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RAILWAY RECRUITMENT BOARD



CIVIL
ENGINEERING

STRENGTH OF MATERIALS

SELF STUDY MATERIAL

CIVIL ENGINEERING FOR ALL

STRENGTH OF MATERIALS

■ CONCEPTS OF STRESS AND STRAIN

STRESS : When some external forces are applied to a body, then the body offers internal resistance to these forces. The magnitude of the internal resisting force is numerically equal to the applied forces. These internal resisting force per unit area is called "stress".

STRAIN : It is defined as change in length per unit length. The strain may be tensile or compressive depending upon whether the length increases (under tensile load) or decreases (under compressive load). It is a dimensionless quantity.

EQUATIONS OF STATIC EQUILIBRIUM :

Whenever a solid body under the action of various forces is in static equilibrium or in the state of rest then the algebraic sum of the components of these forces along any chosen three perpendicular axes reference are separately zero. In addition, the algebraic sum of the moments of these forces about any point is also zero. It is not rotating. Mathematically, these results are expressed in the form of following four equations :

$$\Sigma X = 0$$

$$\Sigma Y = 0$$

$$\Sigma Z = 0$$

$$\Sigma M = 0$$

where, ΣX , ΣY and ΣZ represent the algebraic sum of the components of force along X, Y and Z direction respectively and ΣM the algebraic sum of moments of these individuals forces about any point.

UNITS OF STRESS AND STRAIN :

Stress : The stress is defined as the internal force which is resisting to the applied force of deformation per unit area. Therefore, the stress is usually measured in **Newtons/Metre Square (N/m²)**. It is also called **Pascal (Pa)**. Since Pascal is a very small unit, it is not uncommon to use Mega Pascal (**MPa**) or Giga Pascal (**GPa**) as unit of stress, where $1 \text{ MPa} = 10^6 \text{ Pa}$ and $1 \text{ GPa} = 10^9 \text{ Pa}$.

Strain : It is defined as the change in length (or elongation in length) per unit length. Mathematically

$$\text{it is defined as strain} = \frac{\text{Change in length (l)}}{\text{Unit length (L)}} = \frac{l}{L}$$

which is unitless Quantity.

ELASTIC LIMIT : Considering a tension test of a specimen of round section being stretched in a **UTM**. The load recorded by the universal testing machine is the reaction of the test specimen. At room temperature, the load extension diagram in general, at strain rate less than $2 \times 10^{-3}/\text{sec}$ will tend to follow the curve of the adjacent fig. I. Which is typical for an annealed metal like aluminium or copper.

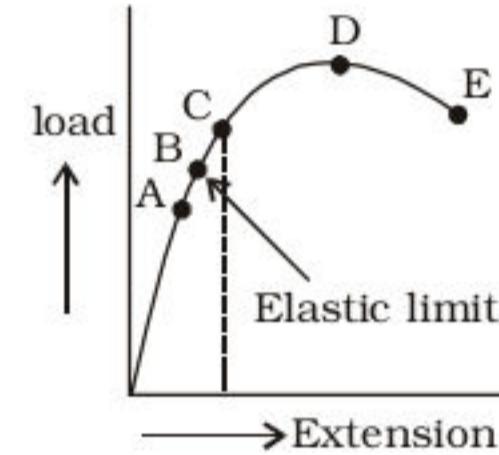


Fig. I

Initially, the relation between the load and extension is essentially linear—that is, portion OA of the curve, where A defines the limit of proportionality. On further straining the relation between load and extension linear—that is portion OA of the curve, where A defines the limit of proportionality. On further straining, the relation between the load and extension no longer remains linear but the material is still elastic. That is if the load is released the specimen will revert back to its original length. The maximum load which can be applied without causing permanent deformation is known as "Elastic Limit" represented by point B in the Fig-I attached. Usually the points A and B are very near to each other and the real difference can be achieved only by using highly sensitive measuring device. At the point B ends the elastic straining and the inelastic or plastic deformation ensues. Beyond the point B there is a general permanent extension of the test piece until the load attains a maximum value, point D, corresponding to which the stress is called the "Ultimate tensile stress" σ_{ult} ($= P_{max}/A_o$).

TERMINOLOGIES:

Conventional or Engineering strain : It is defined as the change in length per unit original length. By definition we have :

$$\epsilon = \frac{l - l_0}{l_0} = \int_{l_0}^l \frac{dl}{l_0} = \frac{1}{l_0} \int_{l_0}^l dl.$$

where, l = Changed or deformed length.

l_0 = Original length

and dl = Change in length.

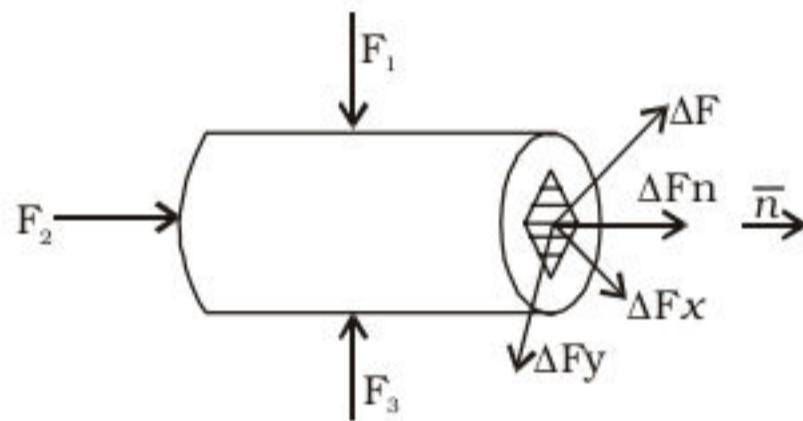
NATURAL STRAIN : It is defined as change in length per unit instantaneous length. By definition we have, Natural strain

$$- = \int_0^l \frac{dl}{l} = \ln \frac{l}{l_0} = l_0 n (1 + E)$$

STRENGTH OF MATERIALS

NORMAL STRAIN : It is the strain produced under the action of direct or normal stress.

SHEAR STRESS : The force ΔF may be resolved into infinite number of components in the plane containing area ΔA , because there are infinite number of directions in plane containing area ΔA which are perpendicular to the unit normal \bar{n} . However, in three dimensional co-ordinate system, we are left with only two directions x and y perpendicular to each other as shown in the diagram above. The shear stress is defined as :



Defining stress at a point

$$\tau_x = \lim_{\Delta a \rightarrow 0} \frac{F_x}{A} = \frac{dF_x}{dA}$$

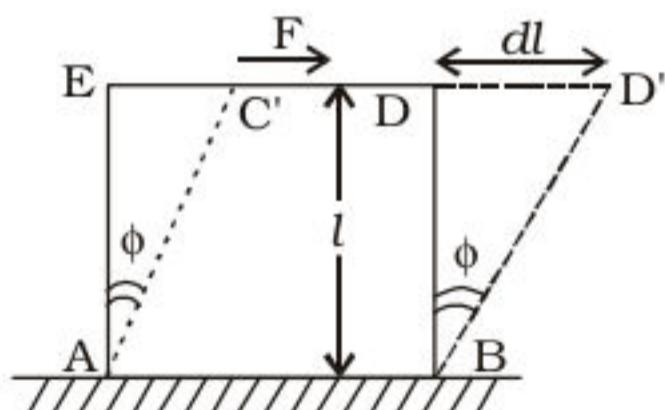
$$\tau_y = \lim_{\Delta a \rightarrow 0} \frac{F_y}{A} = \frac{dF_y}{dA}$$

SHEAR STRAIN : It is the strain produced under the action of shear stresses. The shear strain is measured by the change in angle. As per attached diagram if dl is the change in the length of face CD under the action of shear force F , then by definition, Shear strain $\gamma = \tan \phi$

For small strains,

$$\tan \phi = \phi, \text{ thus } \phi = \frac{dl}{l}$$

GAUGE LENGTH : It is the portion of the test specimen over which extension or deformation is measured.



Shear Strain

Percentage Elongation : It is the change in length per unit original length of the test specimen expressed as a percentage, i.e.,

Percentage elongation

$$= \frac{dl}{l} \times 100.$$

Percentage reduction of area : It is defined as the change in area per unit original area expressed as a percentage, i.e.,

$$\text{Percentage reduction area} = \left(\frac{A_0 - A}{A_0} \right) \times 100.$$

Poisson's Ratio : When a material is subjected to longitudinal deformation then the lateral dimensions also change. The ratio of lateral strain to longitudinal strain is a constant quantity called Poisson's Ratio and is designated by ν (called ν) or $1/m$.

$$\therefore \nu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

Superficial strain : It is defined as change in area of cross-section per unit original area, i.e., superficial strain

$$\epsilon_s = \frac{A - A_0}{A_0} = \frac{dA}{A_0}$$

Volumetric strain : If a uniform stress is applied on all three faces of a body, then all the three dimensions of the body will change resulting in change in volume. Thus, volumetric strain,

$$\epsilon_v = \frac{V - V_0}{V_0} = \frac{dV}{V_0}$$

where, V = Final volume, and V_0 = original volume.

HOOK'S LAW : This law states that within elastic limits (proportional), strain is proportional to stress.

MODULUS OF ELASTICITY : Within elastic limits the ratio of normal stress to normal strain is a constant quantity and is defined as the young's modulus of elasticity, i.e., $E = \frac{P}{A} = \frac{Pl_0}{A_0 dl}$

MODULUS OF RIGIDITY : It is defined as the ratio of shearing stress to shearing strain.

$$\text{i.e., } G = \frac{P}{A}$$

BULK MODULUS : It is defined as the ratio of uniform stress intensity to volumetric strain, within the elastic limits and denoted by K . Thus,

$$K = \frac{P}{\epsilon_v}$$

PROOF STRESS : It is the maximum stress which can be applied to a material without allowing the material to fail.

FACTOR OF SAFETY : Because of uncertainties of loading conditions, a factor of safety is introduced, defined as the ratio of maximum stress to allowable or working stress. The maximum stress is generally taken as the yield stress for ductile materials. This is also called the "Factor of ignorance."

$$\text{Factor of Safety} = \frac{\text{Ultimate Stress}}{\text{Permissible Stress}}$$

STRENGTH OF MATERIALS

FREE BODY DIAGRAM : The free body diagram of an element of a member in equilibrium is the diagram of only that member or element, as if made from the rest, with all the internal and external force.

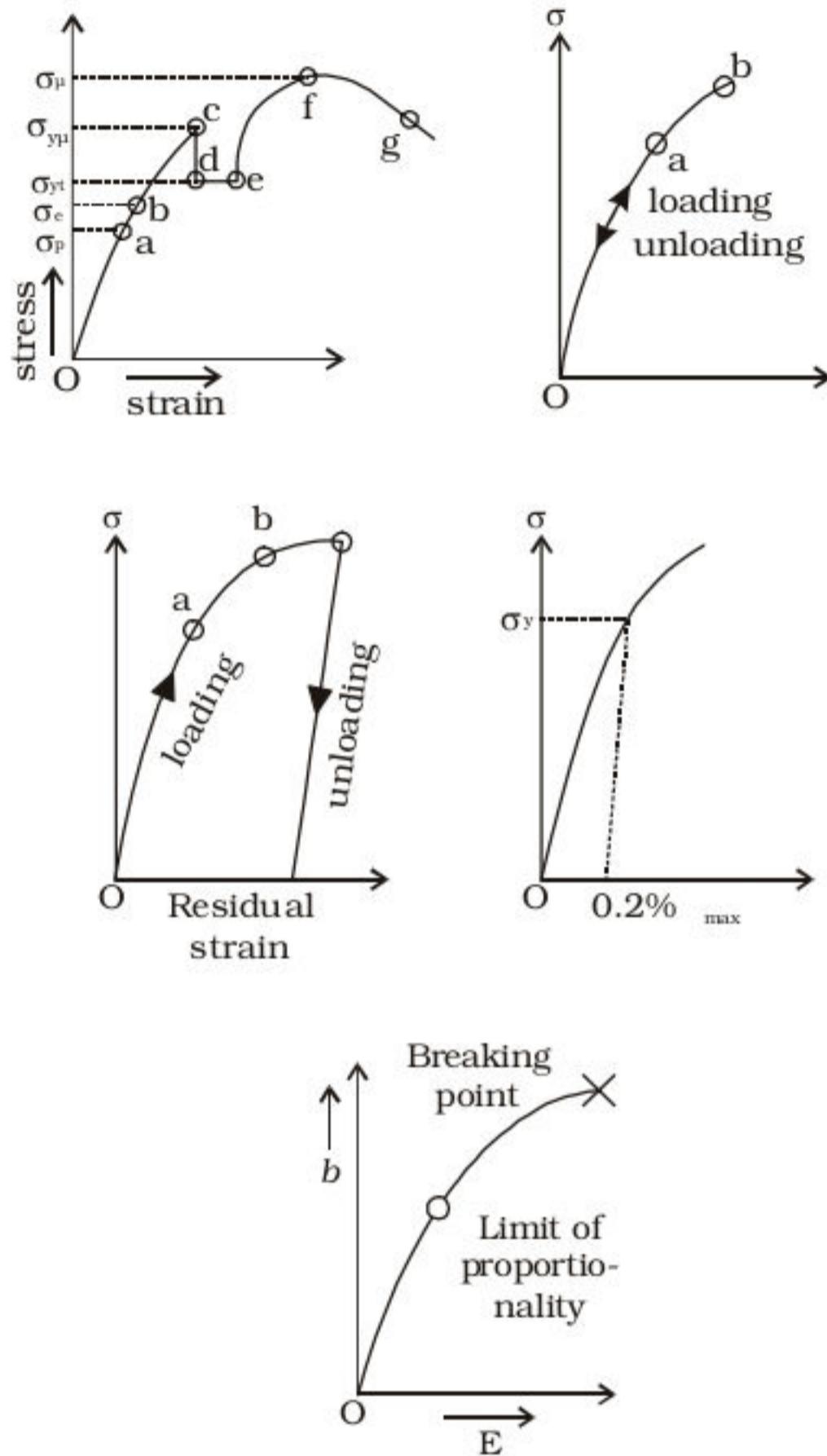


Figure ...I, II, III, IV V

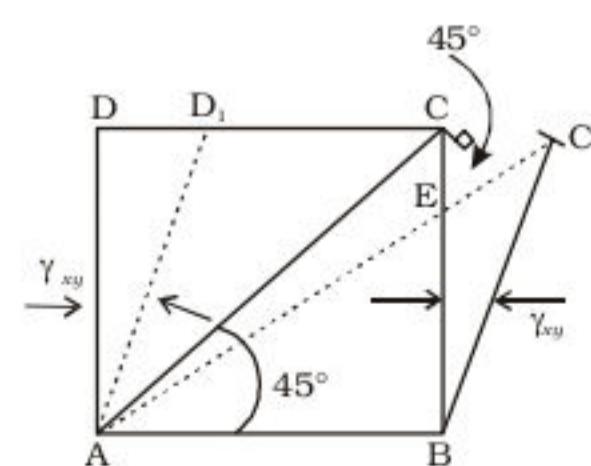
STRESS - STRAIN DIAGRAM : The above diagrams fig I shows the stress- strain diagram for a ductile material like mild-steel. The curve starts from the origin O showing thereby that there is no initial stress or strain in the test specimen. Upto point 'a' **Hook's law** is obeyed and stress is proportional to strain. Therefore, Oa is a straight Line and point a is called the **Limit of proportionality** and the stress at a point is called the **proportionality Limit** stress, σ_p . The portion of the diagram between ab is not a straight Line but upto point b, the material remains elastic, i.e., on removal of the load, no permanent set is formed and the path is retraced. The point b is called the **elastic limit** point and the stress corresponding to that is called the elastic limit stress σ_e . In actual practice, the points a and b are so close to each other that it becomes difficult to differentiate between them.

