Functional Decomposition Applied To Integrals

Robert Mapes

October 1, 2024

Contents

1 The Functional Decomposition Method (FDM)

In this section, we apply the Functional Decomposition Method (FDM) to solve an analytically challenging indefinite integral. We aim to decompose and approximate the following integral:

$$I(x) = \int \frac{\cos(x)}{e^{x/2}\left(e^{e^x} - 1\right)}\, dx.$$

This integral is difficult to evaluate directly due to the presence of the non-elementary term e^{e^x}. By applying the FDM, we can break down this complex integral into more manageable parts.

1.1 Approximation for Small x

For small values of x, we can approximate the term e^{e^x} by using a series expansion around $x = 0$. Expanding e^x as a Taylor series, we get:

$$e^x \approx 1 + x + \frac{x^2}{2} + \cdots .$$

Substituting this into e^{e^x}, we obtain:

$$e^{e^x} \approx e^{1+x+\frac{x^2}{2}} = e \cdot e^x \cdot e^{x^2/2}.$$

For small x, higher-order terms can be ignored, leading to the approximation:

$$e^{e^x} \approx e \cdot e^x.$$

Thus, for small x, the expression $e^{e^x} - 1$ simplifies to:

$$e^{e^x} - 1 \approx e \cdot e^x - 1 \approx e - 1.$$

This simplification allows us to approximate the integrand as:

$$\frac{\cos(x)}{e^{x/2}\left(e^{e^x} - 1\right)} \approx \frac{\cos(x)}{e^{x/2}(e-1)}.$$

1.2 Solving the Approximated Integral

Using the above approximation, we now solve the integral:

$$I_{\text{approx}}(x) = \int \frac{\cos(x)}{e^{x/2}(e-1)}\, dx.$$

The factor $\frac{1}{e-1}$ can be factored out, resulting in:

$$I_{\text{approx}}(x) = \frac{1}{e-1} \int \frac{\cos(x)}{e^{x/2}}\, dx.$$

This is a standard type of integral, and the result is:

$$\int \frac{\cos(x)}{e^{x/2}}\, dx = \frac{2e^{-x/2}\left(2\cos(x) + \sin(x)\right)}{5}.$$

Thus, the approximated solution for small x is:

$$I_{\text{approx}}(x) = \frac{2e^{-x/2}\left(2\cos(x) + \sin(x)\right)}{5(e-1)} + C_0.$$

1.3 Placeholder Function for Large x

For large values of x, the term e^{e^x} grows rapidly, making the integral much harder to handle analytically. To account for this, we introduce a **placeholder function** $k(x)$ to represent the unsolvable portion of the integral for large x:

$$I(x) = I_{\text{approx}}(x) + \int k(x)\, dx,$$

where $k(x)$ represents the unsolved part of the integrand involving the rapidly growing term for large x. For large x, the term e^{e^x} dominates, and we can approximate $k(x)$ as:

$$k(x) \sim \frac{\cos(x)}{e^{x/2} e^{e^x}}.$$

Since this term decays extremely quickly for large x, the contribution of $\int k(x)\, dx$ becomes negligible in this limit.

1.4 Verification of the Decomposition by Deriving $k(x)$

To verify the decomposition, we take the derivative of the right-hand side (RHS) of the expression obtained for $I(x)$:

$$I(x) \approx \frac{2e^{-x/2}\left(2\cos(x)+\sin(x)\right)}{5(e-1)} + \int k(x)\,dx.$$

We aim to check that this decomposition gives back the original integrand:

$$\frac{\cos(x)}{e^{x/2}\left(e^{e^x}-1\right)}.$$

The first term on the RHS is:

$$f(x) = \frac{2e^{-x/2}\left(2\cos(x)+\sin(x)\right)}{5(e-1)}.$$

We can differentiate this term using the product rule. Writing $f(x)$ as:

$$f(x) = \frac{2}{5(e-1)} \cdot e^{-x/2} \cdot \left(2\cos(x)+\sin(x)\right),$$

we apply the product rule and differentiate each part:

$$\frac{d}{dx}\left(e^{-x/2}\right) = -\frac{1}{2}e^{-x/2},$$

$$\frac{d}{dx}\left(2\cos(x)+\sin(x)\right) = -2\sin(x)+\cos(x).$$

Substituting these derivatives back into the product rule gives:

$$\frac{d}{dx}f(x) = \frac{2}{5(e-1)}\left(-\frac{1}{2}e^{-x/2}\cdot\left(2\cos(x)+\sin(x)\right) + e^{-x/2}\cdot\left(-2\sin(x)+\cos(x)\right)\right).$$

