A METHOD OF MEASURING THE RESISTIVITY AND HALL COEFFICIENT ON LAMELLAE OF ARBITRARY SHAPE

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Resistivity and Hall-coefficient measurements at different temperatures play an important part in research on semiconductors, such as germanium and silicon [1]), for it is from these quantities that the mobility and concentration of the charge carriers are found.

Such measurements are commonly carried out with a test bar as illustrated in *fig. 1*. The resistivity is found directly from the potential difference and the distance between the contacts O and P, the current i and the dimensions of the bar. To determine the Hall coefficient the bar is subjected to a magnetic field B applied at right angles to the direction of the current and to the line connecting the diametrically opposite contacts O and Q. From the potential difference thus produced between these latter contacts the Hall coefficient is derived. (The relation between the Hall coefficient and the change in electric potential distribution due to a magnetic field will be explained presently.)

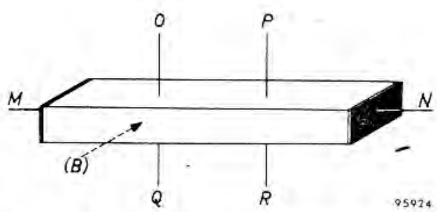


Fig. 1. Classical form of sample used for resistivity and Hall-coefficient measurements. The test bar is provided with current contacts M and N and voltage contacts O, P, Q and R. The fourth voltage contact R serves for check measurements.

In measurements performed at low temperatures — e.g. in liquid nitrogen — point contacts possess resistances of the order of megohms, in consequence of which the voltages cannot be determined with sufficient accuracy. In such cases "bridge-shaped" samples are used as illustrated in *fig. 2*. The voltage and current contacts here have a relatively large surface area, and hence the contact resistances are low.

The methods referred to are based on the fact that the geometry of the sample ensures a simple pattern of virtually parallel current stream-lines. Formulae have been devised to correct for the deviation from parallelism in fig. 2, caused by the finite width of the arms. A drawback of the bridge-shaped sample is that it is rather difficult to make, having to be cut out of the brittle semiconductor material with an ultrasonic tool. There is therefore a considerable risk of breakage, particularly when the arms are made narrow.

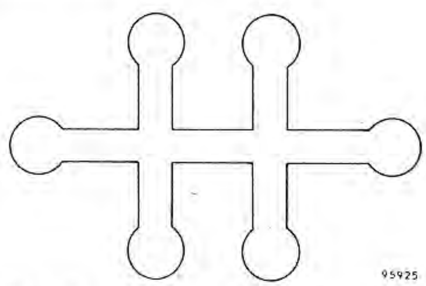


Fig. 2. The bridge-type sample, which is provided with relatively large contact faces to reduce contact resistances. This form is of special importance in measurements at low temperatures.

In the following we shall describe a method of performing resistivity and Hall-coefficient measurements on lamellae of arbitrary shape [2]). The lamella must not, however, contain any (geometrical) holes.

New method of measuring resistivity

We take a flat lamella, completely *free of holes*, and provide it with four *small* contacts, M, N, O and P, at arbitrary places *on the periphery* (*fig. 3*). We apply a current i_{MN} to contact M and take it off at contact N. We measure the potential difference $V_P - V_O$ and define:

$$R_{MN,OP} = \frac{V_P - V_O}{i_{MN}}.$$

Analogously we define:

$$R_{NO,PM} = \frac{V_M - V_P}{i_{NO}}.$$

The new method of measurement is based on the theorem that between $R_{MN,OP}$ and $R_{NO,PM}$ there exists the simple relation:

$$\exp\left(-\frac{\pi d}{\varrho} R_{MN,OP}\right) + \exp\left(-\frac{\pi d}{\varrho} R_{NO,PM}\right) = 1, \quad (1)$$

where d is the thickness of the lamella and ϱ the

[1]) See e.g. C. Kittel, Introduction to solid state physics, 2nd edition, Wiley, New York 1956, Chapter 13, p. 347 et seq.

