EXPONENTIAL OF KAPRANOV'S FORMAL SPECTRUM

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Abstract. We describe a relationship between the formal spectrum introduced by Kapranov [2] and a space $NC\mathrm{Spec}(R)$ associated to a ring R in [3], in the case where R is an NC-nilpotent algebra.

1. Introduction

In [3], a functor $NC\mathrm{Spec}$ was defined from the opposite of the category of rings to the category of ringed spaces. In order to see whether this functor is meaningful or not, it is important to be able to calculate examples of $NC\mathrm{Spec}(R)$ and also to compare $NC\mathrm{Spec}(R)$ to existing notions of (noncommutative) spectrum for various classes of rings. The aim of the present note is to do this for one class of rings introduced by Kapranov in [2], the so-called *NC-nilpotent* algebras. These algebras are close to being commutative, in the sense that an ideal generated by high enough iterated commutators and commutator products vanishes. In [3, Section 8.1], it was shown that when R is a commutative ring, $NC\mathrm{Spec}(R)$ may be described as a certain "exponential" of the space $\mathrm{Spec}(R)$ equipped with its usual base of open sets. We show that the same holds when R is NC-nilpotent.

We adopt the following conventions in this paper. All rings have an identity element and ring homomorphisms preserve the identity elements. We include the zero ring as a ring. It is a final object in the category of rings.

2. Kapranov's formal spectrum

Everything in this section is due to Kapranov [2]. Let $\mathbb{C}$ be a field and let R be a $\mathbb{C}$–algebra, by which we mean that R is a ring together with a ring homomorphism $\mathbb{C} \to R$. For $m \geq 1$, define R_m^{Lie} to be the $\mathbb{C}$–linear span of the iterated commutators $[R, [R, \cdots, [R, R],], \cdots]$ with $m-1$ instances of the Lie bracket. For $d \geq 0$, define the two-sided ideal of R,

$$F^d R = \sum_m \sum_{i_1+i_2+\cdots+i_m-m=d} R R_{i_1}^{Lie} R R_{i_2}^{Lie} R \cdots R R_{i_m}^{Lie} R.$$

Then $F^0 R \supset F^1 R \supset F^2 R \supset \cdots$ is a filtration of R called the *Helton-Howe commutator filtration* [1]. In particular, $F^1 R$ is the two-sided ideal of R generated by all commutators. The quotient $R/F^1 R$ is called the commutativisation of R and is denoted R_{ab}.

Definition 2.1. [2, Definition 1.1.5] *An algebra R is called* NC-complete *if the natural map $R \to \varprojlim (R/F^d R)$ is an isomorphism. An algebra R is called* NC-nilpotent *if $F^d R = 0$ for some $d \geq 1$.*

NC-nilpotent algebras may also be described in terms of central extensions.

Definition 2.2. *If R is an algebra, a central extension of R is a surjective algebra homomorphism $\theta : R' \to R$ such that $I = \ker(\theta) \subset Z(R')$ and $I^2 = 0$.*

Proposition 2.3. [2, Proposition 1.2.3] *An algebra R is NC-nilpotent if and only if R is isomorphic to an iterated central extension of a commutative algebra.*

If R is NC-complete (and in particular if R is NC-nilpotent) the ringed space $\mathrm{Spf}(R)$ is defined as follows. The underlying topological space of $\mathrm{Spf}(R)$ is $X_{ab} := \mathrm{Spec}(R_{ab})$, the set of prime ideals of the commutative ring R_{ab}. A sheaf $\mathcal{O}_K$ is defined on this space in the following way. Given $d \geq 1$, let $\pi_d : R/F^dR \to R_{ab}$ denote the quotient map. For $g \in R_{ab}$, let S_g denote the multiplicative subset $\bigcup_{k \geq 0} \pi_d^{-1}(g^k)$. It is shown in [2, 2.1.6] that S_g is an Ore set in R/F^dR. Let $(R/F^dR)[S_g^{-1}]$ denote the localisation of R/F^dR at this Ore set. For each d, the quotient map $R/F^{d+1}R \to R/F^dR$ induces a map $(R/F^{d+1}R)[S_g^{-1}] \to (R/F^dR)[S_g^{-1}]$ and the sections of the sheaf $\mathcal{O}_K$ over the basic open set $D(g) = \{p : g \notin p\}$ of X_{ab} are defined to be

$$\mathcal{O}_K(D(g)) = \varprojlim \left(\frac{R}{F^dR}[S_g^{-1}] \right).$$

There is a unique sheaf $\mathcal{O}_K$ on X_{ab} satisfying $\mathcal{O}_K(D(g)) = \varprojlim \left(\frac{R}{F^dR}[S_g^{-1}] \right)$ for all $g \in R_{ab}$, where $D(g) = \{P \in \mathrm{Spec}(R_{ab}) : g \notin P\}$.

