THESIS PROGRESS
Structural Transitions in Optimal Networks

September 11, 2024 | Aarathi Parameswaran |

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Outline

- Introduction: Optimal Networks
- Corson and Katifori Model
- Extension of Corson Model
- Rectangular Network
- Square Network
- Triangular Network
- Next Steps
- Conclusion

Introduction

Optimal Networks

- Optimal Networks: supply/transport network
- Applications: infrastructure design in man-made networks
  - Energy grids
  - Water system, gas grids
  - Transportation systems: subways, metro
- Natural networks:
  - Vascular systems
  - River basins
- Study the structure of such networks

Introduction

Corson and Katifori Model

- The fluctuating sink model:
  - One source node, others are sinks.
  - The outflow at the sinks fluctuates.
- Idea: Fluctuations bring about different classes of structures of optimal networks.
- Optimization problem: minimizing dissipation, subject to a resource constraint.

$$\langle D \rangle = \sum_{l=1}^{M} \frac{\langle F_l^2 \rangle}{k_l}$$

$$\sum_{l \in E} k_l^{\gamma} \leq K^{\gamma}$$

Corson and Katifori Model

- Minimization: iterative, self-consistent approach.
- Initial random guess for edge weights.
- Relation between edge weights and flows used to update the weights:

$$k_l = \frac{\langle F_l^2 \rangle^{1/(1+\gamma)}}{\left(\sum_{e \in E} \langle F_e^2 \rangle^{\gamma/(1+\gamma)}\right)^{1/\gamma}}$$

- The moments of the flows are determined by Kirchhoff's laws:
  - KCL: sum of currents flowing into a node is equal to the sum of currents flowing out of the node.
  - KVL: sum of potential differences around a loop is zero.

Extension of the Corson Model

- Including multiple sources
- More sources $\rightarrow$ more variability
- Add Dirichlet noise to the sources $X_i \sim \text{Dir}(\alpha)$

$$P_i = -\frac{1}{N_s} \sum_{i=N_s+1}^{N} P_i + K \left(\frac{1}{N} - X_i \right)$$

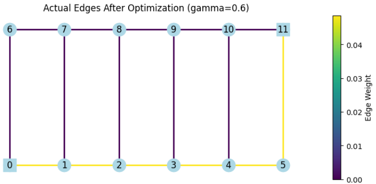
- Dirichlet variables sum to unity
- Changing the parameters or scaling factor in the Dirichlet variables allows you to change correlations between the sources

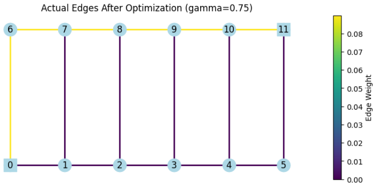
Rectangular Network

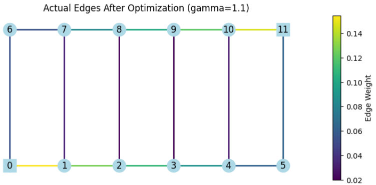
- Rudimentary model to observe symmetry breaking
- Start with a 2×6 grid
- Try and identify symmetry breaking transition using Corson's self-consistency approach
- Start with two sources at two opposite ends