Beyond the point b, the material goes to plastic stage until the **upper yield point 'c'** is reached. At this point, the cross-sectional area of the material starts decreasing and the stress decreases to a lower value to a point d, called the **lower yield point**. Corresponding to point c, the stress is known as upper yield stress σ_{yu} and corresponding to point d, the stress is known as yield point stress, σ_y . At point d the specimen elongates by a considerable amount without any increase in stress and upto point e. The portion de is called the yielding of the material at constant stress. From point e onwards, the strain hardening phenomenon becomes predominant and the strength of material increases thereby requiring more stress for deformation, until f is reached. Point f is called the ultimate point and the stress corresponding to this point is called the **ultimate stress σ_u** . It is the maximum stress to which the material can be subjected in a simple tensile Test. At point f, the necking of the material begins and the cross-sectional area starts decreasing at a rapid rate. Due to this local necking, the stress in the material goes on decreasing inspite of the fact that actual stress intensity goes on increasing. Ultimately the specimen breaks at point g, known as the breaking point and the corresponding stress is called the **nominal breaking stress** based upon the original area of cross-section at the neck. The initial portions of the diagram are shown in fig II and III.

Sometimes it is not possible to locate the yield point quite accurately in order to determine the yield strength of the material. For such materials, the yield point stress is defined at some particular value of the permanent set. It has been observed that if load is removed in the plastic range then the unloading path line is parallel to the straight portion of the stress-strain diagram in fig III. The commonly used value of permanent set for determining the value of yield strength for mild steel is 0.2 per cent of the maximum strain as shown in fig IV.

BRITTLE MATERIALS : The stress-strain diagram for brittle material like cast iron is shown in fig-V. There is very little elongation and reduction in area of the specimen for such materials. The yield point is not marked at all. The straight line portion of the diagram is also very small.

RELATION BETWEEN ELASTIC CONSTANTS : Consider a cube of Unit size acted upon by simple shear stress τ_{xy} as shown in the adjacent diagram.



STRENGTH OF MATERIALS

$$\text{shear strain} = \gamma_{xy} = \frac{CC_1}{BC} = \frac{xy}{G}$$

From C drop CE \perp AC, so that AC = AE Elongation of diagonal AC,

$$EC_1 = CC_1 \cos 45^\circ = \frac{CC_1}{\sqrt{2}}$$

$$\text{Now } AC = \sqrt{2} AB.$$

$$\begin{aligned} \text{Strain in the diagonal} &= \frac{EC_1}{AC} = \frac{CC_1}{\sqrt{2}} : \frac{1}{\sqrt{2}AB} \\ &= \frac{1}{2} : \frac{CC_1}{AB} = \frac{1}{2} \cdot \frac{xy}{G}, [\because AB = BC] \end{aligned}$$

Thus the strain in the diagonal of a cube subjected to simple shear is half the amount of the shear strain. Now if the cube is subjected to simple shear, then the normal stress on the diagonal of the cube is equal to the shear stress, i.e.,

$$\sigma_n = xy$$

$$\therefore \text{Strain in the diagonal} = \frac{1}{2} \cdot \frac{n}{G}$$

where, σ_n is the intensity of the equal and opposite direct stress on planes in the direction of the diagonals. If σ_n acted alone in the direction AC, then the resultant strain would be $\sigma_n E$, but there is an equal and opposite direct stress in the direction of the diagonal BD. The strain caused by it in the direction of AC.

$$\frac{n}{E}.$$

Total strain in the direction of the diagonal AC

$$= \frac{n}{E} - \frac{n}{E}$$

$$= \frac{n}{E} + (1+v)$$

$$\therefore \frac{1}{2} \frac{n}{G} = \frac{n}{E} (1+v)$$

$$\text{or, } G = \frac{E}{2(1+v)} \quad \dots\dots\dots (I)$$

Now, considering a cube under an equal compressive stress σ_n action on all its faces, we have :

$$\text{Bulk or Volume Strain} : \epsilon_v = \frac{n}{K}$$

where, K = bulk Modulus

$$\text{Total strain along the edge} = \frac{n}{E} (1-2v)$$

$$\therefore \epsilon_u = 3 \frac{n}{E} (1-2v)$$

$$\text{Hence, } \frac{n}{K} = \frac{3}{E} \frac{n}{(1-2v)}$$

$$\text{or, } E = 3K(1-2v) \quad \dots\dots\dots (II)$$

From equation (I) and (II) above we get

$$2G(1+v) = 3K(1-2v)$$

$$\text{or, } 2G + 2Gv = 3K - 6Kv$$

$$\text{or, } v(2G + 2G) = 3K - 2G$$

$$\therefore v = \frac{(3K - 2G)}{(2G + 6K)} \quad \dots\dots\dots (III)$$

$$\text{Now from equation (I) above we have : } v = \frac{E - 2G}{2G}$$

$$\text{or, } \frac{E - 2G}{2G} = \frac{3K - 2G}{2G + 6K}$$

$$\text{On solving we get } \frac{E}{2G} = 1 + \frac{3K - 2G}{2G + 6K}$$

$$= \frac{E}{2G} = \frac{9K}{2G + 6K} = \frac{9K}{2(G + 3K)}$$

$$\therefore E = \frac{9KG}{G + 3K} \quad \dots\dots\dots (IV)$$

Also the following equation hold good :

$E = 2G(1+v)$ derivation from equation (I)
above and $E = 3K(1-2v)$

Example : A M.S. rod 20 mm diameter is subjected to an axial pull of 50 kN. Determine the tensile stress induced in the road and the elongation if the unloaded length is 5m. $E = 210 \text{ GN/m}^2$.

Solution : Given, $d = 20 \text{ mm}$; $P = 50 \text{ kN}$; $l = 5\text{m}$.
Area of cross-section of the rod

$$A = \frac{\pi}{4} d^2$$

$$= \frac{\pi}{4} (20)^2 \times 10^{-6}$$

$$\text{Stress, } \sigma = \frac{P}{A} = \frac{50 \times 10^3}{314 \times 10^{-6}}$$

$$= 159.155 \text{ MN/m}^2$$

$$\text{Elongation, } \delta = \frac{Pl}{AE} = \frac{50 \times 10^3}{314 \times 10^{-6}} \frac{5 \times 10^3}{210 \times 10^9}$$

$$= 3.789 \text{ mm.} \quad \dots\dots\dots \text{Ans.}$$

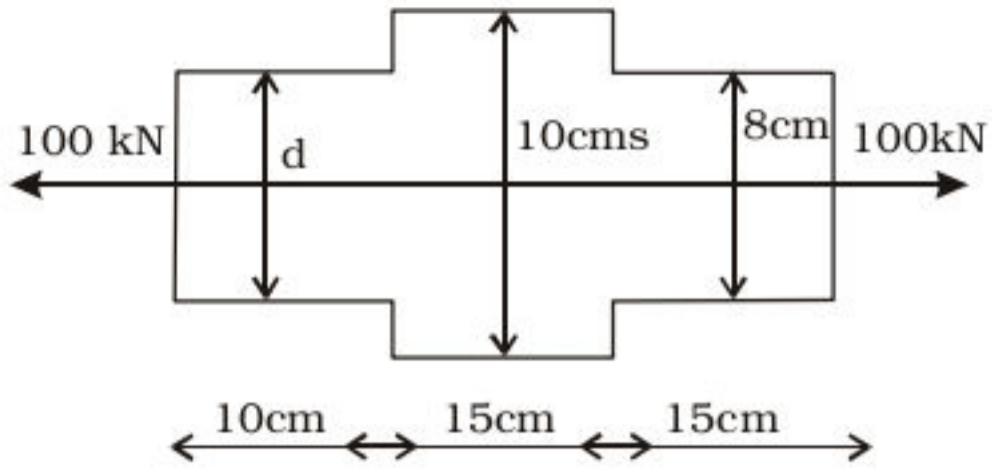
Example : Around bar as shown in the figure below is subjected to an axial tensile load of 100 kN. what must be the diameter 'd' if the stress there is to be 100 MN/m²? Find also the total elongation. $E = 200 \text{ GPa}$.

Solution :

$$\text{We know stress} = \frac{P}{(\pi/4)d^2}$$

Putting the values we get

$$100 \times 10^6 = \frac{100 \cdot 10^3}{(\pi/4)d^2}$$



$$\therefore d = \text{Diameter} = \sqrt{\frac{4}{10^3}} = 0.03568 \text{ m} = 35.68 \text{ mm.}$$

$$\text{Total Elongation, } \Delta l = \frac{P}{E} \left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} \right]$$

$$\Delta l = \frac{100 \cdot 10^3}{200 \cdot 10^9} \left[\frac{0.10}{10^3} + \frac{0.15}{\frac{4}{4} (100)^2} + \frac{0.15}{\frac{4}{4} (80)^2} + \frac{0.15}{\frac{4}{4} (80)^2} \right] \times \frac{1}{10^6}$$

$$\Delta l = \frac{10^4}{2} [1 \ 0.191 \ 0.299]$$

$$\Delta l = \frac{1.490 \cdot 10^4}{2}$$

$$= 0.0745 \text{ mm.}$$

Example : A piece of material 15 cms long by 2.5 cms square, is in compression under a load of 100 kN. If the modulus of elasticity of the material is 105 GPa and Poisson's Ratio is 0.25, Find the alteration in length if all lateral strain is prevented by the application of uniform lateral external pressure of suitable intensity.

$$\text{Solution : } \epsilon_x = \frac{P}{A} = \frac{-100 \cdot 10^3}{2.5 \cdot 2.5 \cdot 10^2} = -160 \text{ MPa}$$

$$\text{Now } \epsilon_x = \frac{1}{E} \left[\epsilon_x - (\epsilon_y - \epsilon_z) \right]$$

$$\epsilon_y = \frac{1}{E} \left[\epsilon_y - (\epsilon_z - \epsilon_x) \right]$$

$$\epsilon_z = \frac{1}{E} \left[\epsilon_z - (\epsilon_x - \epsilon_y) \right]$$

Since the strain in the y and z directions are prevented, therefore $\epsilon_y = \epsilon_z = 0$

$$\therefore \sigma_y - v(\sigma_z + \sigma_x) = \sigma_z - v(\sigma_x + \sigma_y)$$

$$\text{or, } \sigma_y - v\sigma_z = \sigma_z - v\sigma_y$$

$$\sigma_y(1+v) = \sigma_z(1+v)$$

$$\therefore \sigma_y = \sigma_z$$

$$\text{Now } \sigma_y - v(\sigma_z + \sigma_x) = 0$$

$$\text{or, } (1-v)\sigma_y - v\sigma_y = 0$$

$$\text{or, } \sigma_y = \left[\frac{1}{(1-v)} \right] \sigma_x$$

$$\therefore \epsilon_x = \frac{1}{E} [\epsilon_x - 2\epsilon_y]$$

$$\epsilon_x = \frac{1}{E} \left\{ \epsilon_x - \left[\frac{2}{1-v} \right] \epsilon_x \right\}$$

$$\epsilon_x = \frac{\epsilon_x}{E} \left[1 - \frac{2}{1-v} \right] \quad \dots\dots(1)$$

Now putting the values from above in equation no (1) we get

$$\epsilon_x = \frac{-160}{105 \cdot 10^3} \left[1 - \frac{2}{0.75} \cdot 0.0625 \right]$$

$$\epsilon_x = \frac{-160}{105 \cdot 10^3} \left[\frac{5}{6} \right]$$

$$\epsilon_x = -0.143 \times 10^{-2}$$

$$\text{Contraction in length} = l \times \epsilon_x = 15 \times 0.143 \times 10^{-2} = 2.145 \times 10^{-2} = 0.02145 \text{ cm.}$$

PROOF RESILIENCE :

The maximum strain energy stored in a body is known as proof resilience. The strain energy stored in the body will be maximum when the body is stressed upto elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained upto elastic limit.

MODULUS OF RESILIENCE :

It is defined as the proof resilience of a material per unit volume. It is important property of a material Mathematically:

$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}$$

■ BENDING MOMENTS AND SHEAR FORCE DIAGRAM

BEAM : Any member of a machine or structure whose one dimension is very large as compared to the other two dimensions and which can take lateral forces in the axial plane is called a beam. The lateral forces acting on the beam shall cause bending in it. Beams may be classified as simply supported beam, fixed beam, continuous beam, over-hanging beam and cantilever type, also inverted beam.

Otherwise also a member of a machine or structure under the action of lateral forces in the axial plane is called a beam. The effect of these forces will be to cause bending of the member. Depending on the end conditions there are different types of beams. A beam with both ends rigidly fixed is known as fixed or encas-

STRENGTH OF MATERIALS

tre beam. If one of the ends is rigidly fixed and the other is absolutely free it is known as cantilever beam. If both the supports at the ends are such that the beam is capable of taking any angular position at the end and not the lateral movement, the beam is called to be a freely or simply supported beam; in actual practice this requirement is met by providing a well lubricated hinge at one end and a support on rollers at the other.

