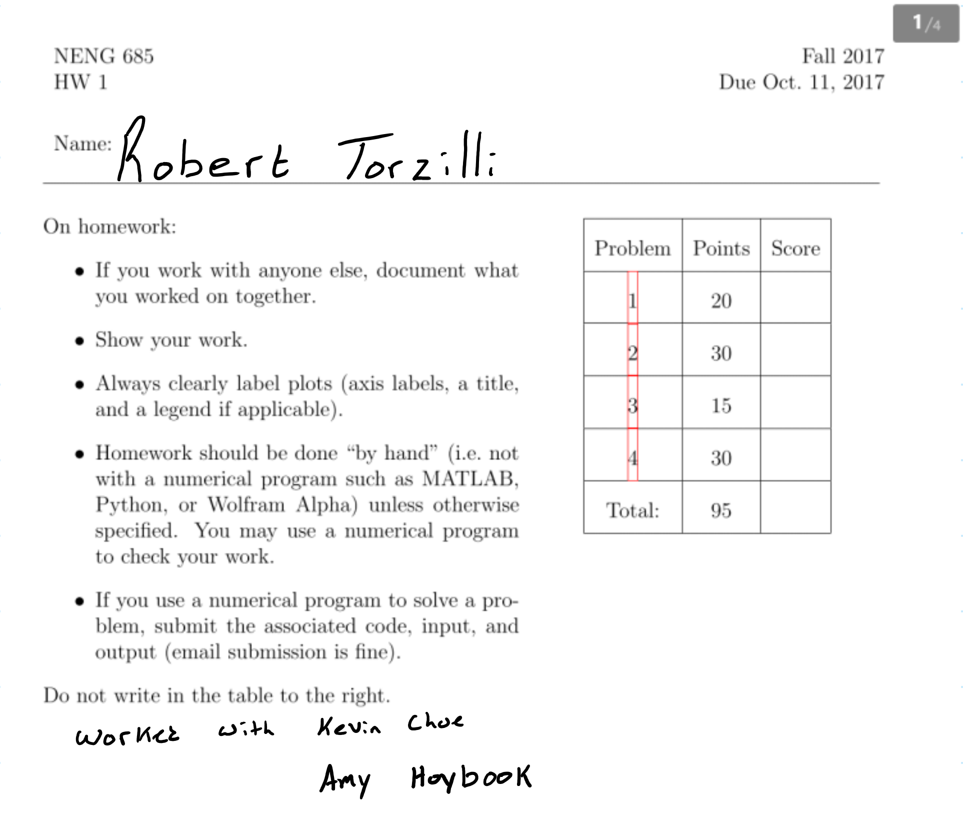

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NENG 685
HW 1

Fall 2017
Due Oct. 11, 2017

Name: Robert Torzilli

On homework:

- If you work with anyone else, document what you worked on together.
- Show your work.
- Always clearly label plots (axis labels, a title, and a legend if applicable).
- Homework should be done "by hand" (i.e. not with a numerical program such as MATLAB, Python, or Wolfram Alpha) unless otherwise specified. You may use a numerical program to check your work.
- If you use a numerical program to solve a problem, submit the associated code, input, and output (email submission is fine).

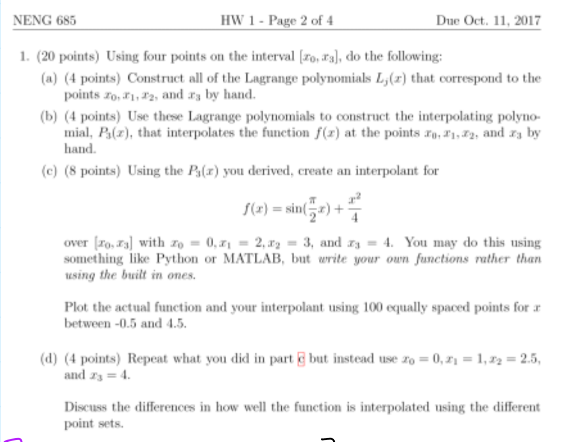
Do not write in the table to the right.

