The Fundamental Equation Of Dynamics (Part - 1)

Q. 59. An aerostat of mass m starts coming down with a constant acceleration w. Determine the ballast mass to be dumped for the aerostat to reach the upward acceleration of the same magnitude. The air drag is to be neglected.

Ans. Let R be the constant upward thurst on the aerostat of mass m, coming down with a constant acceleration w. Applying Newton's second law of motion for the aerostat in projection form

$F_y = mw_y$

mg - R - mw   (1)

Now, if Am be the mass, to be dumped, then using the Eq. $F_y = mw_y$

$R - (m - \Delta m)g = (m - \Delta m)w,$ (2)

From Eqs. (1) and (2), w e get, $\Delta m = \dfrac{2mw}{g+w}$

Q. 60. In the arrangement of Fig. 1.9 the masses m_0, m_1, and m_2 of bodies are equal, the masses of the pulley and the threads are negligible, and there is no friction in the pulley. Find the acceleration w with which the body m_0 comes down, and the tension of the thread binding together the bodies m_1 and m_2, if the coefficient of friction between these bodies and the horizontal surface is equal to k. Consider possible cases.

Ans. Let us write the fundamental equation of dynamics for all the three blocks in terms of projections, having taken the positive direction of x and y axes as shown in Fig; and using the fact that kinematical relation between the accelerations is such that the blocks move with same value of acceleration (say w)
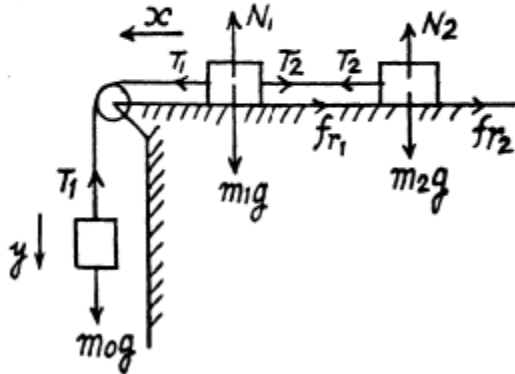
$m_0 g - T_1 = m_0 w$ (1)

$T_1 - T_2 - km_1 g = m_1 w$ (2)

and $T_2 - km_2 g = m_2 w$ (3)

The simultaneous solution of Eqs. (1), (2) and (3) yields,

$$w = g\frac{[m_0 - k(m_1 + m_2)]}{m_0 + m_1 + m_2}$$

and

$$T_2 = \frac{(1+k)m_0}{m_0 + m_1 + m_2}m_2 g$$

As the block m_0 moves down with acceleration w, so in vector form

$$\vec{w} = \frac{[m_0 - k(m_1 + m_2)]\vec{g}}{m_0 + m_1 + m_2}$$

Q. 61. Two touching bars 1 and 2 are placed on an inclined plane forming an angle α with the horizontal (Fig. 1.10). The masses of the bars are equal to m_1 and m_2, and the coefficients of friction between the inclined plane and these bars are equal to k_1 and k_2 respectively, with k1 > k 2. Find:

(a) the force of interaction of the bars in the process of motion;
(b) the minimum value of the angle α at which the bars start sliding down.
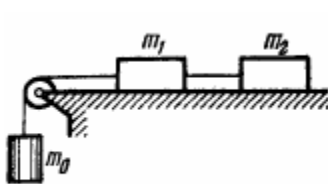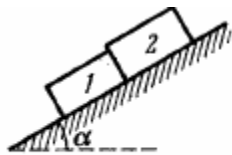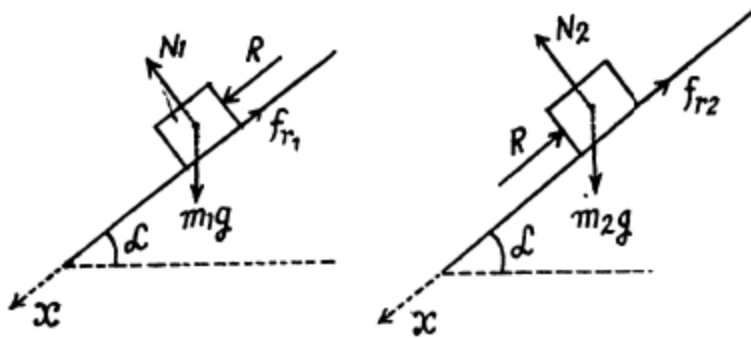

Fig. 1.9. Fig. 1.10.

Ans. Let us indicate the positive direction of Jt-axis along the incline (Fig.). Figures show the force diagram for the blocks.

Let, R be the force of interaction between the bars and they are obviously sliding down with the same constant acceleration w.

Newton's second law of motion in projection form along x-axis for the blocks gives :

$m_1 g \sin \alpha - k_1 m_1 g \cos \alpha + R = m_1 w$ (1)

$m_2 g \sin \alpha - R - k_2 m_2 g \cos \alpha = m_2 w$ (2)

Solving Eqs. (1) and (2) simultaneously, we get

$w = g \sin \alpha - g \cos \alpha \dfrac{k_1 m_1 + k_2 m_2}{m_1 + m_2}$ and

$R = \dfrac{m_1 m_2 (k_1 - k_2) g \cos \alpha}{m_1 + m_2}$

(b) when the blocks just slide down the plane, w = 0 , so from Eqn. (3)

$g \sin \alpha - g \cos \alpha \dfrac{k_1 m_1 + k_2 m_2}{m_1 + m_2} = 0$

or, $(m_1 + m_2) \sin \alpha = (k_1 m_1 + k_2 m_2) \cos \alpha$

Hence $\tan \alpha = \dfrac{(k_1 m_1 + k_2 m_2)}{m_1 + m_2}$

Q. 62. A small body was launched up an inclined plane set at an angle α = 15° against the horizontal. Find the coefficient of friction, if the time of the ascent of the body is η = 2.0 times less than the time of its descent.

Ans. Case 1. When the body is launched up :

Let k be the coefficeint of friction, u the velocity of projection and l the distance traversed along the incline. Retarding force on the block = mg sin α + k mg cos α and hence the retardation = g sin α + k g cos α .

Using the equation of particle kinematics along the incline,

$0 - u^2 - 2 (g \sin \alpha + k g \cos \alpha) l$

or, $l = \dfrac{u^2}{2\,(g\sin\alpha + k\,g\cos\alpha)}$ (1)

and $0 = u - (g\sin\alpha + k\,g\cos\alpha)\,t$

or, $u = (g\sin\alpha + k\,g\cos\alpha)\,t$ (2

Using (2) in (1) $l = \dfrac{1}{2}(g\sin\alpha + k\,g\cos\alpha)\,t^2$ (3)

Case (2). When the block comes downward, the net force on the body - mg sin a - km g cos a and hence its acceleration - g sin a - k g cos a Let, t be the time required then

$l = \dfrac{1}{2}(g\sin\alpha - k\,g\cos\alpha)\,t'^2$ (4)

From Eqs. (3) and (4)

$\dfrac{t^2}{t'^2} = \dfrac{\sin\alpha - k\cos\alpha}{\sin\alpha + k\cos\alpha}$

But $\dfrac{t}{t'} = \dfrac{1}{\eta}$ (according to the question),

Hence on solving we get

$k = \dfrac{(\eta^2 - 1)}{(\eta^2 + 1)}\tan\alpha = 0{\cdot}16$

Q. 63. The following parameters of the arrangement of Fig. 1.11 are available: the angle α which the inclined plane forms with the horizontal, and the coefficient of friction k between the body m_1 and the inclined plane. The masses of the pulley and the threads, as well as the friction in the pulley, are negligible. Assuming both bodies to be motionless at the initial moment, find the mass ratio m_2/m_1 at which the body m_2

(a) starts coming down;
(b) starts going up;
(c) is at rest.

Ans. At the initial moment, obviously the tension in the thread connecting m_1 and m_2 equals the weight of m_2.

(a) For the block m_2 to come down or the block m_1 to go up, the conditions is

$m_2 g - T \geq 0$ and $T - m_1 g\sin\alpha - fr \geq 0$

where T is tension and f_r is friction which in the limiting case equals $k m_1 g \cos\alpha$. Then
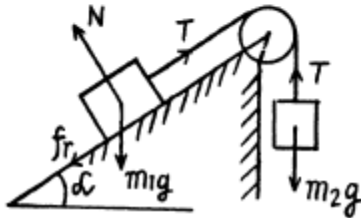
or $\quad m_2 g - m_1 \sin\alpha > k m_1 g \cos\alpha$

or $\quad \dfrac{m_2}{m_1} > (k \cos\alpha + \sin\alpha)$

(b) Similarly in the case

$m_1 g \sin\alpha - m_2 g > fr_{lim}$

or, $\quad m_1 g \sin\alpha - m_2 g > k m_1 g \cos\alpha$

or, $\quad \dfrac{m_2}{m_1} < (\sin\alpha - k \cos\alpha)$

(c) For this case, neither kind of motion is possible, and fr need not be limiting.

Hence, $\quad (k \cos\alpha + \sin\alpha) > \dfrac{m_2}{m_1} > (\sin\alpha - k \cos\alpha)$

Q. 64. The inclined plane of Fig. 1.11 forms an angle α = 30° with the horizontal. The mass ratio $m_2/m_1 = \eta = 2/3$. The coefficient of friction between the body m_1 and the inclined plane is equal to k = 0.10. The masses of the pulley and the threads are negligible. Find the magnitude and the direction of acceleration of the body m_2 when the formerly stationary system of masses starts moving.

Ans. From the conditions, obtained in the previous problem, first we will check whether the mass m_2 goes up or down.
Here, $m_2/ m_1 = \eta > \sin\alpha + k \cos\alpha$, (substituting the values). Hence the mass m_2 will come down with an acceleration (say w). From the free body diagram of previous problem,

$$m_2 - g - T = m_2 w \qquad (1)$$

and $T - m_1 g \sin \alpha - k m_1 g \cos \alpha = m_1 w \qquad (2)$

Adding (1) and (2), we get,

$$m_2 g - m_1 g \sin \alpha - k m_1 g \cos \alpha = (m_1 + m_2) w$$

$$w = \frac{(m_2/m_1 - \sin \alpha - k \cos \alpha) g}{(1 + m_2/m_1)} = \frac{(\eta - \sin \alpha - k \cos \alpha) g}{1 + \eta}$$

Substituting all the values, $w = 0.048\,g \approx 0.05\,g$

As m_2 moves down with acceleration of magnitude $w = .05\,g > 0$, thus in vector form acceleration of m_2:

$$\vec{w_2} = \frac{(\eta - \sin \alpha - k \cos \alpha)\,\vec{g}}{1 + \eta} = 0.05\,\vec{g}.$$

Q. 65. A plank of mass m_1 with a bar of mass m_2 placed on it lies on a smooth horizontal plane. A horizontal force growing with time t as F = at (a is constant) is applied to the bar. Find how the accelerations of the plank w_1 and of the bar w_2 depend on t, if the coefficient of friction between the plank and the bar is equal to k. Draw the approximate plots of these dependences.

Ans. Let us write the Newton's second law in projection form along positive x-axis for the plank and the bar

$$fr - m_1 w_1, \quad fr = m_2 w_2 \qquad (1)$$

At the initial moment, fr represents the static friction, and as the force F grows so does the friction force fr, but up to it's limiting value

$$fr = fr_{s(max)} = kN = k m_2 g.$$

Unless this value is reached, both bodies moves as a single body with equal acceleration. But as soon as the force fr reaches the limit, the bar starts sliding over the plank i.e. $w_1 \geq w_2$.
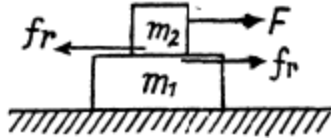
Substituting here the values of w_1 and w_2 taken from Eq. (1) and taking into account that

$f_r = k\, m_2\, g$, we obtain, $(at - km_2\, g)/m_2 \geq \dfrac{km_2}{m_1} g$, were the sign "="

corresponds to the moment $t = t_0$

(say)

Hence, $\quad t_0 = \dfrac{k\, g\, m_2\, (m_1 + m_2)}{a\, m_1}$

If $t \leq t_0$, then $\dot{w}_1 = \dfrac{km_2\, g}{m_1}$ (constant). and

$w_2 = (at - km_2\, g)/m_2$

On this basis w_1 (r) and w_2 (t), plots are as shown in the figure of answersheet.

Q. 66. A small body A starts sliding down from the top of a wedge (Fig. 1.12) whose base is equal to l = 2.10 m. The coefficient of friction between the body and the wedge surface is k = 0.140. At what value of the angle a will the time of sliding be the least? What will it be equal to?

Ans. Let us designate the x-a x is (Fig.) and apply $F_x = m\, w_x$ for body A

$mg \sin \alpha - k\, m\, g \cos \alpha = m\, w$
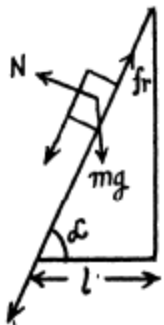
or, $\quad w = g \sin \alpha - k\, g \cos \alpha$

Now, from kinematical equation :

$l \sec \alpha = 0 + (1/2)\, w\, t^2$

or, $\quad t = \sqrt{2\, l \sec \alpha / (\sin \alpha - k \cos \alpha)\, g}$

$= \sqrt{2\, l / (\sin 2\, \alpha / 2 - k \cos^2 \alpha)\, g}$

(using Eq. (1)).

for t_{min},
$$\frac{d\left(\dfrac{\sin 2\alpha}{2} - k\cos^2\alpha\right)}{d\alpha} = 0$$

i.e. $\dfrac{2\cos 2\alpha}{2} + 2k\cos\alpha\sin\alpha = 0$

or, $\tan 2\alpha = -\dfrac{1}{k} \Rightarrow \alpha = 49°$

and putting the values of α , k and l in Eq . (2) we get $t_{min} = 1$s.

