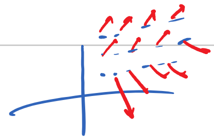
Cauchy-Goursat

Wednesday, August 17, 2022 1:30 PM

$\varphi : \bar{U} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$



$\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$

FT complex functions

- $\bar{U}$ simply connected
- f continuous on $\bar{U}$
- F analytic on $\bar{U}$ w/ $F' = f$
- C_1 contained in $\bar{U}$

$\int_{C_1} f(z)\,dz = F(b) - F(a)$

7.2. **The Cauchy-Goursat theorem.** We've already mentioned conservative vector fields, so lets remind ourselves of what that means. In two dimensions, a **vector field** $\varphi : U \to \mathbb{R}^2$ on an open set $U \subset \mathbb{R}^2$ is **conservative** if there exists a scalar function $\Phi : U \to \mathbb{R}$ of class C^1 so that

$$\varphi = \nabla\Phi;$$

that is, φ is the gradient of Φ. Such a vector field has path independent line integrals - that is, the value of

$$\int_C \varphi \cdot d\mathbf{s}$$

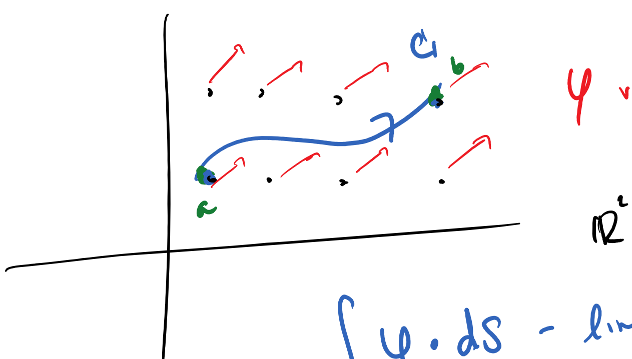
depends only on where C begins and ends (and indeed is equal to $\Phi(b) - \Phi(a)$ if a, b are the initial and final points of C). The converse is also true - if a vector field on $U \subset \mathbb{R}^2$ has path independent line integrals, then it is conservative.

This structure carries over into complex analysis as a relationship between analyticity and path independent integrals (via the Cauchy-Riemann equations). Let us consider what should happen if we evaluate an analytic function on a closed loop.

$\nabla\Phi = \varphi$

$\Phi_x = \varphi_1 \quad \Phi_y = \varphi_2$

$\varphi = (\varphi_1, \varphi_2)$ conservative.

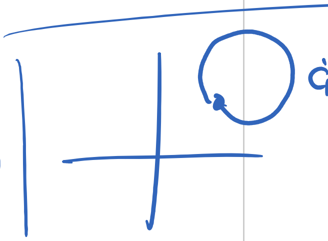


φ vector field.

$\int_{C_1} \varphi \cdot ds$ - line integral of a vector field

$\int_{C_1} \varphi \cdot ds = \int_{C_2} \varphi \cdot ds = \Phi(b) - \Phi(a)$

φ conservative



$\oint_{C_1} \varphi \cdot ds$?

$= \Phi(a) - \Phi(a)$

$= 0$

Example 7.9. *Consider*

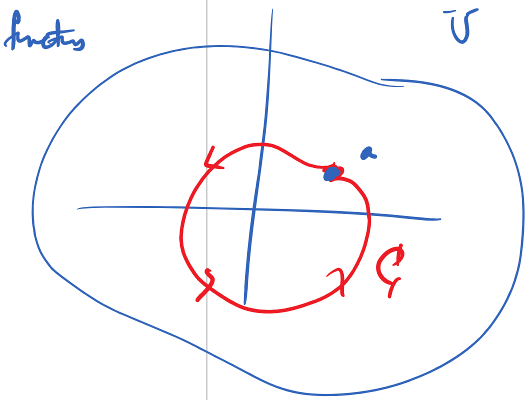
$$\oint_C f(z)\,dz$$

where f is analytic on a simply connected open set containing C.

Assume f has an antiderivative F also analytic on U.

$$\oint_C f(z)\,dz = F(a) - F(a) \quad \text{by FT complex functions}$$

$$= 0$$



Let's formalize this.