Problem	Points	Score
1	20	
2	30	
3	15	
4	30	
Total:	95	

Worked with Kevin Choe
Amy Hoybook

1

Monday, October 2, 2017 11:04 AM



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1. (20 points) Using four points on the interval $[x_0, x_3]$, do the following:
   (a) (4 points) Construct all of the Lagrange polynomials $L_j(x)$ that correspond to the points x_0, x_1, x_2, and x_3 by hand.
   (b) (4 points) Use these Lagrange polynomials to construct the interpolating polynomial, $P_3(x)$, that interpolates the function $f(x)$ at the points x_0, x_1, x_2, and x_3 by hand.
   (c) (8 points) Using the $P_3(x)$ you derived, create an interpolant for

   $$f(x) = \sin(\frac{\pi}{2}x) + \frac{x^2}{4}$$

   over $[x_0, x_3]$ with $x_0 = 0, x_1 = 2, x_2 = 3$, and $x_3 = 4$. You may do this using something like Python or MATLAB, but *write your own functions rather than using the built in ones.*

   Plot the actual function and your interpolant using 100 equally spaced points for x between -0.5 and 4.5.

   (d) (4 points) Repeat what you did in part c but instead use $x_0 = 0, x_1 = 1, x_2 = 2.5$, and $x_3 = 4$.

   Discuss the differences in how well the function is interpolated using the different point sets.

1] $[x_0, x_1, x_2, x_3]$

a] $f(x) = \sum_{i=1}^{n} y_i L_i(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$

$$\boxed{\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\ L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\ L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}}$$

b] $P_3(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$

$$= f(x_0) L_0(x) + f(x_1) L_0(x) + f(x_2) L_2(x) + f(x_3) L_3(x)$$

$$\boxed{\begin{aligned} P_3(x) = f(x_0)\left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right] + f(x_1)\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right] \\ + f(x_2)\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right] + f(x_3)\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right] \end{aligned}}$$

c] attached

} same code

d] attached

The largest % difference ended up at the ends at 8.8%

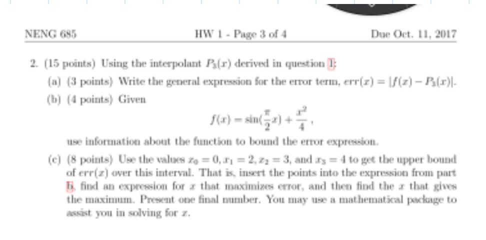
This was later verified with the built in function

While we expected little change, to have nearly indentical graphs indicates the new set was not different enough

Additionally this match could be due to how zoomed the graph is or the fact I am running 3.6.3

2

Tuesday, October 10, 2017 7:26 AM



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2. (15 points) Using the interpolant $P_3(x)$ derived in question 1:
   (a) (3 points) Write the general expression for the error term, $err(x) = |f(x) - P_3(x)|$.
   (b) (4 points) Given
   $$f(x) = \sin(\frac{\pi}{2}x) + \frac{x^2}{4},$$
   use information about the function to bound the error expression.
   (c) (8 points) Use the values $x_0 = 0, x_1 = 2, x_2 = 3$, and $x_3 = 4$ to get the upper bound of $err(x)$ over this interval. That is, insert the points into the expression from part b, find an expression for x that maximizes error, and then find the x that gives the maximum. Present one final number. You may use a mathematical package to assist you in solving for x.

a] $err(x) = |F(x) - P_3(x)|$ / derivative From given Thm the Form takes

$$\boxed{err(x) = e = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)}$$

b) ① solve the err(x) using the given $F(x) = \sin\left(\frac{\pi x}{2}\right) + \frac{x^2}{4}$ then bind the error

We know $n = 3$ for this problem From $P_3(x)$

$$e = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \Rightarrow \frac{f^{IV}(\xi)}{4!} (X - X_0)(X - X_1)(X - X_2)(X - X_3)$$

① $$\frac{f^{IV}(\xi)}{4!} = \frac{1}{4!} \frac{d^4}{dx^4}\left[\sin\left(\frac{\pi}{2} x\right) + \frac{x^2}{4}\right]$$

$$= \frac{1}{24}\left[\left(\frac{\pi}{2}\right)^4 \sin\left(\frac{\pi}{2} x\right)\right]$$

since $\frac{d^3}{dx^3} \frac{x^2}{4} = 0$

$\frac{d^4}{dx^4}(\sin Ax) = A^4 \sin(Ax)$

① $$= \frac{\pi^4}{384} \sin\left(\frac{\pi}{2} x\right)$$

$$e = \frac{\pi^{-4}}{384} \sin\left(\frac{\pi}{2} x\right) \left[(X - X_0)(X - X_1)(X - X_2)(X - X_3)\right]$$