Definition 2.4 (Kapranov)**.** *The ringed space $(X_{ab}, \mathcal{O}_K)$ is called the* formal spectrum *of R.*

In particular, if R is NC-nilpotent then R is NC-complete, and in this case $\mathcal{O}_K(D(g)) = R[S_g^{-1}]$ for each g.

3. Another notion of noncommutative spectrum

Now we describe the space to which we want to compare $\mathrm{Spf}(R)$. If R is a ring and $A \subset R$, we define $loc(R, A)$ by the following universal property:

$loc(R, A)$ is a ring together with a ring homomorphism $\alpha_A : R \to loc(R, A)$ such that $\alpha_A(a)$ is a unit for all $a \in A$ and such that whenever S is a ring and $\theta : R \to S$ a ring homomorphism such that $\theta(a)$ is a unit for all $a \in A$, then there exists a unique $\theta' : loc(R, A) \to S$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha_A"] \arrow[d, "\theta"'] & loc(R, A) \arrow[dl, "\theta'"] \\ S & \end{tikzcd}$$

commute.

Such a ring $loc(R, A)$ always exists. It may be defined as the ring generated by the elements of R together with symbols a^{-1} for $a \in A$, factored by the ideal generated by the relations of R together with $aa^{-1} = a^{-1}a = 1$ for $a \in A$. Notice in particular that $loc(R, A)$ is a $\mathbb{C}$–algebra if R is.

Let $L_0(R)$ denote the set of finite subsets of R. Define a preorder on $L_0(R)$ by $A \preceq B$ if $\alpha_B(a)$ is a unit for all $a \in A$. Define an equivalence relation on $L(R)$ by $A \sim A'$ if $A \preceq A'$ and $A' \preceq A$. Denote the equivalence class of A by R_A. Then the associated poset of equivalence classes $L(R)$ is a topological space with the topology whose basic open sets are $U_E = \{R_A : R_A \geq R_E\}$ for $E \in L_0(R)$. Let $S(L(R))$ denote the set of closed irreducible subsets C of $L(R)$ and define a topology on $S(L(R))$ by taking a base to be the open sets $\widetilde{U_E} = \{C : R_E \in C\}$. A sheaf of rings may be defined on $S(L(R))$ by $\mathcal{O}(\widetilde{U_E}) = loc(R, E)$.

If R is a ring then we will denote the space $S(L(R))$ together with the sheaf of rings $\mathcal{O}$ by $NC\mathrm{Spec}(R)$.

We will use the following theorem, which is the main result of [3].

Theorem 3.1. [3, Theorem 7.7] $R \mapsto NC\mathrm{Spec}(R)$ *is a faithful contravariant functor from the category of rings and ring homomorphisms to the category of ringed spaces.*

Next, we recall how $NC\mathrm{Spec}(R)$ is related to R in the case where R is commutative.

4. Exponentiation

We now describe the results of [3] which show how to calculate $NC\mathrm{Spec}(R)$ when R is a commutative ring.

Let $\mathcal{T}$ be the category whose objects are pairs $(X, \mathcal{B})$ where X is a T_0 topological space and $\mathcal{B}$ is a base of X, with the properties that (1) $X \in \mathcal{B}$ and (2) $\mathcal{B}$ is closed under finite intersections. A morphism $f : (X, \mathcal{B}) \to (Y, \mathcal{C})$ in $\mathcal{T}$ is a continuous function $f : X \to Y$ such that $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$.

