

Notes on functional analysis

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Chapter 1

The spectral theorem for compact operators

1.1 The spectral theorem in finite dimensions

The point of these notes is to generalize one of the most useful theorems in undergraduate linear algebra to a very general setting. Along the way, we'll look at some very modern developments in analysis and the study of spaces of functions.

Recall that a complex $n \times n$ matrix A is called **normal** if $AA^* = A^*A$ where $*$ represents the conjugate transpose. The spectral theorem for normal matrices says that A can be **diagonalized** as

$$A = UDU^*$$

where D is a diagonal matrix of eigenvalues of A and U is a unitary matrix of associated eigenvectors.

The big idea of diagonalization is that operators (maps represented by square matrices) can be understood by looking at their action on **invariant subspaces**, which are spaces that a matrix leaves “pointing in the same direction”. This is a major simplification in the understanding of (normal) linear maps - every normal linear map is (up to a change of basis) scalar multiplication on orthogonal spaces.

A slightly more sophisticated way of presenting this idea is to use **projection operators**.

Definition 1.1.1 Let \mathcal{H} be a finite dimensional Hilbert space, and $V \subset \mathcal{H}$ a linear subspace. Let $\{v_1, \dots, v_k\}$ be an orthonormal basis for V . The **projection operator of \mathcal{H} onto V** , denoted P_V , is given by

$$P_V h = \sum_{i=1}^k \langle h, v_i \rangle v_i \in V$$

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With this language, diagonalization becomes the decomposition of operators into sums of projections. That is, given an operator A on a (complex) finite dimensional Hilbert space \mathcal{H} , denote by (λ_i, E_i) the i th eigenvalue and corresponding eigenspace. Recall that each eigenspace is $E_i = \ker(A - \lambda_i)$. A

normal matrix will always have $\mathcal{H} = \bigoplus_{i=1}^k E_i$ and we can write

$$A = \sum_{i=1}^k \lambda_i P_i. \quad (1.1.1)$$

where P_i is the projection operator onto $\ker(A - \lambda_i)$.

A special case is that of hermitian matrices (that is, matrices that have the property that $A = A^*$), in which case the eigenvalues of A are real.

1.2 Projections and invariant subspaces

The content of this section is motivated by the idea of eigenspaces in finite dimensions. Eigenspaces are invariant under the action of their operators (since they are mapped into themselves) and moreover, eigenspaces of distinct eigenvalues are orthogonal to each other. When we diagonalize, we're essentially pulling an operator apart into projections onto its eigenspaces. All of these ideas are useful in the Hilbert space setting.

Definition 1.2.1 An operator E in $\mathbf{B}(\mathcal{H})$ is called **idempotent** if $E^2 = E$. An idempotent P is called a **projection** if $\ker P = (\text{ran } P)^\perp$. \diamond

Operators defined this way act precisely as you should expect - a projection produces the “orthogonal” piece of a general vector in \mathcal{H} restricted to the subspace $\text{ran } P$. Of course, this must be made rigorous.

Proposition 1.2.2 Let E be a non-zero idempotent on \mathcal{H} . The following are equivalent:

1. E is a projection.
2. E is the orthogonal projection of \mathcal{H} onto $\text{ran } E$.
3. $\|E\| = 1$.
4. E is self-adjoint.
5. E is normal.
6. $\langle Eh, h \rangle \geq 0$ for all $h \in \mathcal{H}$.

The proof will be left as an exercise. (To be revisited in a future draft.)

As in finite dimensions, the range of a projection can be used to decompose a space. Let P be a projection with $\text{ran } P = \mathcal{M}$ and $\ker P = \mathcal{N}$. Kernel and range are closed subspaces, and thus Hilbert spaces in their own right. Then construct the space $\mathcal{M} \oplus \mathcal{N}$. It is straightforward that $\mathcal{M} \oplus \mathcal{N}$ is isomorphic to \mathcal{H} , and in the usual abuse of notation, we'll just say $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, or that \mathcal{H} has been orthogonally decomposed.

Definition 1.2.3 If \mathcal{M}_i is a collection of pairwise orthogonal subspaces of \mathcal{H} , then set

$$\bigoplus_i \mathcal{M}_i = \bigvee_i \mathcal{M}_i.$$

If \mathcal{M}, \mathcal{N} are closed linear subspaces of \mathcal{H} , then set the **orthogonal difference** of \mathcal{M} and \mathcal{N} to be

$$\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^\perp.$$

\diamond

Now we'll identify the sort of special spaces that can be used to pull operators apart.

