Theme: differentiable once means infinitely differentiable

2.1. **Motivation.** The theme is completely untrue for real valued functions. Real valued functions are terrible! The Weierstrass function $w(x)$ is a function that is continuous everywhere and differentiable nowhere on the real line (look it up).

$w(x)$ is continuous everywhere, and differentiable nowhere.

$W(x) = \int_0^x w(t)\,dt \qquad W'(x) = w(x)$ by FTC

holomorphic – differentiable once on a disk of radius $r > 0$,

2.2. Analytic functions.

Definition 2.1. *An* **analytic function** $f : \mathbb{C} \to \mathbb{C}$ *is a function with a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

on a set $S = \{z : |z-a| < r\}$ *where* $r > 0$.

The central result underlying all of complex analysis is that f is holomorphic at a point is equivalent to f is analytic at a point . That is, in complex analysis, we have the seemingly absurd statement that a once differentiable function on a disk has a convergent power series on the disk. We'll have to split this into two parts.

That holomorphic implies analytic is a consequence of Taylor's theorem (but we'll need beefier tools to get it, so we'll prove it later).

An outline of analytic implies holomorphic:

Weierstrass M-test: The M-test is a natural way of testing a series of functions for absolute and uniform convergence (convergence that preserves the properties of the functions in the series)

Root test: The root test gives a method for computing the radius of convergence of a power series.

Infinitely differentiable theorem: Using the root test, we can show that taking the formal derivative of an analytic function gives a convergent power series for the derivative, and indeed that the resulting function is itself analytic with the same radius of convergence.

convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$ converges on a disk S $r>0$,

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

also converges on same disk.

Theorem 2.2 (Weierstrass M-test). *If $|c_n(z-a)^n| \le M_n$ for all n for $|z-a| \le r$ and if $\sum M_n < \infty$ then $\sum c_n(z-a)^n$ converges uniformly and absolutely in $\{z : |z-a| < r\}$.*

$|c_1(z-a)^1| \le M_1$ and $|c_2(z-a)^2| \le M_2$

$\forall n \in \mathbb{N}$ and $\forall z \in |z-a| < r$,

Pf: If $M > N$ then

$$|S_M(z) - S_N(z)| = \left|\sum_{n=N+1}^{M} c_n(z-a)^n\right|$$

$$\le \sum_{n=N+1}^{M} M_n.$$

Since $\sum M_n < \infty$, then $\sum_{n=N+1}^{M} M_n \to 0$ as $M, N \to \infty$. $\{S_N\}$ is a Cauchy sequence.

S_N converges uniformly on $|z-a| < r$.

Same argument shows absolute convergence.

Theorem 2.3 (root test)**.** *Suppose $\sum c_n(z-a)^n$ is a formal power series. Let*

$$\boxed{R = \lim_{n\to\infty} |a_n|^{-1/n}.}$$

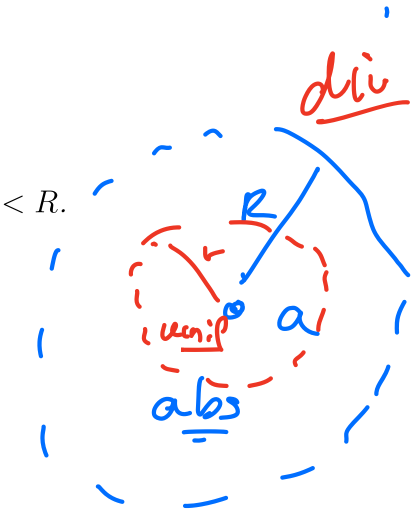
c_n

Then the following hold:

(a) The series converges absolutely in $\{z : |z-a| < R\}$;

(b) The series converges uniformly in $\{z : |z-a| \leq r$ for all $r < R$.

(c) The series diverges on $\{z : |z-a| > R\}$.



R is called radius of convergence.

Pf: apply the M-test

see proof in Abbott's Understanding Analysis.

