THEORY OF STRUCTURES

SIMPLE STRESS

- Strength of materials deals with the relations between externally applied loads and their effects on bodies.
- If the x-axis is normal to the section, the section is known as the x-surface or, more briefly, the x face.
- The first subscript denotes the face on which the component acts; the second subscript indicates the direction of the particular component thus P is the force on the x fase acting in the y direction.
- The purpose of studying strength of materials is to ensure that the strutures used will be safe against the maximum internal effects that may be produced by any combination of loading.
- Density of steel = 7850 kg/m^3
- Young modules of elasticity of steel = 200 GPa = 2×10^5 MPa.
- Rigidity modulus (G) = 80 GPa = 0.8×10^5MPa.
- Stress, $\sigma = \dfrac{P}{A}$ or $\dfrac{dP}{dA}$
- One pascal = 1 N/m^2
- PSI = Pond/inch^2
- KIP = Kilo Pound
- KSI = 1000 pond/inch
- PSf = Pond/ft^2
- The condition under which the stress is constant or uniform is know as **simple stress**.

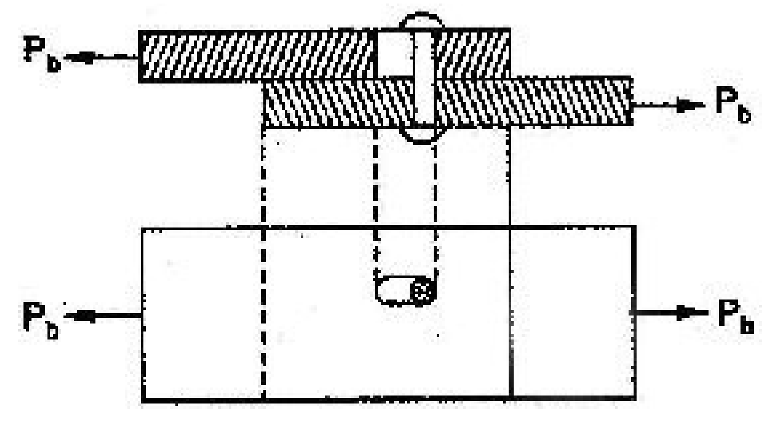
  Shearing stress or Tangential stress : Shreaing stress is caused by force acting along or parallel to the area resisting the forces.

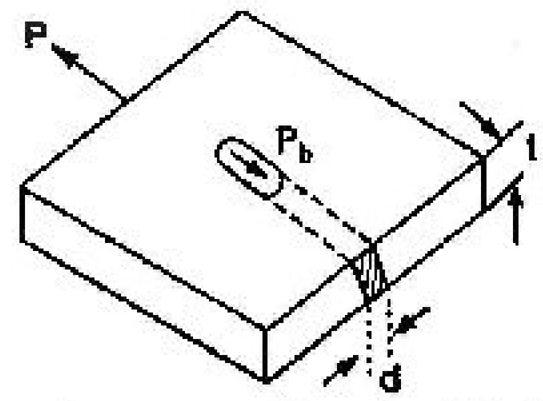
  - The shear occurs over an area parallel to the applied load. This called **direct shear** in contrast to the induced shear.
  - A uniform shearing stress will exist when the resultant shearing force V passes through the centroid of the cross section being sheared.

$$\tau = \frac{V}{A}$$

Bearing Stress

- Bearing stress differs from compressive stress in that the compress stress is the internal stress caused by a compressive force, whereas the bearing stress is a contact pressure between separate b....
- The result of an excessive bearing stress is to cause yielding of the plate or the rivet or both.





The bearing stress σ_b uniformly distributed over a reduced area equal to the projected area of the riset hole.

$$P_b = A_b\sigma_b = (td)\sigma_b$$

STRESS AND STRAIN IN TWO DIMENSIONS

Stress at a Point :

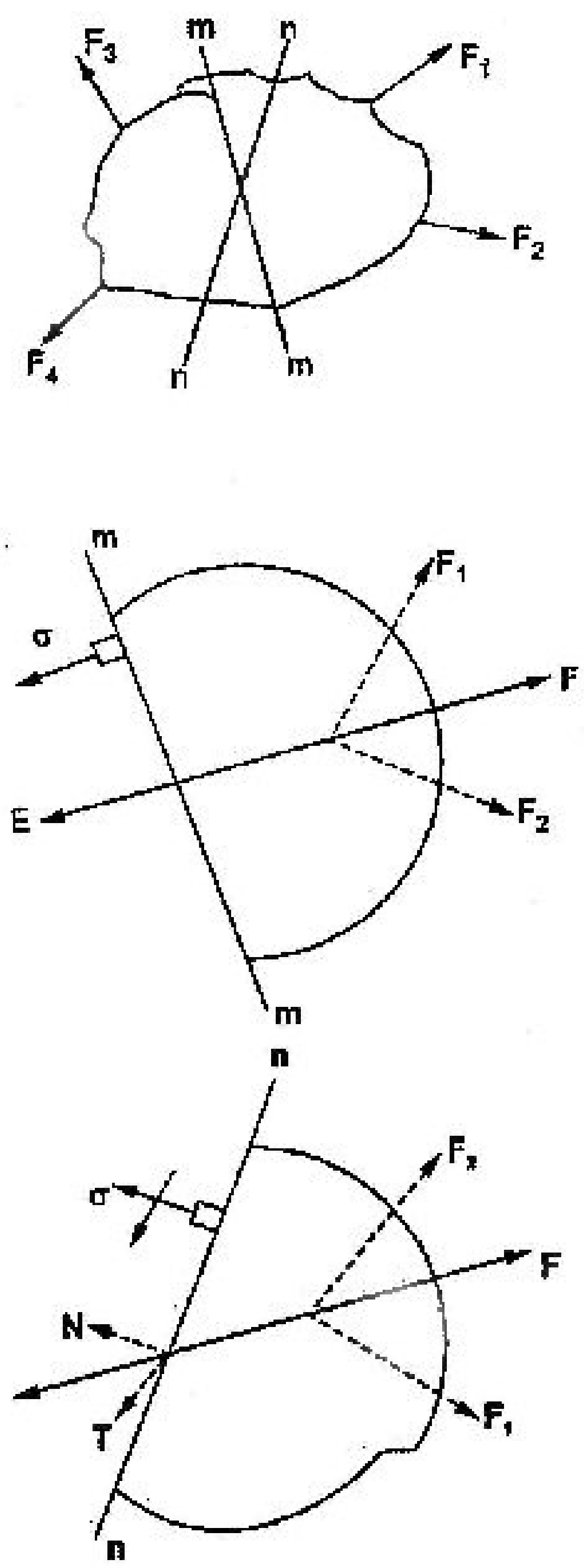
The average stress over an area is obtained by dividing the force by the area over which it acts.

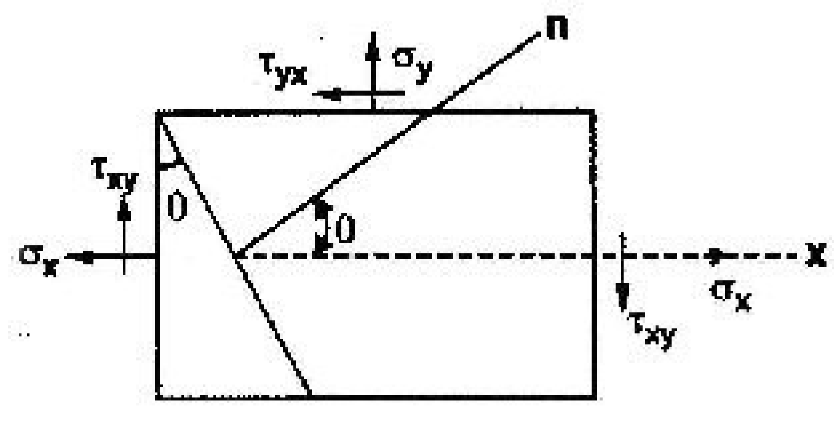
If the average stress is constant over the area, the stress is said to be uniform.

