Mandelbrot and Julia sets

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1 Introduction

In this study, I aim to study Mandelbrot and Julia sets and their fractal nature and relevance to nonlinear dynamics as an example of a chaotic system that can be generated from a simple iterative polynomials. This is also an example of a complex nonlinear system, which we did not explore too much through the course, which made it interesting to study.

2 Methodology

In this study, one of the main focuses is on fractals, which is a key part of chaos and nonlinear dynamics. The sets described in the study are examples of chaotic systems, with sensitive dependence to initial conditions. As you change the value of a constant you get a different type of set.
The research involved a literature review of existing sources on the the topics. Most of the analytical component involved simple numerical iteration of a complex polynomial. The computation was done on Python, based on existing resources tweaked to my convenience.

3 Understanding Dimensions

Dimension has several meanings across different realms in mathematics (and life). It is primarily used in the traditional sense often indicates dimensions like that of space or an object.
Mathematically, it could also mean the number of variables in a dynamical system.
But there is no proper understanding of what it actually means when a line is one dimensional or a plane is two dimensional. It is often associated with the number of ways you can move on this particular object, but this is not always accurate.
The idea of self-similarity helps explain this. Self similarity is the property of an object to be exactly or approximately similar to a part of itself.
Take the following example:

> Consider a square. It can be broken into four self-similar squares or nine self-similar squares. These have magnifications of 2 and 3. A pattern arises wherein the square can be broken down into n square similar copies and when magnified n times retains the original figure.

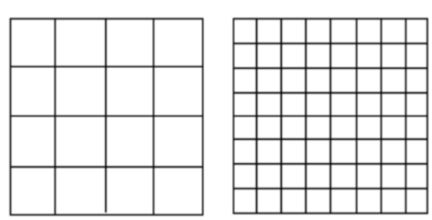

Figure 1: A square broken into N^2 self similar pieces with magnification factor N

This gives a new way of specifying the dimension of an object. It is the exponent of the number of self-similar pieces with a magnification factor n into which the figure may be broken. This is also explicitly called a fractal dimension.

Self -similarity is related to the notion of dimension this way. One of the most important characteristic of a fractal is its property of self-similarity. The name 'fractal' comes from the property that fractal objects have 'fractional dimension'.

4 Fractals

Fractals are found everywhere, in nature in the snowflakes or lightning or even in vegetables like the Romanesco cauliflower. They are irregular geometric objects.

Figure 2: The Romanesco cauliflower with fractal properties

The earliest known fractals were based on simple geometric rules or patterns: like the Cantor set, Sierpinski triangle and Koch snowflake.
The term fractal was coined in 1975 by Benoit Mandelbrot and had the following definitions:

a set with fractional Hausdorff dimension.

And was later changed to:

a set having Hausdorff dimension strictly greater than its topological dimension.

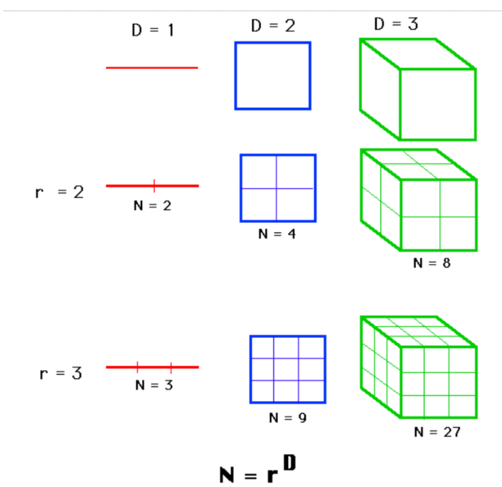

Figure 3: Dimensions of an object

In the example of breaking down a square, suppose you take an object with a Euclidean dimension D, and reduce its linear size by $1/r$ in the spatial direction. Its measure would increase to $N = r^D$ times the original. Taking log on both sides you get: $log(N) = Dlog(r)$
Typically, in objects we've seen D is an integer, making them one-dimensional, two-dimensional or 3-dimensional. But D need not always be an integer, it could be fraction, which is what gives rise to fractal geometry. This is known as the Hausdorff dimension, named after German mathematician Felix Hausdorff. Some fractals have integer Hausdorff dimension. Because of all the exceptions, there is no one true definition of a fractal, only a list of characteristics that can be satisfied to determine fractal nature. In the words of Mandelbrot:

> *"...to use fractal without a pedantic definition, to use fractal dimension as a generic term applicable to all the variants."*

5 The Mandelbrot set

In 1979, Benoit Mandelbrot came up with the Mandelbrot set. The Mandelbrot set belongs to a family of nonlinear fractals: nonlinear fractals come from nonlinear equations.
The Mandelbrot set is the most famous and widely studied fractal.