Theorem 2.4 (Infinitely differentiable theorem). *Any power series $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$ with radius of convergence $R>0$ is holomorphic at all points in $\{z:|z-a|<R\}$, with*

$$f'(z)=\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}.$$

Furthermore, $f'(z)$ is also analytic with radius of convergence R.

$$\underbrace{\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}} = \underbrace{\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}}$$

$\frac{\text{limit}}{\text{difference quotient}}$ $\qquad$ $\frac{\text{convergent power series}}{}$

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idea $\qquad \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} nc_n(z-a)^n$

show that difference gets small as $h\to 0$.

how? Turn the difference into a power series, apply root test).

analytic – convergent power series $\Rightarrow$ infinitely differentiable

where are we going to get power series?

2.3. **The extension theorem.** Combining the root test and the M-test allows us to say that any function with a real power series with a positive radius of convergence extends to a complex power series with the same radius of convergence.

Theorem 2.5. *Any real-valued analytic function*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with a positive or infinite radius of convergence R can be extended to a complex analytic function

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

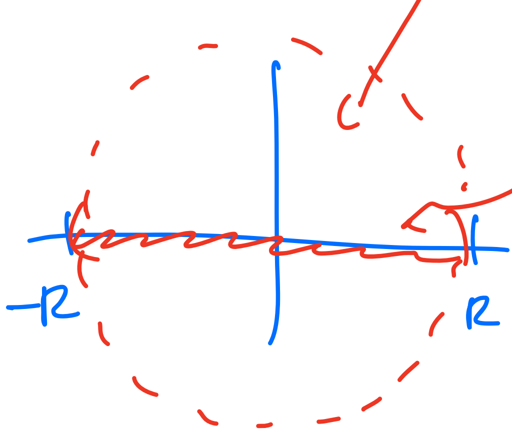
where $f(z)$ has the same radius of convergence; that is, as a complex function, f is analytic on the set $\{z : |z-a| < R\}$.

WLOG, let $a=0$. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R. (real interval $(-R,R)$)

root test $R = \dfrac{1}{\lim_{n\to\infty} |c_n|^{1/n}}$

complex analytic

real analytic



inherit coefficients and radius of convergence from $\sum c_n x^n$ and use root test and M-test to get $\sum c_n z^n$

2.4. **Things to do with the extension theorem.**

Ex: $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $R = \infty$

$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $R = \infty$

$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ $R = \infty$

$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $R = \infty$.

$f(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$f(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