If such a beam has ends extending beyond the supports, it is called an overhanging beam. A beam which is supported at more than two supports is called a continuous beam.

As beams are invariably horizontal, and external forces are weights, the beams will be taken horizontal and the external forces as vertical, although the same conclusions will hold good in other cases. Members of structures and machines transmit axial loads in addition to lateral forces acting upon them. A concentrated or point load is one which is applied to the beam through a knife edge. A uniformly distributed load is that which is spread evenly over a certain length of the beam so that the load carried by a portion of the beam is proportional to the length of that portion. A distributed load will be called varying load when the rate of loading varies from one point to another.

SHEARING FORCE : The shearing force at any point along the loaded beam is the algebraic sum of all the vertical forces acting to one side of the point. Thus the beam shown in the figure enclosed. The shear force at cross-section $x-x$ as measured from left hand side is $F_1 = R_1 - W_1 - W_2$ which is equal to the shear force $F_2 = W_3 - R_2$

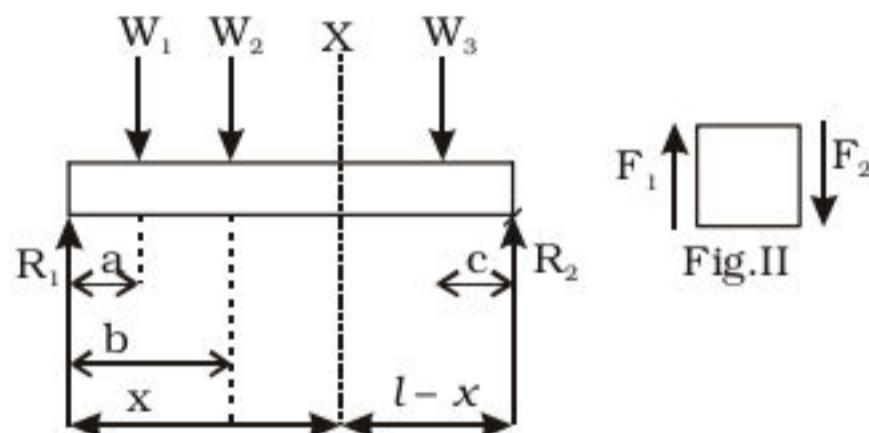


Fig.I

as measured from right hand side, i.e., $F_1 = F_2$, since the beam is in equilibrium. A thin slice of the beam at section $x-x$ subjected to these shear forces is shown in fig-II shear force is assumed to be positive if it produces a clockwise moment and negative if it produces an anticlockwise moment.

BENDING MOMENT : The bending Moment at any point along a loaded beam is the algebraic sum of the moments of all the vertical forces acting to one side of the point about the point. In fig-I, the clockwise moment at cross-section $x-x$ is :

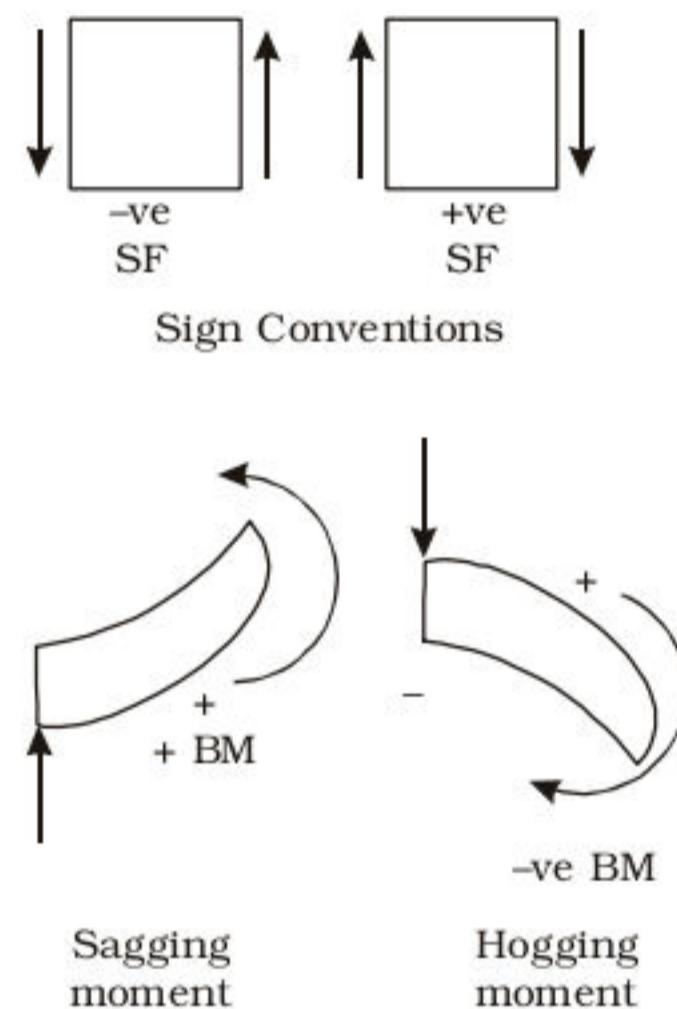
$$M_1 = R_1 x - W_1 (x-a) - W_2 (x-b)$$

whereas anticlockwise moment is $M_2 = R_2 (l-x) - W_3 (l-x-c)$ for equilibrium of the beam $M_1 = M_2$.

Clockwise moments are assumed to be positive and anticlockwise moments are assumed as negative due to all loads acting on the left of a section.

BENDING MOMENT AND SHEAR FORCE DIAGRAM :

In drawing the Bending Moment Diagrams (B.M.D.) and shearing force diagram (S.F.D.),



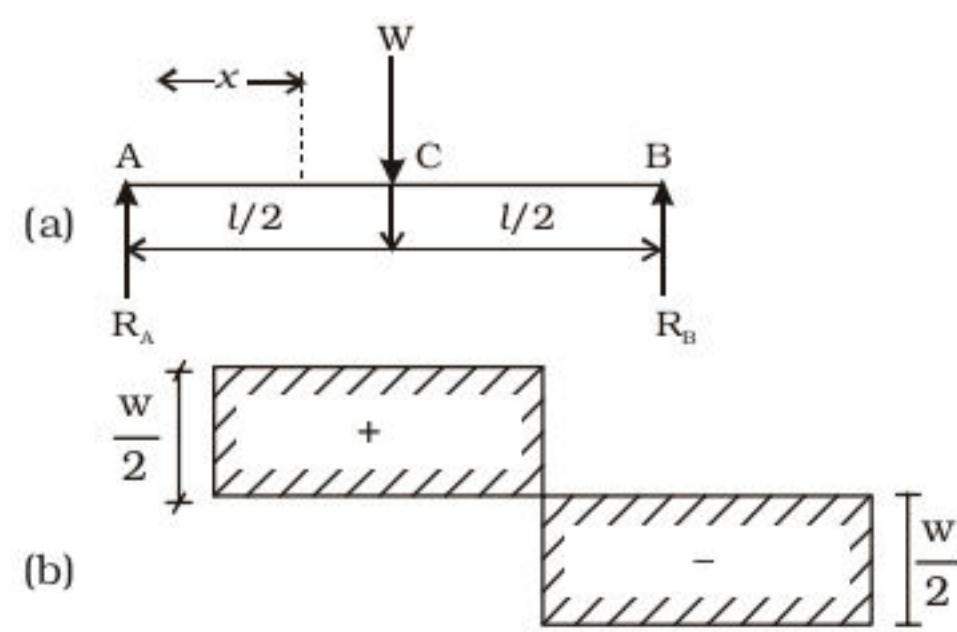
The following sign conventions are given in the enclosed figure are followed. The shear force is positive if the net resultant external force to the left of a section is upwards. Positive bending moment produces compression on the top fibres of the beam and negative bending moment produces tension on the top fibres of the beam.

1. Simply supported Beam carrying a concentrated Load at mid-span.

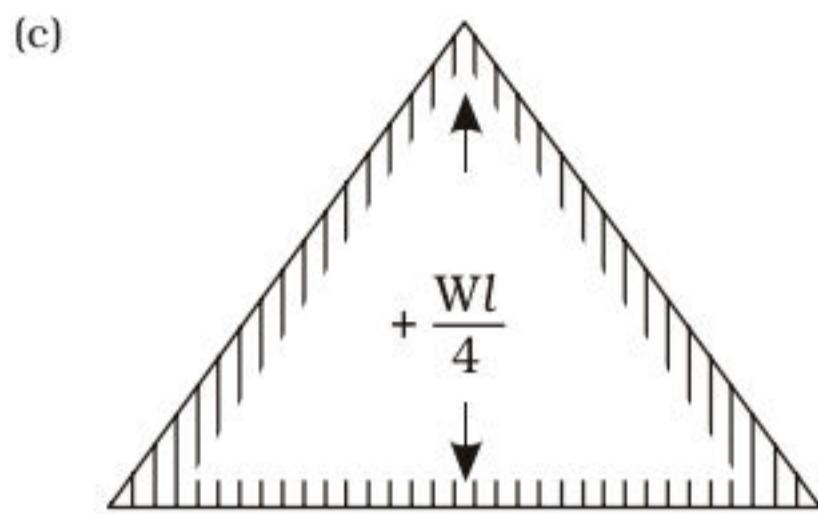
Taking moments we get

$$R_A \times l = W \times \frac{l}{2}$$

$$\therefore R_A = R_B = \frac{W}{2}$$



Shearing Force Diagram



Bending moment Diagram

Taking a section at a distance x from R_A for values of x lying between R_A and W , the shear force is $\left(W - \frac{W}{2}\right) = \frac{W}{2}$, and is negative in nature. Hence, the S.F. diagram is shown in fig- (b).

Similarly, for values of x lying between R_A and W , the bending moment at any point is $\frac{W}{2}x$ and is positive in nature. This bending moment at $x_2 = \frac{l}{2}$ i.e., at the midspan and is given by

$$M_{\max} = \frac{Wl}{4}$$

For Values of x lying between W and R_B , the Bending Moment is $\frac{W}{2}x - W\left(x - \frac{l}{2}\right) = -\frac{Wx}{2} + \frac{Wl}{2}$

$$= \frac{W}{2}(l - x)$$

This bending moment is also positive. The Bending moment diagram is shown in fig-(c).

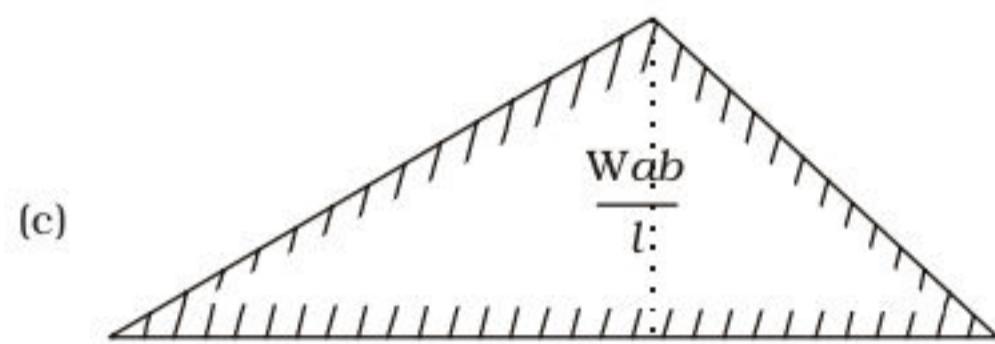
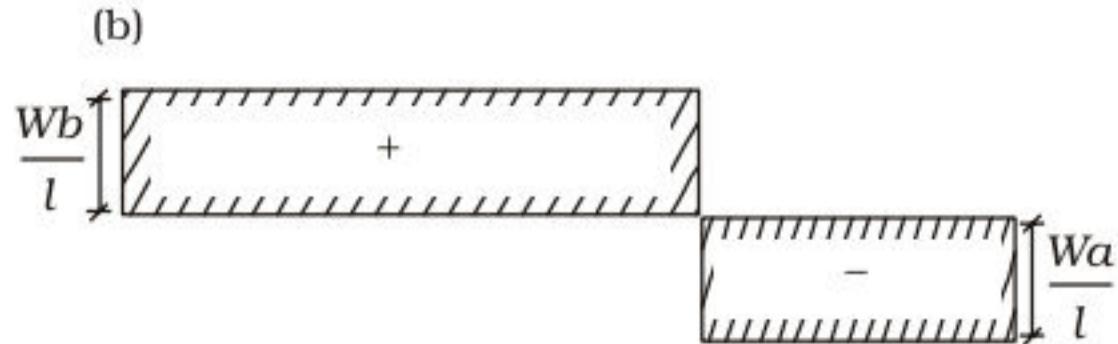
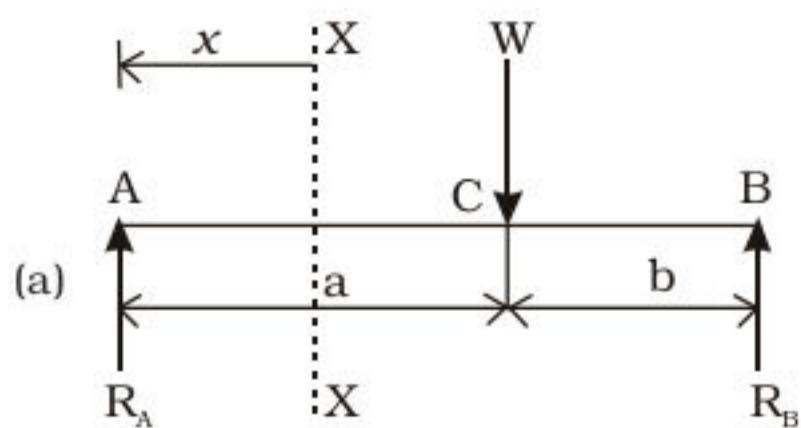
Simply supported beam carrying a concentrated load not a Mid span,

Consider a beam simply supported at the ends and carrying a concentrated load W at a distance a from left hand support as shown in fig (a). Taking the moments about R_B , we have :

$$R_A(a + b) = Wb$$

$$\therefore R_A = \frac{Wb}{a + b}$$

$$\text{and } R_B = \frac{Wa}{a + b}$$



Take a section at a distance x from R_A . When x lies between R_A and W , the shear force is R_A and is positive in nature. When x lies between W and R_B , the shear force is

$$W - R_A = \frac{Wa}{(a - b)} = \frac{Wa}{l}$$

and is negative in nature. Hence, the S.F. diagram is as shown in diagram (b)

Similarly for x lying between R_A and W , the bending moment is

$$M = R_A x$$

$$\text{or, } M = \frac{Wb}{(a - b)} x = \frac{Wbx}{l}$$

This is the equation of a straight line. When $x = a$, the bending moment is maximum and becomes,

$$M_{\max} = \frac{Wab}{(a - b)} = \frac{Wab}{l}$$

This is a positive bending moment. For x lying between W and R_B , the bending moment is

$$M = R_A x - W(x - a)$$

$$= \frac{Wb}{(a - b)} \cdot x - W(x - a) = Wa - \frac{Wa}{(a - b)} \cdot x$$

$$= Wa \left(1 - \frac{x}{l}\right)$$

This also represents a straight line. When $x = a$,

$$M = \frac{Wab}{(a - b)}$$
 and when $x = a + b$, $M = 0$. This bending moment is also positive. Hence the B.M. diagram is also shown in fig - (c).

