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ANALYSIS OF SYMMETRICAL VAN DER PAUW STRUCTURES WITH FINITE CONTACTS

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Abstract—A theory is given for Van der Pauw structures with contacts of non-zero lengths which are invariant for rotations of 90°. A general theorem has been derived. For a certain class of structures an analytical solution of the Laplace equation is obtained instead of using finite difference equations. The method leans heavily on Schwarz-Christoffel transformation techniques.

NOTATION

- R_s sheet resistance
- V voltage
- I current
- $C(\lambda)$ coefficient depending on λ
- λ,μ ratio between sum of lengths of contacts and length of boundary of sheet
- ρ specific resistivity of sample
- d thickness of sample
- $K(k), K'(k)$ complete elliptic integrals with modulus k
- t, z complex variables
- $F(\phi,k)$ elliptic integral of the first kind with argument ϕ and modulus k
- $cn(u,k)$ Jacobian elliptic function with argument u and modulus k
- K $K(k_0)$
- k_0 $\frac{1}{2}\sqrt{2}$
- $C_0(\lambda)$ coefficient for square with contacts in the middle of each side
- $C_1(\mu)$ coefficient for unity circle
- $C_2(\lambda)$ coefficient for square with dual contacts
- $C_3(\lambda)$ coefficient for symmetric octagon.

1. INTRODUCTION

Van der Pauw has developed a method of measuring the sheet resistance of a four-terminal conducting sheet of arbitrary shape[1]. Assuming the terminals to be point contacts at the periphery of the structure, he proved a general theorem for such structures which yields an analytical expression for the sheet resistance R_s. In the special case that the structure is invariant for a rotation of 90°, the formula of Van der Pauw is

$$R_s = \frac{\pi}{\ln 2}\frac{V}{I} \tag{1}$$

where V is the voltage between the two voltage contacts and I is the current flowing through the sample.

From now on structures which are invariant for rotations of 90° will be called symmetrical structures. Formula (1) is no longer true for symmetrical Van der Pauw structures with contacts of finite lengths. Until now such structures have been analysed numerically by Chwang *et al.*[2] and David *et al.*[3].

It is clear that an expression analogous to (1) is valid for any symmetrical structure, viz.:

$$R_s = C(\lambda)\frac{V}{I} \tag{2}$$

where λ is the ratio between the sum of the lengths of the contacts and the length of the boundary of the sheet. The coefficient $C(\lambda)$ is a function of λ and depends on the shape of the structure itself.

It will be shown in this paper that $C(\lambda)$ can be calculated analytically for several practical configurations avoiding the problem of solving the Laplace equation numerically by means of finite difference equations. In fact in our treatment we apply conformal transformations.

Related investigations on Hall devices were carried out by Wick[4] and Haeusler *et al.*[5]. Their approach involves a transformation in which the structure is mapped directly on to a semi-infinite plane. This makes it necessary to deal with four line contacts located along the real axis. In contrast to this, however, it will be shown that the symmetry of the structure allows a treatment by means of two separate structures, each of which having two line contacts. Besides, integrals which occur in our analysis can be calculated without using iteration techniques.

In the next section we shall derive a general theorem concerning the sheet resistance R_s. In Section 3 the theorem will be applied to the square structure with contacts in the middle of each side (see Fig. 7 of [3]). In Section 4 some further properties of this structure will be derived.

However, a direct application of the theorem to other structures has not yet been found. In order to obtain further results one has to use auxiliary transformations, e.g. the unity circle has to be mapped on to a square (see Section 5). Having obtained the results for the circle, we proceed with relating two other structures, i.e. the square with dual contacts and the octagon with four axes of symmetry to the unity circle (see Sections 6 and 7).

2. A THEOREM FOR SYMMETRICAL VAN DER PAUW STRUCTURES WITH FINITE CONTACTS

Consider the symmetrical Van der Pauw structure of Fig. 1(a). The four contacts have equal lengths. The current I enters the sample at contact 1 and leaves it at contact 2. In order to find an expression for the sheet resistance R_s it is necessary to determine the potential difference $V_4 - V_3$ between the contacts 4 and 3 due to the current I.

In Figs. 1(b) and (c) the same structure is considered but now with different current distributions. These distributions are chosen in such a way that superposition yields the current distribution of Fig. 1(a). This is achieved by introducing equal currents of value $\frac{1}{2}I$ at each contact (see Figs. 1(b), (c)).

