1. Limits and Complex derivatives

1.1. Complex analysis extends calculus. The big theme of this workshop is

complex analysis is an extension of freshman calculus.

We'll need to develop the complex versions of the basic tools of calculus:

- limits,
- derivatives,
- integrals.

$x, y \in \mathbb{R}$

1.2. Complex numbers. A complex number is of the form $z = x + iy$, where $i^2 = -1$ is the imaginary unit.

complex conjugate::

$z = x + iy$ $\qquad \bar{z} =$ "z bar"

$$\bar{z} = \overline{(x+iy)} = x - iy$$

ex: $\overline{(2-3i)} = 2+3i$

real part::

$z = x + iy$ $\qquad \operatorname{Re} z = x$

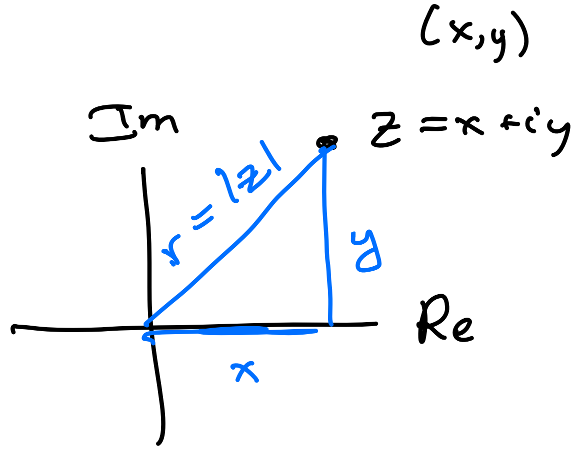
$$\operatorname{Re} z = \frac{z + \bar{z}}{2}$$

imaginary part::

$z = x + iy$

$\operatorname{Im} z = y$

$$\operatorname{Im} z = \frac{z - \bar{z}}{2i}$$



modulus:: absolute value

$$|z| = \sqrt{x^2+y^2}$$

$$z \cdot \bar{z} = x^2+y^2 = |z|^2$$

Basic facts:

- $|zw| = |z|\,|w|$
- $-|z| \leq \operatorname{Re} z \leq |z|$
- $-|z| \leq \operatorname{Im} z \leq |z|$
- $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$

estimates

Theorem 1.1 (Triangle inequalities)**.** *Let $z, w \in \mathbb{C}$. Then*

$$|z+w| \leq |z| + |w|$$

(hammer of analysis)

and

$$|z-w| \geq ||z| - |w||$$

(Reverse Δ inequality)

Pf: let $z, w \in \mathbb{C}$. WTS: $|z+w| \leq |z| + |w|$

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + \underbrace{z\bar{w} + \bar{z}w}_{} + w\bar{w}$$

$$= |z|^2 + 2\operatorname{Re} z\bar{w} + |w|^2$$

$$\leq |z|^2 + 2|z\bar{w}| + |w|^2$$

$$= |z|^2 + 2|z||w| + |w|^2$$

$$= (|z|+|w|)^2$$

check

$$\boxed{z\bar{w} + \overline{(z\bar{w})} = 2\operatorname{Re}(z\bar{w})}$$

$$\boxed{|w| = |\bar{w}|}$$

check

implies that $|z+w| \le |z|+|w|$.

1.3. **Complex functions.** A complex function $f : \mathbb{C} \to \mathbb{C}$ can be thought of as a map

$$f : x + iy \mapsto \underline{u(x, y)} + i\underline{v(x, y)}.$$

$\operatorname{Re} f = u \qquad \operatorname{Im} f = v$

complex disks:

$D_r(a) = \{ z : |z-a| < r \}$ (open disk)

disk of radius r centered at $z=a$

1.4. **Complex limits.**

Definition 1.2. *Let f be defined on $D_r(a)$ for $r > 0$. Then $\lim_{z\to a} f(z) = L$ means that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - a| < \delta$.*

Notice - this is the same definition from calculus! We should expect the limit theorems from calculus to hold. (See Johnston Theorem 1.1.2).