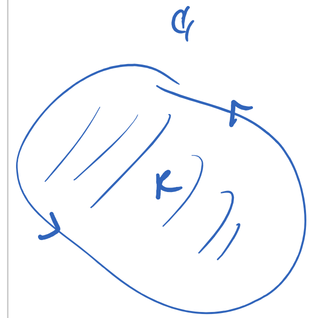
Theorem 7.10 (Cauchy-Goursat). *Let $f : U \to \mathbb{C}$ be analytic on a simply connected domain U, and let C be a piecewise smooth simple closed curve in U. Then*

$$\boxed{\oint_C f(z)\, dz = 0.}$$

Proof. Green's theorem! □

$$\oint_{C_1} L\,dx + M\,dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dA$$

Green's theorem



$f = u + iv$ analytic $\to$ u, v are class C^1 (continuous first partial derivatives)

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist

CRE

$\boxed{\text{p. 113}}$

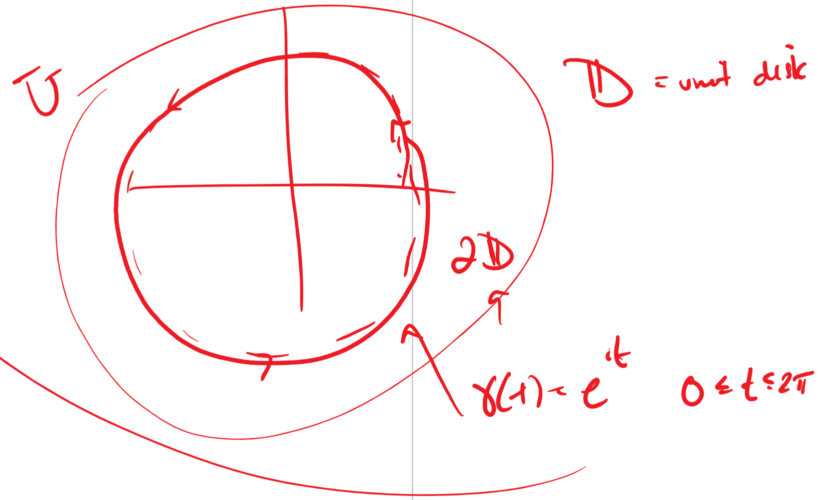
Example 7.11. *Evaluate* $\oint_{\partial\mathbb{D}}(z^2+1)\,dz$.

$$\oint_{\partial\mathbb{D}} z^2+1 \, dz$$

$= 0$

z^2+1 analytic everywhere.

$\partial\mathbb{D}$ - unit circle in $\mathbb{C}$ is simple closed curve.



integral over a simple closed curve of an analytic function on a disk containing G.

$$\oint_{\partial D} z^2+1 \, dz \qquad z(t) = e^{it} \quad 0 \le t \le 2\pi$$

$$dz = ie^{it}\,dt$$

$$\int_0^{2\pi} \left(e^{2it}+1\right) ie^{it}\,dt$$

$$\int_0^{2\pi} ie^{3it} + ie^{it}\,dt = \frac{i}{3i}e^{3it} + \frac{i}{i}e^{it} \Big|_0^{2\pi} = 0$$

One useful consequence of the Cauchy-Goursat theorem is that integration over a simple closed curve is always equivalent to integration over a circle.

