Value Function Methods

CS 285

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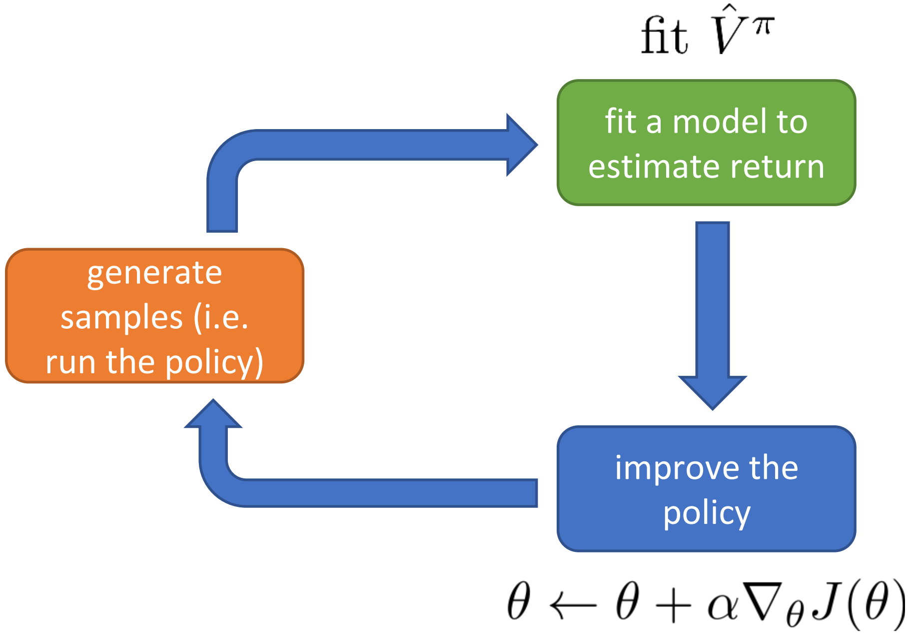
UC Berkeley

Recap: actor-critic

batch actor-critic algorithm:

1. sample $\{\mathbf{s}_i, \mathbf{a}_i\}$ from $\pi_\theta(\mathbf{a}|\mathbf{s})$ (run it on the robot)
2. fit $\hat{V}^\pi_\phi(\mathbf{s})$ to sampled reward sums
3. evaluate $\hat{A}^\pi(\mathbf{s}_i, \mathbf{a}_i) = r(\mathbf{s}_i, \mathbf{a}_i) + \hat{V}^\pi_\phi(\mathbf{s}'_i) - \hat{V}^\pi_\phi(\mathbf{s}_i)$
4. $\nabla_\theta J(\theta) \approx \sum_i \nabla_\theta \log \pi_\theta(\mathbf{a}_i|\mathbf{s}_i)\hat{A}^\pi(\mathbf{s}_i, \mathbf{a}_i)$
5. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$



Can we omit policy gradient completely?

$A^\pi(\mathbf{s}_t, \mathbf{a}_t)$: how much better is $\mathbf{a}_t$ than the average action according to π

$\arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t)$: best action from $\mathbf{s}_t$, if we then follow π

at *least* as good as any $\mathbf{a}_t \sim \pi(\mathbf{a}_t|\mathbf{s}_t)$

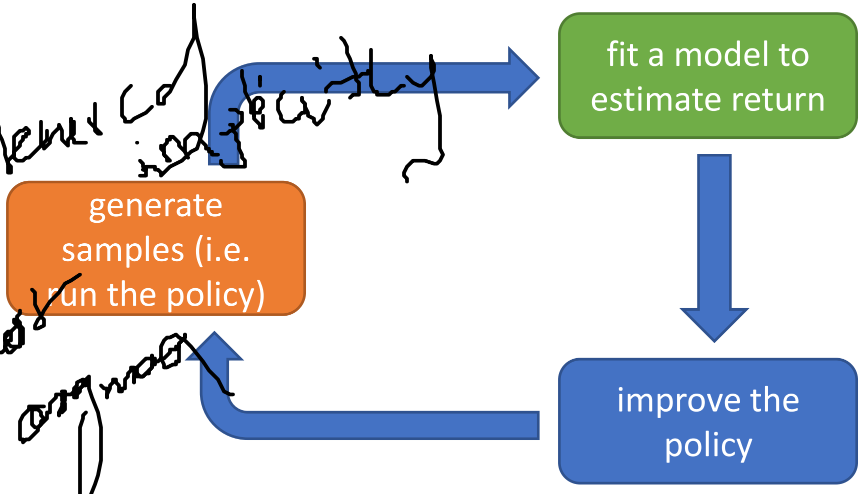
regardless of what $\pi(\mathbf{a}_t|\mathbf{s}_t)$ is!

forget policies, let's just do this!

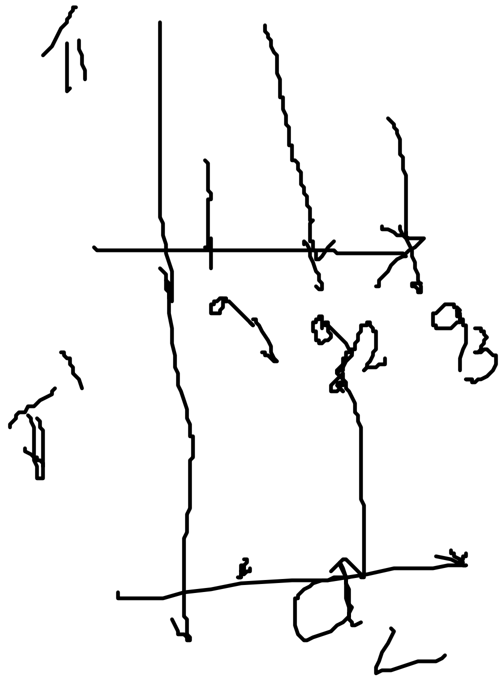
$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

as good as π
(probably better)

fit A^π (or Q^π or V^π)



