DIFFERENTIAL EQUATION

1. INTRODUCTION

Differential equation constitute a very important part of mathematics as it has many applications in real life. Various laws of physics are often in the form of equations involving rate of change of one quantity with respect to another. As the mathematical equivalent of a rate is a derivative, differential equation arise very naturally in real life and methods for solving them acquire paramount importance.

Definition

An equation involving the dependent variable and independent variable and also the derivatives of the dependable variable is known as differential equation.

For example:

(i) $\frac{dy}{dx} = \frac{x}{y^{1/3}(1+x^{1/3})}$ (ii) $\frac{d^2y}{dx^2} = -p^2y$

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = 3\frac{d^2y}{dx^2}$ (iv) $x^2\left(\frac{dy}{dx}\right)^2 = y^2+1$

Differential equations which involve only one independent variable are called ordinary differential equation.

2. ORDER AND DEGREE OF DIFFERENTIAL EQUATIONS

2.1 Order

The order of a differential equation is the order of the highest derivative involved in the differential equation

For example:

(i) $\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)^2 + 4x = 0$ is the differential equation of the first order because maximum derivative of y with respect to x is $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2} = -p^2y$ is the differential equation of the second order because maximum derivative of y w.r.t x is $\frac{d^2y}{dx^2}$

(iii) $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of the third order because maximum derivative of y w.r.t x is $\frac{d^3y}{dx^3}$

2.2 Degree

The degree of a differential equation is the degree of the highest differential coefficient when the equation has been made rational and integral as far as the differential coefficients are concerned.

For example:

(i) $\frac{dy}{dx} = \frac{x}{y^{1/3}(1+x^{1/3})}$ is the differential equation of first degree, because power of the highest order derivative $\frac{dy}{dx}$ is 1.

(ii) $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of second degree, because power of highest order deriavative $\frac{d^3y}{dx^3}$ is 2.

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{2/3} = 3\frac{d^2y}{dx^2}$ is the differential equation of third degree, because power of highest order devivative $\frac{d^2y}{dx^2}$ is 3 (after cubing)

Illustration 1: Find the order and degree of the following differential equations.

(i) $\sqrt{\dfrac{d^2y}{dx^2}} = \sqrt[3]{\dfrac{dy}{dx}+3}$

(ii) $\dfrac{d^2y}{dx^2} = \left\{1+\left(\dfrac{dy}{dx}\right)^4\right\}^{5/3}$

(iii) $y = px + \sqrt{a^2p^2+b^2}$ where $p = \dfrac{dy}{dx}$

Sol. (i) The given differential equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}+3\right)^2$$

Hence order = 2, degree = 3

(ii) The given differential equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left[1+\left(\frac{dy}{dx}\right)^4\right]^5$$

Hence order = 2, degree = 3

(iii) The given differential equation can be written as

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2 + b^2$$

Hence order = 1, degree = 2

3. FORMATION OF ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constants from a relation in the variables and constants. Consider an equation containing n arbitrary constants. Differentiating this equation n times we get n additional equations containing n arbitrary constants and derivatives. Eliminating n arbitrary constants from the above (n + 1) equations, we obtain differential equation involving nth derivative.

Thus if an equation contains n arbitrary constants, the resulting differential equation obtained by eliminating these constants will be a differential equation of nth order. i.e., an equation of the form

$$\phi\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ ....\frac{d^ny}{dx^n}\right) = 0$$

Illustration 2 : Find the differential equation of the family of all circles which pass through the origin and whose centre lie on y–axis

Sol. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

If it passes through (0, 0), then c = 0

∴ The equation of circle is $x^2 + y^2 + 2gx + 2fy = 0$

Since the centre of the circle lies on y–axis then g = 0

∴ The equation of the circle is

$x^2 + y^2 + 2fy = 0$ ...(i)

This represents family of circles.

Differentiating, we get

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0 \qquad ...(ii)$$

From (i) and (ii), we get

or, $(x^2 - y^2)\dfrac{dy}{dx} - 2xy = 0$ Which is the required differential equation.

4. SOLUTION OF A DIFFERENTIAL EQUATION

The solution of the differential equation is a relation is a relation between the independent and dependent variable free from derivatives satisfying the given differential equation.

