MATH 610

Programming Assignment #4

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FEM for fourth order problems using piecewise Hermite polynomials

1 Specifications

Now we consider the following model problem for the unknown function $u(x)$:

$$\big(k(x)u''\big)'' - (p(x)u')' + \gamma(x)u = f(x), \qquad x \in (0,1)$$

where the coefficients are given and satisfy the property

$$K \geq k(x) \geq k_0 > 0, \quad P \geq p(x) \geq 0, \quad Q \geq \gamma(x) \geq 0,$$

where K, P, and Q are constants. Here, k is the coefficient of the flexural rigidity, p is the axial force, and q is the elastic property of the support.

Write a program for solving two-point boundary value problems for fourth order ordinary differential equations by the Ritz-Galerkin method using piecewise Hermite polynomials of degree 3. Submit a report with graphs of the results and tables of the error in the L^2, H^1, H^2, and maximum norms.

- Use 5, 10, 20, and 40 finite elements. For computing the element bending, stiffness, and mass matrices, use the exact matrices or use quadrature. Plot the error as a function of the independent variable.
- In a table, give the error in the L^2, H^1, H^2, and maximum norms.

1.1 Problems

For the first set of problems, the four boundary conditions are

$$u(0) = u''(0) = 0$$
$$u(1) = u''(1) = 0.$$

This boundary value problem describes the transverse deformation of a simply supported elastic beam.

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = \sin(\pi x)$, (exact solution $u(x) = \frac{1}{\pi^4}\sin(\pi x)$).
2. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = 60x$, (exact solution $u(x) = \frac{x}{6}(7 - 10x^2 + 3x^4)$).
3. $k(x) = 1/(4\pi^4)$, $p(x) = 0$, $\gamma(x) = 1$, $f(x) = 1$, (exact solution $u(x) = 1 + \cos(\pi x)(Ae^{\pi x} + Be^{-\pi x})$), $A = \frac{1}{e^\pi - 1}$, $B = -\frac{e^\pi}{e^\pi - 1}$.

For the second set of problems, the four boundary conditions are

$$u(0) = u'(0) = 0$$
$$u(1) = u'(1) = 0.$$

Here, the beam has clamped ends.

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = \sin(\pi x)$, (exact solution $u(x) = \frac{x^2}{\pi^3} - \frac{x}{\pi^3} + \frac{1}{\pi^4}\sin(\pi x)$).
2. $k(x) = 1/(4\pi^4)$, $p(x) = 0$, $\gamma(x) = 1$, $f(x) = 1$, (exact solution $u(x) = 1+$ $\cos(\pi x)(Ae^{\pi x} + Be^{-\pi x})$- $\sin(\pi x)(Ae^{\pi x} - Be^{-\pi x})$), $A = \frac{1}{e^\pi - 1}$, $B = -\frac{e^\pi}{e^\pi - 1}$.
3. Come up with your own problem of this type.

2 Preliminaries

Without considering boundary conditions, we have for our strong form

$$\big(k(x)u''\big)'' - (p(x)u')' + \gamma(x)u = f(x), \qquad x \in (0,1).$$

We first multiply the bilinear and linear form by a test function v that exists in the same space V_h as u, then we integrate both forms. We must integrate the first term by parts twice to arrive at the correct variational form. After this integration by parts is performed, we arrive at

$$\begin{aligned}\big(k(1)u''(1)\big)' v(1) - \big(k(0)u''(0)\big)' v(0) - k(1)u''(1)v'(1) + k(0)u''(0)v'(0) \\ - p(1)u'(1)v(1) + p(0)u'(0)v(0) \\ + \int_0^1 k(x)u''v'' + p(x)u'v' + \gamma(x)uvdx = \int_0^1 f(x)vdx.\end{aligned}$$

Both sets of four boundary conditions eliminate the boundary terms in this variational form, so we can write our final variational form as

$$\int_0^1 k(x)u''v'' + p(x)u'v' + \gamma(x)uvdx = \int_0^1 f(x)vdx, \;\; \forall v \in V = \{H^2(0,1); v(0) = v(1) = 0\}$$

for the first set of boundary conditions with u_h being the best possible approximation to u and

$$\int_0^1 k(x)u''v'' + p(x)u'v' + \gamma(x)uvdx = \int_0^1 f(x)vdx, \;\; \forall v \in V = \{H^2(0,1); v(0) = v'(0) = v(1) = v'(1) = 0\}$$

for the second set of boundary conditions.