Q. 67. A bar of mass m is pulled by means of a thread up'an inclined plane forming an angle α with the horizontal (Fig. 1.13). The coefficient of friction is equal to k. Find the angle β which the thread must form with the inclined plane for the tension of the thread to be minimum. What is it equal to?
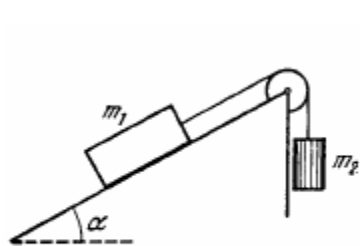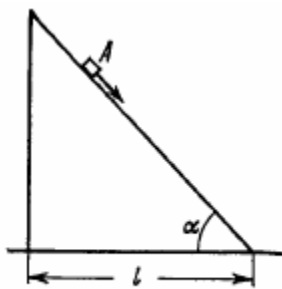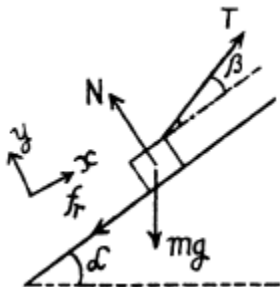

Fig. 1.11.

Fig. 1.12.

Ans. Let us fix the x - y co-ordinate system to the wedge, taking the x - axis up, along the incline and the y - axis perpendicular to it (Fig.).

Let us apply Newton's second law in projection form along x and y axis for the bar :

$T \cos \beta - m g \sin \alpha - fr = 0$ (1)

$T \sin \beta + N - m g \cos \alpha = 0$

or, $N = m g \cos \alpha - T \sin \beta$ (2)

But $f_r = kN$ and using (2) in (1), we get

$T = m g \sin \alpha + k m g \cos \alpha / (\cos \beta + k \sin \beta)$ (3)

For T_{min} the value of $(\cos \beta + k \sin \beta)$ should be maximum

So, $\dfrac{d (\cos \beta + k \sin \beta)}{d \beta} = 0$ or $\tan \beta = k$

Putting this value of β in Eq. (3) we get,

$T_{min} = \dfrac{m g (\sin \alpha + k \cos \alpha)}{1 / \sqrt{1 + k^2} + k^2 / \sqrt{1 + k^2}} = \dfrac{m g (\sin \alpha + k \cos \alpha)}{\sqrt{1 + k^2}}$

Q. 68. At the moment t = 0 the force F = at is applied to a small body of mass m resting on a smooth horizontal plane (a is a constant).
The permanent direction of this force forms an angle α with the horizontal (Fig. 1.14). Find:
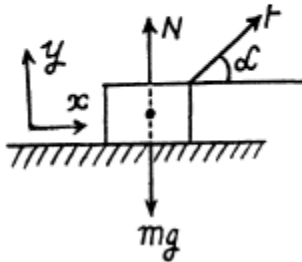
(a) the velocity of the body at the moment of its breaking off the plane;
(b) the distance traversed by the body up to this moment.

Ans. First of all let us draw the free body diagram for the small body of mass m and indicate x - axis along the horizontal plane and y - axis, perpendicular to it, as shown in the figure.
Let the block breaks off the plane at $t = t_0$ i.e. $N = 0$

So, $N = m g - a t_0 \sin \alpha = 0$

or, $t_0 = \dfrac{mg}{a \sin \alpha}$ (1)

From $F_x = m w_x$, for the body under investigation :

$m \, dv/dt = at \cos \alpha$; Integrating within the limits for v (t)

$$m \int_0^v dv_x = a \cos \alpha \int_0^t t \, dt \quad \text{(using Eq. 1)}$$

So, $v = \dfrac{ds}{dt} = \dfrac{a \cos \alpha}{2m} t^2$ (2)

Integrating, Eqn. (2) for s (t)

$$s = \dfrac{a \cos \alpha}{2m} \dfrac{t^3}{3} \quad (3)$$

Using the value of t = t$_o$ from Eq. (1), into Eqs. (2) and (3)

$$v = \dfrac{m \, g^2 \cos \alpha}{2 \, a \sin^2 \alpha} \quad \text{and} \quad s = \dfrac{m^2 g^3 \cos \alpha}{6 \, a^2 \sin^3 \alpha}$$

Q. 69. A bar of mass m resting on a smooth horizontal plane starts moving due to the force F = mg/3 of constant magnitude. In the process of its rectilinear motion the angle α between the direction of this force and the horizontal varies as α = as, where a is a constant, and s is the distance traversed by the bar from its initial position. Find the velocity of the bar as a function of the angle α.
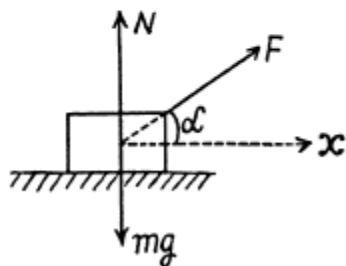
Ans. Newton's second law of motion in projection form, along horizontal or x - axis i.e.

$F_x = m w_x$ gives.

$F \cos (as) = m v \dfrac{dv}{ds}$ (as α = as)

or, $F \cos (as) \, ds = m v \, dv$

Integrating, over the limits for v (s)

$$\frac{F}{m}\int_0^\infty \cos(as)\,ds = \frac{v^2}{2}$$

or $\qquad v = \sqrt{\dfrac{2F\sin\alpha}{m\,a}}$

$= \sqrt{2g\sin\alpha/3\,a} \quad \left(\text{using } F = \dfrac{mg}{3}\right)$

which is the sought relationship.

Q. 70. A horizontal plane with the coefficient of friction k supports two bodies: a bar and an electric motor with a battery on a block. A thread attached to the bar is wound on the shaft of the electric motor. The distance between the bar and the electric motor is equal to l. When the motor is switched on, the bar, whose mass is twice as great as that of the other body, starts moving with a constant acceleration w. How soon will the bodies collide?

Ans. From the Newton's second law in projection from :

For the bar,
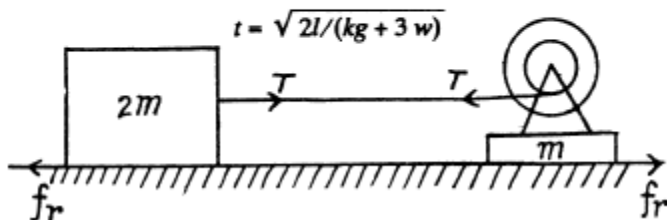
$T - 2\,kwg - (2\,m)\,w \qquad (1)$

For the motor,

$T - kmg = m\,w' \qquad (2)$

Now, from the equation of kinematics inthe frame of bar or motor

$l = \dfrac{1}{2}(w + w')\,t^2 \qquad (3)$

From (1), (2) and (3) wc get on eliminating T and W

$t = \sqrt{2l/(kg + 3\,w)}$

Q. 71. A pulley fixed to the ceiling of an elevator car carries a thread whose ends are attached to the loads of masses m₁ and m₂. The car starts going up with an acceleration w₀. Assuming the masses of the pulley and the thread, as well as the friction, to be negligible find:

(a) the acceleration of the load m₁ relative to the elevator shaft and relative to the car;
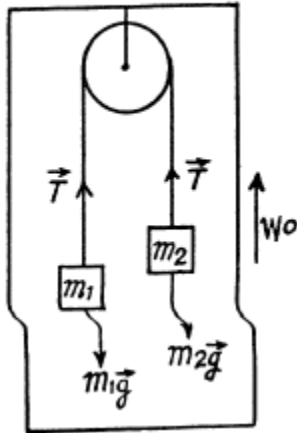**(b) the force exerted by the pulley on the ceiling of the car.**

Ans. Let us write Newton's second law in vector from $\vec{F} = m\vec{w}$, for both the blocks (in the frame ofjround).

$$\vec{T} + m_1\vec{g} = m_1\vec{w_1} \qquad (1)$$
$$\vec{T} + m_2\vec{g} = m_2\vec{w_2} \qquad (2)$$

These two equations contain three unknown quantities $\vec{w_1}, \vec{w_2}$ and $\vec{T}$. The third equation is provided by the kinematic relationship between the accelerations :

$$\vec{w_1} = \vec{w_0} + \vec{w'}, \; \vec{w_2} = \vec{w_0} - \vec{w'} \qquad (3)$$

where $\vec{w'}$ is th acceleration of the mass m₁ with respect to the pulley or elevator car. Summing up termwise the lelt hand and the right-hand sides of these kinematical equations, we get

$$\vec{w_1} + \vec{w_2} = 2\vec{w_0} \qquad (4)$$

The simultaneous solution of Eqs* (1), (2) and (4) yield s

$$\vec{w_1} = \frac{(m_1 - m_2)\vec{g} + 2m_2\vec{w_0}}{m_{1+}m_2}$$

Using this result in Eq. (3) , we get,

$$\vec{w} = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w_0}) \quad \text{and} \quad \vec{T} = \frac{2m_1m_2}{m_1 + m_2}(\vec{w_0} - \vec{g})$$

Using the results in Eq. (3) we get $\quad \vec{w} = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w_0})$

(b) obviously the force exerted by the pulley on the celing of the car

$$\vec{F} = -2\vec{T} = \frac{4m_1m_2}{m_1 + m_2}(\vec{g} - \vec{w_0})$$

Note : one could also solve this problem in the frame of elevator car.

Q. 72. Find the acceleration w of body 2 in the arrangement shown in Fig. 1.15, if its mass is η times as great as the mass of bar l and the angle that the inclined plane forms with the horizontal is equal to α. The masses of the pulleys and the threads, as well as the friction, are assumed to be negligible. Look into possible cases.

Ans. Let us write Newton's second law for both, bar 1 and body 2 in terms of projection having taken the positive direction of x_1 and x_2 as shown in the figure and assuming that body 2 starts sliding, say, upward along the incline
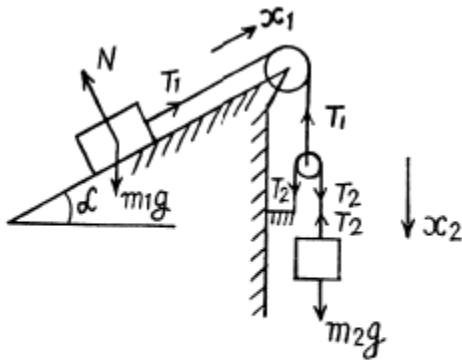
$$T_1 - m_1 g \sin \alpha = m_1 w_1 \qquad (1)$$

$$m_2 g - T_2 = m_2 w \qquad (2)$$

For the pulley, moving in vertical direction from the equation $F_x = m\, w_x$

$$2T_2 - T_1 = (m_p)w_1 = 0$$

(as mass of the pulley $m_p = 0$)

Or $T_1 = 2T_2$ (3)

As the length of the threads are constant, the kinematical relation ship of accelerations becomes

$w = 2w_1$ (4)

$$w = \frac{2g\left(2\dfrac{m_2}{m_1} - \sin\alpha\right)}{\left(4\dfrac{m_2}{m_1} + 1\right)} = \frac{2g(2\eta - \sin\alpha)}{(4\eta + 1)}$$

As $\eta > 1$, w is directed vertically downward, and hence in vector form

$$\vec{w} = \frac{2\vec{g}(2\eta - \sin\alpha)}{4\eta + 1}$$

Q. 73. In the arrangement shown in Fig. 1.16 the bodies have masses m_0, m_1, m_2 ,the friction is absent, the masses of the pulleys and the threads are negligible. Find the acceleration of the body m_1. Look into possible cases.

Ans. Let us write Newton's second law for masses m_1 and m_2 and moving pully in vertical direction along positive x - axis (Fig.) :
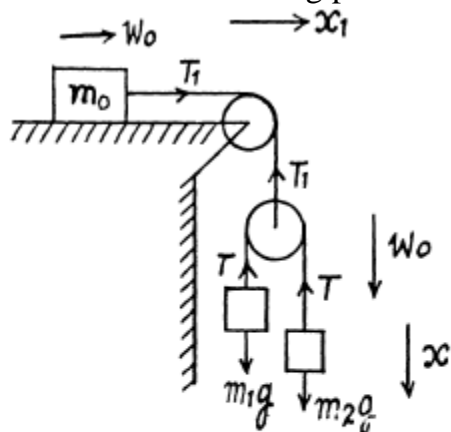
$m_1 g - T = m_1 w_{1x}$ (1)

$m_2 g - T = m_2 w_{2x}$ (2)

$T_1 - 2T = 0$ (as $m = 0$)

or $T_1 = 2T$ (3)

Again using Newton's second law in projection form for mass m_0 along positive x_1 direction
(Fig.), we get

$T1 = m_0\ w_0$ (4)

The kinematical relationship between the accelerations of masses gives in terms of projection on the x – axis

$W_{1x} + w_{2x} = 2w_0$ (5)

Simultaneous solution of the obtained five equations yields

$$w_1 = \frac{[4\,m_1\,m_2 + m_0\,(m_1 - m_2)]\,g}{4\,m_1\,m_2 + m_0\,(m_1 + m_2)}$$

In vector form

$$\vec{w}_1 = \frac{[4\,m_1\,m_2 + m_0\,(m_1 - m_2)]\,\vec{g}}{4\,m_1\,m_2 + m_0\,(m_1 + m_2)}$$

Q. 74. In the arrangement shown in Fig. 1.17 the mass of the rod M exceeds the mass m of the ball. The ball has an opening permitting it to slide along the thread with some friction. The mass of the pulley and the friction in its axle are negligible. At the initial moment the ball was located opposite the lower end of the rod. When set free, both bodies began moving with constant accelerations. Find the friction force between the ball and the thread if t seconds after the beginning of motion the ball got opposite the upper end of the rod. The rod length equals l.

