Structural Transitions in Optimal Networks

Aarathi Parameswaran and Dirk Witthaut

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1 Background

1.1 Introduction

The structure and robustness of network is a central theme in various scientific disciplines, including biophysics [1], geophysics [2, 3] and several technical disciplines [4, 5]. Statistical physics provides the essential tools to understand the design principles of such networks [6]. This thesis is devoted to the analysis of structural transitions in models of optimal networks.

A rather well explored topic is the transition between radial and meshed networks that can be attributed to the trade-off between cost and resilience [7, 8, 9]. Cost limits the number of connections in the network, as resources are generally scarce. Resilience requires additional connections to cope with damages or perturbations. A cornerstone model in the study of such transitions are optimal Kirchhoff networks [10, 11, 12] that will be discussed in detail below.

This thesis aims to generalize the theory of structural transitions in optimal networks both in term of the underlying models and in terms of potential transitions. For instance, it will analyze the occurrence of community and core-periphery structures or spontaneous symmetry breaking.

1.2 Optimal Kirchhoff Networks

A cornerstone model in the study of optimal networks has been introduced in [10, 11] based on ideas of linear flow networks. We first note that Kirchhoff's circuit laws are recovered from the optimization problem

$$\begin{aligned} &\min_{\vec{F}} D(\vec{F}) = \sum_{e \in E} \frac{F_e^2}{k_e} \\ &s.t. \sum_{e \text{ incident on } n} F_e = P_n \qquad \forall \text{ nodes}\, n, \end{aligned} \tag{1}$$

where P_n denotes the source strength at node n, F_e denotes the flow over the oriented edge e and k_e its capacity. That is, given the sources $\vec{P}$, the flows F_e minimize the dissipation D. Generalizing this idea, one searches for the network structure that minimizes the dissipation D, averaged over different scenarios for

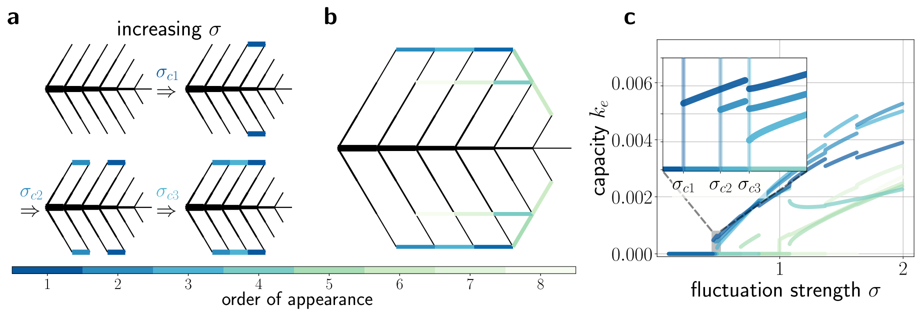

Figure 1: **Emergence of loops in optimal Kirchhoff networks for increasing fluctuations.** We compute the network structure that minimizes the dissipation $\langle D \rangle$ for a fixed budget $K = 1$ and $\gamma = 0.85$, where the set of potential edges E' is given by a triangular lattice. All nodes but one are assumed to be fluctuating sinks with injection $P_n \sim \mathcal{N}(\mu, \sigma)$, $\mu = -1$. The leftmost node serves as a source ensuring that $\sum_n P_n = 0$. (a) Optimal network geometry for different value of the fluctuation strength σ. (b) Order of appearance of the loops as σ increases. (c) Capacities k_e in the optimal network as a function of σ. Notably, the values of k_e are discontinuous when a new loop emerges. Figure taken from [13].

the sources P_n. Technically, the capacities k_e now become optimization variables subject to a budget constraint.

This approach has proven highly successful to explain the morphogenesis of networks in nature. For instance, it has been shown that this model readily explains the structural pattern of the vascular network of leafs [1]. A main feature of the model is the occurrence of a phase transition from radial to meshed networks [12].

1.3 Previous Work of the Network Science group at UoC

The network science group at UoC has contributed in various ways to the theory of optimal networks. Here we summarize some important recent articles to provide an overview of the general background of the thesis.