To solve this problem, we first substitute for u_h and v our vector of basis functions into the variational form. For both problems, we have

$$V^T \left[\underbrace{\int_0^1 k(x)\Phi''(x)\Phi''^T(x)dx}_{A_g^2} + \underbrace{\int_0^1 p(x)\Phi'(x)\Phi'^T(x)dx}_{A_g^1} + \underbrace{\int_0^1 \gamma(x)\Phi(x)\Phi^T(x)dx}_{A_g^0} \right] U = V^T \underbrace{\int_0^1 f\Phi(x)dx}_{F},$$

where A_g^2 is the global bending matrix, A_g^1 is the global stiffness matrix, and A_g^0 is the global mass matrix. To form these global matrices, we must first construct the elemental matrices. This is done through transformation from an arbitrary element $x \in [x_{i-1}, x_i]$ to a reference element $s \in [-1, 1]$. The transformation has been performed for the mass and stiffness elemental matrices in past assignments, but we have for the transformation of the bending matrix

$$\begin{aligned}\left(\frac{du_h}{dx}\right)' = \left(\frac{d\tilde{u_h}}{ds}\frac{ds}{dx}\right)' = \left(\frac{d^2\tilde{u_h}}{ds^2}\frac{ds}{dx}\right)\frac{ds}{dx} + \frac{d\tilde{u_h}}{ds}\cancelto{0}{\frac{d^2 s}{dx^2}} \\ \frac{d^2u_h}{dx^2} = u_h'' = \tilde{u_h}''\frac{4}{h^2}.\end{aligned}$$

$$u_h''^2 = \tilde{u_h}''^2 \frac{16}{h^4}.$$

If we substitute the transformed matrices, we have

$$V^T \left[\underbrace{\frac{h}{2}\frac{16}{h^4}\int_{-1}^{1} k(s)\Phi''(s)\Phi''^T(s)ds}_{A_e^2} + \underbrace{\frac{h}{2}\frac{4}{h^2}\int_{-1}^{1} p(s)\Phi'(s)\Phi'^T(s)ds}_{A_e^1} + \underbrace{\frac{h}{2}\int_{-1}^{1} \gamma(s)\Phi(s)\Phi^T(s)ds}_{A_e^0} \right] U$$
$$= V^T \underbrace{\frac{h}{2}\int_{-1}^{1} f\Phi(s)ds}_{F},$$

with

$$\Phi(s) = \begin{bmatrix} \left(\frac{1-s}{2}\right)^2 (2+s) \\ \left(\frac{1-s}{2}\right)^2 \left(\frac{s+1}{2}\right) h \\ \left(\frac{1+s}{2}\right)^2 (2+s) \\ \left(\frac{1+s}{2}\right)^2 \left(\frac{s-1}{2}\right) h \end{bmatrix}.$$

Our basis functions look as so, but on the element $[-1, 1]$:

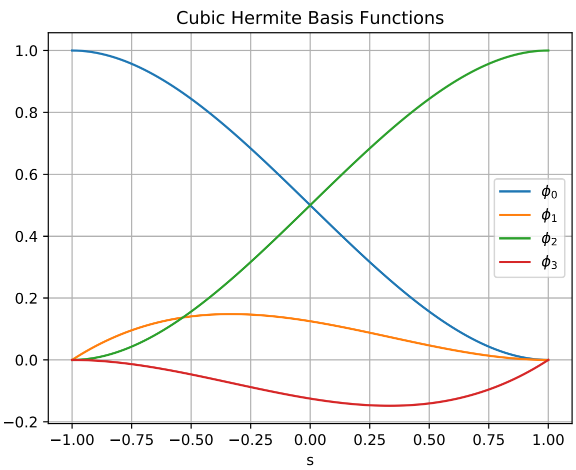

Figure 1: Cubic Hermite basis functions on the reference element $[-1, 1]$.

