**4.** Случайная величина распределена равномерно на отрезке $[\theta, 2\theta]$.

a) По выборке объема $n$ найти оценки параметра $\theta$ методом моментов и методом максимального правдоподобия. Рассмотрим еще одну оценку $\tilde{\theta}_3 = \frac{1}{5}(x_{\min} + 2x_{\max})$.

b) Проверить оценки на несмещенность и состоятельность. Исправить эти оценки, если необходимо.

c) Какая из исправленных оценок более эффективна? Сравнить асимптотическую эффективность оценок?

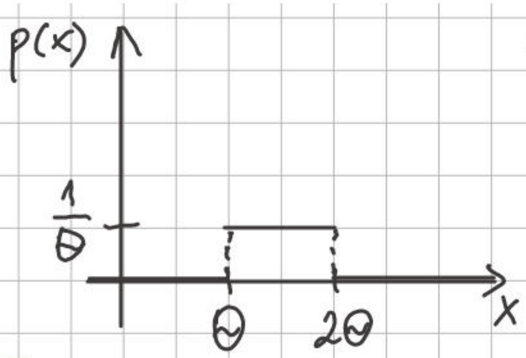
d) Построить точный доверительный интервал для параметра $\theta$.

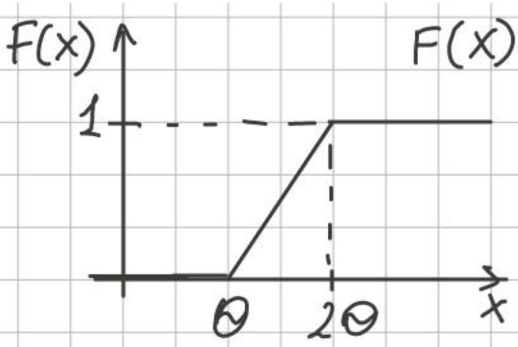
e) Построить асимптотический доверительный интервал для параметра $\theta$.

f) Сгенерируйте выборку объема $n = 100$ для некоторого значения параметра $\theta$. Вычислите указанные выше доверительные интервалы для доверительной вероятности 0.95.

g) Численно постройте бутстраповский доверительный интервал.

h) Сравнить все интервалы.




$$F(x) = \frac{x - \Theta}{\Theta}$$

a ОММ:

$$\alpha_1 = \int_{-\infty}^{+\infty} x\, p(x, \Theta)\, dx = \int_{\Theta}^{2\Theta} x \frac{1}{\Theta} dx = \frac{x^2}{2\Theta}\Big|_{\Theta}^{2\Theta} = \frac{3}{2}\Theta$$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

$$\tilde{\alpha}_1 = \alpha_1$$

$$\overline{x} = \frac{3}{2}\Theta$$

$$\boxed{\tilde{\Theta}_1 = \frac{2}{3}\overline{x}}$$

ОМП:

$$L(\Theta) = \prod_{i=1}^{n} p(x_i, \Theta) = \begin{cases} \frac{1}{\Theta^n}, & x_i \in [\Theta, 2\Theta]\ \forall i \\ 0, & \exists x_i : x_i \notin [\Theta, 2\Theta] \end{cases}$$

$$L \to \max \Rightarrow \begin{cases} \Theta < x_{\min} \\ x_{\max} < 2\Theta \end{cases} \Rightarrow \Theta \in \left[\frac{x_{\max}}{2}; x_{\min}\right]$$

$L \to max \Rightarrow \Theta \to min$

$$\boxed{\tilde{\Theta}_2 = \frac{X_{max}}{2}}$$

b • $\tilde{\Theta}_1 = \frac{2}{3}\overline{x}$

$$M[\tilde{\Theta}_1] = \frac{2}{3} M[\overline{x}] = \frac{2}{3}\frac{1}{n}\sum M[x_i] =$$

$$= \frac{2}{3}\cdot\frac{1}{n}\cdot n\cdot M[\xi] = \frac{2}{3}\cdot\frac{3}{2}\Theta = \Theta \Rightarrow \text{несмещ.}$$

$$D[\tilde{\Theta}_1] = \frac{4}{9n^2} D[\sum x_i] = \frac{4}{9n^2}\cdot n\cdot D\xi = \frac{4}{9n} D\xi$$

$$M[\xi^2] = \int_\Theta^{2\Theta} x^2\cdot\frac{1}{\Theta}dx = \frac{x^3}{3\Theta}\Big|_\Theta^{2\Theta} = \frac{7}{3}\Theta^2$$

$$D\xi = \frac{7}{3}\Theta^2 - \frac{9}{4}\Theta^2 = \frac{\Theta^2}{12}$$

$$D[\tilde{\Theta}_1] = \frac{4}{9\cdot 12n}\Theta^2 = \frac{1}{27n}\Theta^2 \xrightarrow[n\to\infty]{} 0$$

