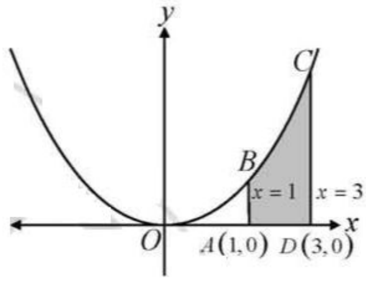
AREA OF BOUNDED REGIONS (XII, R. S. AGGARWAL)

EXERCISE 17 (Pg. No.: 884)

1. Find the area of the region bounded by the curve $y=x^2$, the x-axis and the line $x=1$ and $x=3$

Sol. $x^2=y$ is a upward parabola with its vertex at the origin and $x=1, x=3$ is the line parallel to the y-axis, at a distance $x=1$ to $x=3$ units from it.



Also, $x^2=y$ contains only even power of x.

So, it is symmetrical about the y-axis is

$\therefore$ Required area = area of $ABCD$

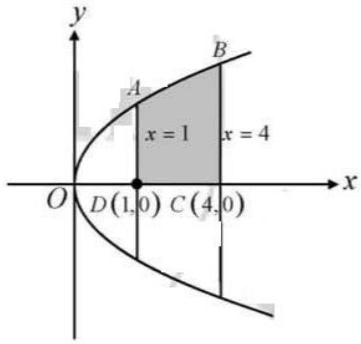
$$=\int_1^3 x^2\,dx=\left[\frac{x^3}{3}\right]_1^3=\left(\frac{27}{3}-\frac{1}{3}\right)=9-\frac{1}{3}=\frac{26}{3} \text{ sq. units.}$$

2. Find the area of the region bounded by the parabola $y^2=4x$ the x-axis and the line $x=1$ and $x=4$.

Sol. $y^2=4x$ is a right handed Parabola with its vertex at the origin and $x=1$, $x=4$ is the line parallel to y-axis at a distance of $x=1$ to $x=4$ units.

Also $y^2=4x$ contains only even power of y.

So, it is symmetrical about the x-axis



$\therefore$ Required area = area of $ABCD$

$$\Rightarrow \int_1^4 y\,dx=\int_1^4 \sqrt{4x}\,dx=2\int_1^4 \sqrt{x}\,dx$$

$$\Rightarrow 2.\left[\frac{x^{3/2}}{3/2}\right]_1^4=\frac{4}{3}\left[(4)^{3/2}-(1)^{3/2}\right]=\frac{4}{3}[8-1]$$

$$\Rightarrow \frac{4}{3}.7=\frac{28}{3} \text{ sq. units.}$$

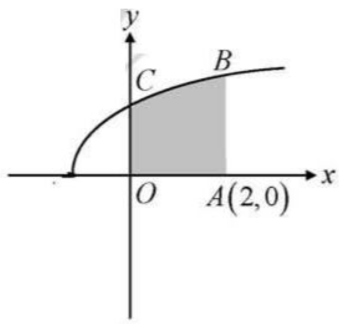
3. Find the area under the curve $y=\sqrt{6x+4}$ (above the x-axis) from $x=0$ to $x=2$

Sol. The equation of the curve $y^2=6x+4$, is a parabola with its vertex at $x=-2/3$

$x=2$ is the line parallel to y-axis at a distance of 2 units.

Also, $y^2=6x+4$, contains only even power of y

So, it is symmetrical about the x-axis



$$\therefore \text{ required area}=\int_0^2 \sqrt{6x+4}\,dx=\left[\frac{(6x+4)^{3/2}}{(3/2).6}\right]_0^2=\frac{1}{9}\left[(6x+4)^{3/2}\right]_0^2$$

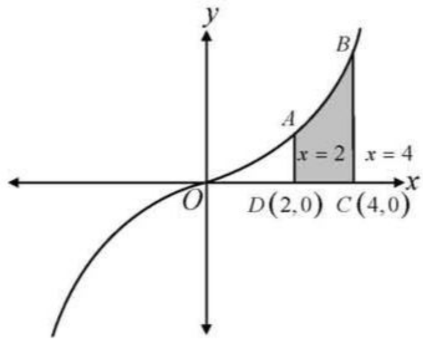
$$=\frac{1}{9}\left[(6\times 2+4)^{3/2}-(6\times 0+4)^{3/2}\right]=\frac{1}{9}\left[(16)^{3/2}-(4)^{3/2}\right]=\frac{1}{9}[64-8]=\frac{56}{9} \text{ sq. units.}$$

4. Determine the area enclosed by the curve $y=x^3$, and the line $y=0$, $x=2$ and $x=4$

Sol. The equation of the curve $y=x^3$, with the vertex at origin and the $x=2, x=4$ is the line parallel to

y-axis at a distance $x=2$ to $x=4$ units.

Required area $=$ area of $ABCD$



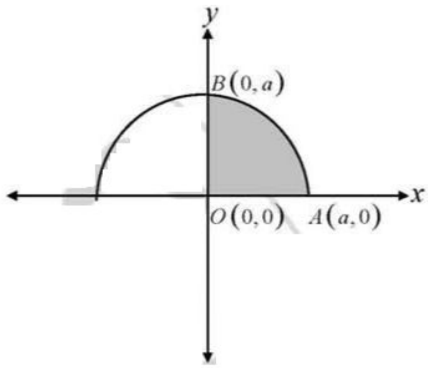
$$=\int_2^4 y\,dx=\int_2^4 x^3\,dx=\left[\frac{x^4}{4}\right]_2^4$$

$$=\frac{1}{4}\left[(4)^4-(2)^4\right]$$

$$=\frac{1}{4}[256-16] \qquad =\frac{1}{4}\times 240=60 \text{ sq. units.}$$

5. Determine the area under the curve $y=\sqrt{a^2-x^2}$, included between the line $x=0$ and $x=4$

Sol. The given equation of the curve, $y=\sqrt{a^2-x^2}$



$$y^2=a^2-x^2, \quad y^2+x^2=a^2 \quad \ldots(1)$$

$x^2+y^2=a^2$ represents a circle with its center at $0(0, 0)$ and radius is a units.

Also $x^2+y^2=a^2$ contains only even power of x & y

So, it is symmetrical about the x-axis and y-axis.

Required area $=$ area of OAB

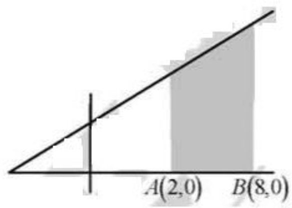
$$=\int_0^a y\,dx=\int_0^a \sqrt{a^2-x^2}\,dx=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\left\{\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right\}-\left\{\frac{0}{2}\sqrt{a^2-0}+\frac{a^2}{2}\sin^{-1}(0)\right\}$$

$$=\frac{a^2}{2}\sin^{-1}(1)=\frac{a^2}{2}\times\frac{\pi}{2}=\frac{\pi a^2}{4} \qquad \therefore \text{ Required area}=\frac{\pi a^2}{4} \text{ sq. units.}$$

6. Using integration, find the area of the region bounded by the line $2y=5x+7$, the x-axis, and the line $x=2$ and $x=8$

Sol. The given equation of the line $2y=5x+7$.



$$\therefore \quad y=\frac{5x+7}{2} \qquad \ldots(1)$$

$$\Rightarrow -5x+2y=7 \quad \Rightarrow \quad \frac{x}{-7/5}+\frac{y}{7/2}=1$$

Since Required area $=\int_2^8 y\,dx=\int_2^8 \frac{5x+7}{2}dx$

$$=\frac{1}{2}\int_2^8 (5x+7)\,dx=\frac{1}{2}\left[5\frac{x^2}{2}+7x\right]_2^8=\frac{1}{2}\left[\left(5.\frac{64}{2}+56\right)-\left(5.\frac{4}{2}+14\right)\right]=\frac{1}{2}\left[(160+56)-(10+14)\right]$$

$$=\frac{1}{2}[216-24]=\frac{1}{2}\times 192=96 \text{ sq. units.}$$

7. Find the area of the region bounded by the curve $y^2=4x$ and the line $x=3$

Sol. $y^2=4x$ is a right handed parabola with its vertex at the origin and $x=3$ is the line parallel to y-axis at a distance of $x=3$ units.