Thus the solution of dy/dx = m could be obtained by simply integrating both sides i.e., y = mx + c, where c is arbitrary constant.

(a) **General solution (or complete premitive)**

The general solution of a differential equation is the relation between the variables (not involving the derivatives) which contain the same number of the arbitrary constants as the order of the differential equation.

Thus the general solution of the differential equation $\dfrac{d^2y}{dx^2} = 4y$ is y = A sin2x + B cos2x, where A and B are the constants.

(b) **Particular solution or Integral**

A solution which is obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution.

Illustration 3: Show that $v = \frac{A}{r} + B$ is the general solution of the second order differential equation $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$, where A and B are arbitrary constant.

Sol. Given $v = \frac{A}{r} + B$

Differentiating twice $\frac{d^2v}{dr^2} = \frac{2A}{r^3}$ ...(i)

From (i) $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = \frac{2A}{r^3} + \frac{2}{r}\left(-\frac{A}{r^2}\right) = \frac{2A}{r^3} - \frac{2A}{r^3} = 0$

Putting A = 4, B = 5 in $v = \frac{A}{r} + B$ we get a particular solution of the differential equation

$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$ is $v = \frac{4}{r} + 5$.

Illustration 4: Show that $y = ae^x + be^{2x} + ce^{-3x}$ is a solution of the equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$

Sol. We have

$y = ae^x + be^{2x} + ce^{-3x}$ ...(i)

Differentiating, we get

$y_1 = ae^x + 2be^{2x} - 3ce^{-3x}$ ...(ii)

$\therefore$ $y_1 - y = be^{2x} - 4ce^{-3x}$ ...(iii)

Differentiating (ii), we get

$y_2 = ae^x + 4be^{2x} + 9ce^{-3x}$

$\therefore$ $y_2 - y_1 = 2be^{2x} + 12ce^{-3x}$ ...(iv)

Now, (iv) – 2 (iii) $\Rightarrow$ $y_2 - y_1 - 2(y_1 - y) = 20ce^{-3x}$

or, $y_2 - 3y_1 + 2y = 20ce^{-3x}$ ...(v)

Differentiating, $y_2 = ae^x + 4be^{2x} + 9ce^{-3x}$, we get

$y_3 = ae^x + 8be^{2x} - 27ce^{-3x}$

Now $y_3 - 3y_2 + 2y_1 = -60ce^{-3x}$ ...(vi)

And (vi) + 3(v) $\Rightarrow$ $y_3 - 3y_2 + 2y_1 + 3(y_2 - 3y_1 + 2y) = 0$

or, $y_3 - 7y_1 + 6y = 0$

i.e., $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$, which is the required differential equation

5. METHOD OF SOLVING AN EQUATION OF THE FIRST ORDER AND FIRST DEGREE

A differential equation of the first order and first degree can be written in the form

$$\frac{dy}{dx} = f(x, y)$$

or, M dx + N dy = 0, where M and N are functions of x and y

Method – 1

(i) Variable Separation:

The general form of such an equation is

$f(x)dx + f(y)dy = 0$ ...(i)

Integrating, we get

$\int f(x)dx + \int f(y)dy = c$ which is the solution of (i)

(ii) Solution of differential equation of the type $\frac{dy}{dx} = f(ax + by + c)$:

Consider the differential equation $\frac{dy}{dx} = f(ax + by + c)$ ...(i)

Where f(ax + by + c) is some function of ax + by + c.

Let z = ax + by + c

$\therefore$ $\frac{dz}{dx} = a + b\frac{dy}{dx}$

or, $\frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$

From (i) $\frac{\frac{dz}{dx} - a}{b} = f(z)$ or, $\frac{dz}{dx} = b\,f(z) + a$

or, $$\frac{dz}{b\,f(z)+a} = dx \qquad ...(ii)$$

In the differential equation (ii), the variables x and z are separated.