Definition 1.2.4 If $A \in \mathbf{B}(\mathcal{H})$ and \mathcal{M} is a closed linear subspace of \mathcal{H} , then we say \mathcal{M} is an **invariant subspace** of A if $A\mathcal{M} \subseteq \mathcal{M}$. If in addition $A\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$, we say that \mathcal{M} **reduces** A or that \mathcal{M} is a **reducing subspace** of A . \diamond

One of the nice consequences of the existence of an invariant or reducing subspace is that we get information about block operator representation of operators on \mathcal{H} . First, suppose that $\mathcal{H} = \mathcal{M} + \mathcal{M}^\perp$. Then any $h \in \mathcal{H}$ can be written as $h = u + v$ where $u \in \mathcal{M}$ and $v \in \mathcal{M}^\perp$. So for $A \in \mathbf{B}(\mathcal{H})$, we can think about the action of A on the components of \mathcal{H} by

$$Ah = \begin{bmatrix} X & W \\ Z & Y \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $X \in \mathbf{B}(\mathcal{M})$, $W \in \mathbf{B}(\mathcal{M}^\perp, \mathcal{M})$, $Z \in \mathbf{B}(\mathcal{M}, \mathcal{M}^\perp)$, and $Y \in \mathbf{B}(\mathcal{M}^\perp)$.

In the representation above, if \mathcal{M} is invariant for A , then Z must be 0. If \mathcal{M} is reducing for A , then we get further that $W = 0$. That is, a reducing subspace essentially diagonalizes an operator by reducing it to smaller operators.

Proposition 1.2.5 Suppose that $A \in \mathbf{B}(\mathcal{H})$, \mathcal{M} is a closed linear subspace of \mathcal{H} and that $P = P_{\mathcal{M}}$. The following are equivalent.

1. \mathcal{M} is invariant for A .
2. $PAP = AP$.
3. The block Z as in the block representation above is the 0 operator.

The following statements are also equivalent.

1. \mathcal{M} reduces A .
2. $PA = AP$.
3. In the block representation above, both W, Z are 0 operators.
4. \mathcal{M} is invariant for both A and A^* .

1.3 Compact operators

The object of study in this section is essentially the most direct generalization of matrices to infinite dimensions, for reasons that will become apparent.

Definition 1.3.1 Suppose \mathcal{H} is a Hilbert space. A linear map $T : \mathcal{H} \rightarrow \mathcal{H}$ is **compact** if the image of the unit ball in \mathcal{H} under T , denoted $T(\text{ball } \mathcal{H})$, has compact closure in \mathcal{H} . \diamond

Denote by $\mathbf{B}_0(\mathcal{H}, \mathcal{K})$ the set of compact operators from \mathcal{H} to \mathcal{K} , and $\mathbf{B}_0(\mathcal{H}) = \mathbf{B}_0(\mathcal{H}, \mathcal{H})$.

Proposition 1.3.2 $\mathbf{B}_0(\mathcal{H}, \mathcal{K}) \subset \mathbf{B}(\mathcal{H}, \mathcal{K})$.

Proof. Let $T \in \mathbf{B}_0(\mathcal{H}, \mathcal{K})$. Since $T(\text{ball } \mathcal{H})$ is compact, there must exist a constant C so that $T(\text{ball } \mathcal{H}) \subset B_{\mathcal{K}}(C) = \{k \in \mathcal{K} : \|k\| \leq C\}$. But then $\|T\| \leq C$, and so $T \in \mathbf{B}(\mathcal{H})$. \blacksquare

Proposition 1.3.3 $\mathbf{B}_0(\mathcal{H}, \mathcal{K})$ is a linear space.

Proposition 1.3.4 *If T_n is a sequence in $\mathbf{B}_0(\mathcal{H}, \mathcal{K})$ and T is an operator in $\mathbf{B}(\mathcal{H}, \mathcal{K})$ such that $\|T_n - T\| \rightarrow 0$, then T is compact.*

Proposition 1.3.5 *If $A \in \mathbf{B}(\mathcal{H})$, $B \in \mathbf{B}(\mathcal{K})$, and $T \in \mathbf{B}_0(\mathcal{H}, \mathcal{K})$, then TA and BT are compact.*

A special family of compact operators is even more restricted by the size of their ranges. Say that an operator is of **finite rank**, if $\text{ran } T$ is finite dimensional (that is, the closed linear span of a finite set of vectors). Let $\mathbf{B}_{00}(\mathcal{H}, \mathcal{K})$ be the set of continuous finite rank operators. It is straightforward that \mathbf{B}_{00} is a linear space and that $\mathbf{B}_{00} \subset \mathbf{B}_0$. It turns out to be the case that compact operators are (norm) limits of operators of finite rank.

Theorem 1.3.6 *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. The following are equivalent.*

1. T is compact.
2. T^* is compact.
3. There is a sequence T_n of operators of finite rank such that $\|T_n - T\| \rightarrow 0$.