We also have h as the width of the element on the global domain and A_e^x as the elemental bending, stiffness, or mass matrix. Globalizing the matrices entails adding each successive 4x4 elemental matrix to the diagonal of the global matrix of zeros in the location where the first two diagonal elements overlap the last two diagonal elements of the the previous elemental matrix on the diagonal of the global matrix. In the case of Programming Assignment 3, only two degrees of freedom existed at each element boundary, so only the first and last elements of the successive elemental matrices overlapped. In this case, we have four DOFs at each element boundary, so we must construct the global matrices accordingly. For a two element domain, a global matrix would resemble

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & & \\ e_{21} & e_{22} & e_{23} & e_{24} & & \\ e_{31} & e_{32} & e_{33}+e_{11} & e_{34}+e_{12} & e_{13} & e_{14} \\ e_{41} & e_{42} & e_{43}+e_{21} & e_{44}+e_{22} & e_{23} & e_{24} \\ & & e_{31} & e_{32} & e_{33} & e_{34} \\ & & e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}.$$

For the right hand side, the same procedure is conducted for the 4 element column vectors. Once we have performed the globalization, we can solve for U in the system

$$\left(A_g^2 + A_g^1 + A_g^0\right) U = F.$$

However, the boundary conditions must be applied to account for the boundary terms in the variational form. In the first case, only the first row, first column, <u>second to last row and second to last column</u> of A_g must be zeroed. We must recognize that the solution vector U has length $2(N+1)$, as we have both a degree of freedom for the value at the node of one of the node's associated basis functions and a degree of freedom for the derivative of the node's other associated basis function at the node. Once we have zeroed these rows and columns, we must replace the diagonal elements of these rows and columns with ones. Finally, we must zero the first and second to last value in the F matrix. Then we can solve the system.

For the second case, both the value of u_h at the boundaries and the derivative of u_h at the boundaries must be zero, so we need to zero the first two rows, the first two columns, the last two rows, and the last two columns of A_g. The elements on the diagonal of the zeroed rows must be set to one, and the associated elements in the F matrix must be set to zero. The system can then be solved for U.

Finally, to compute the norms, we have

$$||e||_{L^2} = \sqrt{\int_0^1 \left(u_h(x) - w(x)\right)^2 dx},$$

$$||e||_{H^1} = \sqrt{\int_0^1 \left(u_h'(x) - w'(x)\right)^2 + \left(u_h(x) - w(x)\right)^2 dx},$$

$$||e||_{H^2} = \sqrt{\int_0^1 \left(u_h''(x) - w''(x)\right)^2 + \left(u_h'(x) - w'(x)\right)^2 + \left(u_h(x) - w(x)\right)^2 dx},$$

$$||e||_{L^\infty} = \max_x |u_h(x) - w(x)|.$$

where $w(x)$ is the exact analytical solution to the problem, and u_h is the linear combination of the constants in U with the basis functions. These norms are computed with 6 point Gauss Legendre quadrature across each element, which are each first transformed to the reference element [-1,1]. Then, the contributions from each element to the norms are summed over the entire domain. The derivative transformation takes place in the norm calculations as well, so the differences must also be multiplied by the correct factor of $2/h$ and the Jacobian $h/2$.

3 Problem 1 Results

For all problems in Problem 1, we have boundary conditions

$$u(0) = u''(0) = 0$$
$$u(1) = u''(1) = 0.$$

3.1 P1.1

Our conditions are

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = \sin(\pi x)$, (exact solution $u(x) = \frac{1}{\pi^4}\sin(\pi x)$).