If the stress is not uniform, then stress at a point is determined. So, stress at a point defines the uniform stress distributed over a differential area (or permitting the area enclosing the point to approach zero as a limit).

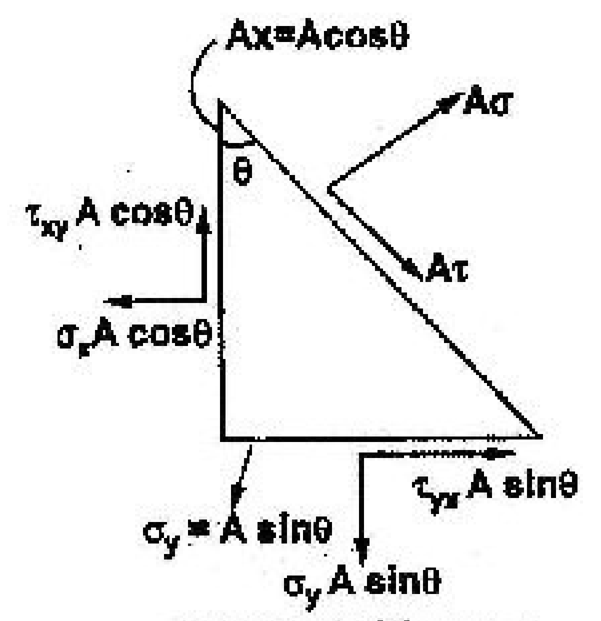
Variation of Stress with Inclination of Element :

Magnitude and type of stress depend on the inclination of an element.





Original State of Stress



Free Body Diagram

There are two sections m-m and n-n passing through the body on which several forces acts. Section m-m is perpendicular to the resultant F of F_1 and F_2 and section *n-n* is inclinded to F.

Thus, element in Fig. (b) is subjected only to a normal stress while element in Fig. (*c*) is subjected to both normal and shearing stresses.

Note: Thus, at same position in a stressed body, the stresses on an element vary with the orientation of the element.

Methods to Determine the position and Magnitude of Maximum Normal and Shearing Stresses:

1. Analytical Method
2. Graphical Method (Based on Mohr's circle)

1. Analytical Method:

When stress variations are determined analytically, a plane is passed that cuts the original element into two parts and conditions of equilibrium are applied to either part.

Figure shows the normal and shearing stress components acting on the plane whose normal *n* makes an angle θ with the X-axis. A is the area of inclined faces.

Applying the conditions of equilibrium to axis choosen as in FDB

$$\Sigma F_n = 0$$

or $$A\sigma = (\sigma_x A\cos\theta)\cos\theta + (\sigma_y A\sin\theta)\sin\theta - (\tau_{xy} A\cos\theta)\sin\theta - (\tau_{xy} A\sin\theta)\cos\theta$$

and $$\Sigma F_t = 0$$

or $$A\tau = (\sigma_x A\cos\theta)\sin\theta - (\sigma_y A\sin\theta)\cos\theta + (\tau_{xy} A\cos\theta)\cos\theta - (\tau_{xy} A\sin\theta)\sin\theta$$

On simplification, we get

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \quad ...(i)$$

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad ...(ii)$$

Planes defining maximum or minimum normal stresses are

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad ...(iii)$$

Plane of maximum shearing stress is

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad ...(iv)$$

Note :

(i) From equation (*iii*) and (*iv*), we can conclude that planes on which maximum and minimum normal stresses occur are 90° apart.

(ii) Also planes on which the maximum on plane shearing stress occurs are also found to be 90° apart.

(iii) Also maximum and minimum normal stresses occur on plane of zero shearing stress.

$$\tan 2\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$$

(iv) The planes of maximum in-plane shearing stress are inclined at 45° with the principal planes.

(v) Maximum and minimum normal stresses are called the principal stresses and the planes on which they act are called principal planes.

Principal Stresses :

$$\sigma_1 \text{ or } \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \quad ...(v)$$

Maximum Shearing Stress :

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \quad ...(vi)$$

Also

$$\tau_{max} = \pm\frac{\sigma_1 - \sigma_2}{2} \quad ...(vii)$$

2. Graphical Method (Mohr's Circle) :

From Previous equations (*i*) and (*ii*), we can write it in a form of circle as

$$\sigma - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \quad ...(i)$$

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad ...(ii)$$

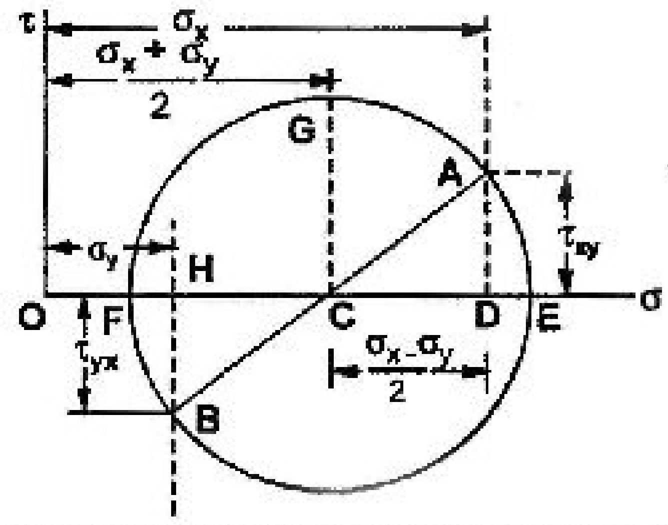
By squaring the equations and simplifying, we get

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2 \quad ...(iii)$$

It is the form of $(\sigma - \tau)^2 + \tau^2 = R^2$, *i.e* of a circle.

Here radius, $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$

Centre is offset a distance $C = \frac{\sigma_x + \sigma_y}{2}$, rightward from the origin. The figure of Mohr's circle is given below.



RULE FOR DRAWING MOHR'S CIRCLE

1. On rectangular $\sigma - \tau$ axis, plot the points having co-ordinates $A(\sigma_x, \tau_{xy})$ and $B(\sigma_y, \tau_{yx})$. Assume tension as plus, compression as minus, and shearing stress as plus when its moment about centre is clockwise.
2. Draw a straight line joining these points which gives the diameter of a circle whose centre is an σ-axis.
3. The radius of the circle to any point on its circumference represents the axis directed normal to the plane whose stress components are given by the co-ordinates of that point.
4. The angle between the radii to selected points on Mohr's circle is twice the angle between the normals to the actual planes represented by these points.

ABSOLUTE MAXIMUM SHEARING STRESS

The maximum shearing stress is not necessarily equal to the maximum in-plane shearing stress.

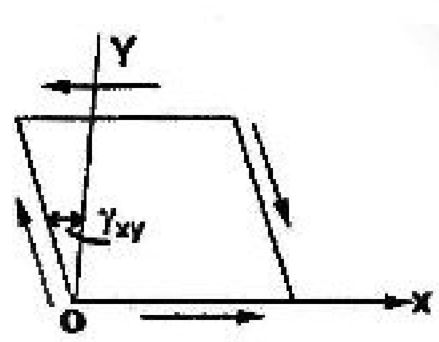
Calculation is done as:

1. If σ_1 and σ_2 have the same sign (both tension or both compression) the absolute maximum shearing stress equals either $\frac{|\sigma_1|}{2}$ or $\frac{|\sigma_2|}{2}$ whichever is larger.
2. If σ_1 and σ_2 have different signs the absolute maximum shearing stress equals the maximum in-plane shearing stress $\frac{|\sigma_1 - \sigma_2|}{2}$.
3. The maximum shearing stress again equals the radius of the largest Motor's circle, that is the largest of the following values :