Proof. First, note that (3) \Rightarrow (1) follows from Proposition 1.3.4.

We will show that (1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).

(1) \Rightarrow (3): We use the fact that since T is compact, the closure of $T(\text{ball } \mathcal{H})$ contains a countable dense subset (one can construct this set using the fact that the image of the ball under T is totally bounded). Then $\mathcal{L} = \text{cl } \text{ran } T$ is a separable subspace of \mathcal{K} . Let $\{e_j\}$ be a basis for \mathcal{L} , and let P_n be the projection of \mathcal{K} onto the closed linear span of the first n basis vectors; that is, P_n is the projection onto $\vee\{e_j : 1 \leq j \leq n\}$. Now set $T_n = P_n T$. Each T_n is a finite rank operator.

We will show that $\|T_n - T\| \rightarrow 0$ by establishing a uniform bound on $\|T_n h - Th\|$ for $\|h\| \leq 1$. Choose $\varepsilon > 0$. By compactness, we can find a finite collection of vectors h_1, \dots, h_m so that

$$T(\text{ball } \mathcal{H}) \subset \bigcup_{j=1}^m B(Th_j, \varepsilon/3).$$

For $h \in \text{ball } \mathcal{H}$, choose h_j so that $\|Th - Th_j\| \leq \varepsilon/3$. Then for any $n \in \mathcal{N}$,

$$\begin{aligned} \|T_n h - Th\| &\leq \|T_n h - T_n h_j\| + \|T_n h_j - Th_j\| + \|Th_j - Th\| \\ &\leq \|P_n(Th - Th_j)\| + \|T_n h_j - Th_j\| + \|Th_j - Th\| \\ &\leq 2\|Th_j - Th\| + \|T_n h_j - Th_j\| \\ &\leq \frac{2\varepsilon}{3} + \|T_n h_j - Th_j\|. \end{aligned}$$

Now, we show that we can find n_0 so that $\|T_n h_j - Th_j\| \leq \varepsilon/3$. Certainly $Th_j \in \mathcal{L}$, and so can be expanded as $Th_j = \sum_{k=1}^{\infty} \langle Th_j, e_k \rangle e_k$. On the other hand,

$$T_n h_j = P_n Th_j = \sum_{k=1}^n \langle Th_j, e_k \rangle e_k$$

Together, this gives $\|T_n h_j - Th_j\| \rightarrow 0$, and so the desired n_0 exists. Since h was an arbitrary vector with $\|h\| \leq 1$, this bound is uniform, and so in the operator norm we have $\|T_n - T\| \rightarrow 0$.

3 \Rightarrow 2 : Suppose that T_n is a sequence in $\mathbf{B}_{00}(\mathcal{H}, \mathcal{K})$ with $\|T_n - T\| \rightarrow 0$. Then $\|T_n^* - T^*\| = \|T_n - T\| \rightarrow 0$. But T_n^* is also in $\mathbf{B}_{00}(\mathcal{H}, \mathcal{K})$ (Exercise). Since 3 \Rightarrow 1, T^* is compact.

2 \Rightarrow 1: (Exercise) ■

We used an important convergence theorem about finite rank projections in the proof above, so we record the following corollary of the proof.

Proposition 1.3.7 *If $T \in \mathbf{B}_0(\mathcal{H}, \mathcal{K})$, then $\text{clran } T$ is separable. If $\{e_j\}$ is a basis for $\text{clran } T$ and P_n is the projection of \mathcal{K} onto $\vee\{e_j : 1 \leq j \leq n\}$, then*

$$\|P_n T - T\| \rightarrow 0.$$

This allows a method for constructing compact operators, at least in separable Hilbert spaces, in a manner suggestive of finite dimensional eigenvalues.

Proposition 1.3.8 *Let \mathcal{H} be a separable Hilbert space with basis $\{e_j\}$. Let $\{\alpha_n\}$ be a sequence in \mathbb{F} with $M = \sup_n \{|\alpha_n|\} < \infty$. Define an operator A on the basis $\{e_j\}$ by $Ae_j = \alpha_j e_j$ for all j . Then A extends by linearity to a bounded operator on \mathcal{H} with $\|A\| = M$. The operator A is compact if and only if $\alpha_n \rightarrow 0$.*

We are now ready to make explicit the apparent connection to eigenvalues of matrices acting on finite dimensional spaces.

Definition 1.3.9 Suppose that $A \in \mathbf{B}(\mathcal{H})$. A scalar $\alpha \in \mathbb{F}$ is called an **eigenvalue** of A if $\ker(A - \alpha) \neq \{0\}$. A nonzero vector h in $\ker(A - \alpha)$ is called an **eigenvector** of A . We recover the usual relationship $Ah = \alpha h$ for such a pair. Let $\sigma_p(A)$ denote the set of eigenvalues of A . \diamond

The p in the notation $\sigma_p(A)$ refers to the **point spectrum** of more general operators, which we'll talk about later.