From the symmetry it is clear that there is a symmetrically situated line in Fig. 1(b) (the dotted line) such that the current lines do not intersect this line. If the structure is mapped on to a circle (such that the origin is mapped on to the centre) the dotted line is mapped on to a straight line through the centre and lies symmetrically with respect to the contacts. Similarly the structure of Fig. 1(c) is divided into four equal parts. By considering the conformal mapping mentioned it is found that $V_4' = V_1'$, $V_3' = V_2'$, $V_4'' = V_2''$ and $V_3'' = V_1''$. Then

$$V_4 - V_3 = V_1' - V_2' + V_2'' - V_1'' \tag{3}$$

where V_i' and V_i'' are the potentials of contacts i in Fig. 1(b) and (c) respectively.

In order to calculate $V_1' - V_2'$ one only has to consider one half of the structure of Fig. 1(b) (see Fig. 2). This means that the number of contacts is now 2 instead of 4. By a conformal transformation from the complex z-plane (Fig. 2) on another complex plane, which we shall call

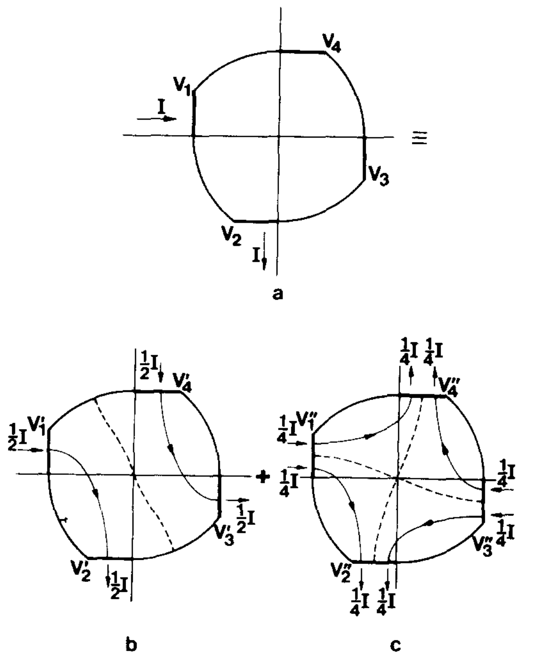

Fig. 1 Reduction to two simpler potential problems.

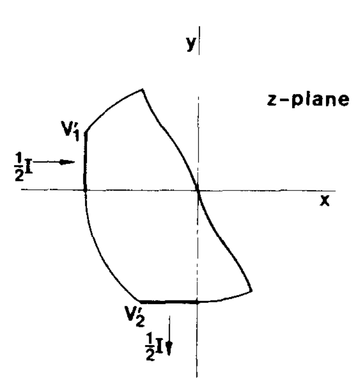

Fig 2 Structure with two contacts First case

the t-plane, the sample of Fig. 2 is mapped on to the upper half-plane of the t-plane (Fig. 3).

Theoretically, it is always possible to find a transformation in such a way that the corresponding contacts are lying symmetrically with respect to the imaginary s-axis. A solution of the Laplace equation is known[6] for the structure of Fig. 3. By using Van der Pauw's reasoning (see Section 2 of [1]) it is found that if

$$V_2' - V_1' = 2K(k_1) \tag{4}$$

then

$$\frac{\rho}{d}\frac{1}{2}I = K'(k_1) \tag{5}$$

where $k_1 = a_1/b_1$; $K(k_1)$ and $K'(k_1)$ are complete elliptic integrals, d is the thickness of the sample and ρ its specific resistivity. From (4) and (5) one derives

$$\frac{V_2' - V_1'}{I} = \frac{\rho}{d}\frac{K(k_1)}{K'(k_1)}. \tag{6}$$

The potential problem of the structure of Fig. 1(c) can be treated in the same way. One now has to analyse one quarter (see Fig. 4). Again the sample of Fig. 4 is mapped on to the upper half-plane of another complex plane. It is easily seen that in this case

$$\frac{V_2'' - V_1''}{I} = \frac{\rho}{2d}\frac{K(k_2)}{K'(k_2)} \tag{7}$$

where $k_2 = a_2/b_2$ and a_2 and b_2 correspond to a_1 and b_1 (see Fig. 3). From (3), (6) and (7) we derive the following theorem for $R_s = \rho/d$:

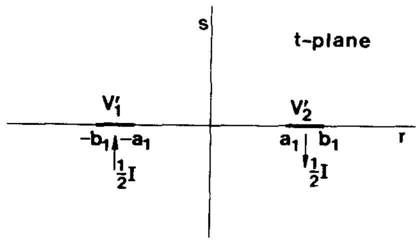

Fig 3. Structure coinciding with the upper half-plane

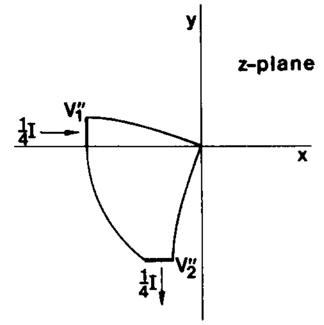

Fig. 4. Structure with two contacts. Second case.

