uniformly distributed load). The deflection curve for the beam is shown by dotted line. The deflection curve is having convexity upwards over the intermediate supports, and concavity upwards over the mid of the span. Hence there will be hogging moments (i.e., negative) over the intermediate supports and sagging moments (i.e., positive) over the mid of the span. The end supports of a simply supported continuous beam will not be subjected to any bending moment. But the end support of fixed continuous beam will be subjected to fixing moments. If the moments over the intermediate supports are known, then the B.M. diagram can be drawn.

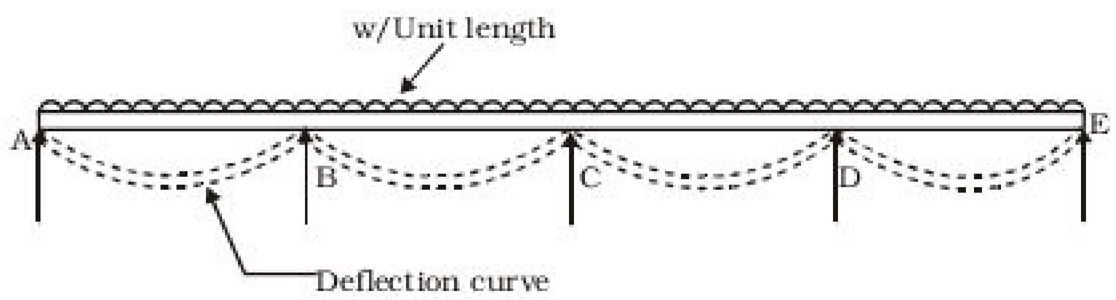


Fig.

The Fig. shows a simply supported continuous beam. In this figure the end supports at A and E will not be subjected to any bending moment. Hence in this case $M_A - M_E = 0$.

Fig. shows a continuous beam with fixed ends at A and E. Here the end supports at A and E will be subjected to fixing moments. Hence $M_A$ and $M_K$ will not be zero.

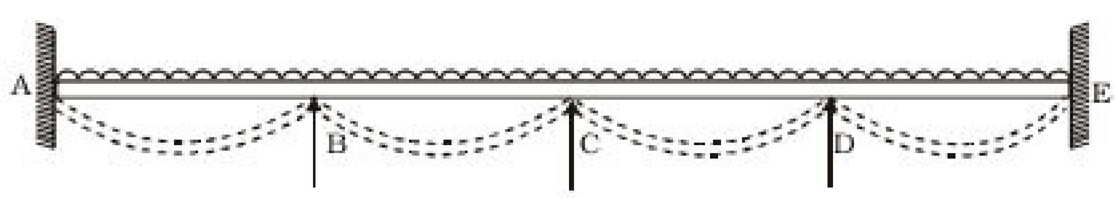


Fig. (*a*)

**BENDING MOMENT DIAGRAM FOR CONTINUOUS BEAMS**

In above Art. it is mentioned that if the moments over the intermediate supports of a continuous beam are known, then the B.M. diagram can be drawn easily. The moments over the intermediate supports are determined by using Clapeyron's theorem of three moments which states that:

If BC and CD are any two consecutive span of a continuous beam subjected to an external loading, then the moments $M_B$, $M_C$ and $M_D$ at the supports B, C and D are given by,

$$M_B.L_1 + 2M_C(L_1 + L_2) + M_D.L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

where

$L_1$ = Length of span BC

$L_2$ = Length of span CD

$a_1$ = Area of B.M. diagram due to vertical loads on span BC

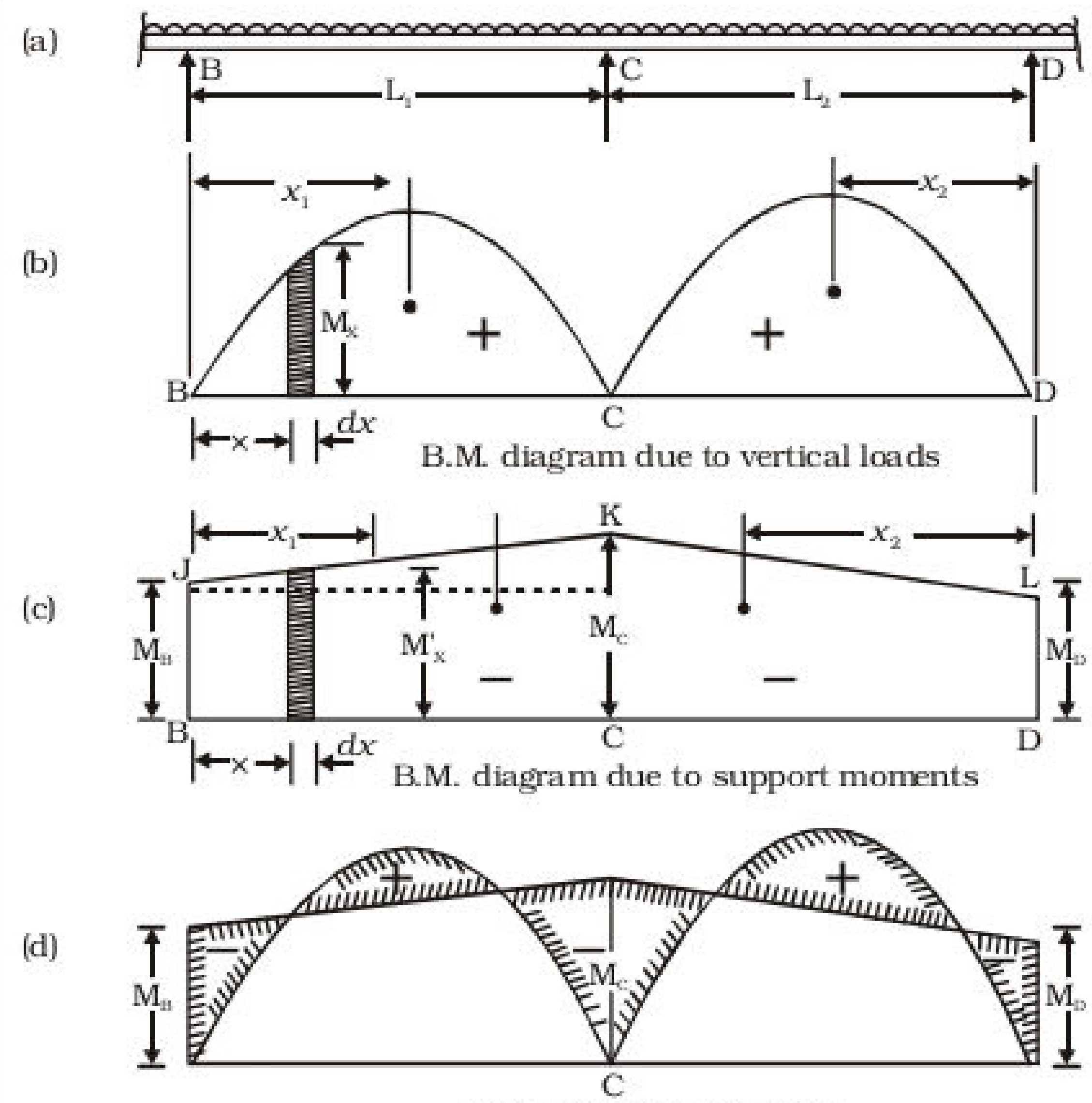
$a^2$ = Area of B.M. diagram due to vertical loads on span CD

$\bar{x}_1$ = Distance of C.G. of the B.M. diagram due to vertical loads on BO from B

$\bar{x}_2$ = Distance of C.G. of the B.M. diagram due to vertical loads on CD from D.