$\pi \leftarrow \pi'$



Policy iteration

High level idea:

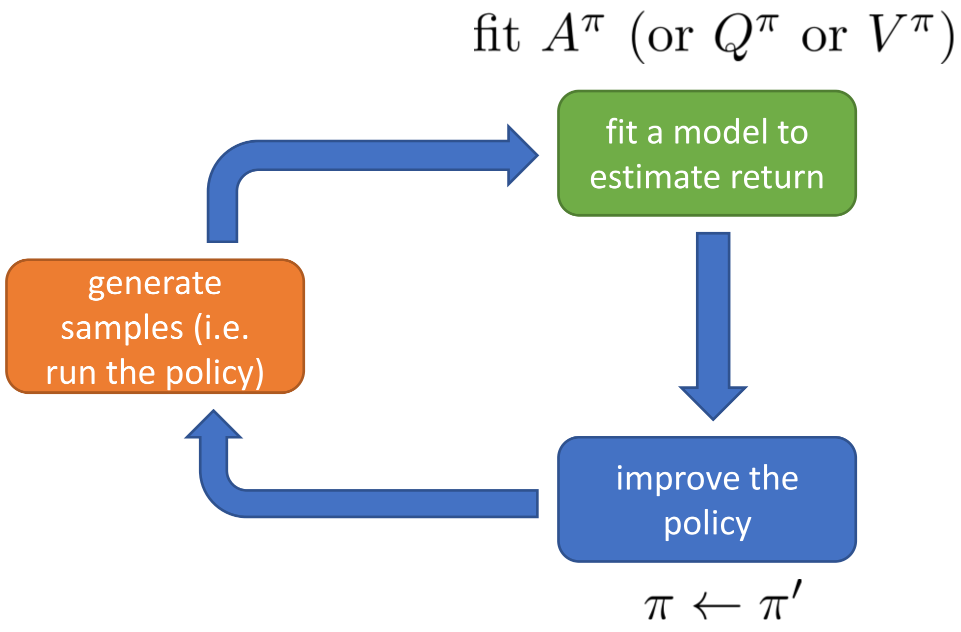
policy iteration algorithm:

1. evaluate $A^\pi(\mathbf{s}, \mathbf{a})$ $\longleftarrow$ how to do this?
2. set $\pi \leftarrow \pi'$

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

as before: $A^\pi(\mathbf{s}, \mathbf{a}) = r(\mathbf{s}, \mathbf{a}) + \gamma E[V^\pi(\mathbf{s}')] - V^\pi(\mathbf{s})$

let's evaluate $V^\pi(\mathbf{s})$!



Dynamic programming $\Rightarrow$ one way of evaluate $\pi(s)$

Let's assume we know $p(\mathbf{s}'|\mathbf{s},\mathbf{a})$, and $\mathbf{s}$ and $\mathbf{a}$ are both discrete (and small)

0.2	0.3	0.4	0.3
0.3	0.3	0.5	0.3
0.4	0.4	0.6	0.4
0.5	0.5	0.7	0.5

16 states, 4 actions per state

can store full $V^\pi(\mathbf{s})$ in a table!

$\mathcal{T}$ is $16 \times 16 \times 4$ tensor

bootstrapped update: $V^\pi(\mathbf{s}) \leftarrow E_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}[r(\mathbf{s},\mathbf{a}) + \gamma E_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}[V^\pi(\mathbf{s}')]]$

just use the current estimate here

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t,\mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

deterministic policy $\pi(\mathbf{s}) = \mathbf{a}$

No. $E_a \sim \pi(a|s)$

simplified: $V^\pi(\mathbf{s}) \leftarrow r(\mathbf{s},\pi(\mathbf{s})) + \gamma E_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\pi(\mathbf{s}))}[V^\pi(\mathbf{s}')]$

Policy iteration with dynamic programming

policy iteration:

1. evaluate $V^\pi(\mathbf{s})$
2. set $\pi \leftarrow \pi'$

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

policy evaluation:

$$V^\pi(\mathbf{s}) \leftarrow r(\mathbf{s}, \pi(\mathbf{s})) + \gamma E_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\pi(\mathbf{s}))}[V^\pi(\mathbf{s}')]$$

0.2	0.3	0.4	0.3
0.3	0.3	0.5	0.3
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16 states, 4 actions per state

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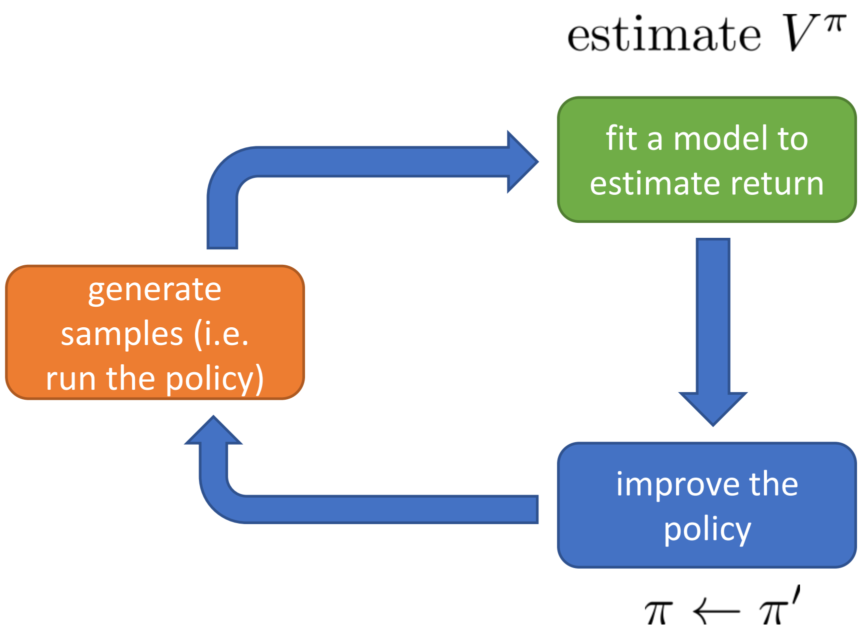