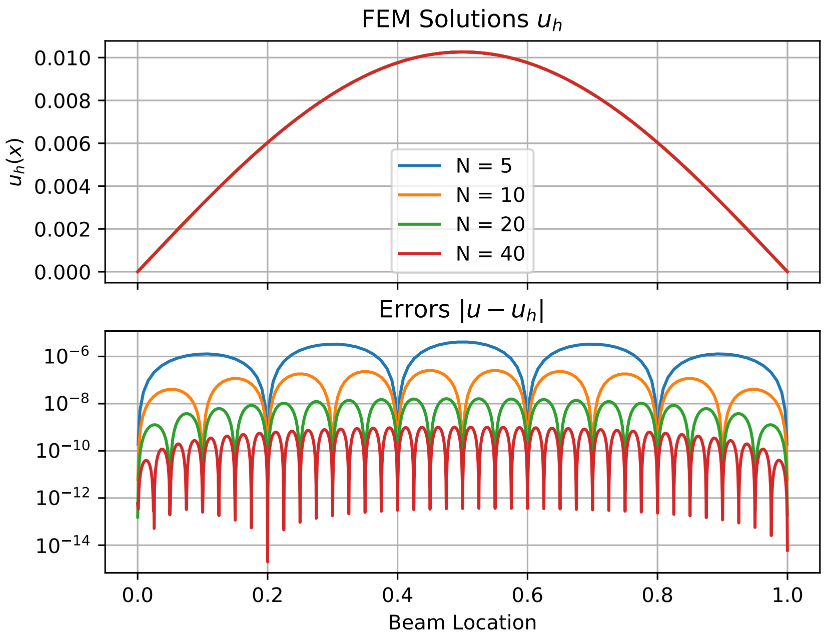

Figure 2: FEM approximations with their errors plotted at 20 quadrature points within each element.

Here, we see the errors are minimized at the nodes; this makes sense, as u_h itself is an interpolant based at the nodes. Also, the error for the 40 element discretization appears to reach an minimum at an arbitrary beam location of 0.2. However, this error value is of magnitude 10^{-14}, which is more precise than is possible with current machine precision. So, this arbitrary minimum should be given no significance.

Table 1: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	1.866e-06	3.239e-05	1.049e-03	3.681e-06	2.000e-01
10	1.172e-07	4.062e-06	2.632e-04	2.286e-07	1.000e-01
20	7.333e-09	5.081e-07	6.586e-05	1.443e-08	5.000e-02
40	4.585e-10	6.353e-08	1.647e-05	9.042e-10	2.500e-02

We see in the table that

$$||e||_{L^2} \leq ||e||_{L^\infty} \leq ||e||_{H^1} \leq ||e||_{H^2},$$

which is expected.

3.2 P1.2

Our conditions are

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = 60x$, (exact solution $u(x) = \frac{x}{6}(7 - 10x^2 + 3x^4)$).

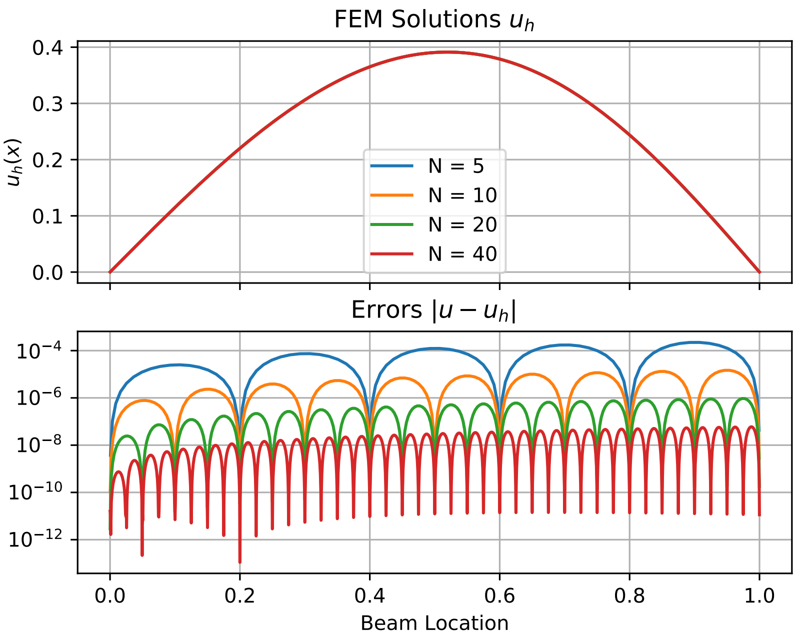

Figure 3: FEM approximations with their errors plotted at 20 quadrature points within each element.

