Math 231 **Pigeonhole!** **Activity Day** 9

Suppose 11 balls must be placed into 10 bins (initially empty). What conclusion can you make about the distribution of the balls into the bins? Put your answer into Question 1 of the Canvas Quiz.

Now suppose 21 balls must be placed into 10 bins (initially empty). What conclusion can you make about the distribution of the balls into the bins? Put your answer into Question 2 of the Canvas Quiz.

Now suppose 51 balls must be placed into 10 bins (initially empty). What conclusion can you make about the distribution of the balls into the bins? Put your answer into Question 3 of the Canvas Quiz.

THE BASIC PIGEONHOLE PRINCIPLE: Let $n \in \mathbb{N}$. If more than n objects are placed into n categories then at least one category contains more than one object.

THE STRONG PIGEONHOLE PRINCIPLE: Let $m, n \in \mathbb{N}$. If more than nm objects are placed into n categories then at least one category has more than m objects.

Show that if you pick five pairwise distinct integers from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ then two of them must add up to nine. Put your proof into Question 4 of the Canvas Quiz.

Hint: Consider the four categories determined by the subsets $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$ and $\{4, 5\}$.

Theorem: For any $n \in \mathbb{N}$, any set of n non-negative integers contains a non-empty subset whose sum is a multiple of n.

Fill-in the details of the following proof of this, putting the missing details on line i of the proof into line i of Question 5 of the Canvas Quiz:

Proof: Let $n \in \mathbb{N}$ and $S = \{k_1, k_2, ..., k_n\}$ be a non-empty **...** of n **...**

For each $i \in \{1, 2, ..., n\}$, define $S_i = \{k_1, k_2, ..., k_i\}$, which is a **...** of S, and

define r_i to be the remainder of $(k_1 + k_2 + \cdots + k_i)$ **...** n.

So for each $i \in \{1, 2, ..., n\}$, $r_i \in$ **...** is the remainder of the sum of elements of **...**

Suppose $\exists j \in \{1, 2, ..., n\}$ such that $r_j = 0$. Then the **...** of the elements

of the subset **...** is a multiple of **...**

Now suppose $\forall j \in \{1, 2, ..., n\}$, $r_j \neq$ **...** Then $r_1, r_2, ..., r_n \in$ **...** Since the

list $r_1, r_2, ..., r_n$ is **...** terms in length and consists of terms chosen from a set of **...** elements

it follows that by the Pigeonhole Principle $\exists \ell, m \in \{1, 2, ..., n\}$ such that **...** and $\ell < m$.

Then both $k_1 + k_2 + \cdots + k_\ell$ and $k_1 + k_2 + \cdots + k_m$ have the **...** when divided by n.

Consider that $(k_1 + k_2 + \cdots + k_m) - (k_1 + k_2 + \cdots + k_\ell) =$ **...** is the difference of two integers

whose **...** are the **...** when divided by n. Hence the sum of the elements

of the subset **...** must be a **...** of n. This completes the proof $\square$

Produce a set of 4 non-negative integers such that every non-empty subset has a sum that is not a multiple of 5. Explain why this doesn't contradict the theorem on the previous page! Put your answer into Question 6 of the Canvas Quiz.

Find the minimum value of n such that a group of n people will contain at least 3 people who were born in a common month. Put your answer into Question 7 of the Canvas Quiz.

Find the minimum value of k such that k people voting for one of 5 candidates in a primary election will cause at least one candidate to get to at least 1000 votes in the election. Put your answer into Question 8 of the Canvas Quiz.

Suppose a building contains x classrooms and each classroom is given a distinct two-digit number. What is the minimum value of x such that at least two classrooms have consecutive numbers? Put your answer into Question 9 of the Canvas Quiz.

BONUS EXERCISE: Assume "friends" is an irreflexive relationship (no one is friends with themselves) and a symmetric relationship (if person x is a friend of person y then person y is a friend of person x).

Show that for any group of two or more people, there must be at least two people who have the same number of friends within the group.

Put your proof into Question 10 of the Canvas Quiz.