The equation is known as the equation of three moments or Clapeyron's equation.

**Derivation of Clapeyron's Equation of three Moments.** Fig. shows the length BCD (two consecutive spans) of a continuous beam which is shown in Fig. Let MB, Mc and MD are the support moments at B. C and D respectively.



Resultant B.M. Diagram

Fig.

Let $L_1$ = Length of span BC

$L_2$ = Length of span CD

$a_1$ = Area of B.M. diagram due to vertical loads on span BC

$a_2$ = Area of B.M. diagram due to vertical loads on span CD

$a_1'$ = Area of B.M. diagram due to support moments $M_B$ and $M_c$

$a_2$ = Area of B.M. diagram due to support moments $M_C$ and $M_D$

$\bar{x}_1$ = Distance of C.G. of B.M. diagram due to vertical loads on BC

$\bar{x}_2$ = Distance of C.G. of B.M. diagram due to vertical loads on CD

$\bar{x}_1$ = Distance of C.G. of B.M. diagram due to support moments on BC

$\bar{x}_2$ = Distance of C.G. of B.M. diagram due to support moments on CD.

Fig. (*b*) and (*c*) shows the B.M. diagrams due to vertical loads and due to supports moments respectively.

(*i*) Cosider the span BC

Let

$M_x$ = B.M. due to vertical loads at a distance *x* from B (sagging)

$M_x$ = B.M. due to support moments at a distance *x* from B (hogging)

$\therefore$ Net B.M. at a distance *x* from B is given by.

$$EI\frac{d^2y}{dx^2} = M_x - M'_x$$

Multiplying by $x$ to both sides, we get

$$EI.x.\frac{d^2y}{dx^2} = x.M_x - M'_x$$

Intergrating from zero to $L_1$, we get

$$\int_0^{L_1} EI.x.\frac{d^2y}{dx^2}.dx = \int_0^{L_1} x.M_x.dx - \int_0^{L_1} x.M_x'.dx$$

or $$EI\left[x\frac{dy}{dx} - y\right]_0^{L_1} = a_1\bar{x}_1 - a_1'\bar{x}_1 \qquad ...( i )$$

($\because$ $M_x.dx$ = Area of B.M. diagram of length $dx$. And $x.\ M_x.dx$ = Moment of area of B.M. diagram of length $dx$ about B.)

Here $\int_0^{L_1} x.M_x.dx = a_1\bar{x}_1$. And so on)

Substituting the limits in L.H.S. of equation ( $i$ ), we have

$$EI\left[\left\{L_1\left(\frac{dy}{dx}\right)_{at\,C} - yc\right\} - \left\{0\times\left(\frac{dy}{dx}\right)_{at\,B} - y_B\right\}\right]$$

$$= a_1\bar{x}_1 - a_1'\bar{x}_1'$$

or $$EI\left[(L_1.\theta_C - y_C) - (0 - y_B)\right] = a_1\bar{x}_1 - a_1'\bar{x}_1'.$$

$$\left[\because \left(\frac{dy}{dx}\right)_{at\,C} = \theta_C\right]$$

But deflection at B and C are zero. Hence $y_B = 0$ and $y_C = 0$. Hence above equation becomes as

$$\left[EI.L_1.\theta_C = a_1\bar{x}_1 - a_1'\bar{x}_1'\right]$$

But

$a_1$ = Area of B.M. diagram due to supports moments
= Area of trapezium BCKJ
$= \frac{1}{2}(M_B + M_C) \times L_1$

and $\bar{x}_1'$ = Distnace of C.G. of area BCKJ from

$$= \frac{M_B.L_1.\frac{L^1}{2} + \frac{1}{2}\times(M_c - M_B).L_1\times\frac{2L_1}{3}}{M_B.L_1 + \frac{1}{2}(M_C - M_B).L_1}$$

$$= \frac{M_{B.}\frac{L_1}{2} + (M_C - M_B)\times\frac{L_1}{3}}{M_B + (M_C - M_B).\frac{1}{2}} = \frac{\frac{3M_BL_1 + 2L_1(M_C - M_B)}{6}}{\frac{2M_B + M_C - M_B}{2}}$$

$$= \frac{\frac{L_1}{3}\left[3M_B + 2M_C - 2M_B\right]}{M_B + M_C} = \left(\frac{M_B + 2M_C}{M_B + M_C}\right)\times\frac{L_1}{3}$$

Substituting the values of $\bar{a}_1$ and $\bar{x}_1$ in equation (ii), we get

$$EI.L_1\theta_C = a_1\bar{x}_1 - \frac{1}{2}(M_B + M_C).L_1 + \left(\frac{M_B + 2M_C}{M_B + M_C}\right)\times\frac{L_1}{3}$$

$$= a_1\bar{x}_1 - \frac{L_1^2}{6}(M_B + 2M_C)$$

$$6EI.\theta_C = \frac{6a_2\bar{x}_1}{L_1} - L_1(M_B + 2M_C) \;...(iii)$$

(ii) Consider the span CD

Similarly considering the span CD and taking D as origin and $x$ positive to the left, it can be shown that

$$6EI.\theta_C = \frac{6a_2\bar{x}_1}{L_1} - L_2(M_D + 2M_C)$$

[In the above case the slope at C (i.e., $\theta_C$) will have opposite sign than that given by equation ($iii$). The reason is that the direction of $x$ from B for the span BC, and from D for span CD are in the opposite direction].

Hence the above equation becomes as

$$\therefore -6EI\theta_C = \frac{6a_2\bar{x}_1}{L_2} - L_2(M_D + 2M_C) \qquad ...(iv)$$

Adding equation ($iii$) and ($iv$), we get

$$0 = \frac{6a_1\bar{x}_1}{L_1} - L_1(M_B + 2M_C) + \frac{6a_2\bar{x}_2}{L_2} - L_2 + (M_D + 2M_C)$$