If $(X, \mathcal{B})$ is an object of $\mathcal{T}$, we make the following construction. Let $\mathcal{P}(X)$ be the set of all subsets of X. For $B \in \mathcal{B}$, let $\widetilde{B}_0 = \{A \in \mathcal{P}(X) : A \subset B\}$. Then $\{\widetilde{B}_0 : B \in \mathcal{B}\}$ is a base for a topology on $\mathcal{P}(X)$. Now define an equivalence relation on $\mathcal{P}(X)$ by $A \sim A'$ if and only if $A \subset B \iff A' \subset B$ for all $B \in \mathcal{B}$. The quotient space $\mathcal{P}(X)/\sim$ has a base of open sets $\widetilde{B}$ for $B \in \mathcal{B}$, where $\widetilde{B}$ is the set of equivalence classes $[A]$ with $A \in \widetilde{B}_0$. If we write $\widetilde{\mathcal{B}} = \{\widetilde{B} : B \in \mathcal{B}\}$, then $(\mathcal{P}(X)/\sim, \widetilde{\mathcal{B}})$ is an object of $\mathcal{T}$. We denote it by $E(X, \mathcal{B})$.

Proposition 4.1. [3, Section 8.1] $E : \mathcal{T} \to \mathcal{T}$ *is a functor, and for each* $(X, \mathcal{B})$*, there is a natural map* $i : (X, \mathcal{B}) \to E(X, \mathcal{B})$*. We call* $E(X, \mathcal{B})$ *the* exponential *of* $(X, \mathcal{B})$.

Example 4.2. $E(X, \mathcal{B})$ depends very much on the choice of the base $\mathcal{B}$. For example, if X is discrete and we choose $\mathcal{B} = \mathcal{P}(X)$, then the exponential is just the set $\mathcal{P}(X)$ with the discrete topology. But if the base is $\mathcal{B} = \{\{x\} : x \in X\} \cup \{X\}$, then $E(X, \mathcal{B})$ is the space X plus two extra points, one of which is generic and the other of which belongs to exactly one open set.

If $(X, \mathcal{B})$ is an object of $\mathcal{T}$ and X is equipped with a sheaf of rings $\mathcal{O}$, then there is a natural sheaf of rings $i_*\mathcal{O}$ on $E(X, \mathcal{B})$, which we denote by $E\mathcal{O}$. In this way, we may extend E to ringed spaces. The main result of [3, Section 8] is the following.

Theorem 4.3. [3, Proposition 8.8] *Let R be a commutative ring and let $\mathcal{B} = \{D(g) : g \in R\}$ be the usual basis of* $\mathrm{Spec}(R)$*. Then there is an isomorphism of ringed spaces, natural in R,*

$$NC\mathrm{Spec}(R) \cong E(\mathrm{Spec}(R), \mathcal{B}).$$

In this case, the points of $E(X, \mathcal{B})$ may be identified with unions $\cup_\lambda P_\lambda$ of prime ideals of R, and if $B = D(g)$ then $\widetilde{B} = \{\cup_\lambda P_\lambda : g \notin \cup_\lambda P_\lambda\}$.

The aim of the present paper is to extend Theorem 4.3 to NC-nilpotent algebras.

Theorem 4.4. *Let R be an NC-nilpotent algebra. Let* $\mathrm{Spf}(R)$ *denote the formal spectrum of R with the base $\mathcal{B} = \{D(g) : g \in R_{ab}\}$. Then there is a natural isomorphism of ringed spaces*

$$NC\mathrm{Spec}(R) \cong E(\mathrm{Spf}(R), \mathcal{B}).$$

Theorem 4.4 will be proved in stages.

5. Central extensions and localisation

Our first aim is to show that the construction $R \mapsto L(R)$ defined in Section 3 is invariant under central extensions.

Proposition 5.1. *Let R be an algebra and*

$$0 \longrightarrow I \longrightarrow R' \xrightarrow{\theta} R \longrightarrow 0$$

be a central extension. Then the induced map $\widehat{\theta} : L(R') \to L(R)$ is a homeomorphism.

The map $\widehat{\theta}$ is defined by $\widehat{\theta}(R'_A) = R_{\theta(A)}$ for A a finite subset of R'. See [3, Section 7].

In order to prove the proposition, we require the following standard lemmas.