Unlike linear operators on finite dimensional spaces, compact operators need not possess eigenvalues. The prototypical example is the **Volterra operator**.

Proposition 1.3.10 *Let $\mathcal{H} = L^2[0, 1]$. For $t \in [0, 1]$, let $V : L^2 \rightarrow L^2$ be the operator*

$$V(f)(x) = \int_0^x f(y) dy.$$

The operator V is compact. However, V has no eigenvalues.

The whole thrust of our conversation is diagonalization of operators in terms of eigen-like structures, and here we've identified a compact operator that cannot be. It's worth asking why. Note that $V^*(f)(x) = \int_x^1 f(y) dy$; that is, V isn't self-adjoint (in fact, V is also not normal). Keep that in mind as we proceed.

If, however, a compact operator does have an eigenvalue, nice things happen.

Proposition 1.3.11 *If $T \in \mathbf{B}_0(\mathcal{H})$ and $\lambda \in \sigma_p(T)$ with $\lambda \neq 0$, then the eigenspace $\ker(T - \lambda)$ is finite-dimensional.*

Proof. Proceed by contradiction. Suppose that $\ker(T - \alpha)$ is infinite-dimensional and so contains an infinite orthonormal sequence. Because T is compact, Te_n has a convergent subsequence Te_{n_k} . Then as a sequence, $\{Te_{n_k}\}$ is Cauchy. But for any $n_k \neq n_j$,

$$\|Te_{n_k} - Te_{n_j}\|^2 = \|\lambda e_{n_k} - \lambda e_{n_j}\|^2 = 2|\lambda|^2 > 0.$$

This contradicts the Cauchyness of $\{Te_{n_k}\}$. We conclude that $\ker(T - \alpha)$ is finite-dimensional. \blacksquare

Proposition 1.3.12 *If $T \in \mathbf{B}_0(\mathcal{H})$, $\lambda \neq 0$, and $\inf\{\|(T - \lambda)h\| : \|h\| = 1\} = 0$ then $\lambda \in \sigma_p(T)$.*

Proof. The hypotheses imply that there exists a sequence of unit vectors h_n so that $\|(T - \lambda)h_n\| \rightarrow 0$. By compactness, this sequence has a subsequence h_{n_k}

and a vector f so that $\|Th_{n_k} - f\| \rightarrow 0$. Combining these two facts, we can use an algebra trick to get

$$\begin{aligned} h_{n_k} &= \lambda^{-1} \lambda h_{n_k} = \lambda^{-1} \lambda (h_{n_k} - \lambda^{-1} Th_{n_k} + \lambda^{-1} Th_{n_k}) \\ &= \lambda^{-1} [(\lambda - T)h_{n_k} + Th_{n_k}] \\ &\rightarrow \lambda^{-1} [0 + f] = \lambda^{-1} f. \end{aligned}$$

Since the h_n are unit vectors, so is any limit point of the sequence, and so $1 = \|\lambda^{-1} f\| = |\lambda|^{-1} \|f\|$, and so $\|f\| \neq 0$. Furthermore, $Th_{n_k} \rightarrow \lambda^{-1} T f$. Since we also have $Th_{n_k} \rightarrow f$, it must be that $f = \lambda^{-1} T f$, or rather that $T f = \lambda f$. That is, $f \neq 0$, $f \in \ker(T - \lambda)$, which gives that $\lambda \in \sigma_p(T)$. ■

1.4 The spectral theorem for compact self-adjoint operators

Our first diagonalization result is a special case, but one that sheds light on the (quite a bit) more complicated general case. This is a good case to get to grips with the point of the theorem because it is so similar to the finite dimensional case (compare with the statement in (1.1.1)).

Theorem 1.4.1 *Let T be a compact self-adjoint operator on a Hilbert space \mathcal{H} . Then T has only countable distinct eigenvalues. Enumerating the eigenvalues as $\{\lambda_1, \lambda_2, \dots\}$, let P_n denote the projection of \mathcal{H} onto $\ker(T - \lambda_n)$. Then $P_n P_m = P_m P_n = 0$ when $n \neq m$, each λ_n is real, and*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where the sum converges in the norm of $\mathbf{B}(\mathcal{H})$.

