Homework 08 Root Finding

1. Use zero-, first-, second- and third-order Taylor series expansions to predict $f(2)$

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

```
%Initialization Workspace
clear
clc
close
hold on
lineStyle = {':','-','--'};

%Initial Function
syms x
function_x(x) = 25*(x^3) - 6*(x^2) + 7*x - 88
```

function_x(x) = $25\,x^3 - 6\,x^2 + 7\,x - 88$

```
trueValue_2 = function_x(2)
```

trueValue_2 = 102

Using a base point at x = 1 and step size = 0.1, 0.5, 1 with **Backward Divided Difference.**

$$f^{(n)}\left(x\right) = \frac{\nabla_h^n[f](x)}{h^n} = \frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x-kh),$$

```
%Initial Condition
STEP_SIZE = [0.1, 0.5, 1];
x_i = 1;
x_target = 2;
```

Calculate the error for each of the Taylor series expansions

$$f\left(x_{i+1}\right) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_i)}{n!}(x_{i+1}-x_i)^n,$$

```
%Error
syms trueValue approximationValue
error(trueValue,approximationValue) = ((trueValue - approximationValue)/trueValue);
```

$$\text{error} = \frac{\text{trueValue} - \text{approximatedValue}}{\text{trueValue}}$$

Zero-Order Taylor Series Expansion

$$f(x_{i+1}) \cong f(x_i)$$

```
order = 0;

for i = 1:size(STEP_SIZE,2)
    h = STEP_SIZE(i);

    approximationValue = taylorSeries_dividedDifference_backward(function_x, ...
        x_i,x_target,order,h);

    l0 = plot([x_i x_target],[function_x(x_i) approximationValue], 'r', 'LineWidth', 1.5);
    plot(x_target,approximationValue, 'o', 'MarkerFaceColor', 'r','MarkerEdgeColor','none')

    fprintf(['Approximation Value (Taylor + Backward Zero-Order Step %0.2f)' ...
        '  : %0.2f \t error : %0.2f\n'], ...
        STEP_SIZE(i), approximationValue, error(trueValue_2,approximationValue))
end
```

```
Approximation Value (Taylor + Backward Zero-Order Step 0.10) : -62.00   error : 1.61
Approximation Value (Taylor + Backward Zero-Order Step 0.50) : -62.00   error : 1.61
Approximation Value (Taylor + Backward Zero-Order Step 1.00) : -62.00   error : 1.61
```

One-Order Taylor Series Expansion

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

```
order = 1;
for i = 1:size(STEP_SIZE,2)
    h = STEP_SIZE(i);

    approximationValue = taylorSeries_dividedDifference_backward(function_x, ...
        x_i,x_target,order,h);

    l1(i) = plot([x_i x_target],[function_x(x_i) approximationValue], 'g', 'LineWidth', 0.5*i);
    plot(x_target,approximationValue, 'o', 'MarkerFaceColor', 'g','MarkerEdgeColor','none')

    fprintf(['Approximation Value (Taylor + Backward First-Order Step %0.2f)' ...
        '  : %0.2f \t error : %0.2f\n'], ...
        STEP_SIZE(i), approximationValue, error(trueValue_2,approximationValue))
end
```

```
Approximation Value (Taylor + Backward First-Order Step 0.10) : 1.35   error : 0.99
Approximation Value (Taylor + Backward First-Order Step 0.50) : -20.25   error : 1.20
Approximation Value (Taylor + Backward First-Order Step 1.00) : -36.00   error : 1.35
```

Second-Order Taylor Series Expansion

$$f\left(x_{i+1}\right) \cong f\left(x_i\right) + f'\left(x_i\right)\left(x_{i+1} - x_i\right) + \frac{f''(x_i)(x_{i+1} - x_i)^2}{2!}$$

```
order = 2;

for i = 1:size(STEP_SIZE,2)
```

```
    h = STEP_SIZE(i);

    approximationValue = taylorSeries_dividedDifference_backward(function_x, ...
        x_i,x_target,order,h);

    l2(i) = plot([x_i x_target],[function_x(x_i) approximationValue], 'b', 'LineWidth', 0.5*i);
    plot(x_target,approximationValue, 'o', 'MarkerFaceColor', 'b','MarkerEdgeColor','none')

    fprintf(['Approximation Value (Taylor + Backward Second-Order Step %0.2f)' ...
        '  :%0.2f \t error : %0.2f\n'], ...
        STEP_SIZE(i), approximationValue, error(trueValue_2,approximationValue))
end
```

```
Approximation Value (Taylor + Backward Second-Order Step 0.10) :62.85   error : 0.38
Approximation Value (Taylor + Backward Second-Order Step 0.50) :11.25   error : 0.89
Approximation Value (Taylor + Backward Second-Order Step 1.00) :-42.00   error : 1.41
```