Integrating, we get

$$\int \frac{dx}{b\,f(z)+a} = \int dx + c$$

or, $$\int \frac{dx}{b\,f(z)+a} = x + c\text{, where } z = ax + by + c$$

This represents the general solution of the differential equation (i)

Illustration 5. Solve $(x-y)^2 \frac{dy}{dx} = a^2$

Sol. Putting x – y = v

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx} \Rightarrow dx = \frac{v^2}{v^2 - a^2}dv$$, variable have been separated

Integrating, we get $\int dx = \int \frac{v^2}{v^2-a^2}dv$

or, $$2y + k = a \log \frac{x-y-a}{x-y+a}$$

Illustration 6. Solve, $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Sol. Let z = x + y

$$\therefore \frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\frac{dz}{dx} - 1 = \sin z + \cos z$$

or, $$dx = \frac{dz}{\sin z + \cos z + 1}$$

Integrating, we get

$$\int dx = \int \frac{dz}{\sin z + \cos z + 1} = \int \frac{dt}{t+1}\text{, putting } t = \tan\frac{z}{2}$$

i.e., $x + c = \log|t+1|$ This is the required general solution.

(iii) Solution of differential equation of the type $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, **where** $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Here $$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \text{ where } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \qquad ...(i)$$

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$ (say)

$\therefore \quad a_1 = \lambda a_2,\ b_1 = \lambda b_2$

From (i), $$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2 y + c_1}{a_2x+b_2y+c_2}$$

$$= \frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2} \qquad ...(ii)$$

Let $z = a_2x + b_2y$

$$\therefore \frac{dz}{dx} = a_2 + b_2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{dz}{dx} - a_2}{b_2} \qquad ...(iii)$$

From (ii) and (iii), we get $$\frac{\frac{dz}{dx} - a_2}{b_2} = \frac{\lambda z + c_1}{z + c_2}$$

or, $$\frac{dz}{dx} = \frac{b_2(\lambda z + c_1)}{z+c_2} + a_2 = \frac{\lambda b_2 z + b_2c_1 + a_2z + a_2c_2}{z+c_2}$$

or, $$dx = \frac{z+c_2}{(\lambda b_2 + a_2)z + b_2c_1 + a_2c_2}dz$$, where x and z are seperated

Integrating, we get

$$x + c = \int \frac{z+c_2}{(\lambda b_2+a_2)z + b_2c_1 + a_2c_2}dz \text{ where } z = a_2x + b_2y$$

Method – 2

(i) Homogeneous differential equation:

A function f(x, y) is called homogeneous function of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example:

(a) $f(x, y) = x^2y^2 - xy^3$ is a homogeneous function of degree four, since

$f(\lambda x, \lambda y) = (\lambda^2 x^2)(\lambda^2 y^2) - (\lambda x)(\lambda^3 y^3)$

$= \lambda^4 (x^2y^2 - xy^3)$

$= \lambda^4 f(x, y)$

(b) $f(x, y) = x^2 e^{x/y} + \dfrac{x^3}{y} + y^2 \log\left(\dfrac{y}{x}\right)$ is a homogeneous function of degree two, since

$$f(\lambda x, \lambda y) = (\lambda^2 x^2) e^{\lambda x/\lambda y} + \frac{\lambda^3 x^3}{\lambda y} + (\lambda^2 y^2) \log\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \lambda^2 \left[x^2 e^{x/y} + \frac{x^3}{y} + y^2 \log\left(\frac{y}{x}\right)\right]$$

$$= \lambda^2 f(x, y)$$

A differential equation of the form $\dfrac{dy}{dx} = f(x, y)$, where f(x, y) is a homogoneous polynomial of degree zero is called a homogeneous differential equation. Such equations are solved by substituting $v = \dfrac{y}{x}$ or $\dfrac{x}{y}$ and then seperating the variables.

Illustration 7 : Solve $\dfrac{dy}{dx} = \dfrac{y(2y - x)}{x(2y + x)}$ ...(i)

Sol. Since each of the functions y(2y – x) and x(2y + x) is a homogeneous function of degree 2, so the given equation is a homogeneous differential equation.