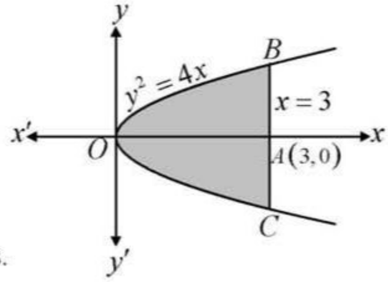
Also, $y^2=4x$ contains only even power of y

So, it is symmetrical about the x-axis.

Required area

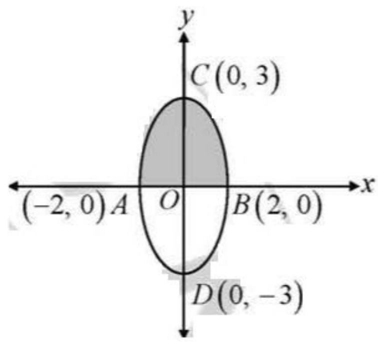
$$= \text{area of } OAB + \text{area of } OAC = 2\,\text{area of } OAB = 2\int_0^3 y\,dx$$

$$= 2\int_0^3 2\sqrt{x}\,dx = 4\left[\frac{x^{3/2}}{3/2}\right]_0^3 = \frac{8}{3}\left[(3)^{3/2} - (0)^{3/2}\right] = \frac{8}{3}.3\sqrt{3} = 8\sqrt{3} \text{ sq. units.}$$



8. Evaluate the area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ above the x-axis.

Sol. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is a right handed ellipse with its vertex at the origin and the point of intersection $(\pm 2, 0)$ and $(0, \pm 3)$. Also $\frac{x^2}{4} + \frac{y^2}{9} = 1$ contains only even power of x and y so, it is symmetrical about the x-axis, y-axis respectively.



Required area $= 2\text{area of } OBC = 2\int_0^2 y\,dx = 2\int_0^2 \frac{3}{2}\sqrt{4-x^2}$

$$= 3\int_0^2 \sqrt{(2)^2 - (x)^2}\,dx = 3\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

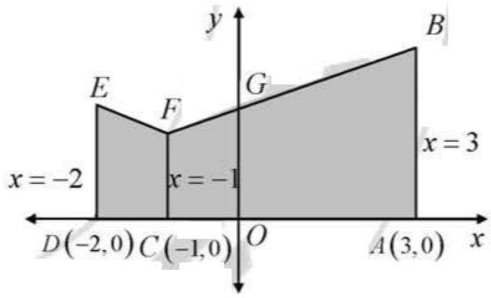
$$= 3\left[\left\{\frac{2}{2}\sqrt{4-4} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right)\right\} - \left\{\frac{0}{2}\sqrt{4-0} + 2\sin^{-1}(0)\right\}\right]$$

$$= 3\left[2\sin^{-1}(1)\right] = 3.2.\frac{\pi}{2} = 3\pi \text{ sq. units.}$$

9. Using integration find the area of the region bounded by the lines $y = 1 + |x+1|$, $x = -2$, $x = 3$ and $y = 0$

Sol. The given equation of the curve $y = 1 + |x+1|$

$$= \begin{cases} 1 + (x+1), & \text{when } x+1 \geq 0 \\ 1 - (x+1), & \text{when } x+1 < 0 \end{cases} = \begin{cases} x+2, & \text{if } x \geq -1 \\ -x, & \text{if } x < -1 \end{cases}$$



Required area $= \int_{-2}^{3} y\,dx = \int_{-2}^{-1} y\,dx + \int_{-1}^{3} y\,dx$

$$= \int_{-2}^{-1} (-x)\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$= \left[-\frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3} = -\left[\frac{1}{2} - 2\right] + \left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right]$$

$$= -\left[\frac{-3}{2}\right] + \left[\frac{21}{2} + \frac{3}{2}\right] = \frac{3}{2} + \frac{24}{2} = \frac{3}{2} + 12 = \frac{27}{2} \text{ sq. units.}$$

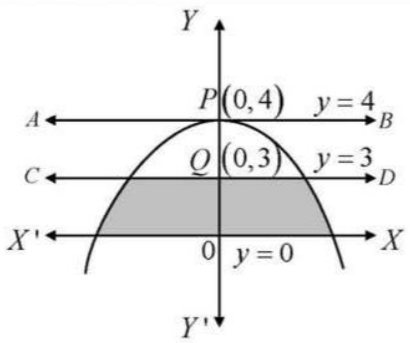
10. Find the area bounded by the curve $y = 4 - x^2$, the y-axis and the lines $y = 0$, $y = 3$

Sol. $y = 4 - x^2$ or $x^2 = 4 - y$

The curve is symmetrical about y-axis and the curve does not lie beyond the line $y = 4$ because when $y > 4$, x^2 is negative and x becomes imaginary. The shape of the curve is as shown in fig.

Required area = the area between the line $y = 0$ (x-axis) and the line $y = 3$, is shown as shaded region

$$= \int_0^3 x\,dy = \int_0^3 \sqrt{4-y}\,dy$$



$$= \left(\frac{-2}{3}\right)\left[(4-y)^{3/2}\right]_0^3$$

$$= \frac{-2}{3}\left[(4-3)^{3/2} - (4-0)^{3/2}\right]$$

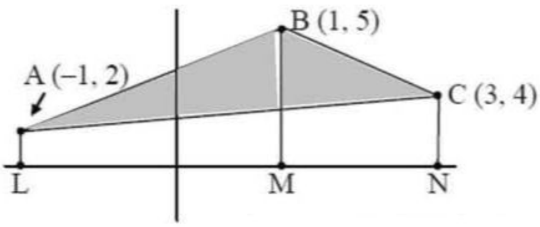
$$= \frac{-2}{3}[1-8] = \frac{-2}{3}(-7) = \frac{14}{3} \text{ sq. units.}$$

11. Using integration, find the area of the region bounded by the triangle whose vertices are $A(-1,2), B(1,5)$ and $C(3,4)$

Sol. Let the co-ordinates of A,B and C are $(-1,2)$, $(1,5)$ and $(3,4)$ respectively

we have to find the area of ΔABC

Find equation of line AB $\quad y-5 = \left(\frac{2-5}{-1-1}\right)\cdot(x-1)$



$$y-5 = \frac{3}{2}(x-1)$$

$$2y-10 = 3x-3$$

$$3x-2y+7=0$$

$$y = \frac{3x+7}{2}$$

Equation of BC $\quad y-4 = \left(\frac{5-4}{1-3}\right)\cdot(x-3)$

$$y-4 = \frac{1}{-2}(x-3)$$

$$2y-8 = -x+3$$

$$x+2y-11=0 \qquad \ldots \text{(ii)}$$

$$y = \frac{11-x}{2}$$

Equation of AC $\quad y-4 = \left(\frac{2-4}{-1-3}\right)\cdot(x-3)$

$$y-4 = \frac{1}{2}(x-3) \Rightarrow 2y-8 = x-3$$

$$x-2y+5=0 \qquad \ldots\ldots \text{(iii)}$$

$$\Rightarrow y = \frac{x+5}{2}$$

So, required area $= \int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{11-x}{2}\right)dx - \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$

$$= \frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1} + \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}+7\right)-\left(\frac{3}{2}-7\right)\right] + \frac{1}{2}\left[\left(33-\frac{9}{2}\right)-\left(11-\frac{1}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$$

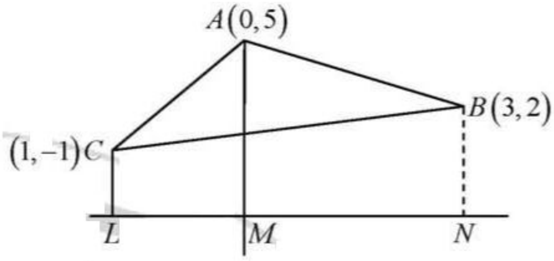
$=\frac{1}{2}[14+22-4-24]=\frac{1}{2}[36-28]=4$ square unit

12. Using integration, find the area of ΔABC, the equation of whose sides AB, BC and AC are given by $y=4x+5,\ x+y=5$ and $4y=x+5$ respectively.

Sol. Given, $4x-y=-5$ ...(1)

$x+y=5$ ...(2)

$x-4y=-5$ ...(3)



From equation (1) & (2), we get, $5x=0 \Rightarrow x=0$.

So, $y=5$. So, the required point $A(0, 5)$.

From equation (2) & (3), we get, $5y=10$

$\therefore\ y=2$ and $x=3$

So, the required point $B(3, 2)$.