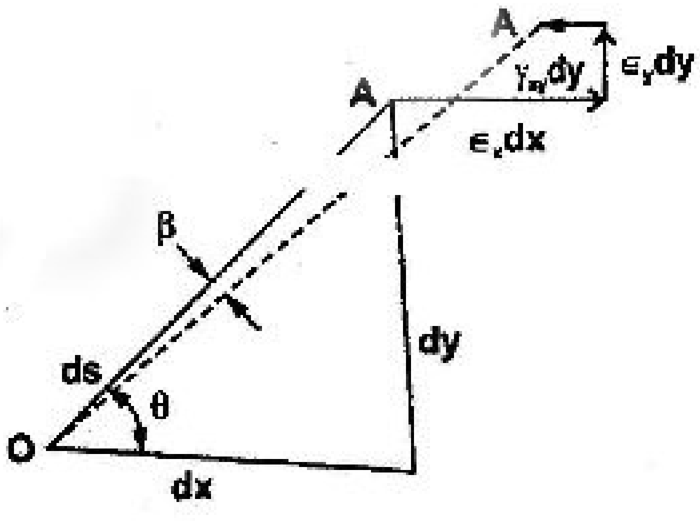
$$\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}$$

TRANSFORMATION OF STRAIN COMPONENTS

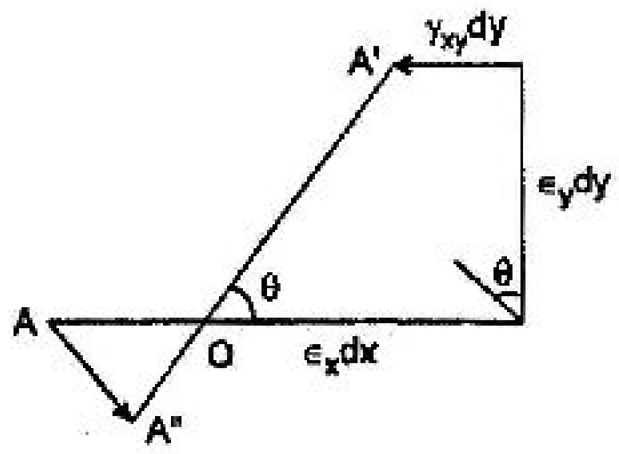
The shearing stress distort the element through the shearing strain γ_{xy} as shown.



(a)



(b)



(c)

The effect of these strain is shown in Fig. (*b*). The movement of A is exaggerated in Fig. (*c*).

The parallel component A″A' represent the increase in length of OA whereas the perpendicular component AA" causes the change β in the angular position of OA.

$$A''A' = \epsilon_x\, dx \cos\theta + \epsilon_y\, dy \sin\theta - \gamma_{xy}\, dy \cos\theta \quad ...(i)$$

$$\epsilon_a = \frac{A''A'}{ds}$$

$$= \frac{\epsilon_x dx \cos\theta}{ds} + \frac{\epsilon_y\, dy \sin\theta}{ds} - \frac{\gamma_{xy} dy \cos\theta}{ds} \quad ...(ii)$$

$$\frac{dx}{ds} = \cos\theta \text{ and } \frac{dy}{ds} = \sin\theta$$

$$\epsilon_a = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta - \gamma_{xy} \sin\theta \,.\, \cos\theta$$

$$\text{Hence } \epsilon_a = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$AA'' = \epsilon_x\, dx \sin\theta - \epsilon_y\, dy \cos\theta - \gamma_{xy}\, dy \sin\theta$$

$$\beta = \frac{AA''}{ds} = \epsilon_x \sin\theta \,.\, \cos\theta - \epsilon_y \sin\theta \,.\, \cos\theta - \gamma_{xy}\sin^2\theta$$

$$\beta' = -\epsilon_x \sin\theta \,.\, \cos\theta + \epsilon_y \sin\theta \,.\, \cos\theta - \gamma_{xy}\cos^2\theta$$

$$\gamma_{ab} = \beta - \beta'$$

$$\frac{1}{2}\gamma_{ab} = \frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta + \frac{1}{2}\gamma_{xy}\cos 2\theta$$

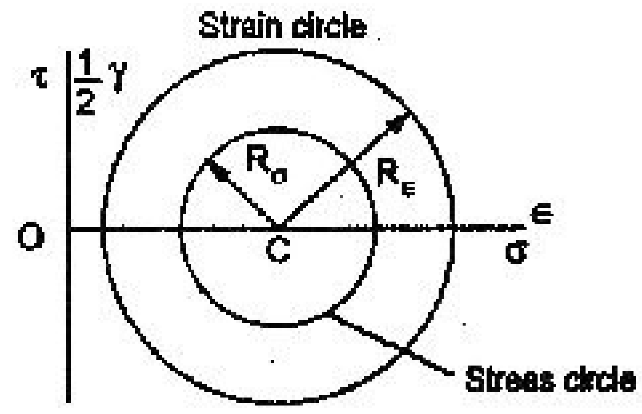
Note : Normal and shearing strains can be represented by a Mohr's circle for strain, constructed in the same manner as Mohr's circle for stress except that half values of shearing strain are plotted instead of shear stress.

A Mohr's circle for strain can be transformed into a concentric Mohr's circle for stress by means of the scale transformations.

$$R_\sigma = R_\epsilon \left(\frac{E}{1+v}\right)$$

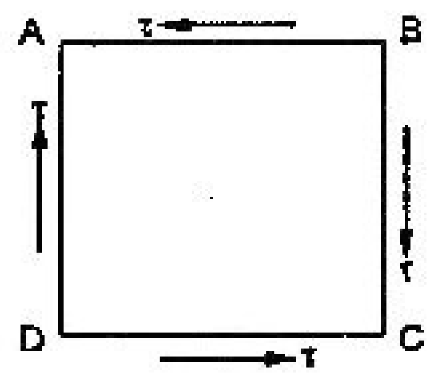
$$(OC)_\sigma = (OC)_\sigma \frac{E}{1-v}$$

where, R_σ and R_ϵ are respectively the radii of the stress and strain circles in the figure $(OC)_\sigma$ and $(OC)_\epsilon$ are respectively the stress and strain co-ordinates of the centres of the concentric circles.

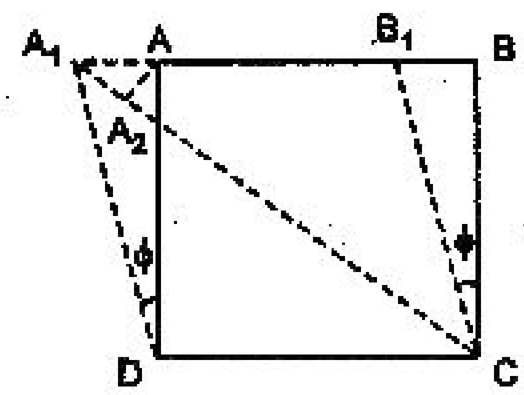


RELATION BETWEEN ELASTIC CONSTANTS

Relation between E and C :



(a)



(b)

Consider a square block of unit thickness and side a is subjected to a shear stress τ Fig. (*b*) is the shape after deformation. Diagonal AC is elongated and BD is shortened.