∴ Putting y = vx

Differentiating w.r.t x, we get $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

From (i),

$$v + x\frac{dv}{dx} = \frac{vx(2vx - x)}{x(2vx + x)} = \frac{v(2v - 1)}{2v + 1}$$

$$\Rightarrow \quad 2\,dv + \frac{dx}{x} + 2\frac{dx}{x} = 0$$

Integrating, we get

$2v + \log v + \log x^2 = \log k$ or, $xy = ke^{-2y/x}$

(ii) Differential equation reducible to homogeneous forms:

Equation of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ where $\dfrac{a}{a'} \neq \dfrac{b}{b'}$

can be reduced to homogeneous form by changing the variables x, y to x', y' by equations $x = x' + h$ and $y = y' + k$ where h and k are constants to be chosen so as to make the given equation homogeneous, we have

$dx = dx'$ and $dy = dy'$

∴ The given equation becomes

$$\frac{dy'}{dx'} = \frac{a(x' + h) + b(y' + k) + c}{a'(x' + h) + b'(y' + k) + c'}$$

$$= \frac{ax' + by' + (ah + bk + c)}{a'x' + b'y' + (a'h + b'k + c')}$$

Now, we choose h and k so that

$$ah + bk + c = 0$$

and $\quad a'h + b'k + c' = 0$

From these equation we get the values of of h and k in terms of the coefficients.

Then the given equation reduces to

$$\frac{dy'}{dx'} = \frac{ax' + by'}{a'x' + b'y'}$$

Which is the homogeneous form.

Method – 3

(i) Linear differential equation:

A differential equation is said to be linear if the dependent variable y and its derivative occur in the first degree.

An equation of the form $\frac{dy}{dx} + Py = Q$ ...(i)

where P and Q are functions of x only or constant is called a linear equation of the first order

Similarly $\frac{dx}{dy} + Px = Q$ is a linear differential equation

where P and Q are functions of y only. To get the general solution of the above equations we shall determine a function R of x called Integrating function (I.F). We shall multiply both sides of the given equation by R

$\therefore$ where, $R = e^{\int P\,dx} = I.F$ ...(iii)

From (i) and (iii), we get

$$e^{\int P\,dx}.\frac{dy}{dx} + Pye^{\int P\,dx} = Q.e^{\int P\,dx} \qquad \text{or,}$$

$$\frac{d}{dx}\left(ye^{\int P\,dx}\right) = Q.e^{\int P\,dx}$$

Integrating, we get

$ye^{\int P\,dx} = \int Q.e^{\int P\,dx}dx + c$ is the required solution.

Note: We remember the solution of the above equation as

$$y(I.F) = \int Q(I.F)dx + c$$

Illustration 8 : Solve $2x\frac{dy}{dx} = y + 6x^{5/2} - 2\sqrt{x}$

Sol. The given equation can be written as

$$\frac{dy}{dx} + \left(\frac{-1}{2x}\right)y = 3x^{3/2} - \frac{1}{\sqrt{x}} \qquad \text{...(i)}$$

This is the form of $\frac{dy}{dx} + Py = Q$

Hence $I.F = e^{\int \frac{-1}{2x}dx} = e^{-\frac{1}{2}\log x} = \frac{1}{\sqrt{x}}$ ...(ii)

From (i) and (ii), we get $y = \frac{3}{2}x^{5/2} - \sqrt{x}\log x + c\sqrt{x}$

(ii) Differential equation reducible to the linear form:

Sometimes equations which are not linear can be reduced to the linear form by suitable transformation.

Here, $f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x)$ ...(i)

Let, $f(y) = u \quad \Rightarrow \quad f'(y)\,dy = du$

Then (i) reduces to

$\frac{du}{dx} + uP(x) = Q(x)$ Which is of the linear differential equation form.

Illustration 9: Solve $\sec^2\theta\,d\theta + \tan\theta(1 - r\tan\theta)\,dr = 0$

Sol. The given equation can be written as

$$\frac{d\theta}{dr} + \frac{\tan\theta}{\sec^2\theta} = \frac{r\tan^2\theta}{\sec^2\theta}$$

or, $$\left(\frac{\sec^2\theta}{\tan^2\theta}\right)\frac{d\theta}{dr} + \frac{1}{\tan\theta} = r$$

or, $$\operatorname{cosec}^2\theta\frac{d\theta}{dr} + \cot\theta = r \qquad \text{...(i)}$$

Let $\cot\theta = u$

$\Rightarrow$ $-\operatorname{cosec}^2\theta\,d\theta = du$

Then (i) reduces to

$$-\frac{du}{dr} + u = r \qquad \text{or,} \qquad \frac{du}{dr} - u = -r \qquad \text{...(ii)}$$

Which is a linear differential equation.