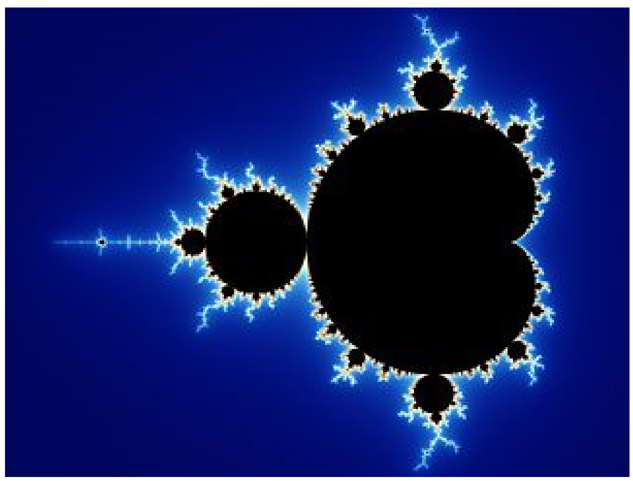

Figure 4: The Mandelbrot set

Formally, it is the set of complex numbers for which an infinite sequence of numbers remains bounded, i.e. it is the set of complex numbers c for which the function $f(z) = z^2 + c$ which does not diverge when iterated from $z = 0$. The formal definition:

> Consider a set of complex quadratic polynomials $P_c : z \leftarrow z^2 + c$. The Mandelbrot set M is the set of all c-values such that the sequence $0, P_c(0), P_c(P_c(0))...$ does not escape to infinity.

For example:

$$z_0 = 1$$

$$z_1 = 1^2 + 1 = 2$$

$$z_2 = 2^2 + 1 = 5$$

$$z_3 = 5^2 + 1 = 26$$

$$z_4 = 26^2 + 1 = 677$$

$$.$$

$$.$$

Here it diverges to infinity.
Another example:

$$z_0 = i$$
$$z_1 = i^2 + i = -1 + i$$
$$z_2 = (-1 + i)^2 + i = -2i + i = -i$$
$$z_3 = (-i)^2 + i = -1 + i$$
$$z_4 = (-1 + i)^2 + i = -i$$
$$.$$
$$.$$

Here it converges.
If you take a complex number such as: $a_0 = 1/3 + 5i/9$:

$$a_0 = 1/3 + 5i/9 = 0.3333 + 0.5556i$$
$$a_1 = (1/3 + 5i/9)^2 - 0.75 = 0.9475 + 0.3704i$$
$$a_2 = 0.0106 + 0.7019i$$
$$a_3 = -1.2424 + 0.0149i$$
$$a_4 = 0.7933 - 0.0371i$$
$$a_5 = -0.1221 + 0.5891i$$

In this case, you can't tell whether the series converges or diverges, hence this must be numerically simulated by a computer.
The M set is bounded by a closed disk around the origin of radius 2. It can be proven that for complex numbers z of $|z| > 2$ diverge.

$$z_0 = [0, 0]; |[0, 0]| = 0$$
$$z_1 = [0, 0]^2 + [-1.5, 1] = [-1.5, 1]; |[-1.5, 1]|1.803$$
$$z_2 = [-1.5, 1]^2 + [-1.5, 1] = [-0.25, -2]; |[-0.25, -2]| = 2.016 > 2$$

z_2 diverges, so it is not a part of the Mandelbrot set.
The boundary of the Mandelbrot set is a fractal curve of infinite complexity, portions of which when blown up give more detailed, smaller replicas of the whole set.
Around certain points, called Misiurewicz points, it exhibits infinite self-similarity. Because of its self-similar nature, the Mandelbrot set and its patterns seem like they are predictable when in actuality, it is a chaotic system. Points outside the bounded region are said to be 'escaping'.