We notice here we have a forcing function that is not symmetric over (0,1); our beam is experiencing more deformation on the right half of the beam. So, our errors generally rise with x.

Table 2: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	9.155e-05	1.589e-03	5.143e-02	2.012e-04	2.000e-01
10	5.743e-06	1.990e-04	1.290e-02	1.323e-05	1.000e-01
20	3.593e-07	2.490e-05	3.227e-03	8.478e-07	5.000e-02
40	2.246e-08	3.112e-06	8.068e-04	5.364e-08	2.500e-02

3.3 P1.3

Our conditions are

1. $k(x) = 1/(4\pi^4)$, $p(x) = 0$, $\gamma(x) = 1$, $f(x) = 1$, (exact solution $u(x) = 1 + \cos(\pi x)(Ae^{\pi x} + Be^{-\pi x})$), $A = \frac{1}{e^\pi - 1}$, $B = -\frac{e^\pi}{e^\pi - 1}$.

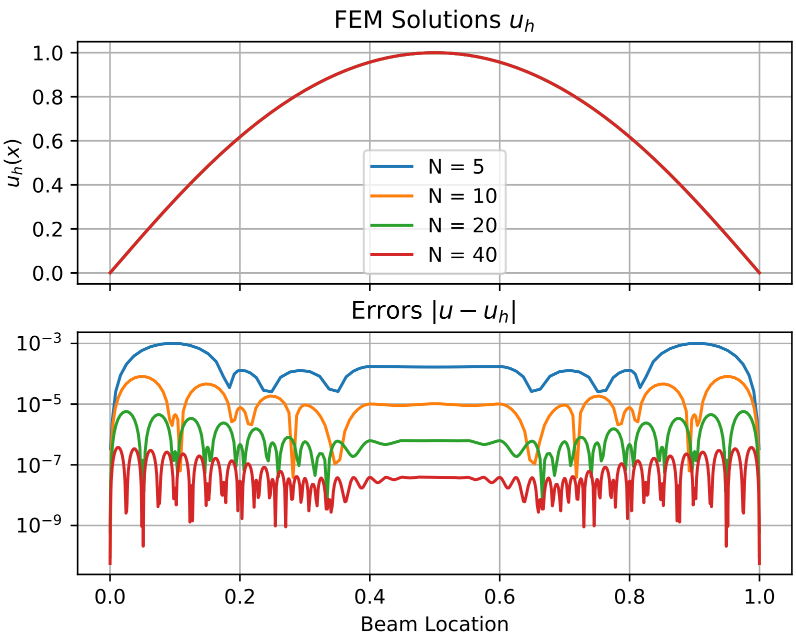

Figure 4: FEM approximations with their errors plotted at 20 quadrature points within each element.

In this problem, the constant on the 4th derivative problem is almost zero, so we should have

$$u_h(x) \approx f(x) = 1.$$

This is why we see a constant error when $u_h(x)$ is near 1 on the domain.

Table 3: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	4.140e-04	7.825e-03	2.560e-01	9.311e-04	2.000e-01
10	2.739e-05	1.021e-03	6.629e-02	7.388e-05	1.000e-01
20	1.735e-06	1.289e-04	1.671e-02	5.126e-06	5.000e-02
40	1.088e-07	1.615e-05	4.187e-03	3.364e-07	2.500e-02

4 Problem 2 Results

For problem 2, we have the boundary conditions

$$u(0) = u'(0) = 0$$
$$u(1) = u'(1) = 0.$$

4.1 P2.1

Our conditions are

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = \sin(\pi x)$, (exact solution $u(x) = \frac{x^2}{\pi^3} - \frac{x}{\pi^3} + \frac{1}{\pi^4}\sin(\pi x)$).