So, $I.F = e^{\int -1\,dr} = e^{-r}$ ...(iii)

Form (ii) and (iii), we get

$ue^{-r} = -\int re^{-r}dr = re^{-r} + \int e^{-r}dr$, by parts

$= re^{-r} - e^{-r} + c$ or, $u = r - 1 + ce^{r}$

or, $\cot\theta = r - 1 + ce^{r}$

(iii) Extended form of linear equations :

Bernoulli's equation:

An equation of the form $\frac{dy}{dx} + Py = Qy^{n}$, where P and Q are function of x alone or constants and n is constant, other than 0 and 1, is called a Bernoulli's equation.

Here $\frac{dy}{dx} + Py = Qy^{n}$

Dividing by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + P.\frac{1}{y^{n-1}} = Q$$

Putting $\frac{1}{y^{n-1}} =$ and differentiating w.r.t x,

we get $-\frac{(n-1)}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{1}{y^n}\frac{dy}{dx} = \frac{-1}{n-1}\frac{dv}{dx}$

or, $\frac{dv}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}$, the equation becomes

$$\frac{dv}{dx} + (1-n)Pv = Q(1-n)$$

Which is a linear equation with v as independent variable.

Illustration 10 : Solve $\cos^2 x\frac{dy}{dx} - y\tan 2x = \cos^4 x$, where $|x|\ \ \frac{\pi}{4}$ and $y\left(\frac{\pi}{4}\right) = \frac{3\sqrt{3}}{8}$

Sol. The given equation can be written as

$$\frac{dy}{dx} - y\tan 2x\sec^2 x = \cos^2 x$$

This is the form of $\frac{dy}{dx} + Py = Q$

Here $P = -\tan 2x \sec^2 x$, $Q = \cos^2 x$

$\therefore$ $\int P\,dx = -\int \tan 2x\sec^2 x\,dx$

$$= -\int \frac{2\tan x}{1-\tan^2 x}\sec^2 x\,dx$$

$$= \int \frac{dt}{t} \quad \text{Putting } 1 - \tan^2 x = t$$

$\therefore$ $-2\tan x\sec^2 x\,dx = dt$

$= \log t = \log(1 - \tan^2 x)$

$\therefore$ I. F. $= e^{\int P\,dx} = e^{\log(1-\tan^2 x)} = 1 - \tan^2 x$

$\therefore$ The solution is

$$y(1-\tan^2 x) = \int (1-\tan^2 x)\cos^2 x\,dx + c$$

$$= \int \cos 2x\,dx + c = \frac{\sin 2x}{2} + c \qquad ...(i)$$

Given, At $x = \frac{\pi}{6}$, $y = \frac{3\sqrt{3}}{8}$, then $\frac{3\sqrt{3}}{8}\left(1 - \frac{1}{3}\right) = \frac{\sqrt{3}}{4} + c$

or $c = 0$

Hence from (i),

$$y(1-\tan^2 x) = \frac{\sin 2x}{2}$$

or $$y = \frac{\sin 2x}{2(1-\tan^2 x)}$$

Method – 4

Exact differential equation:

A differential equation is said to be exact if it can be derived from its solution (primitive) directly by differentiation, without any elimination, multiplication etc.

For example, the differential equation $x\,dy + y\,dx = 0$ is an exact differential equation as it is derived by direct differentiation for its solution, the function $xy = c$

Illustration 11 : Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$

Sol. The given equation can be written as

$y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$

or, $(y\,dx + x\,dy) + xy\,(y\,dx - x\,dy) = 0$

or, $d\,(xy) + xy\,(y\,dx - x\,dy) = 0$

Dividing by x^2y^2, we get

$$\frac{d(xy)}{x^2y^2} + \frac{y\,dx - x\,dy}{xy} = \qquad \text{or,}$$

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} =$$

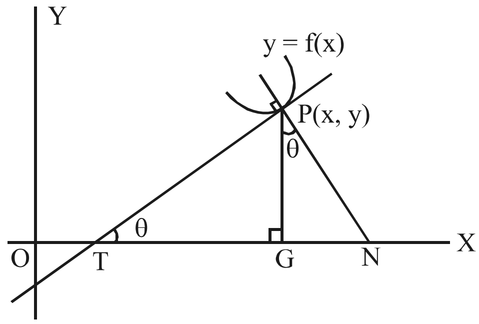
Integrating, we get

$$-\frac{1}{xy} + \log x - \log y = c$$

Which is the required solution.

