proving CRE

throughout : $z = x+iy$

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

$$u, v : \mathbb{R}^2 \to \mathbb{R}$$

u, v, u_x, u_y, v_x, v_y are continuous

Goal: f complex differentiable at z

$$\iff$$

$$u_x = v_y \text{ and } u_y = -v_x$$

at z.

Assume f is complex differentiable at z (in an open disk centred at z)

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

① assume h is a real number

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{f(x+iy+h) - f(x+iy)}{h}$$

$$= \lim_{h\to 0} \frac{f((x+h)+iy) - f(x+iy)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y) - u(x,y) - iv(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i\,\frac{v(x+h,y) - v(x,y)}{h}$$

$$= u_x(x,y) + i\,v_x(x,y)$$

$$\boxed{f'(z) = u_x(x,y) + i\,v_x(x,y)}$$

$$\lim_{ih\to 0} \frac{f(z+ih) - f(z)}{ih} = v_y(x,y) - i\,u_y(x,y)$$

$$= f'(z)$$

So $u_x + i v_x = v_y - i u_y$

$$\boxed{u_x = v_y \text{ and } v_x = -u_y}$$

CRE.

also $f'(z) = u_x + i v_x$

$= v_y - i u_y$ useful formulas!

Goal: use $u_x = v_y$ and $u_y = -v_x$

$$\Rightarrow \lim_{\Delta z \to 0} \underbrace{\frac{f(z+\Delta z) - f(z)}{\Delta z}}_{} = f'(z)$$

Tools: linear approximation

Mean value theorem.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = U(\Delta z) + i V(\Delta z)$$

$$\boxed{U(\Delta z) = \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta z}}$$

$$V(\Delta z) = \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta z}$$

$$\sigma(\Delta z) = \frac{\boxed{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}}{\Delta z} \approx u_x$$

add/subtract

$$+ \frac{\boxed{u(x, y+\Delta y) - u(x,y)}}{\Delta z} \approx u_y$$

$x < x^* < x+\Delta x$

$$\approx \frac{u_x(x^*, y+\Delta y)\Delta x}{\Delta z}$$

$y < y^* < y+\Delta y$

$$+ \frac{u_y(x, y^*)\Delta y}{\Delta z}$$

$$\sigma(\Delta z) = \frac{u_x(x^*, y+\Delta y)\Delta x}{\Delta z} + \frac{u_y(x, y^*)\Delta y}{\Delta z} + \epsilon_1(\Delta z) + \epsilon_2(\Delta z)$$

$\epsilon_1, \epsilon_2 \to 0$

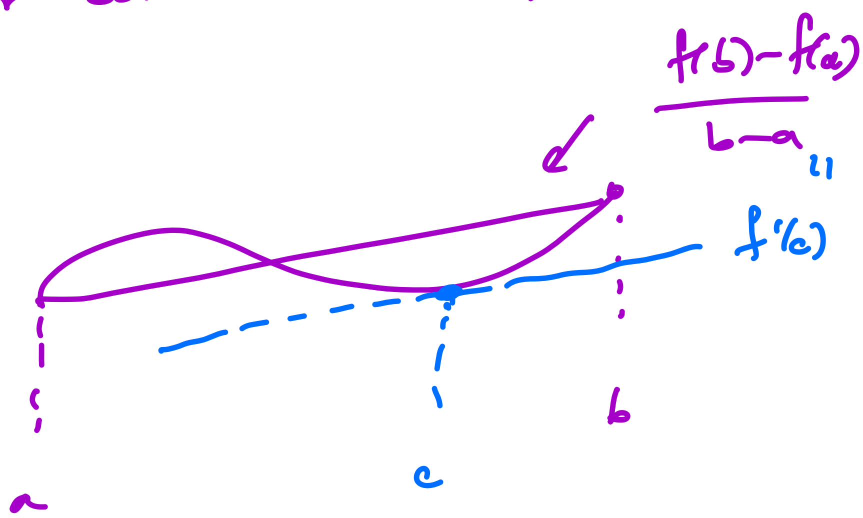
as $\Delta z \to 0$

Mean value theorem:

f is a function and f' is continuous

then $f(b) - f(a) = f'(c)\,(b-a)$

for some $c \in (a,b)$



$u(x,y) \longrightarrow u(x+\Delta x, y)$

$$\underline{u(x+\Delta x, y) - u(x,y)} = u_x(x^*, y)\,\Delta x$$

"b−a"

$(x+\Delta x - x) = \Delta x$

Mean value theorem

$x^* \in (x, x+\Delta x)$

continuity. If f is continuous

then $\lim_{h \to 0} f(x+h) = f(\lim_{h \to 0} x+h) = f(x)$

$u_y(x, y+\Delta y) \to u_y(x,y)$

as $\Delta y \to 0$

if u_y is continuous

WTS: $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ converges as $\Delta z \to 0$

if so, is equal to $f'(z)$

Assume u, v, u_x, u_y, v_x, v_y are continuous

Assume u, v satisfy CRE at all z in some

disk. $u_x = v_y$ $\quad$ $u_y = -v_x$

$f(z) = u(x,y) + i v(x,y)$

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = U(\Delta z) - V(\Delta z)$$

$$U(\Delta z) = \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta z}$$

$$V(\Delta z) = \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta z}$$

$$U(\Delta z) = \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta z}$$ add/subtract

$$= \frac{[u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x,y)]}{\Delta z}.$$

MVT, exist $x^* \in (x, x+\Delta x)$

$y^* \in (y, y+\Delta y)$

$$= \frac{u_x(x^*, y+\Delta y)\Delta x}{\Delta z} + \frac{u_y(x, y^*)\Delta y}{\Delta z}$$

$x^* \to x$ as $\Delta x \to 0$

$y^* \to y$ as $\Delta y \to 0$

$\Delta z = \Delta x + i\Delta y$ if Δz is small

$U(\Delta z)$ close to $\dfrac{u_x(x,y)\Delta x}{\Delta z} + \dfrac{u_y(x,y)\Delta y}{\Delta z}$

$V(\Delta z)$ close to $\dfrac{v_x(x,y)\Delta x}{\Delta z} + \dfrac{v_y(x,y)\Delta y}{\Delta z}$

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = U(\Delta z) + iV(\Delta z)$$

$$\underset{\text{small } \Delta z}{\approx} \frac{u_x(x,y)\Delta x}{\Delta z} + \boxed{\frac{u_y(x,y)\Delta y}{\Delta z}} + i\left(\frac{v_x(x,y)\Delta x}{\Delta z} + \boxed{\frac{v_y(x,y)\Delta y}{\Delta z}}\right)$$

invoke CRE

$$\frac{u_x(x,y)\Delta x}{\Delta z} - \frac{v_x(x,y)\Delta y}{\Delta z} + i\left(\frac{v_x(x,y)\Delta x}{\Delta z} + \frac{u_x(x,y)\Delta y}{\Delta z}\right)$$

$$= u_x(x,y)\left(\underbrace{\frac{\Delta x + i\Delta y}{\Delta z}}_{1}\right) + i\, v_x(x,y)\left(\frac{\Delta x - i\Delta y}{\Delta z}\right)$$

$$u_x(x,y)\left(\frac{\Delta z}{\Delta z}\right) + i\, v_x(x,y)\left(\frac{\Delta z}{\Delta z}\right)$$

$$\lim_{\Delta z \to 0} \longrightarrow \boxed{u_x(x,y) + i\, v_x(x,y)}$$

$$= f'(z)$$ (converges so $f'(z)$ exists)

If $u_x = v_y$ and $u_y = -v_x$

then (assuming all hypotheses)

f is complex differentiable.