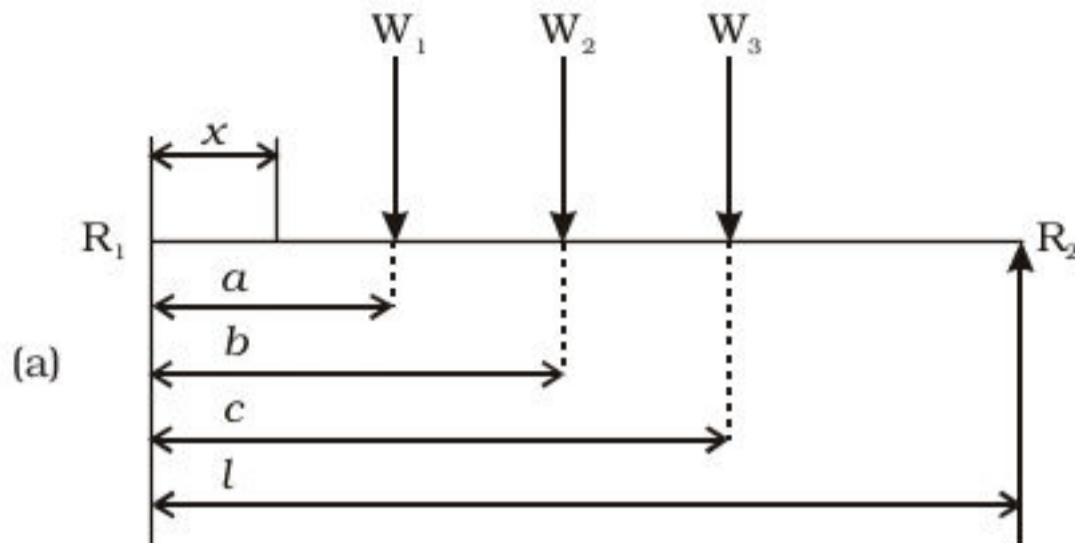
Simply supported Beam Carrying Many Concentrated Loads :

Consider a beam simply supported at the ends and carrying many concentrated loads as shown in fig (a). Taking the moments about R_2 , We get :

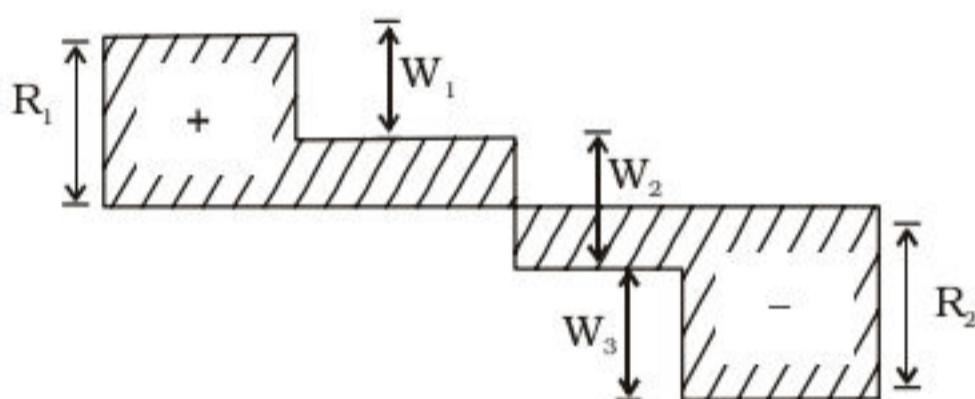
$$R_1 \times l = W_1(l-a) + W_2(l-b) + W_3(l-c)$$

STRENGTH OF MATERIALS

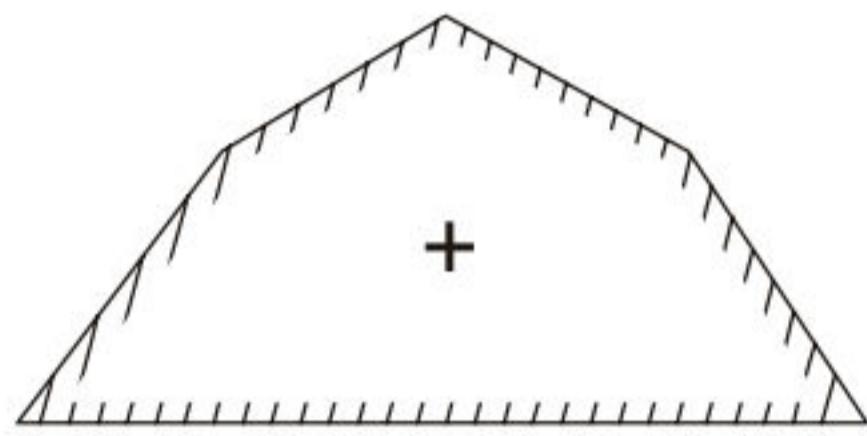
$$\therefore R_1 = W_1 \left(1 - \frac{a}{l}\right) + W_2 \left(1 - \frac{b}{l}\right) + W_3 \left(1 - \frac{c}{l}\right)$$



(b)



(c)



B.M. Diagram

$$\text{or, } R_1 = (W_1 + W_2 + W_3) - \frac{1}{l} (W_1 a + W_2 b + W_3 c)$$

$$\text{and } R_2 = (W_1 + W_2 + W_3) - R_1$$

Let us consider a section at a distance x from R_1 . When x lies between R_1 and W_1 , the shear force is R_1 and is positive in nature. When x lies between W_1 and W_2 , the shear force is $(R_1 - W_1)$. Between R_1 and W_1 and W_2 and W_3 , the shear force remains constant. Shear force between W_1 and W_2 will remain positive, if $W_1 < R_1$, when x lies between W_2 and W_3 , the shear force is $(R_1 - W_1 - W_2)$. We have assumed that $(W_1 + W_2) > R_1$ and hence the shear force becomes negative between W_2 and W_3 . When x lies between W_3 and R_2 , the shear force is $R_1 - W_1 - W_2 - W_3 = R_2$ and is negative in nature. Hence, the shear force diagram is as shown in fig (b). Similarly for x lying between R_1 and W_1 , the bending moment is $M = R_1 x$.

$$\text{At } x = a, M = R_1 a.$$

The variation of bending moment is linear in nature and it is positive, for x lying between W_1 and W_2 ,

$$M = R_1 x - W_1(x - a)$$

At $x = b$, the bending moment becomes,
 $M = R_1 b - W_1(b - a)$
 and as per assumptions made, this is the maximum bending moment.

For x lying between W_2 and W_3 , the bending moment is

$$M = R_1 x - W_1(x - a) - W_2(x - b)$$

At $x = c$, the bending moment becomes :

$$M = R_1 c - W_1(c - a) - W_2(c - b)$$

And for x lying between W_3 and R_2

$$M = R_1 x - W_1(x - a) - W_2(x - b) - W_3(x - c)$$

$$= (R_1 - W_1 - W_2 - W_3)x + (W_1 a + W_2 b + W_3 c)$$

At $x = l$

$$M = (R_1 - W_1 - W_2 - W_3)l + (W_1 a + W_2 b + W_3 c)$$

$$= (W_1 + W_2 + W_3)l - (W_1 a + W_2 b + W_3 c)$$

$$- (W_1 + W_2 + W_3)l + (W_1 a + W_2 b + W_3 c) = 0$$

The B.M. diagram is shown in fig (c)

Simply supported Beam carrying a Uniformly distributed load :

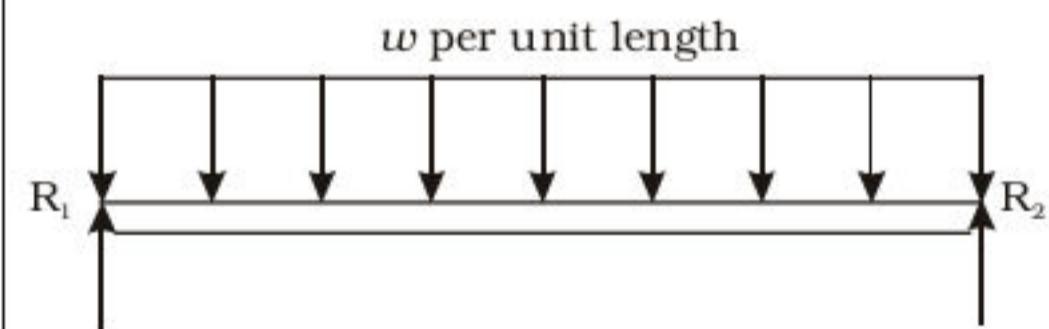
Consider a simply supported beam carrying uniformly distributed load of intensity w per unit length over its whole span as shown in fig (a).

Taking moments about R_2 , we have

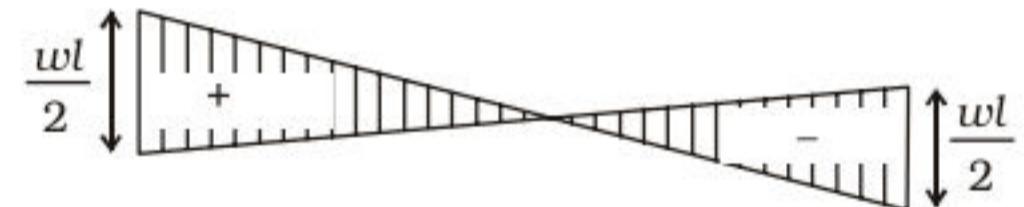
$$R_1 \times l = wl \times \frac{l}{2}$$

$$\therefore R_1 = \frac{wl}{2} \text{ and } R_2 = \frac{wl}{2}$$

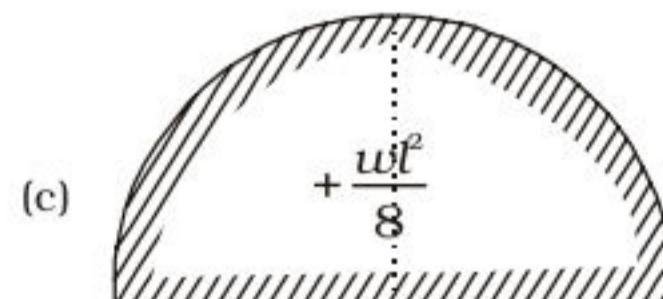
(a)



(b)



S.F. Diagram



B.M. Diagram

Consider a section at a distance x from R_1 . Shear force at this cross-section is

$$F = R_1 - wx = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right)$$

STRENGTH OF MATERIALS

This represents the equation of a straight line. At $x = \frac{l}{2}$, the shear force is zero and is positive in nature upto this length. At $x = l$, the shear force $F = -\frac{wl}{2}$, i.e., it becomes negative. Hence, the shear force diagram is shown in fig - (b).

The bending moment at a section x from R_1 is :

$$\begin{aligned} M &= R_1 x - wx \cdot \frac{x}{2} \\ &= R_1 x - \frac{wx^2}{2} \\ &= \frac{wl}{2} x - \frac{wx^2}{2} \quad [\because R_1 = \frac{wl}{2} \text{ from above}] \\ &= \frac{w}{2} (lx - x^2) \end{aligned}$$

This represents the equation of parabola. Hence the variation of bending moment is parabolic in nature.

$$\text{At } x = 0, \quad M = 0$$

$$\text{At } x = \frac{l}{2}, \quad M_{\max} = \frac{wl^2}{8}$$

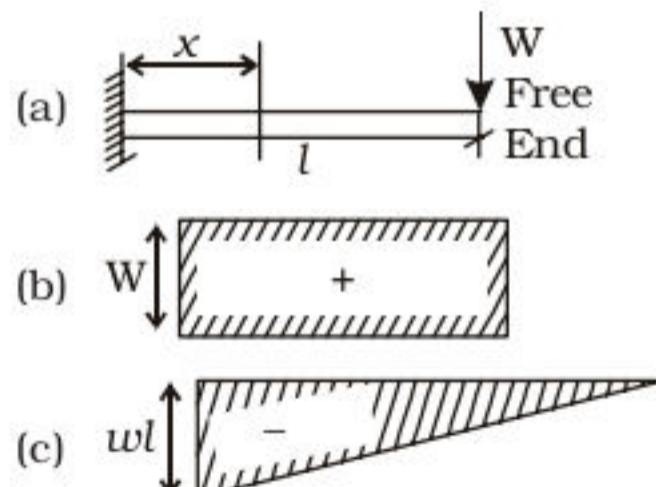
and is the maximum bending moment.

$$\text{At } x = l, M = 0.$$

The bending moment remains positive on the whole span of the beam. Hence the bending moment diagram is as shown in figure (c).

CANTILEVERS CARRYING A CONCENTRATED LOAD AT FREE END

Consider a cantilever beam of span l and carrying a concentrated load W at the free end as shown in figure (a). Consider a section at a distance ' x ' from the fixed end. Shear force at any section is equal to W and is positive in nature. Hence the S.F. diagram is as shown in figure (b).