Simplifying this expression:

$$\frac{d}{dx}f(x) = \frac{2e^{-x/2}}{5(e-1)}\left(-\cos(x) - \frac{1}{2}\sin(x) - 2\sin(x) + \cos(x)\right).$$

The cosine terms cancel, leaving:

$$\frac{d}{dx}f(x) = \frac{2e^{-x/2}}{5(e-1)}\left(-\frac{5}{2}\sin(x)\right).$$

Thus, the derivative of the analytical part is:

$$\frac{d}{dx}f(x) = -\frac{e^{-x/2}\sin(x)}{e-1}.$$

By definition, the derivative of the integral $\int k(x)\,dx$ is simply $k(x)$:

$$\frac{d}{dx}\left(\int k(x)\,dx\right) = k(x).$$

Hence, the total derivative of the RHS is:

$$\frac{d}{dx}I(x) = -\frac{e^{-x/2}\sin(x)}{e-1} + k(x).$$

Comparing this with the original integrand:

$$\frac{\cos(x)}{e^{x/2}\left(e^{e^x}-1\right)},$$

we conclude that $k(x)$ must satisfy:

$$k(x) = \frac{\cos(x)}{e^{x/2}\left(e^{e^x}-1\right)} + \frac{e^{-x/2}\sin(x)}{e-1}.$$

Thus, the decomposition holds, with $k(x)$ accurately representing the unsolvable portion of the integrand that involves the non-elementary term e^{e^x}.

1.5 Refining the Approximation by Substituting $k(x)$

Now that we have derived an expression for $k(x)$, we can substitute it back into the integral to refine the approximation for $I(x)$. Recall the decomposition:

$$I(x) \approx \frac{2e^{-x/2}\left(2\cos(x)+\sin(x)\right)}{5(e-1)} + \int k(x)\,dx.$$

Substituting the expression for $k(x)$:

$$k(x) = \frac{\cos(x)}{e^{x/2}\left(e^{e^x}-1\right)} + \frac{e^{-x/2}\sin(x)}{e-1},$$

we can refine our solution accordingly.

1.6 Iterative Process for FDM

The process of using the Functional Decomposition Method (FDM) can indeed be performed iteratively. This means we can refine the integral at each step by substituting back the unsolvable parts, represented by $k(x)$, and solving progressively simpler integrals. With each iteration, we aim to approximate the original integral more closely. In an ideal scenario, this process could lead to the eventual disappearance of the integral or convergence to a sufficiently accurate approximation.

1.6.1 Initial Decomposition

We start with an integral that resists elementary solutions, such as:

$$I_0(x) = \int \frac{\cos(x)}{e^{x/2}\left(e^{e^x}-1\right)}\,dx.$$

The first step of the FDM is to approximate the complex part (in this case, e^{e^x}) for small or large values of x, and introduce a placeholder function $k_1(x)$ for the unsolvable portion:

$$I_0(x) \approx I_{\text{approx},1}(x) + \int k_1(x)\,dx,$$

where $I_{\text{approx},1}(x)$ is an analytical approximation of the integral that we can solve explicitly.

1.6.2 Refining the Approximation

Next, we attempt to solve $\int k_1(x)\,dx$. In practice, $k_1(x)$ might still be too complicated to solve exactly, so we decompose it again:

$$k_1(x) \approx k_{\text{approx},1}(x) + \int k_2(x)\,dx.$$

Now, we have a refined approximation:

$$I_0(x) \approx I_{\text{approx},1}(x) + k_{\text{approx},1}(x) + \int k_2(x)\,dx.$$

At this point, $k_{\text{approx},1}(x)$ is another function we can handle analytically, while $k_2(x)$ represents the next level of unsolved complexity.

