THESIS PROGRESS
Structural Transitions in Optimal Networks

November 21, 2024 | Aarathi Parameswaran |

Outline

Introduction

Optimal Networks

- Optimal Networks: supply/transport network
- Applications: infrastructure design in man-made networks
  - Energy grids
  - Water system, gas grids
  - Transportation systems: subways, metro
- Natural networks:
  - Vascular systems
  - River basins
- Study the structure of such networks

Introduction

Corson and Katifori Model

- The fluctuating sink model:
  - One source node, others are sinks.
  - The outflow at the sinks fluctuates.
- Idea: Fluctuations bring about different classes of structures of optimal networks.
- Optimization problem: minimizing dissipation, subject to a resource constraint.

$$\langle D \rangle = \sum_{l=1}^{M} \frac{\langle F_l^2 \rangle}{k_l}$$

$$\sum_{l \in E} k_l^{\gamma} \leq K^{\gamma}$$

Corson and Katifori Model

- Minimization: iterative, self-consistent approach.
- Initial random guess for edge weights.
- Relation between edge weights and flows used to update the weights:

$$k_l = \frac{\langle F_l^2 \rangle^{1/(1+\gamma)}}{\left(\sum_{e \in E} \langle F_e^2 \rangle^{\gamma/(1+\gamma)}\right)^{1/\gamma}}$$

- The moments of the flows are determined by Kirchhoff's laws:
  - KCL: sum of currents flowing into a node is equal to the sum of currents flowing out of the node.
  - KVL: sum of potential differences around a loop is zero.
- When the flow through the network is stationary, for $\gamma < 1$, the local minima of the dissipation are spanning trees and for $\gamma > 1$, there is a single global minimum, all edges have non-zero conductance.

Extended Corson and Katifori Model

- Including multiple sources
- More sources $\rightarrow$ more variability
- Add Dirichlet noise to the sources $X_i \sim \mathrm{Dir}(\alpha)$

$$P_i = -\frac{1}{N_s} \sum_{i=N_s+1}^{N} P_i + K \left(\frac{1}{N} - X_i \right)$$

- Dirichlet variables sum to unity
- Changing the parameters or scaling factor in the Dirichlet variables allows you to change correlations between the sources

Symmetry Breaking in Optimal Networks

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Symmetry breaking in optimal transport networks

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Engineering multilayer networks that efficiently connect sets of points in space is a crucial task in all practical applications that concern the transport of people or the delivery of goods. Unfortunately, our current theoretical understanding of the shape of such optimal transport networks is quite limited. Not much is known about how the topology of the optimal network changes as a function of its size, the relative efficiency of its layers, and the cost of switching between layers. Here, we show that optimal networks undergo sharp transitions from symmetric to asymmetric shapes, indicating that it is sometimes better to avoid serving a whole area to save on switching costs. Also, we analyze the real transportation networks of the cities of Atlanta, Boston, and Toronto using our theoretical framework and find that they are farther away from their optimal shapes as traffic congestion increases.

Aims

To extend this to optimal Kirchhoff networks obtained from this Corson and Katifori model, demonstrating numerically and analytically the phenomenon of symmetry breaking in such networks.
Obtain analytical solutions and numerical simulations of two types of networks that belong to distinct symmetry classes:

- Square Network
- Triangular Network
- Ladder Network

Square Network

Mirror symmetry Two sources, passive sinks.