Strain of AC = Strain in the length of AC due to diagonal tensile stress on the plane BD + strain in length of AC due to diagonal compressive stresses on the plane.

$$= \frac{\tau}{E} + \frac{\tau}{mE} = \frac{\tau}{E}\left(1 + \frac{1}{m}\right)$$

where, $\frac{1}{m}$ = Poisson's ratio

From geometry :

Increase in length of the diagonal AC = $A_1C - AC$

Let AA_2 be perpendicular to A_1C or AC = A_2C

$\therefore$ Increase in length of the diagonal

$$AC = CA_1 - CA_2 = A_1A_2 = AA_1 \cos AA_1A_2$$

But $\angle AA_1A_2 \approx \angle BAC = 45°$

$\therefore$ Increase in length of the diagonal AC

$$= AA_1 \cos 45° = \frac{AA_1}{\sqrt{2}}$$

But shear strain = ϕ

$$= \frac{AA_1}{AD} = \frac{AA_1}{a}$$

or $AA_1 = C\phi$

$\therefore$ Increase in AC = $\dfrac{a\phi}{\sqrt{2}}$

Length of the diagonal AC = $a\sqrt{2}$

Strain of diagonal AC = $\dfrac{a\phi}{\sqrt{2}} \cdot \dfrac{1}{a\sqrt{2}} = \dfrac{\phi}{2}$

But $\dfrac{\phi}{\tau}$ = C = modulus of rigidity $\phi = \dfrac{\tau}{C}$

$\therefore$ Strain of diagonal = $\dfrac{\phi}{2} = \dfrac{\tau}{E}\left(1 + \dfrac{1}{m}\right)$

$$E = 2C\left(1 + \frac{1}{m}\right)$$

Relation between E, K and C :

$$E = \frac{9\,KC}{3K + C}$$

Poisson's Ratio, Biaxial and Triaxial Deformations:

If a bar is subjected to axial tension, there is a reduction in the transverse dimensions. According to Poisson's the ratio of the unit deformation or strain in these directions is constant for stresses within the proportional limit. Poisson's Ratio is denoted as ν.

$$\nu = \frac{1}{m}$$

$$\nu = \frac{\epsilon_y}{\epsilon_x} = \frac{\epsilon_z}{\epsilon_x}$$

Poisson's ratio permits to extend Hooke's law of uniaxial stress to the case of biaxial stress.

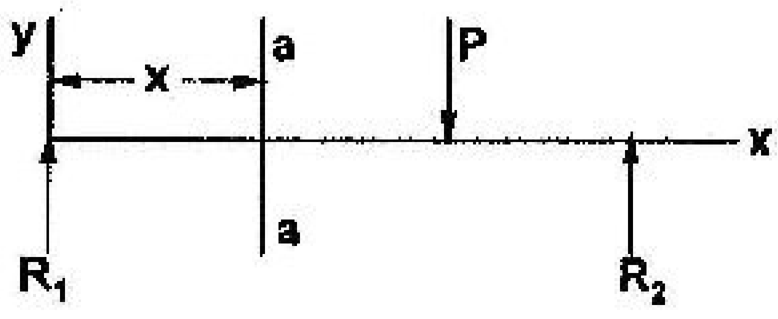
Thus,
$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E}$$

Also,
$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned}\right\}$$

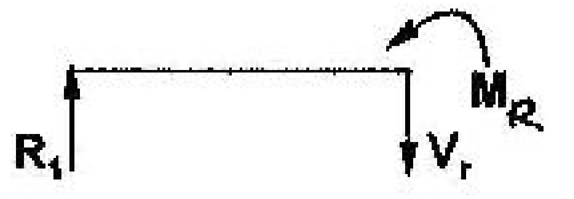
Note : Common value of υ are 0.25 to 0.30 for steel, approximately 0.33 for most other metals, and 0.20 for concrete.

BENDING MOMENTS AND SHEAR FORCE

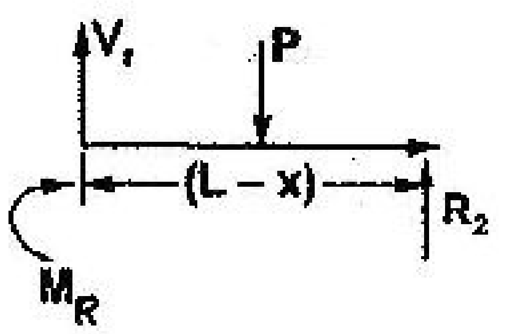
Shearing Force:



(a)



(b)



(c)

Consider a beam carries a concentrated load P and it is in equilibrium under the reactions R_1 and R_2. Assume a cutting plane *a-a* at a distance *x* from R_1. The F.B.D. shows that the externally applied load is R_1. To maintain equilibrium in this segment, the fibre in section *a-a* must supply the resisting forces necessary to satisfy the equilibrium conditions. Here external load is vertical so $\Sigma F_x = 0$ is automatically satisfied.

To satisfy $\Sigma F_y = 0$, the vertical unbalance caused by R_1 requires the fibre in section *a-a* to create a resisting force. This is V_r called the resisting shearing force.

For loading V_r is numerically equal to R_1,but if additional load had been applied between R_1 and section *a-a* the net vertical unbalance would be found from the summation of their vertical components. This net vertical unbalance is called shearing force in the beam.

Thus, it can be expressed mathematically as

$$V = (\Sigma F_y)_L$$

L is the vertical summation includes only the external loads acting on the beam segment to the left of the section being considered.

Bending Moment:

For complete equilibrium of the free body diagram, the summation of moments must also balance. Here R_1 and V_r are equal, thereby producing a couple M that is equal to R_1x and is called the bending moment because it tends to bend the beam. Mathematically

$$M = (\Sigma M)_L = (\Sigma M)_R$$

It is defined as summation of moments about the centroidal axis of any selected section of all the loads acting either to the left or to the right side of the section.

Sign of Bending Moment:

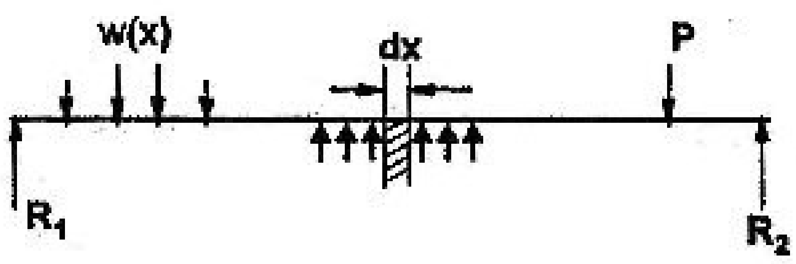
Upward acting external forces cause positive bending moment with respect to any section, downward forces cause negative bending moments.

positive bending *negative bending*

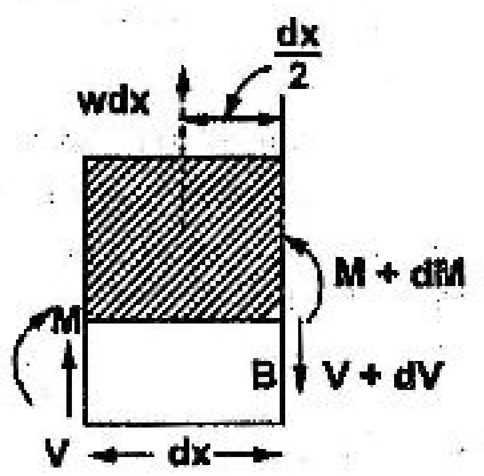
Note : We never think about whether a moment is clockwise or counterclockwise, upward acting forces always cause positive bending moments regardless of whether they act to the left or to the right of the exploratory section.

Relation Between Load, Shear and Moment:

Consider a beam carrying an arbitrary loading. The free body diagram of a segment of this beam of length dx is shown in Fig. (*b*).



(a)



(b)

Applying the conditions of static equilibrium, $\Sigma F_y = 0$

$\therefore \quad V + wdx - (V + dV) = 0$

or $\quad dV = wdx \quad ...(i)$

From a moment summation about point B, we have

$$M + Vdx + (wdx)\frac{dx}{2} - (M + dM) = 0$$

Neglecting square term,

$$dM = Vdx \quad ...(ii)$$

Integrating equation (*i*), we get

$$\int_{v_1}^{v_2} dv = \int_{x_1}^{x_2} wdx$$

or $\quad V_2 - V_1 = \Delta V = (\text{Area})_{\text{Load}} \quad ...(a)$

Integrating, we get

$$\int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} Vdx$$

or $\quad M_2 - M_1 = \Delta M = (\text{Area})_{\text{Shear}} \quad ...(b)$

Also, $\quad w = \dfrac{dV}{dx}$ = slope of shear diagram ...(*c*)

$V = \dfrac{dM}{dx}$ = slope of moment diagram ...(*d*)