– состоят. по дост. условию

• $\tilde{\Theta}_2 = \frac{X_{max}}{2}$

$$M[\tilde{\Theta}_2] = \frac{1}{2}M[X_{max}] = \frac{1}{2}\int_\Theta^{2\Theta} y\cdot n\cdot \Phi^{n-1}(y)\, q(y)\, dy =$$

$$= \frac{1}{2}\int_\Theta^{2\Theta} yn\left(\frac{y-\Theta}{\Theta}\right)^{n-1}\frac{1}{\Theta}dy = \frac{1}{2}\frac{n}{\Theta^n} - \left\{\frac{y}{\Theta} - 1 = t\right\} =$$

$$= \frac{n\Theta}{2}\int_0^1 (1+t)t^{n-1}dt = \frac{\Theta(2n+1)}{2(n+1)} \text{ – смещенная}$$

Исправим: $\tilde{\Theta}_2' = \frac{2(n+1)}{2n+1}\tilde{\Theta}_2$

$$M[\tilde{\Theta}_2^2] = \frac{1}{4}M[X_{max}^2] = \frac{1}{4}\int_\Theta^{2\Theta} y^2 n\,\Phi^{n-1}(y)q(y)dy =$$

$$= \frac{1}{4}\int_\Theta^{2\Theta} y^2 n\left(\frac{y-\Theta}{\Theta}\right)^{n-1}\cdot\frac{1}{\Theta}dy =$$

$$\frac{n}{4\Theta^n}\int_\Theta^{2\Theta} y^2(y-\Theta)^{n-1}dy = \frac{n}{4\Theta^n}\Big(y^2\frac{(y-\Theta)^n}{n}\Big|_\Theta^{2\Theta} -$$

$$-\int_\Theta^{2\Theta}\frac{2y}{n}(y-\Theta)^n dy = \frac{1}{4\Theta^n}\Big(4\Theta^2\cdot\Theta^n - 2\Big(\frac{y(y-\Theta)^{n+1}}{n+1}\Big|_\Theta^{2\Theta} -$$

$$-\int_\Theta^{2\Theta}\frac{(y-\Theta)^{n+1}}{n+1}dy\Big)\Big) = \Theta^2 - \frac{2\cdot 2\Theta\cdot\Theta^{n+1}}{4\Theta^n(n+1)} + \frac{2(y-\Theta)^{n+2}}{4\Theta^n(n+1)(n+2)}\Big|_\Theta^{2\Theta} =$$

$$= \frac{n\Theta^2}{n+1} + \frac{\Theta^2}{2(n+1)(n+2)}$$

$$D[\tilde{\Theta}_2] = \frac{n\Theta^2}{n+1} + \frac{\Theta^2}{2(n+1)(n+2)} - \frac{(2n+1)^2\Theta^2}{4(n+1)^2} =$$

$$= \frac{n\Theta^2}{4(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$ – состоятельная

$$D[\tilde{\Theta}_2'] = \frac{4(n+1)^2 n}{4(n+1)^2(n+2)(2n+1)^2} = \frac{n\Theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$

– состоятельная

- $\tilde{\Theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

$$M[X_{min}] = \int_\Theta^{2\Theta} y\, n(1-\varPhi(y))^{n-1}\varphi(y)dy =$$

$$= \frac{1}{\Theta}n\int_\Theta^{2\Theta} y\Big(1-\frac{y-\Theta}{\Theta}\Big)^{n-1}dy = \frac{1}{\Theta}n\int_\Theta^{2\Theta} y\Big(2-\frac{y}{\Theta}\Big)^{n-1}dy =$$

$$= \frac{n}{\Theta}\Big(y\frac{(-\Theta)}{n}\Big(2-\frac{y}{\Theta}\Big)^n\Big|_\Theta^{2\Theta} + \int_\Theta^{2\Theta}\frac{\Theta}{n}\Big(2-\frac{y}{\Theta}\Big)^n dy =$$

$$= 1\cdot\Theta + \frac{(2-\frac{y}{\Theta})^{n+1}}{n+1}(-\Theta)\Big|_\Theta^{2\Theta} = \frac{n+2}{n+1}\Theta$$

$$M[\tilde{\Theta}_3] = \frac{1}{5}M[X_{min}] + \frac{2}{5}M[X_{max}] =$$

$$= \frac{\Theta}{5}\Big(\frac{n+2}{n+1} + \frac{2(2n+1)}{n+1}\Big) = \frac{5n+4}{5(n+1)}\Theta$$ – смещенная