Theorem

The sheet resistance R_s of a Van der Pauw structure with finite contacts, which is invariant for rotations of 90° is

$$R_s = \left[\frac{K(k_1)}{K'(k_1)} - \frac{K(k_2)}{2K'(k_2)}\right]^{-1} \frac{V}{I} \tag{8}$$

where $V = V_4 - V_3$.

If one can determine the values of k_1 and k_2, the problem of calculating the sheet resistance R_s from measurements on a certain structure is solved. It turns out that a lot of structures admit of such a solution by means of an analytical treatment followed by relatively simple numerical calculations.

3. THE SQUARE STRUCTURE WITH CONTACTS IN THE MIDDLE OF EACH SIDE

To illustrate our method, we shall apply the above theorem to a sample in the form of a square. The contacts are lying in the middle of each side (Fig. 5(a); see [3] Fig. 7). The two reduced structures are shown in Figs. 5(b) and (c) (compare Figs. 2 and 4). The angular points of the triangle are situated in $z = \pm 1$ and $z = -i$, those of the little square in $z = 0$ $z = (1-i)/2$, $z = (-1-i)/2$ and $z = -i$. Evidently this choice does not mean a restriction.

By applying the transformation of Schwarz–Christoffel (e.g. [7]) the triangle is mapped on to the upper half-plane (Fig. 6). One finds

$$z = A_1 \int_0^t \frac{dp}{p^{1/2}(1-p^2)^{3/4}} + A_2. \tag{9}$$

If point $z = -i$ is mapped on point $t = 0$ and $z = 1$ on $t = 1$, then the constants A_1 and A_2 are fixed. In order to calculate k_1, we are only interested in real values of t such that $0 < t < 1$. By substituting $p = w^2/(1+w^4)^{1/2}$ the transformation becomes

$$z = 2A_1 J + A_2 \quad (0 < t < 1) \tag{10}$$

where

$$J = \int_0^q \frac{dw}{(1+w^4)^{1/2}}.$$

†Formula 3 143 no. 1 in [8] contains a minor error.

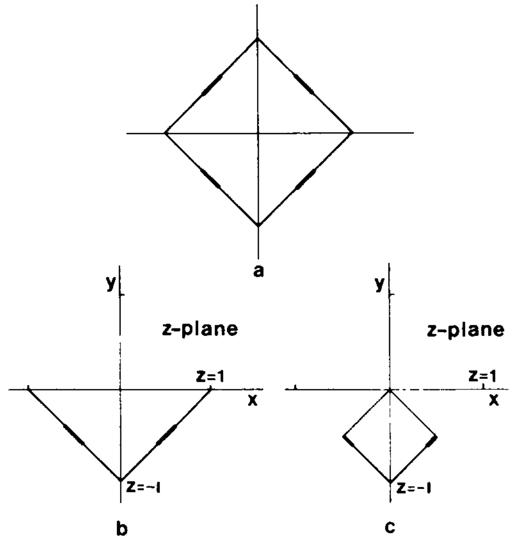

Fig. 5(a). Square with contacts in the middle of each side (b) Reduced structure. First case (c) Reduced structure. Second case.

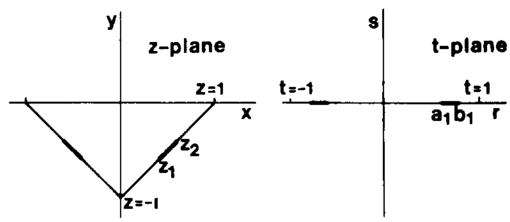

Fig. 6. Mapping of triangle in z-plane on to upper half-plane of t-plane. The ends z_1 and z_2 of one contact correspond to the ends a_1 and b_1 in the t-plane

Here $q = t^{1/2}/(1-t^2)^{1/4}$. If $0 < q < 1$ or $0 < t < \frac{1}{2}\sqrt{2}$, then (see [8], p. 239†)

$$J = \tfrac{1}{2}K - \tfrac{1}{2}F(\psi, k_0)$$

where $k_0 = \frac{1}{2}\sqrt{2}$, $K = K(k_0)$ and $\psi = \arccos\{t^{1/2}(1-t^2)^{1/4}\sqrt{2}\}$. The function $F(\phi, k)$ is the elliptic integral of the first kind

$$F(\phi, k) = \int_0^{\sin\phi} \frac{dx}{(1-x^2)^{1/2}(1-k^2x^2)^{1/2}}.$$

If $q > 1$, then

$$J = \tfrac{1}{2}K + \tfrac{1}{2}F(\psi, k_0).$$

Further, it turns out by straightforward calculations that $A_1 = (1+i)/(2K)$ and $A_2 = -i$. Formula (10) takes the form

$$z = \frac{1-i}{2} \mp \frac{1+i}{2K} F(\psi, k_0) \quad \begin{pmatrix} 0 < t < \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} < t < 1 \end{pmatrix}. \tag{11}$$