The bending moment at any section is given by :

$$M = W(l - x)$$

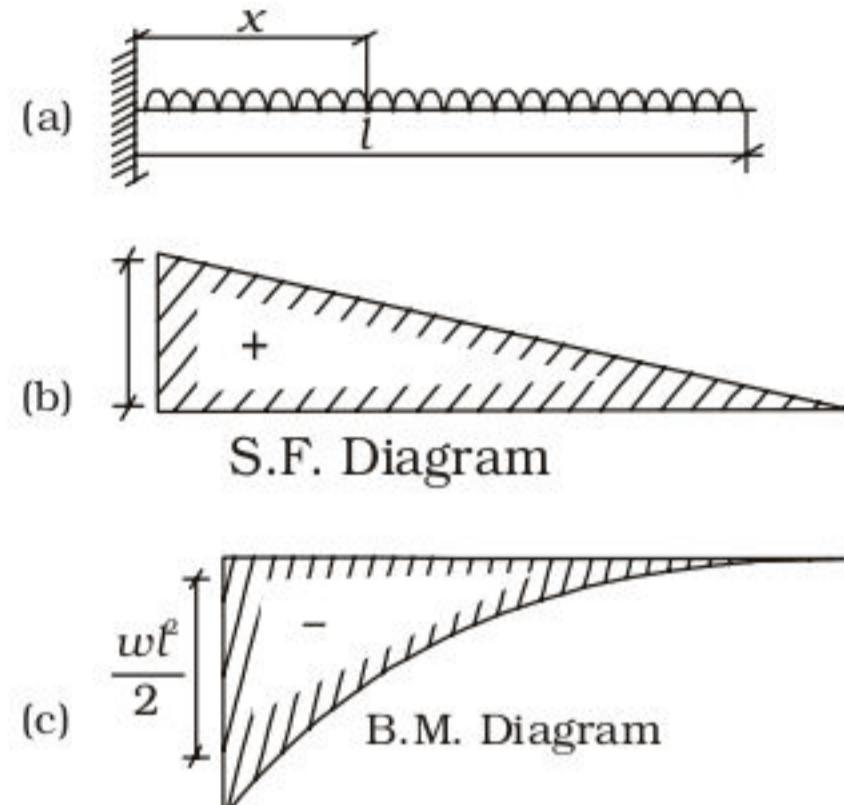
$$\text{At } x = l, M = 0$$

$$\text{and at } x = 0, M = wl$$

The bending moment is negative in nature and varies linearly as shown in figure (c).

Cantilever carrying Uniformly distributed load

Consider a cantilever beam of span l and carrying a uniformly distributed load of intensity w per unit length over its whole span as shown in fig (a). Consider a section at a distance x from the fixed end. The shear force at this section is :



$$F = w(l - x)$$

At $x = l$, $F = 0$ and
at $x = 0$, $F = wl$.

This is a positive shear force and varies linearly. Hence, the S.F. diagram is as shown in fig (b).

The bending moment at any section is,

$$M = w(l - x) \cdot \frac{(l - x)}{2}$$

$$= \frac{w}{2} (l - x)^2$$

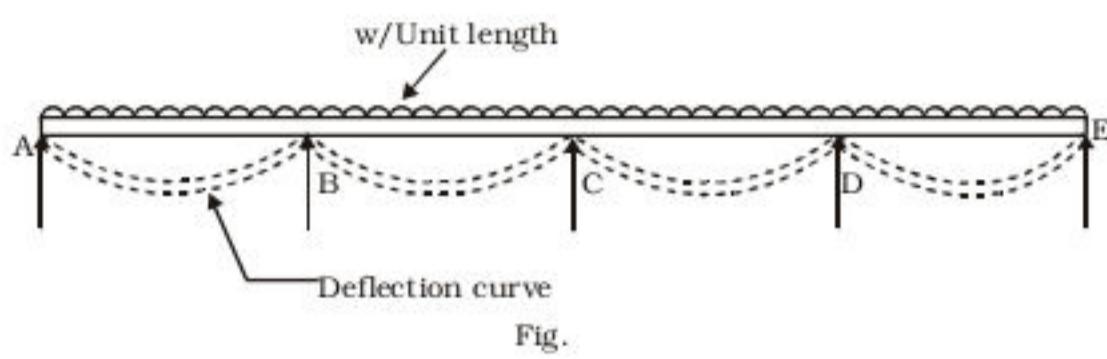
$$\text{At } x = l, M = 0$$

$$\text{and at } x = 0, M = \frac{wl^2}{2}$$

This is a negative bending moment and varies parabolically as shown in fig (c).

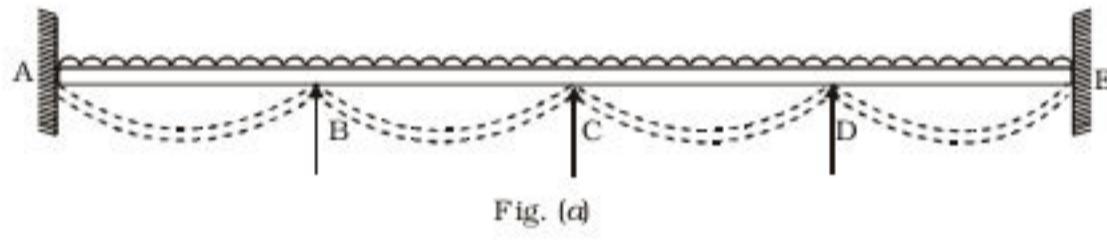
CONTINUOUS BEAMS

Continuous beam is a beam which is supported on more than two supports. Fig. shows such a beam, which is subjected to some external loading (here a uniformly distributed load). The deflection curve for the beam is shown by dotted line. The deflection curve is having convexity upwards over the intermediate supports, and concavity upwards over the mid of the span. Hence there will be hogging moments (i.e., negative) over the intermediate supports and sagging moments (i.e., positive) over the mid of the span. The end supports of a simply supported continuous beam will not be subjected to any bending moment. But the end support of fixed continuous beam will be subjected to fixing moments. If the moments over the intermediate supports are known, then the B.M. diagram can be drawn.



The Fig. shows a simply supported continuous beam. In this figure the end supports at A and E will not be subjected to any bending moment. Hence in this case $M_A = M_E = 0$.

Fig. shows a continuous beam with fixed ends at A and E. Here the end supports at A and E will be subjected to fixing moments. Hence M_A and M_E will not be zero.



BENDING MOMENT DIAGRAM FOR CONTINUOUS BEAMS

In above Art. it is mentioned that if the moments over the intermediate supports of a continuous beam are known, then the B.M. diagram can be drawn easily. The moments over the intermediate supports are determined by using Clapeyron's theorem of three moments which states that:

If BC and CD are any two consecutive span of a continuous beam subjected to an external loading, then the moments M_B , M_C and M_D at the supports B, C and D are given by,

$$M_B \cdot L_1 + 2M_C(L_1 + L_2) + M_D \cdot L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

where

L_1 = Length of span BC

L_2 = Length of span CD

a_1 = Area of B.M. diagram due to vertical loads on span BC

a^2 = Area of B.M. diagram due to vertical loads on span CD

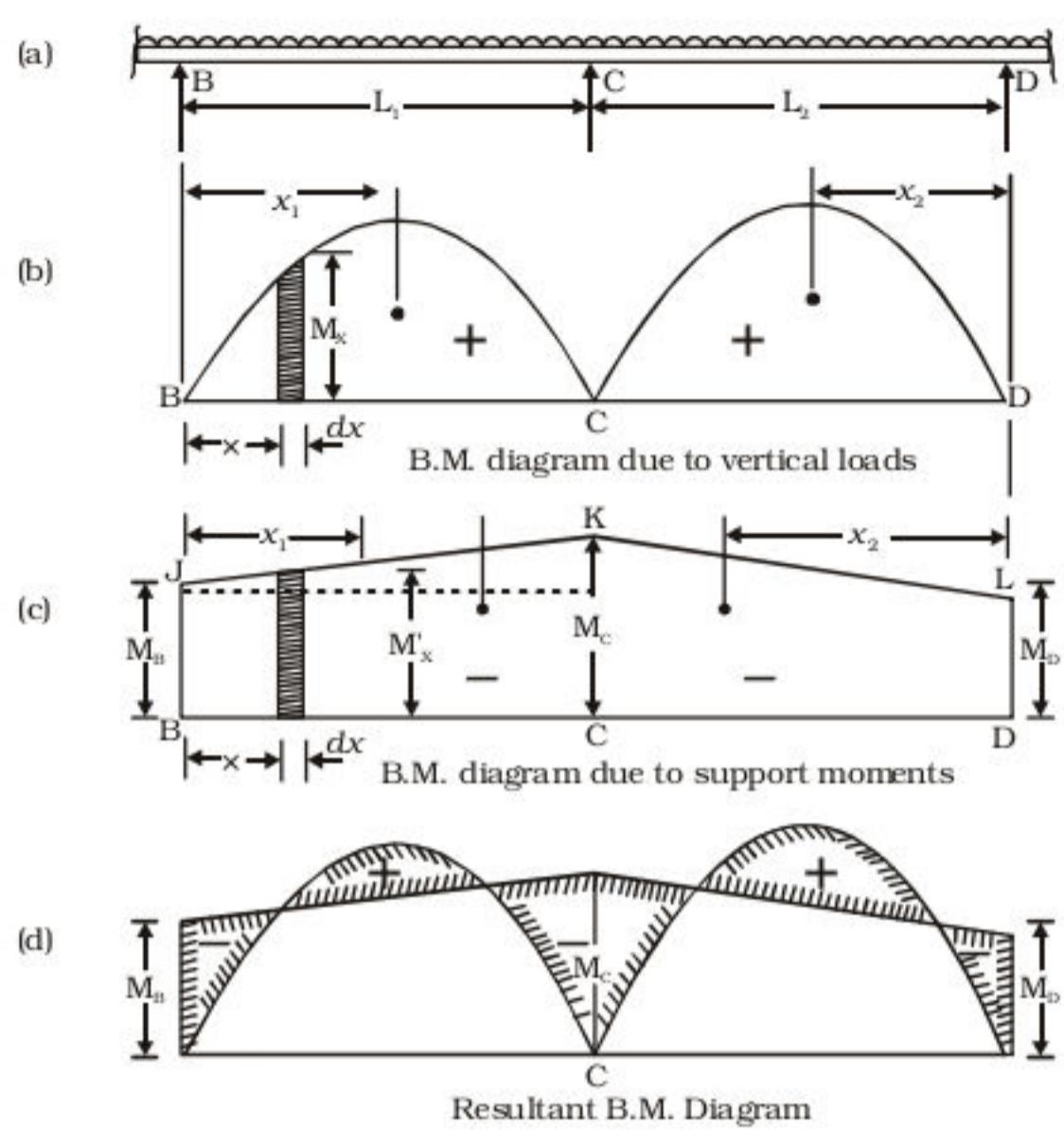
\bar{x}_1 = Distance of C.G. of the B.M. diagram due to vertical loads on BC from B

\bar{x}_2 = Distance of C.G. of the B.M. diagram due to vertical loads on CD from D.

The equation is known as the equation of three moments or Clapeyron's equation.

Derivation of Clapeyron's Equation of three Moments.

Fig. shows the length BCD (two consecutive spans) of a continuous beam which is shown in Fig. Let M_B , M_C and M_D are the support moments at B, C and D respectively.



Let L_1 = Length of span BC

L_2 = Length of span CD

a_1 = Area of B.M. diagram due to vertical loads on span BC

a_2 = Area of B.M. diagram due to vertical loads on span CD

a'_1 = Area of B.M. diagram due to support moments M_B and M_c

a'_2 = Area of B.M. diagram due to support moments M_c and M_D

\bar{x}_1 = Distance of C.G. of B.M. diagram due to vertical loads on BC

\bar{x}_2 = Distance of C.G. of B.M. diagram due to vertical loads on CD

\bar{x}'_1 = Distance of C.G. of B.M. diagram due to support moments on BC

\bar{x}'_2 = Distance of C.G. of B.M. diagram due to support moments on CD.

Fig. (b) and (c) shows the B.M. diagrams due to vertical loads and due to supports moments respectively.

(i) Consider the span BC

Let

M_x = B.M. due to vertical loads at a distance x from B (sagging)

M'_x = B.M. due to support moments at a distance x from B (hogging)

∴ Net B.M. at a distance x from B is given by.

$$EI \frac{d^2y}{dx^2} = M_x - M'_x$$

Multiplying by x to both sides, we get

$$EI \cdot x \cdot \frac{d^2y}{dx^2} = x \cdot M_x - M'_x$$

Integrating from zero to L_1 , we get

$$\int_0^{L_1} EI \cdot x \cdot \frac{d^2y}{dx^2} \cdot dx = \int_0^{L_1} x \cdot M_x \cdot dx - \int_0^{L_1} x \cdot M'_x \cdot dx$$

$$\text{or } EI \left[x \frac{dy}{dx} - y \right]_0^{L_1} = a_1 \bar{x}_1 - a_1' \bar{x}_1 \quad \dots(i)$$

($\because M_x \cdot dx$ = Area of B.M. diagram of length dx . And $x \cdot M'_x \cdot dx$ = Moment of area of B.M. diagram of length dx about B.)

$$\text{Here } \int_0^{L_1} x \cdot M_x \cdot dx = a_1 \bar{x}_1. \text{ And so on}$$

Substituting the limits in L.H.S. of equation (i), we have

$$\begin{aligned} & EI \left[\left\{ L_1 \left(\frac{dy}{dx} \right)_{at C} - y_C \right\} - \left\{ 0 \times \left(\frac{dy}{dx} \right)_{at B} - y_B \right\} \right] \\ &= a_1 \bar{x}_1 - a_1' \bar{x}_1' \\ \text{or } & EI [(L_1 \cdot \theta_C - y_C) - (0 - y_B)] = a_1 \bar{x}_1 - a_1' \bar{x}_1'. \\ & \quad \left[\because \left(\frac{dy}{dx} \right)_{at C} = \theta_C \right] \end{aligned}$$

But deflection at B and C are zero. Hence $y_B = 0$ and $y_C = 0$. Hence above equation becomes as

$$[EI \cdot L_1 \cdot \theta_C = a_1 \bar{x}_1 - a_1' \bar{x}_1']$$

But

$$\begin{aligned} a_1 &= \text{Area of B.M. diagram due to supports moments} \\ &= \text{Area of trapezium BCKJ} \\ &= \frac{1}{2} (M_B + M_C) \times L_1 \end{aligned}$$

and \bar{x}_1' = Distance of C.G. of area BCKJ from

$$\begin{aligned} &= \frac{M_B \cdot L_1 \cdot \frac{L_1}{2} + \frac{1}{2} \times (M_C - M_B) \cdot L_1 \times \frac{2L_1}{3}}{M_B \cdot L_1 + \frac{1}{2} (M_C - M_B) \cdot L_1} \\ &= \frac{M_B \cdot \frac{L_1}{2} + (M_C - M_B) \times \frac{L_1}{3}}{M_B + (M_C - M_B) \cdot \frac{1}{2}} = \frac{\frac{3M_B L_1 + 2L_1(M_C - M_B)}{6}}{\frac{2M_B + M_C - M_B}{2}} \\ &= \frac{\frac{L_1}{3} [3M_B + 2M_C - 2M_B]}{M_B + M_C} = \left(\frac{M_B + 2M_C}{M_B + M_C} \right) \times \frac{L_1}{3} \end{aligned}$$

Substituting the values of a_1 and \bar{x}_1' in equation (ii), we get

$$\begin{aligned} EI \cdot L_1 \cdot \theta_C &= a_1 \bar{x}_1 - \frac{1}{2} (M_B + M_C) \cdot L_1 + \left(\frac{M_B + 2M_C}{M_B + M_C} \right) \times \frac{L_1}{3} \\ &= a_1 \bar{x}_1 - \frac{L_1^2}{6} (M_B + 2M_C) \\ 6EI \cdot \theta_C &= \frac{6a_1 \bar{x}_1}{L_1} - L_1 (M_B + 2M_C) \dots(iii) \end{aligned}$$

(ii) Consider the span CD

Similarly considering the span CD and taking D as origin and x positive to the left, it can be shown that

$$6EI \cdot \theta_C = \frac{6a_2 \bar{x}_1}{L_1} - L_2 (M_D + 2M_C)$$

[In the above case the slope at C (i.e., θ_C) will have opposite sign than that given by equation (iii). The reason is that the direction of x from B for the span BC, and from D for span CD are in the opposite direction].