5.1 Generating the Mandelbrot set

As seen in the previous case, the Mandelbrot set is generated by using the iteration function $z_{n+1} \leftarrow z_n^2 + c$ for $n = 0, 1, 2, 3, ...$ Starting with $z_0 = 0$, square its value and add c. c is the position of the initial point. This gives a new value of z, and the process is repeated.

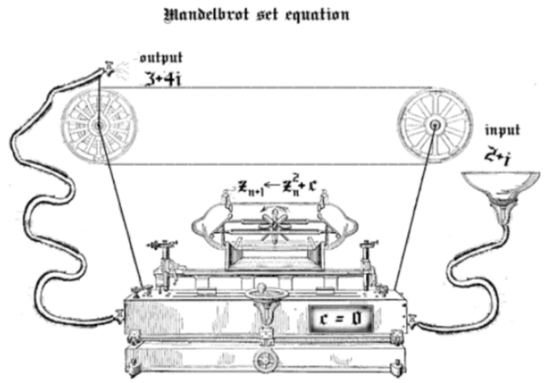

Figure 5: Iteration schematic of the Mandelbrot set

In the visualisation, the Mandelbrot set is evident with one colour, while the numbers outside the set are coloured with a gradient depending on how fast they diverge.

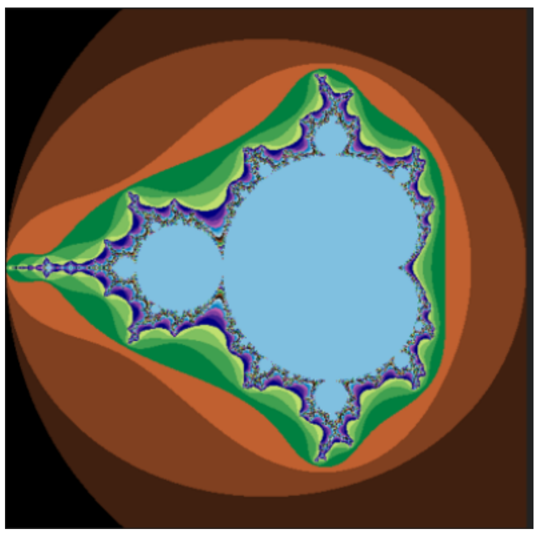

Figure 6: The Mandelbrot set

6 Julia sets

In 1915, mathematician Gaston Julia studied the iteration of polynomials and rational functions and in 1918 he published a paper on the same. When he studied the iterative properties of complex polynomials. He was interested in the iterative properties of: $z^4 + \frac{z^3}{z-1} + \frac{z^2}{z^3+4z^2+5} + c$ and came up with the concept of Julia sets. When c is held constant and the initial value of z is varied, you obtain the corresponding Julia set for the point c. It is the boundary of the set of complex numbers that do not diverge on iterations.

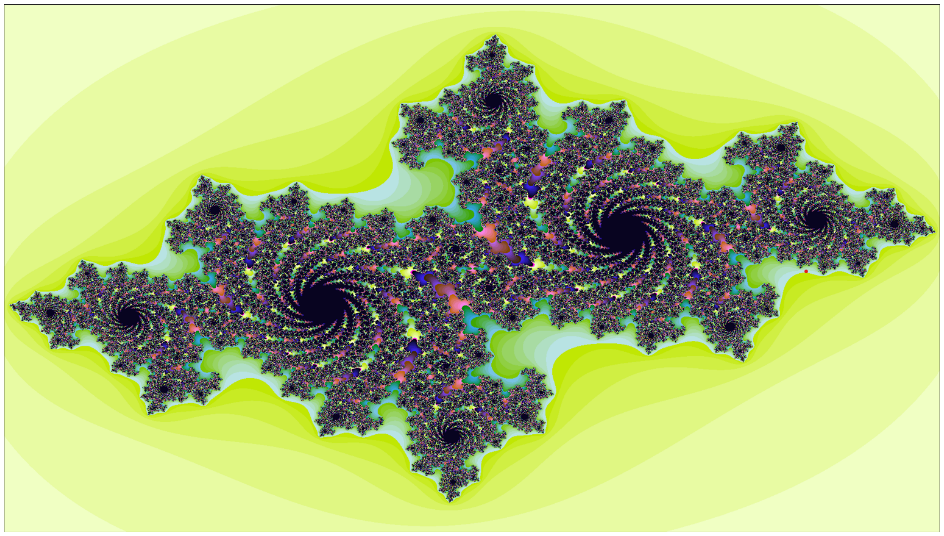

Figure 7: The Julia set

There are two types of Julia sets:

6.1 Connected Julia sets

When a Julia set is connected, there is a path from every complex number in the set to every other without leaving the set. These can be seen as being a part of the bigger Mandelbrot set. A Julia set with values of c chosen from within the Mandelbrot set is connected.