If $t = \frac{1}{2}\sqrt{2}$ then $F(\psi, k_0) = 0$ and $z = (1-i)/2$. We are now in a position to calculate k_1. One has

$$cn(u, k_0) = \cos\psi = t^{1/2}(1-t^2)^{1/4}\sqrt{2} \tag{12}$$

where $u = F(\psi, k_0)$ and $cn(u, k_0)$ is a Jacobian elliptic function with argument u and modulus k_0.

Denoting the ends of the contact on the line connecting the points $z = -i$ and $z = 1$ by z_1 and z_2 (Fig. 6), we have

$$\left.\begin{matrix} z_1 \\ z_2 \end{matrix}\right\} = \frac{1-i}{2} \mp \frac{1+i}{2}\lambda \tag{13}$$

Since $z = z_1$ and $z = z_2$ correspond to $t = a_1$ and $t = a_2$ respectively, it follows from (11), (12) and (13) that a_1 and b_1 satisfy the equation

$$4t^4 - 4t^2 + cn^4(K\lambda, k_0) = 0.$$

We find

$$k_1 = \frac{a_1}{b_1} = \left[\frac{1 - \sqrt{1 - cn^4(K\lambda, k_0)}}{1 + \sqrt{1 - cn^4(K\lambda, k_0)}}\right]^{1/2}$$

The second step is the determination of k_2. By applying the transformation of Schwarz–Christoffel to the structure of Fig. 5(c), a mapping is obtained on to the upper half-plane of a t-plane (Fig 7):

$$z = B_1 \int_0^t \frac{\mathrm{d}p}{p^{1/2}(1-p^2)^{1/2}} + B_2. \tag{14}$$

This can be written for positive values of t as (e g. [8], p. 219)

$$z = B_1\sqrt{2}F(\eta, k_0) + B_2 \qquad (0 < t < 1) \tag{15}$$

where $\eta = \arcsin\{2t/(t+1)\}^{1/2}$. If $z = -i$ corresponds to $t = 0$ and $z = (1-i)/2$ to $t = 1$, one finds $B_1 = (1+i)/(2K\sqrt{2})$ and $B_2 = -i$ From (15) one can calculate $t = a_2$ which corresponds to $z = z_1$ (see Fig. 7). Thus k_2 is known, since $k_2 = a_2/b_2 = a_2$.

The constants k_1 and k_2 being calculated for the square with contacts in the middle of each side (Fig. 5), the coefficient $C(\lambda)$ in formula (8) can also be determined:

$$C(\lambda) = \left[\frac{K(k_1)}{K'(k_1)} - \frac{K(k_2)}{2K'(k_2)}\right]^{-1}. \tag{16}$$

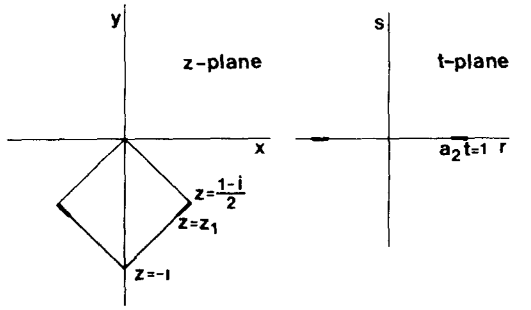

Fig. 7. Mapping of reduced square in z-plane on to upper half-plane of t-plane. The ends of one contact z_1 and $(1-i)/2$ correspond to the ends $t = a_2$ and $t = 1$