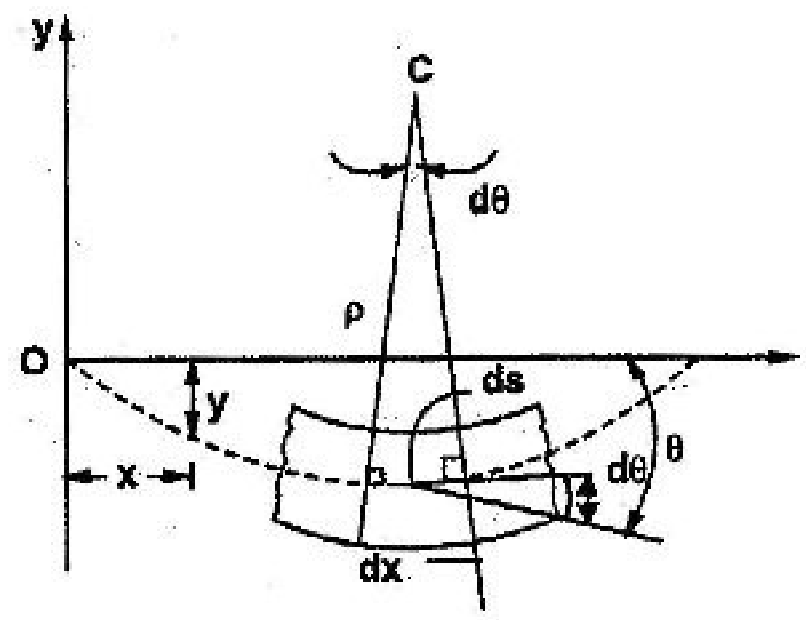
BEAM DEFLECTIONS

The design of a beam is determined by its rigidity rather than by its strength.

Methods to Determine Beam Deflections:

Double Integration Method:

Elastic curve-The edge view of the neutral surface of a deflected beam is called the elastic curve of the beam.



The deflections are assumed to be so small that there is no appreciable difference between the original length of the beam and the projection of its deflected length.

$$\tan\theta \approx \theta = \frac{dy}{dx}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

It is evident that, $\quad ds = \rho d\theta$

or $$\frac{1}{\rho} = \frac{d\theta}{ds} \approx \frac{d\theta}{dx}$$

or $$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

But from flexure formula, $\frac{1}{\rho} = \frac{M}{EI}$

or $$EI\frac{d^2y}{dx^2} = M \qquad ...(i)$$

EI is called the flexural rigidity of the beam is constant along the beam.

If equation (i) is integrated, we get

$$EI\frac{dy}{dx} = \int M dx + C_1$$

This is slope equation specifying the slope or value of $\frac{dy}{dx}$ at any point.

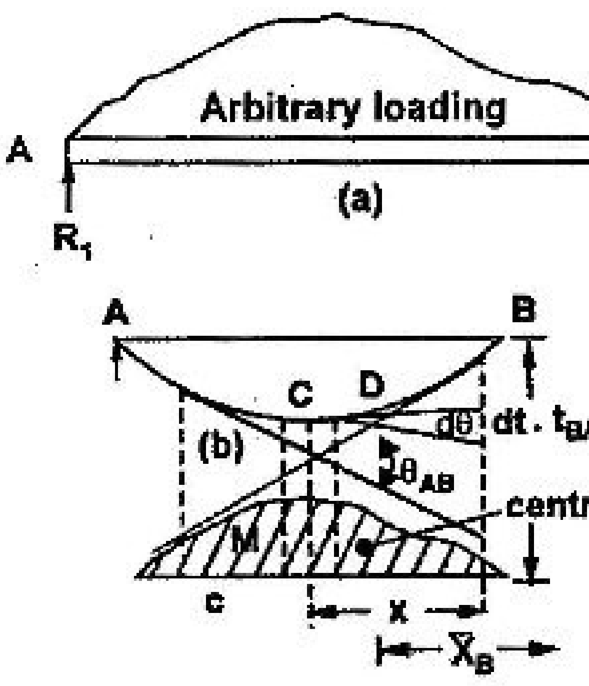
Integrating again, we have

$$EIy = \iint M dx + C_1 x + C_2$$

This is the required equation of the elastic curve.

Area Moment Method : In this method of determining slopes and deflections in beams involves the area of the moment diagram and also the moment of that area.

Let $d\theta$ be the angle of deflection between points C and D on the beam.



We know $$\frac{1}{\rho} = \frac{M}{EI} \qquad ...(i)$$

Since $$ds = \rho d\theta$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} = \frac{M}{EI}$$

or $$d\theta = \frac{M}{EI} ds \qquad ...(ii)$$

$\therefore$ $$d\theta = \frac{M}{EI} dx$$

or $$\theta_{AB} = \int_{\theta_A}^{\theta_B} d\theta = \frac{1}{EI}\int_{x_A}^{x_B} M dx \qquad ...(iii)$$

$$dt = x d\theta$$

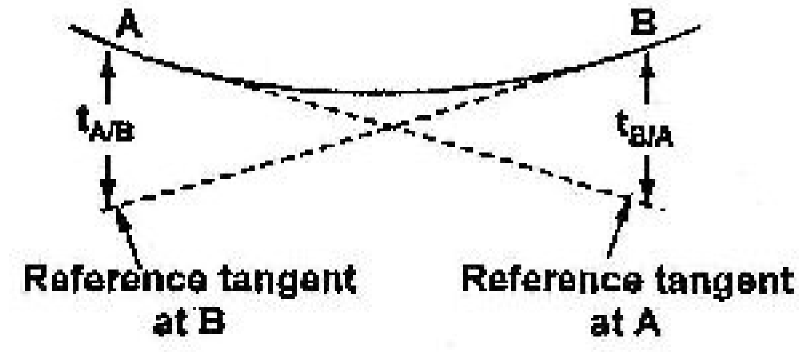
$$t_{B/A} = \int dt = \int x d\theta$$

$$t_{B/A} = \frac{1}{EI}\int_{x_A}^{x_B} x(M dx) \qquad ...(iv)$$

The length $t_{B/A}$ is known as the deviation of B from a tangent drawn at A.

or $$\boxed{\theta_{AB} = \frac{1}{EI}(\text{area})_{AB}}$$

or $$\boxed{t_{B/A} = \frac{1}{EI}(\text{area})_{BA} . \bar{x}_B}$$



where, $\bar{x}_B$ = Moment arm of the area measured from B.

Sign Convention: The deviation at any point is positive if the point lies above the reference tangent from which the deviation is measured, and negative if the point lies below the reference tangent.

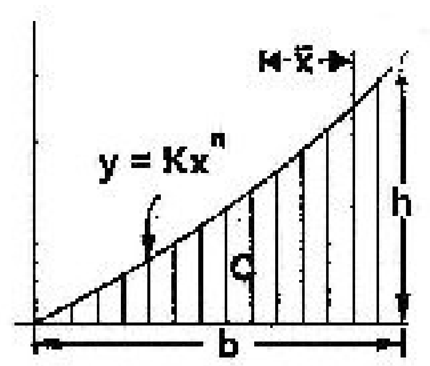
Moment Diagram By Parts: To apply the theorems of the area-moment method, we should be able to compute easily and accurately the area under any part of a moment diagram, and also the moment of such an area about any axis.