Lemma 5.2. *Let R be a ring and let $I \subset R$ be a two-sided ideal such that $I^2 = 0$. Let $x \in R$. Then if $x + I$ is a unit in R/I then x is a unit in R.*

Proof. Suppose $x + I$ is a unit in R/I. Then there exists $u \in R$ such that $ux + I = 1 + I$ and $xu + I = 1 + I$. So there are $i, j \in I$ with $ux = 1 + j$ and $xu = 1 + i$. The latter equation implies $xu(1 - i) = (1 + i)(1 - i) = 1$ because $i^2 = 0$. The former implies that $(1 - j)ux = 1$. Therefore, x is both a left and a right unit in R, so x is a unit in R. □

Lemma 5.3. *Let R' be a ring and let I be an ideal of R' such that $I^2 = 0$ and $I \subset Z(R')$. Let $A \subset R'$ and let $\alpha_A : R' \to loc(R', A)$ be the localisation. Let J be the two-sided ideal of $loc(R', A)$ generated by $\alpha_A(I)$. Then $J^2 = 0$.*

Proof. If $i \in I$ then $\alpha_A(i)$ commutes with $\alpha_A(r)$ for all $r \in R'$. Therefore, it also commutes with $\alpha_A(a)^{-1}$ for $a \in A$. So $\alpha_A(i)$ is central. Since $I^2 = 0$, we get $\alpha_A(i) r \alpha_A(j) = 0$ for all $i, j \in I$ and all $r \in loc(R', A)$. Therefore $J^2 = 0$. □

5.1. **Proof of Proposition 5.1.** Given a central extension

$$0 \longrightarrow I \longrightarrow R' \xrightarrow{\theta} R \longrightarrow 0$$

let $\lambda : R \to R'$ be a linear map which splits θ, ie. $\theta\lambda = \mathrm{id}_R$. The map $\widehat{\theta} : L(R) \to L(R')$ is defined by $\widehat{\theta}(R'_A) = R_{\theta(A)}$. We define a function $\widehat{\lambda}$ from the set $L_0(R)$ of finite subsets of R to $L(R')$ by $\widehat{\lambda}(R_A) = R'_{\lambda(A)}$. Now we show that $\widehat{\lambda}$ is preorder-preserving. Let A, B be finite subsets of R and suppose $A \preceq B$. Then $\alpha_B(a)$ is a unit for all $a \in A$. We must show that $\alpha_{\lambda(B)}(\lambda(a))$ is a unit in $loc(R, \lambda(B))$ for all $a \in A$.

If $x, y \in R$ then $\lambda(xy) - \lambda(x)\lambda(y) \in \ker(\theta) = I$. Therefore, if we let J denote the two-sided ideal of $loc(R', \lambda(B))$ generated by $\alpha_{\lambda(B)}(I)$ then the composition

$$R \xrightarrow{\lambda} R' \xrightarrow{\alpha_{\lambda(B)}} loc(R', \lambda(B)) \longrightarrow loc(R', \lambda(B))/J$$

is a ring homomorphism, which we denote by $\overline{\alpha_{\lambda(B)}\lambda}$. Now, if $b \in B$ then $\overline{\alpha_{\lambda(B)}\lambda}(b) = \alpha_{\lambda(B)}(\lambda(b)) + J$ is a unit, so by the universal property of $loc(R, B)$, there is a unique ring homomorphism $\gamma : loc(R, B) \to loc(R', \lambda(B))/J$ making the following diagram commute.

$$\begin{tikzcd} R \arrow[r, "\overline{\alpha_{\lambda(B)}\lambda}"] \arrow[d, "\alpha_B"'] & loc(R', \lambda(B))/J \\ loc(R, B) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

By our assumption, $\alpha_B(a)$ is a unit for all $a \in A$. Therefore, so is $\gamma\alpha_B(a) = \overline{\alpha_{\lambda(B)}\lambda}(a) = \alpha_{\lambda(B)}(\lambda(a)) + J$. By Lemma 5.3, $J^2 = 0$ and therefore by Lemma 5.2, $\alpha_{\lambda(B)}(\lambda(a))$ is a unit in $loc(R', \lambda(B))$ as required.

We have shown that $\widehat{\lambda}$ preserves the preorder $\preceq$, and so it induces a continuous map $L(R) \to L(R')$ which we also denote by $\widehat{\lambda}$. Now, $\widehat{\theta}\widehat{\lambda} = \mathrm{id}_{L(R)}$, while if $A \subset R'$, we have $\widehat{\lambda}\widehat{\theta}(R_A) = R_{\lambda\theta(A)}$. It therefore suffices to show that $R_{\lambda\theta(A)} = R_A$ for all finite subsets $A \subset R'$.