†Numerical calculations have been carried out on a Burroughs 7700 computer

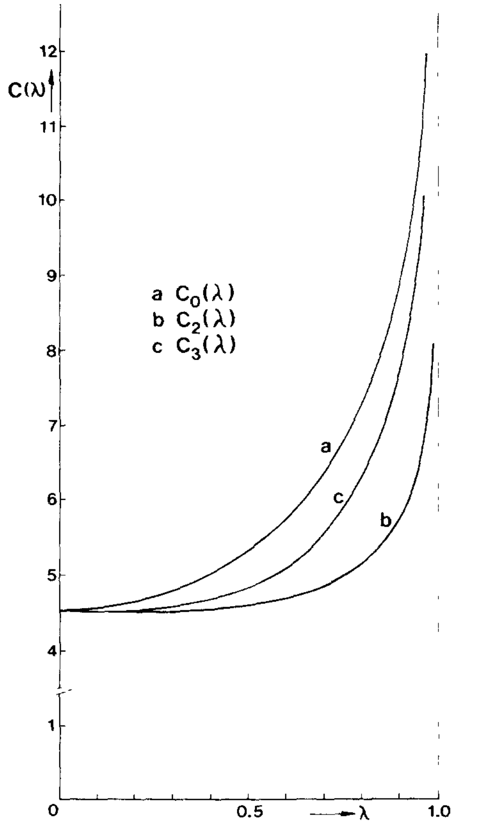

Fig 8 $C(\lambda)$ vs λ for several structures, where λ is the ratio between the total length of the contacts and the length of the boundary (a) Square with contacts in the middle of each side (Fig 5) (b) Square with dual contacts (Fig 11b) (c) Octagon (Fig 12b)

From now on we denote by $C_0(\lambda)$ the function $C(\lambda)$ that belongs to this structure. In Fig. 8 the coefficient† $C_0(\lambda)$ is given as a function of λ. There is a good agreement between the data of David *et al* (Fig. 7 of [3]) and ours. In the next section we shall discuss the function $C_0(\lambda)$ if λ tends to zero and if λ tends to one

4. ASYMPTOTIC BEHAVIOUR OF $C_0(\lambda)$ FOR A SQUARE WITH FINITE CONTACTS IN THE MIDDLE OF EACH SIDE

Evidently if the contacts approach point contacts, i.e. if λ tends to zero, then $C_0(\lambda)$ tends to $\pi/\ln 2$ Also, if the contacts are approaximately as long as the sides, i.e if λ tends to one, then $C_0(\lambda)$ can be expressed by a simple formula. Both statements, which will be proved below, supply us with a tool to check our results.

Starting with the first statement, one has to consider both mappings expressed by (9) and (14). From (9) it is easy to obtain the magnitude $|\mathrm{d}z/\mathrm{d}t|$ of the mapping.

$$\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| = \frac{1}{K\sqrt{2}}\left|\frac{1}{t^{1/2}(1-t^2)^{3/4}}\right|.$$

Introducing finite differences Δz and Δt, one finds, if $t = \frac{1}{2}\sqrt{2}$,

$$\left|\frac{\Delta z}{\Delta t}\right| = \frac{\sqrt{2}}{K}.$$

Let $\delta = |\Delta z| = |z_2 - z_1|$. Then the length of the contact in the t-plane is

$$|\Delta t| = \frac{K}{\sqrt{2}}\,\delta.$$

Thus, if λ tends to zero, one finds (see Fig. 6)

$$k_1 = a_1/b_1 = 1 - K\delta.$$

The complementary modulus k' being defined by $k' = (1-k^2)^{1/2}$, one obtains

$$k_1'^2 = 2\,K\delta. \tag{17}$$

From (14) one has

$$\Delta z = z_2 - z_1 = B_1 \int_{1-\Delta t}^{1} \frac{\mathrm{d}p}{p^{1/2}(1-p^2)^{1/2}}$$

where $z_2 = (1-i)/2$ is mapped on to $t = 1$ (see Fig. 7). In the neighbourhood of $t = 1$ this gives $\Delta z = B_1\sqrt{2}(\Delta t)^{1/2}$, or, substituting the value of B_1 and taking absolute values

$$|\Delta t| = 2K^2|\Delta z|^2.$$

Then, since in this case $|\Delta z| = \frac{1}{2}\delta$, the relationship between $|\Delta t|$ and δ is known and

$$k_2 = 1 - \tfrac{1}{2}K^2\delta^2$$

or

$$k_2'^2 = K^2\delta^2. \tag{18}$$

We now need some properties of the function $K(k)$. If k tends to one, then ([9], formula 773.3)

$$K(k) = \ln\frac{4}{k'} + \frac{1}{4}\left(\ln\frac{4}{k'} - 1\right)k'^2 + \cdots \quad (k' \downarrow 0).$$

Further

$$K'(k) = K(k') = \int_0^1 \frac{\mathrm{d}t}{(1-t^2)^{1/2}(1-k'^2t^2)^{1/2}}.$$

Using these results together with (17) and (18), we obtain

$$\lim_{\lambda \downarrow 0} C_0(\lambda) = \frac{\pi}{\ln 2}$$

where λ tends to zero through positive values. By means of more accurate calculations it is found that

$$C_0(\lambda) = \frac{\pi}{\ln 2} + \frac{1}{8}\frac{\pi K^2}{(\ln 2)^2}\lambda^2 + O(\lambda^3) \quad (\lambda \downarrow 0) \tag{19}$$

or

$$C_0(\lambda) = 4.53236 + 2.80971\,\lambda^2.$$

This formula yields an approximation with an error less than 1% for $\lambda < 0.4$ ($K = K(\frac{1}{2}\sqrt{2}) = 1.85407$).