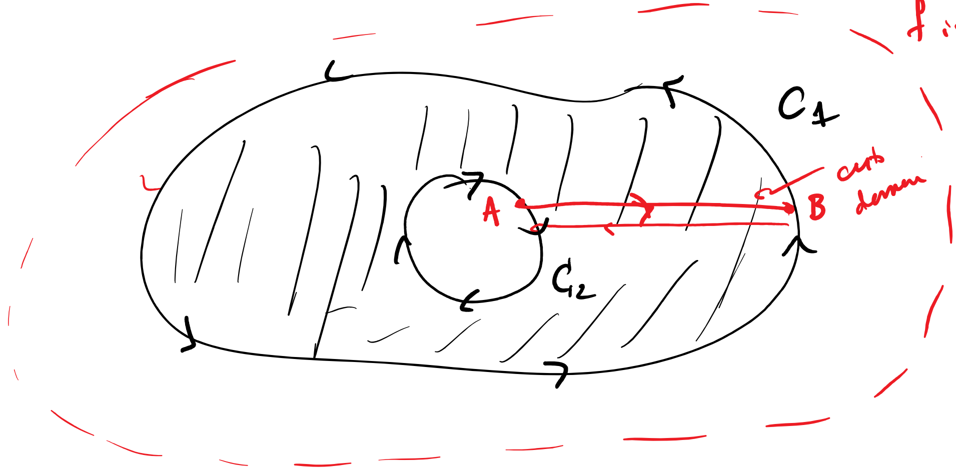
Theorem 7.12. *Suppose that C_1 and C_2 are simple closed curves with one in the interior of the other, and suppose $f(z)$ is a function analytic on a domain U containing both curves. Then*

$$\oint_{C_1} f\,dz = \oint_{C_2} f\,dz.$$

Circles are easy to parametrize

Proof. Do the picture! □

U simply connected, f is analytic on U.



by Cauchy-Goursat

- FTComplex functions
- Cauchy Goursat
- Circle contour

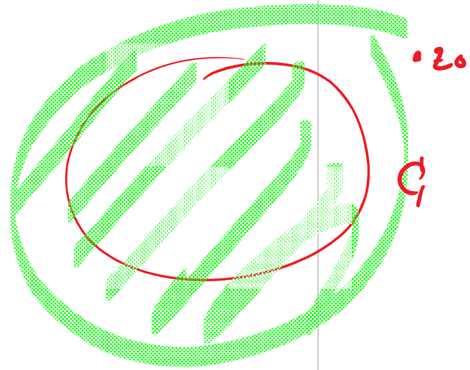
Example 7.13. *Evaluate*

$$\oint_C \frac{1}{z - z_0}\, dz$$

for z_0 in the exterior of C and for z_0 in the interior of C.

$$\oint_{C_1} \frac{1}{z-z_0}\, dz$$

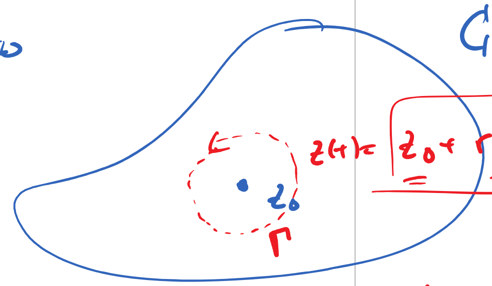
If z_0 is outside C_1



$f(z) = \frac{1}{z - z_0}$ is analytic on U of C_1

Cauchy Goursat $\rightarrow$ $\oint_{C_1} \frac{1}{z-z_0} dz = 0$

z_0 inside C_1. $\quad f(z) = \frac{1}{z-z_0}$



$z(t) = \boxed{z_0 + re^{it}}$, $0 \le t \le 2\pi$

$dz = ire^{it} dt$

$$\oint_{C_1} \frac{1}{z-z_0} dz = \oint_{\Gamma} \frac{1}{z-z_0} dz$$

$$= \int_0^{2\pi} \frac{1}{z_0 + re^{it} - z_0}\, ire^{it} dt$$

$$= \int_0^{2\pi} i\, dt = 2\pi i$$

We end with the other half of the Fundamental Theorem for complex functions, and a corollary for analytic functions. A **primitive** of a complex function f is a function F so that $F' = f$ (that is, an antiderivative).

FTCFA

$\int_C f\,dz = F(b) - F(a)$

if $F' = f$, F analytic.

Theorem 7.14. *An analytic function f on a simply connected domain U has a primitive F.*

$F' = f$

Corollary 7.15. *If f is analytic on a simply connected domain U and $a, b \in U$ are joined by a contour $C \subset U$, then*

$$\int_C f(z), dz = F(b) - F(a)$$

where F is any antiderivative of f over U.

"The FTC"

Proof. Technical. To be addressed in a separate video. $\square$

FT of analytic functions.