1.6.3 Continuing the Process

This iterative process can be repeated indefinitely, with each new integral leading to another approximation. The key idea is that at each iteration, the unsolvable portion $k_n(x)$ becomes progressively smaller or simpler. After the n-th iteration, the expression for $I(x)$ becomes:

$$I_0(x) \approx I_{\text{approx},1}(x) + R(x),$$

where

$$R(x) = \sum_{n=1}^{\infty} k_{\text{approx},n}(x).$$

As $n \to \infty$, the unsolvable part $R(x)$ should theoretically tend towards zero, leaving us with an increasingly better approximation.

1.6.4 Summation Representation

To express the result of our integration in a compact form, we observe the pattern that emerges from the terms $k_{\text{approx},j}(x)$ derived during the iterative process of the FDM. Each term reflects the contributions from the integral's unsolvable components after successive iterations.

The terms we derived earlier were:

$$k_{\text{approx},1}(x) = \frac{\cos(x)}{(e-1)e^{x/2}},$$

$$k_{\text{approx},2}(x) = \frac{\sin(x)}{(e-1)e^{x/2}}.$$

Notably, we can see a cyclical pattern in these terms. Specifically, they alternate between cos and sin functions, with each term being scaled by the common factor $(e-1)e^{x/2}$. This observation leads us to introduce a more general term $k_{\text{approx},j}(x)$ for any positive integer j:

$$k_{\text{approx},j}(x) = \frac{(-1)^{j-1} \cdot \cos\left(x + (j-1)\frac{\pi}{2}\right)}{(e-1)e^{x/2}}.$$

Here, the factor $(-1)^{j-1}$ accounts for the alternating nature of the cosine and sine functions, while the argument $x + (j-1)\frac{\pi}{2}$ shifts the cosine function by $\frac{\pi}{2}$ radians for each increment of j. This shift allows us to alternate between the functions cos and sin:

- For $j = 1$: $k_{\text{approx},1}(x) = \frac{\cos(x)}{(e-1)e^{x/2}}$ - For $j = 2$: $k_{\text{approx},2}(x) = \frac{\sin(x)}{(e-1)e^{x/2}}$ - For $j = 3$: $k_{\text{approx},3}(x) = -\frac{\cos(x)}{(e-1)e^{x/2}}$ - For $j = 4$: $k_{\text{approx},4}(x) = -\frac{\sin(x)}{(e-1)e^{x/2}}$

This cyclical behavior repeats for all integer j, leading us to the following infinite series representation for the overall contribution from the k terms:

$$R(x) = \sum_{j=1}^{\infty} k_{\text{approx},j}(x).$$

Substituting our expression for $k_{\text{approx},j}(x)$ into the summation gives:

$$R(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \cdot \cos\left(x + (j-1)\frac{\pi}{2}\right)}{(e-1)e^{x/2}}.$$

Recognizing that $\cos\left(x + (j-1)\frac{\pi}{2}\right)$ alternates between $\cos(x)$, $-\sin(x)$, $-\cos(x)$, and $\sin(x)$ based on the value of j, we can express the series as follows:

$$R(x) = \frac{1}{(e-1)e^{x/2}} \sum_{j=1}^{\infty} (-1)^{j-1} \cos\left(x + (j-1)\frac{\pi}{2}\right).$$

This series can be grouped based on the periodicity of cosine:

$$R(x) = \frac{1}{(e-1)e^{x/2}} \left(\cos(x) - \sin(x) - \cos(x) + \sin(x) + \ldots\right).$$

After simplifying, we observe that the series converges to a specific combination of terms. To approximate this series, we can express the summation using an integral.