Исправим: $\tilde{\Theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\Theta}_3$ - несмещенная

$$D[\tilde{\Theta}_3] = D[\tfrac{1}{5}X_{min}] + D[\tfrac{2}{5}X_{max}] -$$

$$- cov(\tfrac{1}{5}X_{min}, \tfrac{2}{5}X_{max}) = \frac{1}{25}D[X_{min}] + \frac{4}{25}D[X_{max}] -$$

$$- cov(\tfrac{1}{5}X_{min}, \tfrac{2}{5}X_{max})$$

$$cov(\tfrac{1}{5}X_{min}, \tfrac{2}{5}X_{max}) = \frac{2}{25} \cdot cov(X_{min}, X_{max})$$

$$M[X_{min}^2] = \int_{\Theta}^{2\Theta} y^2 n(1-\Phi(y))^{n-1} \varphi(y)dy =$$

$$= \frac{n}{\Theta}\int_{\Theta}^{2\Theta} y^2\left(2-\frac{y}{\Theta}\right)^{n-1} dy = \frac{n}{\Theta}\Bigg(-y^2 \frac{\Theta\left(2-\frac{y}{\Theta}\right)^n}{n}\Bigg|_{\Theta}^{2\Theta} +$$

$$+ \int_{\Theta}^{2\Theta} 2\Theta y \frac{\left(2-\frac{y}{\Theta}\right)^n}{n} dy\Bigg) = \Bigg(\Theta^2 + 2\Bigg(\frac{-\Theta y\left(2-\frac{y}{\Theta}\right)^{n+1}}{n+1}\Bigg|_{\Theta}^{2\Theta} +$$

$$+ \frac{\Theta}{n+1}\int_{\Theta}^{2\Theta}\left(2-\frac{y}{\Theta}\right)^{n+1} dy\Bigg)\Bigg) = \frac{n^2+5n+8}{(n+1)(n+2)}\Theta^2$$

$$D[X_{min}] = \frac{n^2+5n+8}{(n+1)(n+2)}\Theta^2 - \frac{n^2+4n+4}{(n+1)^2}\Theta^2 = \frac{n\Theta^2}{(n+1)^2(n+2)}$$

$$M[X_{min}X_{max}] = \iint_{-\infty}^{+\infty} \overset{X_{min}}{y}\, \overset{X_{max}}{z}\, g(y,z)\,dy\,dz = \int_{\Theta}^{2\Theta} dz \underbrace{\int_{\Theta}^{z} yz\,g(y,z)\,dy}_{J}$$

$$J = \int_{\Theta}^{z} yz\,g(y,z)\,dy = \int_{\Theta}^{z} yzn(n-1)\left(\frac{z-\Theta}{\Theta} - \frac{y-\Theta}{\Theta}\right)^{n-2}\frac{1}{\Theta^2}dy =$$

$$= \int_{\Theta}^{z} zn(n-1)y\frac{(z-y)^{n-2}}{\Theta^n}dy = \frac{zn(n-1)}{\Theta^n}\Bigg(-\frac{y(z-y)^{n-1}}{n-1}\Bigg|_{\Theta}^{z} + \frac{1}{n-1}\int_{\Theta}^{z}(z-y)^{n-1}dy =$$

$$= \frac{zn}{\Theta^n}\Bigg(\Theta(z-\Theta)^{n-1} - \frac{(z-y)^n}{n}\Bigg|_{\Theta}^{z}\Bigg) = \frac{n}{\Theta^{n-1}}z(z-\Theta)^{n-1} + \frac{z(z-\Theta)^n}{\Theta^n}$$