Even simpler dynamic programming

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

$$A^\pi(\mathbf{s},\mathbf{a}) = r(\mathbf{s},\mathbf{a}) + \gamma E[V^\pi(\mathbf{s}')] - V^\pi(\mathbf{s})$$

$$\arg\max_{\mathbf{a}_t} A^\pi(\mathbf{s}_t, \mathbf{a}_t) = \arg\max_{\mathbf{a}_t} Q^\pi(\mathbf{s}_t, \mathbf{a}_t)$$

$$Q^\pi(\mathbf{s},\mathbf{a}) = r(\mathbf{s},\mathbf{a}) + \gamma E[V^\pi(\mathbf{s}')] \quad \text{(a bit simpler)}$$

skip the policy and compute values directly!

value iteration algorithm:

1. set $Q(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q(\mathbf{s},\mathbf{a})$

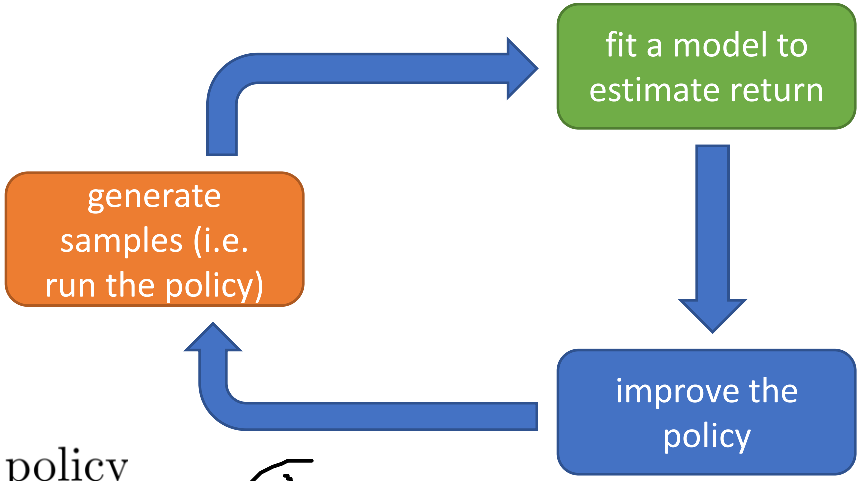
$\mathbf{a}$

$\mathbf{s}$				
	$Q(\mathbf{s},\mathbf{a})$		$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$
	$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$		$Q(\mathbf{s},\mathbf{a})$
		$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$
	$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$	
	$Q(\mathbf{s},\mathbf{a})$		$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$
	$Q(\mathbf{s},\mathbf{a})$	$Q(\mathbf{s},\mathbf{a})$		$Q(\mathbf{s},\mathbf{a})$

$$\arg\max_{\mathbf{a}} Q(\mathbf{s},\mathbf{a}) \rightarrow \text{policy}$$

approximates the new value!

$$Q^\pi(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma E_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}[V^\pi(\mathbf{s}')]$$



$$V^\pi(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q^\pi(\mathbf{s},\mathbf{a})$$

max Q

Q(s,a)

$\pi(s) \leftarrow \arg\max_a Q(s,a)$

$$Q(s,a) \rightarrow \arg\max_a Q(s,a) \rightarrow \pi$$

$$Q(s,a) = r(s,a) + \gamma\left[\max_{a'} Q(s,a)\right]$$

$$\max_a Q(s,a) \xrightarrow{\text{update}} V(s)$$

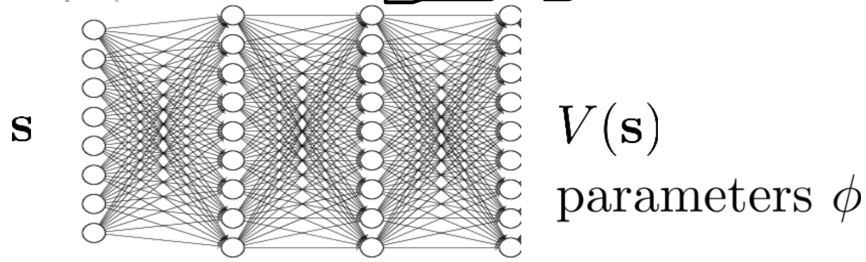
Fitted Value Iteration & Q-Iteration

Fitted value iteration

how do we represent $V(\mathbf{s})$?

big table, one entry for each discrete $\mathbf{s}$

neural net function $\boxed{V : \mathcal{S} \rightarrow \mathbb{R}}$



$$\mathcal{L}(\phi) = \frac{1}{2}\left\| V_\phi(\mathbf{s}) - \max_{\mathbf{a}} Q^\pi(\mathbf{s},\mathbf{a}) \right\|^2$$