[2]) L. J. van der Pauw, A method of measuring specific resistivity and Hall effect of discs of arbitrary shape, Philips Res. Repts. **13**, 1-9, 1958 (No. 1).

resistivity of the material. If d and the "resistances" $R_{MN,OP}$ and $R_{NO,PM}$ are known, then (1) yields an equation in which ϱ is the only unknown quantity.

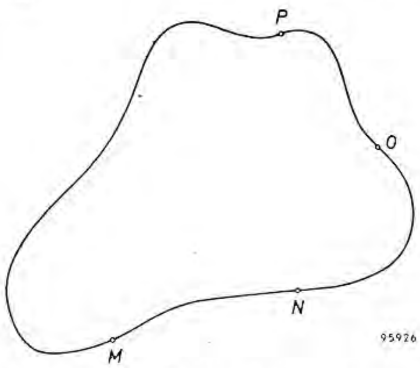


Fig. 3. A flat lamella of arbitrary shape, with four contacts M, N, O and P on the periphery, as used in the new method of measuring resistivity. The Hall coefficient can also be measured on a sample of this kind.

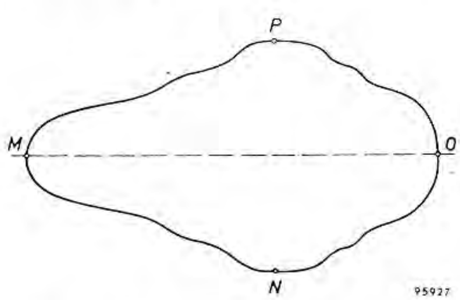
The situation is particularly straightforward if the sample possesses a line of symmetry. In that case, M and O are placed on the line of symmetry, while N and P are disposed symmetrically with respect to this line (*fig. 4*). Then:

$$R_{NO,PM} = R_{MN,OP}, \quad \dots\dots \quad (2)$$

which may be seen as follows. From the reciprocity theorem for passive fourpoles, we have quite generally that $R_{NO,PM} = R_{PM,NO}$ (interchange of current and voltage contacts), and it follows from the symmetry that $R_{PM,NO} = R_{MN,OP}$. Hence we arrive at (2); ϱ can then easily be found from (1):

$$\varrho = \frac{\pi d}{\ln 2} R_{MN,OP}. \quad \dots\dots \quad (3)$$

In this case a single measurement suffices.



. 4. The resistivity measurement is simplified if the sample a line of symmetry. If two of the contacts are situated he line of symmetry and the two others are symmetrically ted with respect to this line, one measurement is sufficient ive the required resistivity.

In the general case it is not possible to express ϱ explicitly in known functions. The solution can, however, be written in the form

$$\varrho = \frac{\pi d}{\ln 2} \frac{R_{MN,OP} + R_{NO,PM}}{2} f, \quad \dots \quad (4)$$

where f is a factor which is a function only of the ratio $R_{MN,OP}/R_{NO,PM}$, as plotted in *fig. 5*. Thus, to determine ϱ, we first calculate $R_{MN,OP}/R_{NO,PM}$, read from fig. 5 the corresponding value of f and then find ϱ from (4).

Photographs of samples as used for the old and for the new method are shown in *fig. 6*.

The complete proof of the theorem underlying the measurement of ϱ is given in the paper quoted in footnote [2]). The proof consists of two parts. First of all, relation (1) is developed for a special case, the case of a lamella in the form of an infinite half-plane, provided with four contacts at the