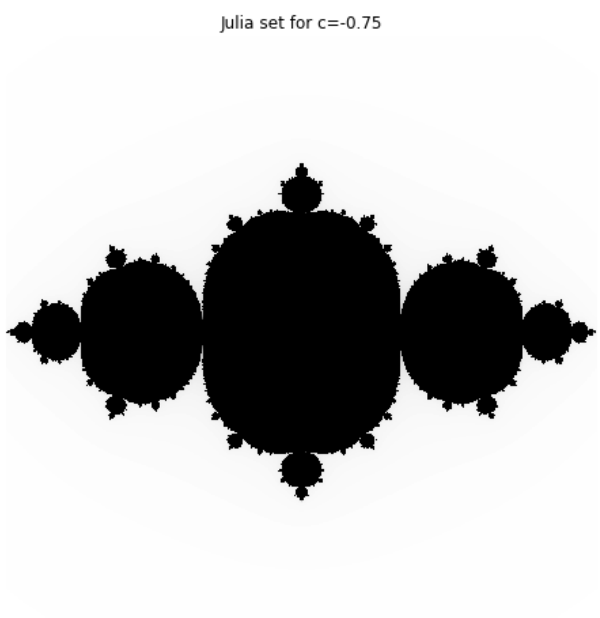

Figure 8: A connected Julia set

6.2 Disconnected Julia sets

When a Julia set is fully disconnected, every connected subset has one element. There is always a complex number which is not in the set between two complex numbers in the set. A path between these sets cannot be drawn. A Julia set with values of c chosen outside the Mandelbrot set are disconnected. The disconnected sets are also known as Cantorian dust.

Figure 9: A disconnected Julia set

The Mandelbrot set can also be defined as the set of complex numbers for which the Julia set is connected.

6.3 Generating a Julia set

The Julia set fractals are generated by initializing z and iterating it with $z = z^2 + c$, for a given value of c that gives rise to a specific Julia set. There are many different Julia sets based on different values of c. The following images represent Julia sets for different values of c.