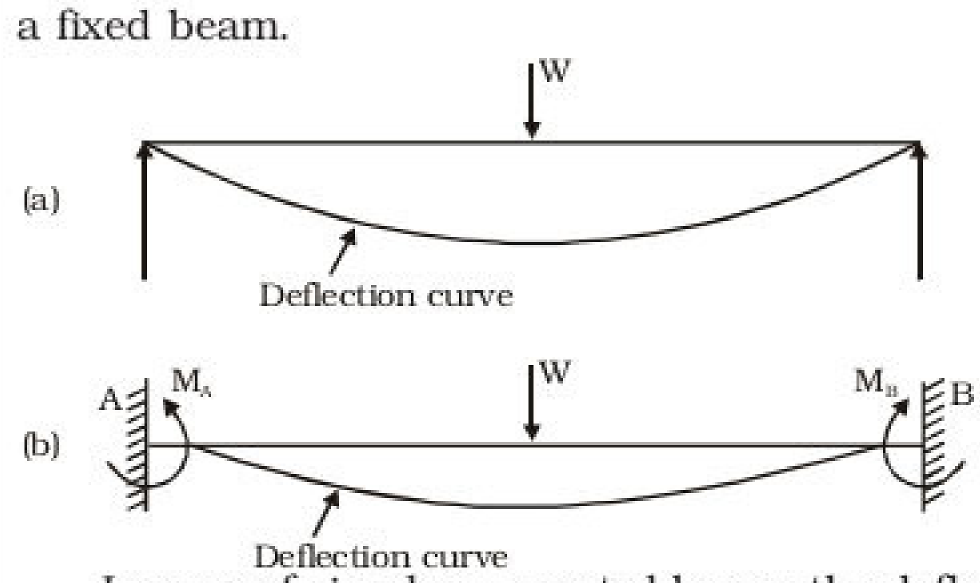
$$= \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2} - L_1M_B - 2L_1M_C - L_2M_D - 2L_2M_C$$

$$L_1.M_B + L_2M_D + 2M_C(L_1 + L_2) = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

$$M_BL_1 + 2M_C(L_1 + L_2) + M_DL_2 = \frac{6a_1x_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

**FIXED BEAM:-**

A beam whose both ends are fixed is known as a fixed beam. Fixed beam is also called a built-in or encaster beam. In case of a fixed beam both its ends are rigidly fixed and the slope and deflection at the fixed ends are zero. But the fixed ends are subjected to end moments. Hence end moments are not zero in case of a fixed beam.




In case of simply supported beam, the deflection is zero at the ends. But the slope is not zero at the ends as shown in Fig. (a).

In case of fixed beam, the deflection and slope are zero at the fixed ends as shown in Fig. (b). The slope will be zero at the ends if the deflection curve is horizontal at the ends. To bring the slope back to zero (i.e., to make the deflection curve horizontal at the fixed ends), the end moments $M_A$ and $M_B$ will be acting in which $M_A$ will be acting anti-clockwise and $M_B$ will be acting clockwise as shown in Fig. (b).

A beam which is supported on more than two supports is known as continuous beam. This section deals with the fixed beams and continuous beam. In case of fixed beams the B.M. diagram, slope and deflection for various types of loading such as point loads, uniformly distributed load and combination of point load and u.d.l., are discussed. In case of continuous beam, Clapeyron's equation of three moments and application of this equation to the continuous beam of simply supported ends and fixed ends are explained.

**BENDING MOMENT DIAGRAM FOR FIXED BEAMS**:

Fig. 15.1 (c) shows a fixed beam AB of length L subjected to two loads W and 2W at a distance of $\frac{L}{4}$ form each ends.

Let $R_A$ = Reaction at A

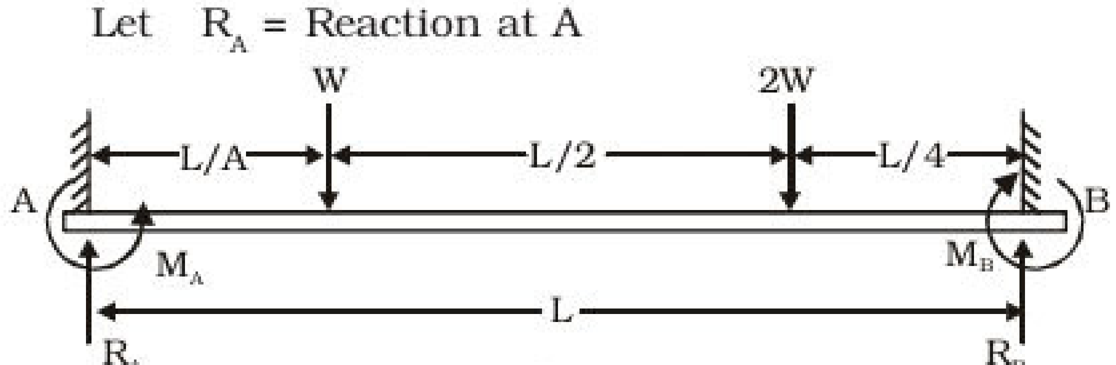



Fig.

$R_B$ = Reaction at B
$M_A$ = Fixed end moment at A
$M_B$ = Fixed end moment at B

The above four quantities i.e., $R_A$, $R_B$, $M_A$ and $M_B$ are unknown.

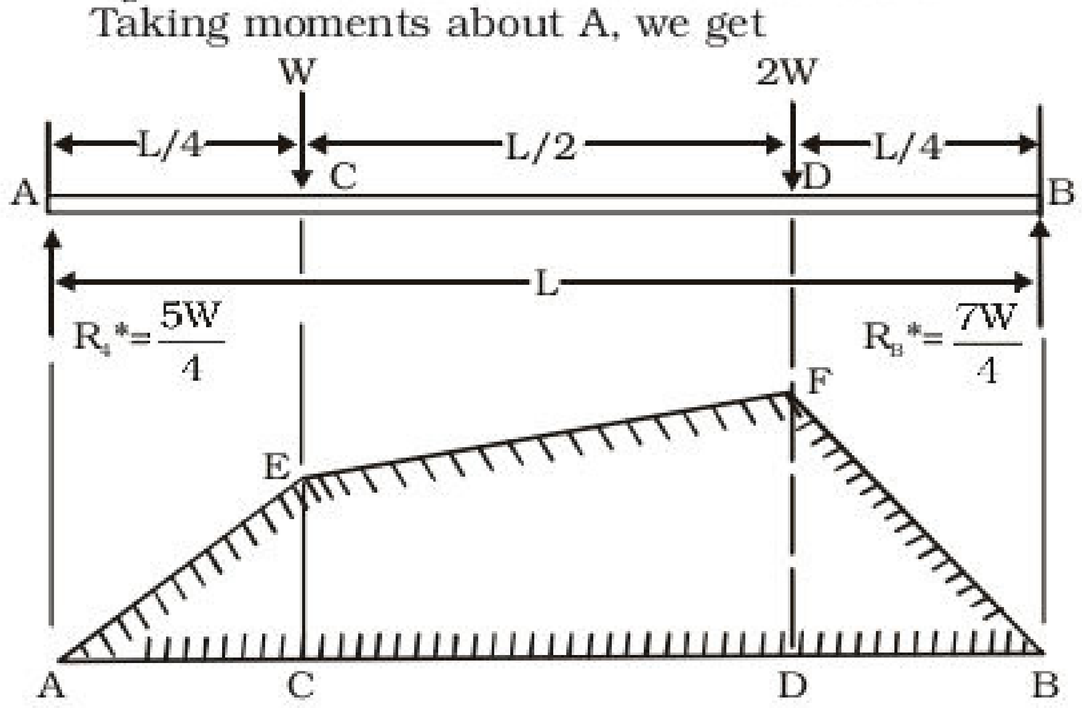
The values of $R_A$, $R_B$, $M_A$ and $M_B$ are calculated by analysing the given beam in the following two stages :

(i) A simply supported beam subjected to given vertical lnads as shown in Fig.