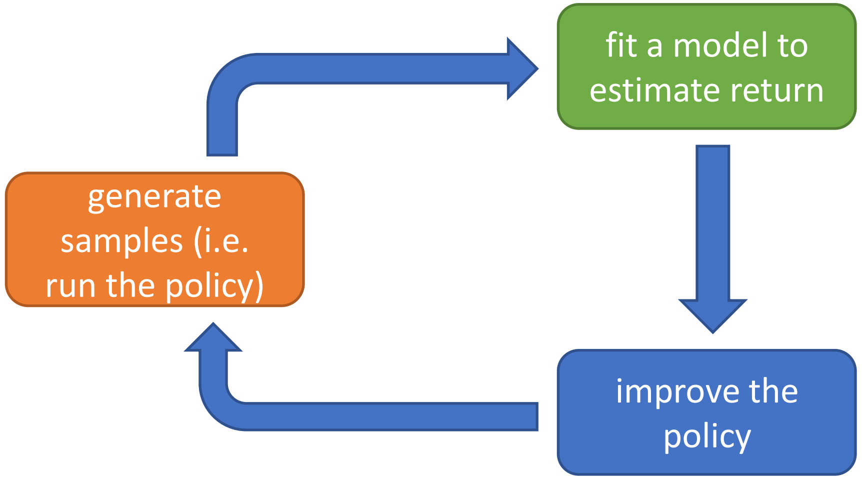
$$\mathbf{s} = 0 : V(\mathbf{s}) = 0.2$$
$$\mathbf{s} = 1 : V(\mathbf{s}) = 0.3$$
$$\mathbf{s} = 2 : V(\mathbf{s}) = 0.5$$

$$|\mathcal{S}| = (255^3)^{200\times 200}$$

(more than atoms in the universe)

curse of dimensionality

$$Q^\pi(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma E_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}[V^\pi(\mathbf{s}')]$$



$$V^\pi(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q^\pi(\mathbf{s},\mathbf{a})$$

fitted value iteration algorithm:

1. set $\mathbf{y}_i \leftarrow \max_{\mathbf{a}_i}(r(\mathbf{s}_i,\mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}_i')])$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2}\sum_i \|V_\phi(\mathbf{s}_i) - \mathbf{y}_i\|^2$

What if we don’t know the transition dynamics?

fitted value iteration algorithm:

1. set $\mathbf{y}_i \leftarrow \max_{\mathbf{a}_i}(r(\mathbf{s}_i, \mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}_i')])$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|V_\phi(\mathbf{s}_i) - \mathbf{y}_i\|^2$

need to know outcomes for different actions!

Back to policy iteration…

policy iteration:

1. evaluate $Q^\pi(\mathbf{s}, \mathbf{a})$
2. set $\pi \leftarrow \pi'$

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} Q^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

policy evaluation:

$$\cancel{V^\pi(\mathbf{s}) \leftarrow r(\mathbf{s}, \pi(\mathbf{s})) + \gamma E_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s}, \pi(\mathbf{s}))}[V^\pi(\mathbf{s}')]}$$

you don't need this

$$Q^\pi(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \gamma E_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s}, \mathbf{a})}[Q^\pi(\mathbf{s}', \pi(\mathbf{s}'))]$$

a

can fit this using samples

Can we do the "max" trick again?

policy iteration:

1. evaluate $V^\pi(\mathbf{s})$
2. set $\pi \leftarrow \pi'$

fitted value iteration algorithm:

1. set $\mathbf{y}_i \leftarrow \max_{\mathbf{a}_i}(r(\mathbf{s}_i, \mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}_i')])$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|V_\phi(\mathbf{s}_i) - \mathbf{y}_i\|^2$

forget policy, compute value directly

can we do this with Q-values **also**, without knowing the transitions?

fitted Q iteration algorithm:

1. set $\mathbf{y}_i \leftarrow r(\mathbf{s}_i, \mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}_i')]$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i\|^2$

approxiate $E[V(\mathbf{s}_i')] \approx \max_{\mathbf{a}'} Q_\phi(\mathbf{s}_i', \mathbf{a}_i')$

doesn't require simulation of actions!

$s_i\ a_i\ s'$

+ works even for off-policy samples (unlike actor-critic)

+ only one network, no high-variance policy gradient

- no convergence guarantees for non-linear function approximation (more on this later)

Fitted Q-iteration

full fitted Q-iteration algorithm:

1. collect dataset $\{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)\}$ using some policy

$K\times$

2. set $\mathbf{y}_i \leftarrow \underline{r(\mathbf{s}_i, \mathbf{a}_i)} + \gamma \underline{\max_{\mathbf{a}'_i} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)}$
3. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i\|^2$