From equation (3) & (1), we get, $-15x=15$

$\therefore\ x=-1$ and $y=1$. So, the point $C(-1, 1)$. $\therefore$ Required area $=\int_{-1}^{0} y_{AC}dx+\int_{0}^{3} y_{AB}dx-\int_{-1}^{3} y_{BC}dx$

$$=\int_{-1}^{0}(4x+5)dx+\int_{0}^{3}(-x+5)dx-\frac{1}{4}\int_{-1}^{3}(x+5)dx=\left[4\frac{x^2}{2}+5x\right]_{-1}^{0}+\left[\frac{-x^2}{2}+5x\right]_{0}^{3}-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

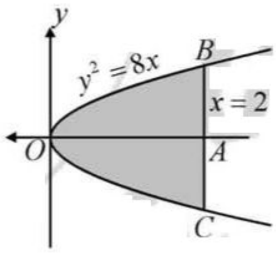
$$=\left[0-(2-5)\right]+\left[\left(-\frac{9}{2}+15\right)-0\right]-\frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]=3+\frac{21}{2}-6=\frac{21}{2}-3=\frac{15}{2}\text{ sq. units.}$$

13. Using integration, find the area of region bounded between the line $x=2$ and the parabola $y^2=8x$.

Sol. $y^2=8x$ is a right handed parabola with its vertex at the origin and $x=2$ is the line parallel to y-axis at a distance of $x=2$ units.

Also, $y^2=8x$ contains only even power of y.

So, it is symmetrical about the x-axis.



Required area

$=$ area of $OAB+$ area of $OAC=2$ area of OAB

$$=2\int_{0}^{2} y\,dx=2\int_{0}^{2}2\sqrt{2}\sqrt{x}\,dx=4\sqrt{2}.\left[\frac{x^{3/2}}{3/2}\right]_0^2=\frac{8\sqrt{2}}{3}\left[2^{3/2}-0\right]$$

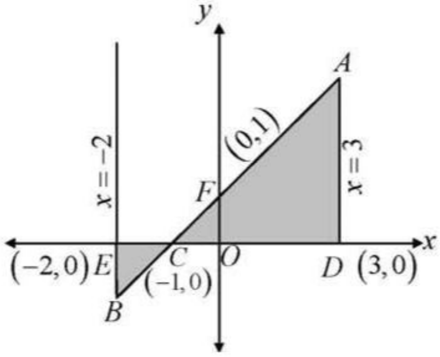
$$=\frac{8\sqrt{2}}{3}\left[2\sqrt{2}-0\right]=\frac{32}{3}\text{ sq. units}$$

14. Using integration, find the area of the region bounded by the line $y-1=x$ and the x-axis, and the ordinate $x=-2$ and $x=3$

Sol. The given equation of the line

$y-1=x$ ...(1)

$\therefore\ y=x+1 \Rightarrow -x+y=1 \quad \Rightarrow \frac{-x}{1}+\frac{y}{1}=1$



Let AB be the given line, intersecting the x-axis at $C(-1, 0)$

$\therefore$ Required area

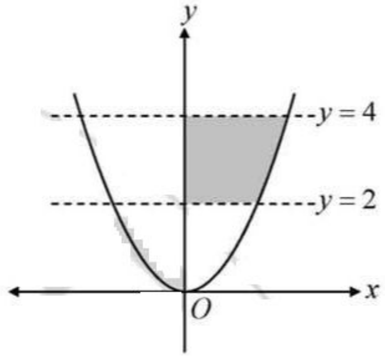
$=$(area of $CDAC+$ area of $CBEC$)$=\int_{-1}^{3} y\,dx+\int_{-2}^{-1}(-y)\,dx$

$$= \int_{-1}^{3}(x+1)\,dx + \int_{-2}^{-1} -(x+1)\,dx = \left[\frac{x^2}{2}+x\right]_{-1}^{3} - \left[\frac{x^2}{2}+x\right]_{-2}^{-1}$$

$$= \left[\left(\frac{9}{2}+3\right)-\left(\frac{1}{2}-1\right)\right]-\left[\left(\frac{1}{2}-1\right)-(2-2)\right]=\left[\frac{15}{2}+\frac{1}{2}\right]-\left[\frac{-1}{2}\right]=8+\frac{1}{2}=\frac{17}{2}=8.5 \text{ sq. units}$$

15. Sketch the region lying in the first quadrant and bounded by $y=4x^2$, $x=0$, $y=2$ and $y=4$. Find the area of the region using integration.

Sol. $x^2=\frac{1}{4}y$ is a upward parabola with its vertex at the origin and $y=2$, $y=4$ is the line parallel to the x-axis at a distance $y=2$ to $y=4$ units from it.



Also, $x^2=\frac{1}{4}y$ contains only even power of y.

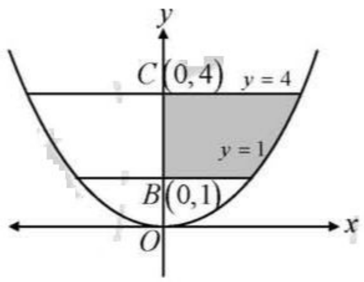
So it is symmetrical about the y-axis

Required area $= \int_{2}^{4} x\,dy = \int_{2}^{4} \frac{1}{2}\sqrt{y}\,dy = \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_{2}^{4}$

$$= \frac{1}{3}\left[(4)^{3/2}-(2)^{3/2}\right] = \frac{1}{3}\left[8-2\sqrt{2}\right] = \frac{8-2\sqrt{2}}{3} \text{ sq. units.}$$

16. Sketch the region lying in the first quadrants and bounded by $y=9x^2$, $x=0$, $y=1$ and $y=4$, find the area of the region, using integration.

Sol. $x^2=\frac{1}{9}y$ is a upward parabola with its vertex at the origin and $y=1$, $y=4$ is the line parallel to x-axis at a distance $y=1$ to $y=4$ units from it.



Also, $x^2=\frac{1}{9}y$ contains only even power of y.

So it is symmetrical about the y-axis.

$\therefore$ Required area

$$= \int_{1}^{4} x\,dy = \int_{1}^{4} \frac{1}{3}\sqrt{y}\,dy = \frac{1}{3}\left[\frac{y^{3/2}}{3/2}\right]_{1}^{4} = \frac{2}{9}\left[(4)^{3/2}-(1)^{3/2}\right] = \frac{2}{9}[8-1] = \frac{14}{9} \text{ sq. units.}$$

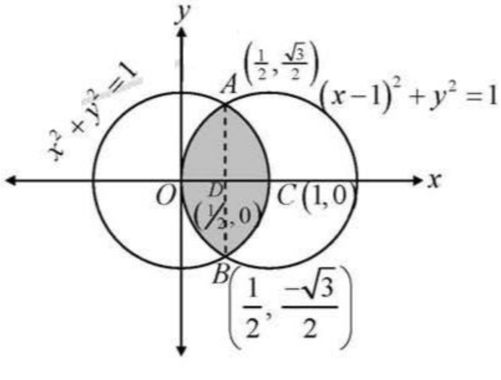
17. Find the area of the region enclosed between the circle $x^2+y^2=1$ and $(x-1)^2+y^2=1$

Sol. The circle $x^2+y^2=1$ ...(1) and $(x-1)^2+y^2=1$ ...(2)

Intersection at the point obtained by solving (1) and (2), from (1), we get,

$y^2=1-x^2$, put in (2) $(x-1)^2+1-x^2=1 \quad \Rightarrow (x-1)^2-x^2=0$

$\Rightarrow (x-1-x)(x-1+x)=0 \quad \Rightarrow -2x+1=0 \quad \Rightarrow x=\frac{1}{2}$



Now, From (1), $x=\frac{1}{2} \quad \Rightarrow y=\pm\frac{\sqrt{3}}{2}$

$\therefore$ (1) and (2) intersection at $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

and $B\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$ centre of first circle is $(0, 0)$

and radius $= 1$. Also, centre of second circle is $(1, 0)$ and radius $= 1$.

Also, both the circle are symmetrical about x-axis. $\therefore$ Required area is shown shaded.

