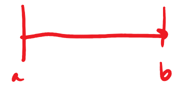
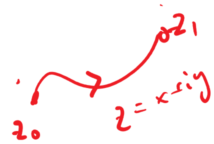
Complex contour integration

Sunday, August 14, 2022 11:00 AM

$\int_a^b f(x)\,dx$





7. Complex line integrals

Complex integrals are defined over curves, just as in multivariable calculus. The idea, once again, is to create an object that we can approach with the theory of real calculus.

Definition 7.1. *A* **smooth curve** *(also called a differentiable curve) is the graph $z = x+iy$ in $\mathbb{C}$ of a parametric function*

$$x = x(t), y = y(t), \boxed{a \le t \le b}$$

where $x(t)$ and $y(t)$ are $\boxed{\text{of class } C^1}$ (that is, they are continuous and have continuous derivatives).

$x = x(t)$
$y = y(t)$

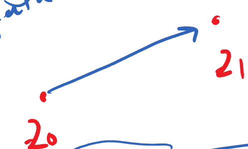
$x(a), y(a) \to z_0 = x(a)+iy(a)$

$x(b), y(b) \to z_1 = x(b)+iy(b)$

Just as in multivariable calculus, we define the integral of a complex function over a curve by composition.

Some examples:

straight line path



$z_0 = x_0+iy_0$

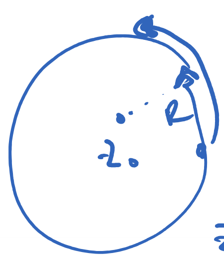
$z_1 = x_1+iy_1$

$\boxed{z(t) = z_0 + t(z_1 - z_0)}$ $\quad 0 \le t \le 1$

$z(0) = z_0$

$z_1(1) = z_1$

circle



$x(t) = x_0 + R\cos t$

$y(t) = y_0 + R\sin t$

$\mathbb{R}^2$

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$z(t) = (x_0+R\cos t) + i(y_0 + R\sin t)$

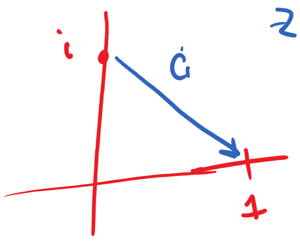
$0 \le t \le 2\pi$

Definition 7.2. *The* **contour integral** *of a complex function $f(z)$ over a smooth curve C is defined by*

$$\int_C f(z)\,dz = \int_a^b f(z(t))\boxed{z'(t)dt} = \int_a^b f(x(t)+iy(t))(x'(t)+iy'(t))\,dt.$$

This is justified, as the function $f(x(t)+iy(t))(x'(t)+iy'(t))$ can be sorted into real and imaginary parts, which gives two integrals of real-valued functions.

Example 7.3. *Integrate $f(z) = z+1$ over the straight line curve in $\mathbb{C}$ from i to 1.*

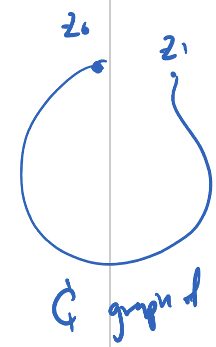


$z(t) = i + t(i-1) \qquad 0 \le t \le 1$

$z'(t) = i - 1$

$$\int_C z+1\,dz = \int_0^1 \Big(\underbrace{(i + t(i-1))}_{z} \underbrace{+1}_{+1} \Big)(i-1)\,dt$$

algebra $\displaystyle\int_0^1 2 - 2it\,dt = \underline{2 - i}$



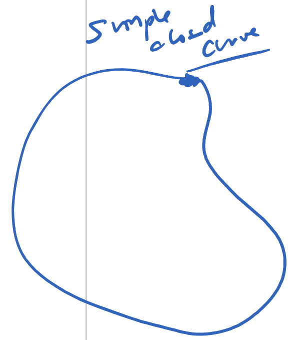
C graph of $z(t)$

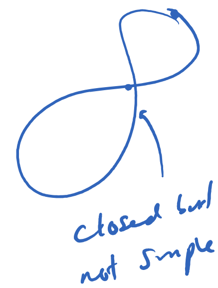
$dz = \frac{d}{dt}(z(t)) = z'(t)\,dt$

$F = f(z)\,z'(t) \qquad \mathrm{Re}\,F + i\,\mathrm{Im}\,F$

$\int F\,dt = \int \mathrm{Re}\,F\,dt + i\int \mathrm{Im}\,F\,dt$

One of the most important types of curves in complex integration is called a **simple closed curve**. Closed means that the curve begins and ends at the same point. Simple means that the curve doesn't intersect itself. Just as in line integration from real calculus, the direction of integration matters. We set the standard positive **orientation** of a closed curve to be the direction that keeps the interior of the curve to the left as the parametrization travels the curve. That is, a curve is positively oriented if it is traveled in a counterclockwise direction about the area it encloses.