Theorem 1.3 (Example limit theorem)**.** *Suppose that*

$$\underline{\lim_{z\to a} f(z) = L} \text{ and } \underline{\lim_{z\to a} g(z) = M}.$$

Then

$$\lim_{z\to a} f(z)g(z) = LM.$$

Pf: assume $f(z) \to L$, $g(z) \to M$. $z \to a$. WLOG, let $a=0$.

Let $\varepsilon > 0$ be given. WTS, $\exists \delta > 0$ so that $|fg - LM| < \varepsilon$ when $|z| < \delta$.

Let δ_f be such that $|f(z) - L| < \frac{\varepsilon |L|}{\varepsilon + 2|ML|}$ * when $|z| < \delta_f$ *

Let δ_g be such that $|g(z) - M| < \frac{\varepsilon}{2|L|}$ * when $|z| < \delta_g$ **

Let $\delta = \min \delta_f, \delta_g$.

so $|fg - LM| = |fg - Lg + Lg - LM| \le |fg - Lg| + |Lg - LM|$

$= |g||f - L| + |L||g - M|$

$= |g-M+M|\,|f-L| + |c|\,|g-M| \le (|g-M|+|M|)|f-L| + |c|\,|g-M|$

$\le \left(\frac{\varepsilon}{2|c|}+|M|\right)\left(\frac{\varepsilon|c|}{\varepsilon+2|M||c|}\right)+|c|\frac{\varepsilon}{2|c|} \le \varepsilon.$

1.5. The complex derivative.

Definition 1.4. *Let f be a complex function. Then define the* **derivative of** f **at** $z=a$ *by*

$$f'(z)=\frac{df}{dz}:=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \quad \text{when the limit exists.}$$

ex. try to figure out.

Example 1.5. *Complex functions can be differentiable at only a point.* $\boxed{\text{Consider } f(z)=|z|^2.}$

Definition 1.6. *A function f is called* **holomorphic at** $z=z_0$ *if f is (complex) differentiable at every point in some disk centered at z_0. A function that is holomorphic at every point in its domain is called* **holomorphic**. *A function that is holomorhpic at every complex number is called* **entire**.

holomorphic – differentiable on a small disk.

Example 1.7. *The function $f(z)=z^2$ is differentiable for all $z\in\mathbb{C}$.*

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{(z+h)^2-z^2}{h}$$

$$=\lim_{h\to 0}\frac{z^2+2zh+h^2-z^2}{h}$$

$$=\lim_{h\to 0}\frac{2zh+h^2}{h}$$

$$=\lim_{h\to 0} 2z+h = 2z.$$

$f(z)=z^2$

$f'(z)=2z$

$$\lim_{h\to 0}\frac{\overline{(z+h)}-\overline{z}}{h}=\lim_{h\to 0}\frac{\overline{h}}{h} \qquad h=x+iy$$

real axis, $h=x$ — imaginary axis $h=iy$

$$\lim_{x\to 0}\frac{\overline{x}}{x}=\lim_{x\to 0}\frac{x}{x}=1. \qquad \lim_{y\to 0}\frac{\overline{iy}}{iy}=\lim_{y\to 0}\frac{-iy}{iy}=-1$$

different limits on different paths, no limit.

Example 1.8. *The function $f(z) = \overline{z}$ is differentiable for no value of z.*

Just like the limit theorems, the derivative theorems of calculus carry forward into complex analysis. See Johnston Theorem 1.1.3.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Theorem 1.9 (Example derivative theorem)**.**

$$\frac{d}{dz} z^n = n z^{n-1}$$

$$\lim_{h\to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h\to 0} \frac{z^n + n z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \dots h^n - z^n}{h}$$

$$= \lim_{h\to 0} \frac{n z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \dots + h^n}{h}$$

$$= \lim_{h\to 0} n z^{n-1} + \binom{n}{2} z^{n-2} h + \dots + h^{n-1}$$

$$= n z^{n-1}$$