$\therefore$ Required area

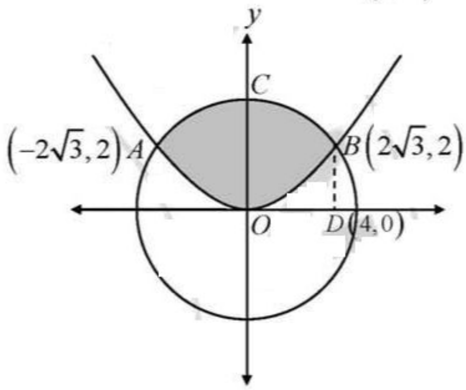
$= \text{area } OACB = 2\,(\text{area } OAC) = 2\,[\text{Area } OAD + \text{area } DCA]$

$$= 4\left[\int_{1/2}^{1} \sqrt{1-x^2}\, dx\right] = 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}(x)\right]_{1/2}^{1} = 4\left[\frac{1}{2}\sin^{-1}(1) - \frac{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= 4\left[\frac{1}{2}.\frac{\pi}{2} - \frac{\sqrt{3}}{8} - \frac{1}{2}.\frac{\pi}{6}\right] = 4\left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] = 4\left[\frac{2\pi}{12} - \frac{\sqrt{3}}{8}\right] = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{sq. units.}$$

18. Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Also, find the area of the region, using integration.

Sol. $x^2 = 6y$ is upward parabola with vertex $(0, 0)$ and $x^2 + y^2 = 16$ is a circle with centre $(0, 0)$ and radius 4. The given curve are



$x^2 + y^2 = 16 \quad \ldots(1)$

and $x^2 = 6y \quad \ldots(2)$

By, Solving equation (1) & (2), we get, the point

$A(-2\sqrt{3}, 2)$ and $B(2\sqrt{3}, 2)$

$\therefore$ Required area

$= 2\,(\text{area } OBCO) = 2\,[\text{area } ODBCO - \text{area } ODBO]$

$$= 2\left\{\int_0^{2\sqrt{3}} \sqrt{16-x^2}\, dx - \int_0^{2\sqrt{3}} \frac{x^2}{6} dx\right\}$$

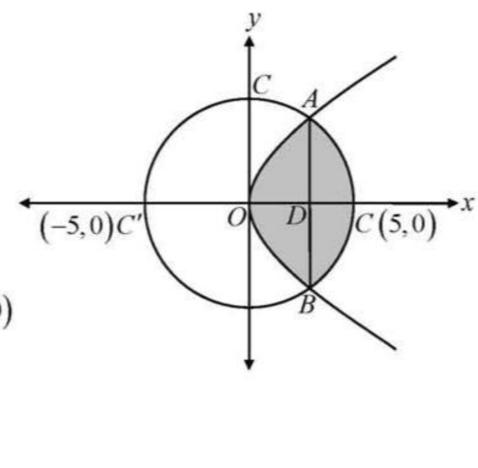
$$= 2\left\{\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^{2\sqrt{3}} - \frac{1}{6}\left[\frac{x^3}{3}\right]_0^{2\sqrt{3}}\right\} = 2\left\{\left[\frac{2\sqrt{3}}{2}.2 + \frac{16}{2}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{1}{18}.24\sqrt{3}\right]\right\}$$

$$= 4\sqrt{3} + 16\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{8}{3}\sqrt{3} = \frac{4\sqrt{3}}{3} + 16.\frac{\pi}{3} = \frac{16\pi + 4\sqrt{3}}{3} \text{ sq. units.}$$

19. Sketch the region common to the circle $x^2 + y^2 = 25$ and the parabola $y^2 = 8x$. Also find the area of the region, using integration.

Sol. $y^2 = 8x$ is right hand parabola with vertex $(0, 0)$ and $x^2 + y^2 = 25$ is a circle with centre $(0, 0)$ and radius 5 units.

$x^2 + y^2 = 25 \quad \ldots(1) \quad$ and $y^2 = 8x \quad \ldots(2)$



On solving (1) and (2), we get,

$x^2 + 8x - 25 = 0 \quad \Rightarrow x = \dfrac{-8 \pm \sqrt{64+100}}{2}$

$\Rightarrow x = -4 + \sqrt{41}$ (rejecting –ve value)

By, Putting $y = 0$ in (1), we get, $x = \pm 5$

Thus, the circle (1) cuts the x-axis at $C(5, 0)$ and $C'(-5, 0)$

$\therefore$ Required area

$= 2\,[\text{area of } ODCAO] = 2\,[\text{area of } ODAO + \text{area of } ADCA]$

$$= 2\left[\int_0^a \sqrt{8x}\,dx + \int_a^5 \sqrt{25-x^2}\,dx\right] = \left\{2\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^a + \left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_a^5\right\}$$

$$= \left\{\frac{8\sqrt{2}}{3}.a^{\frac{3}{2}} + \frac{25\pi}{2} - a\sqrt{25-a^2} - 25\sin^{-1}\left(\frac{a}{5}\right)\right\} \text{sq units, where } a = -4+\sqrt{41}$$

20. Draw a rough sketch of the region $\{(x, y): y^2 \le 3x, 3x^2+3y^2 \le 16\}$ and find the area enclosed by the region, using the method of integration.

Sol. $R = \{(x, y): y^2 \le 3x,\ 3x^2+3y^2 \le 16\}$

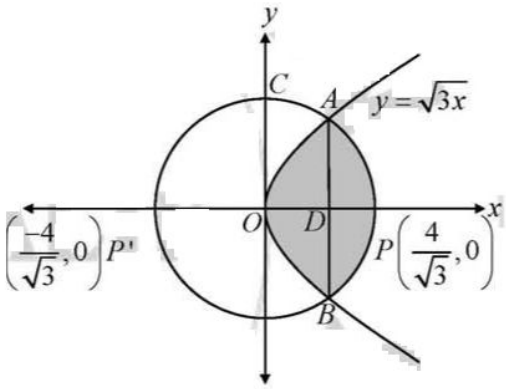
$= \{(x, y): y^2 \le 3x\} \cap \{(x, y): 3x^2+3y^2 \le 16\} = R_1 \cap R_2$

Where $R_1 = \{(x, y): y^2 \le 3x\}$ represents the region inside the parabola, $y^2 = 3x$ with vertex $(0, 0)$ and x-axis as its axis and $R = \{(x, y): 3x^2+3y^2 \le 16\}$ represent the interior of the circle $3x^2+3y^2 = 16$ with centre $(0, 0)$ and radius $= \frac{4}{\sqrt{3}}$

Thus the region R which is intersection of R_1 and R_2 is shown shaded in the figure.

$3x^2+3y^2 = 16 \qquad \ldots(1)$

$y^2 = 3x \qquad \ldots(2)$



By solving equation (1) & (2), we get,

$3x^2+9x-16 = 0 \quad \Rightarrow x = \frac{-9\pm\sqrt{273}}{6}$

$x = \frac{-9+\sqrt{273}}{6}$ (rejecting $-ve$ value)

By, Putting $y = 0$ in (1), we get, $x = \pm\frac{4}{\sqrt{3}}$.

Thus, the circle (1) cuts the x-axis at

$P\left(\frac{4}{\sqrt{3}}, 0\right)$ and $P'\left(-\frac{4}{\sqrt{3}}, 0\right)$

Required area

$= 2\,[\text{Area of } ODPAO] = 2\,[\text{area of } ODAO + \text{area of } ADPA]$

$$= 2\left[\int_0^a \sqrt{3x}\,dx + \int_a^{4/\sqrt{3}} \sqrt{\frac{16}{3}-x^2}\,dx\right] = 2\left\{\sqrt{3}.\left[\frac{x^{3/2}}{3/2}\right]_0^a + \left[\frac{x}{2}\sqrt{\frac{16}{3}-x^2} + \frac{16}{3}\sin^{-1}\left(\frac{x}{4/\sqrt{3}}\right)\right]_a^{4/\sqrt{3}}\right\}$$

$$= \left\{\frac{4}{\sqrt{3}}a^{3/2} + \frac{8\pi}{3} - a\sqrt{\frac{16}{3}-a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}\,a}{4}\right)\right\} \text{sq units, where } a = \frac{-9+\sqrt{273}}{9}$$

21. Draw a rough sketch and find the area of the region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$ using the method of integration.

Sol. The given parabola are

$y^2 = 4x \qquad \ldots(1)$

and $x^2 = 4y \qquad \ldots(2)$

In order to find the points of intersection of the given curves, we solve (1) and (2).