Consider the beam AB as simply supported.

Let $R_A^*$ = Reaction at A due to vertical loads
$R_B^*$ = Reaction at B due to vertical loads.

Taking moments about A, we get




B.M. diagram considering beam as simply supported
Fig.

$$R_B^* \times L = W \times \frac{L}{4} + 2W \times \frac{3L}{4}$$

$$\therefore \qquad R_B^* = \frac{W}{4} + \frac{6W}{4} = \frac{7W}{4}$$

and $\quad RA^*$ = Total load – $R_B^*$

$$= 3W - \frac{7W}{4} = \frac{5W}{4}$$

B.M. at A = 0, B.M. at B = 0.

$$\text{B.M. at C} = \frac{5W}{4} \times \frac{L}{4} = \frac{5WL}{16}$$

$$\text{B.M. at D} = \frac{7W}{4} \times \frac{L}{4} = \frac{5WL}{16}$$

Now B.M. diagram can be drawn as shown in Fig. (b). In this case, B.M. at any point is a sagging (+ve) moment.

(ii) A simply supported beam subjected to end moments only (without given loading) as shown in Fig.

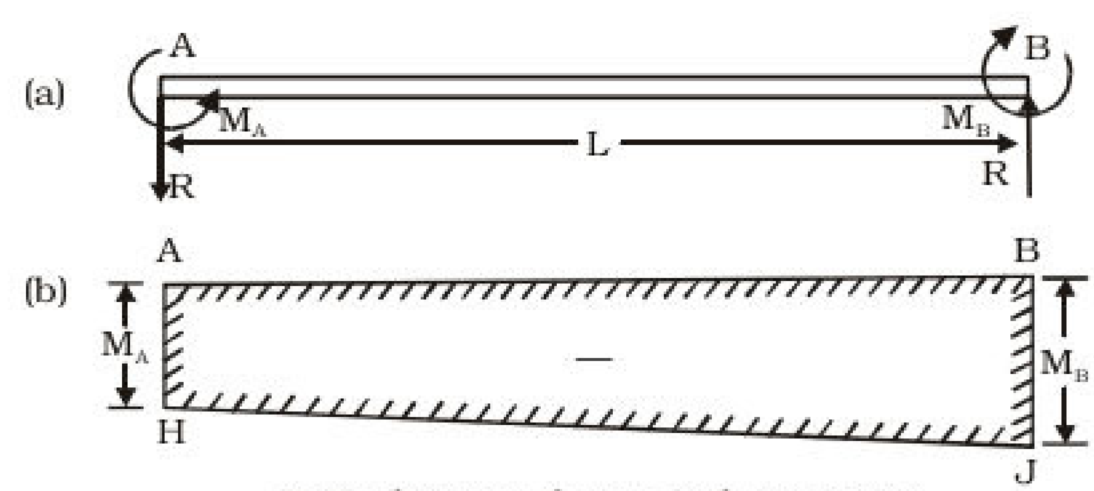
Let MA = Fixed end moment at A
MB = Fixed end moment at B
R = Reaction* at each end due to these moments.

As the vertical loads acting on the beam are not symmetrical fthey are W at distance L/4 from A and 2W at a distance L/4 from B). the fixed end moments will be different.

Suppose $M_B$ is more than $M_A$ and reaction R at B is acting upwards. Then reaction R at A will be acting downwards as there is no other load on the beam. ($\Sigma Fy = 0$). Taking moments about A, we get clockwise moment at A = Anti-clockwise moment at A.

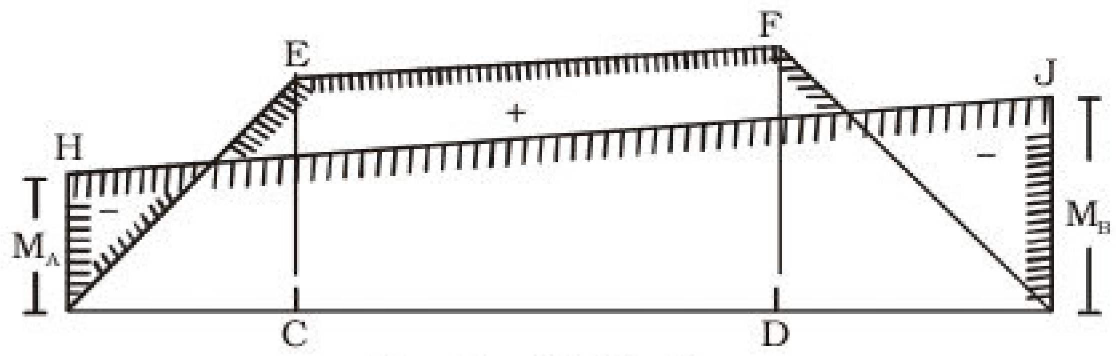
$$M_B = M_A + R.L$$

$$\therefore \qquad R = \frac{M_B - M_A}{L} \qquad \text{...(A)}$$




B.M. diagram due to end moments

As MB has been assumed more than $M_A$, the R.H.S. of equation (A) will be positive. This means the magnitude of reaction R at B is positive. This also means that the direction of reaction R at B is according to our assumption. Hence the reaction R will be upwards at B and downwards at A as shown in Fig. (a). The B.M. diagram for this condition is shown in Fig. (b). In this case, B.M. at any point is a hogging (-ve) moment.



Resultant B.M. diagram
Fig.

Now the final reactions RA and RB are given by

$R_A = R_A^* - R$

and $RB = RB^* + R$

In the above two equations, $R_A^*$ and $R_B^*$ are already calculated. They are : $R_A^* = 5W/4$ and $R_B^* = 7W/4$. But the value of R is in terms of $M_B$ and $M_A$. It is given by $R = (M_B-M_A)/L$. Hence to find the value of ft, we must calculate the value of $M_B$ and $M_A$ first.

**To find the values of $M_4$ and $M_B$**

Let $M_x$ = B.M. at any section at a distance x from A due to vertical loads

$M_x'$= B.M. at any section at a distance x from A due to end moments.

The resultant B.M. at any section at a distance $x$ from A

$= Mx - M_x'$

($M_x$ is +ve but $M_x'$ is -ve)

But B.M. at any section is also equal to $EI\dfrac{d^2y}{dx^2}$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = M_x - M_x'$$

Integrating the above equation for the entire length, we get

$$EI\left[\frac{dy}{dx}\right]_0^L = \int_0^L Mxdx - \int_0^L M_x'dx$$

But $\dfrac{dy}{dx}$ represents the slope. And slope at the fixed ends i.e., at A and B are zero. The above equation can be written as

$$EI\left[\left(\frac{dy}{dx} at\ x = L\right) - \left(\frac{dy}{dx} at\ x = 0\right)\right]$$

$$= \int_0^L M_x.dx - \int_0^L M_x'dx$$

$$EI[0-0] = \int_0^L M_x.dx - \int_0^L M_x'dx$$

$$\text{or } 0 = \int_0^L M_x.dx - \int_0^L M_x'dx \qquad ...(ii)$$

Now $\int_0^L M_x.dx$ represents the area of B.M. diagram due to vertical loads and $\int_0^L M_x'.dx$ represents the area of B.M. diagram due to end moments.