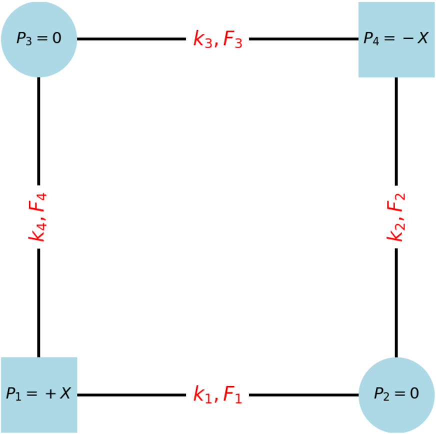

Figure: Schematic of a simple square network

Square Network

Numerical Simulation

Observe symmetry breaking transition, occurring at a trivial value of gamma = 1

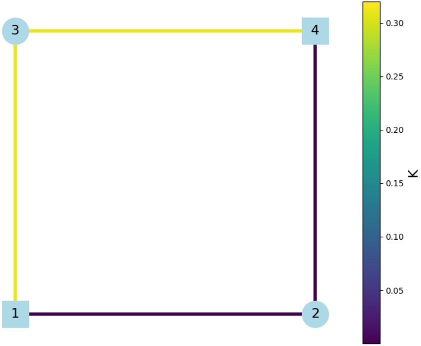

Figure: Optimal capacities for $\gamma = 0.6$ and $\gamma = 1.2$

Square Network
Analytical Solution

- System reduces to two variables
- Using Kirchhoff's laws and the resource constraint:

$$\langle D \rangle = 2\,\langle X^2 \rangle \frac{1}{k_1 + \left[\frac{1}{2} - k_1^{\gamma}\right]^{\frac{1}{\gamma}}}$$

- In the two regimes:

$$\langle D_a \rangle = 2^{1+\frac{1}{\gamma}}\,\langle X^2 \rangle$$

$$\langle D_s \rangle = 4^{\frac{1}{\gamma}}\,\langle X^2 \rangle$$

$$k_a^* = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}$$

$$k_s^* = \left(\frac{1}{4}\right)^{\frac{1}{\gamma}}$$

Square Network

Analytical Dissipation

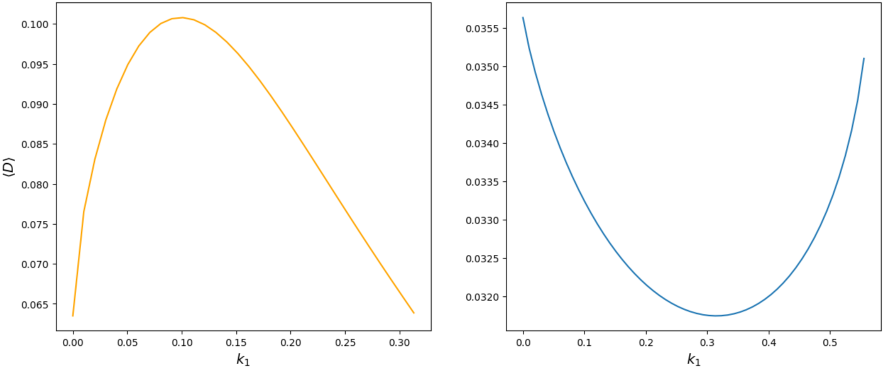

Figure: Analytically obtained dissipation for the asymmetric and symmetric regimes