1.4.1 Proof of the spectral theorem for compact self-adjoint operators

Proposition 1.4.2 *If A is a normal operator and $\lambda \in \mathbb{F}$, then $\ker(A - \lambda) = \ker(A - \lambda)^*$ and $\ker(A - \lambda)$ is a reducing subspace for A .*

Proof. Since A is normal, it is easy to see that so must be $A - \lambda$. By [provisional cross-reference: prop-normaladnorm], we have that $\|(A - \lambda)h\| = \|(A - \lambda)^* h\|$ for all h . But then $h \in \ker(A - \lambda)^*$ if and only if $h \in \ker(A - \lambda)$.

If $h \in \ker(A - \lambda)$, then $Ah = \lambda h \in \ker(A - \lambda)$ and $A^* h = \bar{\lambda} h \in \ker(A - \lambda)$. We conclude that $\ker(A - \lambda)$ is a reducing subspace for A . ■

The next result is similar to the finite dimensional fact that eigenspaces of distinct eigenvalues are orthogonal.

Proposition 1.4.3 *If A is a normal operator and λ, μ are distinct eigenvalues of A then $\ker(A - \lambda) \perp \ker(A - \mu)$.*

Proof. Let $h \in \ker(A - \lambda)$ and $g \in \ker(A - \mu)$. By Proposition 1.4.2, we have that $A^* g = \bar{\mu} g$. Then

$$\begin{aligned} \lambda \langle h, g \rangle &= \langle Ah, g \rangle \\ &= \langle h, A^* g \rangle \\ &= \langle h, \bar{\mu} g \rangle \\ &= \mu \langle h, g \rangle. \end{aligned}$$

Thus, $(\lambda - \mu) \langle h, g \rangle = 0$. Since λ, μ are distinct, we must have $h \perp g$ and thus that $\ker(A - \lambda) \perp \ker(A - \mu)$. ■

As with hermitian matrices, the eigenvalues of self-adjoint operators must be real.

Proposition 1.4.4 *If $A = A^*$ and $\lambda \in \sigma_p(A)$ then λ is a real number.*

Proof. If $Ah = \lambda h$ then $Ah = A^*h = \bar{\lambda}h$ by [Proposition 1.4.2](#). Equating the two expressions for Ah , we get

$$(\lambda - \bar{\lambda})h = 0.$$

Choosing a non-zero h requires that $\lambda - \bar{\lambda} = 0$, or $\lambda = \bar{\lambda}$. ■

The final ingredient we need is to demonstrate that the spectrum of a compact self-adjoint operator contains non-zero elements. In the case of a finite dimensional diagonalizable matrix, we have that the norm of the map induced by the matrix is equal to the magnitude of the largest eigenvalue. The same result holds in the compact self-adjoint case.

Lemma 1.4.5 *If T is a compact self-adjoint operator, then $\|T\|$ or $-\|T\|$ is an eigenvalue of T .*

Proof. If T is the zero operator, then we're done. If $T \neq 0$, there exists a sequence of unit vectors h_n so that

$$|\langle Th_n, h_n \rangle| \rightarrow \|T\|.$$

A limit point argument produces a subsequence h_n so that

$$\langle Th_n, h_n \rangle \rightarrow \lambda$$

where $|\lambda| = \|T\|$.

Since $|\lambda| = \|T\|$, we get

$$\begin{aligned} 0 &\leq \|(T - \lambda)h_n\|^2 \\ &= \|Th_n\|^2 - 2\lambda \langle Th_n, h_n \rangle + \lambda^2 \\ &= 2\lambda^2 - 2\lambda \langle Th_n, h_n \rangle. \end{aligned}$$

But $\langle Th_n, h_n \rangle \rightarrow \lambda$, so the computation above implies that $\|(T - \lambda)h_n\| \rightarrow 0$. Applying [Proposition 1.3.12](#), we conclude that $\lambda \in \sigma_p(T)$. ■

We are now prepared to prove the spectral theorem for compact self-adjoint operators. Essentially, we're going to use [Lemma 1.4.5](#) to find an eigenvalue, construct a projection onto its eigenspace, and then restrict the operator to the remaining orthogonal complement. Proceeding inductively will give the result. *Proof of Theorem 1.4.1.* Suppose that T is a compact self-adjoint operator on a Hilbert space \mathcal{H} .

By [Proposition 1.4.4](#) and [Lemma 1.4.5](#), there exists a real number λ_1 in $\sigma_p(T)$ with $|\lambda_1| = \|T\|$. Let $E_1 = \ker(T - \lambda_1)$ and $P_1 : \mathcal{H} \rightarrow E_1$ be the projection onto E_1 . Denote by \mathcal{H}_2 the orthogonal complement of E_1 in \mathcal{H} . By [Proposition 1.4.2](#), E_1 reduces T , and so \mathcal{H}_2 reduces T . Then define T_2 to be the restriction of T to \mathcal{H}_2 ; that is $T_2 = T|_{\mathcal{H}_2} = TP_1$. Using block operator representation of T on $\mathcal{H} = E_1 \oplus \mathcal{H}_2$, it is easy to see that T_2 is a compact self-adjoint operator on \mathcal{H}_2 .