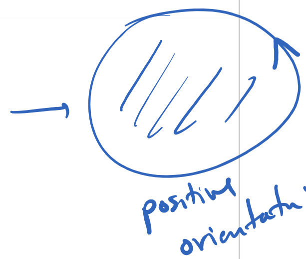




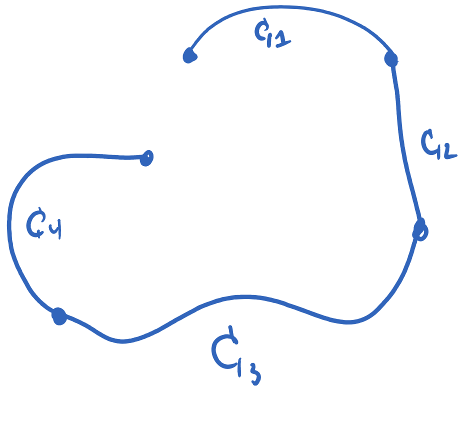
Integrals around positively oriented closed curves are so important that they get their own symbol: $\oint_C f(z)\,dz$.

Note 7.4. We should be very careful here. That simple closed curves even have a well-defined notion of inside and outside is the subject of a theorem called the Jordan Curve Theorem, which has (perhaps surprisingly) a rather involved proof.

$$\oint_C f(z)\,dz$$



Another thing we can do with parametrized curves is link them together, as long as the orientations match. If the pieces are smooth, such a curve is called **piecewise smooth**. Integrals over piecewise curves are defined in the obvious way:



piecewise smooth

$$\int_C f(z)\,dz = \int_{C_1} f + \int_{C_2} f + \int_{C_3} f + \int_{C_4} f$$

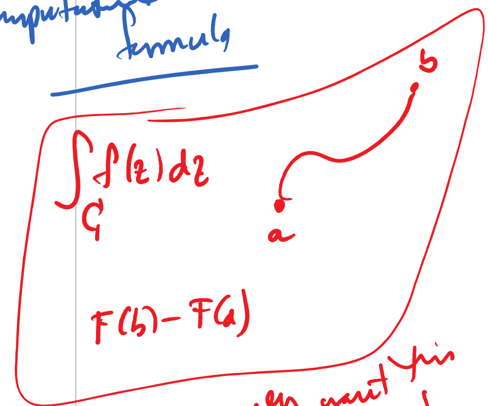
$$= \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

7.1. **The fundamental theorem of complex functions.** The fundamental theorem of calculus is a profoundly important result that connects antidifferentiation to the evaluation of a definite integral. One part of the theorem asserts that for a function f which is the derivative of a continuous function F defined on an interval $[a, b]$, one has

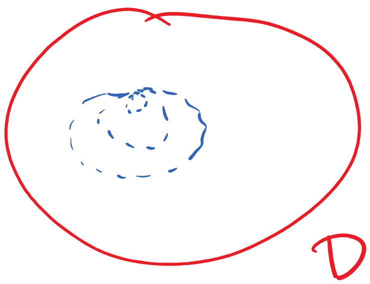
$$\boxed{\int_a^b f(x)\,dx} = F(b) - F(a).$$

amazingly useful computational formula

It turns out to be the case that a similar result holds for complex functions with appropriate hypotheses (hypotheses that allow the application of the real FTC). This is remarkably useful when evaluating contour integrals, because it obviates the need to choose a parametrization!. One important detail is that the complex Fundamental Theoerm applies to functions on **simply connected** domains, which roughly speaking are domains without holes. It will be important for us to consider non-simply connected domains, so this distinction matters.



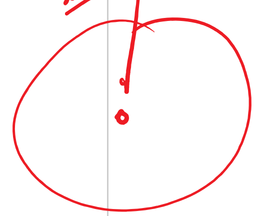
Really want this to be true!