Let

$a$ = Area of B.M. diagram due to vertical loads

$a'$ = Area of B.M. diagram due to end moments.

Then $\int_0^L M_x.dx = a$ and $\int_0^L M_x'.dx = a'$

Substituting these values in equation (*ii*), we get

$0 = a - a^1$

or $a = a^1$

The above equation shows that area of B.M. diagram due to vertical loads is equal to the area of B.M. diagram due to end moments.

Again consider the equation ( *i* )

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

Multiplying the above equation by $x$, we get

$$EI.x.\frac{d^2y}{dx^2} = xM_x - x.M_x'$$

Integrating for the whole length of the beam i.e., from 0 to L, we get

$$\int_0^L EI.x\frac{d^2y}{dx^2}.dx = \int_0^L x.M_x.dx - \int_0^L x.M_x'dx$$

$$EI\int_0^L x.\frac{d^2y}{dx^2}.dx = \int_0^L x.M_x.dx - \int_0^L x.M_x'.dx \qquad ...(iii)$$

In the above equation, $M_x.dx$ represents the area of B.M. diagram due to vertical loads at a distance $x$ from the end A. And the term ($x.M_x.dx$) represents the moment of area of B.M.

$$\therefore \qquad \int_0^L x.M_x'.dx$$

Diagram about the end A. Hence $\int_0^L x.M_x'.dx$ represents the moment of the total area of B.M.

diagram due to vertical loads about A, and it is equal to total area of B.M. diagram due to vertical loads multiplied by the distance of C.G. of area from A,

$$\therefore \qquad \int_0^L x.M_x dx.d\bar{x}$$

where $\bar{x}$ = Distance of the C.G. of B.M. diagram due to vertical loads.

Similarly $\int_0^L x.M_x dx = d'\bar{x}'$

where $\bar{x}$ = Distance of the C.G. of B.M. diagram due to end moments. Substituting the above values in equation (*iii*), we get

$$EI\int_0^L x.\frac{d^2y}{dx^2}.dx = a\bar{x} - a'\bar{x}'$$

or $$EI\int_0^L \left[x\frac{dy}{dx} - y\right]_0^L = a\bar{x} - a'\bar{x}'$$

$$\left[\because \frac{d}{dx}\left(x\frac{dy}{dx} - y\right) = \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) - \frac{dy}{dx} = x\frac{d^2y}{dx^2}\right]$$

or $$EI\left[\left(\frac{xdy}{dx} - y\right)_{at\ x=L} - \left(x\frac{dy}{dx} - y\right)_{at\ x=0}\right] = a\bar{x} - a'\bar{x}'$$

or $$EI\left[(L\theta_B - y_B) - (0\times\theta_A - y_A)\right] = a\bar{x} - a'\bar{x}'.$$

Since, slope and deflection at A and B are zero, hence $\theta_A$, $\theta_B$, $y_A$ and $y_B$ are zero.

$\therefore$ $0 = a\bar{x} - a'\bar{x}'$

or $a\bar{x}$ - $a'\bar{x}'$

But from equation we have

$a = a'$

$\bar{x} = \bar{x}'$

Hence the distance of C.G. of B.M. diagram due to vertical loads from A is equal to distance of C.G. of B.M. diagram due to end moments from A.

Now by using equations the unknowns $M_A$ and $M_B$ can be calculated

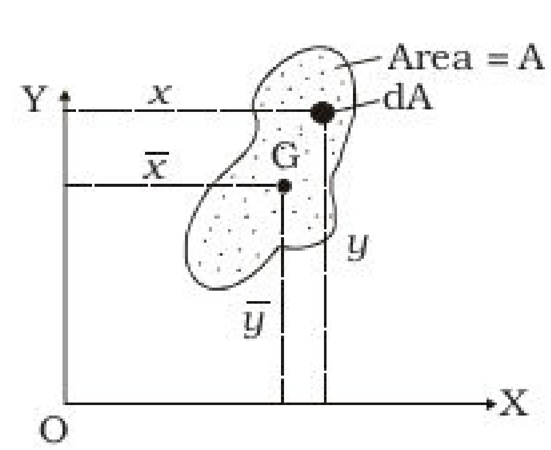
This also means that $M_A$ and $M_B$ can be calculated by

(i) equating the area of B.M. diagram due to vertical loads to the area of B.M. diagram due to end moments.

(ii) equating the distance of C.G. of B.M. diagram due to vertical loads to the distance C.G. of B.M. diagram due to end moments. The distance of C.G. must be taken from the same end in both cases.

**Moment of Inertia product of Inertia Principle Axis :**

◆ **Centroid :**

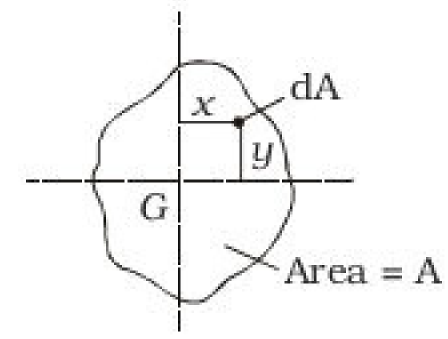



$$\bar{x} = \frac{\int x.dA}{\int dA} = \frac{\Sigma A.x}{\Sigma.A} = \bar{y} = \frac{A_1\bar{x}_1 + A_2\bar{x}_2 + \ldots A_n\bar{x}_n}{A_1 + A_2 + \ldots A_n}$$

$$\bar{y} = \frac{\int y dA}{\int dA} = \frac{\Sigma Ay}{\Sigma A} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2 + \ldots A_n\bar{y}_n}{A_1 + A_2 + \ldots A_n}$$

◆ **Moment of Inertia :**

Let XX and YY are centroidal axis of area A consider, any elemental area dA having co-ordinates ($x$, $y$) with respect to origin at centroid.




M.I (area M.I) is defined as 2[nd] moment of Inertia about an axis. It is always non-zero and non (–ve) quantity [I > 0].

The M.I about centroidal axis XX is ;

$$I_{xx} = I_{xsey} = \int_0^A y^2.dA$$

$$I_{yy} = I_{ysey} = \int_0^A x^2.dA$$

◆ **Product of Inertia (I × $y$) :**

It is moment of area about both axis

$$I_{xy} = I_{xysey} = \int_0^A x.y.dA$$

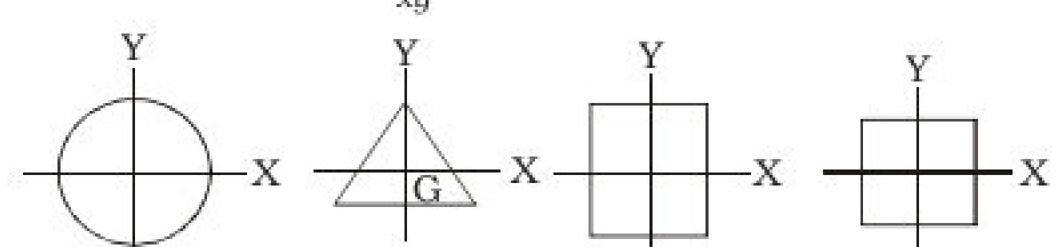
It may be > Zero, = Zero, < Zero.