Hence the above equation becomes as

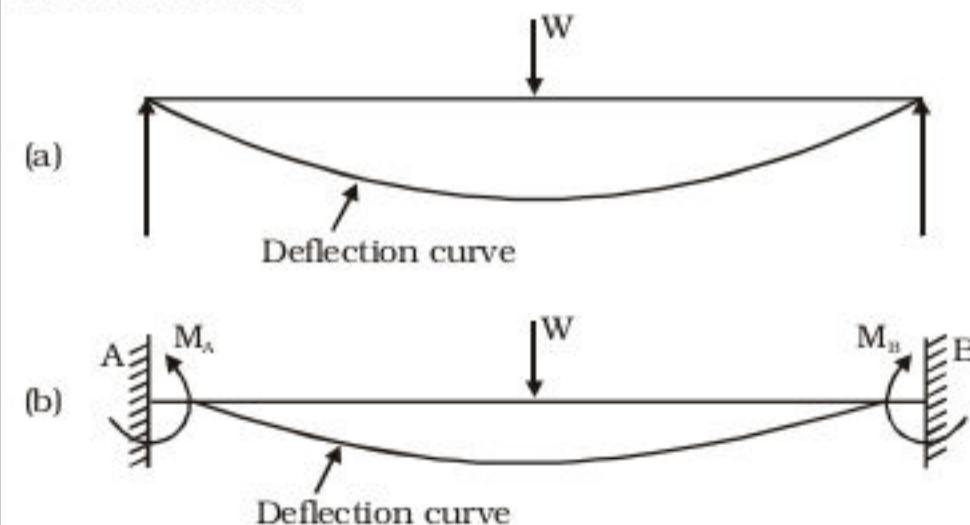
$$\therefore -6EI \cdot \theta_C = \frac{6a_2 \bar{x}_1}{L_1} - L_2 (M_D + 2M_C) \dots(iv)$$

Adding equation (iii) and (iv), we get

$$\begin{aligned} 0 &= \frac{6a_1 \bar{x}_1}{L_1} - L_1 (M_B + 2M_C) + \frac{6a_2 \bar{x}_2}{L_2} - L_2 (M_D + 2M_C) \\ &= \frac{6a_1 \bar{x}_1}{L_1} + \frac{6a_2 \bar{x}_2}{L_2} - L_1 M_B - 2L_1 M_C - L_2 M_D - 2L_2 M_C \\ L_1 \cdot M_B + L_2 \cdot M_D + 2M_C(L_1 + L_2) &= \frac{6a_1 \bar{x}_1}{L_1} + \frac{6a_2 \bar{x}_2}{L_2} \\ M_B L_1 + 2M_C(L_1 + L_2) + M_D L_2 &= \frac{6a_1 \bar{x}_1}{L_1} + \frac{6a_2 \bar{x}_2}{L_2} \end{aligned}$$

FIXED BEAM:-

A beam whose both ends are fixed is known as a fixed beam. Fixed beam is also called a built-in or en-caster beam. In case of a fixed beam both its ends are rigidly fixed and the slope and deflection at the fixed ends are zero. But the fixed ends are subjected to end moments. Hence end moments are not zero in case of a fixed beam.



In case of simply supported beam, the deflection is zero at the ends. But the slope is not zero at the ends as shown in Fig. (a).

In case of fixed beam, the deflection and slope are zero at the fixed ends as shown in Fig. (b). The slope

STRENGTH OF MATERIALS

will be zero at the ends if the deflection curve is horizontal at the ends. To bring the slope back to zero (i.e., to make the deflection curve horizontal at the fixed ends), the end moments M_A and M_B will be acting in which M_A will be acting anti-clockwise and M_B will be acting clockwise as shown in Fig. (b).

A beam which is supported on more than two supports is known as continuous beam. This section deals with the fixed beams and continuous beam. In case of fixed beams the B.M. diagram, slope and deflection for various types of loading such as point loads, uniformly distributed load and combination of point load and u.d.l., are discussed. In case of continuous beam, Clapeyron's equation of three moments and application of this equation to the continuous beam of simply supported ends and fixed ends are explained.

BENDING MOMENT DIAGRAM FOR FIXED BEAMS:

Fig. 15.1 (c) shows a fixed beam AB of length L subjected to two loads W and 2W at a distance of $\frac{L}{4}$ from each ends.

Let R_A = Reaction at A

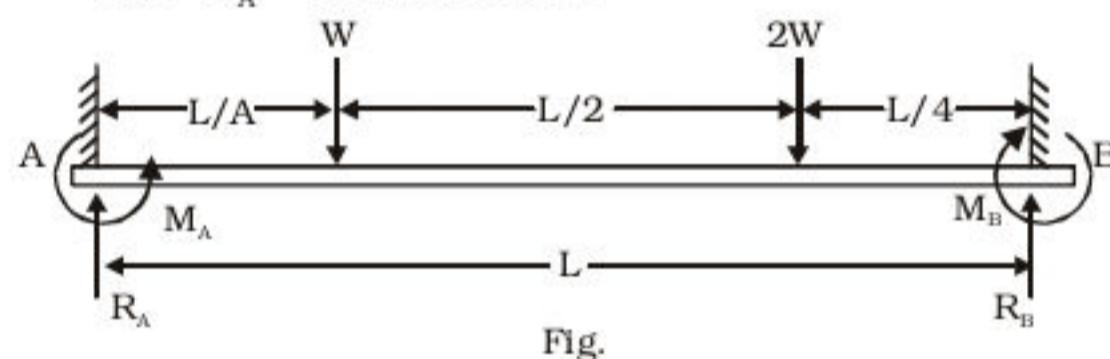


Fig.

R_B = Reaction at B

M_A = Fixed end moment at A

M_B = Fixed end moment at B

The above four quantities i.e., R_A , R_B , M_A and M_B are unknown.

The values of R_A , R_B , M_A and M_B are calculated by analysing the given beam in the following two stages :

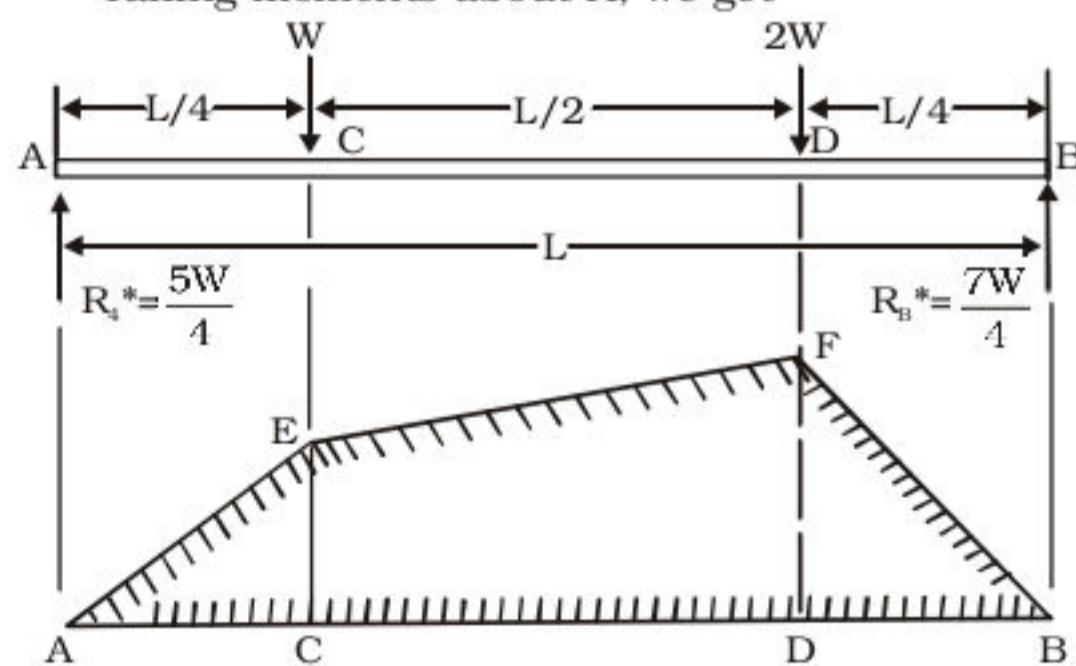
(i) A simply supported beam subjected to given vertical loads as shown in Fig.

Consider the beam AB as simply supported.

Let R_A^* = Reaction at A due to vertical loads

R_B^* = Reaction at B due to vertical loads.

Taking moments about A, we get



B.M. diagram considering beam as simply supported
Fig.

$$R_B^* \times L = W \times \frac{L}{4} + 2W \times \frac{3L}{4}$$

$$\therefore R_B^* = \frac{W}{4} + \frac{6W}{4} = \frac{7W}{4}$$

$$\text{and } RA^* = \text{Total load} - R_B^*$$

$$= 3W - \frac{7W}{4} = \frac{5W}{4}$$

$$\text{B.M. at A} = 0, \quad \text{B.M. at B} = 0.$$

$$\text{B.M. at C} = \frac{5W}{4} \times \frac{L}{4} = \frac{5WL}{16}$$

$$\text{B.M. at D} = \frac{7W}{4} \times \frac{L}{4} = \frac{7WL}{16}$$

Now B.M. diagram can be drawn as shown in Fig. (b). In this case, B.M. at any point is a sagging (+ve) moment.

(ii) A simply supported beam subjected to end moments only (without given loading) as shown in Fig.

Let MA = Fixed end moment at A

MB = Fixed end moment at B

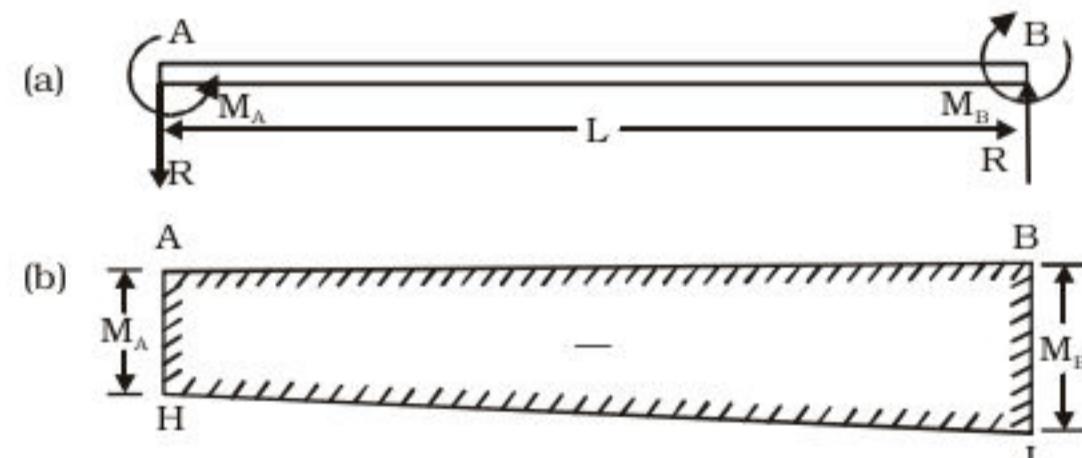
R = Reaction* at each end due to these moments.

As the vertical loads acting on the beam are not symmetrical (they are W at distance $L/4$ from A and 2W at a distance $L/4$ from B), the fixed end moments will be different.

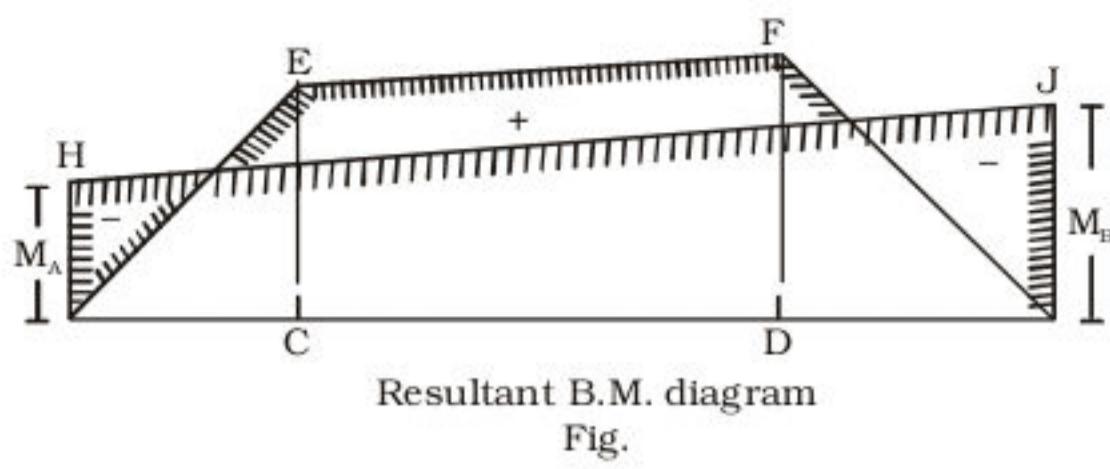
Suppose M_B is more than M_A and reaction R at B is acting upwards. Then reaction R at A will be acting downwards as there is no other load on the beam. ($\Sigma F_y = 0$). Taking moments about A, we get clockwise moment at A = Anti-clockwise moment at A.

$$M_B = M_A + RL$$

$$\therefore R = \frac{M_B - M_A}{L} \quad \dots(A)$$



As M_B has been assumed more than M_A , the R.H.S. of equation (A) will be positive. This means the magnitude of reaction R at B is positive. This also means that the direction of reaction R at B is according to our assumption. Hence the reaction R will be upwards at B and downwards at A as shown in Fig. (a). The B.M. diagram for this condition is shown in Fig. (b). In this case, B.M. at any point is a hogging (-ve) moment.