Fig. 1.17.   Fig. 1.18.   Fig. 1.19.
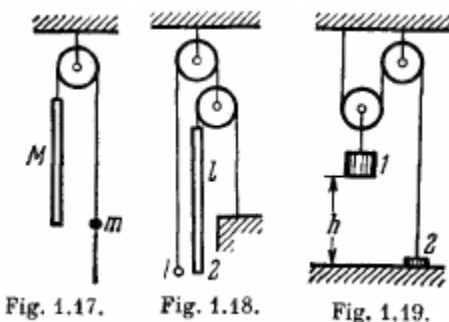
Ans. As the thread is not tied with m, so if there were no friction between the thread and the ball m, the tension in the thread would be zero and as a result both bodies will have free fall motion. Obviously in the given problem it is the friction force exerted by the ball on the thread, which becomes the tension in the thread. From the condition or language of the problem $w_M > w_m$ and as both are directed downward so , relative acceleration of $M = w_M > w_m$ and is directed downward. Kinematical equation for the ball in the frame of rod in projection form along upward direction gives :
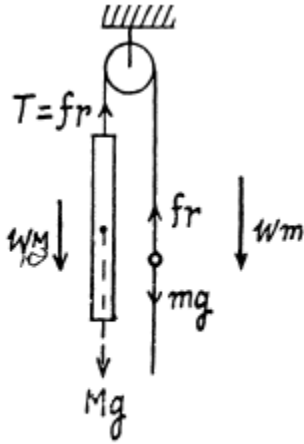
$$l = \frac{1}{2}(w_M - w_m)t^2 \qquad (1)$$

Newton's second law in projection form along vertically down direction for both, rod and ball gives,

$$Mg - fr = M w_M \qquad (2)$$
$$mg - fr = m w_m \qquad (3)$$

Multiplying Eq. (2) by m and Eq. (3) by M and then subtracting Eq. (3) from (2) and after using Eq. (1) we get

$$fr = \frac{2lMm}{(M-m)t^2}$$

The Fundamental Equation Of Dynamics (Part - 2)

Q. 75. In the arrangement shown in Fig. 1.18 the mass of ball 1 is $\eta = 1.8$ times as great as that of rod 2. The length of the latter is $l = 100$ cm. The masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod?

Ans. Suppose, the ball goes up with accleration w_1 and the rod comes down with the acceleration w_2. As the length of the thread is constant,

$$2 = w_2 \quad (1)$$

From Newton's second law in projection form along vertically upward for the ball and vertically downward for the rod respectively gives,

$$T - mg = mw_1 \quad (2)$$

and $$Mg - T' = Mw_2 \quad (3)$$

but $T = 2T$ (because pulley is massless)   (4)

From Eqs. (1), (2), (3) and (4)

$$w_1 = \frac{(2M - m)g}{m + 4M} = \frac{(2 - \eta)g}{\eta + 4} \text{ (in upward direction)}$$

and $$w_2 = \frac{2(2 - \eta)g}{(\eta + 4)} \text{ (downwards)}$$
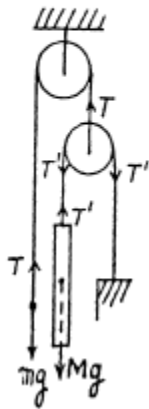
From kinematical equation in projection form, we get

$$l = \frac{1}{2}(w_1 + w_2)\, t^2$$

as, w_1 and w_2 are in the opposite direction.

Putting the values of w_1 and w_2, the sought time becomes

$$t = \sqrt{2l(\eta + 4)/3(2 - \eta)g} = 1{\cdot}4\,s$$

Q. 76. In the arrangement shown in Fig. 1.19 the mass of body 1 is $\eta = 4.0$ times as great as that of body 1. The height h = 20 cm. The masses of the pulleys and the threads, as well as the friction, are negligible. At a certain moment body 2 is released and the arrangement set in motion. What is the maximum height that body 2 will go up to?

Ans. Using Newton's second law in projection form along x - axis for the body 1 and along negative x - axis for the body 2 respectively, we get

$$m_1 g - T_1 = m_1 w_1 \qquad\qquad (1)$$
$$T_2 - m_2 g = m_2 w_2 \qquad\qquad (2)$$

For the pulley lowering in downward direction from Newton's law along x axis,

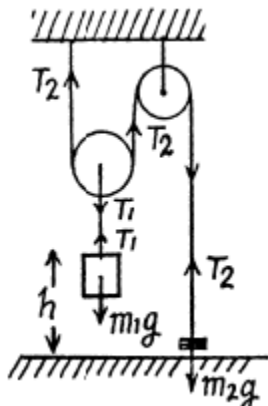$$T_1 - 2T_2 = 0 \text{ (as pulley is mass less)}$$

or, $\qquad\qquad T_1 = 2T_2 \qquad\qquad (3)$

As the length of the thread is constant so,

$$w_2 = 2w_1 \qquad (4)$$

$$w_2 = \frac{2(m_1 - 2m_2)g}{4m_2 + m_1} = \frac{2(\eta - 2)}{\eta + 4} \quad \left(\text{as } \frac{m_1}{m_2} = \eta\right)$$

The simultaneous solution of above equations yields,

$$v_2^2 = 2\,w_2\,(2h) = 2\left[\frac{2\,(\eta-2)\,g}{\eta+4}\right]2h = \frac{8\,h\,(\eta-2)\,g}{\eta+4} \quad (5)$$

Obviously during the time interval in which the body 1 comes to the horizontal floor covering the distance h, the body 2 moves upward the distance 2h. At the moment when the body 2 is at the height 2h from the floor its velocity is given by the expression :

$$h' = \frac{v_2^2}{2g} = \frac{4h\,(\eta-2)}{\eta+4} \qquad H = 2h + h' = 2h + \frac{4h\,(\eta-2)}{(\eta+4)} = \frac{6\,\eta\,h}{\eta+4}$$

After the body m_1 touches the floor the thread becomes slack or the tension in the thread zero, thus as a result body 2 is only under gravity for it's subsequent motion.

Owing to the velocity v_2 at that moment or at the height 2h from the floor, the body 2 further goes up under gravity by the distance,

$$h' = \frac{v_2^2}{2g} = \frac{4h\,(\eta-2)}{\eta+4}$$

Thus the sought maximum height attained by the body 2 :

$$H = 2h + h' = 2h + \frac{4h\,(\eta-2)}{(\eta+4)} = \frac{6\,\eta\,h}{\eta+4}$$

Q. 77. Find the accelerations of rod A and wedge B in the arrangement shown in Fig. 1.20 if the ratio of the mass of the wedge to that of the rod equals η, and the friction between all contact surfaces is negligible.

Ans. Let us draw free body diagram of each body, i.e. of rod A and of wedge B and also draw the kinemetical diagram for accelerations, after analysing the directions of motion of A and B. Kinematical relationship of accelarations is :

$$\tan \alpha = \frac{w_A}{w_B} \quad (1)$$

Let us write Newton's second law for both bodies in terms of projections having taken positive directions of y and x axes as shown in the figure.
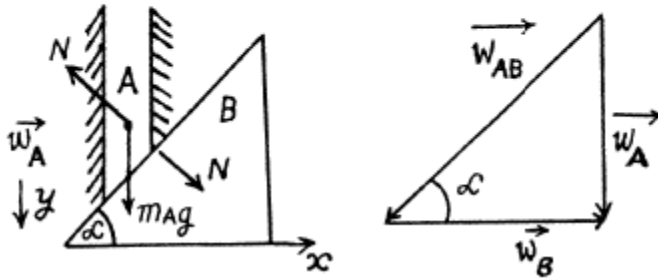
$$m_A\,g - N\cos\alpha = m_A\,w_A \quad (2)$$

and $N \sin\alpha = m_B\,w_B \quad (3)$

Simultaneous solution of (1), (2) and (3) yields :

$$w_A = \frac{m_A\, g \sin \alpha}{m_A \sin \alpha + m_B \cot \alpha \cos \alpha} = \frac{g}{(1 + \eta \cot^2 \alpha)} \quad \text{and}$$

$$w_B = \frac{w_A}{\tan \alpha} = \frac{g}{(\tan \alpha + \eta \cot \alpha)}$$

Note : We may also solve this problem using conservation of mechanical energy instead of Newton's second law.

Q. 78. In the arrangement shown in Fig. 1.21 the masses of the wedge M and the body m are known. The appreciable friction exists only between the wedge and the body m, the friction coefficient being equal to k. The masses of the pulley and the thread are negligible. Find the acceleration of the body m relative to the horizontal surface on which the wedge slides.
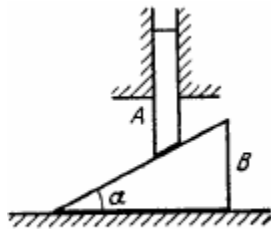

Fig. 1.20.
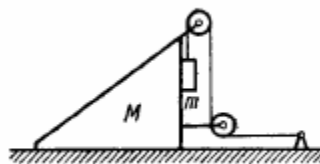
Fig. 1.21.

Ans. Let us draw free body diagram of each body and fix the coordinate system, as shown in the figure. After analysing the motion of M and m on the basis of force diagrams, let us draw the kinematical diagram for accelerations (Fig.).
As the length of threads are constant so,

$ds_{mM} = ds_M$ and as $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions that why

$$\left|\vec{w}_{mM}\right| = \left|\vec{w}_M\right| = w \text{ (say) and}$$

$$\vec{w}_{mM} \uparrow\uparrow \vec{v}_M \text{ and } \vec{w}_M \uparrow\uparrow \vec{v}_M$$

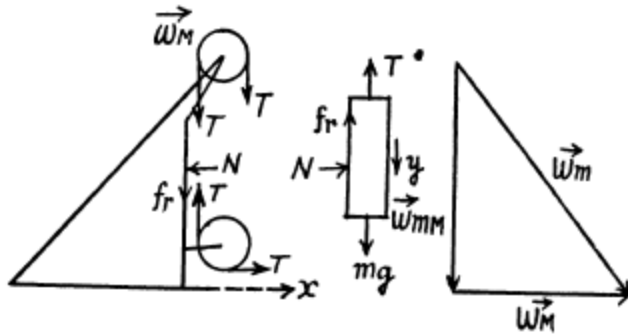$\left|\vec{w}_{mM}\right| = \left|\vec{w}_M\right| = w$ (say) and

$\vec{w}_{mM} \uparrow\uparrow \vec{v}_M$ and $\vec{w}_M \uparrow\uparrow \vec{v}_M$

As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$

so, from the triangle law of vector addition

$w_m = \sqrt{2}\, w$ (1)

From the Eq. $F_x = mw_x$, for the wedge and block :

$T - N = Mw$,   (2)

and $N = mw$ (3)

Now, from the Eq. $F_y = mw_y$, for the block

$mg - T - kN = mw$ (4)

Simultaneous solution of Eqs. (2), (3) and (4) yields :

$$w = \frac{mg}{(km + 2m + M)} = \frac{g}{(k + 2 + M/m)}$$

Hence using Eq. (1)

$$w_m = \frac{g\sqrt{2}}{(2 + k + M/m)}$$

Q. 79. What is the minimum acceleration with which bar A (Fig. 1.22) should be shifted horizontally to keep bodies 1 and 2 stationary relative to the bar? The masses of the bodies are equal, and the coefficient of friction between the bar and the bodies is equal to k. The masses of the pulley and the threads are negligible, the friction in the pulley is absent.

Ans. Bodies 1 and 2 will remain at rest with repect to bar A for $w_{min} \le w \le w_{max}$, where w_{min} is the sought minimum acceleration of the bar. Beyond these limits there will be a relative motion betw een b ar and the bodies.

For $0 \le w \le w_{min}$ the tendency o f body 1 in relation to the bar A is to move towards right and is in the opposite sense for $w \ge w_{max}$ On the basis of above argument the static friction on 2 by A is directed upward and on 1 by A is directed towards le ft for the purpose o f calculating w_{min}.

Let us write Newton's second law for bodies 1 and 2 in terms of projection along positive x - axis (Fig.).

$$T - fr_1 = m\bar{w} \text{ or, } fr_1 = T - mw \quad (1)$$
$$N_2 = mw \quad (2)$$
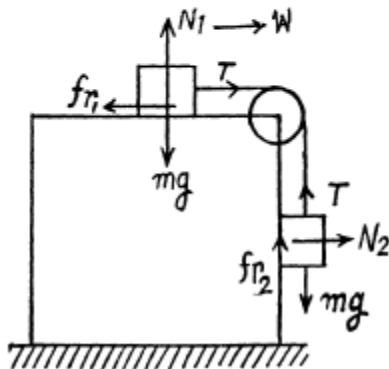
As body 2 has no acceleration in vertical direction, so

$$fr_2 = mg - T \quad (3)$$

From (1) and (3)
$$(fr_1 + fr_2) = m(g - w) \quad (4)$$

But $\quad fr_1 + fr_2 \le k(N_1 + N_2)$

or $\quad fr_1 + fr_2 \le k(mg + mw) \quad (5)$

From (4) and (5)

$m (g - w) \le mk (g + w)$, or $w \ge \dfrac{g(1-k)}{(1+k)}$

Hence

$w_{min} = \dfrac{g(1-k)}{(1+k)}$

Q. 80. Prism 1 with bar 2 of mass m placed on it gets a horizontal acceleration w directed to the left (Fig. 1.23). At what maximum value of this acceleration will the bar be still stationary relative to the prism, if the coefficient of friction between them k< cot α?