parameters

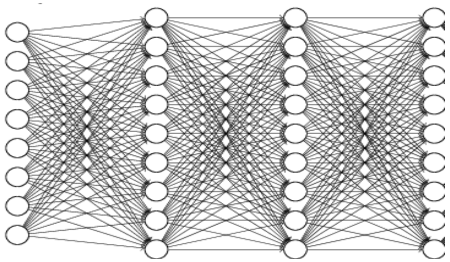
dataset size N, collection policy

iterations K

gradient steps S

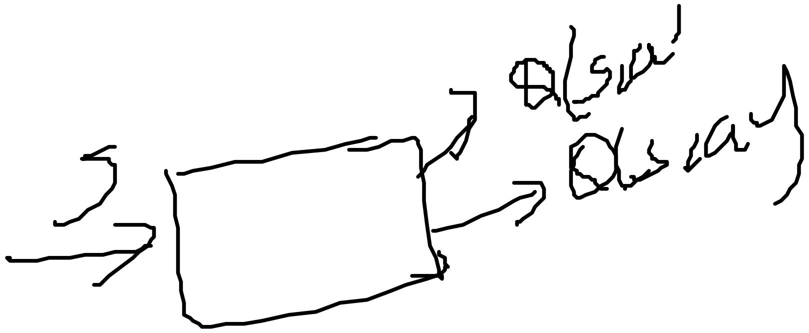
$\mathbf{s}$

$\mathbf{a}$



$Q_\phi(\mathbf{s}, \mathbf{a})$

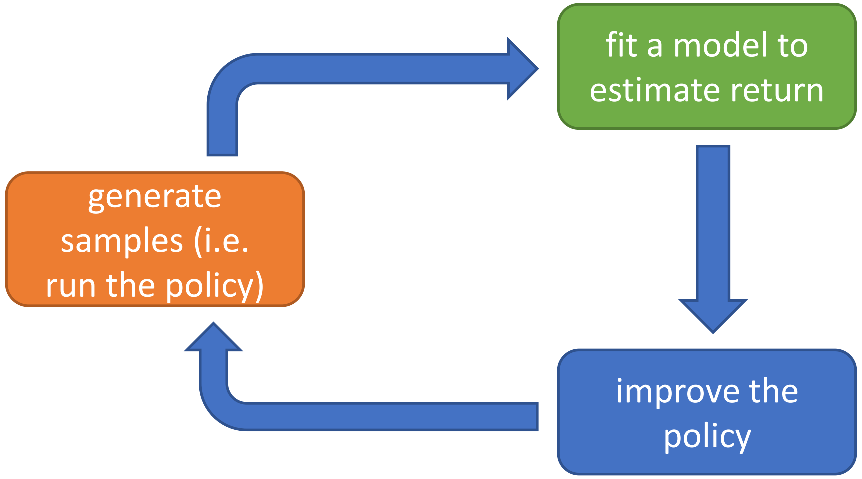
parameters ϕ



Review

- Value-based methods
  - Don't learn a policy explicitly
  - Just learn value or Q-function
- If we have value function, we have a policy
- Fitted Q-iteration

$$Q_\phi(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}',\mathbf{a}')$$



$$\mathbf{a} = \arg\max_{\mathbf{a}} Q_\phi(\mathbf{s},\mathbf{a})$$

From Q-Iteration to Q-Learning

Why is this algorithm off-policy?

full fitted Q-iteration algorithm:

$K\times$

1. collect dataset $\{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)\}$ using some policy
2. set $\mathbf{y}_i \leftarrow r(\mathbf{s}_i, \mathbf{a}_i) + \gamma \max_{\mathbf{a}'_i} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)$
3. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i\|^2$

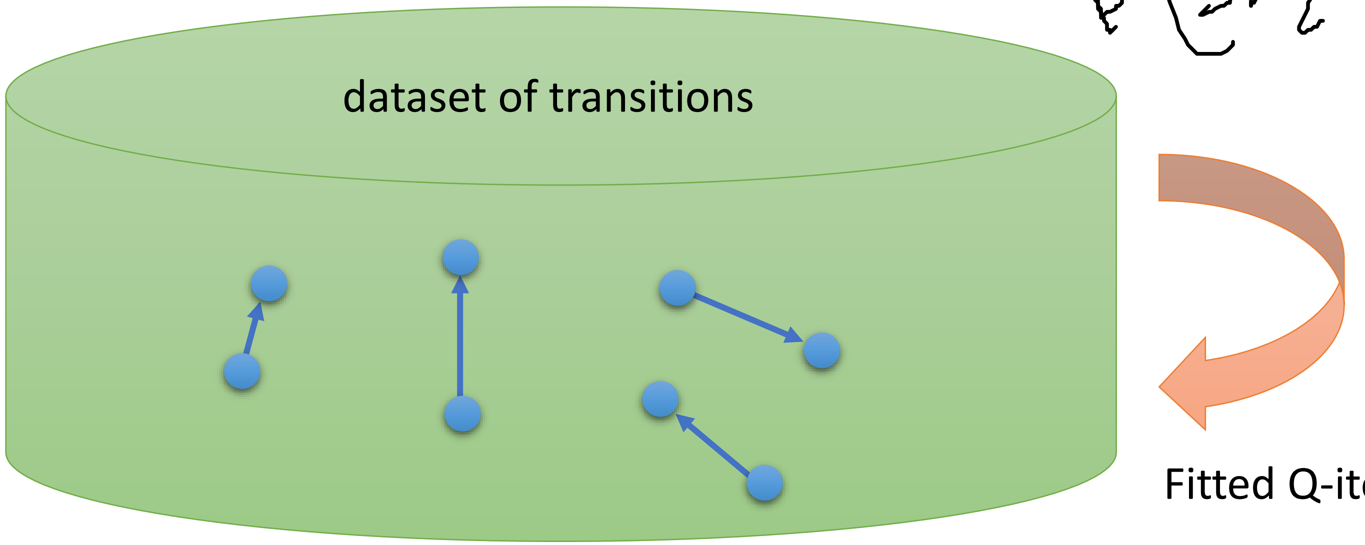
$\pi(a_t|s_t) = \begin{cases} 1 & a_t = \arg\max_{a_t} Q^\pi(s_t) \\ 0 & \end{cases}$

given $\mathbf{s}$ *and* $\mathbf{a}$, transition is independent of π

this approximates the value of π' at $\mathbf{s}'_i$

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} Q^\pi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

$Q_\phi(s_i', \underbrace{\arg\max_a Q_\phi(s_i', a)}_{\text{policy}})$



Fitted Q-iteration

What is fitted Q-iteration optimizing?

full fitted Q-iteration algorithm:

1. collect dataset $\{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)\}$ using some policy

$K\times$

2. set $\mathbf{y}_i \leftarrow r(\mathbf{s}_i, \mathbf{a}_i) + \gamma \max_{\mathbf{a}'_i} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)$ ← this max improves the policy (tabular case)
3. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i\|^2$

error $\mathcal{E}$

Bellman Err

$$\mathcal{E} = \frac{1}{2} E_{(\mathbf{s},\mathbf{a})\sim\beta}\left[\left(Q_\phi(\mathbf{s}, \mathbf{a}) - [r(\mathbf{s}, \mathbf{a}) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}', \mathbf{a}')]\right)^2\right]$$

if $\mathcal{E} = 0$, then $Q_\phi(\mathbf{s}, \mathbf{a}) = r(\mathbf{s}, \mathbf{a}) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}', \mathbf{a}')$

this is an *optimal* Q-function, corresponding to optimal policy π':

$$\pi'(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} Q_\phi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

maximizes reward

sometimes written $Q^\star$ and $\pi^\star$

most guarantees are lost when we leave the tabular case (e.g., use neural networks)