Considering the case that λ tends to one, put $\epsilon = 1 - \lambda$. Then using the same arguments as above, one finds, after a somewhat lengthy but straight-forward calculation,

$$\lim_{\epsilon \to 0} \frac{1}{C_0(\lambda)} = \left[\frac{1}{\ln 4 - \ln K - \ln\epsilon} - \frac{1}{\ln 8 - 2\ln K - 2\ln\epsilon}\right]\frac{\pi}{4}. \tag{20}$$

Taking $\epsilon = 0.01$ and applying the asymptotic formula (20), we find $C_0(0.99) \sim 14.70$ in accordance with results obtained directly from (16), viz. $C_0(0.99) = 14.6983$.

5. CONFORMAL MAPPING OF THE SQUARE ON TO THE UNITY CIRCLE

In order to apply the theorem of Section 2 to other structures than that of Fig. 5, it is necessary to consider the mapping of a square on to the unity circle (Fig. 9) The transformation is ([10], p. 329)

$$z = A\int_0^t \frac{\mathrm{d}w}{(1+w^4)^{1/2}} \tag{21}$$

where it is assumed that $z = 0$ is mapped on $t = 0$ and the integration constant A follows from the choice $z = 1$ if $t = 1$. Since

$$\int_0^1 \frac{\mathrm{d}w}{(1+w^4)^{1/2}} = \frac{1}{2}K \tag{22}$$

the value of A is $A = 2/K$. Assuming $z = z_2$ corresponds to $t = \exp(i\theta)$, one has

$$z_2 = A\int_0^{\exp(i\theta)} \frac{\mathrm{d}w}{(1+w^4)^{1/2}} \quad \left(0 < \theta < \frac{\pi}{4}\right). \tag{23}$$

Again using (22), one finds

$$z_2 = 1 + \frac{2}{K}\int_1^{\exp(i\theta)} \frac{\mathrm{d}w}{(1+w^4)^{1/2}}.$$

Substituting consecutively $w = \exp(i\phi)$ and $\cos(2\phi) = u^2$, the second term can easily be calculated. One obtains

$$z_2 = 1 + \frac{i}{K}F\{\arccos\sqrt{\cos(2\theta)}, k_0\} \tag{24}$$

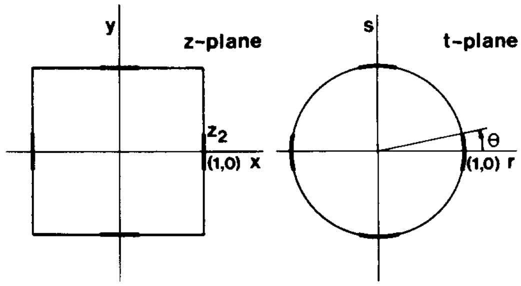

Fig. 9. Conformal mapping of square with finite contacts in the middle of each side on to unity circle.

since

$$\int_x^1 \frac{dt}{(1-t^4)^{1/2}} = \frac{1}{\sqrt{2}} F\{\arccos x, k_0\}.$$

It is easy to see that

$$\lambda = \frac{1}{K} F\{\arccos \sqrt{\cos(2\theta)}, k_0\} \qquad \left(0 < \theta < \frac{\pi}{4}\right)$$

which corresponds in the t-plane to

$$\mu = 4\theta/\pi$$

where μ is defined as the ratio between the length of one contact and the length of one quarter of the circumference of the unity circle. From this it is clear that

$$C_1(\mu) = C_0(\lambda) \tag{25}$$

where $C_1(\mu)$ is defined for the circle structure by

$$R_s = C_1(\mu)\frac{V}{I}.$$

Formula (25) enables us to calculate $C_1(\mu)$ as a function of μ for the circle structure of Fig. 9. The results are given in Fig. 10 and in Table 1.

Note. It can be shown that

$$C_1(\mu) = \frac{\pi}{\ln 2} + \frac{1}{64}\frac{\pi^3}{(\ln 2)^2}\mu^2 \qquad (\mu \downarrow 0) \tag{26}$$
$$= 4.53236 + 1.00837\,\mu^2$$

This formula is accurate within 1% for $\lambda < 0.49$.