Ans. On the basis of the initial aigument of the solution of Q.79, the tendency of bar 2 with respect to 1 will be to move up along the plane.

Let us fix (x - y) coordinate system in the frame o f ground as shown in the figure.

From second law of motion in projection form along y and x axes :

m g cos α - N = m w sin a

or, $N = m (g \cos \alpha - w \sin \alpha)$ (1)
      $m g \sin \alpha + fr = m w \cos \alpha$
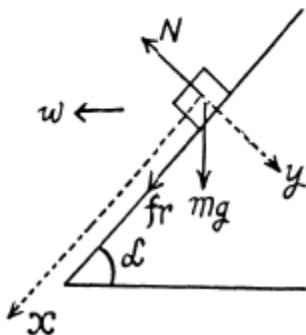or, $fr = m (w \cos \alpha - g \sin \alpha)$ (2)

but $fr \le kN$, so from (1) and (2)
  $(w \cos \alpha - g \sin \alpha) \le k (g \cos \alpha + w \sin \alpha)$
or, $w (\cos \alpha - k \sin \alpha) \le g (k \cos \alpha + \sin \alpha)$

or, $w \le g \dfrac{(\cos \alpha + \sin \alpha)}{\cos \alpha - k \sin \alpha}$,

So, the sought maximum acceleration of the wedge :

$$w_{max} = \frac{(k\cos\alpha + \sin\alpha)g}{\cos\alpha - k\sin\alpha} = \frac{(k\cot\alpha + 1)g}{\cot\alpha - k} \quad \text{where} \quad \cot\alpha > k$$

Q. 81. Prism 1 of mass m_1 and with angle α (see Fig. 1.23) rests on a horizontal surface. Bar 2 of mass m_2 is placed on the prism. Assuming the friction to be negligible, find the acceleration of the prism.

Ans. Let us draw the force diagram of each body, and on this basis we observe that the prism moves towards right say with an acceleration $\vec{w_1}$ and the bar 2 of mass moves down the plane with respect to 1, say with

acceleration $\vec{w_{21}}$, then, $\vec{w_2} = \vec{w_{21}} + \vec{w_1}$ (Fig.)

Let us write Newton's second law for both bodies in projection form along positive y_2 and x_1 axes as shown in the Fig.

$$m_2 g\cos\alpha - N = m_2 w_{2(y_2)} = m_2\left[w_{21(y_2)} + w_{1(y_2)}\right] = m_2\left[0 + w_1\sin\alpha\right]$$

or, $m_2 g\cos\alpha - N = m_2 w_1\sin\alpha$ (1)

and $N\sin\alpha = m_1 w_1$ (2)

Solving (1) and (2), we get

$$w_1 = \frac{m_2 g\sin\alpha\cos\alpha}{m_1 + m_2\sin^2\alpha} = \frac{g\sin\alpha\cos\alpha}{(m_1/m_2) + \sin^2\alpha}$$
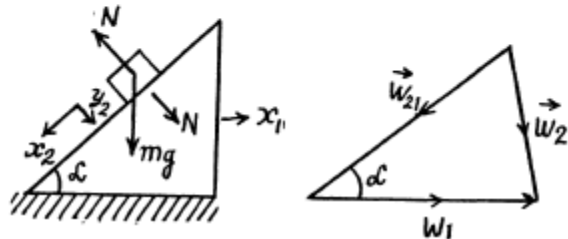
Q. 82. In the arrangement shown in Fig. 1.24 the masses m of the bar and M of the wedge, as well as the wedge angle α, are known. The masses of the pulley and the thread are negligible. The friction is absent. Find the acceleration of the wedge M.
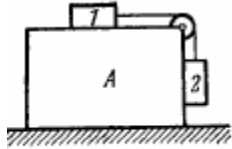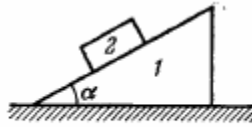

Fig. 1.22.   Fig. 1.23.

Ans. To analyse the kinematic relations between the bodies, sketch the force diagram of each body as shown in the figure.

On the basis of force diagram, it is obvious that the wedge M will move towards right and the block will move down along the wedge. As the length of the thread is constant, the distance travelled by the block on the wedge must be equal to the distance travelled by the wedge on the flo&r. Hence

$ds_{mM} = ds_M$. As $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions and acceleration that's why

$\vec{w}_{mM} \uparrow\uparrow \vec{v}_{mM}$ and $\vec{w}_M \uparrow\uparrow \vec{v}_M$ and $w_{mM} = w_M = w$ (say) and accordingly the diagram of kinematical dependence is shown in figure.
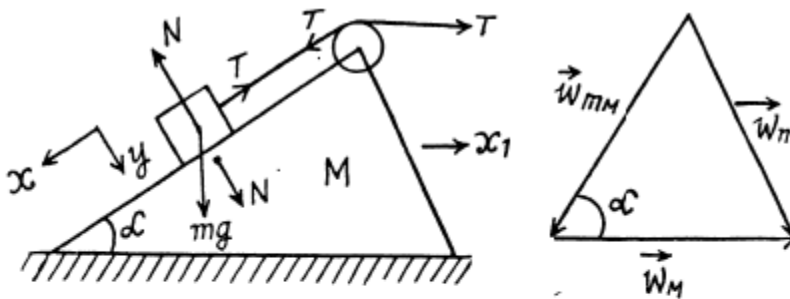
As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$, so from triangle law o f vector addition.

$w_m = \sqrt{w_M^2 + w_{mM}^2 - 2 w_{mM} w_M \cos\alpha} = w\sqrt{2(1-\cos\alpha)}$ (1)

From $F_X = m\, w_X$, (for the wedge),

$T - T\cos\alpha + N\sin\alpha = M w$ (2)

For the bar m let us fix (x - y) coordinate system in the frame of ground Newton 's law in projection form along x and y axes (Fig.) gives

$$mg \sin\alpha - T = m\, w_{m\,(x)} = m\left[w_{mM\,(x)} + w_{M\,(x)}\right]$$

$$= m\left[w_{mM} + w_M \cos(\pi - \alpha)\right] = m\, w\,(1 - \cos\alpha) \qquad (3)$$

$$mg \cos\alpha - N = m\, w_{m\,(y)} = m\left[w_{mM\,(y)} + w_{M\,(y)}\right] = m\left[0 + w \sin\alpha\right] \qquad (4)$$

Solving the above Eqs. simultaneously, we get

$$w = \frac{mg \sin\alpha}{M + 2m\,(1 - \cos\alpha)}$$

Note : We can study the motion of the block m in the frame of wedge also, alternately we may solve this problem using conservation of mechanical energy.

Q. 83. A particle of mass m moves along a circle of radius R. Find the modulus of the average vector of the force acting on the particle over the distance equal to a quarter of the circle, if the particle moves

(a) uniformly with velocity v;
(b) with constant tangential acceleration w_τ the initial velocity being equal to zero.
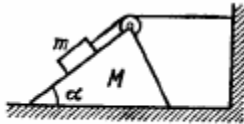
Fig. 1.24.

Ans. Let us sketch the diagram for the motion of the particle of mass m along the circle of radius R and indicate x and y axis, as shown Jn the figure.

(a) For the particle, change in momentum $\Delta \vec{p} = mv\,(-\vec{i}) - m\,v\,(\vec{j})$

so , $|\Delta \vec{p}| = \sqrt{2}\, m\, v$

and time taken in describing quarter of the circle,

$$\Delta t = \frac{\pi R}{2v}$$

Hence, $<\vec{F}> = \dfrac{|\Delta \vec{p}|}{\Delta t} = \dfrac{\sqrt{2}\, mv}{\pi R/2v} = \dfrac{2\sqrt{2}\, m\, v^2}{\pi R}$

(b) In this case

$$\vec{p_i} = 0 \text{ and } \vec{p_f} = m\,w_t\,t\,(-\vec{i}),$$

so $\quad |\Delta\vec{p}| = m\,w_t\,t$

Hence, $|<\vec{F}>| = \dfrac{|\Delta\vec{p}|}{t} = m\,w_t$

Q. 84. An aircraft loops the loop of radius R = 500 m with a constant velocity v = 360 km per hour. Find the weight of the flyer of mass m = 70 kg in the lower, upper, and middle points of the loop.

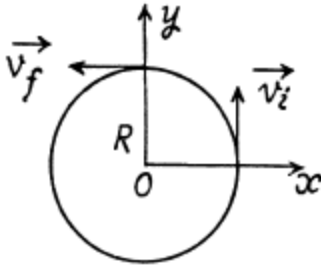**Ans.** While moving in a loop, normal reaction exerted by the flyer on the loop at different points and uncom pensated weight if any contribute to the weight of flyer at those points,

(a) When the aircraft is at the lowermost point, Newton's second law of motion in projection form $F_n = mw_n$ gives

$$N - mg = \dfrac{m\,v^2}{R}$$

or, $N = mg + \dfrac{m\,v^2}{R} = 2{\cdot}09 \text{ kN}$

(b) When it is at the upper most point, again from $F_n - mw_n$ we get
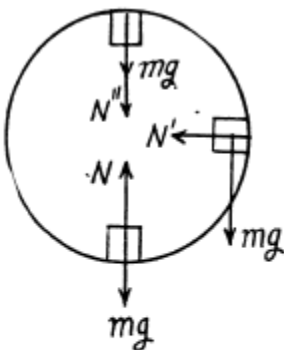
$$N'' + mg = \dfrac{mv^2}{R}$$

$$N'' = \dfrac{m\,v^2}{R} - m\,g = 0{\cdot}7 \text{ kN}$$

(c) When the aircraft is at the middle point of the loop, again from $F_n = m\,w_n$

$$N' = \frac{m\,v^2}{R} = 1.4\ kN$$

The uncompensated weight is mg. Thus effective weight

$= \sqrt{N'^2 + m^2 g^2} = 1.56\ kN$ acts obliquely.

Q. 85. A small sphere of mass m suspended by a thread is first taken aside so that the thread forms the right angle with the vertical and then released. Find:

(a) the total acceleration of the sphere and the thread tension as a function of θ, the angle of deflection of the thread from the vertical;
(b) the thread tension at the moment when the vertical component of the sphere's velocity is maximum;
(c) the angle θ between the thread and the vertical at the moment when the total acceleration vector of the sphere is directed horizontally.

Ans. Let us depict the forces acting on the small sphere m, (at an arbitrary position when the thread makes an angle θ from the vertical) and write equation $\vec{F} = m\,\vec{w}$ via projection on A A the unit vectors $\hat{u}_t$ and $\hat{u}_n$. From $F_t = m\,w_t$, we have

$$mg\sin\theta = m\frac{dv}{dt}$$

$$= m\frac{v\,dv}{ds} = m\frac{v\,dv}{l(-d\theta)}$$

(as vertical is refrence line of angular position)

or $v\,dv = -gl\sin\theta\,d\theta$

Integrating both the sides :

$$\int_0^v v\,dv = -g\,l\int_{\pi/2}^{\theta} \sin\theta\,d\theta$$

or, $\dfrac{v^2}{2} = g\,l\cos\theta$

Hence $\dfrac{v^2}{l} - 2\,g\cos\theta = w_n$.(1)
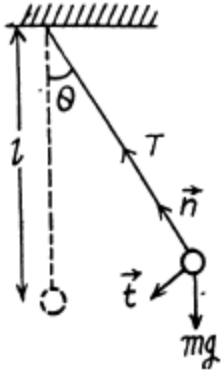
(Eq. (1) can be easily obtained by the conservation of mechanical energy).
From

From $F_n = m\,w_n$

$$T - m\,g\cos\theta = \dfrac{m\,v^2}{l}.$$

Using (1) we have

       $T = 3\,mg\cos\theta$ (2)

Again from the Eq. $F_t = m\,w_t$:

$mg\sin\theta = m\,w_t$ or $w_t = g\sin\theta$ (3)

Hence $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(g\sin\theta)^2 + (2g\cos\theta)^2}$ (using 1 and 3)

   $= g\sqrt{1 + 3\cos^2\theta}$

(b) Vertical component of velocity, $v_y = v\sin\theta$

So, $v_y^2 = v^2\sin^2\theta = 2g\,l\cos\theta\sin^2\theta$ (using 1)

For maximum v_y or v_y^2, $\dfrac{d\,(\cos\theta\sin^2\theta)}{d\theta} = 0$

which yields $\cos\theta = \dfrac{1}{\sqrt{3}}$

Therefore from $T = 3mg\,\dfrac{1}{\sqrt{3}} = \sqrt{3}\,mg$

(c) We have We have $\vec{w} = w_t \hat{u}_t + w_n \hat{u}_n$ thus $w_y = w_{t(y)} + w_{n(y)}$

But in accordance with the problem $w_y = 0$

So, $w_{t(y)} + w_{n(y)} = 0$

or, $g \sin \theta \sin \theta + 2g \cos^2 \theta (-\cos \theta) = 0$

or, $\cos \theta = \dfrac{1}{\sqrt{3}}$ or, $\theta = 54 \cdot 7°$

Q. 86. **A ball suspended by a thread swings in a vertical plane so that its acceleration values in the extreme and the lowest position are equal. Find the thread deflection angle in the extreme position.**

Ans. The ball has only normal acceleration at the lowest position and only tangential acceleration at any of the extreme position. Let v be the speed of the ball at its lowest position and l be the length of the thread, then according to the problem

$$\dfrac{v^2}{l} = g \sin \alpha \qquad (1)$$

where α is the maximum deflection angle

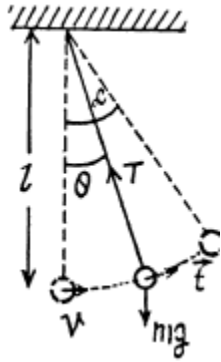From Newton's law in projection form : $F_t = m w_t$

or, $-mg \sin \theta = mv \dfrac{dv}{l \, d\theta}$

$-gl \sin \theta \, d\theta = v \, dv$

On integrating both the sides within their limits.

$$-gl \int_0^{\alpha} \sin \theta \, d\theta = \int_v^0 v \, dv$$

or, $v^2 = 2gl\,(1 - \cos\alpha)$ (2)

Note : Eq. (2) can easily be obtained by the conservation of mechanical energy of the ball in the uniform field of gravity.
From Eqs. (1) and (2) with $\theta = \alpha$

$$2gl\,(1 - \cos\alpha) = lg\cos\alpha$$
or, $\cos\alpha = \dfrac{2}{3}$ so, $\alpha = 53°$

Q. 87. A small body A starts sliding off the top of a smooth sphere of radius R. Find the angle θ (Fig. 1.25) corresponding to the point at which the body breaks off the sphere, as well as the break-off velocity of the body.