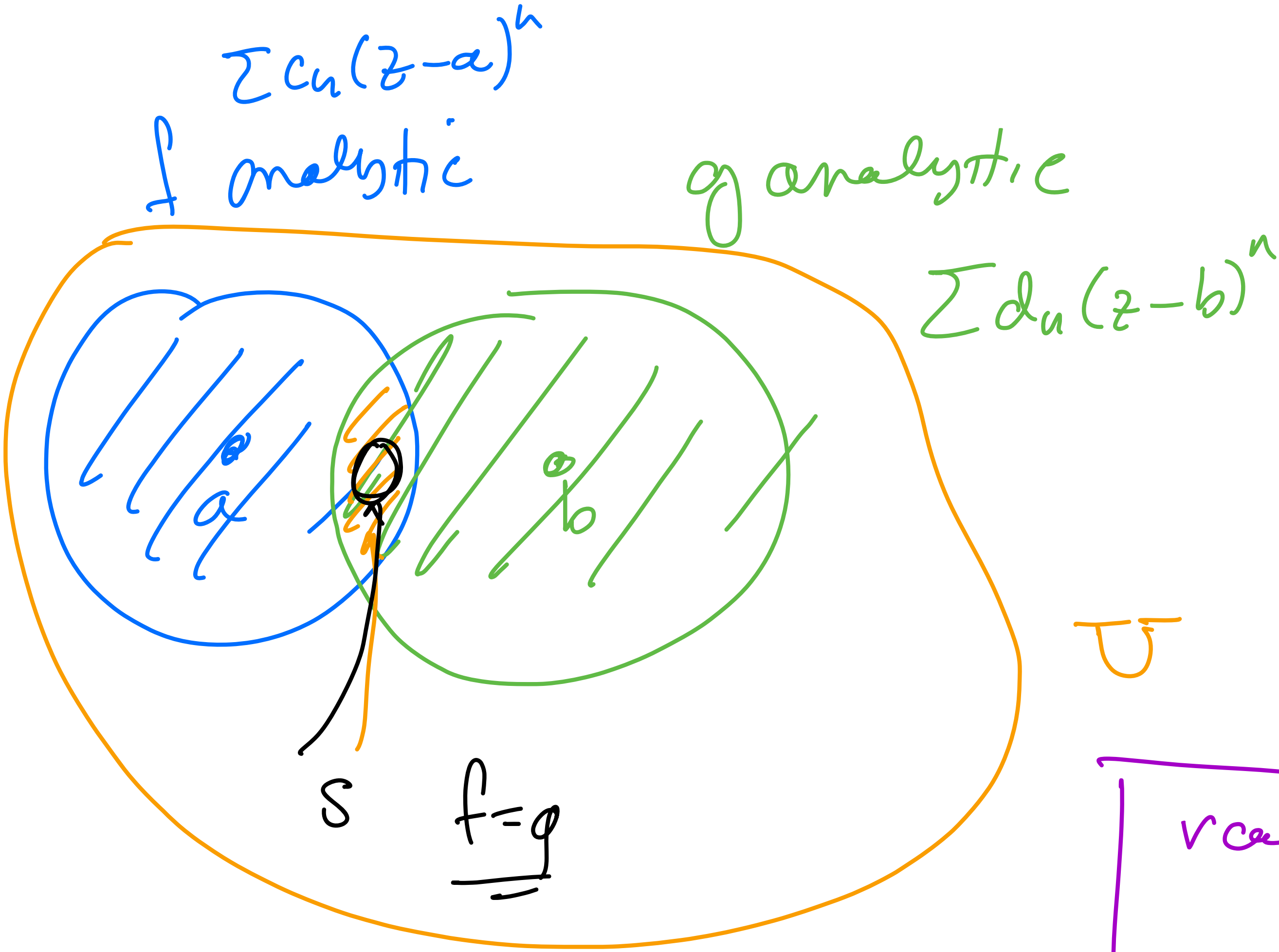
$\cos^2 x + \sin^2 x = 1$

$\cos^2 z + \sin^2 z = 1$

identities extend

2.5. **The identity theorem.** Complex analytic functions turn out to be a much smaller family of functions that the usual families we deal with in real calculus (which isn't surprising, given that we're asking them to be infinitely differentiable!) If two analytic functions agree on some open disk in their domain, they are the same function everywhere.

Theorem 2.6. *If $\boxed{f = g}$ on some non-empty open disk S contained in a set U where f and g are both analytic, then $f \equiv g$ on U.*



If $f=g$ on S

then $f=g$ on all of U.

analytic – convergent power series

analytic function is infinitely differentiable

extension theorem: real analytic ⇒ complex analytic identity theorem: $f=g$ small S $\Rightarrow$ $f=g$ on big U

3. Complex exponentials

f,g are analytic.

Now that we have the extension theorem, we can start using it to rebuild calculus in the context of complex variables. We begin with one of the most important functions, the natural exponential. In real variables, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note that this also gives us a (weird if you think about it) identity on power series arising from the identity $e^{a+b} = e^a e^b$.

$$\sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{n=0}^{\infty} \frac{b^n}{n!}$$

The extension theorem implies that this identity holds for complex numbers with some interesting consequences.

Theorem 3.1. *If $z = x + iy$,*

$$e^z = e^x(\cos x + i \sin y)$$