There are two principles

1. The resultant bending at any section caused by any load system is the algebric sum of the bending moments at that section caused by each load acting separately.

$$M = (\Sigma M)_L = (\Sigma M)_R$$

2. The moment effect of any single specified loading is always some variation of the general equation.



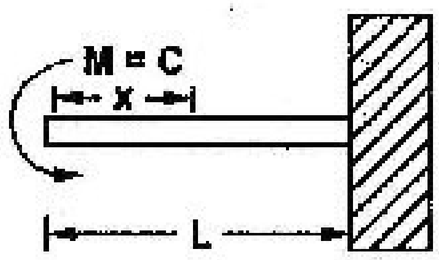
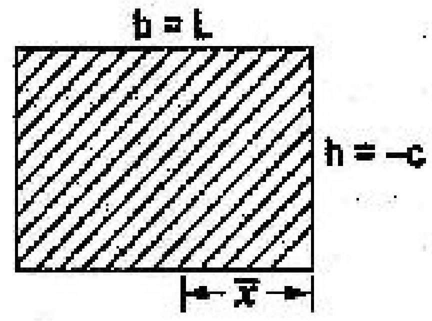
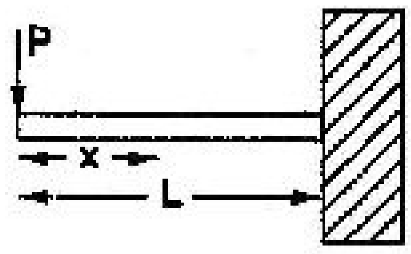
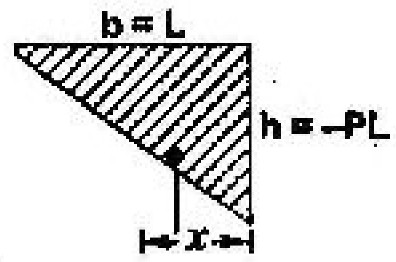
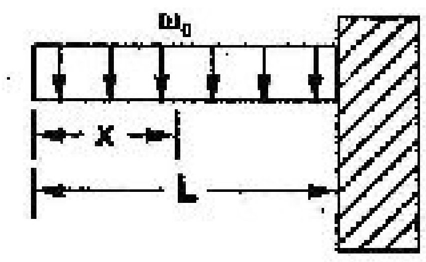
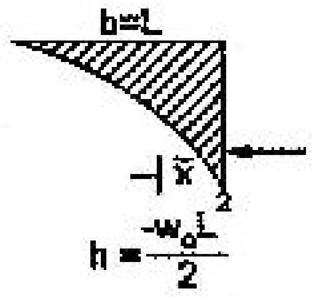
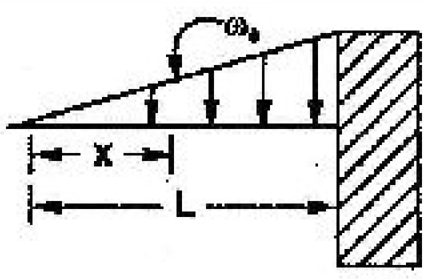
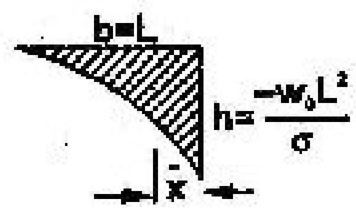
$$y = Kx^n\delta$$

$$\text{Area} = \frac{1}{(n+1)} b . h$$

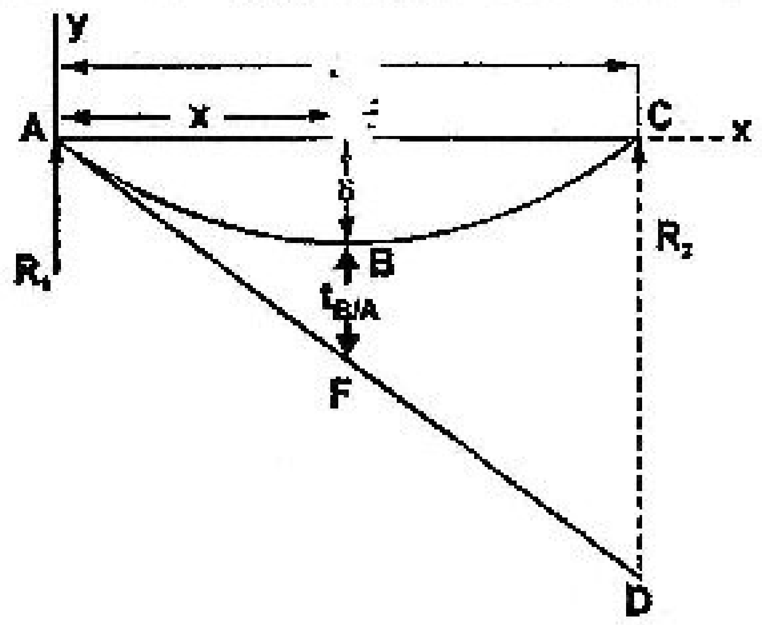
$$\bar{x} = \frac{1}{n+2} b$$

where, b = base, h = height

Some Examples of Cantilever Loading :

Type of loading	Cantilever beam	Moment equation	Moment diagram	Area	$\bar{x}$
Couple		$M = -C$	 $\frac{1}{1}bh$	$\frac{1}{2}b$	
Concentrated		$M = -Px$	 $\frac{1}{2}bh$	$\frac{1}{3}b$	
Uniformly distributed		$M = -\frac{w_0}{2}x^2$	 $\frac{1}{3}bh$	$\frac{1}{4}b$	
Uniformly varying		$M = \frac{w_0}{\sigma L}x^3$	 $\frac{1}{4}bh$	$\frac{1}{5}b$	

Deflection in Simply Supported Beams:



(*i*) Compute $t_{C/A}$ $\quad t_{C/A} = \frac{1}{EI}(\text{area})_{CA} \cdot \bar{x}_C$

(*ii*) From similar triangles $\quad EF = \frac{x}{L} \cdot t_{C/A}$

(*iii*) Compute $t_{B/A}$ $\quad t_{B/A} = \frac{1}{EI}(\text{area})_{BA} \cdot \bar{x}_B$

(*iv*) Since, EF is the sum of δ and $t_{B/A}$, the value of 8 is given by

$$\delta = EF - t_{B/A}$$

COLUMNS

A column is a compression member that under gradually increasing loads it fails by buckling at loads considerably less than those required to cause failure by crushing.

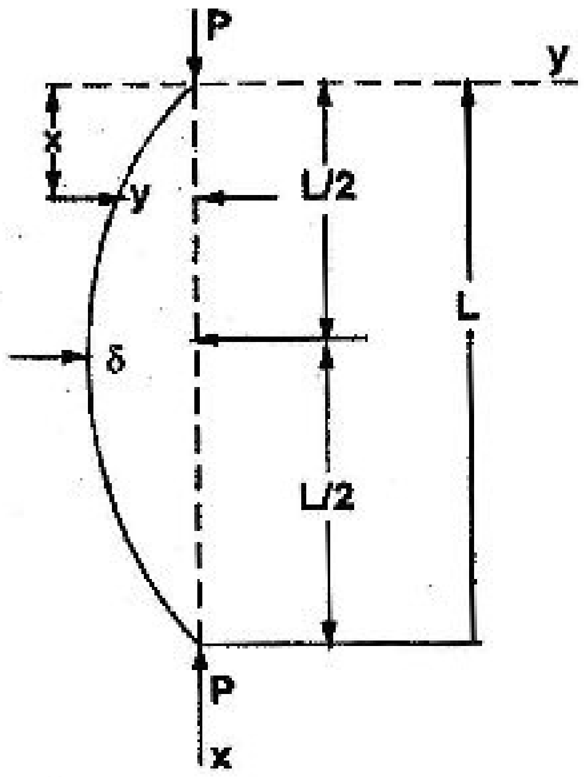
A compression member is generally considered to be a column when its unsupported length is more than 10 times its lateral dimension.

Long column fails by buckling, intermediate by a combination of crushing and buckling, short compression blocks by crushing.

Note : An ideal column is homogenous that is initially straight and is subjected to axial compressive loads. However, actual columns have small imperfections of material and fabrication as well as unavoidable accidental eccentricities of load. The initial crookedness of the column, together with the placement of the load, causes an intermediate eccentricity, *e* with respect to the centroid of a, typical section.