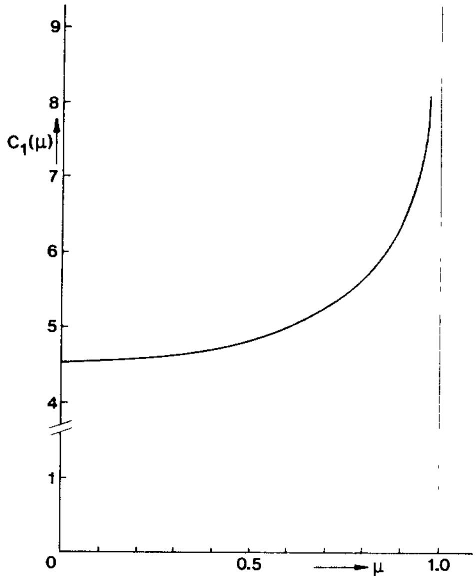

Fig 10. The coefficient $C_1(\mu)$ as a function of the ratio μ for the circle structure (Fig 9) with four equal finite contacts.

6. THE SQUARE WITH DUAL CONTACTS

In Fig. 11 two square structures are shown. The second structure has its contacts complementary to those of the first. By the same conformal transformation both structures are mapped on to a circle (see (24)). For the square of Fig. 11(b) it is easily found that

$$\lambda = 1 - \frac{1}{K} F\{\arccos \sqrt{\cos(2\theta)}, k_0\} \qquad \left(0 < \theta < \frac{\pi}{4}\right)$$

Table 1

θ (degrees)	μ	$C_1(\mu)$	θ (degrees)	μ	$C_1(\mu)$
1	0.02222	4.533	24	0.5333	4.877
2	0.04444	4.534	25	0.5556	4.914
3	0.06667	4.537	26	0.5778	4.954
4	0.08889	4.540	27	0.6000	4.996
5	0.1111	4.545	28	0.6222	5.043
6	0.1333	4.551	29	0.6444	5.093
7	0.1556	4.557	30	0.6667	5.149
8	0.1778	4.565	31	0.6889	5.209
9	0.2000	4.574	32	0.7111	5.276
10	0.2222	4.584	33	0.7333	5.350
11	0.2444	4.595	34	0.7556	5.432
12	0.2667	4.607	35	0.7778	5.524
13	0.2889	4.621	36	0.8000	5.628
14	0.3111	4.636	37	0.8222	5.747
15	0.3333	4.652	38	0.8444	5.884
16	0.3556	4.670	39	0.8667	6.047
17	0.3778	4.689	40	0.8889	6.244
18	0.4000	4.710	41	0.9111	6.490
19	0.4222	4.733	42	0.9333	6.814
20	0.4444	4.758	43	0.9556	7.283
21	0.4667	4.784	44	0.9778	8.104
22	0.4889	4.813	44.5	0.9889	8.943
23	0.5111	4.844			

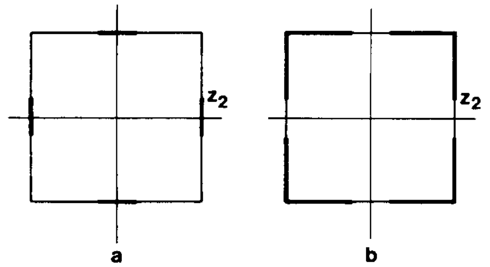

Fig 11. Squares with dual contacts.

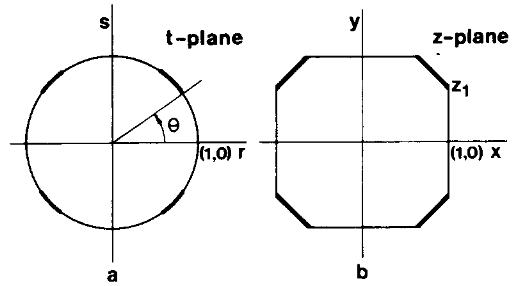

Fig. 12. Conformal mapping of symmetric octagon on to unity circle

which corresponds to

$$\mu = 1 - \frac{4\theta}{\pi}$$

for the circle structure. We remark that z_2 is mapped on the point $t = \exp(i\theta)$ in the t-plane. Denoting $C(\lambda)$ for this structure by $C_2(\lambda)$, we have $C_2(\lambda) = C_1(\mu)$ (see (25)), but now $C_1(\mu)$ is known. In Fig. 8 the coefficient $C_2(\lambda)$ as a function of λ is given for the structure of Fig. 11(b).

Note. It can be shown that

$$C_2(\lambda) = \frac{\pi}{\ln 2} + \frac{1}{64}\frac{\pi}{(\ln 2)^2}K^4\lambda^4 \quad (\lambda \downarrow 0) \tag{27}$$

$$= 4.53236 + 1.20732\,\lambda^4.$$

This formula is accurate within 1% for $\lambda < 0.7$.