Note error is bound with respect to $-AX \leq e \leq AX$

Since $\sin\left(\frac{\pi}{2} x\right) = \begin{cases} 0 & X = \text{even, whole} \\ 1 & X = \text{odd, whole} \end{cases}$

② we bind error

$$-\frac{\pi^4}{384} \sin\left(\frac{\pi}{2} x\right) \left[(x - x_0)(x - x_1)(x - x_2)(x - x_3)\right] \leq e \leq \frac{\pi^4}{384} \sin\left(\frac{\pi}{2} x\right) \left[(x - x_0)(x - x_1)(x - x_2)(x - x_3)\right]$$

Where x is bound $-A \le x \le A$

where $A = \pm 1, \pm 3, \pm 5, \ldots \pm a$

Thus ② is reduced to

$$-\frac{\pi}{384}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \le e \le \frac{\pi^4}{384}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

c) given $x = [0, 2, 3, 4]$

Find x that gives e_{max}

Find x that gives maximum

From b) we know that e_{max} will occur for some x in $|3$ to $3.\overline{9}|$ since 4 will collapse the function to $\emptyset$

and the other data points are smaller in terms of absolute values

Using the non-reduced max since symetry

$$e \le \frac{\pi^4}{384}\sin\left(\frac{\pi}{2}x\right)\left[(x)(x-2)(x-3)(x-4)\right]$$

Solution from code (attached)

$e_{max} \approx 0.238839$

when

$x = 3.9999$ →

Python's comparison could not go far enough. When the below x value was used directly we showed a larger number 4 decimals after the written values here

Solution from Wolfram Alpha

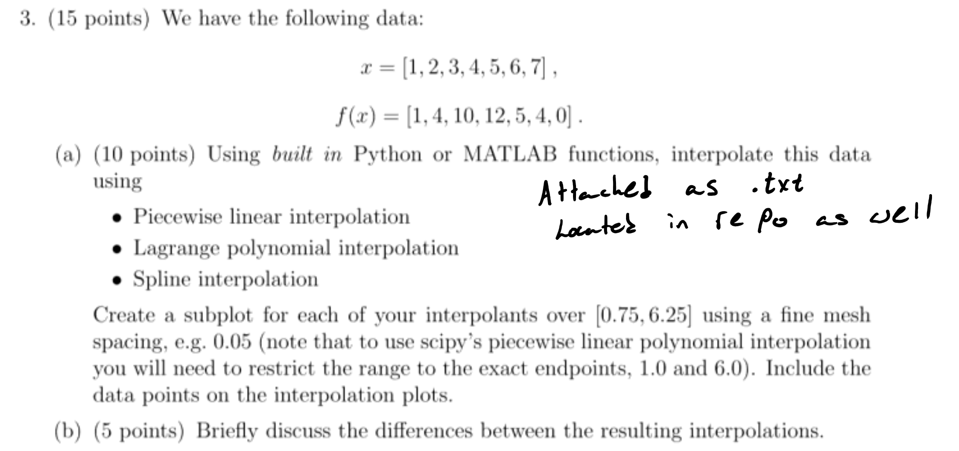
$e_{max} \approx 0.238839$

at

$x = 3.45321$

3

Tuesday, October 10, 2017 7:26 AM



3. (15 points) We have the following data:

$$x = [1, 2, 3, 4, 5, 6, 7] \, ,$$

$$f(x) = [1, 4, 10, 12, 5, 4, 0] \, .$$

(a) (10 points) Using *built in* Python or MATLAB functions, interpolate this data using

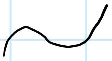
- Piecewise linear interpolation
- Lagrange polynomial interpolation
- Spline interpolation

Create a subplot for each of your interpolants over [0.75, 6.25] using a fine mesh spacing, e.g. 0.05 (note that to use scipy's piecewise linear polynomial interpolation you will need to restrict the range to the exact endpoints, 1.0 and 6.0). Include the data points on the interpolation plots.