If $T_2 = 0$, we're done. If not, we can repeat the process to produce an eigenvalue λ_2 and associated eigenspace E_2 . Continuing, we either terminate with some $T_n = 0$, or we proceed by induction to produce an infinite sequence λ_n . In either case, we end up with a sequence of eigenvalues ordered by magnitude

and pairwise orthogonal eigenspaces.

The last piece to be shown is that the accumulation point of the eigenvalues is precisely 0. The monotone bounded sequence theorem gives a real number α so that $|\lambda_n| \rightarrow \alpha$. We will argue that $\alpha = 0$.

Let e_n be a unit vector in E_n . Since T is compact, the sequence Te_n must have a convergent subsequence, say converging to $h \in \mathcal{H}$, so that $\|Te_{n_j} - h\| \rightarrow 0$. But $\|Te_{n_j}\| = |\lambda_{n_j}| \rightarrow \alpha$. ■

Chapter 2

Topology

2.1 Topological catch-all

Definition 2.1.1 Separation axioms.

- A topological space \mathcal{X} is T_1 if whenever $x \neq y$, there exists an open set containing y but not x .
- A topological space \mathcal{X} is called **Hausdorff** if it is T_2 ; that is, if $x \neq y$ then there are disjoint open sets U, V with $x \in U, y \in V$.
- A topological space \mathcal{X} is called **normal** if it is T_4 ; that is, \mathcal{X} is T_1 and for any disjoint closed sets A, B in \mathcal{X} , there are disjoint open sets U, V with $A \subset U$ and $B \subset V$.

◇

Definition 2.1.2 Continuous functions. Suppose that \mathcal{X}, \mathcal{Y} are topological spaces.

- A function is called **continuous** if $f^{-1}(V)$ is open for every open set $V \subset \mathcal{Y}$.
- A function is called **continuous at x** if $f^{-1}(V)$ is open for every neighborhood V of x .
- A bijective function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called a **homeomorphism** if f, f^{-1} are continuous.

◇

We'll use the notation $C(\mathcal{X}, \mathcal{Y})$ to designate the family of continuous functions $f : \mathcal{X} \rightarrow \mathcal{Y}$.

We'll need a couple of extension theorems. They have names, so you know they must be important.

Theorem 2.1.3 Uryshon's Lemma. *Let X be a normal space. If A, B are disjoint closed sets in X , then there exists $f \in C(X, [0, 1])$ such that $f = 0$ on A and $f = 1$ on B .*

Theorem 2.1.4 *Let X be a normal space. If A is a closed subset of X and $f \in C(A, [a, b])$, there exists a continuous extension $F \in C(X, [a, b])$ so that the restriction $F|_A$ is f .*

2.2 Nets

In general topological spaces, we usually can't get away with thinking about sequences. Instead, we'll capture convergence in topological spaces with nets. A sequence is a function $x : \mathbb{N} \rightarrow \mathcal{X}$ that maps the natural numbers into a space \mathcal{X} . We want to think of nets as functions that map more general index sets into a space. We need to retain a notion of forward progress to address convergence, which we get with the notion of directed sets.

Definition 2.2.1 A **directed set** is a set A with a binary relation \preceq with the properties that

1. (self-relation): $\alpha \preceq \alpha$ for all $\alpha \in A$;
2. (transitivity): if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$;
3. (upper bounds): for any $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

◇

Not every pair of elements in a directed set relate (unlike a totally ordered set), but we do have the upper bound property that will allow us to define convergence.

Definition 2.2.2 A **net** in \mathcal{X} is a function x from a directed set \mathcal{A} into a space \mathcal{X} . We usually write $x(\alpha) = x_\alpha$ and write the net as $\langle x_\alpha \rangle_{\alpha \in \mathcal{A}} = \langle x_\alpha \rangle$. ◇

Definition 2.2.3 Now let \mathcal{X} be a topological space and $E \subset \mathcal{X}$.

- A net $\langle x_\alpha \rangle$ is **eventually** in E if there exists $\alpha_0 \in \mathcal{A}$ such $x_\alpha \in E$ for all $\alpha \succeq \alpha_0$.
- A net $\langle x_\alpha \rangle$ is **frequently** in E if for every $\alpha \in \mathcal{A}$ there exists $\beta \succeq \alpha$ such that $x_\beta \in E$.
- A point $x \in \mathcal{X}$ is a **limit** of $\langle x_\alpha \rangle$ if for every neighborhood U of x , $\langle x_\alpha \rangle$ is eventually in U . In this case, we say that x_α **converges** to x and we write $x_\alpha \rightarrow x$.
- A point $x \in E$ is a **cluster point** of $\langle x_\alpha \rangle$ if for every neighborhood U of x , $\langle x_\alpha \rangle$ is frequently in U .

