Complex Derivatives

A complex function

$$f : \mathbb{C} \to \mathbb{C}$$

takes $z = x + iy$ to $f(z) = u + iv$

and can be thought of as

$$f(x,y) = u(x,y) + i\, v(x,y)$$

$x = \operatorname{Re} z$
$u = \operatorname{Re} f(z)$

$y = \operatorname{Im} z$
$v = \operatorname{Im} f(z)$

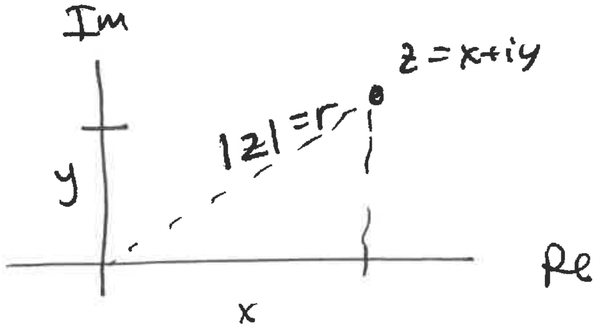
To reader: look at basics of manipulate of complex numbers in 1.1.1.

we're going to look at the calculus of complex functions as an extension of the calculus of real functions.

- limits
- derivatives
- integrals

limits are going to be more complicated, because every complex function is a function of two real variables.

First, we'll need a notion of distance to capture limits.



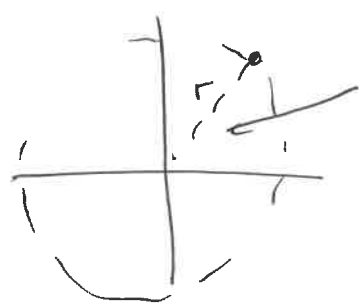
$|z| = |x+iy| = \sqrt{x^2+y^2}$

So

$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$ (very useful!)

Intervals in $\mathbb{R}$ are the key types of sets in calculus.

Def: In $\mathbb{C}$, we use disks of the form $|z|<r$



$|z|<r.$

Def: Let f be defined on $\{z : 0<|z|<r\}$ w/ $r>0$.

$\lim_{z\to 0} f(z) = L$ means

given $\varepsilon>0$, $\exists\,\delta$ such that $0<|z|<\delta \Rightarrow |f(z)-L|<\varepsilon$.

This def naturally holds at any $z=a$:

$\lim_{z \to a} f(z) = L$ means
given $\varepsilon > 0$, $\exists \delta > 0$ such that
$0 < |z-a| < \delta \Rightarrow |f(z) - L| < \varepsilon$.

Note: we already get a hint here that complex analysis is an extension of real calculus: symbolically, this is the same defn!

In fact, as in real analysis, the triangle inequality drives the basic results.

$$|zw| = |z||w|$$

~~Ex: prove~~

Theorem: Let $w, z \in \mathbb{C}$.

A. $|w+z| \le |w| + |z|$

B. $|w-z| \ge \big||w| - |z|\big|$ (sometimes called reverse inequality)

pf: let $w = a+ib$ $z = c+id$

$\cancel{|w+z|^2 = ((a+c)+i(b+d))\ (a+c)^2 + (b+d)^2}$

$\cancel{= (a^2+c^2) + (b^2+d^2)}$

$\cancel{\le a^2 + 2ac + c^2 +}$

④

$$\cancel{(|w|+|z|)^2 = \left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)^2}$$

$$\cancel{= a^2+b^2+c^2+d^2}$$

$$\cancel{+ 2\sqrt{(a^2+b^2)(c^2+d^2)}}$$

$$\cancel{\geq}$$

$$\cancel{|z+w|^2 = (z+w}$$

Basic estimates:

$$-|z| \leq \mathrm{Re}|z| \leq |z|$$

$$-|z| \leq \mathrm{Im}|z| \leq |z|$$

and $|z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

Theorem: let $z, w \in \mathbb{C}$.