Ans. Let us depict the forces acting on the body A(which are the force of gravity $m\vec{g}$ and the normal reaction $\vec{N}$ and write equation $\vec{F} = m\vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$ (Fig.)

From $F_t = mw_t$

$$mg\sin\theta = m\dfrac{dv}{dt}$$
$$= m\dfrac{v\,dv}{ds} = m\dfrac{v\,dv}{R\,d\theta}$$

or, $gR\sin\theta\,d\theta = v\,dv$

or,

Integrating both side for obtaining v (θ)

$$\int_0^\theta g R \sin\theta \, d\theta = \int_0^v v \, dv$$

or, $v^2 = 2gR(1-\cos\theta)$

From $F_n = mw_n$

$$mg\cos\theta - N = m\frac{v^2}{R}$$
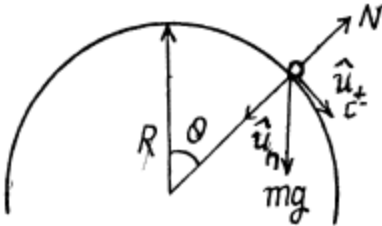
At the mom ent the body loses contact with the surface, N = 0 and therefore the Eq. (2) becomes

$$v^2 = gR\cos\theta$$

where v and θ correspond to the moment when the body loses contact with the surface.

Solving Eqs. (1) and (3) we obtain $\cos\theta = \frac{2}{3}$ or, $\theta = \cos^{-1}(2/3)$ and $v = \sqrt{2gR/3}$.

Q. 88. A device (Fig. 1.26) consists of a smooth L-shaped rod located in a horizontal plane and a sleeve A of mass m attached by a weight- less spring to a point B. The spring stiffness is equal to x. The whole system rotates with a constant angular velocity ω about a vertical axis passing through the point O. Find the elongation of the spring. How is the result affected by the rotation direction?

Ans. At first draw the free body diagram of the device as, shown. The forces, acting on the sleeve are it's weight, acting vertically downward, spring force, along the length of the spring and normal reaction by the rod, perpendicular to its length.
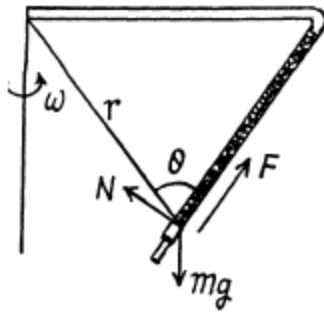Let F be the spring force, and Δl be the elongation.

From, $F_n = mw_n$:

$N\sin\theta + F\cos\theta = m\omega^2 r$ (1)

where $r\cos\theta = (l_0 + \Delta l)$.

Similarly from $F_t = mw_t$

$N \cos \theta - F \sin \theta = 0$ or, $N = F \sin \theta / \cos \theta$ (2)

From (1) and (2)

$F (\sin \theta / \cos \theta) \cdot \sin \theta + F \cos \theta = m \omega^2 r$

$\qquad = m \omega^2 (l_0 + \Delta l) / \cos \theta$

On putting $F = \kappa \Delta l$,

$\kappa \Delta l \sin^2 \theta + \kappa \Delta l \cos^2 \theta = m \omega^2 (l_0 + \Delta l)$

on solving, we get,

$\Delta l = m \omega^2 \dfrac{l_0}{\kappa - m \omega^2} = \dfrac{l_0}{(\kappa / m \omega^2 - 1)}$

and it is independent of the direction of rotation.

Q. 89. A cyclist rides along the circumference of a circular horizontal plane of radius R, the friction coefficient being dependent only on distance r from the centre O of the plane as k = k₀ (1—r/R), where k₀ is a constant. Find the radius of the circle with the centre at the point along which the cyclist can ride with the maximum velocity. What is this velocity?

Ans. According to the question, the cyclist moves along the circular path and the centripetal force is provided by the frictional force. Thus from die equation $F_n = m w_n$

$fr = \dfrac{m v^2}{r}$ or $kmg = \dfrac{m v^2}{r}$

or, $k_0 \left(1 - \dfrac{r}{R}\right) g = \dfrac{v^2}{r}$ or $v^2 = k_0 (r - r^2/R) g$ (1)

For v_{max}, we should have $\dfrac{d \left(r - \dfrac{r^2}{R}\right)}{dr} = 0$

or, $1 - \dfrac{2r}{R} = 0$,   so $r = R/2$

Hence $v_{max} = \dfrac{1}{2} \sqrt{k_0 gR}$

Q. 90. A car moves with a constant tangential acceleration $w_\tau = 0.62$ m/s^2 along a horizontal surface circumscribing a circle of radius R = 40 m. The coefficient of sliding friction between the wheels of the car and the surface is k = 0.20. What distance will the car ride without sliding if at the initial moment of time its velocity is equal to zero?

Ans. As initial velocity is zero thus

$$v^2 = 2 w_t s$$

As $w_t > 0$ the speed of the car increases with time or distance. Till the moment, sliding starts, the static friction provides the required centripetal acceleration to the car.

Thus $fr = mw$, but $fr \le kmg$

So, $w^2 \le k^2 g^2$ or, $w_t^2 + \dfrac{v^2}{R} \le k^2 g^2$

or, $v^2 \le (k^2 g^2 - w_t^2) R$

Hence $v_{max} = \sqrt{(k^2 g^2 - w_t^2) R}$

so, from Eqn. (1), the sought distance $s = \dfrac{v_{max}^2}{2 w_t} = \dfrac{1}{2} \sqrt{\left(\dfrac{kg}{w_t}\right)^2 - 1} = 60$ m.

The Fundamental Equation Of Dynamics (Part - 3)

Q. 91. A car moves uniformly along a horizontal sine curve y = a sin (x/α), where a and α are certain constants. The coefficient of friction between the wheels and the road is equal to k. At what velocity will the car ride without sliding?

Ans. Since the car follows a curve, so the maximum velocity at which it can ride without sliding at the point of minimum radius of curvature is the sought velocity and obviously in this case the static friction between the car and the road is limiting.

Hence from the equation $F_n = mw$

$$kmg \geq \frac{mv^2}{R} \quad \text{or} \quad v \leq \sqrt{kRg}$$

so $\quad v_{max} = \sqrt{kR_{min}g}$. $\quad$ (1)

We know that, radius of curvature for a curve at any point (x, y) is given as,

$$R = \left| \frac{[1 + (dy/dx)^2]^{3/2}}{(d^2y)/dx^2} \right| \quad (2)$$

For the given curve,

$$\frac{dy}{dx} = \frac{a}{\alpha} \cos\left(\frac{x}{\alpha}\right) \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{-a}{\alpha^2} \sin\frac{x}{\alpha}$$

Substituting this value in (2) we get,

$$R = \frac{[1 + (a^2/\alpha^2)\cos^2(x/\alpha)]^{3/2}}{(a/\alpha^2)\sin(x/\alpha)}$$

For the minimum R, $\frac{x}{\alpha} = \frac{\pi}{2}$

and therefore, corresponding radius of curvature

$$R_{min} = \frac{\alpha^2}{a} \quad (3)$$

Hence from (1) and (2)

$$v_{max} = \alpha\sqrt{kg/a}$$

Q. 92. A chain of mass m forming a circle of radius R is slipped on a smooth round cone with half-angle θ. Find the tension of the chain if it rotates with a constant angular velocity co about a vertical axis coinciding with the symmetry axis of the cone.

Ans. The sought tensile stress acts on each element of the chain. Hence divide the chain into small, similar elements so that each element may be assumed as a particle. We consider one such element of mass dm, which subtends angle d α at the centre. The chain moves along a circle of known radius R with a known angular speed ω and certain forces act on it We have to find one of these forces.

From Newton's second law in projection form, $F_x = mw_x$ we get

$2 T \sin (d\alpha/2) - dN \cos \theta = dm \, \omega^2 R$
and from $F_z = mw_z$ we get

$dN \sin\theta = gdm$

Then putting $dm = md\alpha/2\pi$ and $\sin (d\alpha/2) = d\alpha/2$ and solving, we get,

$T = \dfrac{m (\omega^2 R + g \cot \theta)}{2 \pi}$
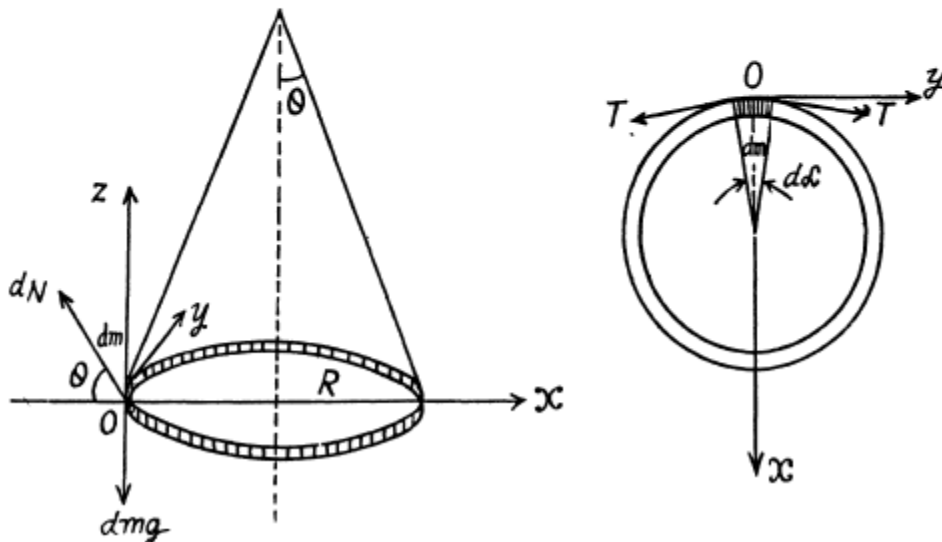
Q. 93. A fixed pulley carries a weightless thread with masses m_1 and m_2 at its ends. There is friction between the thread and the pulley. It is such that the thread starts slipping when the ratio $m_2/m_1 = \eta_0$. Find:

(a) the friction coefficient;
(b) the acceleration of the masses when $m_2/m_1 = \eta > \eta_0$.

Ans. Let, us consider a small element of the thread and draw free body diagram for this element, (a) Applying Newton's second law of motion in projection form, $F_n = mw_n$ for this element,

$(T + dT) \sin (d\,\theta/2) + T \sin (d\,\theta/2) - dN = dm\,\omega^2 R = 0$

or, $\quad 2T \sin (d\,\theta/2) = dN$, [negelecting the term$(dT \sin d\,\theta/2)$]

or, $\quad T d\theta = dN$, as $\sin \dfrac{d\theta}{2} = \dfrac{d\theta}{2}$ $\qquad$ (1)

Also, $\quad d\,fr = k\,dN = (T + dT) - T = dT$ $\qquad$ (2)

From Eqs. (1) and (2),

$kT d\theta = dT$ or $\dfrac{dT}{T} = k\,d\theta$

In this case $Q = \pi$ so,

or, $\quad$ or, $\ln \dfrac{T_2}{T_1} = k\pi$ $\qquad\qquad$ (3)

So, $\quad k = \dfrac{1}{\pi} \ln \dfrac{T_2}{T_1} = \dfrac{1}{\pi} \ln \eta_0$

as $\quad \dfrac{T_2}{T_1} = \dfrac{m_2\,g}{m_1\,g} = \dfrac{m_2}{m_1} = \eta_0$
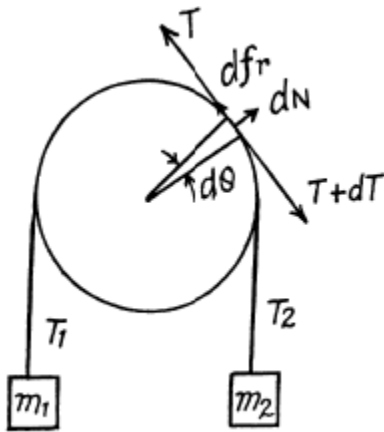
(b) When $\dfrac{m_2}{m_1} = \eta$, Svhich is greater than η_0 the blocks will move with same value of acceleration, (say w) and clearly m2 moves downward. From Newton's second law in projection form (downward for m$_2$ and upward for m$_1$) we get :

$m_2 g - T_2 = m_2 w$ (4)

and $T_1 - m_1 g = m_1 w$ (5)

Also $\dfrac{T_2}{T_1} = \eta_0$ (6)

Simultaneous solution of Eqs. (4), (5) and (6) yields :

$$w = \dfrac{(m_2 - \eta_0 m_1) g}{(m_2 + \eta_0 m_1)} = \dfrac{(\eta - \eta_0)}{(\eta + \eta_0)} g \left(\text{as } \dfrac{m_2}{m_1} = \eta \right)$$

Q. 94. A particle of mass m moves along the internal smooth surface of a vertical cylinder of radius R. Find the force with which the particle acts on the cylinder wall if at the initial moment of time its velocity equals v₀ and forms an angle α with the horizontal.