Long Columns by Euler's Formula:

Analysis of the critical load for long columns was made by Euler. His analysis is based on the differential equations of the elastic curve



$$EI\left(\frac{d^2y}{dx^2}\right) = M$$

$$EI\frac{d^2y}{dx^2} = M = P(-y) = -py \qquad ...(i)$$

Equation (i) is similar to the equation of a simple vibrating body

$$m\frac{d^2x}{dt^2} = -Kx$$

For which the general equation is

$$x = C_1 \sin\left(t\sqrt{\frac{K}{m}}\right) + C_2 \cos\left(t\sqrt{\frac{K}{m}}\right)$$

In a similar fashion equation (i) can be written as

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) \qquad ...(ii)$$

Putting $y = 0$ at $x = 0$, we get $C_2 = 0$

$y = 0$ at $x = L$, we get

$$0 = C_1 \sin\left(L\sqrt{\frac{P}{EI}}\right)$$

$$C_1 \text{ or } L\sqrt{\frac{P}{EI}} = n\pi$$

or $$\boxed{P = n^2\frac{EI\pi^2}{L^2}}$$

Special Cases :

1. For fixed end columns, $P_{cr} = \dfrac{4\pi^2EI}{L^2}$
2. One end fixed and other hinged, $P_{cr} = \dfrac{2\pi^2EI}{L^2}$
3. Both ends hinged, $P_{cr} = \dfrac{\pi^2EI}{L^2}$
4. One end fixed, the other free, $P_{cr} = \dfrac{\pi^2EI}{4L^2}$

Limitations:

(i) The value of I in the column formulas is always the least moment of inertia of the cross section. Any tendency to buckle, therefore occurs about the least axis of inertia of the cross-section.

(ii) Euler's formula also shows that the critical load that causes buckling depends not on the strength of the material, but only with dimensions and modulus of elasticity.

(iii) In order for Euler's formula to be applicable, the stress accompanying the bending that occurs during buckling must not exceed the proportional limit.

(iv) Euler's formula determine critical loads, not working loads.

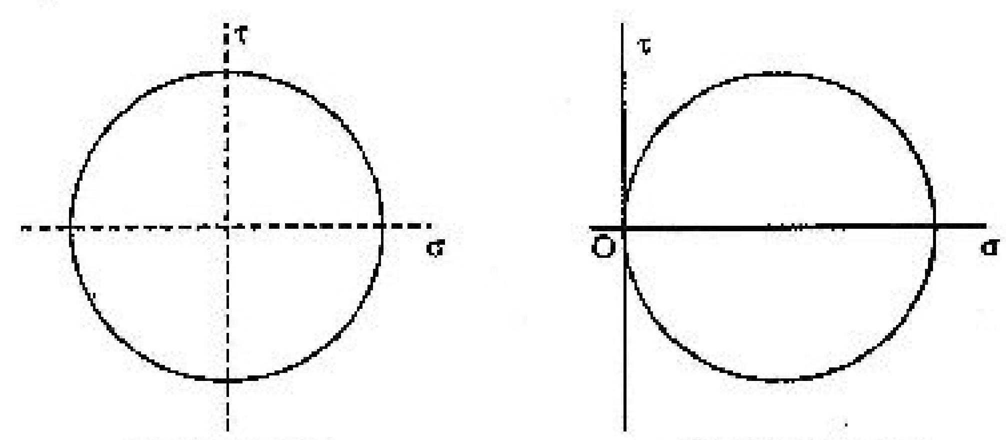
THEORIES OF FAILURE

Various theories of failure have been proposed, their purpose being to establish from the behaviour of a material subjected to simple tension or compression tests, the point at which failure will occur under any type of combined loading.

By failure we mean either yielding or actual rupture, whichever occurs first.

1. The Maximum Stress Theory: The maximum stress theory was proposed by Rankine. It states that failure occurs when the maximum principal stress on an element reaches a limiting value, the limit being the yield point in a simple tension test (or ultimate strength, if the material is brittle).

Mohr's circles for the pure shear and pure tension in parts (a) and (b)



(a) Pure shear *(b)* Pure tension

2. The Maximum Strain Theory : According to the maximum strain theory, a ductile material begins to yield when the maximum principal strain reaches the strain at which yielding occurs in simple tension or when the minimum principal strain equals the yield point strain in simple compression.

If σ_1 and σ_2 are two principal stress and $\sigma_1 > \sigma_2$, then strain in the direction of σ_1 is given by

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{v\sigma_2}{E}$$

or we can write $$\frac{\sigma_1}{E} - \frac{v\sigma_2}{E} = \frac{\sigma_{y.p}}{E}$$

or $$\sigma_1 - v\sigma_2 = \sigma_{y.p}$$

3. The Maximum Shear Stress Theory: It is called Guest theory. According to this theory yielding begins when the maximum shearing stress equals the maximum shearing stress developed at yielding in simple tension.

For biaxial stress system, $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \tau_{y.p}$

4. The Mises Yield Theory: This theory is also known as the maximum shear distortion theory. According to this theory, yielding can occur in a general three-dimensional state of stress when the root mean square of the differences between the principal stresses in equal to the same value in a tensile test.

If σ_1 and σ_2 and σ_3 are the principal stresses and σ_{yp} is the yield strength in simple tension, this concept gives

$$\frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{3}[(\sigma_{yp} - 0)^2 + (0 - 0)^2 + (0 - \sigma_{yp})^2] = \frac{2}{3}\sigma_{yp}^2$$

or $\quad 2\sigma_{yp}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

Note: Out of these theories, experimental work shows best agreement with the Mises yield theory when applied to ductile materials. For rupture in brittle materials, such as cast iron, the maximum stress theory is generally prepared.

TORSION

Assumptions:

(*i*) Circular sections remain circular.

(*ii*) Plane sections remain plane and do not warp.

(*iii*) The projection upon a transverse section of straight radial lines in the section remains straight.

Formula.

$$\frac{I}{I_P} = \frac{\tau}{r} = \frac{G\theta}{l}$$

where, I_p = Polar moment of Inertia

τ = Shear stress

G = Modulus of rigidity

The stress distribution along any radius varies linearly with the radial distance from the axis of the shaft.

Note: Maximum shear stress in rectangular shaft is

$$\tau = \frac{T}{ab^2}\left(3 + 18\frac{b}{a}\right)$$

where, a = Length of Long side

b = Length of Short side

Flexture Formula:

Assumptions in deriving flexture formula:

(*i*) The plane sections of the beam remain plane.

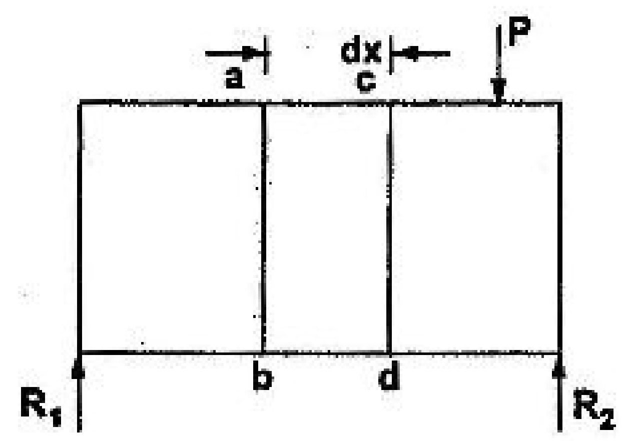
(*ii*) The material of the beam is homogenous and obeys Hooke's law

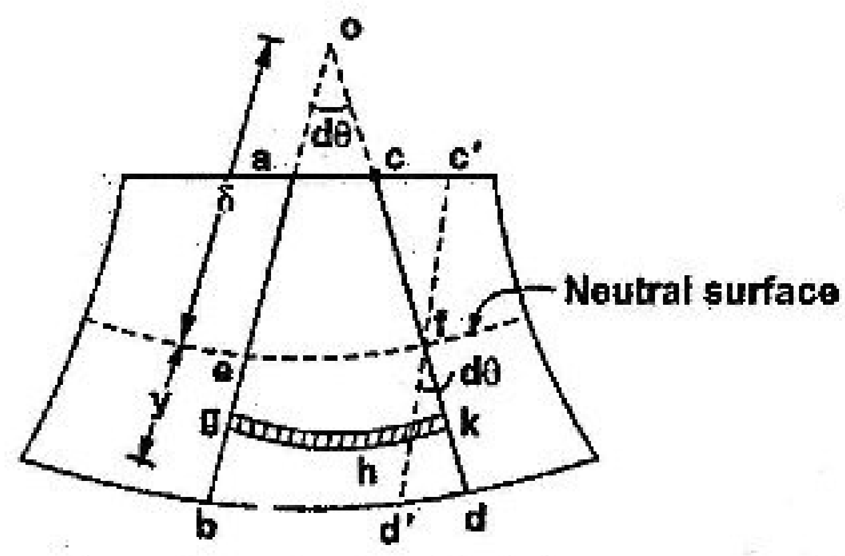
(*iii*) The modulii of elasticity for tension and compression are equal.