$D \subseteq \mathbb{C}$ simply connected if "no holes"

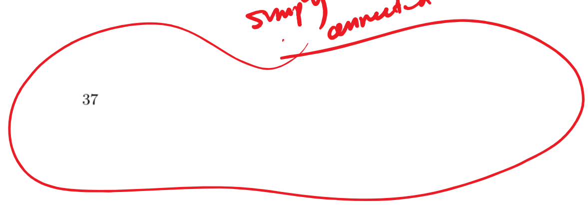
not simply connected

remove a single point



not simply connected.

simply connected



infinitely differentiable

Theorem 7.5. *Suppose that a continuous complex function f is the derivative of a function F analytic on a simply connected domain D so that $F'(z) = f(z)$ for all $z \in D$. Then for any piecewise smooth curve C in D beginning at a and ending at b,*

$$\boxed{\int_C f(z)\,dz = F(b) - F(a).}$$

You might recognize this type of integral as related to the notion of **path independence** from multivariable calculus (once again pointing towards a strong connection between analytic functions and conservative vector fields).

Proof. Essentially, one pulls the path into continuously differentiable parametrizations, applies the Chain Rule, and then applies the real FTC to the real and imaginary parts of the result. □

Real FTC

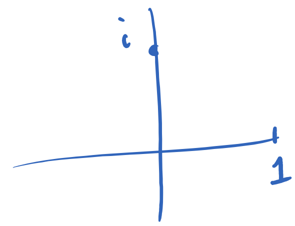
f is the derivative of a continuous function F.

Complex FTC

f cont and F is analytic

then

$\int_{C_1} z+1\,dz$ from $z=i$ to $z=1$



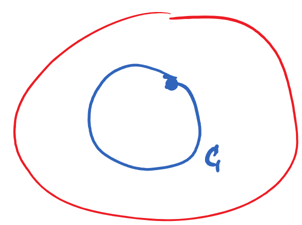
$f(z) = z+1$ is continuous

$F(z) = \frac{1}{2}z^2 + z$ has $F' = f$

F is analytic everywhere.

$$\int_{C_1} z+1\,dz = F(1) - F(i) = \left(\tfrac{1}{2}(1)^2+1\right) - \left(\tfrac{1}{2}(i^2)+i\right)$$
$$= \tfrac{1}{2} + 1 + \tfrac{1}{2} - i = \boxed{2-i}$$

f analytic on D



Q: what is going to happen to $\oint_C f(z)\,dz$?

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Example 7.6. *Evaluate*

$$\int_C z+3\,dz$$

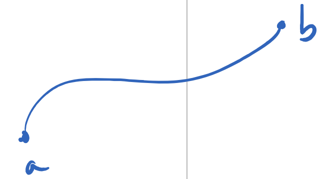
where $C = \{z = x+iy : x(t) = t, y(t) = 2t-1; 0 \le t \le 1\}$.

$F(z) = \frac{1}{2}z^2 + 3z$

analytic everywhere

$F(b) - F(a)$

Apply FT complex function



$$\int_C z+3\,dz = F(1+i) - F(-i)$$

$$= \frac{1}{2}(1+i)^2 + 3(1+i) - \left[\frac{1}{2}(-i)^2 + 3(-i)\right]$$

$$= \frac{7}{2} + 7i$$

✓ algebra

$x(t) = t$ $0 \le t \le 1$

$y(t) = 2t - 1$

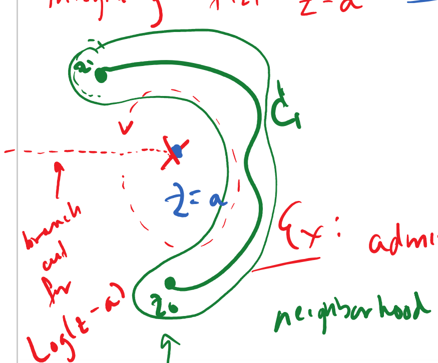
$a = x(0) + iy(0) = 0 + i(-1) = -i$

$b = x(1) + iy(1) = 1 + i1$

integrand $f(z) = \frac{1}{z-a}$ WANT $\mathrm{Log}(z-a)$

principal branch

$-\pi < \mathrm{Arg}(z-a) \le \pi$



C_γ: admissible contour

neighborhood

open, simply connected set U avoids $z=a$ and branch cut.