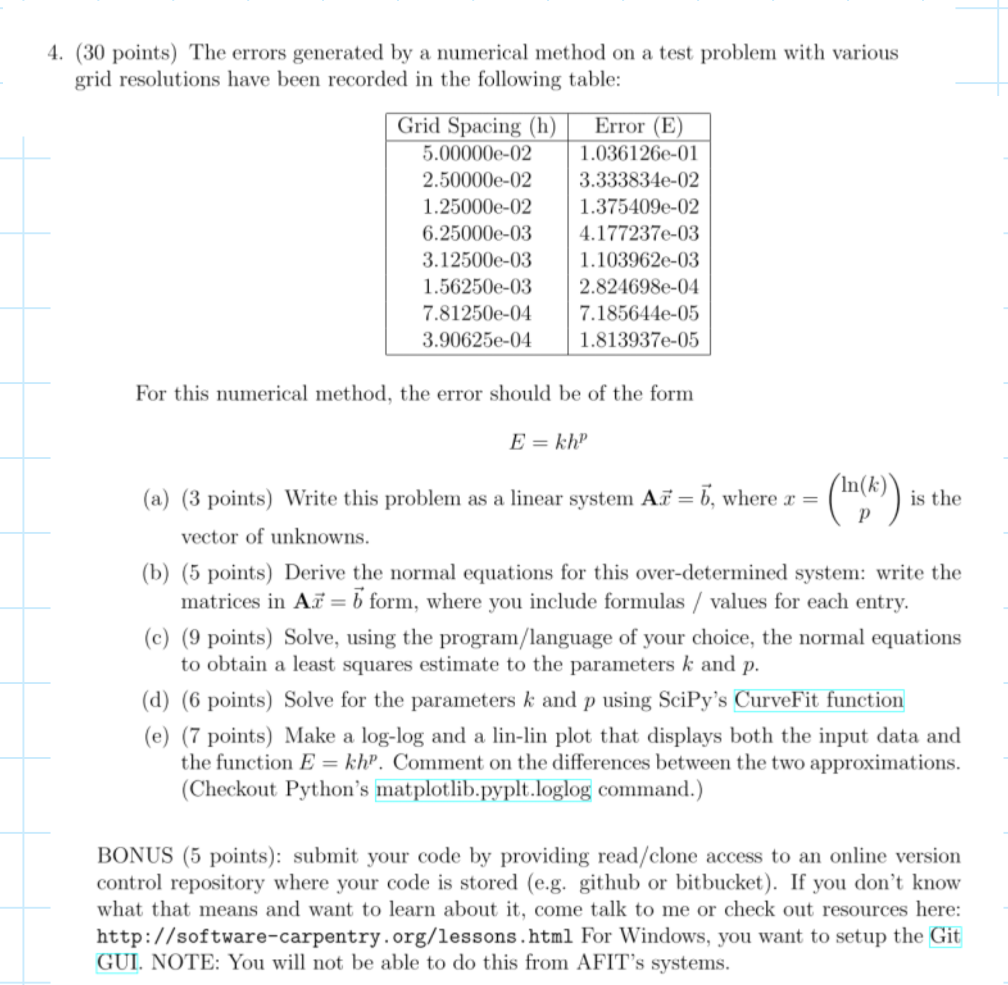
(b) (5 points) Briefly discuss the differences between the resulting interpolations.

Attached as .txt
Located in repo as well

b) The Piecewise Linear Interpolation simply connects straight lines between the given points so its not a very good curve. Doesn't extrapolate

The Lagrange Interpolation took a better curve. Its extrapolation is From a -Y to a +Y 

The spline interpolation is a nice curve Fits like the lagrange but its extrapolation fits better and is not as exagerated in the Y-axis we can see that it would hit the last given point if it etrapolated Further.



4. (30 points) The errors generated by a numerical method on a test problem with various grid resolutions have been recorded in the following table:

Grid Spacing (h)	Error (E)
5.00000e-02	1.036126e-01
2.50000e-02	3.333834e-02
1.25000e-02	1.375409e-02
6.25000e-03	4.177237e-03
3.12500e-03	1.103962e-03
1.56250e-03	2.824698e-04
7.81250e-04	7.185644e-05
3.90625e-04	1.813937e-05

For this numerical method, the error should be of the form

$$E = kh^p$$

(a) (3 points) Write this problem as a linear system $\mathbf{A}\vec{x} = \vec{b}$, where $x = \begin{pmatrix}\ln(k)\\ p\end{pmatrix}$ is the vector of unknowns.

(b) (5 points) Derive the normal equations for this over-determined system: write the matrices in $\mathbf{A}\vec{x} = \vec{b}$ form, where you include formulas / values for each entry.

(c) (9 points) Solve, using the program/language of your choice, the normal equations to obtain a least squares estimate to the parameters k and p.

