Extension Theorem:

Motivation: real valued functions can be terrible.

consider the Weierstrass function $w(x)$ (look it up). It is continuous and nowhere differentiable on the real line.

now, defined

$$\overline{W}(x) = \int_0^x w(t)\,dt.$$

FTC guarantees this function exists, and

$$\overline{W}'(x) = w(x)$$

but $\overline{W}''(x)$ doesn't exist anywhere!

once differentiable means nothing in real variables.

Power series and extension!

An analytic function is a function with a convergent power series $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ on $S = \{z : |z-a| < r\}$ where $r > 0$

Perhaps the most important theorem in complex analysis is that holomorphic on a disk is equivalent to analytic on that disk.

That is, once differentiable implies convergent power series!

~~Outline~~

Theorem: (Weierstrass M-test.)

If $|a_n(z-z_0)^n| \le M_n$ for $|z-z_0| \le r$ and if $\sum M_n < \infty$

then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and uniformly in $\{z : |z-z_0| \le r\}$

Pf: If $M > N$ then

$$|S_M(z) - S_N(z)| = \left|\sum_{n=N+1}^{M} a_n (z-z_0)^n\right| \le \sum_{n=N+1}^{M} M_n$$

since $\sum M_n < \infty$, $\sum_{n=N+1}^{M} M_n \to 0$ as $N, M \to \infty$. So $\{S_N\}$ is Cauchy

converging uniformly. also gives absolute convergence.

Theorem: (root test) Suppose $\sum a_n (z-z_0)^n$ is a formal power series.

Let $R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$.

then $\sum_{n=0}^{\infty} a_n ((z-z_0))^n$

a) conv abs in $\{z : |z-z_0| < R\}$.

b) conv unif in $\{z : |z-z_0| \le r\}$ $\forall r < R$.

c) diverges in $\{z : |z-z_0| > R\}$.

R is called the radius of convergence.

Pf: note that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(which still holds for } |z|<1\text{)}.$$

let $R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$.

If $|z-z_0| \le r < R$, choose r_1 so $r < r_1 < R$.

then $r_1 < \liminf |a_n|^{-\frac{1}{n}}$ and $\exists N$ such that

$r_1 < |a_n|^{-\frac{1}{n}}$ for all $n \geq N$. This implies that $|a_n (z-z_0)^n| < \left(\frac{r}{r_1}\right)^n$

since $r_1 > r$, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{r}{r_1}\right)^n = \frac{1}{1-r/r_1} < \infty$

then the M-test implies uniform and abs convergence on

$\{z : |z-z_0| < r\}$, holds for all $r < R\}$.

the other direction, ~~choose $r > R$~~

if $|z-z_0| > R$, choose r so $R < r < |z-z_0|$

$|a_n|^{-\frac{1}{n}} < r$ for infinitely many n

so $|a_n (z-z_0)^n| > \left(\frac{|z-z_0|}{r}\right)^n$ for infinitely many n.

but $\left(\frac{|z-z_0|}{r}\right)^n \to \infty$ as $n \to \infty$ so

(c) holds.

Lemma: (Infinite differentiability)

Any power series $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$ with $R>0$ is holomorphic at all points in $\{z : |z-a|<R\}$.

$$f'(z)=\sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$ which is also analytic w/ radius R.

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pf: WLOG let $a=0$.

$$\frac{f(z+h)-f(z)}{h}-\sum_{n=1}^{\infty} n c_n z^{n-1}$$

$$=\sum_{n=1}^{\infty} c_n\left(\frac{(z+h)^n-z^n}{h}-nz^{n-1}\right)$$

now, $$\left|\frac{(z+h)^n-z^n}{h}-nz^{n-1}\right|$$

$$=\left|\sum_{k=2}^{n}\binom{n}{k} z^{n-k}h^{k-1}\right|$$

$$\le \sum_{k=2}^{n}\binom{n}{k}\left|z^{n-k}\right|\left|h^{k-1}\right|$$

$$< \sum_{k=2}^{n}\binom{n}{k}|z|^{n-k}|h|\left[\frac{R-|z|}{2}\right]^{k-2}$$

$$\left(\frac{(z+h)^2-z^2}{h}-2z\right)$$
$$=\frac{z^2+2zh+h^2-z^2}{h}-2z$$
$$=2z-h-2z$$
$$=-h.$$