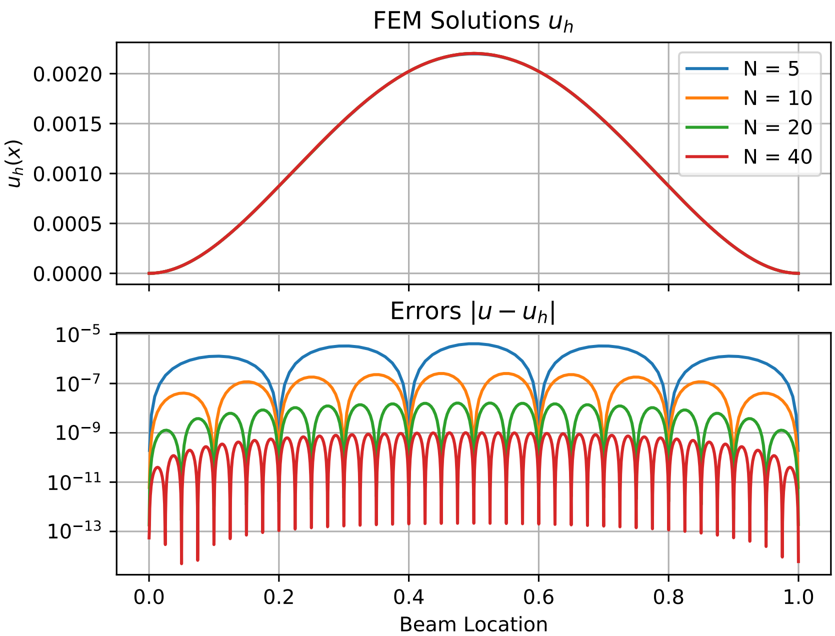

Figure 5: FEM approximations with their errors plotted at 20 quadrature points within each element.

Here we see evidence of the "clamped" ends of the beam, as the derivative is 0 at both ends. Also, we have a symmetric forcing function on (0,1), so the beam deformation is also symmetric.

Table 4: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	1.866e-06	3.239e-05	1.049e-03	3.681e-06	2.000e-01
10	1.172e-07	4.062e-06	2.632e-04	2.286e-07	1.000e-01
20	7.333e-09	5.081e-07	6.586e-05	1.443e-08	5.000e-02
40	4.585e-10	6.353e-08	1.647e-05	9.040e-10	2.500e-02

4.2 P2.2

Our conditions are

1. $k(x) = 1/(4\pi^4)$, $p(x) = 0$, $\gamma(x) = 1$, $f(x) = 1$, (exact solution $u(x) = 1+ \cos(\pi x)(Ae^{\pi x} + Be^{-\pi x})$-$\sin(\pi x)(Ae^{\pi x} - Be^{-\pi x})$), $A = \frac{1}{e^\pi - 1}$, $B = -\frac{e^\pi}{e^\pi - 1}$.

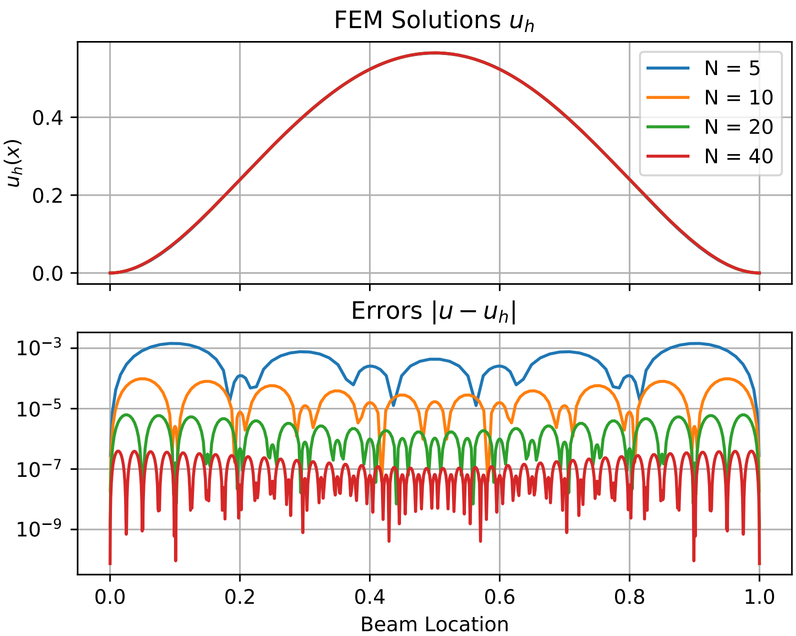

Figure 6: FEM approximations with their errors plotted at 20 quadrature points within each element.