(d) (6 points) Solve for the parameters k and p using SciPy's CurveFit function

(e) (7 points) Make a log-log and a lin-lin plot that displays both the input data and the function $E = kh^p$. Comment on the differences between the two approximations. (Checkout Python's matplotlib.pyplt.loglog command.)

BONUS (5 points): submit your code by providing read/clone access to an online version control repository where your code is stored (e.g. github or bitbucket). If you don't know what that means and want to learn about it, come talk to me or check out resources here: `http://software-carpentry.org/lessons.html` For Windows, you want to setup the Git GUI. NOTE: You will not be able to do this from AFIT's systems.

a) given $E = Kh^P$ write this as a linear Function

$$\overset{\text{some constant}}{A}\vec{x} = \vec{b} \quad \text{wh} \quad \vec{x} = \begin{bmatrix}\ln(K)\\ P\end{bmatrix}$$

E is a Function of h

First attempt to match $\vec{x}$

$$\ln(E) = \ln(Kh^P)$$

$$\ln(E) = \ln(K) + \ln(h^P)$$

$$= \ln(K) + P\ln(h) = A\vec{x}$$

$$\boxed{\ln(E) = P\ln(h) + \ln(K)}$$

b) Find the normal equations

note
$$Y = \ln(E)$$
$$a_1 = p$$
$$X = h(h)$$
$$a_0 = \ln(K) \Rightarrow e^{a_0} = K$$

$$\begin{bmatrix} A & \cdots \\ \vdots & \end{bmatrix} \begin{bmatrix} \ln(K) \\ p \end{bmatrix} = \begin{bmatrix} 1 & \ln(h) \end{bmatrix} \begin{bmatrix} \ln(K) \\ p \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 & \ln(h_0) \\ 1 & \ln(h_1) \\ \vdots & \vdots \\ 1 & \ln(h_n) \end{bmatrix} \begin{bmatrix} \ln(K) \\ p \end{bmatrix} = \begin{bmatrix} \ln(E_0) \\ \ln(E_1) \\ \vdots \\ \ln(E_n) \end{bmatrix}$$

Normal equations where h_0 to h_7 and E_0 to E_7 are found in the given table

$$\begin{bmatrix} 1 & \ln(h_0) \\ 1 & \ln(h_1) \\ 1 & \ln(h_2) \\ 1 & \ln(h_3) \\ 1 & \ln(h_4) \\ 1 & \ln(h_5) \\ 1 & \ln(h_6) \\ 1 & \ln(h_7) \end{bmatrix} \begin{bmatrix} \ln(K) \\ p \end{bmatrix} = \begin{bmatrix} \ln(E_0) \\ \ln(E_1) \\ \ln(E_2) \\ \ln(E_3) \\ \ln(E_4) \\ \ln(E_5) \\ \ln(E_6) \\ \ln(E_7) \end{bmatrix}$$

c) Solving the normal equations

ref "Numerical Methods" by Gilat and Subramaniam

$Y = a_1 X + a_0$ $\qquad$ $a_1 = P$ $\quad$ $a_0 = \ln(K)$

$\ln E = P\ln(h) + \ln(K)$ $\qquad$ $X_i = \ln(h_i)$ $\quad$ $Y_i = \ln E_i$

given

$$E = \sum_{i=1}^{n} \left[Y_i - (a_1 X_i + a_0)\right]^2$$

we know the solutions will take this form for a given linear function

6.14) $a_1 = \dfrac{n S_{xy} - S_x S_y}{n S_{xx} - (S_x)^2}$ $\qquad$ $a_0 = \dfrac{S_{xx} S_y - S_{xy} S_x}{n S_{xx} - (S_x)^2}$

where

n = terms or data points

$S_x = \sum_{i=0}^{n} X_i$ $\quad$ $S_y = \sum_{i=0}^{n} Y_i$ $\quad$ $S_{xy} = \sum_{i=0}^{n} X_i Y_i$ $\quad$ $S_{xx} = \sum_{i=0}^{n} (X_i)^2$

Using a program

$a_0 = \ln(K) \Rightarrow e^{a_0} = K$

$$\boxed{\begin{array}{l} K \approx .99416 \\ a_1 = P \approx 2.0397 \end{array}}$$

See program output for exact values

d) Using Curve Fit

$$\boxed{\begin{array}{l} K \approx 11.4786 \\ P \approx 1.5722 \end{array}}$$

e) The log-log plot is essentially straight with a slight crook in its latter half indicating our data is governed by the Power Law

The standard linear-linear plot is like an exponential Function drawn with piecewise interpolation