Week 2 - Set Theory Discussion Board Exercises

(1) Let $S = \{s, e, t, t, h, e, o, r, y, i, s, f, u, n\}$. Find $|S|$, which is the number of elements in S and called the *cardinality* of S.

(2) Find the union and intersection of the sets $\{1, 3, 5, 7, 9\}$ and $\{2, 3, 5, 7\}$.

(3) Suppose set S has 37 elements, set T has 21 elements, and the union $S \cup T$ has 49 elements. How many elements does the intersection $S \cap T$ have?

(4) At Johnsonville high school, 51 students take Algebra I, 36 take Spanish I and 13 take both. How many are taking Algebra I or Spanish I?

(5) For each integer n let $T_n = \{n, n^3\}$. How many elements are in each of T_{-2}, T_{-1}, T_1 and T_3?

(6) How would the answers change to exercise above if it was n^2 in place of n^3?

(7) Suppose we know that for sets S, T we have that $S \times T = \emptyset$. What can be said about the sets S, T?

(8) List all elements of the power set of $\{1, 2, 3, 4\}$.

(9) List all elements of the direct product $\{1, 2, 3\} \times \{a, b\}$.

(10) Consider the set $S = \{1, 4, 9, 16, ...\}$. Describe this set. How many elements does S have?

(11) Consider the set $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$. Describe this set. How many elements does P have?

(12) Consider the set $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 1 < x^2 + y^2 \leq 4\}$. Draw this set. How many elements does T have?

(13) Consider the set $G = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | 1 < a^2 + b^2 < 9\}$. Draw this set. How many elements does G have?

(14) Consider the set $N = \{x \in \mathbb{R} : x^2 = -1\}$. Describe this set. How many elements does N have?

(15) Consider the set $F = \{A \in \mathscr{P}(\{1, 2, 3, 4, 5, 6\}) : |A| = 5\}$. Describe this set. How many elements does F have?

(16) Consider the set $R = \{z \in \mathbb{R} | a^2 \in \{0, 1, 2, 3, 4\}\}$. Describe this set. How many elements does R have?

(17) Let W be the set consisting of all the letters in the word WORRYWORT. How many elements does W have?

(18) Let Z be the set of two-digit odd positive integers. How many elements does Z have?

(19) For any two finite sets S and T justify that the average of $|S|$ and $|T|$ does not exceed $|S \cup T|$.

(20) Given a family of sets $\{S_i | i \in I\}$ indexed over a set I, is the condition that
$$S_i \cap S_j \neq \emptyset \text{ implies } i = j$$
equivalent to the condition that
$$S_i \cap S_j \neq \emptyset \text{ implies } S_i = S_j?$$
Explain. Hint: Consider $S_k = \{n \in \mathbb{Z} | 0 \leq 2n \leq k\}$ for $k = 1, 2, 3$.

(21) Let S be a non-empty set. Under what conditions on a subset $T \subseteq S$ does $\{T, S \setminus T\}$ form a partition of S? Explain.

(22) Does the family of intervals $\{[n, n+1) | n \in \mathbb{Z}\}$ form a partition of $\mathbb{R}$? Explain.

(23) Does the family of intervals $\{(-n, n) | n \in \mathbb{N}\}$ form a partition of $\mathbb{R}$? Explain.

(24) Let Λ be the set of the 50 states in the USA. Let $\mathcal{F} = \{A, B, C, ..., Z\}$ where A is the set of states in Λ that begin with the letter A, B is the set of states in Λ that begin with the letter B, and so on. Does $\mathcal{F}$ form a partition of Λ? Explain.

(25) Let $\mathcal{F} = \{[1/n, 3 - 1/n] | n \in \mathbb{N}\}$. Determine $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$.

(26) Suppose $\{B_i | i \in I\}$ is a partition of the real number interval $[0, \infty)$. For each $i \in I$, let $S_i = \{x \in \mathbb{R} | x^2 \in B_i\}$. Let $\mathcal{S} = \{S_i | i \in I\}$. Is $\mathcal{S}$ a partition of $\mathbb{R}$?

(27) For each room rented, a beach hotel records a three-symbol code consisting of a $1, 2, 3, 4+$ to specify the number of occupants, an S or N to specify smoking or non-smoking and an S or O to specify street view or ocean view. Let X be the set of all possible codes. Find $|X|$.