Application of differential equations :

In solving some geometrical problems, the following results are very helpful.



Let PT and PN be the tangent and the normal at P(x, y). Let the tangent at P makes an angle θ with the x-axis.

Then the slope of the tangent at $P = \tan\theta = \left(\frac{dy}{dx}\right)_P$

and the slope of the normal at $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$.

Equation of the tangent at P(x, y) is

$$Y - y = \left(\frac{dy}{dx}\right)_P (X - x)$$

Equation of the normal at P(x, y) is

$$Y - y = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(X - x)$$

From Δ PGT $\quad \sin\theta = \frac{PG}{PT} = \frac{y}{PT}$

$\therefore$ $$\boxed{PT = y\,\text{cosec}\theta \text{ (lenght of the tangent)} \quad = y\frac{\sqrt{1+\tan^2\theta}}{\tan\theta} \quad = y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}}$$

And, $\tan\theta = \frac{PG}{TG} = \frac{y}{TG}$

$\Rightarrow$ $\boxed{TG = y\cot\theta \text{ (length of the sub tangent)} = \dfrac{y}{\dfrac{dy}{dx}}}$

From Δ PGN $\quad \cos\theta = \dfrac{PG}{PN} = \dfrac{y}{PN}$

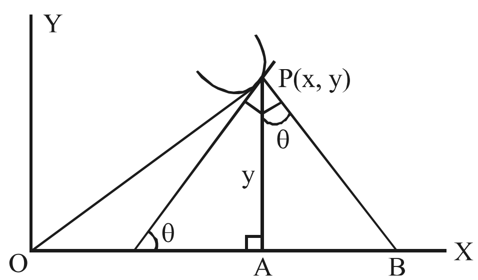
$$\Rightarrow \boxed{\begin{aligned} &PN = y\sec\theta \text{ (length of the normal)} \\ &= y\sqrt{1+\tan^2\theta} \;= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \end{aligned}}$$

$\tan\theta = \dfrac{GN}{y}$

$\Rightarrow$ $\boxed{GN = y\tan\theta = y\dfrac{dy}{dx} \text{ (length of the sub normal)}}$

Illustration 12 : If the length of the sub-normal at any point P on the curve is directly proportional to OP^2, where O is the origin, then form the differential equation of the family of curves and hence find the family of curves.

Sol. Here $AB = y\tan\theta = y\dfrac{dy}{dx}$



Also $OP^2 = x^2 + y^2$

Given, length of the subnormal = k. OP^2

or, $\quad y\dfrac{dy}{dx} = k(x^2 + y^2)$

or, $\quad 2y\dfrac{dy}{dx} - 2ky^2 = 2kx^2 \quad$ ...(i)

Let $\quad y^2 = t \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dt}{dx} \quad$ ...(ii)

From (i) and (ii), we get $\dfrac{dt}{dx} - 2kt = 2kx^2$

Which is a linear differential equation.

$\therefore \quad I.F = e^{\int -2k\,dx} = e^{-2kx}$

$\therefore \quad$ The solution is

$$t\,.\,e^{-2kx} = \int 2kx^2\, e^{-2kx}\, dx + c$$

$$= 2k\left[x^2\frac{e^{-2kx}}{-2k} + \frac{2}{2k}\int x\, e^{-2kx}\, dx\right]$$

$$= 2k\left[x^2\frac{e^{-2kx}}{-2k} + \frac{1}{k}\left\{x\frac{e^{-2kx}}{-2k} + \frac{1}{2k}\int e^{-2kx}\, dx\right\}\right]$$

$$= -x^2 e^{-2kx} - \frac{x\,e^{-2kx}}{k} + \frac{1}{k}\frac{e^{-2kx}}{2k} + c \quad \text{or,}$$

$$y^2 = -x^2 - \frac{x}{k} + \frac{1}{2k^2} + c\, e^{2kx}$$