$\mathrm{Log}(z-a)$ analytic on U

$f(x) = \frac{1}{x}$ not defined at $x=0$

$F(x) = \ln|x| + C$

$F'(x) = f(x)$

$f(x) = \frac{1}{x}$ $x > 0$, $x < 0$

allowed to use FT complex functions

$$\int_{C_1} \frac{1}{z-a}\,dz = \mathrm{Log}(z-a)\Big|_{z_0}^{z_1}$$

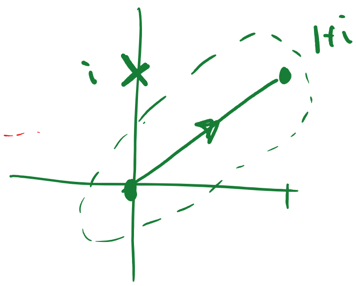
Real calculus techniques apply when using the Fundamental Theorem (as the theorem rests on the real FTC).

Example 7.7. *Evaluate*

straight line

$$\int_C \frac{dz}{z^2+1}$$

on a contour C beginning at $a = 0$ and ending at $b = 1 + i$.



$$\int_C \frac{1}{z^2+1}\,dz = \arctan z \Big|_0^{1+i}$$

$$= \arctan 1+i - \arctan 0 = \arctan 1+i$$

real calculus

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$

holds on all of $\mathbb{R}$

$\frac{1}{z^2+1}$ has an asymptote at $z=i$

$$\boxed{i^2+1 = -1+1 = 0}$$

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The final example is very important and motivates perhaps the central theorem of complex analysis, *Cauchy's integral theorem.*

Example 7.8. *Evaluate*

$$\int_C \frac{dz}{z-a}$$

on the circular arc centered at a *of radius* R *from polar angle* $\boxed{-\pi + \frac{1}{n}}$ *to polar angle* $\boxed{\pi - \frac{1}{n}}$. *Writing* $\boxed{a = a_1 + ia_2}$, *use the parametrization of* C *given by*

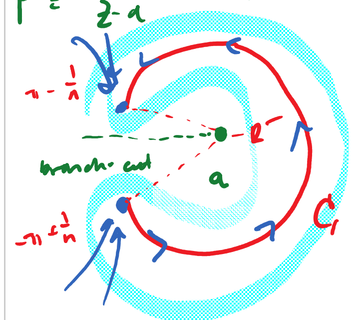
$$z(t) = (a_1 + R\cos t) + i(a_2 + R\sin t)$$

$$= a_1 + ia_2 + R(\cos t + i\sin t) = a + Re^{it}$$

$$z_0 = a + Re^{i(-\pi + \frac{1}{n})} \qquad z_1 = a + Re^{i(\pi - \frac{1}{n})}$$

$F = \mathrm{Log}(z-a)$

$F' = \frac{1}{z-a}$



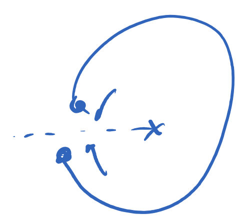
$$\int_{C_1} \frac{1}{z-a}\,dz \overset{FTC}{=} \mathrm{Log}(z-a)\Big|_{a+Re^{i(-\pi+\frac{1}{n})}}^{a+Re^{i(\pi-\frac{1}{n})}}$$

$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

$$= \mathrm{Log}\left(Re^{\boxed{i(\pi - \frac{1}{n})}}\right) - \mathrm{Log}\left(Re^{i(-\pi + \frac{1}{n})}\right)$$

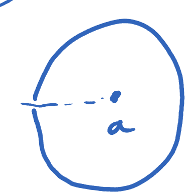
$$= \ln R + i\left(\pi - \tfrac{1}{n}\right) - \left(\ln R + i\left(-\pi + \tfrac{1}{n}\right)\right)$$

$$\boxed{= 2\pi i - i\frac{2}{n}}$$



$$\int_{C_1} \frac{1}{z-a}\,dz = 2\pi i - i\frac{2}{n}$$

$\lim_{n\to\infty}$



$$\lim_{n\to\infty} \int_{C_1} \frac{1}{z-a}\,dz = \lim_{n\to\infty} 2\pi i - i\frac{2}{n} \;\boxed{= 2\pi i}$$

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