This problem is identical to P1.3 with the exception of the boundary conditions. Here, the zero Dirichlet BC's limit the maximum of $u_h(x)$ to about 0.5 at the center of the domain, so we don't see the error plateau at $u_h(x)$ approaches its maximum.

Table 5: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	6.598e-04	1.298e-02	4.183e-01	1.308e-03	2.000e-01
10	4.142e-05	1.618e-03	1.047e-01	8.779e-05	1.000e-01
20	2.592e-06	2.021e-04	2.619e-02	5.599e-06	5.000e-02
40	1.620e-07	2.526e-05	6.548e-03	3.518e-07	2.500e-02

4.3 P2.3

For this problem, we are asked to create a new problem with the same boundary conditions at Problem 2. Our conditions are

1. $k(x) = 1$, $p(x) = \gamma(x) = 0$, $f(x) = 70x^2 - 5$, (exact solution $u(x) = \frac{1}{72}(x-1)^2x^2(14x^2 + 28x + 27)$).

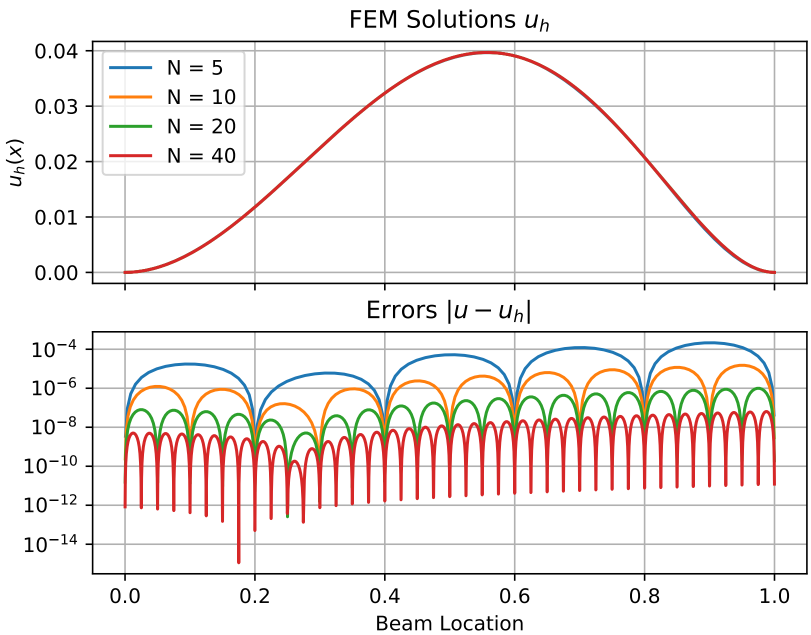

Figure 7: FEM approximations with their errors plotted at 20 quadrature points within each element.

This problem is similar to P1.1, as the forcing function is not symmetric on the domain. But, this figure also shows the tighter restriction of the maximum of $u_h(x)$ with the derivative zero Dirichlet boundary conditions, as the maximum is about one tenth the maximum of the solution in P1.1.

Table 6: Error norms for each domain resolution.

N	L^2	H^1	H^2	L^∞	h
5	7.251e-05	1.258e-03	4.076e-02	1.942e-04	2.000e-01
10	4.592e-06	1.591e-04	1.031e-02	1.355e-05	1.000e-01
20	2.879e-07	1.995e-05	2.586e-03	8.933e-07	5.000e-02
40	1.801e-08	2.495e-06	6.469e-04	5.731e-08	2.500e-02