$$\int_\Theta^{2\Theta}\left(\frac{n}{\Theta^{n-1}} z(z-\Theta)^{n-1} + \frac{z(z-\Theta)^n}{\Theta^n}\right)dz = \left(\frac{nz}{\Theta^{n-1}}\,\frac{(z-\Theta)^n}{n}\Big|_\Theta^{2\Theta} - \right.$$

$$\left. - \frac{n}{\Theta^{n-1}\cdot n}\int_\Theta^{2\Theta}(z-\Theta)^n dz + \frac{z}{\Theta^n}\,\frac{(z-\Theta)^{n+1}}{n+1}\Big|_\Theta^{2\Theta} - \frac{1}{(n+1)\Theta^n}\cdot\int_\Theta^{2\Theta}(z-\Theta)^{n+1}dz\right) =$$

$$2\Theta^2 - \frac{1}{\Theta^{n-1}}\,\frac{(z-\Theta)^{n+1}}{n+1}\Big|_\Theta^{2\Theta} + \frac{2\Theta^2}{n+1} - \frac{(z-\Theta)^{n+2}}{(n+1)(n+2)\Theta^n}\Big|_\Theta^{2\Theta} = \frac{2n+4}{n+1}\Theta^2 - \frac{\Theta^2}{n+1} -$$

$$- \frac{\Theta^2}{(n+1)(n+2)} = \left(\frac{(2n+3)(n+2)-1}{(n+1)(n+2)}\right)\Theta^2 = \frac{(2n^2+7n+5)\Theta^2}{(n+1)(n+2)}$$

$$D[\tilde\Theta_3] = \frac{\Theta^2}{25}\left(\frac{n+n-2(2n^2+7n+5)(n+1)+2(2n+1)(n+2)^2}{(n+1)^2(n+2)}\right) =$$

$$= \frac{2(n-1)\Theta^2}{25(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$ – состоятельная

$$D[\tilde\Theta_3'] = \frac{25(n+1)^2}{(5n+4)^2} D[\tilde\Theta_3] = \frac{2(n-1)\Theta^2}{(5n+4)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

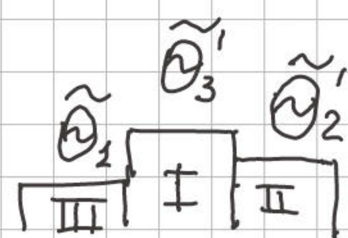
– состоятельная

c) $D[\tilde\Theta_1] = \frac{1}{2+n}\Theta^2 \sim \frac{1}{n}$ – менее эффективна, чем $\tilde\Theta_2'$ и $\tilde\Theta_3'$

$$D[\tilde\Theta_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2} \sim \frac{1}{n^2}$$

$$D[\tilde\Theta_3'] = \frac{2(n-1)\Theta^2}{(5n+4)^2(n+2)} \sim \frac{1}{n^2}$$

Вывод:




$$\frac{n}{(n+2)(2n+1)^2} \qquad \frac{2(n-1)}{(5n+4)^2(n+2)}$$

$$n(25n^2+40n+16) \qquad 2(n-1)(4n^2+4n+4)$$

$$\wr \qquad\qquad \wr$$

$$25n^3 \qquad\qquad 8n^3$$

$\Rightarrow \tilde\Theta_3'$ – наиболее эффективная из всех

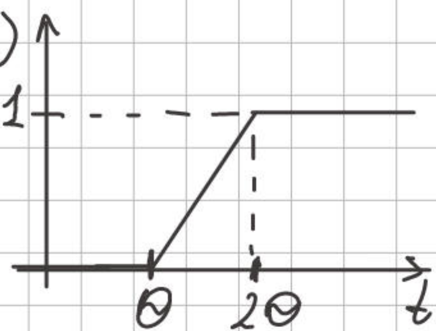
## d Точный доверит. интервал для $\Theta$

$$\xi \in R[\Theta, 2\Theta] \qquad \begin{cases} h = \Theta \\ \tilde{h} = \dfrac{X_{max}}{2} \quad (\text{ОМП}) \end{cases}$$