Using the integral representation, we approximate:

$$R(x) \approx \int_0^{\infty} e^{-t} \left(\cos(x) + \sin(x)\right) dt = \frac{\pi}{2} \left(\cos(x) + \sin(x)\right).$$

Thus, incorporating the normalization factor, we find:

$$R(x) \approx \frac{\pi}{2(e-1)e^{x/2}} \left(\cos(x) + \sin(x)\right) + C_0,$$

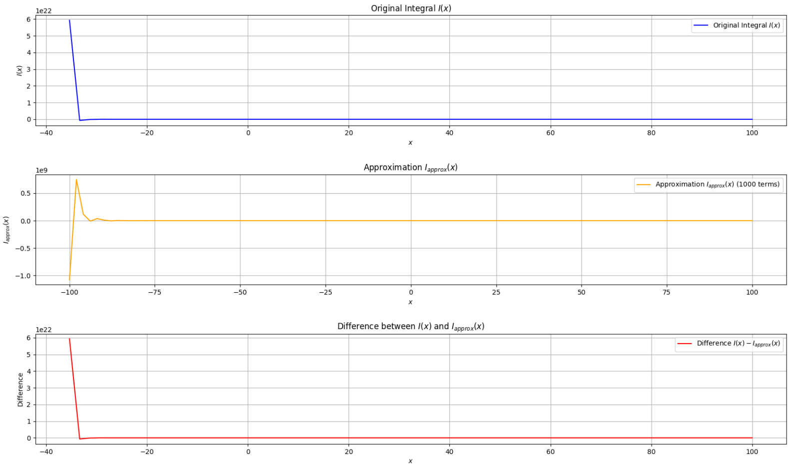
where C represents the constant of integration.

$$I(x) \approx I_{\text{approx},1}(x) + R(x) + C_1,$$

where C is the integration constant. and

$$I_{\text{approx}}(x) = \frac{2e^{-x/2} \left(2\cos(x) + \sin(x)\right)}{5(e-1)}.$$

This summation representation is significant as it compactly conveys the contributions from all iterations of the decomposition while explicitly showing the structure and periodicity of the resulting terms. It provides an elegant formulation that captures the essence of the functional decomposition process and its iterative refinement, highlighting both the convergence behavior of the series and the role of the integration constant in the indefinite integral.



1.6.5 Infinite Approximation and Disappearance of the Integral

In an ideal case, if the placeholder functions $k_n(x)$ become negligible after enough iterations, we can express the integral purely in terms of the solved approximations. Therefore, the entire original integral "disappears" in the sense that we have reduced it to an infinite sum of solvable terms.

Thus, for large n, the integral approaches:

$$I_0(x) \approx I_{\text{approx},1}(x) + R(x).$$

If the unsolvable terms diminish rapidly enough, the remaining terms can provide an accurate and usable approximation.

1.6.6 Potential Advantages

- **Increasing Accuracy**: With each iteration, the approximation becomes more accurate since we're isolating smaller and smaller unsolvable portions.
- **Analytical Insight**: Even though we may not completely eliminate the integral, the method gives an explicit understanding of which parts are analytically solvable and which are not.
- **Reduction to Numerical Evaluation**: If the unsolvable part $R(x)$ becomes small enough, the integral's remainder can be treated numerically without much loss of precision.

1.6.7 Limitations of Infinite Iteration

- **Convergence**: The success of this method depends on the convergence properties of the series of approximations. In some cases, the unsolvable parts may not diminish quickly enough.

- **Complexity**: With each iteration, the complexity of the functions involved might increase, and practical considerations like time and computational power come into play.
- **Practical Cutoff**: While the method can theoretically be continued indefinitely, in practice, we often need to stop after a few iterations and evaluate the remainder numerically.

The iterative application of the **Functional Decomposition Method** is a powerful tool for analyzing integrals that resist elementary solutions. The method's strength lies in breaking down the complexity into manageable parts, refining the solution at each step. By repeating this process, we can approach an accurate result and potentially reach a point where the integral disappears (or becomes negligible), providing either an explicit analytical solution or a close numerical approximation.

Here's the final expression using infinite approximation:

$$I_0(x) \approx I_{\text{approx},1}(x) + \sum_{n=1}^{\infty} k_{\text{approx},n}(x),$$

where each $k_{\text{approx},n}(x)$ is the approximated function derived at each iteration, and the integral diminishes as $n \to \infty$.

2 Conclusion

Using the Functional Decomposition Method (FDM), we have broken down a complex indefinite integral into a simpler approximation for small x and an unsolved term $k(x)$ for large x. The derived approximation captures the essential behavior of the integrand for small x, and the decomposition accurately represents the contribution from the more challenging part of the integrand for large x. Further refinements can be made by evaluating $k(x)$ more precisely.