Simultaneously :

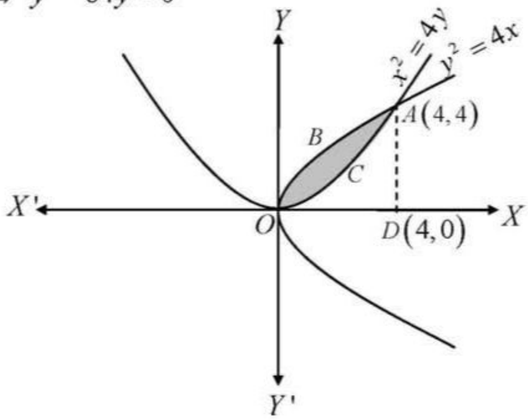
Now, putting $x=\frac{y^2}{4}$ from (1) in (2), we get, $\frac{y^4}{16}=4y \quad \Rightarrow y^4-64y=0$

$\Rightarrow y\left(y^3-64\right)=0 \quad \Rightarrow y=0,\ y=4$

Now, $(y=0,\ x=0)$ and $\left(y=4,\ x=\frac{16}{4}=4\right)$

Thus the points of intersection of the two parabolas are

$0(0,\ 0)$ and $A(4,\ 4)$.



Draw $AD \perp OX$ then point D is $(4,\ 0)$.

Required area

$=$area $OCABO=$(area of $OBADO-$ area $OCADO$)

$$=\int_0^4 y_1\,dx-\int_0^4 y_2\,dx=\int_0^4 2\sqrt{x}\,dx-\int_0^4\frac{x^2}{4}dx=\left[2.\frac{2}{3}.x^{3/2}\right]_0^4-\frac{1}{4}\left[\frac{x^3}{3}\right]_0^4=\left(\frac{32}{3}-\frac{16}{3}\right)=\frac{16}{3}\text{ sq. units.}$$

Hence, the required area is $\frac{16}{3}$ sq. units.

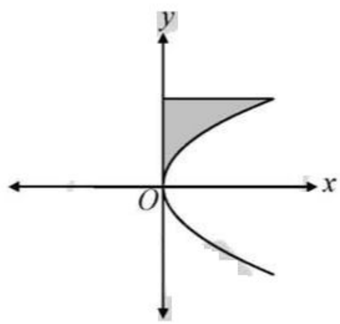
22. Find the integration the area bounded by the curve $y^2=4ax$ and the line $y=2a$ and $x=0$.

Sol. $y^2=4ax$ is a right handed parabola with its vertex at the origin, put $y=2a$ in $y^2=4ax$, then, $x=a$ is the line parallel to y-axis at a distance of $x=a$ units,

Also, $y^2=4ax$ contains only even power of y.

So, it is symmetrical about the x-axis

$y^2=4ax \qquad \ldots(1)$



$\therefore$ Required area

$$=\text{area of } OAB=\int_0^{2a}x\,dy=\int_0^{2a}\frac{y^2}{4a}\,dy$$

$$=\frac{1}{4a}.\left[\frac{y^3}{3}\right]_0^{2a}=\frac{1}{12a}\left[(2a)^3-(0)^3\right]=\frac{8a^3}{12a}=\frac{2a^2}{3}\text{ sq. units.}$$

23. Find the area between the curve $y=\frac{x}{\pi}+2\sin^2 x$, the x-axis and the ordinates $x=0$ and $x=\pi$

Sol. The given equation of the curve $y=\sin^2 x$

same value of x and the corresponding value of y are given below.

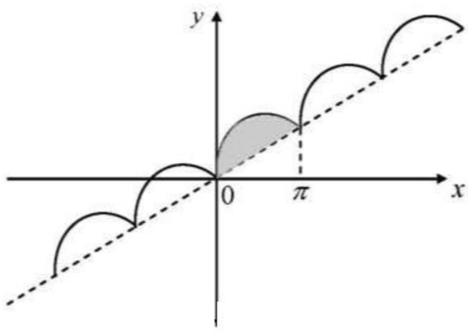
x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
y	0	1/4	1/2	3/4	1	3/4	0

$\therefore$ Required Area

$$=\int_0^{\pi}\left(\frac{x}{\pi}+2\sin^2 x\right)dx=\left[\frac{1}{\pi}.\frac{x^2}{2}+2\int\sin^2 x\,dx\right]_0^{\pi}$$

$$=\left[\frac{x^2}{2\pi}+2\int\frac{1-\cos 2x}{2}dx\right]_0^{\pi}$$

$$=\left[\frac{x^2}{2\pi}+x-\frac{\sin 2x}{2}\right]_0^{\pi}=\left(\frac{\pi^2}{2\pi}+\pi-\frac{\sin 2\pi}{2}\right)-(0)$$



$= \frac{\pi}{2} + \pi - 0 = \frac{3\pi}{2}$ sq units.

24. Find the area bounded by the curve $y = \cos x$, the x-axis and the ordinates $x = 0$ and $x = 2\pi$.

Sol. The given curve is $y = \cos x$.

Some value of x and the corresponding value of y are given below :

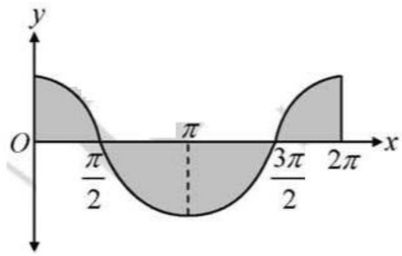
x	0	$\pi/6$	$\pi/2$	π	$3\pi/2$	2π
y	1	$\sqrt{3}/2$	0	-1	0	1

Taking a fixed unit distance for π along the x-axis,

we can plot the points $(0, 1), \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right)$ and $(2\pi, 1)$.

Now, Join these points free hand to obtain a rough sketch of the give curve.

$\therefore$ Required area



$= \{$ area of $OABO +$ area of $BCD +$ area of DEF $\}$

$$= \int_0^{\pi/2} y\,dx + \int_{\pi/2}^{3\pi/2} (-y)\,dx + \int_{3\pi/2}^{2\pi} y\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

$$= (\sin \pi/2 - \sin 0) - [\sin 3\pi/2 - \sin \pi/2] + (\sin 2\pi - \sin 3\pi/2)$$

$$= (1-0) - (-2) + 1 = 1 + 2 + 1 = 4 \text{ sq units.}$$

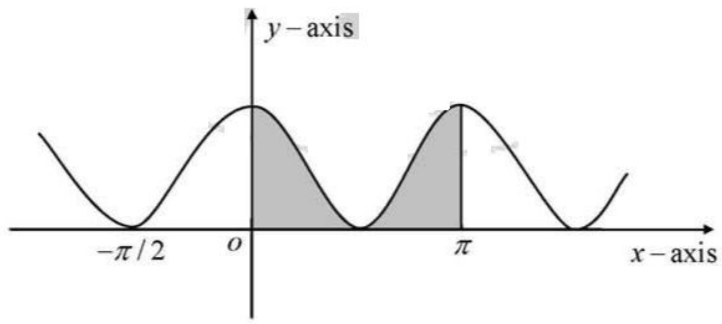
25. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

Sol. The given equation of the curve $y = \cos^2 x$. Some value of x and the corresponding value of y are given below.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
y	1	$3/4$	$1/2$	$1/4$	0	$3/4$	1

Taking the fixed unit distance from π along the x-axis, we can plot the points $(0, 1)$, $\left(\frac{\pi}{6}, \frac{3}{4}\right)$, $\left(\frac{\pi}{4}, \frac{1}{2}\right)\left(\frac{\pi}{3}, \frac{1}{4}\right)\left(\frac{\pi}{2}, 0\right)\left(\frac{2\pi}{3}, \frac{3}{4}\right) \& (\pi, 1)$

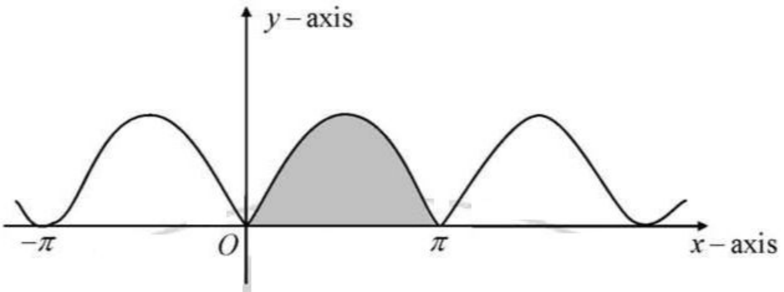
Graph of $\cos^2 x$



The given equation of the curve $y = \sin^2 x$ some value of x and the corresponding value of y are given below.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
y	0	$1/4$	$1/2$	$3/4$	1	$3/4$	0

Graph of $\sin^2 x$



$\therefore$ Required area $= \int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \frac{1-\cos 2x}{2}dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\left[(\pi - 0) - 0\right] = \frac{\pi}{2}$ sq. units.

