6.1. **Harmonic functions.** Analytic functions have a close connection to a family of functions that arise as solutions to equations involving the study of heat and electromagnetism.

Definition 6.5. *Let $u(x, y)$ be a real-valued function of real variables x, y. u is called a* **harmonic function** *if the second derivatives of u exist and are continuous and satisfy the differential equation*

$$u_{xx} + u_{yy} = 0. \tag{6.1}$$

The equation (6.1) is called Laplace's equation. Harmonic functions satisfy an important property that we will see more of in subsequent discussion, the so-called **maximum principle** (which we will not prove here).

Theorem 6.6 (Maximum principle). *A harmonic function $u(x, y)$ defined on a disk D attains its maximum and minimum value on the boundary of the disk unless u is constant.*

Example 6.7. *Show that $u(x, y) = 12x^2y + 15x - 4y^3 + 2xy - 3$ is harmonic for all $z = x + iy$ in $\mathbb{C}$.*

$u_x = 24xy + 15 + 2y$

$u_{xx} = 24y$

$u_y = 12x^2 - 12y^2 + 2x$

$u_{yy} = -24y$

$u_{xx} + u_{yy} = 24y - 24y = 0$

u is harmonic

The next theorem gives the connection between harmonic and analytic functions - analytic functions are built from harmonic pieces!

Theorem 6.8. *A function $f(z) = u(x,y) + iv(x,y)$ analytic on a disk S has u and v harmonic on S.*

Proof. This is essentially Clairaut's theorem together with the Cauchy-Riemann equations. □

$$\left.\begin{aligned} \operatorname{Re} f(z) &= u(x,y) \\ \operatorname{Im} f(z) &= v(x,y) \end{aligned}\right\}$$ harmonic function

$$u_x = v_y \qquad \boxed{u_y = -v_x}$$

Clairaut's theorem: v is class C^2

then $u_{xy} = u_{yx}$

$$u_{xx} = v_{yx} \qquad u_{yy} = -v_{xy}$$

$$u_{xx} = -u_{yy} \qquad \boxed{u_{xx} + u_{yy} = 0}$$

$u \to$ derive $v = u^*$ harmonic conjugate

In the other direction, a single harmonic function u defined on a disk S implies the existence of a partner function v so that we can define an analytic function f from u and v.

Theorem 6.9 (Harmonic conjugates)**.** *For $u(x,y)$ real-valued an harmonic in a disk S there exists a function $v(x,y)$ harmonic on S so that $f(z) = u(x,y) + iv(x,y)$ is analytic on S. The* **harmonic conjugate** *v, also denoted u^* is unique up to an additive constant.*

Proof. The most important step in the proof gives a formula for the harmonic conjugate: Given a harmonic u, choose a point (x_0, y_0) in the disk S. Then

$$\boxed{u^*(x,y) = \int_{x_0}^{x} -u_y(s,y)\,ds + \int_{y_0}^{y} u_x(x_0,t)\,dt.}$$

□

u harmonic on disk S

derive

u^* harmonic on disk S

define $\boxed{f(z) = u(x,y) + i\,u^*(x,y)}$

analytic

Example 6.10. *Find the harmonic conjugate of* $u = x^2 + y^2 + 6x + 2y$.

Via formula:

$$u^* = \int_{x_0}^{x} -u_y(s,y)\,ds + \int_{y_0}^{y} u_x(x_0,t)\,dt$$

(x_0, y_0) base-point

$$u^* = \int_0^x -\boxed{u_y(s,y)}\,ds + \int_0^y u_x(0,t)\,dt$$

$$u = x^2 - y^2 + 6x + 2y$$

$$u_x = 2x - 6 \qquad u_x(0,t) = 6$$

$$u_y = 2y - 2 \qquad u_y(s,y) = 2y + 2$$

$$\boxed{u^* = \int_0^x (2y-2)\,ds + \int_0^y 6\,dt = 2xy - 2x + 6y}$$

$$f(z) = (x^2 - y^2 + 6x + 2y) + i(2xy - 2x + 6y)$$

$$u = x^2 - y^2 + 6x + 2y$$

Exploiting CRE:

$u_x = 2x + 6$ $\qquad$ $u_y = -2y + 2$

$\underline{\underline{v_y = 2x + 6}}$ $\qquad$ $-v_x = -2y + 2$

$v_x = 2y - 2$

$$\boxed{v = \int 2x + 6 \, dy = 2xy + 6y + G(x)}$$

$$v_x = \boxed{2y + 0 + G'(x)}$$

$G'(x) = -2$

$G(x) = -2x$

$$\boxed{v = 2xy + 6y - 2x}$$

point: deep connection between

harmonic functi $u_{xx}+u_{yy}=0$

---

analytic $u_x = v_y$

$u_y = -v_x$

---

conservative vector fields