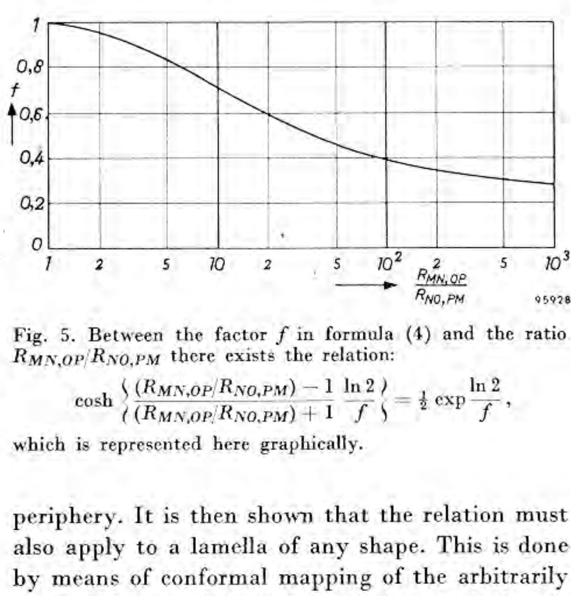


Fig. 5. Between the factor f in formula (4) and the ratio $R_{MN,OP}/R_{NO,PM}$ there exists the relation:

$$\cosh\left\{\frac{(R_{MN,OP}/R_{NO,PM}) - 1}{(R_{MN,OP}/R_{NO,PM}) + 1}\,\frac{\ln 2}{f}\right\} = \tfrac{1}{2}\exp\frac{\ln 2}{f},$$

which is represented here graphically.

periphery. It is then shown that the relation must also apply to a lamella of any shape. This is done by means of conformal mapping of the arbitrarily shaped plate on the infinite half-plane with the aid of complex functions.

We shall consider the first part of the proof in more detail, since it reveals the origin of the exponential functions in (1).

We first consider a lamella which extends to infinity in all directions. To a point M we apply a current $2i$, which flows away from M with radial symmetry into infinity. Let d again be the thickness of the lamella and ϱ the resistivity. Then at a distance r from M the current density is

$$J = 2i/2\pi r d.$$

The field-strength E is radially oriented and according to the generalized form of Ohm's law has the value:

$$E = \varrho J = \varrho i/\pi r d.$$

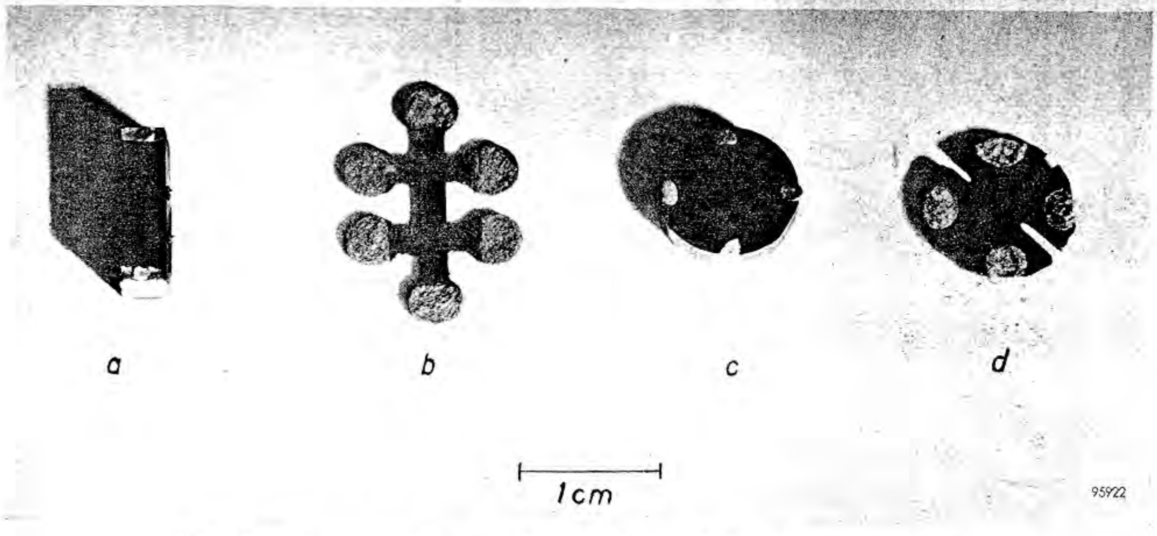


Fig. 6. Some samples of silicon used for resistivity and Hall-coefficient measurements. Samples a and b correspond respectively to figs. 1 and 2. Measurements on samples c and d are possible only by the new method. The incisions in sample d serve to reduce the error caused by the contacts not being infinitely small.