A. $|z+w| \leq |z|+|w|$

B. $|z-w| \geq \big||z|-|w|\big|$

The hammer of analysis!

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pf: $|z+w|^2 = (z+w)\overline{(z+w)}$

$$= (z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w})$$

$$= |z^2|^2 + 2\,\mathrm{Re}(w\bar{z}) + |w|^2$$

$$\leq |z^2| + 2|w||z| + |w|^2$$

$$= (|z|+|w|)^2.$$

B. $|z| = \cancel{|z + w + (-w)|}$ $\quad |z - w + w|$

$\cancel{\le |z + w}$ $\quad \le |z - w| + |w|$

switch z and w to get

$|w| \le |w - z| + |z|$

$|w| - |z| \le |w - z| = |z - w|$

so $\cancel{|z - w|}$

$|z| - |w| \le |z - w|$

and

$|w| - |z| \le |z - w|$

so $\big||z| - |w|\big| \le |z - w|$

Let's find some nails.

~~Theorem~~ Example limit theorem:

Theorem: suppose $\lim_{z \to a} f = L$ and $\lim_{z \to b} g = M$.

~~then~~ $\lim_{z \to a} f \cdot g = LM$.

let $a = 0$ WLOG.

Pf: let $\varepsilon > 0$ be given. ~~δ_f~~

let δ_f be such that $|f(z) - L| < \dfrac{\varepsilon |L|}{\varepsilon + 2|M_L|}$ when $|z| < \delta_f$

δ_g such that $|g(z) - L| < \dfrac{\varepsilon}{2|L|}$ when $|z| < \delta_g$.

Now, let $\delta = \min \delta_f, \delta_g$.

(5A)

then

$$|f(z)g(z) - LM| = |fg - Lg + Lg - LM|$$

$$\leq |f-L||g| + |L||g-M|$$

$$= |f-L|\,|g+M-M| + |L|\,|g-M|$$

$$\leq |f-L|\left(|g-M| + |M|\right) + |L||g-M|$$

$$\leq \frac{\varepsilon|L|}{\varepsilon + 2|ML|}\left(\frac{\varepsilon}{2|L|} + |M|\right) + |L|\frac{\varepsilon}{2|L|}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\blacksquare$

Ex ~~prove the other statements~~
rest as exercises.

Def: complex derivative

$$f'(z) = \frac{df}{dz} := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \quad \text{when it exists}$$

(h is complex, so we're approaching 0 on a collapsing disk)

- complex functions can be differentiable at only a point!

  Ex $f(z) = |z|^2$.

  $f'(0)=0$ but no other derivative exists.

- If the function is differentiable at every point on a disk, much more holds!

a function is called holomorphic at $z=z_0$ if f is (complex) differentiable at every point in a disk centering z_0.

- a function holomorphic at every point on its domain is just called holomorphic

- A function holomorphic everywhere is called entire.

Ex $f(z)=z^2$ is differentiable:

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{z^2+2zh+h^2-z^2}{h}$$

$$= \lim_{h\to 0} 2z+h = \underline{2z}$$

this limit holds $\forall z$, so $f(z)=z^2$ is entire!

Ex $f(z)=\bar{z}$.

$$f'(z)=\lim_{h\to 0}\frac{\overline{z+h}-\bar{z}}{h} = \lim_{h\to 0}\frac{\bar{z}+\bar{h}-\bar{z}}{h} = \lim_{h\to 0}\frac{\bar{h}}{h} =$$

$$\lim_{h\to 0}\frac{x-iy}{x+iy}.$$ on $x=0$,

$$\lim_{y\to 0}\frac{-iy}{iy} = -1$$

on $y=0$,

$$\lim_{x\to 0}\frac{x}{x} = 1$$

limit on two different paths disagree!

because the limit laws hold, and complex derivatives are algebraically similar to real derivatives, the usual differentiation rules carry over.

<u>Ex</u> recall Thm 1.1.3. prove a couple.