Third-Order Taylor Series Expansion

$$f\left(x_{i+1}\right) \cong f\left(x_i\right) + f'\left(x_i\right)\left(x_{i+1} - x_i\right) + \frac{f''\left(x_i\right)\left(x_{i+1} - x_i\right)^2}{2!} + \frac{f'''\left(x_i\right)\left(x_{i+1} - x_i\right)^3}{3!}$$

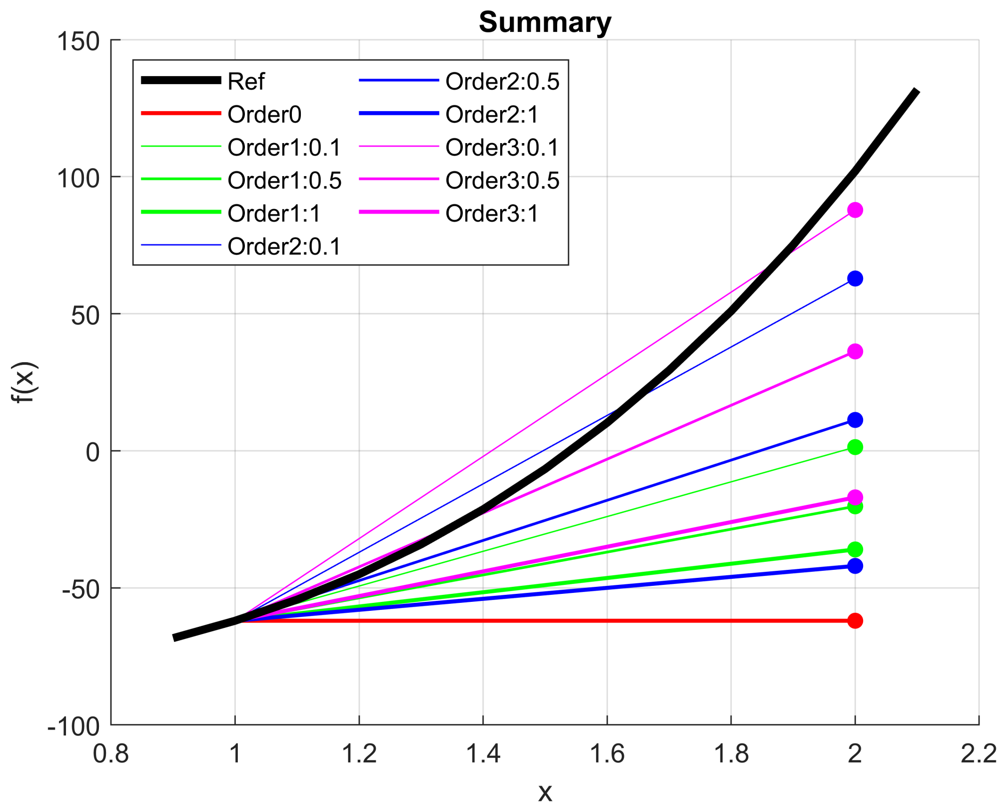
```
order = 3;

for i = 1:size(STEP_SIZE,2)
    h = STEP_SIZE(i);

    approximationValue = taylorSeries_dividedDifference_backward(function_x, ...
        x_i,x_target,order,h);

    l3(i) = plot([x_i x_target],[function_x(x_i) approximationValue], 'm', 'LineWidth', 0.5*i);
    plot(x_target,approximationValue, 'o', 'MarkerFaceColor', 'm','MarkerEdgeColor','none')

    fprintf(['Approximation Value (Taylor + Backward Third-Order Step %0.2f)' ...
        '  : %0.2f \t error : %0.2f\n'], ...
        STEP_SIZE(i), approximationValue, error(trueValue_2,approximationValue))
end
```

```
Approximation Value (Taylor + Backward Third-Order Step 0.10) : 87.85   error : 0.14
Approximation Value (Taylor + Backward Third-Order Step 0.50) : 36.25   error : 0.64
Approximation Value (Taylor + Backward Third-Order Step 1.00) : -17.00   error : 1.17
```