The potential difference between two points O and P lying on a straight line with M (*fig. 7a*) is:

$$V_P - V_O = \int_P^O E\,\mathrm{d}r = \frac{\varrho i}{\pi d} \int_P^O \frac{\mathrm{d}r}{r} = -\frac{\varrho i}{\pi d} \ln \frac{a+b+c}{a+b}.$$

Since no current flows in the direction perpendicular to the line through M, O and P, the result obtained remains valid if we omit the part of the lamella at one side of this line — yielding a half-plane — and if at the same time we halve the current (fig. 7b).

Next we consider the case of c) in fig. 7, where a current i now flows *out* from a point N, that again lies on the same straight line with OP, viz. on the edge of the infinite half-plane. We now find:

$$V_P - V_O = +\frac{\varrho i}{\pi d} \ln \frac{b+c}{b}.$$

Superposition of the cases b) and c) in fig. 7 yields the case d), the current i being introduced at M and taken off at N. The value now assumed by $V_P - V_O$ is found by adding together the two previous results. After dividing by i we then find:

$$R_{MN,OP} = \frac{\varrho}{\pi d} \ln \frac{(a+b)\,(b+c)}{(a+b+c)\,b},$$

or

$$\frac{(a+b+c)b}{(a+b)\,(b+c)} = \exp\left(-\frac{\pi d}{\varrho} R_{MN,OP}\right).$$

Similarly we find:

$$\frac{a\,c}{(a+b)\,(b+c)} = \exp\left(-\frac{\pi d}{\varrho} R_{NO,PM}\right).$$

Addition of the last two equations yields (1).

We shall now explain how formula (4) follows from (1). For simplification we put:

$$\left.\begin{aligned} \pi\,d\,R_{MN,OP} &= x_1\,, \\ \pi\,d\,R_{NO,PM} &= x_2\,. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5)$$

Formula (1) can then be written:

$$e^{-\frac{x_1}{\varrho}} + e^{-\frac{x_2}{\varrho}} = 1\,. \quad \ldots\ldots\ldots \quad (6)$$

Next we write:

$$x_1 = \tfrac{1}{2}\{(x_1+x_2) + (x_1-x_2)\}$$

and $$x_2 = \tfrac{1}{2}\{(x_1+x_2) - (x_1-x_2)\},$$

whereby (6) takes the form:

$$e^{-\frac{x_1+x_2}{2\varrho}} \left(e^{-\frac{x_1-x_2}{2\varrho}} + e^{+\frac{x_1-x_2}{2\varrho}}\right) = 1\,.$$

This is the same as:

$$e^{-\frac{x_1+x_2}{2\varrho}} \cosh \frac{x_1-x_2}{2\varrho} = \tfrac{1}{2}\,. \quad \ldots\ldots \quad (7)$$

The exponent of e in (7) is now written as $-(\ln 2)/f$, i.e. we put:

$$\frac{x_1+x_2}{2\varrho} = \frac{\ln 2}{f}\,. \quad \ldots\ldots\ldots \quad (8)$$

Formula (7) then becomes:

$$e^{-\frac{\ln 2}{f}} \cosh \left\{ \frac{(x_1/x_2)-1}{(x_1/x_2)+1} \times \frac{\ln 2}{f} \right\} = \tfrac{1}{2}\,.$$