Can assume that $|h| < \frac{R-|z|}{2}$ since $h \to 0$.

then
$$\sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} |h| \left[\frac{R-|z|}{2}\right]^{k-2}$$

$$= |h| \left[\frac{R-|z|}{2}\right]^{-2} \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} \left[\frac{R-|z|}{2}\right]^{k}$$

$$\le |h| \left[\frac{R-|z|}{2}\right]^{-2} \underbrace{\sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} \left[\frac{R-|z|}{2}\right]^{k}}_{\text{This is a binomial expansion}}$$

$$= |h| \left[\frac{R-|z|}{2}\right]^{-2} \left[|z| + \frac{R-|z|}{2}\right]^{n}$$

$$= |h| \left[\frac{R-|z|}{2}\right]^{-2} \left[\frac{R+|z|}{2}\right]^{n}$$

so
$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} \right|$$

$$= \left| \sum_{n=1}^{\infty} c_n \left(\frac{(z+h)^n - z^n}{h} - n z^{n-1} \right) \right|$$

$$< |h| \left[\frac{R-|z|}{2}\right]^{-2} \sum_{n=1}^{\infty} |c_n| \left[\frac{R+|z|}{2}\right]^{n} \to 0 \text{ as } h \to 0$$

since $\sum_{n=1}^{\infty} |c_n| \left(\frac{R+|z|}{2}\right)^n$ converges by the root test with same radius R.

Theorem:

If $\sum z_n$ is a complex series and

$\sum |z_n|$ converges, then so does $\sum z_n$.

Theorem (Extension Theorem)

Any real-valued analytic

$$f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad \text{(with real coefficients)}$$

with a ~~finite~~ positive or ∞ radius of convergence

with Int $(x_0 - R, x_0 + R)$ can be extended to

~~analytic~~ $\mathbb{C}$ as

$$f(z) = \sum_{n=0}^{\infty} c_n (z-x_0)^n.$$

$f(z)$ has same radius of convergence so disk of convergence is $\{z : |z-x_0| < R\}$.

Pf: WLOG, set $x_0 = 0$.

Let $f(x) = \sum c_n x^n$ have Interval $(-R, R)$.

Root test gives $R = \dfrac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$

Note that the complex series has the same result because of the M-test.

(6)

Ex: $f(x) = e^x = \sum \frac{x^n}{n!}$ $\quad R=\infty$

so $f(z) = e^z = \sum \frac{z^n}{n!}$ $\quad R=\infty$

$f(x) = \cos x = \sum \frac{(-1)^n}{(2n)!} x^{2n}$ $\quad R=\infty$

$f(z) = \cos z = \sum \frac{(-1)^n}{(2n)!} z^{2n}$ $\quad R=\infty$

$f(x) = \frac{1}{1-x} = \sum x^n$ $\quad R=1$

$f(z) = \frac{1}{1-z} = \sum z^n$ $\quad R=1$

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later, relations about functions w/ power series
continue to hold

The identity theorem.

If $f=g$ on some non-empty open disk S contained in U where f and g are analytic, then $f=g$ on U.

Pf: WLOG, let $g(z)\equiv 0$ (otherwise: consider $f-g$)

set $P=\{z\in U : f^{(k)}(z)=0 \ \forall k\in\mathbb{N}\}$.

P is a closed set because f is ~~continuous~~ analytic (and so continuous)

and P is the inverse image of $\{f(z) : f(z)=0\}$.

For any $w\in P$, Taylor series for ~~f and g agree~~ f and $g(z)\equiv 0$ agree at w and have non-zero R, which means $f\equiv 0$ on some open disk centred at w. So each $w\in P$ is in an open nbhd contained in P. So P is also open.

Since P is nonempty, P must be U. (as an open and closed set)