$\therefore$ Required area $= \int_0^{\pi} \cos^2 x\,dx = \int_0^{\pi} \frac{1+\cos 2x}{2}dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\left[(\pi + 0) - 0\right] = \frac{\pi}{2}$ sq units.

Hence area of $\cos^2 x$ & area of $\sin^2 x$ are similar.

26. Using integration, find the area of the triangle, the equations of whose sides are $y = 2x+1$, $y = 3x+1$ and $x = 4$.

Sol. Let the given equation

$y = 2x+1$ ...(1)

$y = 3x+1$ ...(2)

$x = 4$ ...(3)

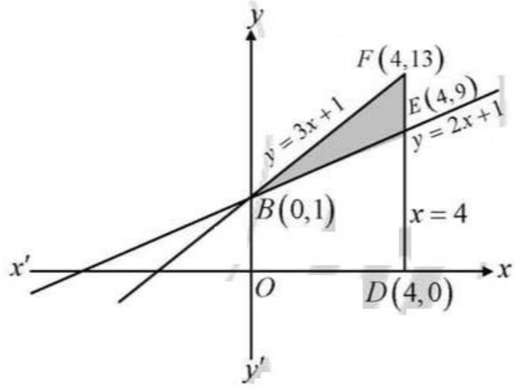
Solving equation (1) & (2) given points $(4, 9)$.

Solving equation (2) & (3) given points $(4, 13)$.

Solving equation (1) & (3) given points $(0, 1)$

and $x = 4$ is the line parallel to y-axis

and passing through $D(4, 0)$.



Required area = area $ODFB$ – area $ODEB$ $= \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx = \left[\frac{3x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$

$= \left[\left\{\frac{3}{2}(4)^2 + 4\right\} - 0\right] - \left[\left\{(4)^2 + 4\right\} - 0\right] = 28 - 20 = 8$ sq. units.

27. Find the area of the region $\{(x, y) : x^2 \le y \le x\}$

Sol. We have

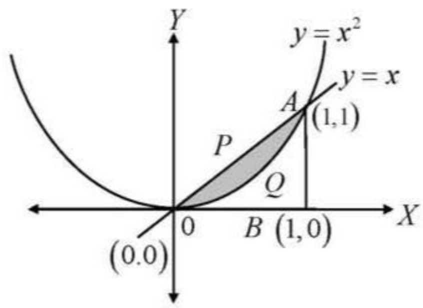
$y = x^2$ ...(1)

and $y = x$ ...(2)

We know that $y = x^2$ is an upward parabola

and the line $y = x$ is passing through origin.

Now, on solving (1) and (2), we get,



$$x^2 = x \quad \Rightarrow x(x-1)=0 \quad \Rightarrow x=0 \text{ or } 1$$

From (2), $x=0 \Rightarrow y=0$ and $x=1 \Rightarrow y=1$

So, the point of intersection of (1) and (2), are $0(0, 0)$ and $A(1, 1)$.

Draw $AB \perp OX$.

$\therefore$ Required area = shaded area shown in fig.

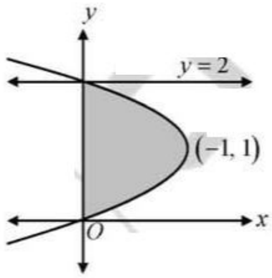
$$= \text{area } OPABO - \text{area } OQABO = \int_0^1 x\,dx - \int_0^1 x^2 dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

Hence, the required area is $\frac{1}{6}$ sq. units.

28. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

Sol. The equation of the given curve is,

$$y^2 = 2y - x$$
$$\Rightarrow y^2 - 2y = -x$$
$$\Rightarrow y^2 - 2y + 1 = -x + 1$$
$$\Rightarrow (y-1)^2 = -(x-1)$$



Clearly, this equation represents a parabola vertex at $(1, 1)$ and open on the left.

By Putting $x=0$ in $y^2 = 2y - x$, we get, $y^2 - 2y = 0 \Rightarrow y = (0, 2)$

So, the curve meets y-axis at $(0, 0)$ and $(0, 2)$.

A rough sketch of the curve is as shown in fig. and the required area is the shaded area.
Here we slice this region into horizontal strips for the approximating rectangle shown in fig.
We have width $= \Delta y$, length $= x$ and area $= x\Delta y$.

The approximately rectangle can move from $y=0$ to $y=2$

$$\therefore \text{ Required area} = \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}$$

29. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$, as x varies from 0 to $\frac{\pi}{2}$, and find the area of the region enclosed between them and the x-axis.

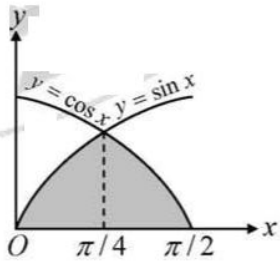
Sol. A rough sketch of $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is shown here with.

$\therefore$ Required area

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = [-\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{4} + \cos 0\right] + \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right]$$

$$= \left(\frac{-1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units.}$$



30. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y = 1$

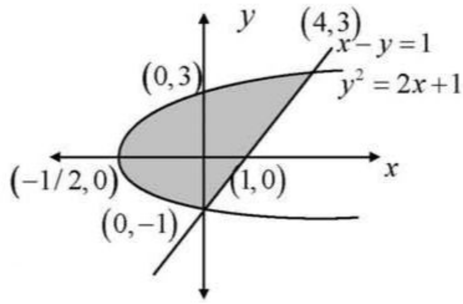
Sol. The equation of the curve is $y^2 = 2x + 1$ and line $x - y = 1$ can be drawn, as shown in the given figure.

Then, we have to find the area of the shaded region.

$y^2 = 2x+1$ ... (1) and $x - y = 1 \Rightarrow x = 1+y$ ... (ii)

By, putting the value of x in equation (i) we get, $y^2 = 2(1+y)+1 \Rightarrow y^2 = 2+2y+1$

$\Rightarrow y^2 - 2y - 3 = 0 \quad \Rightarrow (y+1)(y-3) = 0 \quad \Rightarrow y = -1, 3$



from equation (ii) $x = 1+y$, $x = 0, 4$

$\therefore$ Required Area $2\int_0^1 (y+1)dy + 1\int_1^3 (y+1)dy$

$-\int_1^3 \left(\frac{y^2-1}{2}\right)dy + 2\int_0^1 \left(\frac{y^2-1}{2}\right)dy$

$= \int_{-1}^3 \left[(y+1) - \left(\frac{y^2-1}{2}\right)\right]dy = \frac{1}{2}\int_{-1}^3 \left(2y+2-y^2+1\right)dy$

$= \frac{1}{2}\int_{-1}^3 \left(2y - y^2 + 3\right)dy = \frac{1}{2}\left[\frac{2y^2}{2} - \frac{y^3}{3} + 3y\right]_{-1}^3$

$= \frac{1}{2}\left[9 - \frac{3^3}{3} + 3.3 - \left(1 + \frac{1}{3} - 3\right)\right] = \frac{1}{2}\left[9 - 9 + 9 - \left(\frac{-5}{3}\right)\right] = \frac{1}{2}\left[9 + \frac{5}{3}\right] = \frac{1}{2} \times \frac{32}{3} = \frac{16}{3}$ sq. units.

31. Find the area of the region bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$

Sol. The given curve is, $y = 2x - x^2$ ...(1)

Now $y = 2x - x^2 \quad \Rightarrow x^2 - 2x + 1 = (-y+1)$

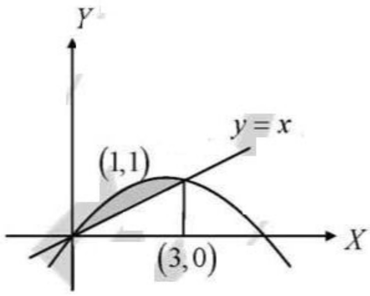
$\Rightarrow (x-1)^2 = -1(y-1) \quad \Rightarrow X^2 = -Y$, where $x - 1 = X$ and $y - 1 = Y$

Clearly, $X^2 = -Y$ is a downward parabola with its vertex at $(X = 0, Y = 0)$.