◇

Now, we'll show that nets have many of the desirable properties that sequences do.

Proposition 2.2.4 If \mathcal{X} is a topological space, $E \subset \mathcal{X}$, and $x \in \mathcal{X}$, then x is an accumulation point of E if and only if there exists a net in $E \setminus \{x\}$ that converges to x . Likewise, $x \in \overline{E}$ if and only if there is a net in E that converges to x .

Proof. If x is a limit point of E , then let \mathcal{N} be the set of neighborhoods of x , directed by reverse inclusion. For each $U \in \mathcal{N}$, choose an element $x_U \in (U \setminus \{x\}) \cap E$. Then by definition, $x_U \rightarrow x$.

On the other hand, if x_α is a net in $E \setminus \{x\}$ and $x_\alpha \rightarrow x$, then for any neighborhood $U \in \mathcal{N}$, the punctured set $U \setminus \{x\}$ contains some point in x_α . Thus, x is a limit point of E .

The second statement follows from the definition and the observation that the closure of a set contains its limit points. ■

We also get a net definition of continuity akin to the typical sequence definition.

Proposition 2.2.5 *Let \mathcal{X}, \mathcal{Y} be topological spaces and $f : \mathcal{X} \rightarrow \mathcal{Y}$. The function f is continuous at $x \in \mathcal{X}$ if and only if for every net $\langle x_\alpha \rangle$ converging to x , the net $\langle f(x_\alpha) \rangle$ converges to $f(x)$.*

Proof. Suppose that f is continuous at x . Then for a neighborhood V of $f(x)$, $f^{-1}(V)$ is a neighborhood of x . Now suppose that a net x_α converges to x . By definition, x_α must eventually be in $f^{-1}(V)$, which implies that $\langle f(x_\alpha) \rangle$ is eventually in V . But this means that $f(x_\alpha) \rightarrow f(x)$, as the choice of V was arbitrary. ■

Finally, we'll state a net version of the theorem for sequences that states that sequential cluster points have convergent subsequences - be careful with this one, because a subnet it isn't quite the direct analogy that it appears to be.

Subnets. There are at least three non-equivalent definitions of subnet. Despite the complications, the idea is generally the same - to recover as many theorems involving convergent sequences as possible. To see more explanation, see [the article on subnets](https://en.wikipedia.org/wiki/Subnet_(mathematics))¹ on Wikipedia.

Definition 2.2.6 A **subnet** of a net $\langle x_\alpha \rangle_{\alpha \in \mathcal{A}}$ is a net $\langle y_\beta \rangle_{\beta \in \mathcal{B}}$ and a map $\beta \mapsto \alpha_\beta$ from $\mathcal{B} \rightarrow \mathcal{A}$ such that the following hold:

- For every $\alpha_0 \in \mathcal{A}$ there exists $\beta_0 \in \mathcal{B}$ such that $\alpha_\beta \succeq \alpha_0$ whenever $\beta \succeq \beta_0$.
- $y_\beta = x_{\alpha_\beta}$

◇

If $\langle x_\alpha \rangle$ converges to a point x , then so too must any subnet $\langle x_{\alpha_\beta} \rangle$

Proposition 2.2.7 *If $\langle x_\alpha \rangle$ is a net in \mathcal{X} then x is a cluster point of x_α if and only if there is a subnet of x_α converging to x .*

Proof. Suppose that $\langle x_\alpha \rangle$ has a subnet $\langle y_\beta \rangle = \langle x_{\alpha_\beta} \rangle$ converging to x . Let U be a neighborhood of x . Now choose β_1 so that $y_\beta \in U$ whenever $\beta \succeq \beta_1$ (which we can do by convergence). For a given α , choose β_2 so that $\alpha_\beta \succeq \alpha$ for $\beta \succeq \beta_2$ (which we can do by the definition of a subnet). Since \mathcal{B} is a directed set, choose $\beta \in \mathcal{B}$ so that $\beta \succeq \beta_1$ and $\beta \succeq \beta_2$. Then $\alpha_\beta \succeq \alpha$ and $x_{\alpha_\beta} = y_\beta \in U$. That is, x_α is frequently in U , and so x is a cluster point of $\langle x_\alpha \rangle$.