(28) One can identify an ordered pair (a, b) with the set $\{\{a\}, \{a, b\}\}$. Find the cardinality of the ordered pairs $(1, 2)$ and $(1, 1)$ identified as sets in this manner. Explain your reasoning.

(29) Let A, B, C be sets. Justify the following so-called distributive property: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(30) Let A, B be sets. Justify $A \cap (A \cup B) = A$.

(31) Let A, B, C be sets. Use DeMorgan's Laws for sets and the distributive property above to justify that $(A \setminus B) \setminus (B \setminus C) = A \setminus C$. Hint: Start with the equality $S \setminus T = S \cap \overline{T}$.

(32) Find sets of integers S_1, S_2, S_3,... such that for every positive integer i, $S_i \supset S_{i+1}$ and $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$.

(33) Let A, B be sets. Find an expression for $(A \times B) \cap (B \times A)$ of the form $C \times C$ for some set C.

(34) Determine if the proposition "Every set is an element of its power set" is true or false, and explain your answer.

(35) Determine if the proposition "Every set is an element of the direct product of the set with itself" is true or false, and explain your answer.

(36) Determine if the proposition "Every set is a subset of its power set" is true or false, and explain your answer.

(37) Determine if the proposition "There exists a set that is a subset of its power set" is true or false, and explain.

(38) Give an example of a set S such that there exists a set T such that $T \in S$ and $T \subseteq S$.

(39) Give an example of a set S such that there exists a set T such that $T \in S$, but $T \not\subseteq S$.

(40) The *symmetric difference* of two sets S and T is all the elements in the union but not in the intersection, denoted $S \Delta T$. Compute $\{5, 10, 15, 20, 25, 30\} \Delta \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$.

(41) Explain why $\{(0, 1), (1, 1), (2, 2), (3, 6), (4, 10), (4, 24), (5, 6)\}$ is NOT a function from the domain $\{0, 1, 2, 3, 4, 5\}$ to any codomain containing at least the numbers 1, 2, 6, 10 and 24.

(42) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) =$ the largest integer less than or equal to x. This function is called the "floor" function. Find $f(1.9)$, $f(2)$ and $f(-3.1)$. Determine the range.

(43) Let $f : \mathbb{Q} \to \mathbb{Q}$ be given by $f(n) = (n/3) - 1$. Demonstrate that this is a bijection and find a rule for the inverse function.

(44) List all functions from $\{1, 2, 3\}$ to $\{1, 2\}$. You may do this by either explicitly writing down all the pairs of each function OR by using arrow diagrams. How many are $1 - 1$? How many are onto?

(45) List all functions f from $\{1, 2, 3\}$ to $\{1, 2, 3\}$ that satisfy $(f \circ f)(x) = x$ for all $x \in \{1, 2, 3\}$. You may do this by either explicitly writing down all the pairs of each function OR by using arrow diagrams. Hint: There are 4 such functions!

(46) Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is an onto function. Explain why the function $g = f \circ f$ must also be onto.

(47) Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = x^n$ where n is a positive integer. For what n is f a bijection?

(48) Let $f : \mathscr{P}(\{1, 2, 3\}) \to \mathbb{Z}$ be given by $f(A) = |A|$. Show that f is not $1 - 1$.

(49) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ that is $1 - 1$ but not onto.

(50) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ that is onto but not $1 - 1$.

(51) Consider the family of sets $\mathcal{Z} = \{Z_k | k \in \mathbb{N}\}$ where $Z_k = \{1, 2, ..., k\}$. Determine $\bigcap \mathcal{Z}$ (the intersection of all sets in the family) and $\bigcup \mathcal{Z}$ (the union of all the sets in the family).

(52) Let $P(x)$ be a proposition function over a domain D. P is said to be "satisfiable" if there exists an x such that $P(x)$ is true. Consider the propositional function $x^2 < x$. Is it satisfiable over the domain of positive real numbers? What about over the domain of positive integers? Explain.

The rest of the exercises deal with the following family of sets:

$\mathcal{D} = \{D_k | k \in \{0, 1, 2, ..., 10\}\}$ where $D_k = \{n \in \mathbb{N} \setminus \{1\} | k$ is a multiple of $n\}$.

(53) Determine D_0, D_1, D_2, D_4, D_5 and D_{10}.

(54) Consider the proposition "For every $j, k \in \{0, 1, 2, ..., 10\}$ if $j \neq k$ then $D_j \neq D_k$." Is it true or false? Explain

(55) Find $|\mathcal{D}|$.