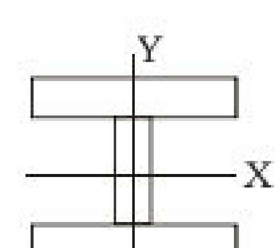
If $xx$ and $yy$ are centroid axis then :

$I_{xy}$ = Centroidal product of Inertia

**Note :**

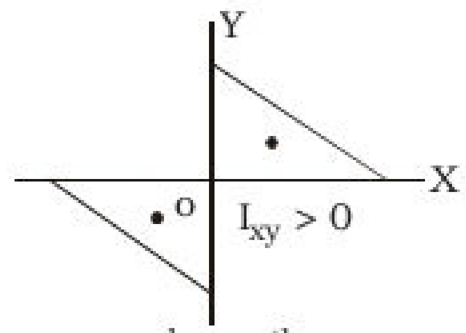
If area is symmetrical either about $x$ or about $y$ or about both then; $I_{xy} = 0$




Here, $I_{xy} = 0$

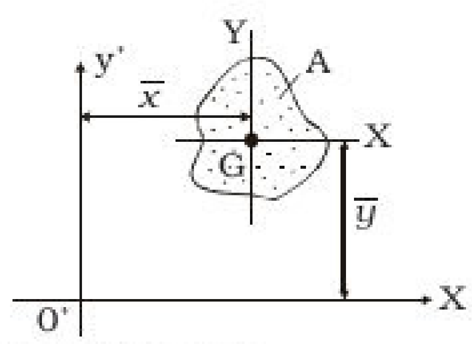
2) If area is located in 1[st] or 3[rd] quadrant, then $I_{xy}$ is (+ve).




3) If area is located in $2^{nd}$ or $4^{th}$ quadrant, then $I_{xy} < 0$ (–ve)

◆ **PARALLEL AXIS THEOREM :**

Let $xx$ and $yy$ are centroidal axis of area A and let $x'x'$ and $y'y'$ area $II^{lr}$ axis to $xx$ and $yy$ respective then ;




$I_{xy}$ = Self product of Inertia

$= \int x \,.\, y .\, dA$

$I_{x'x'} = I_{xx} + A\overline{y}^2$

$I_{y'y'} = I_{yy} + A\overline{x}^2$

$I_{xx} = I_{xself}$ = Self M.I

$= \int x^2 .dA$

$I_{yy}$ = self M.I about xx.

$= \int y^2 .\, dA$

$I_{yy}$ = Self M.I about xx

$= \int x^2 .\, dA$

◆ **PERPENDICULAR AXIS THEOREM :**

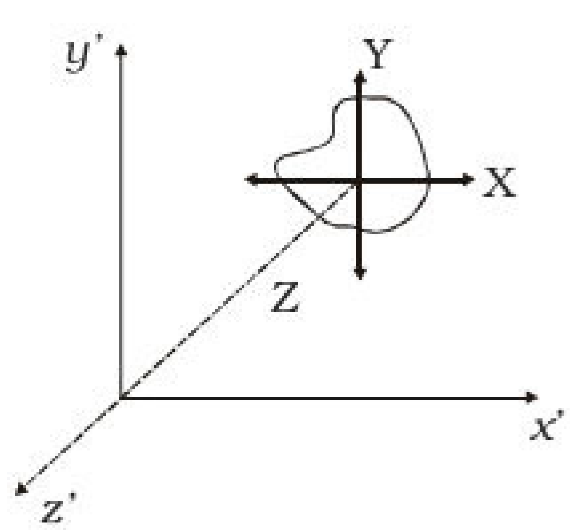
Let $xx$ and $yy$ are transverse centroidal axis and $zz$ is longitudinal or polar centroidal axis which $\perp^{lr}$ to $xx$ and $yy$.

**Hence ;**

$I_{zz} = I_{xx} + I_{yy}$

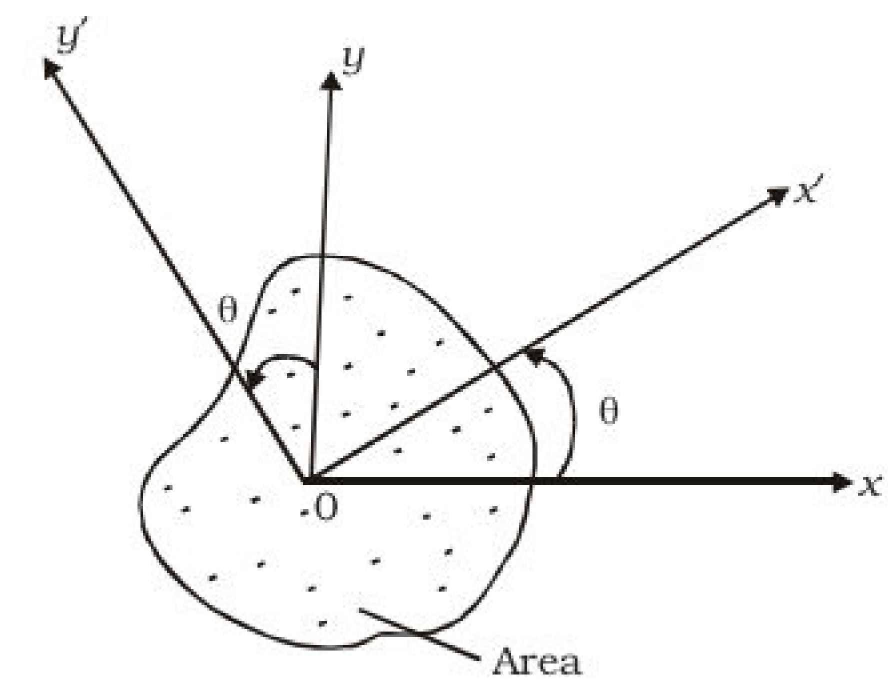
similarly,

$Iz'z' = Ix'x' + Iy'y'$




◆ **Transformation of M.I :** (Texs or quantity can be transformed from one axis to other).

It $xx$ and $yy$ are centroidal axis of area A and $I_{xx}$ and $I_{yy}$ are M.I about $xx$ and $yy$ and let $I_{xy}$ is product of Inertia about $xy$.




Let $x'$ and $y'$ axis are inclined to $x$ and $y$ axis as shown in figure.