The group has provided a detailed analysis of the transition from radial to meshed networks in optimal Kirchhoff networks in the article [13]. Focusing on small networks, Kaiser et al have rigorously proven that loops emerge discontinuously via a saddle-node bifurcation [13] when a system parameter is varied, cf. figure 1. Furthermore, this article provides efficient heuristics to predict where the network becomes meshed, i.e. where new edges are formed.

The impact of fluctuations on the structure of optimal network was analyzed in [14]. While previous research focused almost exclusively on the transition from radial to meshed networks, this article demonstrates a different structural transition. Consider for example the network shown in figure 2 with two fluc-

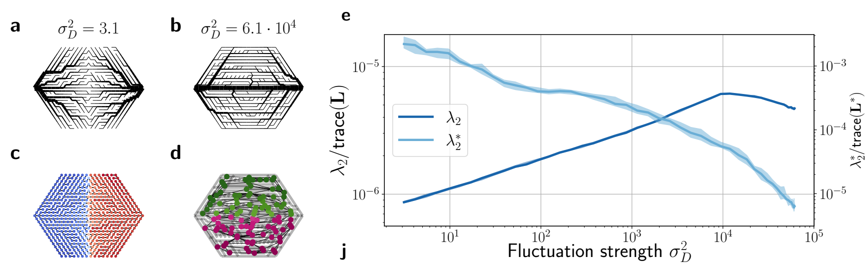

Figure 2: **Structural patterns in optimal networks from non-trivial fluctuation.** We consider a model of optimal networks similar to figure 1. All nodes but two are assumed to be fluctuating sinks with injections $P_n \sim \mathcal{N}(\mu = -1, \sigma = 0.1)$. The leftmost and rightmost vertices are sources, whose inflows P_n are subject to correlated fluctuations described by a Dirichlet distribution with standard deviation σ_D. (a,b) Optimal network geometry for different values of σ_D. (c,d) The optimal networks have a pronounced community structure for weak fluctuations σ_D and a "dual" community structure for strong fluctuations σ_D. Communities are indicated by the colours. (e) Primal and dual community structures characterized by the algebraic connectivity of the primal (λ_2) and the dual graph (λ_2^*) as a function of σ_D . Figure taken from [14].

tuating sources at the outermost nodes and fluctuating sinks at the remaining nodes. Depending on the correlation of the source fluctuations, the optimal network either has a "primary" or "dual" community structure. The term dual here refers to the plane dual of a weighted graph. In this thesis, we want to derive analytic results on how randomness shapes the structure of optimal networks.

Further work includes the rigorous characterization of network structures that impede the spreading of failures [15] and the demonstration and analysis of Braess' paradox in Kirchhoff networks [16].

2 Plan of the thesis

2.1 Initial phase

The initial phase of the thesis aims at learning the necessary analytic and computational methods for the study of optimal networks. To this end, simulations of different flow network models will implemented and compared. Different variants of Kirchhoff networks will be studied:

- Network that do not have to obey Kirchhoff's voltage law are routinely studied in graph theory. Neglecting this constraint can vastly facilitate the optimization problem.
- In many use cases further constraints arise. For instance, the magnitude

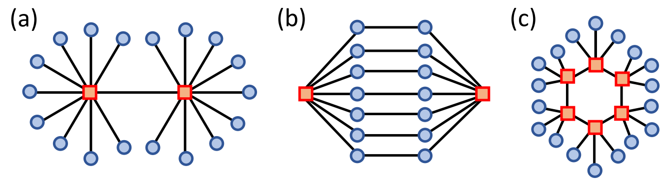

Figure 3: **Elementary network geometries for the study of optimal networks.** (a) In a radial network the the optimal structure can be obtained fully analytically. (b) A highly symmetric network for the study of community formation. (c) A network with a single loop for the study of core-periphery structures. Red squares correspond to fluctuating sources and blue circles to fluctuating sinks.

of the flow on each edge may be limited.

- The cost function may be modified according to reflect different application cases.

