outline of what we were going for

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0 This document

This document signifies that the project isn't done, and will not be done by the time it is due. But we've written this to explain the proof we're trying to formalize, so we can explain exactly where we've succeeded and where we've failed. Hopefully these ideas are worth something.

1 The theorem

The theorem we're trying to formalize is a simplified version of the hairy ball theorem, after the steps we take to extend it. Let $\mathbb{R}^n$ be the usual Euclidean space and $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. The theorem we attempted to formalize was:

Theorem 1.1. *Suppose there is $v : \mathbb{R}^n \to \mathbb{R}^n$ which is smooth such that $\|v(x)\| = \|x\|$ for all x. Then n is even.*

We break the proof into bits. In what follows, we define for $t \in [0, 1]$ the map $f_t : \mathbb{R}^n \to \mathbb{R}^n$ via

$$f_t(x) = x + tv(x).$$

We also define the *homothety with scale r* via

$$h_r(x) = rx.$$

1.1 For sufficiently small t, f_t is injective on compact subsets

Proof. We claim that f_t is anti-Lipschitz on compact subsets for t sufficiently small, and that this implies injectivity. Let $x, y \in A$ for $A \subseteq \mathbb{R}^n$ compact. As v is C^∞, v is C^1, hence locally Lipschitz. Let $K > 0$ be the Lipschitz constant of v on A. Then

$$\|f_t(x) - f_t(y)\| = \|x - y + t(v(x) - v(y))\| \geq \|x - y\| - |t|\|v(x) - v(y)\| \geq (1 - K|t|)\|x - y\|,$$

applying the reverse triangle inequality and Lipschitzness of v. If $t \in (-1/K, 1/K)$, then $1 - K|t| > 0$ and so we have that f_t is anti-Lipschitz. Now, if $x, y \in A$, $t \in (-1/K, 1/K)$ are such that $f_t(x) = f_t(y)$, then we have

$$0 = \|f_t(x) - f_t(y)\| \geq (1 - K|t|)\|x - y\|,$$

so $\|x - y\| = 0$ and hence $x = y$. Job done! □

Successes and headaches. We successfully proved everything in this section with the exception of one thing: to make the rest of the proof work in Lean, we used an assertion that v was Lipschitz globally, because we did not figure out how to deal with the restriction of v to compact sets in time.

1.2 For sufficiently small t, f_t maps S^{n-1} onto $h_{\sqrt{1+t^2}}[S^{n-1}]$

Proof. For $x \in S^{n-1}$, since $x \perp v(x)$ and $\|v(x)\| = 1$ we have

$$\|f_t(x)\| = \|x + tv(x)\| = \sqrt{1+t^2} = h_{\sqrt{1+t^2}}(1),$$

so that $f_t : S^{n-1} \to h_{\sqrt{1+t^2}}[S^{n-1}]$ is well-defined. To prove it's surjective for suff. small t, it suffices to show that

$$\frac{1}{\sqrt{1+t^2}} f_t : S^{n-1} \to S^{n-1}$$

is surjective. Let $y_0 \in S^{n-1}$; we need to find $x_0 \in S^{n-1}$ with $f_t(x_0) = x_0 + tv(x_0) = \sqrt{1+t^2}y_0$. Equivalently, we need to find a fixed point of the map

$$g : x \mapsto \sqrt{1+t^2}y_0 - tv(x).$$

We use Banach's fixed-point theorem. For $x, y \in S^{n-1}$,

$$\|g(x) - g(y)\| = |t|\|v(y) - v(x)\| < \frac{1}{2}\|v(y) - v(x)\|$$

for t sufficiently small [i.e. $t \in (-1/2, 1/2)$]. So g is a contraction. As S^{n-1} is closed in the complete space $\mathbb{R}^n$, S^{n-1} is complete, so by Banach's theorem g has a unique fixed point, and we're done. □

Major headaches: Both of us tried super hard to figure out how to make the restrictions work, which, while super easy on paper, is super finicky in Lean. Unfortunately this crucial step occurs very early, so it was a pretty major roadblock at this stage of the proof for a long while.

When trying to do this more abstractly, we ran into another issue: namely, *what complete space does h map into itself,* and how can we guarantee that the fixed point of h is on S^{n-1}? These issues weren't quite solved.