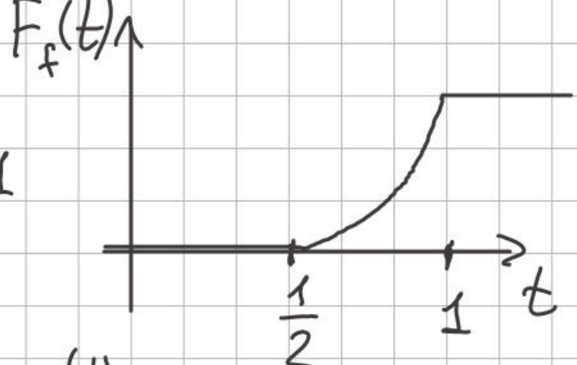
$$\exists f = \frac{\tilde{h}}{h} = \frac{X_{max}}{2\Theta}; \quad f(h, \tilde{h}) \sim q(t) \text{ - не завис. от } h$$

$$F = P(f < t) = P(X_{max} < 2\Theta t) = P(x_i < 2\Theta t;\ i = 1 \div n) =$$
$$= (P(X_i < 2\Theta t))^n = (F(2\Theta t))^n$$

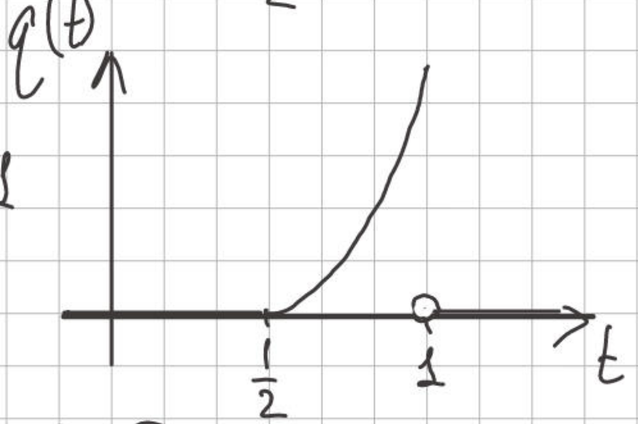
$$F(t) = \begin{cases} 0; & t \leq \Theta \\ \dfrac{t}{\Theta} - 1; & \Theta < t \leq 2\Theta \\ 1; & t > 2\Theta \end{cases}$$




$$F_f(t) = \begin{cases} 0; & t \leq \frac{1}{2} \\ (2t - 1)^n; & \frac{1}{2} < t \leq 1 \\ 1; & t > 1 \end{cases}$$




$$q(t) = \begin{cases} 0; & t \leq \frac{1}{2} \\ 2n(2t-1)^{n-1}; & \frac{1}{2} < t \leq 1 \\ 0; & t > 1 \end{cases}$$




$$P\left(t_1 < \frac{X_{max}}{2\Theta} < t_2\right) = \beta = 0{,}95$$

$n=100$

$$200\int_a^b (2t-1)^{99}\,dt = \left\{\begin{matrix} 2t-1=u \\ du=2dt \end{matrix}\right\} = 100\int_a^b u^{99}\,du = 100\left(\frac{u^{100}}{100}\right)\Big|_a^b =$$

$$= b^{100} - a^{100}$$

$$\int_{\frac{1}{2}}^{t_1} q(t)\,dt = \int_{\frac{1}{2}}^{t_1} 2\cdot 100\cdot(2t-1)^{99}\,dt = 0{,}025 = t_1^{100} - \frac{1}{2^{100}} =>$$

$$\boxed{t_1 = \sqrt[100]{0{,}025 + (0.5)^{100}}}$$

$$\int_{t_2}^{1} q(t)\,dt = \int_{t_2}^{1} 2\cdot 100(2t-1)^{99}\,dt = 0{,}025 = 1 - t_2^{100} =>$$

$$=> \boxed{t_2 = \sqrt[100]{0{,}975}}$$

$$t_1 < \frac{X_{max}}{2\Theta} < t_2 \quad => \quad \boxed{\frac{X_{max}}{2t_2} < \Theta < \frac{X_{max}}{2t_1}};$$

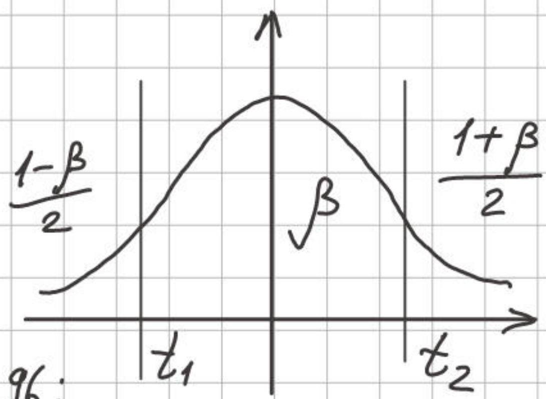
e) Асимпт. доверит. интервал для $\Theta$