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Square Network

Analytical Capacities

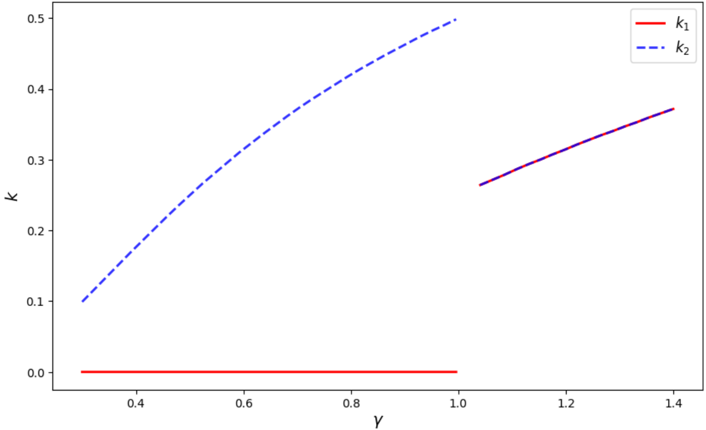


Figure: Analytically obtained values of line capacities

Triangular Network

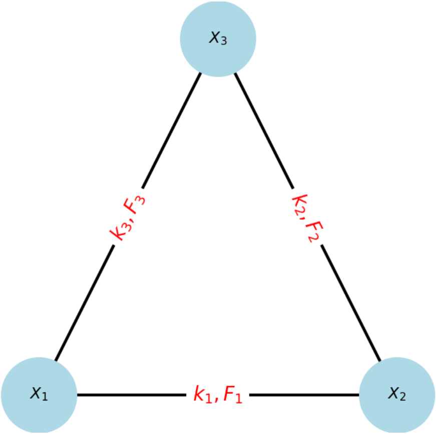
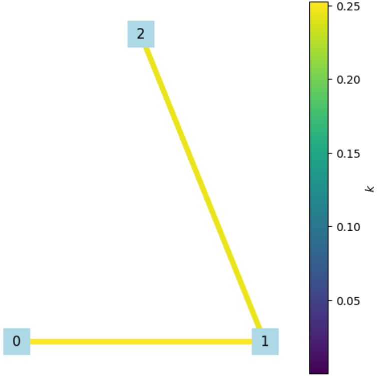


Figure: Schematic of a simple triangular network

Triangular Network

Numerical Simulation



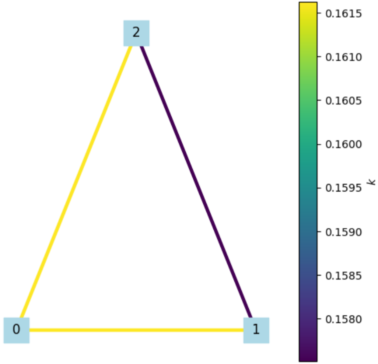


Figure: Optimal capacities for $\gamma = 0.55$ and $\gamma = 0.6$

Triangular Network

Analytical Solution

- Covariance matrix:

$$\Gamma = \frac{\sigma^2}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

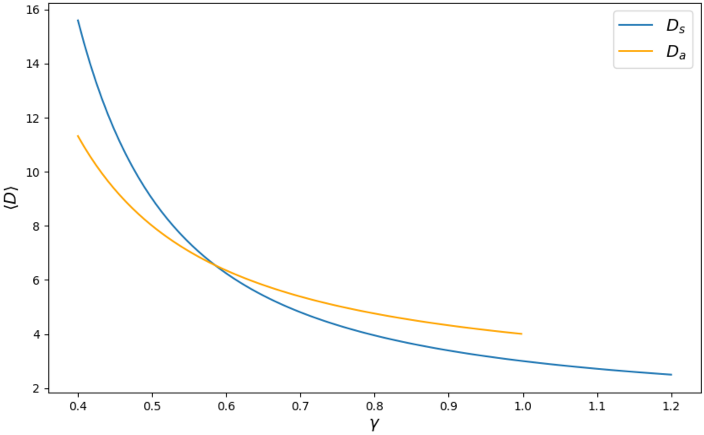
- Analytic calculation of dissipation:

$$\langle D_s \rangle = \frac{1}{k} = \frac{1}{(1/3)^{1/\gamma}}$$

$$\langle D_a \rangle = \frac{2}{k} = \frac{2}{(1/2)^{1/\gamma}}$$

Triangular Network

Analytical Solution



Triangular Network

Numerical Dissipation

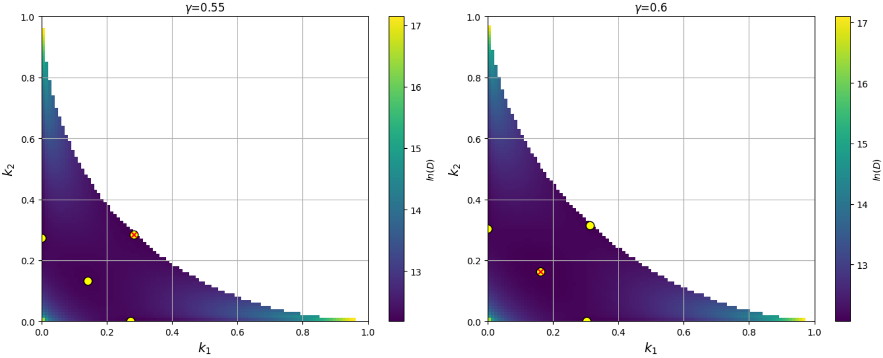


Figure: Optimal capacities for $\gamma = 0.55$ and $\gamma = 0.6$

Triangular Network
Analytical Solution

The capacities are:

$$k_1 = \epsilon$$

$$k_2 = k_3 = \left(\frac{1}{2}(1 - \epsilon^\gamma)\right)^{1/\gamma}$$

By Ohm's law:

$$\vec{F} = K_{\text{diag}} E \vec{\theta}$$

From Power Transfer Distribution Factors:

$$\vec{F} = T\vec{X}$$

From KCL:

$$\vec{X} = E^\top \vec{F}$$

Triangular Network
Continued

The Laplacian matrix is defined as

$$K_{\text{diag}} E := L$$

$$\vec{\theta} = L^{\dagger} \vec{X}$$

$$\vec{F} = K_{\text{diag}} E L^{\dagger} \vec{X}$$

The pseudo-inverse of the Laplacian is calculated by grounding the Laplacian.
The dissipation is:

$$D = \frac{1}{2} \vec{F}^{\top} K_{\text{diag}}^{-1} \vec{F}$$

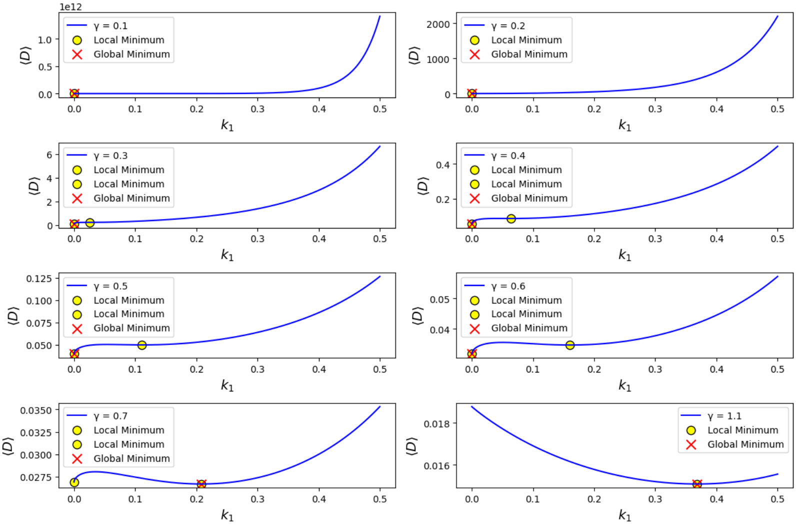
Triangular Network

The obtained expressions for the dissipation are:

$$D = \frac{\epsilon^2 + 4\epsilon k_2 + k_2^2}{2k_2(2\epsilon + k_2)^2} X_2^2 + \frac{2\epsilon^2 + 4\epsilon k_2}{2k_2(2\epsilon + k_2)^2} X_2 X_3 + \frac{2\epsilon^2 + 3\epsilon k_2 + k_2^2}{2k_2(2\epsilon + k_2)^2} X_3^2$$

$$D = \frac{1}{(k_2 + 2\epsilon)^2} \sigma^2 \left(\frac{\epsilon^2}{k_2} + k_2 + 3\epsilon \right)$$

Triangular Network



Ladder Network

- Extension of the square network
- Serves as a rudimentary model of a multilayer network, as a next step
- Use a 2×6 grid