Ans. The force with which the cylinder wall acts on the particle will provide centripetal force necessary for the motion of the particle> and since there is no acceleration acting in the horizontal direction, horizontal component of the velocity will remain constant througout the motion.
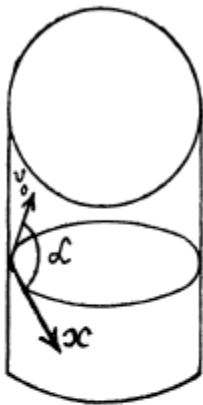
So $v_x = v_0 \cos \alpha$

Using, $F_n = m\, w_n$, for the particle of mass m,

$$N = \dfrac{m\, v_x^2}{R} = \dfrac{m\, v_0^2 \cos^2 \alpha}{R},$$

which is the required normal force.

Q. 95. Find the magnitude and direction of the force acting on the particle of mass m during its motion in the plane xy according to the law x = a sin ωt, y = b cos ωt, where a, b, and ω are constants.

Ans. Obviously the radius vector describing the position of the particle relative to the origin of coordinate is

$$\vec{r} = x\vec{i} + y\vec{j} = a \sin \omega t\, \vec{i} + b \cos \omega t\, \vec{j}$$

Differentiating twice with respect the time :

$$\vec{w} = \frac{d^2\vec{r}}{dt^2} = -\omega^2 (a \sin \omega t\, \vec{i} + b \cos \omega t\, \vec{j}) = -\omega^2 \vec{r} \qquad (1)$$

Thus $\vec{F} = m\vec{w} = -m\omega^2 \vec{r}$

Q. 96. A body of mass m is thrown at an angle to the horizontal with the initial velocity v_0. Assuming the air drag to be negligible, find:

(a) the momentum increment Δp that the body acquires over the first t seconds of motion;
(b) the modulus of the momentum increment Δp during the total time of motion.

Ans. (a) We have $\quad \Delta \vec{p} = \int \vec{F}\, dt$

$$= \int_0^t m\vec{g}\, dt = m\vec{g}t \qquad (1)$$

(b) Using the solution of problem 1.28 (b), the total time of motion $\quad \tau = -\dfrac{2(\vec{v_0} \cdot \vec{g})}{g^2}$

Hence using $t = \tau$ in (1)

$|\Delta \vec{p}| = mg\,\tau$

$= -2m\,(\vec{v_0} \cdot \vec{g})/g \quad (\vec{v_0} \cdot \vec{g} \text{ is } -ve)$

Q. 97. At the moment t = 0 a stationary particle of mass in experiences a time-dependent force F = at (τ — t), where a is a constant vector, τ is the time during which the given force acts. Find:
(a) the momentum of the particle when the action of the force discontinued;
(b) the distance covered by the particle while the force acted.

Ans. From the equation o f the g iven time dependence force $\vec{F} = \vec{a}\, t\,(\tau - t)$ at $t = \tau$, the force vanishes,

(a) Thus $\Delta \vec{p} = \vec{p} = \int\limits_0^\tau \vec{F}\, dt$

or, $\vec{p} = \int\limits_0^\tau \vec{a}\, t\,(\tau - t)\, dt = \dfrac{\vec{a}\,\tau^3}{6}$

but $\vec{p} = m\vec{v}$ so $\vec{v} = \dfrac{\vec{a}\,\tau^3}{6m}$

(b) Again from the equation $\vec{F} = m\vec{w}$

or, $\vec{a}\,t\,(\tau - t) = m\,\dfrac{d\vec{v}}{dt}$

$\vec{a}\,(t\tau - t^2)\, dt = m\, d\vec{v}$

Integrating w ithin the limits for $\vec{v}(t)$,

$\int\limits_0^t \vec{a}\,(t\tau - t^2)\, dt = m\int\limits_0^{\vec{v}} d\vec{v}$

or, $\vec{v} = \dfrac{\vec{a}}{m}\left(\dfrac{\tau t^2}{2} - \dfrac{t^3}{3}\right) = \dfrac{\vec{a}\,t^2}{m}\left(\dfrac{\tau}{2} - \dfrac{t}{3}\right)$

Thus $v = \dfrac{a\,t^2}{m}\left(\dfrac{\tau}{2} - \dfrac{t}{3}\right)$ for $t \le \tau$

Hence distance covered during the tim e interval t = τ,

$s = \int\limits_0^\tau v\, dt$

$= \int\limits_0^\tau \dfrac{a\,t^2}{m}\left(\dfrac{\tau}{2} - \dfrac{t}{3}\right) dt = \dfrac{a}{m}\dfrac{\tau^4}{12}$

Q. 98. At the moment t = 0 a particle of mass m starts moving due to a force F = $F_0 \sin \omega t$, where F_0 and ω are constants. Find the distance covered by the particle as a function of t. Draw the approximate plot of this function.

Ans. We have $F = F_0 \sin \omega t$

or $m\dfrac{d\vec{v}}{dt} = \vec{F_0} \sin \omega t$ or $m\, d\vec{v} = \vec{F_0} \sin \omega t\, dt$

On integrating,

$m\vec{v} = \dfrac{-\vec{F_0}}{\omega} \cos \omega t + C$, (where C is integration constant)

When

$t = 0,\ v = 0$, so $C = \dfrac{\vec{F_0}}{m\omega}$

Hence, $\vec{v} = \dfrac{-\vec{F_0}}{m\,\omega} \cos \omega t + \dfrac{\vec{F_0}}{m\,\omega}$

As $|\cos \omega t| \le 1$ so, $v = \dfrac{F_0}{m\,\omega}(1 - \cos \omega t)$

Thus $s = \displaystyle\int_0^t v\, dt$

$= \dfrac{F_0 t}{m\,\omega} - \dfrac{F_0 \sin \omega t}{m\,\omega^2} = \dfrac{F_0}{m\,\omega^2}(\omega t - \sin \omega t)$

(Figure in the answer sheet).

Q. 99. At the moment t = 0 a particle of mass m starts moving due to a force F = $F_0 \cos \omega t$, where F_0 and ω are constants. How long will it be moving until it stops for the first time? What distance will it traverse during that time? What is the maximum velocity of the particle over this distance?

Ans. According to the problem, the force acting on the particle of mass m is, $\vec{F} = \vec{F_0} \cos \omega t$

So, $\displaystyle\int_0^{\vec{v}} d\vec{v} = \frac{\vec{F_0}}{m}\int_0^t \cos \omega t\, dt$ or $\vec{v} = \frac{\vec{F_0}}{m\omega}\sin \omega t$

Integrating, within the limits.

$\displaystyle\int_0^{\vec{v}} d\vec{v} = \frac{\vec{F_0}}{m}\int_0^t \cos \omega t\, dt$ or $\vec{v} = \frac{\vec{F_0}}{m\omega}\sin \omega t$

It is clear from equation (1), that after starting at t = 0, die particle comes to rest fro the first time at $t = \dfrac{\pi}{\omega}$.

From Eqn. (1), $v = |\vec{v}| = \dfrac{F_0}{m\omega}\sin \omega t$ for $t \le \dfrac{\pi}{\omega}$ (2)

Thus during the time interval t = π/ω, the sought distance

$$s = \frac{F_0}{m w}\int_0^{\pi/\omega} \sin \omega t\, dt = \frac{2F}{m\omega^2}$$

From Eq. (1)

$V_{max} = \dfrac{F_0}{m\omega}$ as $|\sin \omega t| \le 1$

Q. 100. A motorboat of mass m moves along a lake with velocity v_0. At the moment t = 0 the engine of the boat is shut down. Assuming the resistance of water to be proportional to the velocity of the boat F = —rv, find:

(a) how long the motorboat moved with the shutdown engine;

(b) the velocity of the motorboat as a function of the distance covered with the shutdown engine, as well as the total distance covered till the complete stop;

(c) the mean velocity of the motorboat over the time interval (beginning with the moment t = 0), during which its velocity decreases η times.

Ans. (a) From the problem $\vec{F} = -r\vec{v}$ so $m\dfrac{d\vec{v}}{dt} = -r\vec{v}$

Thus $m\dfrac{dv}{dt} = -rv$ [as $\vec{dv} \uparrow\downarrow \vec{v}$]

or, $\dfrac{dv}{v} = -\dfrac{r}{m}\,dt$

On integrating $\ln v = -\dfrac{r}{m}t + C$

But at $t = 0$, $v = v_0$, so, $C = \ln v_0$

or, $\ln \dfrac{v}{v_0} = -\dfrac{r}{m}t$ or, $v = v_0\,e^{-\frac{r}{m}t}$

Thus for $t \to \infty$, $v = 0$

(b) We have $m\dfrac{dv}{dt} = -rv$ so $dv = \dfrac{-r}{m}\,ds$

Integrating within the given limits to obtain v (s):

or, $\displaystyle\int_{v_0}^{v} dv = -\dfrac{r}{m}\int_0^s ds$ or $v = v_0 - \dfrac{rs}{m}$

Thus for $v = 0$, $s = s_{total} = \dfrac{m\,v_0}{r}$

(c) Let we have $\dfrac{m\,dv}{v} = -rv$ or $\dfrac{dv}{v} = \dfrac{-r}{m}\,dt$

or, $\displaystyle\int_0^{v_0/\eta} \dfrac{dv}{v} = \dfrac{-r}{m}\int_0^t dt$, or, $\ln\dfrac{v_0}{\eta v_0} = -\dfrac{r}{m}t$

So $t = \dfrac{-m\,\ln(1/\eta)}{r} = \dfrac{m\ln\eta}{r}$

Now, average velocity over this time interval,

$\langle v \rangle = \dfrac{\int v\,dt}{\int dt} = \dfrac{\displaystyle\int_0^{\frac{m}{r}\ln\eta} v_0 e^{-\frac{n}{m}}\,dt}{\dfrac{m}{r}\ln\eta} = \dfrac{v_0(\eta-1)}{\eta\ln\eta}$

Q. 101. **Having gone through a plank of thickness h, a bullet changed its velocity from v_0 to v. Find the time of motion of the bullet in the plank, assuming the resistance force to be proportional to the square of the velocity.**

Ans. According to the problem

$$m\frac{dv}{dt} = -kv^2 \quad \text{or,} \quad m\frac{dv}{v^2} = -k\,dt$$

Integrating, withing the limits,

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_{0}^{t} dt \quad \text{or,} \quad t = \frac{m}{k}\frac{(v_0-v)}{v_0 v}$$

To find the valufc of k, rewrite

$$mv\frac{dv}{ds} = -kv^2 \quad \text{or,} \quad \frac{dv}{v} = -\frac{k}{m}ds$$

On integrating

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}\int_{0}^{h} ds$$

So, $\quad k = \frac{m}{h}\ln\frac{v_0}{v} \quad$ (2)

Putting the value of k from (2) in (1), we get

$$t = \frac{h(v_0-v)}{v_0 v \ln\frac{v_0}{v}}$$

Q. 102. A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The friction coefficient depends on the distance x covered as k = ax, where a is a constant. Find the distance covered by the bar till it stops, and its maximum velocity over this distance.

Ans. From Newton's second law for the bar in projection from, $F_x = m\,w_x$ along x direction

we get $mg \sin \alpha - kmg \cos \alpha = mw$

or, $v\dfrac{dv}{dx} = g \sin \alpha - ax\, g \cos \alpha$, (as $k = ax$),

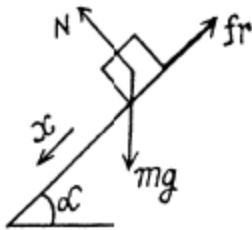or, $v\,dv = (g \sin \alpha - ax\, g \cos \alpha)\,dx$

or, $\displaystyle\int_0^v v\,dv = g\int_0^x (\sin \alpha - x \cos \alpha)\,dx$

So, $\dfrac{v^2}{2} = g\left(\sin \alpha\, x - \dfrac{x^2}{2} a \cos \alpha\right)$ (1)

From (1) $v = 0$ at either

$x = 0$, or $x = \dfrac{2}{a}\tan \alpha$

As the motion of the bar is unidirectional it stops after going through a distance of **2/a tan** α.

From (1), for v_{max},

$\dfrac{d}{dx}\left(\sin \alpha x - \dfrac{x^2}{2}a \cos \alpha\right) = 0$, which yields $x = \dfrac{1}{a}\tan \alpha$

Hence, the maximum velocity will be at the distance, $x = \tan \alpha/a$ Putting this value of x in (1) the maximum velocity,

$v_{max} = \sqrt{\dfrac{g \sin \alpha \tan \alpha}{a}}$

Q. 103. A body of mass m rests on a horizontal plane with the friction coefficient lc. At the moment t = 0 a horizontal force is applied to it, which varies with time as F = at, where a is a constant vector. Find the distance traversed by the body during the first t seconds after the force action began.

Ans. Since, the applied force is proportional to the time and the frictional force also exists, the motion does not start just after applying the force. The body starts its motion when F equals the limiting friction.