2.2 Spontaneous symmetry breaking

A recent article has demonstrated spontaneous symmetry breaking in a model for optimal transportation networks [17]. This work package aims to demonstrate and analyze spontaneous symmetry breaking for Kirchhoff networks. One approach is to consider multiplex or multi-layer network, where capacity can be allocated to either of the layers. We want to demonstrate this transition numerically and provide an analytic understanding inspired by elementary examples. A real-world instance of such a multiplex network ca be found in modern multi-energy networks.

2.3 Fluctuation induced structural transitions

Optimal Kirchhoff networks show a discontinuous phase transition from loopy to meshed (cf. figure 1). Non-trivial fluctuations can drive other transitions as shown in figure 2). Against this background, this work package will address the following questions:

- The transition from primal to dual communities appears as a gradual crossover. Are there setting where fluctuations can drive a sharp transition analog to a physical phase transition?
- Can we extend this approach to other structural patterns, such as the occurrence of core-periphery structures?
- Can we develop an analytic understanding of these transitions beyond numerical simulations? To reach this goal, we will start from elementary

network geometries as sketched in figure 3 and later generalize to more complex settings.

References

[1] Katifori, E., Szöllősi, G. J. & Magnasco, M. O. Damage and fluctuations induce loops in optimal transport networks. *Phys. Rev. Lett.* **104**, 048704 (2010).

[2] Rodríguez-Iturbe, I. & Rinaldo, A. *Fractal river basins: chance and self-organization* (Cambridge University Press, 2001).

[3] Konkol, A., Schwenk, J., Katifori, E. & Shaw, J. B. Interplay of river and tidal forcings promotes loops in coastal channel networks. *Geophysical Research Letters* **49**, e2022GL098284 (2022).

[4] Witthaut, D. *et al.* Collective nonlinear dynamics and self-organization in decentralized power grids. *Reviews of modern physics* **94**, 015005 (2022).

[5] Sterbenz, J. P. *et al.* Resilience and survivability in communication networks: Strategies, principles, and survey of disciplines. *Comput. Netw.* **54**, 1245–1265 (2010).

[6] Havlin, S. *et al.* Challenges in network science: Applications to infrastructures, climate, social systems and economics. *Eur. Phys. J. Spec. Top.* **214**, 273–293 (2012).

[7] Ronellenfitsch, H. & Katifori, E. Phenotypes of Vascular Flow Networks. *Phys. Rev. Lett.* **123**, 248101 (2019).

[8] Ganin, A. A. *et al.* Resilience and efficiency in transportation networks. *Sci. Adv.* **3**, e1701079 (2017).

[9] Avena-Koenigsberger, A. *et al.* Path ensembles and a tradeoff between communication efficiency and resilience in the human connectome. *Brain Struct. Funct.* **222**, 603–618 (2017).

[10] Banavar, J. R., Colaiori, F., Flammini, A., Maritan, A. & Rinaldo, A. Topology of the Fittest Transportation Network. *Phys. Rev. Lett.* **84**, 4745–4748 (2000).

[11] Bohn, S. & Magnasco, M. O. Structure, Scaling, and Phase Transition in the Optimal Transport Network. *Phys. Rev. Lett.* **98**, 088702 (2007).

[12] Corson, F. Fluctuations and redundancy in optimal transport networks. *Phys. Rev. Lett.* **104**, 048703 (2010).

[13] Kaiser, F., Ronellenfitsch, H. & Witthaut, D. Discontinuous transition to loop formation in optimal supply networks. *Nature communications* **11**, 5796 (2020).

[14] Kaiser, F., Böttcher, P. C., Ronellenfitsch, H., Latora, V. & Witthaut, D. Dual communities in spatial networks. *Nature Communications* **13**, 7479 (2022).

[15] Kaiser, F., Latora, V. & Witthaut, D. Network isolators inhibit failure spreading in complex networks. *Nature communications* **12**, 3143 (2021).

[16] Schäfer, B. *et al.* Understanding braess' paradox in power grids. *Nature Communications* **13**, 5396 (2022).

[17] Patwardhan, S., Barthelemy, M., Erkol, S., Fortunato, S. & Radicchi, F. Symmetry breaking in optimal transport networks. *arXiv preprint arXiv:2311.05059* (2023).