Summary Plot

```
x_p = x_i-0.1:0.1:x_target+0.1;

title('Summary');
grid on;
xlabel('x');
ylabel('f(x)');

ref = plot(x_p,function_x(x_p), 'black', 'LineWidth', 3);
legend([ref, l0 ,l1(1) ,l1(2) ,l1(3) ,l2(1) ,l2(2), l2(3), l3(1), l3(2), l3(3)], ...
```

```
    ["Ref","Order0","Order1:0.1","Order1:0.5","Order1:1","Order2:0.1", ...
    "Order2:0.5","Order2:1","Order3:0.1","Order3:0.5","Order3:1"], ...
    'Location','northwest','NumColumns',2)
hold off
```



2. Write a computer program that find the root of

$$f(x) = x^3 - 3x^2 + 1$$

```
%Initialization Workspace
clear
clc
close

%Initial Function
syms function_x x
function_x(x) = (x^3) - 3*(x^2) + 1
```

function_x(x) = $x^3 - 3\,x^2 + 1$

by using both bisection and false-position method in period $[0, 2]$

and terminate the program when approximated

relative error $\varepsilon_a \leq 1\%$ the program must generate a report file,

```
syms epsilon_a approximationValue_previous approximationValue_present
epsilon_a(approximationValue_previous,approximationValue_present) = ( ...
    (approximationValue_present - approximationValue_previous)/ ...
    approximationValue_present)*100;
```

report.txt, that shows in each line:

(1) the number of iterations,

(2) an estimated root in each iteration,

(3) the approximate relative error in each iteration.

```
path = matlab.desktop.editor.getActiveFilename;
index = max(strfind(path,'\'));
filePart = extractBetween(path,1,index);
fileID = fopen(strcat(filePart{1},'report.txt'),'w');

hold on;
```

Bracketing Method: Bisection Method

```

x_lower = 0;
x_upper = 2;

x_root_previous  = 0;
error_approximation = 100;

i = 1;

fprintf(fileID,'Bracketing Method: Bisection Method  \n');

%Algorithm Approximation Value Bisection Method
while error_approximation > 1

    x_root = (x_lower + x_upper)/2;
    approximationValue_root = function_x(x_lower) * function_x(x_root);

    if approximationValue_root < 0
        x_upper = x_root;
    elseif approximationValue_root > 0
        x_lower = x_root;
    end

    error_approximation = abs(epsilon_a(x_root_previous,x_root));
```

```
    mark1 = plot(x_root,approximationValue_root,'o', 'MarkerFaceColor', 'g','MarkerEdgeColor','

    fprintf(fileID,['Iteration : %i \t\t Estimated root : %0.4f \t\t ' ...
        'Approximate Relative Error : %0.4f \n'], ...
        i,x_root,error_approximation);

    i = i + 1;
    x_root_previous = x_root;
end
```

```
Iteration : 1    Estimated root : 1.0000    Approximate Relative Error : 100.0000
Iteration : 2    Estimated root : 0.5000    Approximate Relative Error : 100.0000
Iteration : 3    Estimated root : 0.7500    Approximate Relative Error : 33.3333
Iteration : 4    Estimated root : 0.6250    Approximate Relative Error : 20.0000
Iteration : 5    Estimated root : 0.6875    Approximate Relative Error : 9.0909
Iteration : 6    Estimated root : 0.6563    Approximate Relative Error : 4.7619
Iteration : 7    Estimated root : 0.6406    Approximate Relative Error : 2.4390
Iteration : 8    Estimated root : 0.6484    Approximate Relative Error : 1.2048
Iteration : 9    Estimated root : 0.6523    Approximate Relative Error : 0.5988
```

Bracketing Method: False-Position Method

```
x_lower = 0;
x_upper = 2;

x_root_previous  = 0;
error_approximation = 100;

i = 1;

fprintf(fileID,'\n\nBracketing Method: False-Position Method  \n');

%Algorithm Approximation Value False-Position Method
while error_approximation > 1

    approximationValue_lower = function_x(x_lower);
    approximationValue_upper = function_x(x_upper);

    x_root = (x_lower * approximationValue_upper - x_upper * approximationValue_lower)/( ...
        approximationValue_upper - approximationValue_lower);

    approximationValue_root = function_x(x_root);

    if approximationValue_root < 0
        x_upper = x_root;
    elseif approximationValue_root > 0
        x_lower = x_root;
    end

    error_approximation = abs(epsilon_a(x_root_previous,x_root));

    mark2 = plot(x_root,approximationValue_root,'o', 'MarkerFaceColor', 'r','MarkerEdgeColor','

    fprintf(fileID,['Iteration : %i \t\t Estimated root : %0.4f \t\t ' ...
```