In the other direction, if x is a cluster point of $\langle x_\alpha \rangle$, the idea will be to use the neighborhoods at x as the index. Let \mathcal{N} be the neighborhoods of x . Construct the product of directed sets $\mathcal{N} \times \mathcal{A}$ with the relation $(U, \alpha) \preceq (U_1, \alpha_1)$ when $U \supset U_1$ and $\alpha \preceq \alpha_1$. Then using the upper bound property, for each $(U, \beta) \in \mathcal{N} \times \mathcal{A}$, we can choose $\alpha_{(U, \beta)}$ such that $\alpha_{(U, \beta)} \succeq \beta$ and $x_{\alpha_{(U, \beta)}} \in U$. If $(U', \beta') \succeq (U, \beta)$, we have

$$\alpha_{(U', \beta')} \succeq \beta' \succeq \beta$$

and

$$x_{\alpha_{(U', \beta')}} \in U' \subset U.$$

Thus $\langle x_{\alpha_{(U, \beta)}} \rangle$ is a subnet of $\langle x_\alpha \rangle$ that converges to x . ■

2.3 Compact spaces

Now we'll take a look at how compactness and related results generalize to topological spaces.

Definition 2.3.1 Let \mathcal{X} be a topological space.

- \mathcal{X} is **compact** if every open cover of \mathcal{X} has a finite subcover. That is,

¹[https://en.wikipedia.org/wiki/Subnet_\(mathematics\)](https://en.wikipedia.org/wiki/Subnet_(mathematics))

given a collection of open sets so that $\mathcal{X} = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, there is a finite subset \mathcal{B} of \mathcal{A} so that $X = \bigcup_{\alpha \in \mathcal{B}} U_\alpha$.

- A subset Y of a topological space is **compact** if it is compact in the relative topology. That is, $Y \subset \mathcal{X}$ is compact if and only if for every collection of open sets in \mathcal{X} with $Y \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, there is a finite subset \mathcal{B} of \mathcal{A} so that $Y \subset \bigcup_{\alpha \in \mathcal{B}} U_\alpha$.
- A set $Y \subset \mathcal{X}$ is called **precompact** if its closure is compact.

◇

One useful characterization of compactness is in terms of closed sets. A family of sets $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ has the **finite intersection property** if $\bigcap_{\alpha \in \mathcal{B}} F_\alpha \neq \emptyset$ for any finite subset \mathcal{B} of \mathcal{A} .

Proposition 2.3.2 *A topological space is compact if and only if $\bigcap_{\alpha \in \mathcal{A}} F_\alpha \neq \emptyset$ for every family of sets $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ with the finite intersection property.*

Proof. Suppose a family of closed sets $\{F_\alpha\}$ is given. Let $\{U_\alpha\}$ be the family of open sets given by $U_\alpha = (F_\alpha)^c$. By construction, $\bigcap_{\alpha \in \mathcal{A}} F_\alpha \neq \emptyset$ if and only if $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \neq \mathcal{X}$. Also, $\{F_\alpha\}$ has the finite intersection property if and only if no finite subfamily of $\{U_\alpha\}$ covers \mathcal{X} . ■

Chapter 3

Locally convex spaces

The move from basic to advanced functional analysis in part comes from generalizing the sorts of spaces that we can work on. The appropriate setting for working in weak topologies in Banach spaces is that of *locally convex spaces*, which are topological generalizations of Banach spaces.

3.1 Locally convex spaces

We begin by introducing a special class of vector spaces that are well-suited for analysis. Since we typically want to consider questions of convergence and continuity, we require our vector spaces to come equipped with a topological structure.

Definition 3.1.1 A topological vector space (TVS) is a vector space \mathcal{X} over a field \mathbb{F} and a topology \mathcal{T} so that

1. addition is continuous - that is, the map $(x, y) \mapsto x + y$ is continuous.
2. scalar multiplication is continuous - that is, the map $(\alpha, x) \mapsto \alpha x$ is continuous.

◇

Topological vector spaces include normed spaces (and thus inner product spaces). The other big feature of Euclidean space that we'd like to lift into the general setting is the notion of convexity.

Definition 3.1.2 A locally convex space (LCS) is a TVS with the property that there is a base of the topology consisting of convex sets ◇

One way to create such spaces is by way of seminorms (norm-like functions that may send non-zero vectors to 0).

Definition 3.1.3 A seminorm is a real-valued function $p : \mathcal{X} \rightarrow \mathbb{R}$ satisfying the following properties:

1. Triangle inequality:

$$\forall x, y \in \mathcal{X}, p(x + y) \leq p(x) + p(y)$$

2. Homogeneity:

$$\forall \alpha \in \mathbb{F}, x \in \mathcal{X}, p(\alpha x) = |\alpha| p(x)$$

3. Non-negativity:

$$p(x) \geq 0 \quad \forall x \in \mathcal{X}$$

◇

The basic idea is to use the “balls” defined by the seminorms to generate the topology.

Theorem 3.1.4