Now the final reactions RA and RB are given by

$$\begin{aligned} R_A &= R_A^* - R \\ \text{and} \quad R_B &= R_B^* + R \end{aligned}$$

In the above two equations, R_A^* and R_B^* are already calculated. They are : $R_A^* = 5W/4$ and $R_B^* = 7W/4$. But the value of R is in terms of M_B and M_A . It is given by $R = (M_B - M_A)/L$. Hence to find the value of R, we must calculate the value of M_B and M_A first.

To find the values of M_A and M_B

Let M_x = B.M. at any section at a distance x from A due to vertical loads

M'_x = B.M. at any section at a distance x from A due to end moments.

The resultant B.M. at any section at a distance x from A

$$= M_x - M'_x$$

$(M_x$ is +ve but M'_x is -ve)

But B.M. at any section is also equal to $EI \frac{d^2y}{dx^2}$

$$\therefore EI \frac{d^2y}{dx^2} = M_x - M'_x$$

Integrating the above equation for the entire length, we get

$$EI \left[\frac{dy}{dx} \right]_0^L = \int_0^L M_x dx - \int_0^L M'_x dx$$

But $\frac{dy}{dx}$ represents the slope. And slope at the fixed ends i.e., at A and B are zero. The above equation can be written as

$$EI \left[\left(\frac{dy}{dx} \text{ at } x = L \right) - \left(\frac{dy}{dx} \text{ at } x = 0 \right) \right]$$

$$= \int_0^L M_x dx - \int_0^L M'_x dx$$

$$EI[0 - 0] = \int_0^L M_x dx - \int_0^L M'_x dx$$

$$\text{or } 0 = \int_0^L M_x dx - \int_0^L M'_x dx \quad \dots(ii)$$

Now $\int_0^L M_x dx$ represents the area of B.M. diagram

due to vertical loads and $\int_0^L M'_x dx$ represents the area of B.M. diagram due to end moments.

Let

a = Area of B.M. diagram due to vertical loads

a' = Area of B.M. diagram due to end moments.

$$\text{Then } \int_0^L M_x dx = a \text{ and } \int_0^L M'_x dx = a'$$

Substituting these values in equation (ii), we get

$$0 = a - a'$$

$$\text{or } a = a'$$

The above equation shows that area of B.M. diagram due to vertical loads is equal to the area of B.M. diagram due to end moments.

Again consider the equation (i)

$$EI \frac{d^2y}{dx^2} = M_x - M'_x$$

Multiplying the above equation by x, we get

$$EI \cdot x \cdot \frac{d^2y}{dx^2} = xM_x - xM'_x$$

Integrating for the whole length of the beam i.e., from 0 to L, we get

$$\int_0^L EI \cdot x \cdot \frac{d^2y}{dx^2} dx = \int_0^L x \cdot M_x dx - \int_0^L x \cdot M'_x dx$$

$$EI \int_0^L x \cdot \frac{d^2y}{dx^2} dx = \int_0^L x \cdot M_x dx - \int_0^L x \cdot M'_x dx \quad \dots(iii)$$

In the above equation, $M_x dx$ represents the area of B.M. diagram due to vertical loads at a distance x from the end A. And the term $(x \cdot M'_x dx)$ represents the moment of area of B.M.

$$\therefore \int_0^L x \cdot M'_x dx$$

Diagram about the end A. Hence $\int_0^L x \cdot M'_x dx$

represents the moment of the total area of B.M.

Diagram due to vertical loads about A, and it is equal to total area of B.M. diagram due to vertical loads multiplied by the distance of C.G. of area from A,

$$\therefore \int_0^L x \cdot M'_x dx \cdot \bar{x}$$

where \bar{x} = Distance of the C.G. of B.M. diagram due to vertical loads.

$$\text{Similarly } \int_0^L x \cdot M_x dx = d' \bar{x}$$

where \bar{x} = Distance of the C.G. of B.M. diagram due to end moments. Substituting the above values in equation (iii), we get

$$EI \int_0^L x \cdot \frac{d^2y}{dx^2} dx = a\bar{x} - a'\bar{x}'$$

$$\text{or } EI \int_0^L \left[x \frac{dy}{dx} - y \right] dx = a\bar{x} - a'\bar{x}'$$

$$\left[\because \frac{d}{dx} \left(x \frac{dy}{dx} - y \right) = \left(x \frac{d^2y}{dx^2} + \frac{dy}{dx} \right) - \frac{dy}{dx} = x \frac{d^2y}{dx^2} \right]$$

$$\text{or } EI \left[\left(x \frac{dy}{dx} - y \right)_{at \, x=L} - \left(x \frac{dy}{dx} - y \right)_{at \, x=0} \right] = a\bar{x} - a'\bar{x}'$$

$$\text{or } EI[(L\theta_B - y_B) - (0 \times \theta_A - y_A)] = a\bar{x} - a'\bar{x}'.$$

Since, slope and deflection at A and B are zero, hence θ_A , θ_B , y_A and y_B are zero.

$$\therefore 0 = a\bar{x} - a'\bar{x}'$$

$$\text{or } a\bar{x} = a'\bar{x}'$$

But from equation we have

$$a = a'$$

$$\bar{x} = \bar{x}'$$

Hence the distance of C.G. of B.M. diagram due to vertical loads from A is equal to distance of C.G. of B.M. diagram due to end moments from A.

Now by using equations the unknowns M_A and M_B can be calculated

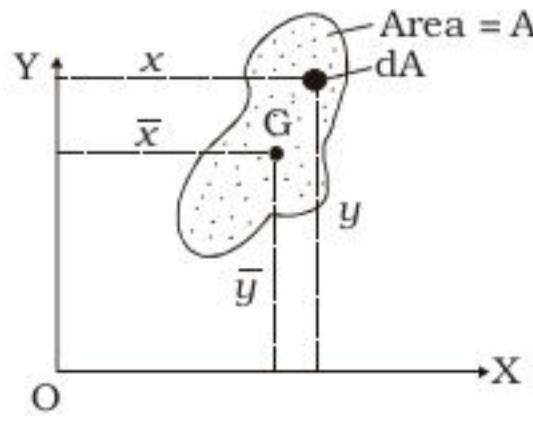
This also means that M_A and M_B can be calculated by

- (i) equating the area of B.M. diagram due to vertical loads to the area of B.M. diagram due to end moments.
- (ii) equating the distance of C.G. of B.M. diagram due to vertical loads to the distance C.G. of B.M. diagram due to end moments. The distance of C.G. must be taken from the same end in both cases.

Moment of Inertia product of Inertia

Principle Axis :

◆ Centroid :

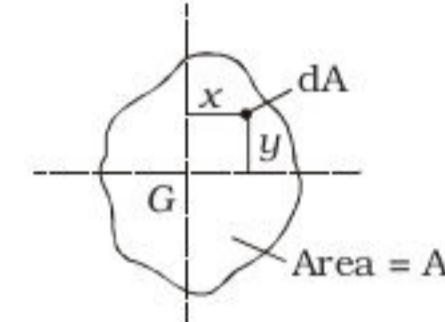


$$\bar{x} = \frac{\int x \cdot dA}{\int dA} = \frac{\Sigma A \cdot x}{\Sigma A} = \bar{y} = \frac{A_1 \bar{x}_1 + A_2 \bar{x}_2 + \dots A_n \bar{x}_n}{A_1 + A_2 + \dots A_n}$$

$$\bar{y} = \frac{\int y \cdot dA}{\int dA} = \frac{\Sigma A \cdot y}{\Sigma A} = \frac{A_1 \bar{y}_1 + A_2 \bar{y}_2 + \dots A_n \bar{y}_n}{A_1 + A_2 + \dots A_n}$$

◆ Moment of Inertia :

Let XX and YY are centroidal axis of area A consider, any elemental area dA having co-ordinates (x, y) with respect to origin at centroid.



M.I (area M.I) is defined as 2nd moment of Inertia about an axis. It is always non-zero and non (-ve) quantity [$I > 0$].

The M.I about centroidal axis XX is ;

$$I_{xx} = I_{xsey} = \int_0^A y^2 \cdot dA$$

$$I_{yy} = I_{ysey} = \int_0^A x^2 \cdot dA$$

◆ Product of Inertia ($I \times y$) :

It is moment of area about both axis

$$I_{xy} = I_{xysey} = \int_0^A x \cdot y \cdot dA$$

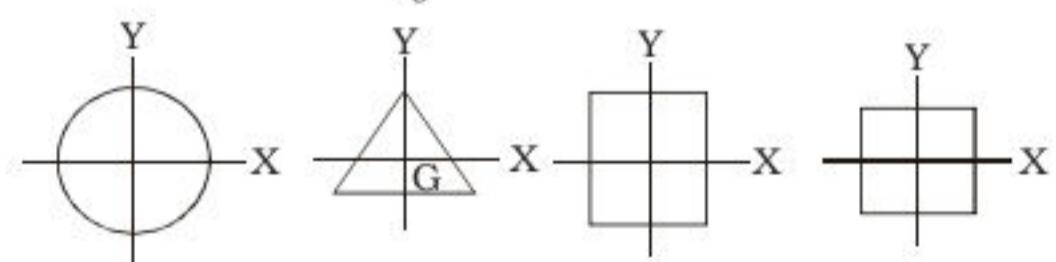
It may be > Zero, = Zero, < Zero.

If xx and yy are centroid axis then :

I_{xy} = Centroidal product of Inertia

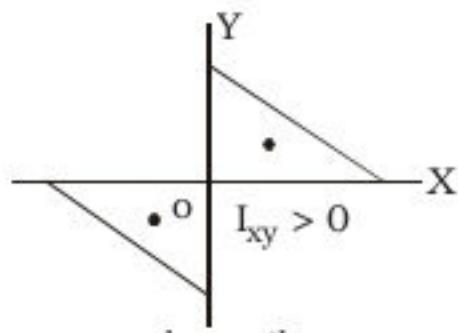
Note :

If area is symmetrical either about x or about y or about both then; $I_{xy} = 0$



Here, $I_{xy} = 0$

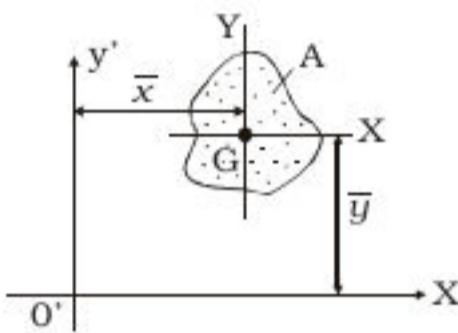
- 2) If area is located in 1st or 3rd quadrant, then I_{xy} is (+ve).



3) If area is located in 2nd or 4th quadrant, then $I_{xy} < 0$ (-ve)

◆ **PARALLEL AXIS THEOREM :**

Let xx and yy are centroidal axis of area A and let $x'x'$ and $y'y'$ area II^{lr} axis to xx and yy respective then ;



I_{xy} = Self product of Inertia

$$= \int x \cdot y \cdot dA$$

$$I_{xx'} = I_{xx} + A\bar{y}^2$$

$$I_{y'y'} = I_{yy} + A\bar{x}^2$$

$$I_{xx} = I_{xself} = \text{Self M.I}$$

$$= \int x^2 \cdot dA$$

I_{yy} = self M.I about xx .

$$= \int y^2 \cdot dA$$

I_{yy} = Self M.I about xx

$$= \int x^2 \cdot dA$$

◆ **PERPENDICULAR AXIS THEOREM :**

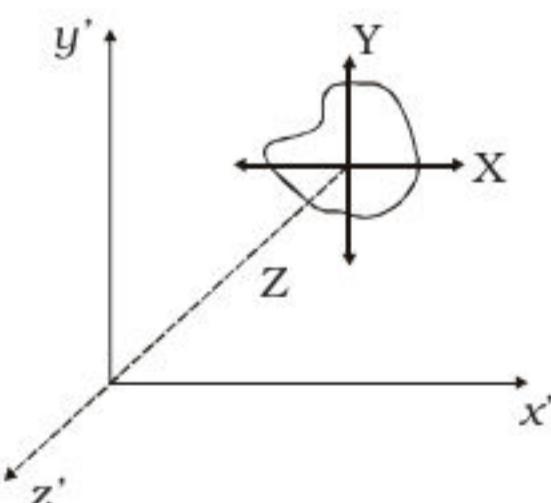
Let xx and yy are transverse centroidal axis and zz is longitudinal or polar centroidal axis which \perp^{lr} to xx and yy .

Hence :

$$I_{zz} = I_{xx} + I_{yy}$$

similarly,

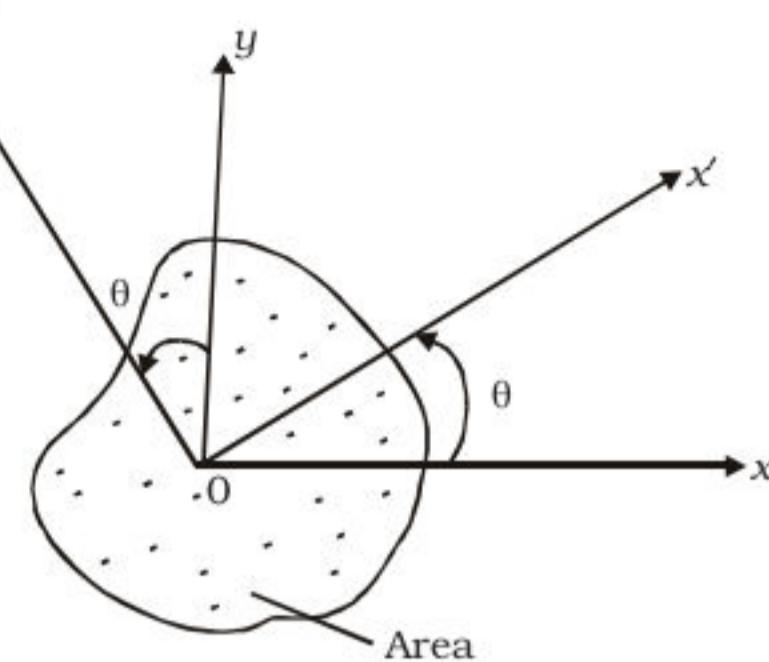
$$Iz'z' = Ix'x' + Iy'y'$$



◆ **Transformation of M.I :** (Tens or quantity can be transformed from one axis to other).