Let $A \subset R'$. If $a \in A$ then $\lambda\theta(a) - a \in \ker(\theta) = I$, so if we denote by J the two-sided ideal of $loc(R', A)$ generated by $\alpha_A(I)$, then $\alpha_A(\lambda\theta(a)) - \alpha_A(a) \in J$. Thus, $\alpha_A(\lambda\theta(a)) + J = \alpha_A(a) + J$ is a unit. So by Lemma 5.3 combined with Lemma 5.2, $\alpha_A(\lambda\theta(a))$ is a unit in $loc(R', A)$ for all $a \in A$, so $\lambda\theta(A) \preceq A$. The converse is true for the same reason, and so $R_{\lambda\theta(A)} = R_A$.

Thus, $\widehat{\lambda}\widehat{\theta} = \mathrm{id}_{L(R')}$ and so $\widehat{\lambda}$ is a continuous inverse to $\widehat{\theta}$. Thus, $\widehat{\theta}$ is a homeomorphism.

Corollary 5.4. *Suppose R is an NC-nilpotent algebra. Then the natural quotient map $R \to R_{ab}$ induces a homeomorphism $S(L(R_{ab})) \to S(L(R))$.*

Proof. If R is NC-nilpotent, then $F^d R = 0$ for large enough d. There is a chain of central extensions.

$$R \longrightarrow R/F^{d-1}R \longrightarrow \cdots \longrightarrow R/F^2R \longrightarrow R_{ab}$$

Proposition 5.1 implies that $L(R) \to L(R_{ab})$ is a homeomorphism. Thus, $S(L(R_{ab})) \to S(L(R))$ is also a homeomorphism. ☐

6. NC-complete algebras

In this section, we take R to be an NC-complete algebra. We construct a map of ringed spaces $E(\mathrm{Spf}(R)) \to NC\mathrm{Spec}(R)$. Later, in Section 7, we will prove that this map is an isomorphism in the special case when R is NC-nilpotent.

Let R be an NC-complete algebra. There is a quotient map $\pi : R \to R_{ab}$ and thus by Theorem 3.1 there is a morphism

$$\pi^* : NC\mathrm{Spec}(R_{ab}) \to NC\mathrm{Spec}(R).$$

We let f be the underlying map of topological spaces of π^*.

Since the underlying space of $NC\mathrm{Spec}(R_{ab})$ is $E(\mathrm{Spec}(R_{ab}))$, we may regard f as a map between the underlying spaces of $E(\mathrm{Spf}(R))$ and $NC\mathrm{Spec}(R)$. We wish to define a map $\widetilde{f} : \mathcal{O} \to f_* E\mathcal{O}_K$ of sheaves on $NC\mathrm{Spec}(R)$. Then $(f, \widetilde{f})$ will be a map of ringed spaces $E(\mathrm{Spf}(R)) \to NC\mathrm{Spec}(R)$.

If $\widetilde{U_A}$ is a basic open subset of $S(L(R))$ with A a finite subset of R, we have $f^{-1}(\widetilde{U_A}) = \widetilde{U_{\pi(A)}}$ by [3, Lemma 7.2]. Regarded as an open subset of $E(\mathrm{Spec}(R_{ab})) = NC\mathrm{Spec}(R_{ab})$, this is $\widetilde{D(g)}$ where $g = \prod_{e \in A} \pi(e)$. To define a map of sheaves $\mathcal{O} \to f_* E\mathcal{O}_K$, we therefore need, for each $A \in L_0(R)$, a ring homomorphism

$$\mathcal{O}(A) = loc(R, A) \to \mathcal{O}_K(D(g)) = \varprojlim_d (R/F^d R)[S_g^{-1}] \tag{1}$$

such that these homomorphisms are compatible with the restriction maps.

For each d, take the composition $R \to R/F^d R \to (R/F^d R)[S_g^{-1}]$. These maps commute with the natural map $(R/F^d R)[S_g^{-1}] \to (R/F^{d-1} R)[S_g^{-1}]$, so they give a map

$$\gamma : R \to \varprojlim_d (R/F^d R)[S_g^{-1}].$$

We now wish to show that $\gamma(e)$ is invertible for each $e \in A$. If $e \in A$, choose $f \in R$ such that $ef + F^1 R = \pi(e) \prod_{e' \in A \setminus \{e\}} \pi(e') = g$. Then $ef + F^d R$ is a unit in $(R/F^d R)[S_g^{-1}]$, so there exists a such that $(e + F^d R)(a + F^d R) = 1$. Similarly, there exists b such that $(b + F^d R)(e + F^d R) = 1$. Therefore, $e + F^d R$ is a unit in $(R/F^d R)[S_g^{-1}]$. It follows that the element $(e + F^d R)_{d \geq 1} \in \prod_{d \geq 1} (R/F^d R)[S_g^{-1}]$ has an inverse $(y_d + F^d R)_{d \geq 1}$. It is easy to see that $(y_d + F^d R)_{d \geq 1}$ belongs to $\varprojlim_d (R/F^d R)[S_g^{-1}]$ and therefore $\gamma(e)$ is a unit for all $e \in E$. Thus, γ induces