Let $I_{x'x'}$ and $I_{y'y'}$ are M.I about $x'$ and $y'$ axis respective and $I_{x'y'}$ is product of Inertia about transformed axis

$$\therefore\ I_{x'x'} = \frac{I_{xx} \quad I_{yy}}{2} + \left(\frac{I_{xx} \quad I_{xy}}{2}\right)\cos 2\theta - I_{xy}\sin 2\theta$$

Put $\theta = \theta + 90°$

$$I_{y'y'} = \left(\frac{I_{xx} + I_{yy}}{2}\right) - \left(\frac{I_{xx} - I_{yy}}{2}\right)\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin\theta + I_{xy}\cos 2\theta$$

**Note :**

1. If the angle θ is such that product of Inertia about $x'$ and $y'$ is zero, then such axis are called Principal axis. Hence, $I_{x'x'}$ and $y'y'$ are called principal moment Inertia.
   Principal moment of Inertia are either maximum or minimum M.I.
2. The angle of principal axis from $x'x$ and $yy$ are given by :
   $I_{x'y'} = 0$
   $$\tan 2\theta = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$
3. The magnitude of principal M.I is given by
   $$I_1/I_2 = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2}$$
4. If an area has axis of symmetry, then product of Inertia about those axis will be zero. Hence, axis of symmetry is always principal axis. The reverse is not compulsary.
5. Circle has infinite principal axis.

◆ **TRUSSES:**

For trusses the total number of unknown include the forces in '$m$' number of bars of the truss and the total '$r$' of support reactions, Since the truss members are assumed to carry only axial forces and lie in the same plane, the force system acting at each joint is coplanar and concurrent. Thus it is only necesaary to sotisfy $\sum F_x = 0$ and $\sum F_y = 0$. If '*j*' number of joints are available, the total number of equations available for solution is 2*j*. This determinocy of the trues can be specfied by comparing (*m* + *r*) with the total number of available of equilibrium. We have

m + r = 2j      staticolly determinate

m + r > 2j      staticolly indeterminate

For truss anolysis it is assumed that:

→ Bars are pin connected

→ Joints are frictionless hinges

→ Loads are applied at the joint only

→ Strees in each member is constant along its length.

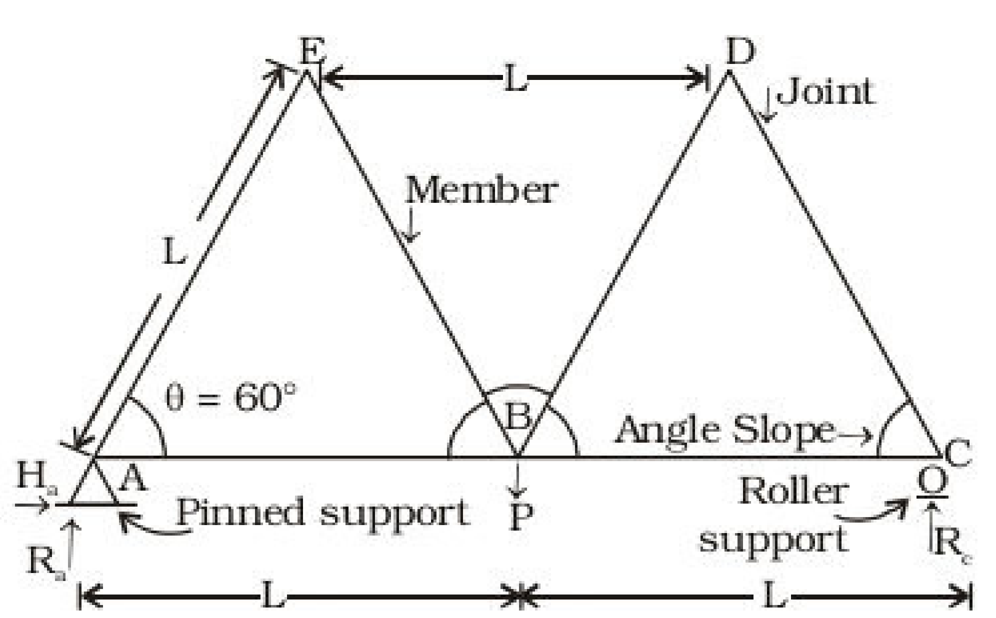
The objective of truss analysis is to determine the reactions and member forces. The method used for carrying out the anolysis with the equations of equilibrium and by considering only parts of the structure through analyzing its free body diagram to solve the unknowns.

**(1) METHOD OF JOINTS FOR TRUSS ANALYSIS:**

We start by assuming that all members are in tension reaction. A tension member experiences pull forces at both ends of the bar and usually denoted by positve (tve) sign. When a member is experiencing a push force at both ends, the bar is said to be in compression mode and designated as negative sign.

In the joints methods, a virtual cut is made around a joint and the cut portion is isolated as a free body diagram. Using the equilibrium equations of $\sum F_x = 0$ and $\sum F_y = 0$, He unknown member forces can be solved. It is assumed that all members are joined in the form of an ideal pin, and that all forces are in tension (the reactions).

An imaginory section may be completely passed around a joint in a truss. The joint has become a free body in equilibrium under the force applied to it. The equation $\sum H = 0$ and $\sum_v = 0$ may be applied to the joint to determine the unknown forces in members meeting there. It is evident that no more than two unknowns can be determined at a joint with these two equations.




A simple truss model supported by pinned and roller support at is end. Each triangle has the same length, L and it is equilateral where degree of angle, θ is 60° on every angle. The support reqction Ra, Rc can be determined by taking a point of moment either at point A or point C, whereas H*a* = 0 ( o other horizontal force).

Here are some simple guidelines for this methods:

(i) Firstly draw the free body diagram (FBD)

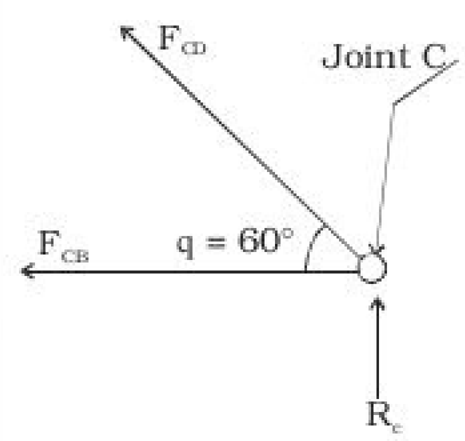
(ii) Solve the reaction of the given structure.

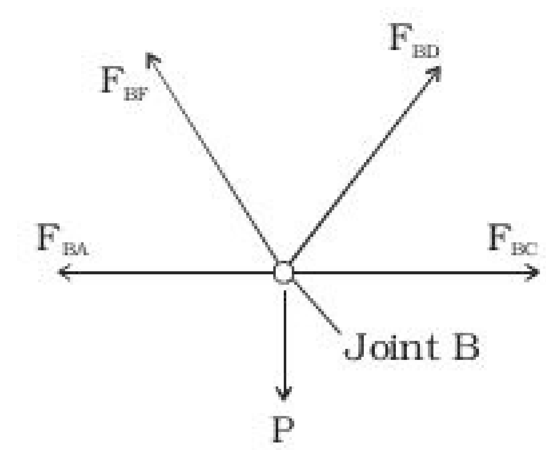
(iii) Select a joint with a minimum number of unknown (not more than 2) and analyse it with $\sum F_x = 0$ & $\sum F_y = 0$

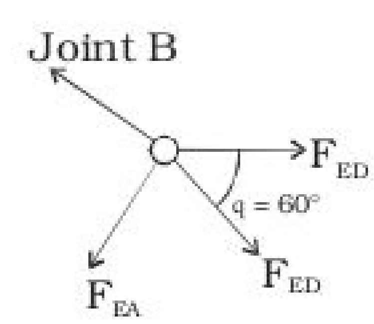
(iv) Proceed to the rest of the joints and again concentrating on joints that have very minimal of unknouns.