Let the motion start after time t_0 , then

$$F = at_0 = kmg \text{ or, } t_0 = \frac{kmg}{a}$$

So, for $t = \leq t_0$, the body remains at rest and for $t > t_0$ obviously

$$\frac{mdv}{dt} = a(t - t_0) \text{ or, } m\,dv = a(t - t_0)\,dt$$

Integrating, and noting $v = 0$ at $t = t_0$, we have for $t > t_0$

$$\int_0^v m\,dv = a\int_{t_0}^t (t - t_0)\,dt \text{ or } v = \frac{a}{2m}(t - t_0)^2$$

Thus $\quad s = \int v\,dt = \frac{a}{2m}\int_{t_0}^t (t - t_0)^2\,dt = \frac{a}{6m}(t - t_0)^3$

Q. 104. A body of mass m is thrown straight up with velocity vo. Find the velocity v' with which the body comes down if the air drag equals kv^2, where k is a constant and v is the velocity of the body.

Ans. While going upward, from Newton's second law in vertical direction :

$$m\frac{v\,dv}{ds} = -(mg + kv^2) \text{ or } \frac{v\,dv}{\left(g + \dfrac{kv^2}{m}\right)} = -ds$$

At the maximum height h, the speed $v = 0$, so

$$\int_{v_0}^0 \frac{v\,dv}{g + (kv^2/m)} = -\int_0^h ds$$

Integrating and solving, we get,

$$h = \frac{m}{2k}\ln\left(1 + \frac{kv_o^2}{mg}\right) \quad (1)$$

When the body falls downward, the net force acting on the body in downward direction equals $(mg - kv^2)$,

Hence net acceleration, in downward direction, according to second law of motion

$$\frac{v\,dv}{ds} = g - \frac{kv^2}{m} \quad \text{or,} \quad \frac{v\,dv}{g - \frac{kv^2}{m}} = ds$$

Thus, $\quad \displaystyle\int_0^{v'} \frac{v\,dv}{g - kv^2/m} = \int_0^{h} ds$

Integrating and putting the value of h from (1), we get,

$$v' = v_0 \Big/ \sqrt{1 + k v_0^2/mg}.$$

Q. 105. A particle of mass m moves in a certain plane P due to a force F whose magnitude is constant and whose vector rotates in that plane with a constant angular velocity ω. Assuming the particle to be stationary at the moment t = 0, find:

(a) its velocity as a function of time;
(b) the distance covered by the particle between two successive stops, and the mean velocity over this time.
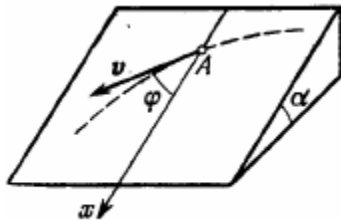

Fig. 1.27.

Ans. Let us fix x - y co-ordinate system to the given plane, taking x-axis in the direction along which the force vector was oriented at the moment $t = 0$, then the fundamental equation of dynamics expressed via the projection on x and y-axes gives,

$$F \cos \omega t = m \frac{d v_x}{dt} \quad (1)$$

and $\quad F \sin \omega t = m \frac{d v_y}{dt} \quad (2)$

(a) Using the condition $v(0) = 0$, we obtain $v_x = \dfrac{F}{m\,\omega}\sin\omega t$ (3)
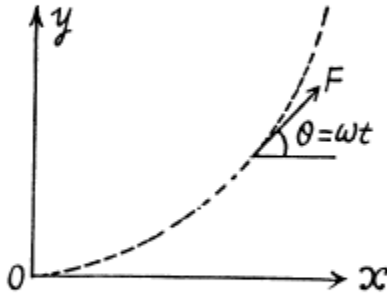
and

$$v_y = \dfrac{F}{m\,\omega}(1 - \cos\omega t) \qquad (4)$$

Hence, $\quad v = \sqrt{v_x^2 + v_y^2} = \left(\dfrac{2F}{m\,\omega}\right)\left|\sin\left(\dfrac{\omega t}{2}\right)\right|$

(b) It is seen from this that the velocity v turns into zero after the time interval Δt, which can be found from the relation ,

$\omega\,\dfrac{\Delta t}{2} = \pi$. Consequently, the sought distance, is

$$s = \int_0^{\Delta t} v\, dt = \dfrac{8F}{m\,\omega^2}$$

Average velocity, $\langle v \rangle = \dfrac{\int v\, dt}{\int dt}$

So, $\langle v \rangle = \int_0^{2\pi/\omega} \dfrac{2F}{m\,\omega}\sin\left(\dfrac{\omega t}{2}\right) dt \,/\, (2\pi\omega) = \dfrac{4F}{\pi\,m\,\omega}$

The Fundamental Equation Of Dynamics (Part - 4)

Q. 106. A small disc A is placed on an inclined plane forming an angle a with the horizontal (Fig. 1.27) and is imparted an initial velocity v_0. Find how the velocity of the disc depends on the angle y if the friction coefficient k = tan α and at the initial moment yo $= = \pi/2$.

Ans. The acceleration of the disc along the plane is determined by the projection of the force of gravity on this plane F_x = mg sin α and the friction force fr = kmg cos α . In our case k = tan α and therefore
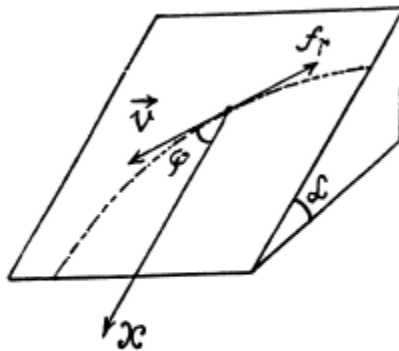
$$fr = F_x = mg \sin \alpha$$

Let us find the projection of the acceleration on the derection of the tangent to the trajectory and on the x-axis :

$$m w_t = F_x \cos \varphi - fr = mg \sin \alpha \, (\cos \varphi - 1)$$
$$m w_x = F_x - fr \cos \varphi = mg \sin \alpha \, (1 - \cos \varphi)$$

It is seen fromthis that $w_t = - w_x$, which means that the velocity v and its projection v_x differ only by a constant value C which does not change with time, i.e.

$$v = v_x + C,$$

where $v_x = v \cos \varphi.$. The constant C is found from the initial condition v = v_0, whence $C = v_0$ since $\varphi = \dfrac{\pi}{2}$ initially. Finally we obtain

$$v = v_0 / (1 + \cos \varphi).$$

In the cource of time $\varphi \to 0$ and $v \to v_0/2$ (Motion then is unaccelerated.)

Q. 107. A chain of length l is placed on a smooth spherical surface of radius R with one of its ends fixed at the top of the sphere. What will be the acceleration w of each element of the chain when its upper end is released? It is assumed that the length of the chain $l < \frac{1}{2}\pi R.$

Ans. Let us consider an element of length ds at an angle qp from the vertical diameter. As the speed of this element is zero at initial instant of time, it's centripetal acceleration is zero, and hence, $dN - \lambda\, ds \cos \varphi = 0,$ where λ is the linear mass density of the chain Let T **and** $T+dT$ be the tension at the upper and the lower ends of ds. we have from, $F_t = m\, w_t$

$$(T + dT) + \lambda\, ds\, g \sin \varphi - T = \lambda ds\, w_t$$

or, $\quad dT + \lambda\, Rd\varphi\, g \sin \varphi = \lambda ds\, w_t$

If we sum the above equation for all elements, the term $\int dT = 0$ because there is no tension at the free ends, so

$$\lambda\, gR \int_0^{l/R} \sin \varphi\, d\varphi = \lambda w_t \int ds = \lambda l\, w_t$$
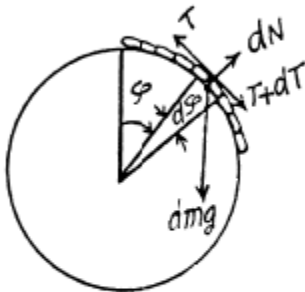
Hence $\quad w_t = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

As $w_n = a$ at initial moment

So, $w = |w_t| = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

Q. 108. A small body is placed on the top of a smooth sphere of radius R. Then the sphere is imparted a constant acceleration w_0 in the horizontal direction and the body begins sliding down. Find:

(a) the velocity of the body relative to the sphere at the moment of break-off;
(b) the angle θ_0 between the vertical and the radius vector drawn from the centre of the sphere to the break-off point; calculate θ_0 for $w_0 = g$.

Ans. In the problem, we require the velocity of the body, realtive to the sphere, which itself moves with an acceleration w_0 in horizontal direction (say towards left). Hence it is advisible to solve the problem in the frame of sphere (non-inertial frame).
At an arbitary moment, when the body is at an angle θ with the vertical, we sketch the force diagram for the body and write the second law of motion in projection form

$F_n = mw_n$

or, $mg \cos\theta - N - mw_0 \sin\theta = \dfrac{mv^2}{R}$ (1)
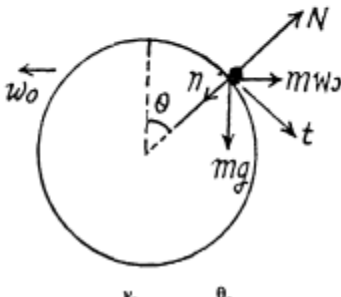
At the break off point, $N = 0$, $\theta = \theta_0$ and let
$v = v_0$ so the Eq. (1) becomes,

$\dfrac{v_0^2}{R} = g \cos\theta_0 - w_0 \sin\theta_0$ (2)

From, $F_t = mw_t$

$mg \sin\theta - mw_0 \cos\theta = m\dfrac{v\,dv}{ds} = m\dfrac{v\,dv}{R\,d\theta}$

or, $v\,dv = R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

Integrating, $\displaystyle\int_0^{v_0} v\,dv = \int_0^{\theta_0} R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

$\dfrac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0 \sin\theta_0$ (3)

Note that the Eq. (3) can also be obtained by the work-energy theorem $A = \Delta T$ (in the frame of sphere)

therefore, $mgR(1 - \cos \theta_0) + mw_0 R \sin \theta_0 = \frac{1}{2}mv_0^2$

[here $mw_0 R \sin \theta_0$ is the work done by the pseudoforce $(-m\vec{w_0})$]

or, $\frac{v_0^2}{2R} = g(1 - \cos \theta_0) + w_0 \sin \theta_0$

Solving Eqs. (2) and (3) we get,

$v_0 = \sqrt{2gR/3}$ and $\theta_0 = \cos^{-1}\left[\frac{2 + k\sqrt{5 + 9k^2}}{3(1 + k^2)}\right]$, where $k = \frac{w_0}{g}$

Hence $\theta_0 \big|_{w_0 - g} = 17°$

Q. 109. A particle moves in a plane under the action of a force which is always perpendicular to the particle's velocity and depends on a distance to a certain point on the plane as $1/r^n$, where n is a constant. At what value of n will the motion of the particle along the circle be steady?

Ans. This is not central force problem unless the path is a circle about the said point. Rather here F_t(tangential force) vanishes. Thus equation of motion becomes,

$v_t = v_0 = $ constant

and, $\frac{mv_0^2}{r} = \frac{A}{r^2}$ for $r = r_0$

We can consider the latter equation as the equilibrium under two forces. When the motion is perturbed, we write $r = r_0 + x$ and the net force acting on the particle is,

$-\frac{A}{(r_0 + x)^n} + \frac{mv_0^2}{r_0 + x} = \frac{-A}{r_0^n}\left(1 - \frac{nx}{r_0}\right) + \frac{mv_0^2}{r_0}\left(1 - \frac{x}{r_0}\right) = -\frac{mv_0^2}{r_0^2}(1 - n)x$

This is opposite to the displacement x, if $n < 1$. $\left(\frac{mv_0^2}{r}\right)$ is an outward directed centrifiigul

force while $\frac{-A}{r^n}$ is thd inward directed external force).

Q. 110. A sleeve A can slide freely along a smooth rod bent in the shape of a half-circle of radius R (Fig. 1.28). The system is set in rotation with a constant angular velocity ω about a vertical axis OO'. Find the angle θ corresponding to the steady position of the sleeve.

Ans. There are two forces on the sleeve, the weight F_1 and the centrifugal force F_2. We resolve both forces into tangential and normal component then the net downward tangential force on the sleeve is,
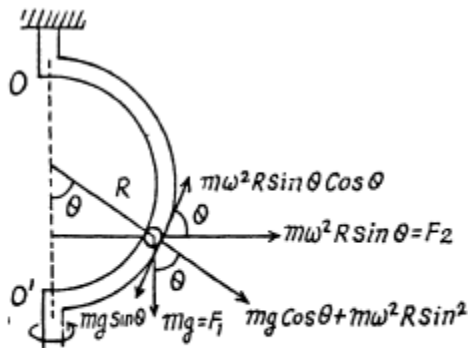
$$mg \sin\theta \left(1 - \frac{\omega^2 R}{g} \cos\theta\right)$$

This vanishes for $\theta = 0$ and for

$\theta = \theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$, which is real if

$\omega^2 R > g$. If $\omega^2 R < g$, then $1 - \frac{\omega^2 R}{g} \cos\theta$

is always positive for small values of 0 and hence the net tangential force near θ = 0 opposes any displacement away from it. θ = 0 is then stable.

If $\omega^2 R > g$, $1 - \frac{\omega^2 R}{g} \cos\theta$ is negative for small

θ near θ = 0 and θ = 0 is then unstable.

However $\theta = \theta_0$ is stable because the force tends to bring the sleeve near the equilibrium position $\theta = \theta_0$

If $\omega^2 R = g$, the two positions coincide and becomes a stable equilibrium point.

Q. 111. A rifle was aimed at the vertical line on the target located precisely in the northern direction, and then fired. Assuming the air drag to be negligible, find how much off the line, and in what direction, will the bullet hit the target. The shot was fired in the horizontal direction at the latitude φ = 60°, the bullet velocity v = 900 m/s, and the distance from the target equals s = 1.0 km.