Online Q-learning algorithms

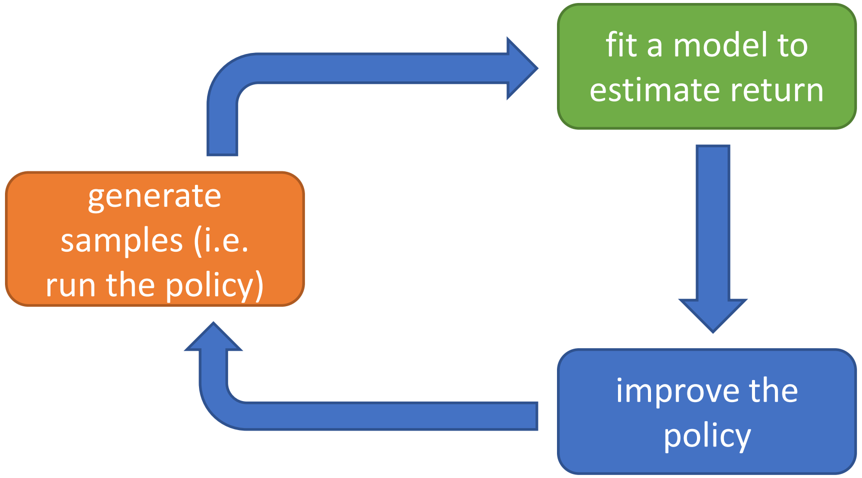
$Q_\phi(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}',\mathbf{a}')$

full fitted Q-iteration algorithm:

1. collect dataset $\{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)\}$ using some policy

$K\times$

2. set $\mathbf{y}_i \leftarrow r(\mathbf{s}_i,\mathbf{a}_i) + \gamma \max_{\mathbf{a}'_i} Q_\phi(\mathbf{s}'_i,\mathbf{a}'_i)$
3. set $\phi \leftarrow \arg\min_\phi \frac{1}{2}\sum_i \|Q_\phi(\mathbf{s}_i,\mathbf{a}_i) - \mathbf{y}_i\|^2$



$\mathbf{a} = \arg\max_{\mathbf{a}} Q_\phi(\mathbf{s},\mathbf{a})$

off policy, so many choices here!

online Q iteration algorithm:

1. take some action $\mathbf{a}_i$ and observe $(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)$
2. $\mathbf{y}_i = r(\mathbf{s}_i,\mathbf{a}_i) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}'_i,\mathbf{a}'_i)$
3. $\phi \leftarrow \phi - \alpha \frac{dQ_\phi}{d\phi}(\mathbf{s}_i,\mathbf{a}_i)(Q_\phi(\mathbf{s}_i,\mathbf{a}_i) - \mathbf{y}_i)$

you can pick a_i → anything you want

Exploration with Q-learning

online Q iteration algorithm:

1. take some action $\mathbf{a}_i$ and observe $(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)$
2. $\mathbf{y}_i = r(\mathbf{s}_i, \mathbf{a}_i) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)$
3. $\phi \leftarrow \phi - \alpha \frac{dQ_\phi}{d\phi}(\mathbf{s}_i, \mathbf{a}_i)(Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i)$

final policy:

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} Q_\phi(\mathbf{s}_t, \mathbf{a}_t) \\ 0 \text{ otherwise} \end{cases}$$

→ exploit No exploration

why is this a bad idea for step 1?

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \begin{cases} 1 - \epsilon \text{ if } \mathbf{a}_t = \arg\max_{\mathbf{a}_t} Q_\phi(\mathbf{s}_t, \mathbf{a}_t) \\ \epsilon/(|\mathcal{A}| - 1) \text{ otherwise} \end{cases}$$

"epsilon-greedy"

$$\pi(\mathbf{a}_t|\mathbf{s}_t) \propto \exp(Q_\phi(\mathbf{s}_t, \mathbf{a}_t))$$

"Boltzmann exploration"

Better

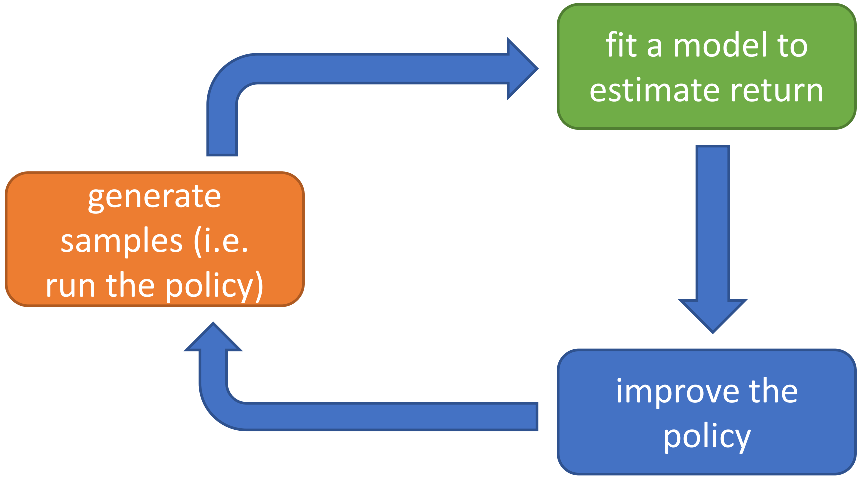
softmax

We'll discuss exploration in detail in a later lecture!

Review

- Value-based methods
  - Don't learn a policy explicitly
  - Just learn value or Q-function
- If we have value function, we have a policy
- Fitted Q-iteration
  - Batch mode, off-policy method
- Q-learning
  - Online analogue of fitted Q-iteration

$$Q_\phi(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}',\mathbf{a}')$$



$$\mathbf{a} = \arg\max_{\mathbf{a}} Q_\phi(\mathbf{s},\mathbf{a})$$

Value Functions in Theory

Value function learning theory

with no V based don't converge

value iteration algorithm:

1. set $Q(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q(\mathbf{s},\mathbf{a})$

0.2	0.3	0.4	0.3
0.3	0.3	0.5	0.3
0.4	0.4	0.6	0.4
0.5	0.5	0.7	0.5

does it converge? and if so, to what?