Indeed, it *can't* be quite solved, because the aforementioned proof from the lecture notes doesn't quite work - Matthew spotted this. Here's Milnor's original proof, which uses additionally the fact that v is equivariant, i.e. $v(rx) = rv(x)$ for all $x \in S^{n-1}$, $r \in \mathbb{R}$. In this proof and the rest of the text, define the annulus

$$A = \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} \leq \|x\| \leq \frac{3}{2} \right\}.$$

Proof. v is Lipschitz on A, with some constant $C > 0$. Assume $v(rx) = rv(x)$ for all $x \in S^{n-1}$. The proof that f_t maps S^n Choose t such that $|t| < \min\{1/3, 1/C\}$. For $y \in S^{n-1}$, we want $x \in S^{n-1}$ such that $\frac{1}{\sqrt{1+t^2}} f_t(x) = y$. Define $g : A \to A$ via

$$g(x) = y - tv(x).$$

g maps $A \to A$, since

$$\|g(x)\| \leq \|y\| + |t|\|v(x)\| = 1 + |t|\|x\| \leq 1 + \frac{1}{3} \cdot \frac{3}{2} = \frac{3}{2}$$

and

$$\|g(x)\| \geq 1 - |t|\|v(x)\| = 1 - |t|\|x\| \geq 1 - \frac{1}{3} \cdot \frac{3}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Also, g is a contraction:

$$\|g(x) - g(y)\| = |t|\|v(x) - v(y)\| \leq C|t|\|x - y\|,$$

and $C|t| < 1$ by our choice of T. As A is a closed subset of the complete space $\mathbb{R}^n$, A is complete, so by Banach's fixed point theorem g has a unique fixed point x. Now, since

$$x = y - tv(x),$$

we have

$$x + tv(x) = y \Rightarrow 1 = \|x + tv(x)\| = \sqrt{\|x\|^2 + t^2\|x\|^2} = \|x\|\sqrt{1+t^2},$$

so $\|x\| = \frac{1}{\sqrt{1+t^2}}$. Scale everything up by $\sqrt{1+t^2}$; equivariance saves the day. □

With one day left we had to redo this.

(Update: This proof has more or less been outlined, but there are a lot of details we were unable to figure out, as per the Zulip message.)

1.3 For sufficiently small t, $\mathcal{L}^n(f_t[A])$ is a polynomial in t.

Proof. Section 1.1, together with smoothness of v, imply that f_t is injective and smooth, hence a valid change of variables. So

$$\mathcal{L}^n(f_t[A]) = \int_{f_t[A]} 1\, dx = \int_A \det Df_t\, dx = \int_A \det(I_n + tDv(x))\, dx$$

which is a polynomial in t since we're not integrating in t and the determinant is a polynomial in the entries. □

Headaches: Now we get to polynomial hell. We've found that proving that something is or isn't a polynomial is surprisingly tricky. The first week of the project was spent, possibly in vain, to understand the *Polynomial* library in mathlib. It was rough. Sorried out.

1.4 $f_t[A] = h_{\sqrt{1+t^2}}[A]$.

Proof. Note that

$$A = \bigcup_{r \in [1/2, 3/2]} \{x \in \mathbb{R}^n \mid \|x\| = r\}$$

by an obvious double containment. □

Successes: Formalizing that "obvious double-containment" was surprisingly challenging, because Lean was being super finicky about unions.

Headaches: Injective functions distributing over images. Should be easy, but we were running out of time.

1.5 For sufficiently small t, $\mathcal{L}^n(h_{\sqrt{1+t^2}}[A]) = (1+t^2)^{n/2}\mathcal{L}^n(A)$.

Proof. For suff. small t, $h_{\sqrt{1+t^2}}$ is injective and smooth, hence a valid COV, and $\det Dh_{\sqrt{1+t^2}} = (1+t^2)^{n/2}$, so by COV we get

$$\mathcal{L}^n(h_{\sqrt{1+t^2}}[A]) = (1+t^2)^{n/2}\mathcal{L}^n(A).$$

□

No time, sorried out.

1.6 $(1+t^2)^{n/2}$ is a polynomial if and only if n is even.

Proof. Write $(1+t^2)^{n/2}$ as a polynomial function $q(t)$. Then $(1+t^2)^n = q(t)^2$. Now, the roots of the LHS are $i, -i$ with multiplicity n, and the roots of the RHS must have even multiplicity. Hence n is even. □

No time, sorried out. Things we find very intuitive are very finicky with Lean.