$\exists\ \beta = 0{,}95$

$\tilde{\Theta}_1 = \frac{2}{3}\bar{X}$ - ОММ ; $\bar{X} = \frac{3}{2}\tilde{\Theta}_1$ ; $M[\xi] = \frac{3}{2}\Theta$

ЦПТ: $\frac{\bar{X} - M\xi}{\sqrt{D\xi}}\sqrt{n} \rightsquigarrow N(0,1)$

$S \xrightarrow{F} \sqrt{D\xi}$




$\frac{3}{2} \cdot \frac{\tilde{\Theta}_1 - \Theta}{S}\sqrt{n} \rightsquigarrow N(0,1)$

$t_1 = U_{\frac{1-\beta}{2}} = -1{,}96$ ; $t_2 = U_{\frac{1+\beta}{2}} = 1{,}96$ ;

$t_1 < \frac{3}{2} \cdot \frac{\tilde{\Theta}_1 - \Theta}{S}\sqrt{n} < t_2$

$\Rightarrow \boxed{\frac{1}{\sqrt{n}} \cdot \frac{2}{3} \cdot S\, t_1 + \tilde{\Theta}_1 < \Theta < \frac{1}{\sqrt{n}} \cdot \frac{2}{3} S\, t_2 + \tilde{\Theta}_1}$

**5.** Случайная величина имеет распределение Парето:

$$p(x) = \begin{cases} \dfrac{\theta - 1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1.$$

a) По выборке объема $n$ найти оценку параметра $\theta$ методом максимального правдоподобия.

b) Построить доверительный интервал для медианы.

c) Найти байесовскую оценку параметра $\theta$ и построить байесовский доверительный интервал для этой оценки, если априорная плотность распределения параметра имеет вид $p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$.

d) Построить асимптотический доверительный интервал для параметра $\theta$.

e) Сгенерируйте выборку объема $n = 100$ для некоторого значения параметра $\theta$. Вычислите указанные выше доверительные интервалы для доверительной вероятности 0.95.

f) Численно постройте бутстраповский доверительный интервал двумя способами, используя параметрический бутстрап и непараметрический бутстрап.

g) Сравнить все интервалы.

a ОМП:

$$\xi \sim p(x, \theta)$$

$$L(\vec{x}_n, \theta) = \prod_{i=1}^{n} p(x_i, \theta) = \begin{cases} \prod_{i=1}^{n} \dfrac{(\theta-1)}{x_i^\theta}; & x_i \geq 1 \\ 0; & x_i < 1 \end{cases}$$

$$\ln L = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i \to \max$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \boxed{\tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1}$$

Проверим: $\dfrac{\partial^2 \ln L}{\partial \theta^2} = -\dfrac{n}{(\theta-1)^2} < 0$ – нашли max

b $\exists\, \beta = 0,95$

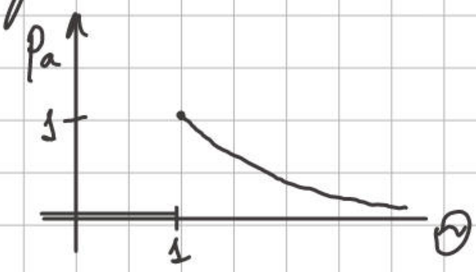
$$\int_1^{med} \frac{\theta - 1}{x^\theta} dx = \int_1^{med} (\theta - 1) x^{-\theta} dx = \frac{\theta - 1}{-(\theta - 1)} x^{-\theta+1} \Big|_1^{med} = \frac{1}{2}$$

$$1 - med^{-\theta+1} = \frac{1}{2} \Rightarrow med = 2^{\frac{1}{\theta - 1}}$$ (при $\theta = 10$ $med \approx 1{,}0801$)