$$\widetilde{f}_A : loc(R, A) \to \varprojlim_d (R/F^d R)[S_g^{-1}] = \mathcal{O}_K(f^{-1}(\widetilde{U_A})).$$

The restriction maps on both sides are the localisation maps. The maps $\widetilde{f}_A$ are compatible with these, and therefore we obtain a map of sheaves on $NC\mathrm{Spec}(R)$,

$$\widetilde{f} : \mathcal{O}_{NC\mathrm{Spec}(R)} \to f_* E\mathcal{O}_K.$$

Proposition 6.1. *If R is NC-complete then there is a natural map of ringed spaces*

$$(f, \widetilde{f}) : E(\mathrm{Spf}(R)) \to NC\mathrm{Spec}(R)$$

which is injective on the underlying topological spaces.

Proof. The naturality is a routine calculation. The injectivity on the underlying spaces follows from the general fact that if $R \to S$ is a surjective ring homomorphism, then the induced map $S(L(S)) \to S(L(R))$ is injective. This is easy to check from the definitions, and we apply it in the special case $S = R_{ab}$. $\square$

7. NC-nilpotent algebras

Our goal is to show that if R is NC-nilpotent then the map $(f, \widetilde{f})$ of Proposition 6.1 is an isomorphism of ringed spaces.

We have already seen in Corollary 5.4 that the underlying map of spaces f is a homeomorphism. It therefore suffices to show that for each basic open subset $\widetilde{U_A}$ of $NC\mathrm{Spec}(R)$, the map

$$\widetilde{f}_A : \mathcal{O}_{NC\mathrm{Spec}(R)}(\widetilde{U_A}) \to \mathcal{O}_{E(\mathrm{Spf}(R))}(f^{-1}(\widetilde{U_A}))$$

is an isomorphism of rings. This map coincides with the natural map

$$loc(R, A) \to loc(R, \bigcup_{k \geq 0} \pi^{-1}\left(\prod_{e \in A} \pi(e)^k\right)). \tag{2}$$

This is because in the case of an NC-nilpotent algebra, the right hand side of Equation (1) is just the localization of R at the set $\bigcup_{k\geq 0} \pi^{-1}(\prod_{e\in A} \pi(e)^k)$. This is the localization *as an algebra*, however it is easy to check that this coincides with the localization as a ring, because they satisfy the same universal property.

To show that (2) is an isomorphism, it suffices to show that if $x \in R$ and $\pi(x) = \prod_{e\in A} \pi(e)^k$ for some $k \geq 0$, then $\alpha_A(x)$ is a unit in $loc(R, A)$.

Note that if $x \in \pi^{-1}\left(\prod_{e\in A} \pi(e)^k\right)$ then there is a vector space splitting $\lambda : R_{ab} \to R$ of the quotient map $\pi : R \to R_{ab}$ such that $\lambda(\prod_{e\in A} \pi(e)^k) = x$. Therefore, if $g = \prod_{e\in A} \pi(e)$, we may write

$$\bigcup_{k\geq 0} \pi^{-1}(x_0^k) = \bigcup_{k\geq 0} \bigcup_{\lambda \in \mathsf{Split}(\pi)} \lambda(g^k)$$

where $\mathsf{Split}(\pi)$ is the set of linear maps which split π. Recall that the *degree* of an NC-nilpotent algebra R is the smallest d such that $F^{d+1}R = 0$. We prove the following proposition by induction on d.

Lemma 7.1. *Let R be an NC-nilpotent algebra. Let $x_0 \in R_{ab}$ and let $\lambda : R_{ab} \to R$ be a linear map which splits $\pi : R \to R_{ab}$. Let $x \in R$ with $\pi(x) = x_0$. Then the canonical map*

$$loc(R, x) \to loc(R, \bigcup_{\lambda} \bigcup_{k\geq 0} \lambda(x_0^k))$$

is an isomorphism.