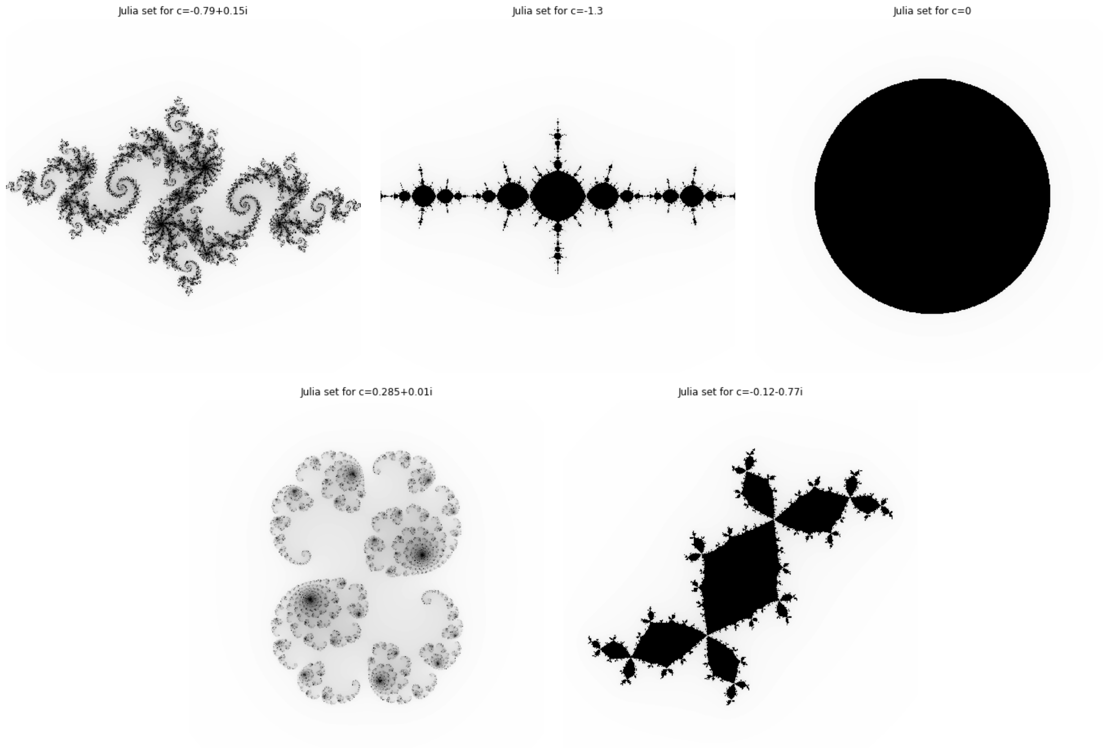

Figure 10: Julia sets for different values of c

6.4 Self-similarity of the Julia sets

Julia sets have the fractal property of being self-similar.In the following example of a Julia set for $c = 0.25 + 0.55i$, as you zoom in you see the self-similar nature, wherein the whole set is repeated infinitely.

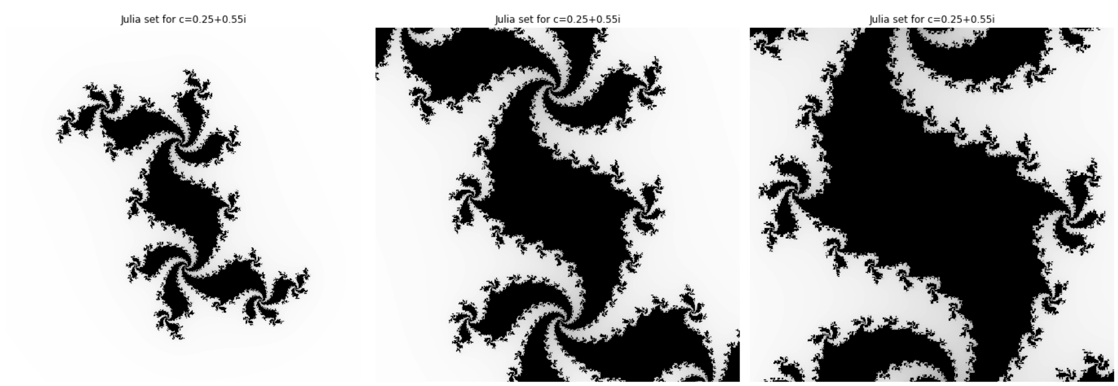

Figure 11: Caption

7 Differences between Mandelbrot and Julia sets

The Mandelbrot set:

- Is created in the parameter space.
- Iterates complex numbers of $z = 0$ for all values of c.

The filled Julia set:

- Is created in the dynamical field/plane.
- Iterates complex numbers of all z for a given value of c.

The Mandelbrot set can be considered an index of the Julia set - it marks the points in the complex plane where the corresponding Julia set is connected and not computable.

8 Future applications

- While the Mandelbrot set does not have any real world applications as such, it gives useful insight into studying the world of fractals, which is important given how prevalent they are.
- Fractal geometry can also be used to create art, because of its beautiful configurations.
- Using the complex polynomial $z = z^2 + c$ is just one way of updating z at each iteration to generate such sets. For example, using $z = z^3 + c$ gives it a 3-way symmetry instead of a 2-way symmetry and creates a new space of possible fractal shapes to explore.

9 Conclusion

The Mandelbrot set is popular because of its fractal nature, aesthetic appeal and as an example of a complex, chaotic structure that arises from simple rules - in this case from the iteration of a polynomial.

> *"The stunning beauty of the images it generates means that its appeal is both emotional and universal...[It] quite literally [has] infinite ramifications"*
> - Edith and Donald Craig, *The Pscyhodynamics of the M-set*

This allows us to study fractal nature from a system that arises out of something as simple as polynomial iterated a number of times, serving as an example of how chaotic systems can be found everywhere. This bears testimony to the behaviour of chaotic systems: complicated systems are not needed to produce complicated behaviour.

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