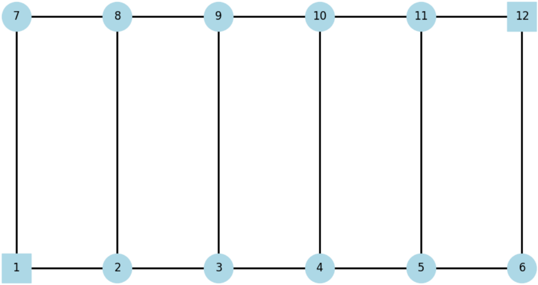


Figure: Schematic of the ladder network

Ladder Network

Numerical Simulation

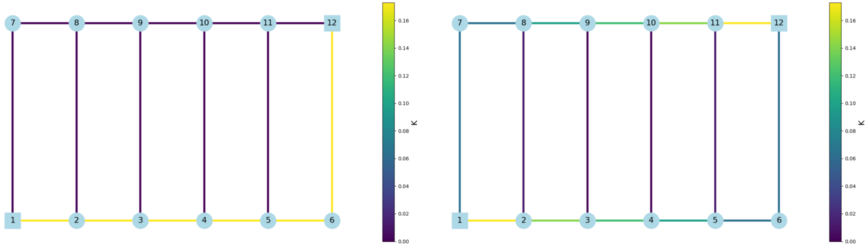


Figure: Optimal capacities for $\gamma = 0.6$ and $\gamma = 1.2$

Ladder Network

Numerical Simulation

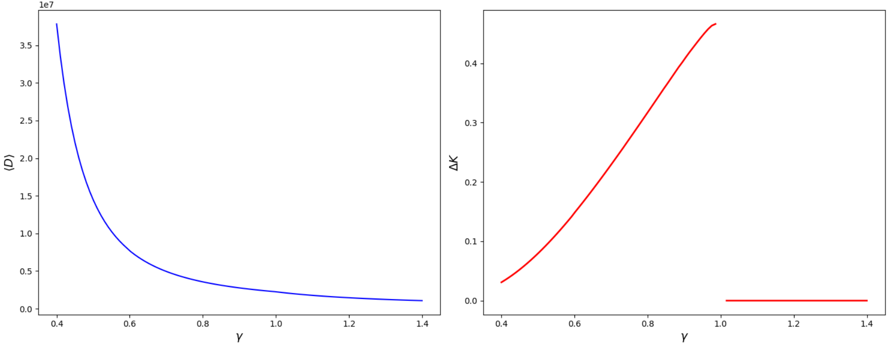


Figure: Dissipation $\langle D \rangle$ and observable $\Delta K = \sum_{\text{upper}} k_j - \sum_{\text{lower}} k_j$

Ladder Network

Numerical Simulation

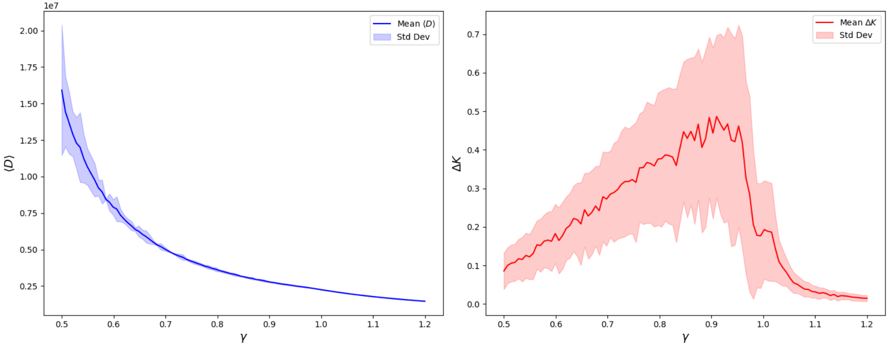


Figure: Objective function $\langle D \rangle$ and observable Δk for 100 different random initial states

Further Aims

Next steps:

- Symmetric fixed point in the triangular network
- Add a fluctuating demand to the passive nodes for the ladder network
- Extend the triangular network to generalize it, again with a fluctuating demand
- Look at the effect of noise
- Extend to multilayer networks

Conclusion

- So far, we have used the Corson and Katifori model to observe symmetry breaking in two types of networks
- Two different routes to symmetry breaking:
  - Bifurcation
  - Exchange of minima
- Generalized this to an extended version of the square network: ladder network
- Next steps to completely understand our two simple models and extend it to networks with non-passive nodes and multilayers.