7. OCTAGON WITH FOUR AXES OF SYMMETRY

In this section we shall show how the method can be applied to other polygons having four axes of symmetry. As an example, the configuration of Fig. 12(b) will be analysed (cf. [2]). Applying the Schwarz–Christoffel transformation of a polygon on to a circle (see [10], p. 329), one obtains

$$z = A\int_0^t [\{p^4 - \exp(4i\theta)\}\{p^4 - \exp(-4i\theta)\}]^{-1/4}\,\mathrm{d}p \tag{28}$$

where again $z = 0$ corresponds to $t = 0$.† Assuming $z = 1$ is mapped on $t = 1$, one has

$$1 = A\int_0^1 \{p^8 - 2p^4\cos(4\theta) + 1\}^{-1/4}\,\mathrm{d}p.$$

The integral can be calculated numerically. The result yields

$$A = A(\theta) \qquad \left(0 < \theta < \frac{\pi}{4}\right).$$

Let $z = z_1$ correspond to $t = t_1 = \exp(i\theta)$. Then

$$z_1 = 1 + A(\theta)J \tag{29}$$

†The relation $t(iz) = it(z)$ is valid, since the structure is invariant for a rotation of 90°.

where

$$J = \int_1^{t_1} \frac{\mathrm{d}p}{p[p^4 - \{\exp(4i\theta) + \exp(-4i\theta)\} + p^{-4}]^{1/4}}.$$

Substituting consecutively $p = \exp(\frac{1}{2}i\psi)$, $w = \cot(2\theta)\tan\psi$ and $w = \sin q$, the integral can be written as

$$J = iG(\theta) \qquad \left(0 < \theta < \frac{\pi}{4}\right)$$

where

$$G(\theta) = \frac{\{\sin(2\theta)\}^{1/2}}{2\sqrt{2}\cos(2\theta)}\int_0^{\pi/2} \mathrm{d}q\,\frac{(\cos q)^{1/2}}{\{1 + \tan^2(2\theta)\sin^2(q)\}^{3/4}}$$

Then from (29)

$$z_1 = 1 + iA(\theta)G(\theta).$$

From Fig. 12(b) one can determine λ:

$$\lambda = \frac{1 - A(\theta)G(\theta)}{1 + (\sqrt{2} - 1)A(\theta)G(\theta)} \tag{30}$$

which corresponds in the t-plane (Fig. 12a) to

$$\mu = 1 - \frac{4\theta}{\pi}.$$

The function $G(\theta)$ can also be calculated numerically. Denoting $C(\lambda)$ for this structure by $C_3(\lambda)$ it is clear that $C_3(\lambda) = C_1(\mu)$. Herewith the problem of calculating $C_3(\lambda)$ for the symmetric octagon of Fig. 12(b) has been solved. In Fig. 8 the function $C_3(\lambda)$ is given. It is in accordance with Fig. 2 of [2].

Note. It can be shown that

$$C_3(\lambda) = \frac{\pi}{\ln 2} + A\lambda^4 \qquad (\lambda \downarrow 0) \tag{31}$$

$$= 4.532 + 9.280\lambda^4.$$

The constant A is a complicated expression involving elliptic integrals. The formula is accurate within one per cent for $\lambda < 0.3$.

8. FURTHER CONFIGURATIONS

From the example of the foregoing section it is clear that other structures like the Greek cross [3] and its dual structure [2] admit of an analytical solution. The method is straightforward. The numerical work needed consists of the calculation of two integrals while using the results for the circle.

In order to apply the method of Section 7, it is only necessary that the structure (including the metal contacts) should be invariant for a rotation of 90° Structures like Fig. 13 belong to this class (cf. [3], Fig. 6). Instead of one parameter θ occurring in the analysis of the structure of Fig. 12(b) one now needs two parameters α and β.

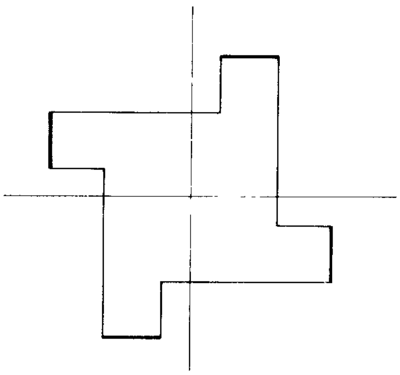

Fig 13 Polygon invariant for rotation of 90°

9. CONCLUSION

A general method is outlined in order to obtain an analytical solution of the potential equation for Van der Pauw structures, if these structures are polygons which are invariant for rotations of 90°. Examples are the square (Figs. 5 and 11) and the octagon (Fig. 12). In view of these examples it is easy to see how the method can be applied to other practical structures The results obtained are corollaries of the theorem proved in Section 2. In all but one case the circle structure of Fig 9 appears to be the key for obtaining numerical results for the sheet resistance without being forced to solve the Laplace equation by means of difference equations. In fact the necessary numerical work is reduced to numerical calculations of definite integrals

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