Посчитаем дов. интервал bootstrap'ом

c) Байес

$$p(\theta)=\begin{cases} e^{1-\theta}, & \theta\geq 1\\ 0, & \theta<1\end{cases}$$ – априорная пл-ость




$$p(\theta\,|\,\vec{x}_n)=\frac{p(\theta)L(\theta)}{P(\vec{x}_n)}=CLP(\theta)\to\max$$

$$L(\theta)=\prod_{i=1}^{n}p(x_i,\theta)=\begin{cases}\prod\limits_{i=1}^{n}\dfrac{(\theta-1)}{x_i^{\theta}}; & x_i\geq 1\\ 0; & x_i<1\end{cases}$$

$$\ln L=n\ln(\theta-1)-\theta\sum_{i=1}^{n}\ln x_i$$

$$\ln p(\theta|\vec{x}_n)=\ln c+\ln L+\ln p(\theta)=\{\text{тк } \theta>1\}$$

$$=\ln c+n\ln(\theta-1)-\theta\sum_{i=1}^{n}\ln x_i+1-\theta$$

$$\frac{\partial\ln p(\theta|\vec{x}_n)}{\partial\theta}=\frac{n}{\theta-1}-\sum_{i=1}^{n}\ln x_i-1=0$$

$$\frac{n}{\theta-1}=1+\sum_{i=1}^{n}\ln x_i$$

$$\boxed{\tilde{\theta}=\frac{n}{1+\sum\limits_{i=1}^{n}\ln x_i}+1}$$

$$p(\theta|\vec{x}_n)=c\cdot(\theta-1)^n\cdot e^{1-\theta}\cdot\prod_{i=1}^{n}\frac{1}{x_i^{\theta}}=c(\theta-1)^n e^{1-\theta}\left(\prod_{i=1}^{n}\frac{1}{x_i}\right)^{\theta}=$$

$$=e\tilde{c}(1-\theta)^{100}\cdot(e\prod x_i)^{-\theta}$$

d Асимпт. довер. интервал для $\Theta$

$$\sqrt{nJ(\Theta)}(\tilde{\Theta}-\Theta) \to N(0,1)$$

$$J(\Theta) = M\left[\left(\frac{\partial \ln p}{\partial\Theta}\right)^2\right] = M\left[\left(\frac{\partial \ln \frac{\Theta-1}{x^{\Theta}}}{\partial\Theta}\right)^2\right] =$$

$$= M\left[\frac{1}{(\Theta-1)^2}\right] - \frac{2}{\Theta-1}M[\ln x] + M[\ln^2 x]$$

$$M[\ln x] = \int_1^{+\infty} \ln x \, \frac{\Theta-1}{\Theta} dx = (\Theta-1)\int_1^{+\infty} \ln x \cdot x^{-\Theta} dx =$$

$$= (\Theta-1)\left(\frac{\ln \cdot x^{-\Theta+1}}{-\Theta+1}\Big|_1^{+\infty} + \int_1^{+\infty} \frac{x^{-\Theta}}{\Theta-1}dx = 0 + \frac{x^{-\Theta+1}}{\Theta+1}\Big|_1^{+\infty} = \frac{1}{\Theta-1}\right.$$

$$M[\ln^2 x] = \int_1^{+\infty} \ln^2 x \, \frac{\Theta-1}{x^{\Theta}} dx = (\Theta-1)\int_1^{+\infty} \ln^2 x \cdot x^{-\Theta} dx =$$

$$= (\Theta-1)\left(\frac{\ln^2 x \cdot x^{-\Theta+1}}{-\Theta+1}\Big|_1^{+\infty} - \int_1^{+\infty} \frac{2\ln x \cdot x^{-\Theta}}{-\Theta+1}dx = 0 + 2(\Theta-1)\int_1 \ln x \cdot x^{-\Theta} dx = \frac{2}{(\Theta-1)^2}\right.$$

$$J(\Theta) = \frac{1}{(\Theta-1)^2} - \frac{2}{(\Theta-1)^2} + \frac{2}{(\Theta-1)^2} = \frac{1}{(\Theta-1)^2}$$

$$\frac{\sqrt{n}(\tilde{\Theta}-\Theta)}{\tilde{\Theta}-1} \sim N(0,1)$$

$$\tilde{\Theta} - \frac{\tilde{\Theta}-1}{\sqrt{n}}t_1 < \Theta < \tilde{\Theta} - \frac{\tilde{\Theta}-1}{\sqrt{n}}t_2$$

$$\boxed{\tilde{\Theta} - \frac{\tilde{\Theta}-1}{\sqrt{n}}u_{\frac{1-\beta}{2}} < \Theta < \tilde{\Theta} - \frac{\tilde{\Theta}-1}{\sqrt{n}}u_{\frac{1+\beta}{2}}}$$