Proof. If $d = 0$ then R is commutative and λ is the identity. Then $loc(R, x) = loc(R, \{x^k : k \geq 0\})$ which proves the base step.

For the inductive step, suppose R is NC-nilpotent of degree d. Let $\theta : R \to R/F^dR$ be the quotient map, so that θ is a central extension, and let $\rho : R/F^dR \to R_{ab}$ be the quotient map, so that $\rho\theta = \pi$. Let

$\lambda : R_{ab} \to R$ be any splitting of $\rho\theta$, so that $\rho\theta\lambda = \mathrm{id}_{R_{ab}}$. Then $\theta\lambda$ splits ρ. Let $\mathsf{Split}(\rho)$ be the set of linear maps $R_{ab} \to R/F^dR$ which split ρ. Then by the induction hypothesis,

$$loc(R/F^dR, \theta(x)) = loc(R/F^dR, \bigcup_{k\geq 0} \bigcup_{\gamma\in\mathsf{Split}(\rho)} \gamma(x_0^k)). \tag{3}$$

We wish to show that $\alpha_x(\lambda(x_0^k))$ is invertible in $loc(R, x)$.

Let $I = \ker(\theta)$ and let J be the two-sided ideal of $loc(R, x)$ generated by $\alpha(I)$. Let $\mu : R/F^dR \to R$ be a vector space splitting of θ. Then there is a ring homomorphism $\overline{\alpha_x\mu} : R/F^dR \to loc(R, x)/J$ defined by $\overline{\alpha_x\mu}(s) = \alpha_x(\mu(s)) + J$.

Now let $k \geq 0$. Since $\alpha_x(x)$ is a unit, so is $\alpha_x(x) + J$. Since $x - \mu\theta(x) \in I$, $\alpha_x(\mu\theta(x)) + J$ is a unit. Therefore, $\overline{\alpha_x\mu}(\theta(x))$ is a unit. Then (3) implies that $\overline{\alpha_x\mu}(\theta\lambda(x_0^k))$ is a unit because $\theta\lambda \in \mathsf{Split}(\rho)$. This is the same as saying that $\alpha_x(\mu\theta\lambda(x_0^k)) + J$ is a unit. But again, $\alpha_x(\mu\theta\lambda(x_0^k)) + J = \alpha_x(\lambda(x_0^k)) + J$, so by Lemma 5.2 combined with Lemma 5.3, $\alpha_x(\lambda(x_0^k))$ is a unit in $loc(R, x)$. □

7.1. **Proof of Theorem 4.4.** We have already reduced this to showing that the canonical map (2) is an isomorphism. If $A = \{e_1, e_2, \ldots, e_n\}$ is a finite subset of R and $x := e_1e_2\cdots e_n$, then there are ring homomorphisms α, β and γ making a commutative diagram

$$\begin{tikzcd}
 & loc(R,x) \arrow[rr, "\beta"] \arrow[dr, "\alpha"] & & loc(R, \bigcup_\lambda \bigcup_{k\geq 0} \lambda(g^k)) \\
R \arrow[ur, "\alpha_x"] \arrow[rr, "\alpha_A"'] & & loc(R,A) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where γ is the map in Equation (2). Lemma 7.1 states that β is an isomorphism. This implies that α is injective. But α is also surjective. This is because $loc(R, A)$ is generated by R and $\alpha_A(e_i)^{-1}$ for $e_i \in A$, and the map $\alpha : loc(R, x) \to loc(R, A)$ takes elements of R to their images in $loc(R, A)$ and takes $\alpha_x(x)^{-1}$ to $\alpha_A(x)^{-1} = \alpha_A(e_n)^{-1} \cdots \alpha_A(e_1)^{-1}$. From this it follows that $\alpha_A(e_i)^{-1}$ is in the image of α for every i, and therefore the image of α is the whole of $loc(R, A)$. Therefore, α is an isomorphism and hence so is $\gamma = \beta\alpha^{-1}$. This completes the proof.

Remark 7.2. Lemma 7.1 has two algebraic consequences. It shows that every universal localisation of an NC-nilpotent algebra at a finite set of elements may be obtained by inverting a product of these elements (as is the case for commutative rings). It also shows that the Ore localisation defined by Kapranov in [2, Proposition 2.1.5] may be obtained by inverting a single element.

References

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