Now, $X = 0, Y = 0 \Rightarrow x - 1 = 0$

and $y - 1 = 0 \Rightarrow x = 1$ and $y = 1$

Thus the vertex of the parabola is $A(1, 1)$, we know that $y = -x$ is the line passing through the origin and making an angle of 45° with x-axis.



$y = 2x - x^2$ ...(2)

$y = x$ ...(3)

By, Solving equation (1) & (2), given point $x = 0, 3$ & $y = 0, -3$.

$\therefore$ Required area $\int_0^3 \left(2x - x^2\right)dx + \int_0^3 x\,dx$

$= \int_0^3 \left(2x - x^2\right)dx - \int_0^3 -x\,dx = \left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^2}{2}\right]_0^3 = (9-9) + \frac{9}{2} = \frac{9}{2}$ sq. units.

32. Find the area of the region bounded by the curve $(y-1)^2 = 4(x+1)$ and the straight line $y = x - 1$

Sol. The equation of the curve is $(y-1)^2 = 4(x+1)$ and the line $y = x - 1$ can be drawn, as shown in the given figure.

Then, we have to find the area of the shaded region

$(y-1)^2 = 4(x+1)$ ... (1) and $y = x - 1 \Rightarrow y + 1 = x$ ... (ii)

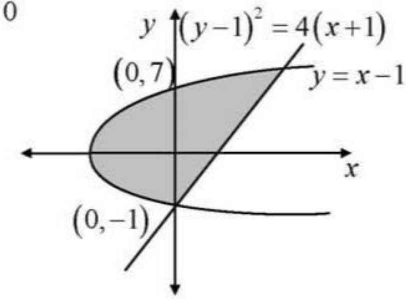
Now, Putting the value of x in equation (i) we get,

$(y-1)^2 = 4(y+1+1) \Rightarrow y^2-2y+1 = 4y+8 \quad \Rightarrow y^2-6y-7=0$

$\Rightarrow (y+1)(y-7)=0 \Rightarrow y=-1, 7$

Now, from equation (ii)

$x = y+1,\ y=-1,\ x=0\ \&\ y=7,\ x=8$



$\therefore$ Required Area

$$= \int_{-1}^{7} \left\{ (y+1) - \frac{(y-1)^2-4}{4} \right\} dy$$

$$= \left[\frac{y^2}{2} + y \right]_{-1}^{7} - \frac{1}{4}\int_{-1}^{7} \left\{ (y-1)^2 - 4 \right\} dy$$

$$= \left[\left\{ \frac{(7)^2}{2} + 7 \right\} - \left\{ \frac{(-1)^2}{2} + (-1) \right\} \right] - \frac{1}{4}\int_{-1}^{7} \left(y^2 - 2y + 1 - 4 \right) dy$$

$$= \left[\left(\frac{49}{2} + 7 \right) - \left(\frac{1}{2} - 1 \right) \right] - \frac{1}{4}\int_{-1}^{7} \left(y^2 - 2y - 3 \right) dy = \left[\frac{63}{2} + \frac{1}{2} \right] - \frac{1}{4}\left[\frac{y^3}{3} - \frac{2y^2}{2} - 3y \right]_{-1}^{7}$$

$$= 32 - \frac{1}{4}\left[\frac{y^3}{3} - y^2 - 3y \right]_{-1}^{7} = 32 - \frac{1}{4}\left[\left\{ \frac{(7)^3}{3} - (7)^2 - 3\times 7 \right\} - \left\{ \frac{(-1)^3}{3} - (-1)^2 - 3(-1) \right\} \right]$$

$$= 32 - \frac{1}{4}\left[\left(\frac{343}{3} - 49 - 21 \right) - \left(-\frac{1}{3} - 1 + 3 \right) \right] = 32 - \frac{1}{4}\left[\left(\frac{343}{3} - 70 \right) - \left(-\frac{1}{3} + 2 \right) \right]$$

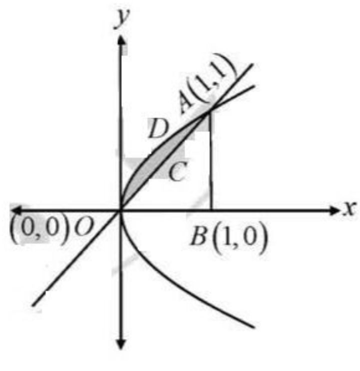
$$= 32 - \frac{1}{4}\left[\frac{133}{3} - \frac{5}{3} \right] = 32 - \frac{1}{4}.\frac{128}{3} = 32 - \frac{32}{3} = \frac{64}{3} \text{ sq units}$$

33. Find the area of the region bounded by the curve $y = \sqrt{x}$ and the line $y = x$

Sol. To get the point of intersection of the line $y = x$ and the curve $y^2 = x$, we have to solve the given equations.

Simultaneously and thus we find the co-ordinate $(1, 1)$.

Drawing the perpendicular from A on opposite axis and we find $B(1, 0)$.



$\therefore$ Required area

$$= \int_0^1 y \text{ of parabola} - \int_0^1 y \text{ of line}$$

$$= \int_0^1 \sqrt{x}\, dx - \int_0^1 x\, dx = \left[\frac{x^{3/2}}{3/2} \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$

$$= \frac{2}{3}.\left[(1)^{3/2} - 0 \right] - \left[\frac{1}{2} - 0 \right] = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6} \text{ sq units.}$$

34. Find the area of the region included between the parabola $y^2 = 3x$ and the circle $x^2 + y^2 - 6x = 0$, lying in the first quadrant.

Sol. Equation of the given parabola and circle

$$y^2 = 3x \qquad \ldots(1)$$

and $(x-3)^2+y^2=9 \quad \ldots(2)$

If, equation (1) is a parabola and 0 at the origin and equation (2) is a circle with centre $C(3,\,0)$ and radius 3.

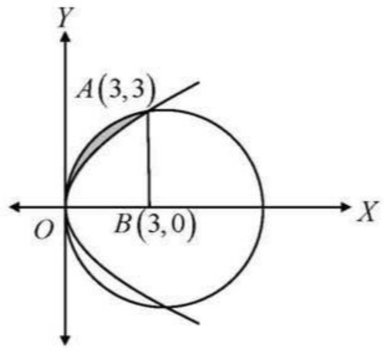
By, solving equation (1) and (2), we have,

$$(x-3)^2+3x=9 \Rightarrow x^2-6x+9+3x=9$$

$$\Rightarrow x^2-3x=0 \Rightarrow x(x-3)=0,\ x=0,\ x=3$$

which gives $y=0,\,3$

Thus, the points of intersection of the given circle area $A(3,\,3)$ and $A'(3,\,-3)$ as shown in figure.



$\therefore$ Required area $=$[area of circle $-$ area of parabola]

$$=\left[\int_0^3\sqrt{9-(x-3)^2}\,dx-\int_0^3\sqrt{3}\sqrt{x}\,dx\right]=\left\{\left[\frac{x-3}{2}\sqrt{9-(x-3)^2}+\frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^3-\sqrt{3}.\left[\frac{x^{3/2}}{3/2}\right]_0^3\right\}$$

$$=\left[\left(0+\frac{9}{2}\sin^{-1}(0)\right)-\left(\frac{9}{2}\sin^{-1}(-1)\right)-\frac{2\sqrt{3}}{3}\left(3^{3/2}\right)\right]=\left[\frac{9}{2}.\frac{\pi}{2}-6\right]=\frac{9\pi}{4}-6 \text{ sq units.}$$

35. Find the area bounded by the curve $y=\cos x$ between $x=0$ to $x=2\pi$.

Sol. The given curve is $y=\cos x$.

Some value of x and the corresponding value of y are given below :

x	0	$\pi/6$	$\pi/2$	π	$3\pi/2$	2π
y	1	$\sqrt{3}/2$	0	-1	0	1

By, Taking a fixed unit distance for π along the x-axis.

Now, we can plot the points $(0,\,1)$, $\left(\frac{\pi}{6},\,\frac{\sqrt{3}}{2}\right)$, $\left(\frac{\pi}{2},\,0\right)$, $(\pi,\,-1)$, $\left(\frac{3\pi}{2},\,0\right)$ and $(2\pi,\,1)$.

Join these points free hand to obtain a rough sketch of the given curve.