It xx and yy are centroidal axis of area A and I_{xx} and

I_{yy} are M.I about xx and yy and let I_{xy} is product of Inertia about xy .



Let x' and y' axis are inclined to x and y axis as shown in figure.

Let $I_{x'x'}$ and $I_{y'y'}$ are M.I about x' and y' axis respective and I_{xy} is product of Inertia about transformed axis

$$\therefore I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{xx} - I_{yy}}{2} \right) \cos 2\theta - I_{xy} \sin 2\theta$$

$$\text{Put } \theta = \theta + 90^\circ$$

$$I_{y'y'} = \left(\frac{I_{xx} + I_{yy}}{2} \right) - \left(\frac{I_{xx} - I_{yy}}{2} \right) \cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin \theta + I_{xy} \cos \theta$$

Note :

1. If the angle θ is such that product of Inertia about x' and y' is zero, then such axis are called Principal axis. Hence, $I_{x'x'}$ and $y'y'$ are called principal moment Inertia.

Principal moment of Inertia are either maximum or minimum M.I.

2. The angle of principal axis from $x'x$ and yy are given by :

$$I_{x'y'} = 0$$

$$\tan 2\theta = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$

3. The magnitude of principal M.I is given by

$$I_1/I_2 = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2} \right)^2 + (I_{xy})^2}$$

4. If an area has axis of symmetry, then product of Inertia about those axis will be zero. Hence, axis of symmetry is always principal axis. The reverse is not compulsory.

5. Circle has infinite principal axis.

◆ **TRUSSES:**

For trusses the total number of unknown include the forces in ' m ' number of bars of the truss and the total ' r ' of support reactions,

STRENGTH OF MATERIALS

Since the truss members are assumed to carry only axial forces and lie in the same plane, the force system acting at each joint is coplanar and concurrent. Thus it is only necessary to satisfy $\sum F_x = 0$ and $\sum F_y = 0$. If j number of joints are available, the total number of equations available for solution is $2j$. This determinacy of the trusses can be specified by comparing $(m + r)$ with the total number of available of equilibrium. We have

$$m + r = 2j \quad \text{statically determinate}$$

$$m + r > 2j \quad \text{statically indeterminate}$$

For truss analysis it is assumed that:

- Bars are pin connected
- Joints are frictionless hinges
- Loads are applied at the joint only
- Stresses in each member is constant along its length.

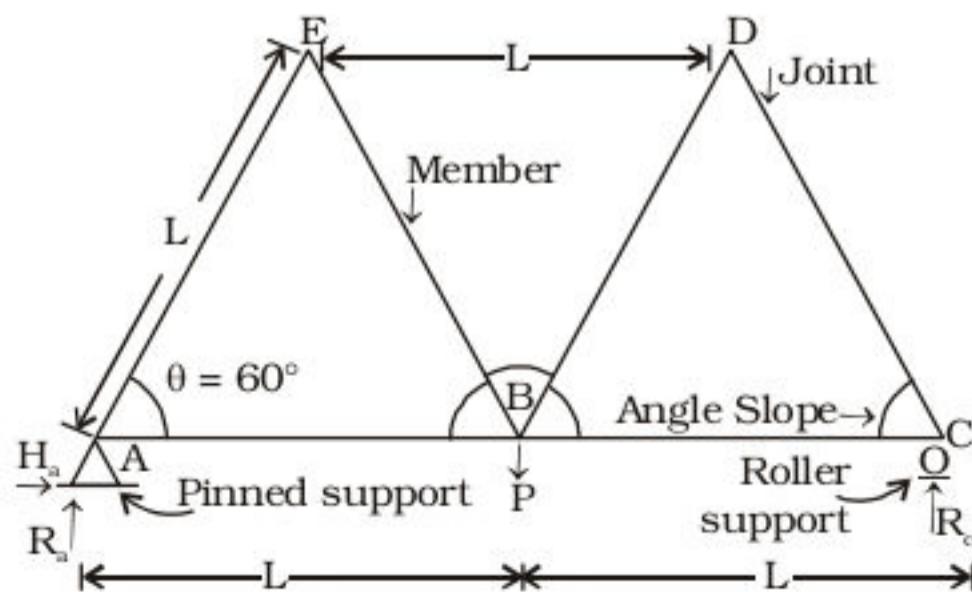
The objective of truss analysis is to determine the reactions and member forces. The method used for carrying out the analysis with the equations of equilibrium and by considering only parts of the structure through analyzing its free body diagram to solve the unknowns.

(1) METHOD OF JOINTS FOR TRUSS ANALYSIS:

We start by assuming that all members are in tension reaction. A tension member experiences pull forces at both ends of the bar and usually denoted by positive (tve) sign. When a member is experiencing a push force at both ends, the bar is said to be in compression mode and designated as negative sign.

In the joints methods, a virtual cut is made around a joint and the cut portion is isolated as a free body diagram. Using the equilibrium equations of $\sum F_x = 0$ and $\sum F_y = 0$, the unknown member forces can be solved. It is assumed that all members are joined in the form of an ideal pin, and that all forces are in tension (the reactions).

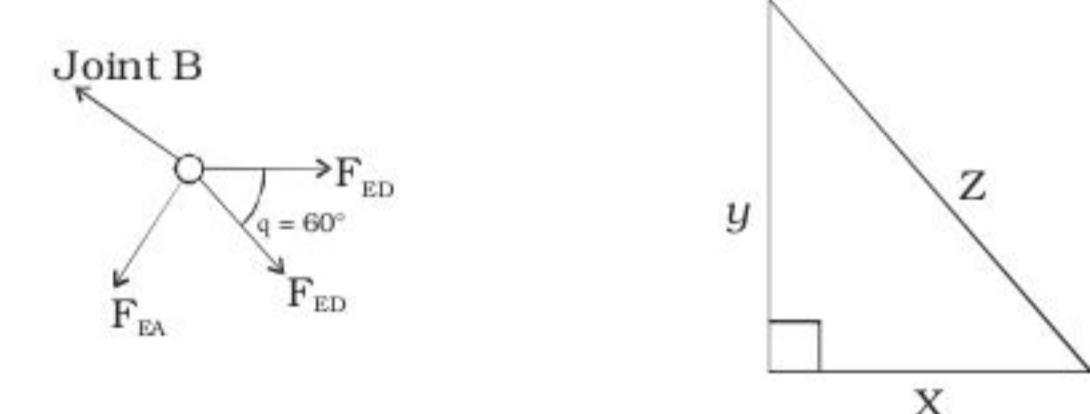
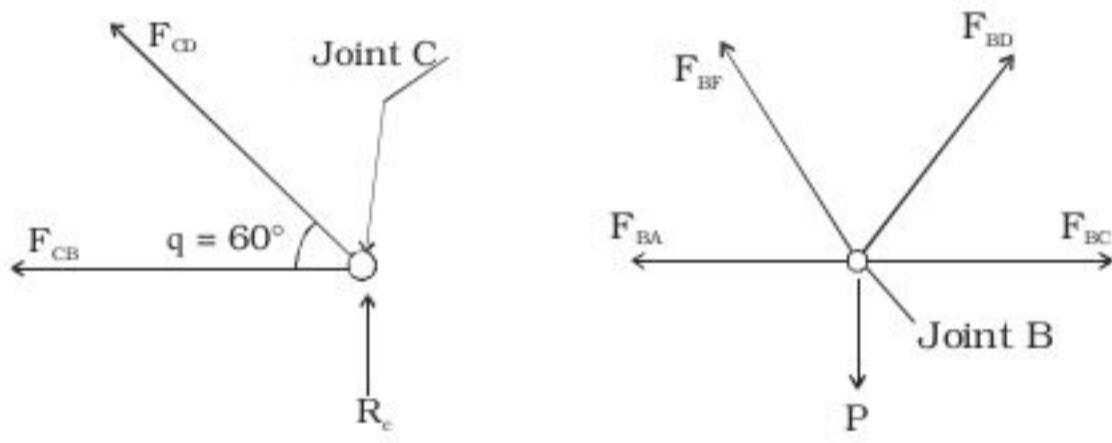
An imaginary section may be completely passed around a joint in a truss. The joint has become a free body in equilibrium under the force applied to it. The equation $\sum H = 0$ and $\sum v = 0$ may be applied to the joint to determine the unknown forces in members meeting there. It is evident that no more than two unknowns can be determined at a joint with these two equations.



A simple truss model supported by pinned and roller support at its end. Each triangle has the same length, L and it is equilateral where degree of angle, θ is 60° on every angle. The support reaction R_a , R_c can be determined by taking a point of moment either at point A or point C, whereas $H_a = 0$ (no other horizontal force).

Here are some simple guidelines for this methods:

- (i) Firstly draw the free body diagram (FBD)
- (ii) Solve the reaction of the given structure.
- (iii) Select a joint with a minimum number of unknown (not more than 2) and analyse it with $\sum F_x = 0$ & $\sum F_y = 0$
- (iv) Proceed to the rest of the joints and again concentrating on joints that have very minimal of unknowns.
- (v) Check number of forces at unused joints with $\sum F_x = 0$ & $\sum F_y = 0$
- (vi) Tabulate the member forces whether it is in tension (tve) or compression (-ve) reaction



The figure showing 3 selected joints at B, C and E. The force in each member can be determined from any joint or point. The best way to start is by selecting the earliest joint like joint Σ where the reaction R_c is already obtained and with only 2 unknown, forces of F_{CB} and F_{CD} . Both can evaluated with $\sum F_x = 0$ and $\sum F_y = 0$ rules. At joint E, there are 3 unknown, forces of F_{EA} , F_{EB} and F_{ED} , which may lead to more complex solution compared to 2 unknown values. For checking purpose joint B is selected to show that the equation of $\sum F_x$ is equal to $\sum F_y$ which leads to zero value $\sum F_x = \sum F_y = 0$. Each members condition should be indicated clearly as whether it is in tension (tve) or in compression (-ve) state).

Trigonometric Functions:

Taking an angle between member X and Z

$$\cos \theta = \frac{X}{Z}$$

$$\sin \theta = \frac{y}{z}$$

$$\tan \theta = \frac{y}{x}.$$

5. Strain Energy Method [Castigianos 2nd theorem]

Assumption :

- i) Material is homogeneous, isotropic & linear elastic in which Hooks law is valid
- ii) Temperature is constant
- iii) Supports are un-yielding

According to this theorem;

"The deflection in any beam, truss or frame at any point in the direction of the force is equal to 1st partial derivative of total S.E w.r.t. that force"



$$\delta_B = \frac{\partial u}{\partial P}$$



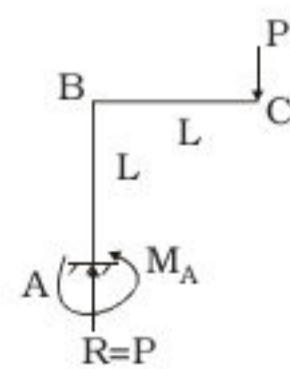
$$\Delta h = \frac{\partial u}{\partial w}$$



$$\theta_B = \frac{\partial u}{\partial m}$$



$$\theta = \text{Twist angle at } B = \frac{\partial u}{\partial T}$$



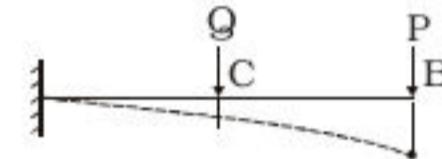
$$\delta_C = \frac{\partial u}{\partial P}$$

Note :

1. The (positive) slope & deflect are those which are in the direction of moment & load respectively
2. If u is only due to BM, then deflection/slope is caused by BM only & if u is function B.M, S.F & Axial force, then deflection is under the combined effect of (BM, S.F & A.F)
3. Special case (a)

Let load is applied as B but deflection is required at other point C, then apply a pseudo load in the direction of desired deflection

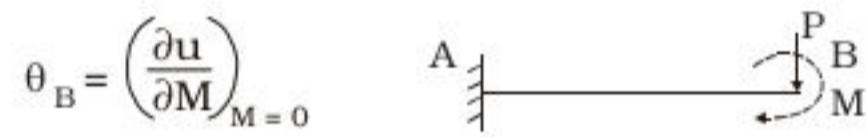
Say, pseudo load is Q



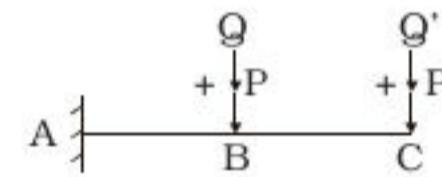
The vertical deflection at C is given by

$$\Delta_C = \left(\frac{\partial u}{\partial Q} \right)_{Q=0}; u = f_n (P \& Q)$$

Special case 1. (b). If slope at any point B is required then apply a pseudo moment at B
Then,



Special case 1. (c).



$$\begin{aligned}\delta_B &= P + Q \\ \delta_C &= P + Q'\end{aligned}$$

4. This method is suitable for prismatic & non prismatic members both & suitable for cantilever frames & arches.

STRAIN ENERGY :

1. Due to B.M $u = \int \frac{M_x^2 ds}{2EI}$

2. Due to Axial force $u = \int \frac{P_x^2 dx}{2AE}$

3. Due to Torque $u = \int \frac{T_x^2 dx}{2GI_p}$

4. Due to S.F $u = \int \frac{S^2 dx}{2A_r G}$

where ;

ds = Any elementary length

EI = Flexural Rigidity

AE = Axial Rigidity

GI_p = Torsional rigidity

A_r = Reduced area in shear

A_r.G = Shear Rigidity



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