6. The Cauchy-Riemann equations

Recall that we don't yet have Taylor's theorem (which will let us show that holomorphic implies analytic). Still, we have an easy condition (related to Clairaut's theorem on mixed partial derivatives from calculus) for characterizing when a complex function is holomorphic (that is, differentiable on a disk).

Theorem 6.1. *Suppose that $u(x, y)$ and $v(x, y)$ are real-valued functions with continuous first partial derivatives on a disk S centered at $z = a$. Then*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

is differentiable on S if and only if $u_x = v_y$ and $u_y = -v_x$.

$\Rightarrow$:

$u(x,y)$ $v(x,y)$ real-valued

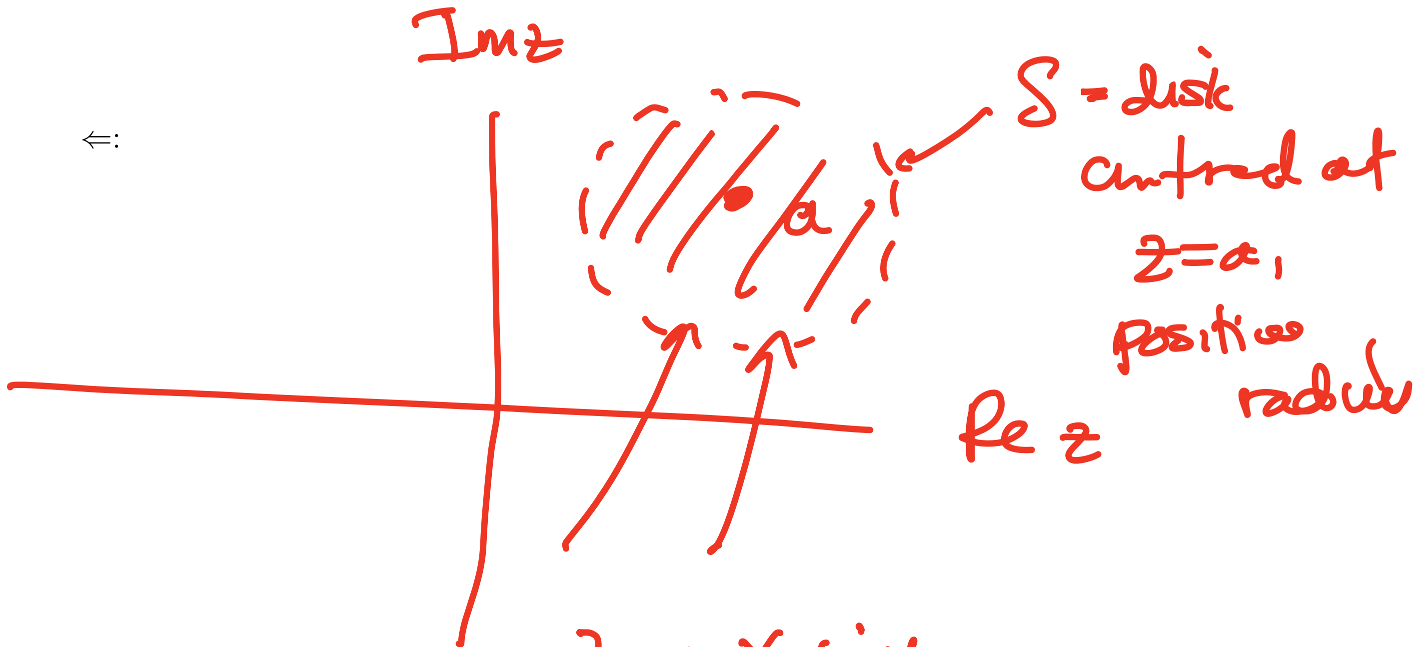
$$\text{CRE} \begin{cases} u_x(x,y) = v_y(x,y) \\ \text{and} \\ u_y(x,y) = -v_x(x,y) \end{cases}$$

complex

$$\boxed{f(z) = u(x,y) + i v(x,y)}$$

f is differentiable on a disk iff u, v satisfy CRE.

$\Leftarrow$:



$z = x+iy$

$u(x,y)$ - real valued

$v(x,y)$ - real valued

u_x, u_y, v_x, v_y are all continuous (also called C^1 functions)

Definition 6.2. *Let $u(x,y), v(x,y)$ be real-valued functions with continuous first partial derivatives. The* **Cauchy-Riemann equations** *(often abbreviated CRE) are the system of differential equations*

$$u_x = v_y$$

$$u_y = -v_x$$

Example 6.3. *Show that $f(z) = z^2$ is analytic using the CRE.*

$f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i\,2xy$

$u(x,y) = x^2 - y^2 = \operatorname{Re} f$

$v(x,y) = 2xy = \operatorname{Im} f$

$u_x = \underline{2x}$ $\quad v_y = \underline{2y}$ $\quad$ equal $\quad u_x = v_y$ ✓

$u_y = \underline{-2y}$ $\quad v_x = \underline{2y}$ $\quad u_y = -v_x$ ✓

Continuous? yes

Example 6.4. *Show that $f(z) = \overline{z}$ is analytic nowhere using the CRE.*

$f(z) = \overline{z} = \overline{(x+iy)} = x - iy$

$u = \operatorname{Re} f = x \qquad v = \operatorname{Im} f = -y$

$u_x = 1 \qquad v_y = -1 \qquad u_x \neq v_y$

$u_y = 0 \qquad v_x = 0 \qquad u_y = -v_x$ ✓

Since CRE are not satisfied,
$f(z) = \overline{z}$ is analytic on no disk.