(*iv*) The beam is straight and of constant cross-section.'

(*v*) The plane of loading must contain a principal axis of the beam cross-section and the load must be perpendicular to the longitudinal axis of the beam.

The stress caused by the bending moment are known as bending or flexture stresses and relation between these stresses and the bending moment is expressed by the flexture formula.





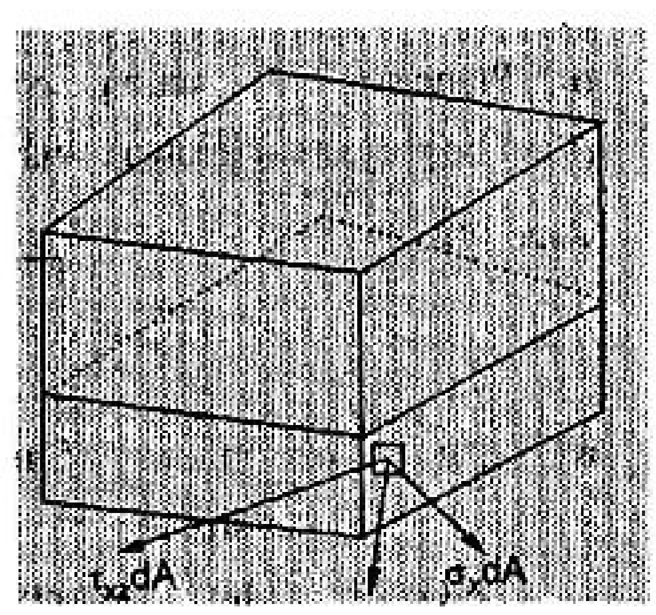
Two sections ab and cd and their exaggerated form are shown. The plane containing fibres like *ef* is called the neutral surface because such fibres remain unchanged in length.

$$\delta = hk = yd\theta$$

$$\in = \frac{\delta}{L} = \frac{yd\theta}{ef} = \frac{yd\theta}{\rho d\theta} = \frac{y}{\rho}$$

Stress in fiber gh is given by

$$\sigma = E\in = \left(\frac{E}{\rho}\right)y$$



Now we apply condition of equilibrium.

$$\Sigma F_x = 0$$

$$\int \sigma_x dA = 0$$

$$\frac{E}{\rho}\int y dA = 0$$

$$\frac{E}{\rho} A\,\bar{y} = 0$$

Thus, neutral axis must contain the centroid of the cross-section area.

$$\Sigma M_y = 0 \qquad \int z(\sigma_x dA) = 0$$

$$\frac{E}{\rho}\int z\, y dA = 0$$

Now,

$$\Sigma M_2 = 0$$

$$M = \int y\,(\sigma_x dA) = \frac{E}{\rho}\int y^2 dA$$

or

$$M = \frac{EI}{\rho}, \text{ where } I = \int y^2 dA$$

or

$$\frac{1}{\rho} = \frac{M}{EI}$$

or

$$\frac{E}{\rho} = \frac{M}{I} = \frac{\sigma}{y}$$

SOLVED EXAMPLES

1. The principal tensile stresses at a point across two perpendicular planes are 80 N/mm² and 40 N/mm².

(*i*) The resultant stress

(*ii*) The obliquity of the resultant plane on a plane at 20° with the major principal axes

(*iii*) Its obliquity the resultant plane on a plane at 20° the intensity of stress which acting alone can produce the same maximum strain (Take Poisson's ratio = 1/4)

Solution : At a plane at 20° with major principal plane

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta$$

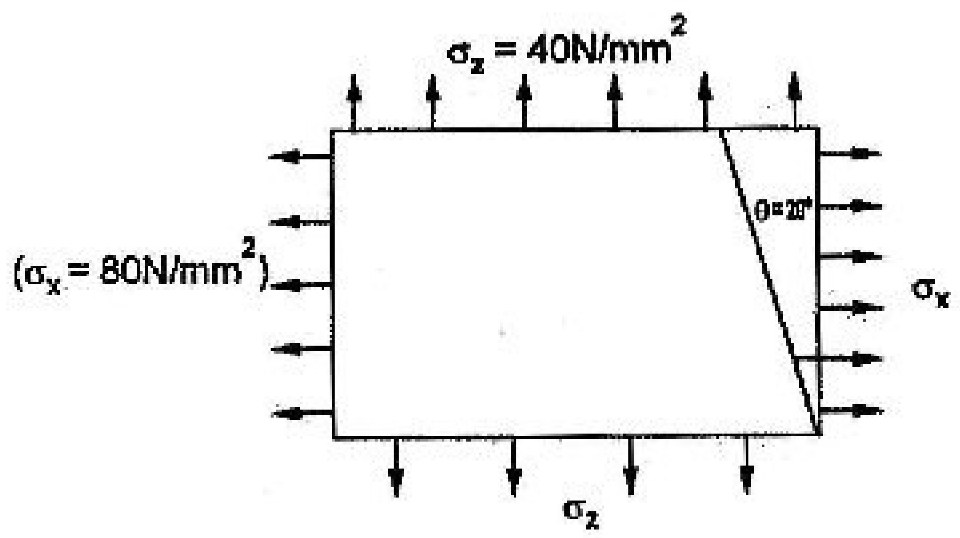
$$= \frac{80+40}{2} + \frac{80-40}{2}\cos 40^\circ$$

$$= 75.32 \text{ N/mm}^2 \text{ (tensile)}$$

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} + \sin 2\theta$$

$$= \frac{80-40}{2}\sin 40^\circ = 20 \sin 40^\circ$$

$$= 12.86 \text{ N/mm}^2$$



(*i*) Resultant stress

$$= \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{(75.32)^2 + (12.86)^2}$$

$$= 76.41 \text{ N/mm}^2$$

(*ii*) Obliquity, $\phi = \tan^{-1}\frac{\sigma_t}{\sigma_n} \tan^{-1}\frac{12.86}{75.32} = 9^\circ 41'$

(*iii*) Maximum strain

$$= \frac{\sigma_1}{E} - \frac{\sigma_2}{mE} = \frac{1}{E}\left[\sigma_1 - \frac{\sigma_2}{m}\right] = \frac{1}{E}\left[80 - \frac{40}{4}\right] = \frac{70}{E}$$

Let σ be the stress acting along which can produce the same maximum strain.

$$\therefore \qquad \frac{\sigma}{E} = \frac{70}{E}$$

or $\qquad \sigma = 70 \text{ N/mm}^2$

2. A rectangular block of material is subjected to a tensile stress of 110 N/mm² on one plane and a tensile stress of 47 N/mm² on a plane at right angles, together with shear stresses of 63 N/mm² on the same planes. Find

(*i*) The direction of the principal planes

(*ii*) The magnitude of the principal stress

(*iii*) The magnitude of the greatest shear stress.

Solution : Let σ_1 and σ_2 be the principal stresses. The inclination of the principal planes with the plane A carrying the tensile of σ = 110 N/mm² is given by

$$\tan 2\theta = \frac{2\tau}{\sigma - \sigma'} = \frac{2 \times 63}{110 - 47}$$

$$\therefore \qquad 2\theta = 63^\circ\,26' \text{ or } 243^\circ\,26'$$

or $\qquad \theta = 31^\circ\,43' \text{ or } 121^\circ\,43'$