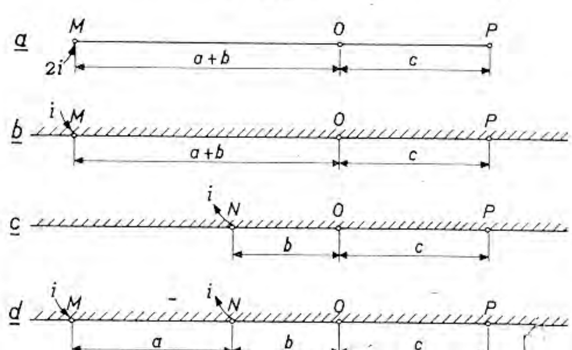


Fig. 7. Illustrating the derivation of formula (1).

This expression represents a relation between f and x_1/x_2, and hence also between f and $R_{MN,OP}/R_{NO,PM}$ (see 5). The relation is shown graphically in fig. 5. By re-writing (8) to give ϱ and substituting for x_1 and x_2 from (5), we find formula (4).

Method of measuring the Hall coefficient

The Hall coefficient, too, can be measured on an arbitrary lamella as in fig. 3. We then apply the current to one of the contacts, say M, and take it off at the contact following the succeeding one, i.e. in our case at O. We measure $R_{MO,NP}$, after which we set up an uniform magnetic induction B at right angles to the surface of the lamella. This changes $R_{MO,NP}$ by an amount $\Delta R_{MO,NP}$. We shall now denote the Hall coefficient R_H and show that it is given by:

$$R_H = \frac{d}{B}\,\Delta R_{MO,NP}, \quad \dots \quad (9)$$

provided that:

a) the contacts are sufficiently small,
b) the contacts are on the periphery,
c) the lamella is of uniform thickness and free of holes.

The validity of formula (9) depends on the distribution of current stream-lines not changing when the magnetic field is applied. With samples of the classical shape of figs. 1 and 2, where the current stream-lines are always parallel to the edges of the sample, there is evidently no change. From the properties of the vector field representing the current density it follows that the same also applies to lamellae of arbitrary shape, provided the above conditions are satisfied [3]).

Under the magnetic induction B, the charge carriers, with charge q, are subjected to a force perpendicular to the stream-lines and perpendicular to the lines of magnetic induction. The magnitude of this force is $F = qvB$, where v is the velocity of the charge carriers. Between v, the concentration n of the charge carriers and the current density J there exists the relation $v = J/nq$. Dividing the force exerted on the charge carriers by their charge q, we see that the effect of the magnetic field is equivalent to an apparent electric field E_H, the Hall electric field, for which we can write [4]):

$$E_H = \frac{1}{nq}\,J\,B\,.$$

E_H is proportional to J and to B; the proportionality constant ($= 1/nq$) is called the Hall coefficient R_H.

Since q is known, one can calculate from R_H the concentration n of the charge carriers.

The fact that the current stream-lines are not affected by the magnetic field implies that after application of the magnetic field the electric field is no longer in the same direction as the current stream-lines, but has acquired a transverse component E_t which exactly compensates the

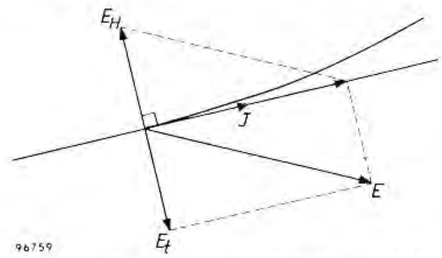


Fig. 8. The resultant of the electrical field-strength E and the Hall field-strength E_H lies in the direction of the current density J. Resolving E in directions perpendicular and parallel to J therefore yields a perpendicular component E_t which in magnitude is equal to E_H.

apparent Hall electric field E_H (*fig. 8*). The change $\Delta(V_P - V_N)$ in the potential difference between P and N is therefore given by integrating E_t from P over a path orthogonal to the current stream-lines to N' across the lamella (*fig. 9*), and thence along the periphery — i.e. along a stream-line — from N' to N. The last portion of the path makes no contribution to the integral; hence

$$\Delta(V_P - V_N) = \int_P^{N'} E_H\,\mathrm{d}s = R_H B \int_P^{N'} J\,\mathrm{d}s = R_H B\,\frac{i_{MO}}{d}\,,$$

where d is again the thickness of the lamella. This expression leads directly to (9).