Bellman

stacked vector of rewards at all states for action $\mathbf{a}$

define an operator $\mathcal{B}$: $\underline{\mathcal{B}V} = \max_{\mathbf{a}} r_{\mathbf{a}} + \gamma \mathcal{T}_{\mathbf{a}} V$

$T_{A,j,i}$

$V^\star = \mathcal{B}V^\star$

matrix of transitions for action $\mathbf{a}$ such that $\mathcal{T}_{\mathbf{a},i,j} = p(\mathbf{s}' = i | \mathbf{s} = j, \mathbf{a})$

$P(s|s')$

$V^\star$ is a *fixed point* of $\mathcal{B}$ $\quad V^\star(\mathbf{s}) = \max_{\mathbf{a}} r(\mathbf{s},\mathbf{a}) + \gamma E[V^\star(\mathbf{s}')]$, so $V^\star = \mathcal{B}V^\star$

always exists, is always unique, always corresponds to the optimal policy

...but will we reach it?

S

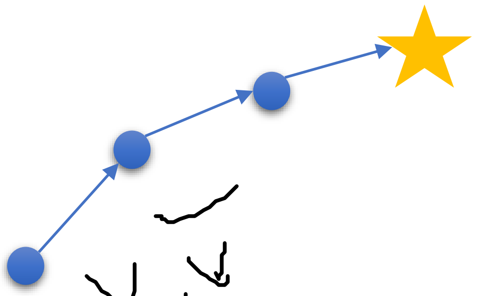
Value function learning theory

value iteration algorithm:

1. set $Q(\mathbf{s},\mathbf{a}) \leftarrow r(\mathbf{s},\mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q(\mathbf{s},\mathbf{a})$

0.2	0.3	0.4	0.3
0.3	0.3	0.5	0.3
0.4	0.4	0.6	0.4
0.5	0.5	0.7	0.5

$V^\star$ is a *fixed point* of $\mathcal{B}$ $\quad$ $V^\star(\mathbf{s}) = \max_{\mathbf{a}} r(\mathbf{s},\mathbf{a}) + \gamma E[V^\star(\mathbf{s}')]$, so $V^\star = \mathcal{B}V^\star$



we can prove that value iteration reaches $V^\star$ because $\mathcal{B}$ is a *contraction*

contraction: for any V and $\bar{V}$, we have $\|\mathcal{B}V - \mathcal{B}\bar{V}\|_\infty \leq \gamma\|V - \bar{V}\|_\infty$

gap always gets smaller by γ!
(with respect to ∞-norm)

$V, \bar{V}$

$\underbrace{\|\mathcal{B}V - \mathcal{B}\bar{V}\|}_{\max} \leq \gamma \underbrace{\|V - \bar{V}\|}_{\max}$

what if we choose $V^\star$ as $\bar{V}$? $\mathcal{B}V^\star = V^\star$!

$\|\mathcal{B}V - V^\star\|_\infty \leq \gamma\|V - V^\star\|_\infty$ $\quad$ $\|\mathcal{B}V - V^\star\| \leq \gamma\|V - V^\star\|$

Non-tabular value function learning

value iteration algorithm (using $\mathcal{B}$):

1. $V \leftarrow \mathcal{B}V$

fitted value iteration algorithm (using $\mathcal{B}$ and Π):

1. $V \leftarrow \Pi\mathcal{B}V$

projection

fitted value iteration algorithm:

1. set $\mathbf{y}_i \leftarrow \max_{\mathbf{a}_i}(r(\mathbf{s}_i, \mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}'_i)])$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2}\sum_i \|V_\phi(\mathbf{s}_i) - \mathbf{y}_i\|^2$

what does this do?

define new operator Π: $\Pi V = \arg\min_{V' \in \Omega} \frac{1}{2}\sum \|V'(\mathbf{s}) - V(\mathbf{s})\|^2$

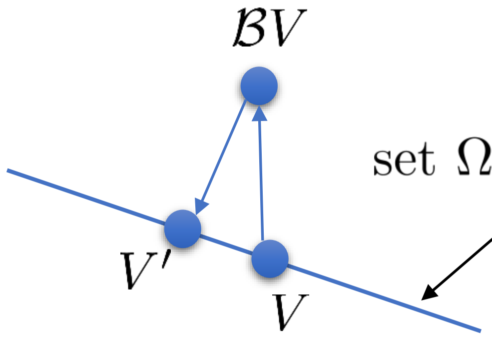
Π is a *projection* onto Ω (in terms of ℓ_2 norm)

updated value function

$$V' \leftarrow \arg\min_{V' \in \Omega} \frac{1}{2}\sum \|V'(\mathbf{s}) - (\mathcal{B}V)(\mathbf{s})\|^2$$

→ hypothesis space

all value functions represented by, e.g., neural nets

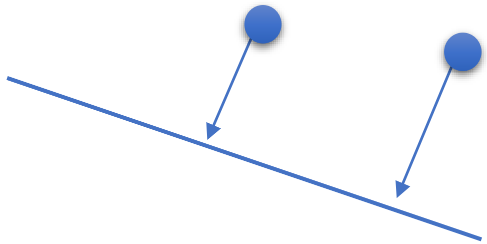


set Ω (e.g., neural nets)

Non-tabular value function learning

fitted value iteration algorithm (using $\mathcal{B}$ and Π):