Ans. Define the axes as shown with z along the local vertical, x due east and y due north. (We assume we are in the northern hemisphere). Then the Coriolis force has the components.

$$\vec{F}_{cor} = -2m\,(\vec{\omega} \times \vec{v})$$

$$= 2m\omega\left[v_y\cos\theta - v_z\sin\theta)\,\vec{i} - v_x\cos\theta\,\vec{j} + v_x\cos\theta\,\vec{k}\right] = 2m\omega\,(v_y\cos\theta - v_z\sin\theta)\,\vec{i}$$

$\ddot{x} = 2m\omega\,(v_y\sin\varphi - v_z\cos\varphi),\ \ddot{y} = 0$ and $\ddot{z} = -g$

Integrating we get $\dot{y} = v$ (constant), $\dot{z} = -gt$

and $\dot{x} = 2\omega\, v\sin\varphi\, t + \omega g\, t^2\cos\varphi$

Finally,

$x = \omega v\, t^2 \sin\varphi + \dfrac{1}{3}\omega g t^3 \cos\varphi$
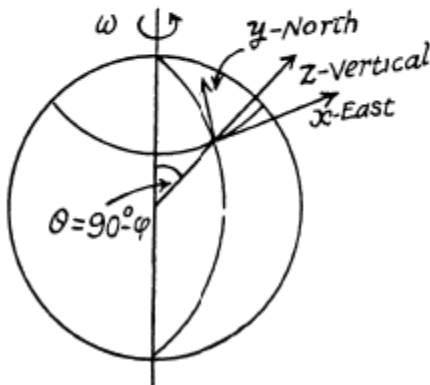
Now $v \gg gt$ in the present case. so,

$x \approx \omega v \sin\varphi \left(\dfrac{s}{v}\right)^2 = \omega\sin\varphi\,\dfrac{s^2}{v}$

≈ 7 cm (to the east).

Q. 112. A horizontal disc rotates with a constant angular velocity ω = 6.0 rad/s about a vertical axis passing through its centre. A small body of mass m = 0.50 kg moves along a diameter of the disc with a velocity v' = 50 cm/s which is constant relative to the disc. Find the force that the disc exerts on the body at the moment when it is located at the distance r = 30 cm from the rotation axis.

Ans. The disc exerts three forces which are mutually perpendicular. They are the reaction of the weight, mgy vertically upward, the Coriolis force 2mv' ω perpendicular to the plane of the vertical and along the diameter, and mω²r outward along the diameter. The resultant force is,

$$F = m\sqrt{g^2 + \omega^4 r^2 + (2v'\omega)^2}$$

Q. 113. A horizontal smooth rod AB rotates with a constant angular velocity ω = 2.00 rad/s about a vertical axis passing through its end A. A freely sliding sleeve of mass m = 0.50 kg moves along the rod from the point A with the initial velocity v₀ = 1.00 m/s. Find the Coriolis force acting on the sleeve (in the reference frame fixed to the rotating rod) at the moment when the sleeve is located at the distance r = 50 cm from the rotation axis.
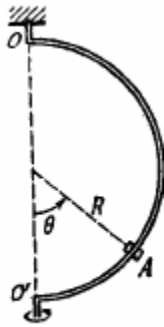

Fig. 1.28.

Ans. The sleeve is free to slide along the rod AB. Thus only the centrifugal force acts on it.

The equation is,

$$m\dot{v} = m\omega^2 r \quad \text{where } v = \frac{dr}{dt}.$$

But $\dot{v} = v\dfrac{dv}{dr} = \dfrac{d}{dr}\left(\dfrac{1}{2}v^2\right)$

so, $\dfrac{1}{2}v^2 = \dfrac{1}{2}\omega^2 r^2 + \text{constant}$

or, $v^2 = v_0^2 + \omega^2 r^2$

v₀ being the initial velocity when r = 0. The Coriolis force is then,

$$2m\omega\sqrt{v_0^2 + \omega^2 r^2} = 2m\omega^2 r\sqrt{1 + v_0^2/\omega^2 r^2}$$

$$= 2\cdot 83 \text{ N on putting the values.}$$

Q. 114. A horizontal disc of radius R rotates with a constant angular velocity ω about a stationary vertical axis passing through its edge. Along the circumference of the disc a particle of mass m moves with a velocity that is constant relative to the disc. At the moment when the particle is at the maximum distance from the rotation axis, the resultant of the inertial forces F$_{in}$ acting on the particle in the reference frame fixed to the disc turns into zero. Find:

(a) the acceleration ω' of the particle relative to the disc;
(b) the dependence of F$_{in}$ on the distance from the rotation axis.

Ans. The disc OBAC is rotating with angular velocity ω about the axis OO' passing through the edge point O. The equation of motion in rotating frame is,

$$m\vec{w'} = \vec{F} + m\omega^2 \vec{R} + 2m\vec{v'} \times \vec{\omega} = \vec{F} + \vec{F}_{in}$$

where $\vec{F}_{in}$ is the resultant inertial force. (pseudo force) which is the vector sum of centrifugal and Coriolis forces.
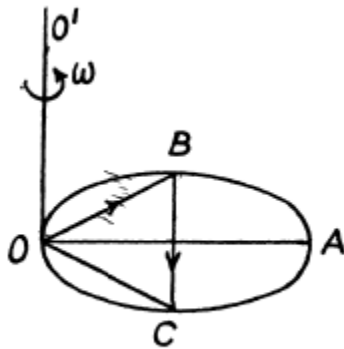
(a) At A, F_{in} vanishes. Thus $0 = -2m\omega^2 R \hat{n} + 2mv' \omega \hat{n}$

Where $\hat{n}$ **is** the inward drawn unit vector to the centre from the point in question, here A.

Thus, $v' = \omega R$

so, $w = \dfrac{v'^2}{\rho} = \dfrac{v'^2}{R} = \omega^2 R.$

(b) At B $\vec{F}_{in} = m\omega^2 \vec{OC} + m\omega^2 \vec{BC}$

its magnitude is $m\omega^2 \sqrt{4R^2 - r^2}$, where $r = OB.$

Q. 115. A small body of mass m = 0.30 kg starts sliding down from the top of a smooth sphere of radius R = 1.00 m. The sphere rotates with a constant angular velocity ω = 6.0 rad/s about a vertical axis passing through its centre. Find the centrifugal force of inertia and the Coriolis force at the moment when the body breaks off the surface of the sphere in the reference frame fixed to the sphere.

Ans. The equation of motion in the rotating coordinate system is,

$$m \vec{w'} = \vec{F} + m\omega^2 \vec{R} + 2m\,(\vec{v'} \times \vec{\omega})$$

Now, $\quad \vec{v'} = R\dot{\theta}\,\vec{e_\theta} + R\sin\theta\,\dot{\varphi}\,\vec{e_\varphi}$

and $\quad \vec{w'} = w'\cos\theta\,\vec{e_r} - w'\sin\theta\,\vec{e_\theta}$

$$\frac{1}{2m}\vec{F_{cor}} = \begin{vmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_\varphi} \\ 0 & R\dot{\theta} & R\sin\theta\,\dot{\varphi} \\ \omega\cos\theta & -\omega\sin\theta & 0 \end{vmatrix}$$
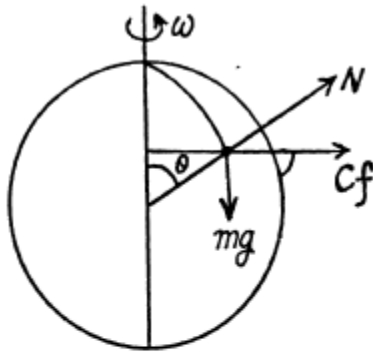
$$= \vec{e_r}\,(\omega R\sin^2\theta\,\dot{\varphi}) + \omega R\sin\theta\cos\theta\,\dot{\varphi}\,\vec{e_\theta} - \omega R\dot{\theta}\cos\theta\,\vec{e_\varphi}$$

Now on the sphere,

$$\vec{v} = (-R\dot{\theta}^2 - R\sin^2\theta\,\dot{\varphi}^2)\,\vec{e_r}$$
$$+ (R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\varphi}^2)\,\vec{e_\theta}$$
$$+ (R\sin\theta\,\ddot{\varphi} + 2R\cos\theta\,\dot{\theta}\,\dot{\varphi})\,\vec{e_\varphi}$$

Thus the equation of motion are,

$$m\,(-R\dot{\theta}^2 - R\sin^2\theta\,\dot{\varphi}^2) = N - mg\cos\theta + m\omega^2 R\sin^2\theta + 2m\omega R\sin^2\theta\,\dot{\varphi}$$

$$m\,(R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\varphi}^2) = mg\sin\theta + m\omega^2 R\sin\theta\cos\theta + 2m\omega R\sin\theta\cos\theta\,\dot{\varphi}$$

$$m\,(R\sin\theta\,\ddot{\varphi} + 2R\cos\theta\,\dot{\theta}\,\dot{\varphi}) = -2m\omega R\dot{\theta}\cos\theta$$
From the third equation, we get, $\dot{\varphi} = -\omega$

A result that is easy to understant by considering the motion in non-rotating frame. The eliminating φ we get,

$mR\,\ddot{\theta}^2 = mg\cos\theta - N$
$mR\,\ddot{\theta} = mg\sin\theta$

Integrating the last equation

$$\frac{1}{2}mR\,\dot{\theta}^2 = mg\,(1-\cos\theta)$$

Hence

$$N = (3-2\cos\theta)\,mg$$

So the body must fly off for $\theta = \theta_0 = \cos^{-1}\frac{2}{3}$, exactly as if the sphere were nonrotating.

Now, at this point $F_{cf} = \text{centrifugal force} = m\omega^2 R\sin\theta_0 = \sqrt{\frac{5}{9}}\,m\omega^2 R$

$$F_{cor} = \sqrt{\omega^2 R^2\dot{\theta}^2\cos^2\theta + (\omega^2 R^2)^2\sin^2\theta}\times 2m$$
$$= \sqrt{\frac{5}{9}(\omega^2 R)^2 + \omega^2 R^2\times\frac{4}{9}\times\frac{2g}{3R}}\times 2m = \frac{2}{3}m\omega^2 R\sqrt{5+\frac{8g}{3\,\omega^2 R}}$$

Q. 116. A train of mass m = 2000 tons moves in the latitude φ = 60° North. Find:

(a) the magnitude and direction of the lateral force that the train exerts on the rails if it moves along a meridian with a velocity v = 54 km per hour;
(b) in what direction and with what velocity the train should move for the resultant of the inertial forces acting on the train in the reference frame fixed to the Earth to be equal to zero.

Ans. (a) When the train is moving along a meridian only the Coriolis force has a lateral component and its magnitude (see the previous problem) is,

$2m \omega v \cos \theta = 2m \omega \sin \lambda$
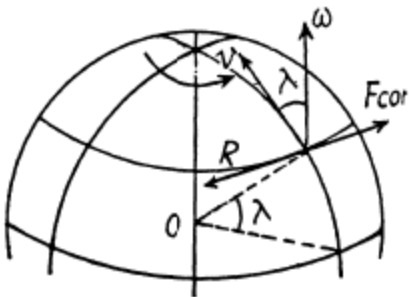
(Here we have put $R \dot{\theta} \to v$)

So, $F_{lateral} = 2 \times 2000 \times 10^3 \times \dfrac{2\pi}{86400} \times \dfrac{54000}{3600} \times \dfrac{\sqrt{3}}{2}$

$= 3.77$ kN, (we write λ for the latitude)

(b) The resultant of the inertial forces acting on the train is,

$$\vec{F}_{in} = -2m\omega R \dot{\theta} \cos \theta \, \vec{e}_\varphi$$
$$+ (m\omega^2 R \sin \theta \cos \theta + 2m \omega R \sin \theta \cos \theta \, \dot{\varphi}) \, \vec{e}_\theta$$
$$+ (m\omega^2 R \sin^2 \theta + 2m \omega R \sin^2 \theta \, \dot{\varphi}) \, \vec{e}_r$$

This vanishes if $\dot{\theta} = 0$, $\dot{\varphi} = -\dfrac{1}{2}\omega$

Thus $\vec{v} = v_\varphi \vec{e}_\varphi$, $v_\varphi = -\dfrac{1}{2}\omega R \sin \theta = -\dfrac{1}{2}\omega R \cos \lambda$

(We write λ for the latitude here)

Thus the train must move from the east to west along the 60th parallel with a speed,

$\dfrac{1}{2}\omega R \cos \lambda = \dfrac{1}{4} \times \dfrac{2\pi}{8.64} \times 10^{-4} \times 6.37 \times 10^6 = 115.8$ m/s ≈ 417 km/hr

Q. 117. At the equator a stationary (relative to the Earth) body falls down from the height h = 500 m. Assuming the air drag to be negligible, find how much off the vertical, and in what direction, the body will deviate when it hits the ground.

Ans. We go to the equation given in 1.111. Here $v_y = 0$ so we can take $y = 0$, thus we get for the motion in the $x\dot{z}$ plane. .

$$\ddot{x} = -2\omega\, v_z \cos\theta$$

and $\quad \ddot{z} = -g$

Integrating, $\quad z = -\dfrac{1}{2} g t^2$

$$\dot{x} = \omega\, g \cos\varphi\, t^2$$

So $\quad x = \dfrac{1}{3}\omega\, g \cos\varphi\, t^3 = \dfrac{1}{3}\omega\, g \cos\varphi \left(\dfrac{2h}{g}\right)^{3/2}$

$$= \dfrac{2\,\omega h}{3} \cos\varphi \sqrt{\dfrac{2h}{g}}$$

There is thus a displacement to the east of

$$\dfrac{2}{3} \times \dfrac{2\pi}{8}\, 64 \times 500 \times 1 \times \sqrt{\dfrac{400}{9\cdot8}} \approx 26 \text{ cm.}$$