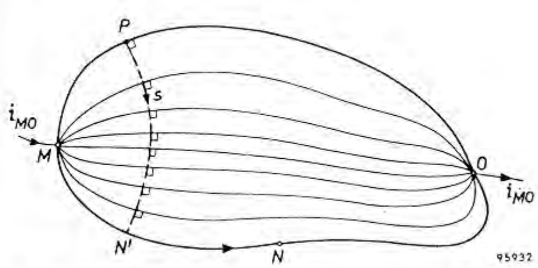


Fig. 9. To calculate by how much the potential difference between points P and N changes when a magnetic field is applied at right-angles to the plane of the sample, the transverse electric field E_t produced by the magnetic field is integrated along the path s which runs from P, orthogonal to the current stream-lines, to N' and thence along the periphery from N' to N.

[3]) The proof of this statement is also indicated in the paper quoted under [2]).

[4]) This formula is not entirely exact because, apart from their ordered motion with velocity v, the electrons also move randomly owing to thermal agitation. More precise calculation shows, however, that the formula given here is a good approximation.

Estimation of errors

In the foregoing we have assumed the contacts to be "sufficiently" small and to be situated on the periphery. To gain an idea of the error made in the calculations when these conditions are not exactly satisfied, we have worked out the error for three cases. For simplicity we consider a circular disc of diameter D with the contacts mutually 90° apart. We assume further that only one of the contacts is not ideal. The three cases are represented in the adjoining *table*, together with the formulae for the relative errors in the resistivity and the Hall coefficient. The cases are:

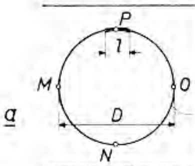
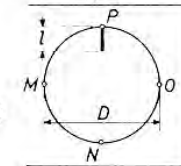
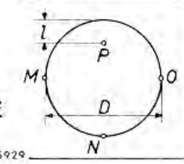
a) One of the contacts has a length l along the periphery.

b) One of the contacts has a length l perpendicular to the periphery.

c) One of the contacts, although a point, is situated at a distance l from the periphery.

In practice, none of the contacts will be ideal. To the first approximation the total error is then equal to the sum of the errors per contact.

Table. The relative errors $\Delta\varrho/\varrho$ and $\Delta R_H/R_H$ in the calculated values of the resistivity and the Hall coefficient for a circular disc of diameter D on which one contact P is non-ideal, in the senses indicated in the sketches.

		$\Delta\varrho/\varrho$	$\Delta R_H/R_H$
a		$\approx \frac{-l^2}{16D^2 \ln 2}$	$\approx \frac{-2l}{\pi^2 D}$
b		$\approx \frac{-l^2}{4D^2 \ln 2}$	$\approx \frac{-4l}{\pi^2 D}$
c		$\approx \frac{-l^2}{2D^2 \ln 2}$	$\approx \frac{-2l}{\pi D}$

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The value of the method described here lies in the fact that, in many cases, the material under investigation is already available in the form of small lamellae (e.g. thin discs sawn from a crystal rod); these samples now need no further preparation and can therefore be used for other purposes after the measurement.

If very small contacts are undesirable, having regard to contact resistances in measurements at low temperature, use can be made of "clover-leaf" samples (see fig. 6*d*), the incisions in which substantially reduce the error due to the finite dimensions of the contacts. The clover-leaf sample thus replaces the bridge-type sample (fig. 6*b*) which is used for the same purpose in the classical method. The clover-leaf sample is easier to make than the bridge-type sample and is also less susceptible to breakage.

L. J. van der PAUW.