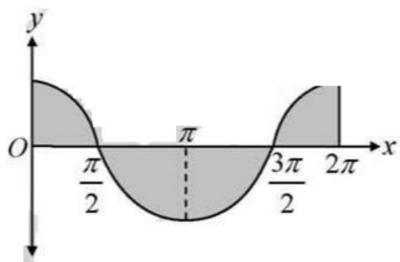
$\therefore$ Required area

$$=\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{3\pi/2}\cos x\,dx+\int_{3\pi/2}^{2\pi}\cos x\,dx$$

$$=[\sin x]_0^{\pi/2}-[\sin x]_{\pi/2}^{3\pi/2}+[\sin x]_{3\pi/2}^{2\pi}$$

$$=(\sin\pi/2-\sin 0)-[\sin 3\pi/2-\sin\pi/2]+(\sin 2\pi-\sin 3\pi/2)$$

$$=(1-0)-(-2)+1=1+2+1=4 \text{ sq units.}$$



36. Using integration, find the area of the region in the first quadrant, enclosed by the x-axis, the line $y=x$ and the circle $x^2+y^2=32$

Sol. The given equation are

$y=x \quad \ldots(1) \qquad$ and $\qquad x^2+y^2=32 \quad \ldots(2)$

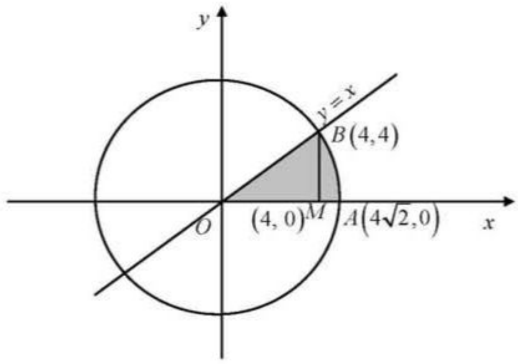
By, solving (1) and (2), we find that the line and the circle meet at $B(4,\,4)$ in the first quadrant.

Draw perpendicular BM to the x-axis. Therefore,

the required area $=$ area of the region $OBMO$ $+$ area of the region $BMAB$.

Now, the area of the region

$\therefore\ OBMO = \int_0^4 y\,dx$



$= \int_0^4 x\,dx$

$= \frac{1}{2}\left[x^2\right]_0^4 = 8 \quad \ldots(3)$

Again, the area of the region $BMAB$

$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$

$= \left[\frac{1}{2}x\sqrt{32-x^2} + \frac{1}{2}\times 32\times \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$

$= \left\{\frac{1}{2}4\sqrt{2}\times 0 + \frac{1}{2}\times 32\times \sin^{-1}(1)\right\} - \left(\frac{4}{2}\sqrt{32-16} + \frac{1}{2}\times 32\times \sin^{-1}\frac{1}{\sqrt{2}}\right)$

$= 8\pi - (8+4\pi) = 4\pi - 8 \quad \ldots(4)$

Adding (3) & (4), we get, the total area $= 8 + 4\pi - 8 = 4\pi$ sq. units.

37. Using integration, find the area of the triangle whose vertices are vertices are $A(1, 0)$, $B(2, 2)$ and $C(3, 1)$.

Sol. The equation of AB is $A(1, 0) \,\&\, B(2, 2)$,

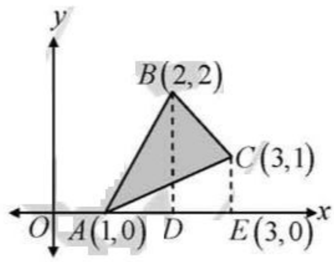
$\Rightarrow\ y - 0 = \frac{2-0}{2-1}(x-1) \Rightarrow y = 2(x-1)$

The equation of BC is $B(2, 2) \,\&\, C(3, 1)$,

$\Rightarrow\ y - 2 = \frac{1-2}{3-2}(x-2) \Rightarrow y - 2 = \frac{-1}{1}(x-2) \Rightarrow y = -x+2+2 \Rightarrow y = -x+4$

The equation of AC is $A(1, 0) \,\&\, C(3, 1)$

$\Rightarrow\ y - 0 = \frac{1-0}{3-1}(x-1) \Rightarrow y = \frac{1}{2}(x-1)$



Draw $BM \perp OX$ and $CM \perp OX$.

$\therefore$ Area of ΔABC

$= \{\text{area of } ABM + \text{area of } BMNC\} - \text{area of } ACN$

$= \int_1^2 y_{AB}\,dx + \int_2^3 y_{BC}\,dx - \int_1^3 y_{AC}\,dx$

$= \int_1^2 2(x-1)\,dx + \int_2^3 (-x+4)\,dx - \int_1^3 \frac{x-1}{2}\,dx = 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[\frac{-x^2}{2} + 4x\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$

$= 2\left[\left\{\frac{(2)^2}{2} - 2\right\} - \left\{\frac{(1)^2}{2} - 1\right\}\right] + \left[\left\{-\frac{(3)^2}{2} + 4\times 3\right\} - \left\{-\frac{(2)^2}{2} + 4\times 2\right\}\right] - \frac{1}{2}\left[\left\{\frac{(3)^2}{2} - 3\right\} - \left\{\frac{(1)^2}{2} - 1\right\}\right]$

$= 2\left[\frac{1}{2}\right] + \left[\frac{15}{2} - 6\right] - \frac{1}{2}\left[\frac{3}{2} - \left(-\frac{1}{2}\right)\right] = 1 + \frac{3}{2} - \frac{1}{2}.2 = 1 + \frac{3}{2} - 1 = \frac{3}{2}$ sq. units.

38. Using integration, find the area of the triangle whose vertices are $A(1, 3)$, $B(2, 5)$ and $C(3, 4)$.

Sol. The equation of AB if, $A(1, 3)$ and $B(2, 5)$

$$\Rightarrow y-3=\frac{5-3}{2-1}(x-1) \Rightarrow y-3=2(x-1) \Rightarrow y=2x-2+3 \Rightarrow y=2x+1$$

The equation of BC if $B(2, 5)$ and $C(3, 4)$

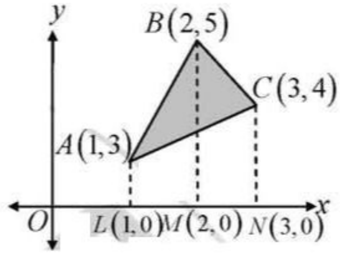
$$\Rightarrow y-5=\frac{4-5}{3-2}(x-2) \Rightarrow y-5=\frac{-1}{1}(x-2) \Rightarrow y=-x+7$$

The equation of AC if $A(1, 3)$ and $C(3, 4)$

$$y-3=\frac{4-3}{3-1}(x-1)$$

$$\Rightarrow y-3=\frac{1}{2}(x-1) \Rightarrow y=\frac{x}{2}-\frac{1}{2}+3$$

$$\Rightarrow y=\frac{x}{2}+\frac{5}{2} \Rightarrow y=\frac{1}{2}(x+5)$$



Draw $AL \perp OX$, $BM \perp OX$ and $CM \perp OX$

Area of $\Delta ABC =$ (area $ALMBA$ + area $BMNCB$) − area $ALNCA$

$$=\int_1^2 y_{AB}+\int_2^3 y_{BC}-\int_1^3 y_{AC}=\int_1^2(2x+1)dx+\int_2^3(-x+7)dx-\int_1^3\frac{x+5}{2}dx$$

$$=\left[2\frac{x^2}{2}+x\right]_1^2+\left[\frac{-x^2}{2}+7x\right]_2^3-\frac{1}{2}\left[\frac{x^2}{2}+5x\right]_1^3=(6-2)+\left(\frac{33}{2}-12\right)-\frac{1}{2}\left[\frac{39}{2}-\frac{11}{2}\right]$$

$$=4+\frac{9}{2}-\frac{1}{2}.14 \quad =4+\frac{9}{2}-7 \quad =\frac{17-14}{2}=\frac{3}{2} \text{ sq. units.}$$

39. Using integration, find the area of the triangular region bounded by the lines $y=2x+1$, $y=3x+1$ and $x=4$.

Sol. Using $y=2x+1$ is the line passing through $A\left(\frac{-1}{2}, 0\right)$ and $B(0, 1)$

And $y=3x+1$ is the line passing through $C\left(\frac{-1}{3}, 0\right)$ and $B(0, 1)$.

Also $x=4$ is the line parallel to y-axis and passing through $D(4, 0)$.

$\therefore$ Required area

$$=(\text{area } ODB)-(\text{area } ODEB)=\int_0^4(3x+1)dx-\int_0^4(2x+1)dx$$

$$=(28-20)=8 \text{ sq. units.}$$