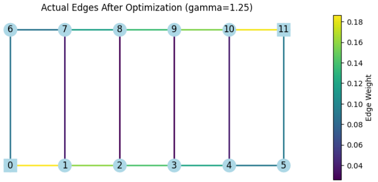
Rectangular Network

Numerics



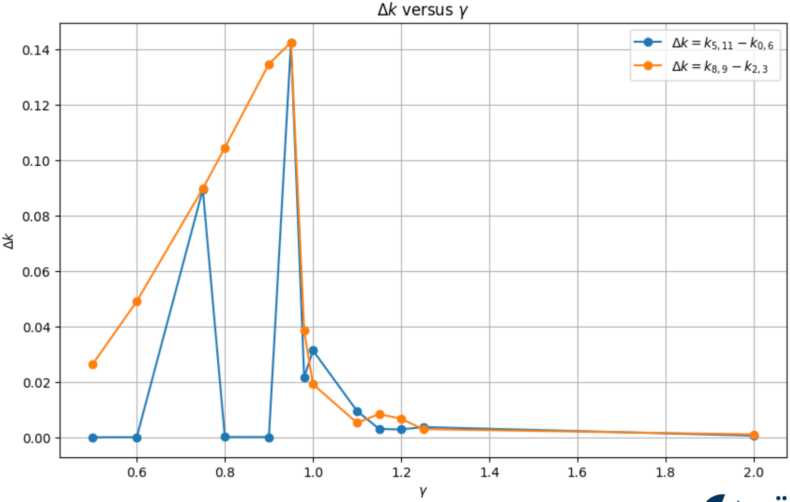






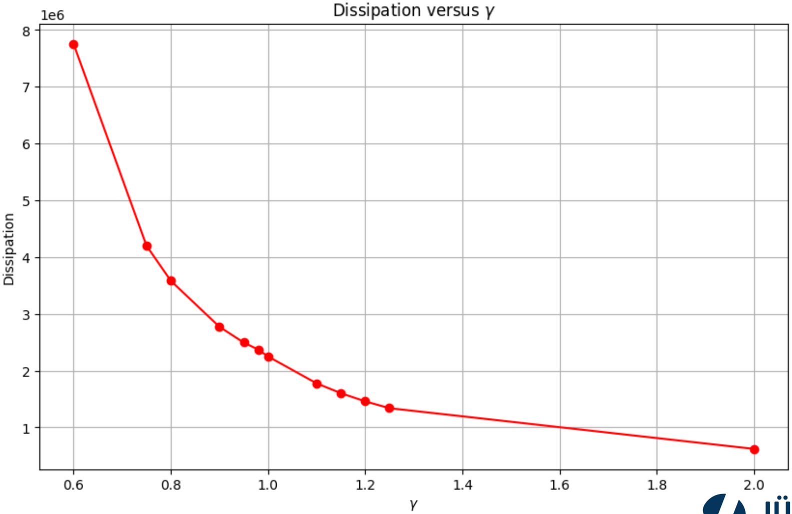
Rectangular Network

Numerics



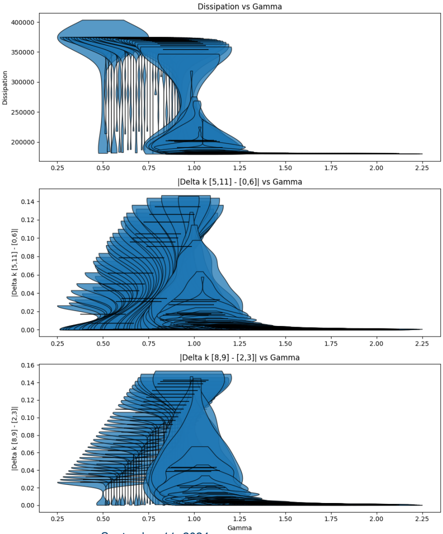
Rectangular Network

Numerics



Rectangular Network

Numerics



Square Network
Analytics

Mirror-symmetry

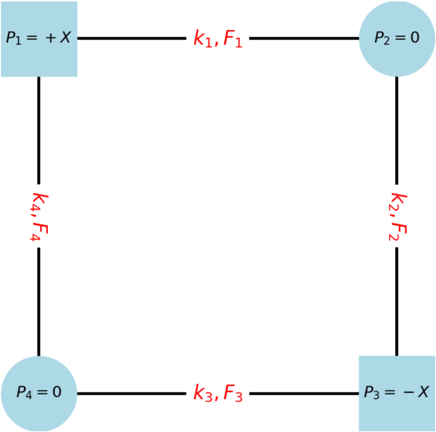


Figure: Simple square grid schematic

Square Network

Analytics

- System reduces to two variables
- Using Kirchhoff's laws:

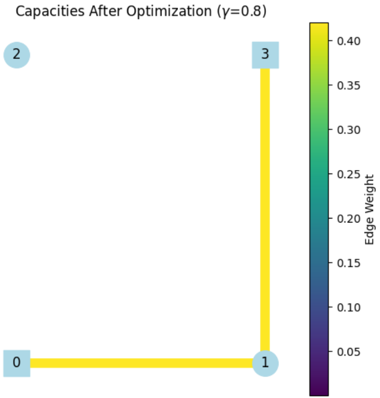
$$\langle D \rangle = 2 \langle X^2 \rangle \frac{1}{k_1 + k_3}$$

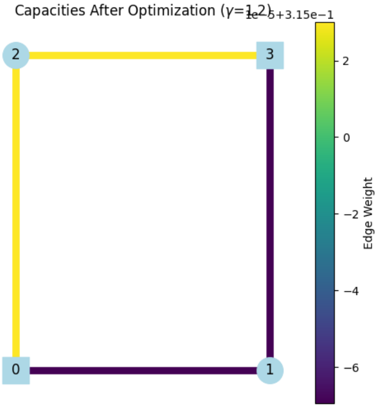
- Using resource constraint:

$$\langle D \rangle = 2 \langle X^2 \rangle \frac{1}{k_1 + \left[\frac{1}{2} - k_1^{\gamma}\right]^{\frac{1}{\gamma}}}$$

Square Network

Numerics





Square Network

Analytics

- In the two regimes:

$$\langle D_a \rangle = 2^{1+\frac{1}{\gamma}} \langle X^2 \rangle$$

$$\langle D_s \rangle = 4^{\frac{1}{\gamma}} \langle X^2 \rangle$$

$$k_a^* = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}$$

$$k_s^* = \left(\frac{1}{4}\right)^{\frac{1}{\gamma}}$$

Square Network

Numerics vs Analytics

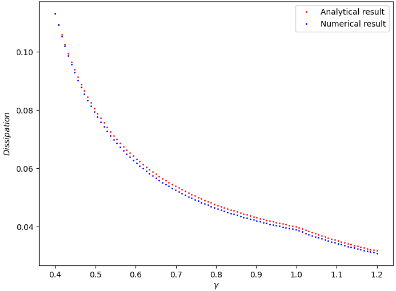


Figure: Dissipation vs γ

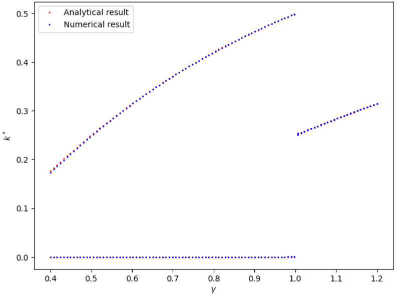


Figure: Optimal edge capacities vs γ

Triangular Network

Analytics

- Rotational Symmetry class

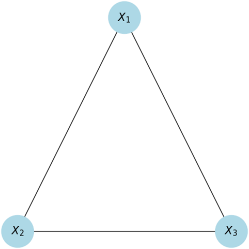


Figure: Schematic of Triangular Network

$$X_j = \mathrm{Dir}(k = 3, \alpha) - \frac{1}{3}$$

Triangular Network

Analytics

- 3 possible configurations of the network
- Covariance matrix:

$$\Gamma = \frac{\sigma^2}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

- Analytic calculation of dissipation:

$$\langle D_s \rangle = \frac{1}{k} = \frac{1}{(1/3)^{1/\gamma}}$$

$$\langle D_a \rangle = \frac{2}{k} = \frac{2}{(1/2)^{1/\gamma}}$$

Triangular Network

Analytics

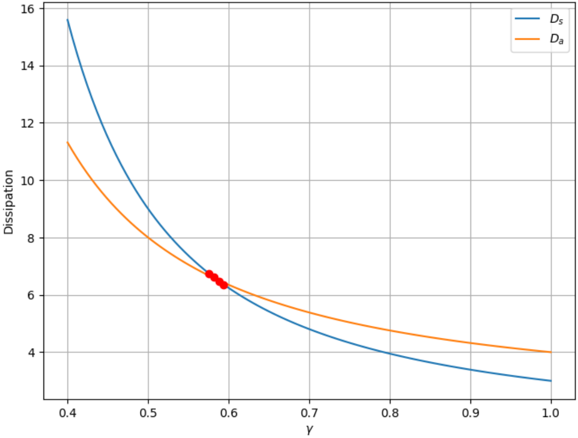
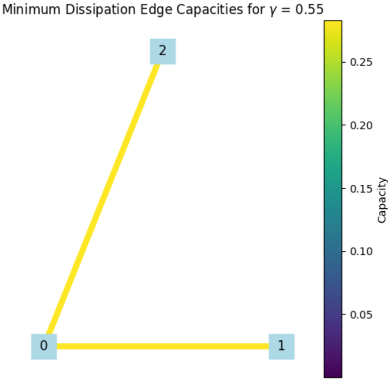
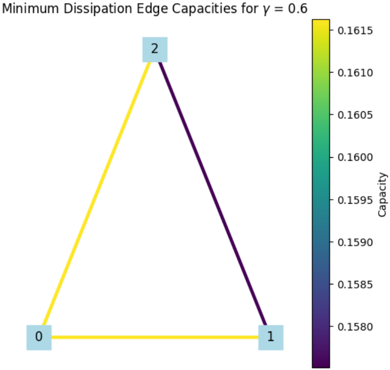


Figure: Analytic calculation of dissipation vs γ

Triangular Network

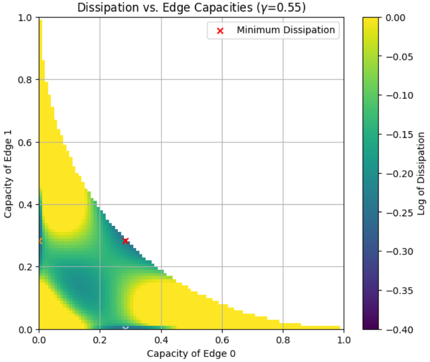
Numerics

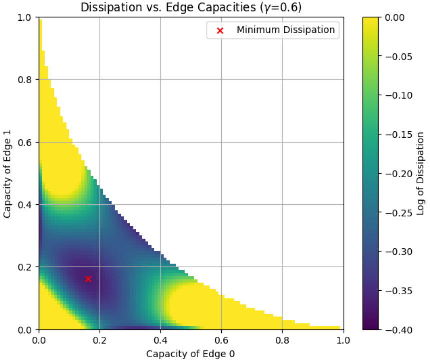




Triangular Network

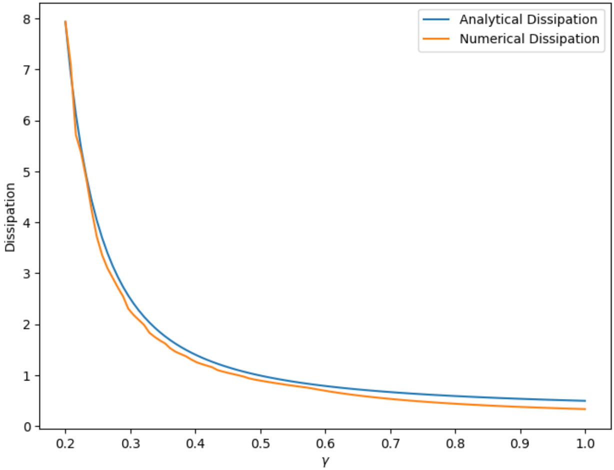
Numerics





Triangular Network

Numerics vs Analytics



Next Steps

- Immediate steps:
  - Test for minima for the triangular network
  - Mapping the bifurcation
  - Extend numerics to larger rings
- Future plans

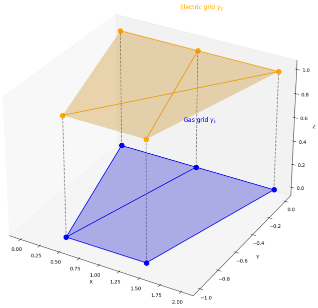


Figure: Multiplex network example

Conclusion

- So far: used Corson's self-consistency approach to observe symmetry breaking in three kinds of networks
- Two different symmetry classes
- (Almost) complete understanding of simple square and triangular networks.
- Aim to extend this to larger networks with slightly more relations between sources and sinks
- Look at the effect of noise on these structures