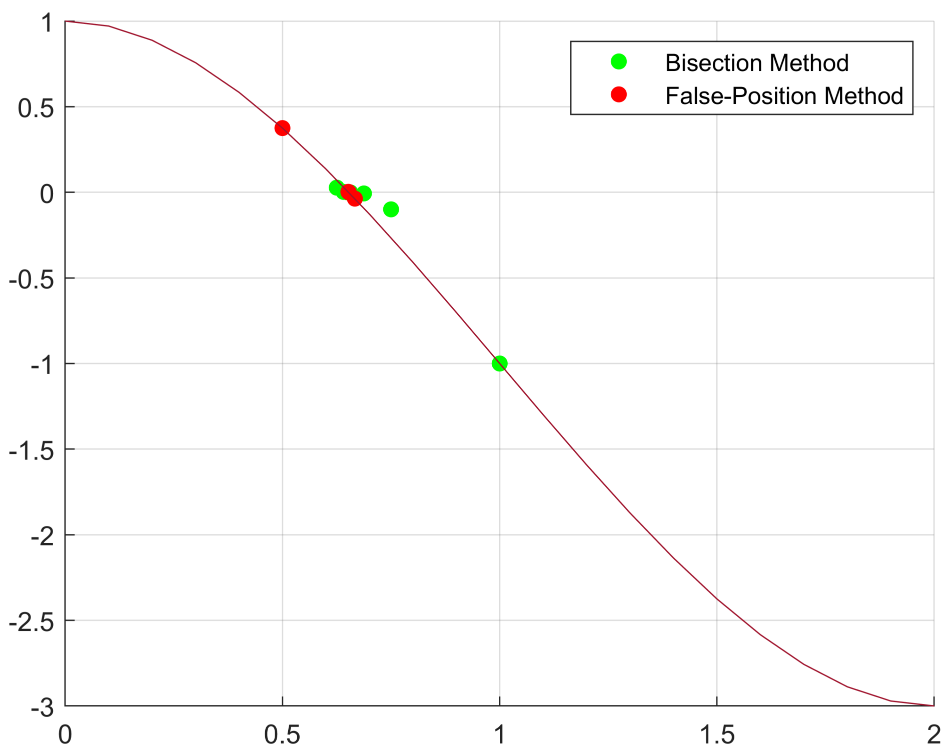
```
        'Approximate Relative Error : %0.4f \n'], ...
        i,x_root,error_approximation);

    i = i + 1;
    x_root_previous = x_root;
end
```

```
Iteration : 1    Estimated root : 0.5000    Approximate Relative Error : 100.0000
Iteration : 2    Estimated root : 0.6667    Approximate Relative Error : 25.0000
Iteration : 3    Estimated root : 0.6517    Approximate Relative Error : 2.2989
Iteration : 4    Estimated root : 0.6527    Approximate Relative Error : 0.1552
```

```
fclose(fileID);
```

Summary Plot

```
plot(0:0.1:2,function_x(0:0.1:2));grid on;hold off;
legend([mark1,mark2],["Bisection Method","False-Position Method"])
```



Appendix

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

```
syms n k
nchoosek(n,k);
```

Backward Divided Difference Taylor Series

$$f\left(x_{i+1}\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)}{n!} (x_{i+1} - x_i)^n,$$

$$f^{(n)}\left(x\right) = \frac{\nabla_h^n[f](x)}{h^n} = \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x - kh),$$

```
function approximationValue = taylorSeries_dividedDifference_backward(function_x, x_i, ...
    x_target, order, h)

%Taylor Series
clear term_taylorSeries dividedDifference_backward_order
for n = 0:order

    %Divided Difference Backward
    clear term_dividedDifference_backward
    for k = 0:n
        term_dividedDifference_backward(k+1) = (((-1)^k) * nchoosek(n,k) * ...
            function_x(x_i - k*h));
    end
    dividedDifference_backward_order(n+1) = sum(term_dividedDifference_backward) / (h^n);
    %Divided Difference Backward END

    term_taylorSeries(n+1) = ( ...
        dividedDifference_backward_order(n+1) / factorial(n)) * ((x_target - x_i)^n);
end

approximationValue = sum(term_taylorSeries);
%Taylor Series END

end
```