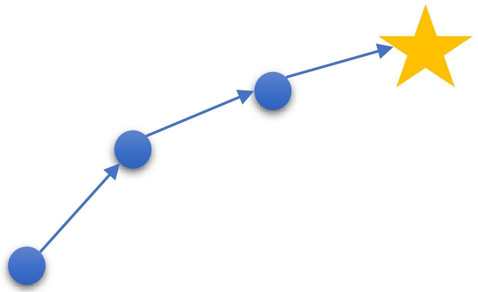
1. $V \leftarrow \Pi \mathcal{B} V$

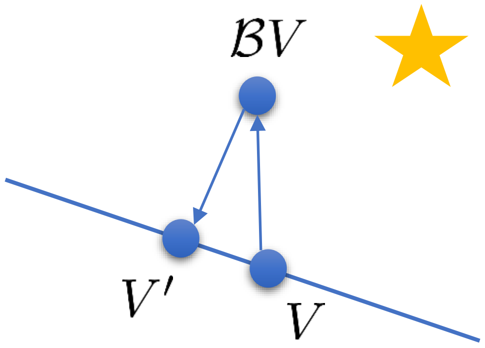


$\mathcal{B}$ is a contraction w.r.t. ∞-norm ("max" norm) $\qquad \|\mathcal{B}V - \mathcal{B}\bar{V}\|_\infty \leq \gamma \|V - \bar{V}\|_\infty$

Π is a contraction w.r.t. ℓ_2-norm (Euclidean distance) $\qquad \|\Pi V - \Pi \bar{V}\|^2 \leq \|V - \bar{V}\|^2$

but… $\Pi \mathcal{B}$ is not a contraction of any kind





Conclusions:

value iteration converges (tabular case)

fitted value iteration does **not** converge

not in general

often not in practice

What about fitted Q-iteration?

fitted Q iteration algorithm:

1. set $\mathbf{y}_i \leftarrow r(\mathbf{s}_i, \mathbf{a}_i) + \gamma E[V_\phi(\mathbf{s}_i')]$
2. set $\phi \leftarrow \arg\min_\phi \frac{1}{2} \sum_i \|Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i\|^2$

define an operator $\mathcal{B}$: $\mathcal{B}Q = r + \gamma \mathcal{T} \max_{\mathbf{a}} Q$

max now after the transition operator

define an operator Π: $\Pi Q = \arg\min_{Q' \in \Omega} \frac{1}{2} \sum \|Q'(\mathbf{s}, \mathbf{a}) - Q(\mathbf{s}, \mathbf{a})\|^2$

fitted Q-iteration algorithm (using $\mathcal{B}$ and Π):

1. $Q \leftarrow \Pi \mathcal{B} Q$

$\mathcal{B}$ is a contraction w.r.t. ∞-norm ("max" norm)

Π is a contraction w.r.t. ℓ_2-norm (Euclidean distance)

$\Pi \mathcal{B}$ is not a contraction of any kind Applies also to online Q-learning

But… it's just regression!

online Q iteration algorithm:

1. take some action $\mathbf{a}_i$ and observe $(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)$
2. $\mathbf{y}_i = r(\mathbf{s}_i, \mathbf{a}_i) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)$
3. $\phi \leftarrow \phi - \alpha \frac{dQ_\phi}{d\phi}(\mathbf{s}_i, \mathbf{a}_i)(Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - \mathbf{y}_i)$

Q learning is not gradient descent

isn't this just gradient descent? that converges, right?

Q-learning is *not* gradient descent!

target value

$$\phi \leftarrow \phi - \alpha \frac{dQ_\phi}{d\phi}(\mathbf{s}_i, \mathbf{a}_i)(Q_\phi(\mathbf{s}_i, \mathbf{a}_i) - [r(\mathbf{s}_i, \mathbf{a}_i) + \gamma \max_{\mathbf{a}'} Q_\phi(\mathbf{s}'_i, \mathbf{a}'_i)])$$

no gradient through target value

residual algorithm

Q learn

A sad corollary

batch actor-critic algorithm:

1. sample $\{\mathbf{s}_i, \mathbf{a}_i\}$ from $\pi_\theta(\mathbf{a}|\mathbf{s})$ (run it on the robot)
2. fit $\hat{V}^\pi_\phi(\mathbf{s})$ to sampled reward sums
3. evaluate $\hat{A}^\pi(\mathbf{s}_i, \mathbf{a}_i) = r(\mathbf{s}_i, \mathbf{a}_i) + \hat{V}^\pi_\phi(\mathbf{s}'_i) - \hat{V}^\pi_\phi(\mathbf{s}_i)$
4. $\nabla_\theta J(\theta) \approx \sum_i \nabla_\theta \log \pi_\theta(\mathbf{a}_i|\mathbf{s}_i) \hat{A}^\pi(\mathbf{s}_i, \mathbf{a}_i)$
5. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

ℓ_∞ contraction $\mathcal{B}$ (but without max)

$$y_{i,t} \approx r(\mathbf{s}_{i,t}, \mathbf{a}_{i,t}) + \gamma \hat{V}^\pi_\phi(\mathbf{s}_{i,t+1})$$

$$\mathcal{L}(\phi) = \frac{1}{2} \sum_i \left\| \hat{V}^\pi_\phi(\mathbf{s}_i) - y_i \right\|^2$$

ℓ_2 contraction Π

fitted bootstrapped policy evaluation doesn't converge!

An aside regarding terminology

V^π: value function for policy π
this is what the critic ~~does~~

$V^\star$: value function for optimal policy $\pi^\star$
this is what value iteration ~~does~~

Review

- Value iteration theory
  - Operator for backup
  - Operator for projection
  - Backup is contraction
  - Value iteration converges
- Convergence with function approximation
  - Projection is also a contraction
  - Projection + backup is **not** a contraction
  - Fitted value iteration does not in general converge
- Implications for Q-learning
  - Q-learning, fitted Q-iteration, etc. does not converge with function approximation
- But we can make it work in practice!
  - Sometimes – tune in next time