1 HW 1

1.1 Exercise 1

In the notes we showed how to prove the following:

$$\mathbb{E}[X] = \int_0^\infty P(X > t)dt$$

This follows from an application of Fubini's theorem applied to the integral:

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty u f_X(u) \\
&= \int_0^\infty \int_0^\infty \mathbb{1}(0 \le t \le u)\, dt\, f_X(u) \\
&= \int_0^\infty \int_0^\infty \mathbb{1}(0 \le t \le u)\, f_X(u) dt \\
&= \int_0^\infty P(X > t)dt
\end{aligned}$$

Use Fubini's theorem to show the following equality:

$$\int_x^\infty (u - x) f_X(u) du = \int_x^\infty S_X(u) du.$$

Use indicator variables.

1.2 Exercise 2

In the notes we showed that when $\mathbb{E}[|X|] < \infty$ that

$$\lim_{u\to\infty} uP(X > u) = 0. \tag{1}$$

Use Markov's inequality:

$$P(X > a) \le \frac{\mathbb{E}[X]}{a}$$

to show that when $\mathbb{E}[X^2] < \infty$ $\lim_{u\to\infty} uP(X > u) = 0$.

1.3 Exercise 3

Suppose we have a survival time, X that is exponentially distributed with rate θ.

$$X \mid \Theta = \theta \sim \text{Exp}(\theta).$$

Further, suppose that Θ is randomly distributed with a Gamma distribution with parameters α, β:

$$f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\beta\theta) \mathbb{1}(\theta > 0).$$

1. What is the survival function of the marginal random variable X? (Hint: calculate $\mathbb{E}_\Theta\left[e^{-\Theta t}\right]$)

2. What is the hazard function for X marginal over Θ?

3. Use R and your favorite plotting package (base R, ggplot2, etc.) to plot the hazard function for several values of α, β.

1.4 Exercise 4

Let $X_i \overset{\text{iid}}{\sim} \text{Exp}(\beta)$ for $i \in \{1, \ldots, n\}$ and let $Y = \sum_{i=1}^n X_i$.

1. Show that $Y \sim \text{Gamma}(n, \beta)$.

2. Show that the survival function for Y can be expressed in terms of the upper incomplete gamma function.

3. Express the mean residual lifetime in terms of the upper incomplete gamma function

As a reminder, the upper incomplete gamma function, $\Gamma(\alpha, x)$ is defined as:

$$\int_x^\infty u^{\alpha-1} e^{-u} du.$$

Note that the order of the arguments differs by source. Some textbooks define it as $\Gamma(x, \alpha)$, whereas Abramowitz and Stegun 1948 defines it as I have above. Either is fine as long as you are consistent in your definition.

1.5 Exercise 5

The mean residual lifetime, or $\mathbb{E}\left[X - x \mid X > x\right]$ of a random variable X is $x + 10$. What is the hazard function and the survival function of X? Hint: Use the definition of MRL in terms of the survival function,

$$\frac{\int_x^\infty S_X(t) dt}{S_X(x)} = x + 10$$

and try to manipulate this expression to get an expression relating the density to the survival function of X.

References

[1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Vol. 55. US Government printing office, 1948.