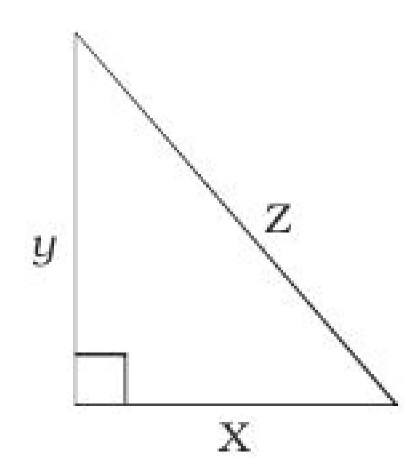
(v) Check number of forces at unused joints with $\sum F_x = 0$ & $\sum F_y = 0$

(vi) Tabulate he member forces whether it is in tension (tve) or compression (-ve) reaction




The figure showing 3 selected joints at B, C and E. The force in each member can be determined from any joint or point. The best way to start is by selecting the eariest joint like joint $\sum$ where the reaction Rc is already obtained and with only 2 unknown, forces of FCB and FCD. Both can evaluated with $\sum Fx = 0$ and $\sum Fy = 0$ rules. At joint E, there are 3 unknown, forces of FEA, FEB and FED, which may lead to more complex solution compared to 2 unknown values. For checking purpose joint B is selected to show hot the equation of $\sum Fx$ is equal to $\sum Fy$ which leads to zero value $\sum Fx = \sum Fy = 0$. Each members condition should be indicated clearly as wheher it is in tension (tve) or ir compression (–ve) state).

Trignometric Functions:
Taking an angle between member X and Z

$\cos\theta = \dfrac{X}{Z}$

$\sin\theta = \dfrac{y}{z}$

$\tan\theta = \dfrac{y}{x}$.

**5. Strain Energy Method [Castiglianos 2nd theorem]**

**Assumption :**

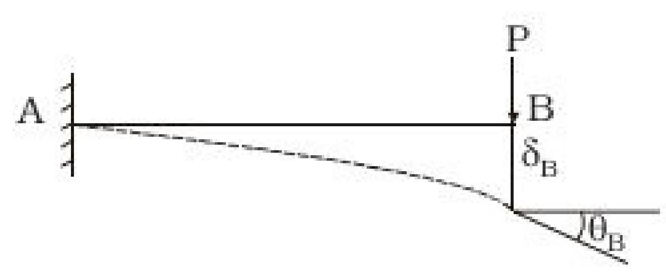
i) Material is homogeneous, isotropic & linear elastic in which Hooks law is valid

ii) Temperature is constant

iii) Supports are un-yielding

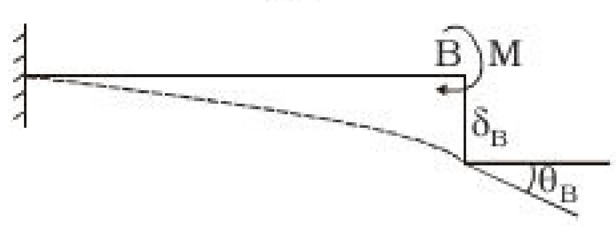
**According to this theorem;**

" The deflection in any beam, truss or frame at any point in the direction of the force is equal to 1st partial derivative of total S.E w.r.t. that force"

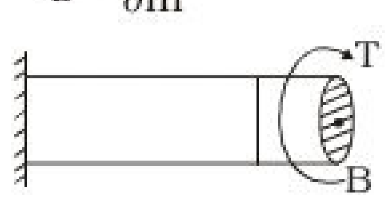



$$\delta_B = \frac{\partial u}{\partial p}$$

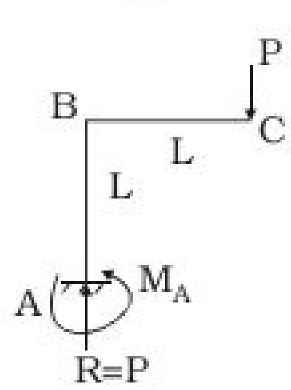

$$\Delta h = \frac{\partial u}{\partial w}$$




$$\theta_B = \frac{\partial u}{\partial m}$$



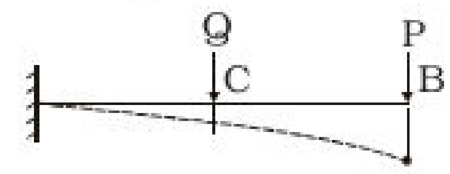

$$\theta = \text{Twist angle at B} = \frac{\partial u}{\partial T}$$




$$\delta_C = \frac{\partial u}{\partial P}$$

**Note :**

1. The (positive) slope & deflect are those which are in the direction of moment & load respectively
2. If u is only due to BM, then deflection/slope is caused by BM only & if u is function B.M, S.F & Axial force, then deflection is under the combined effect of (BM, S.F & A.F)
3. Special cash (a)
   Let load is applied as B but deflection is required at other point C, then apply a pseudo load in the direction of desired deflection
   Say, pseudo load is Q



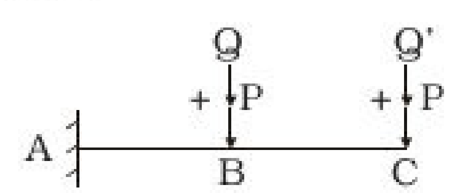

The vertical deflection at C is given by

$$\Delta_C = \left(\frac{\partial u}{\partial Q}\right)_{Q=0} \ ; \ u = f_n\,(P \,\&\, Q)$$

Special case 1. (b). If slope at any point B is required then apply a pseudo moment at B
Then,

$$\theta_B = \left(\frac{\partial u}{\partial M}\right)_{M=0}$$


Special case 1. (c).




$\delta_B = P + Q$
$\delta_C = P + Q$

4. This method is suitable for prismatic & non prismatic members both & suitable for cantilever frames & arches.

**STRAIN ENERGY :**

1. Due to B.M $\quad u = \displaystyle\int \frac{M_x^2\,ds}{2EI}$
2. Due to Axial force $\quad u = \displaystyle\int \frac{P_x^2\,.dx}{2AE}$
3. Due to Torque $\quad u = \displaystyle\int \frac{T_x^2\,dx}{2GI_P}$
4. Due to S.F $\quad u = \displaystyle\int \frac{S^2\,.dx}{2A_r\,.\,G}$

where ;
ds = Any elementary length
EI = Flexural Rigidity
AE = Axial Rigidity
$GI_P$ = Torsional rigidity
$A_r$ = Reduced area in shear
$A_r.G$ = Shear Rigidity