FUNDAMENTAL MATHEMATICS FOR ROBOTICS

Numerical Methods

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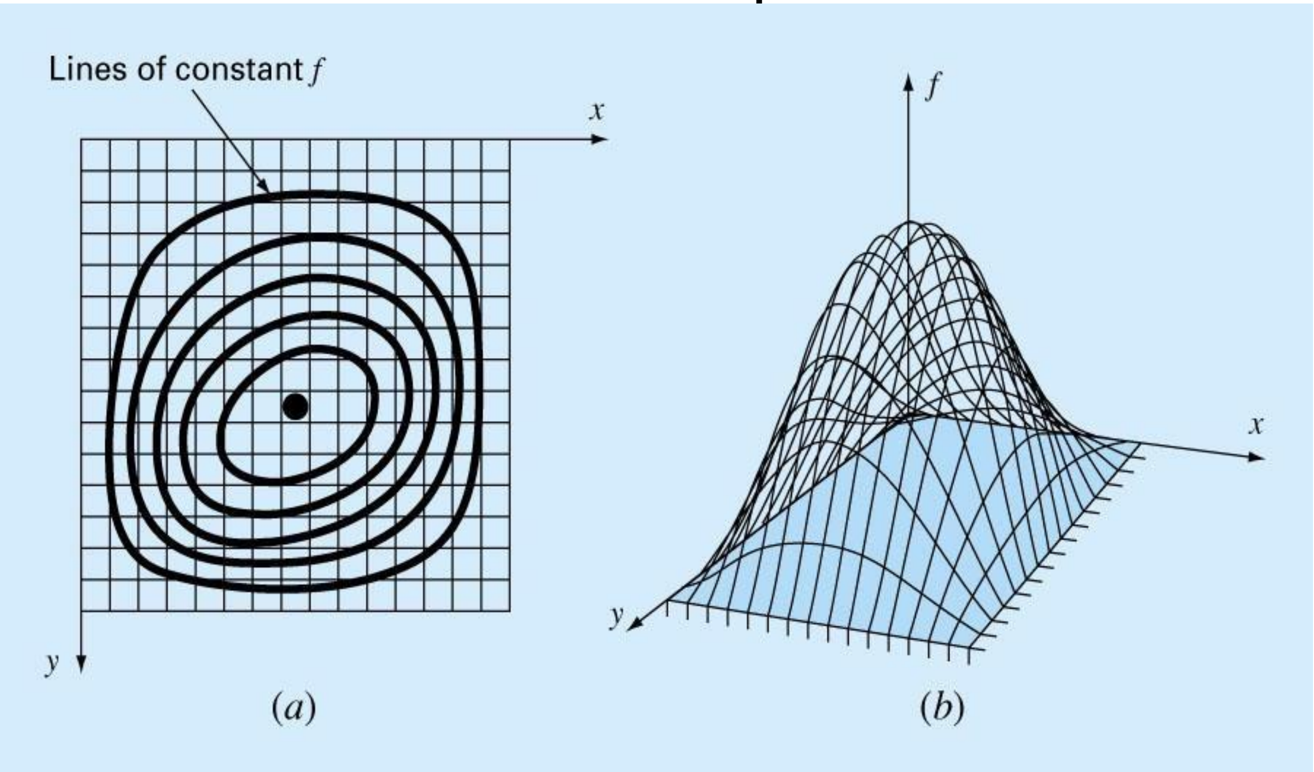
Optimization

Optimization

- One-dimensional Unconstrained Optimization
  - Golden-Section Search
  - Newton's Method
- Multidimensional Unconstrained Optimization



Multidimensional Unconstrained Optimization

- Techniques to find minimum and maximum of a function of several variables

- These techniques are classified as:
  - That require derivative evaluation
    - Gradient or descent (or ascent) methods
  - That do not require derivative evaluation
    - Non-gradient or direct methods

Direct Methods

Direct Methods: Random Searches

- Based on evaluation of the function randomly at selected values of the independent variables
- If a sufficient number of samples are conducted, the optimum will be eventually located.
- Example: maximum of a function

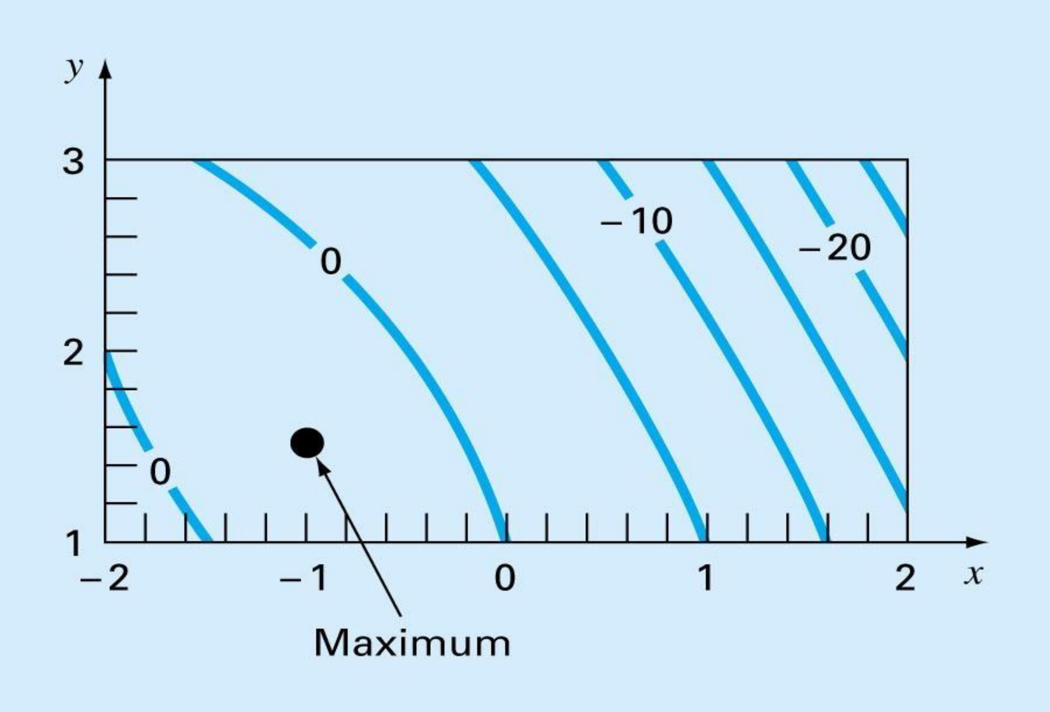
$$f(x, y) = y - x - 2x^2 - 2xy - y^2$$

can be found using a random number generator

Direct Methods: Random Searches

$$0 \le r \le 1 \qquad x = x_l + (x_u - x_l)r \qquad y = y_l + (y_u - y_l)r$$

- Take sufficient number of samples
- Keep track of the maximum value from among random trials



Direct Methods: Random Searches

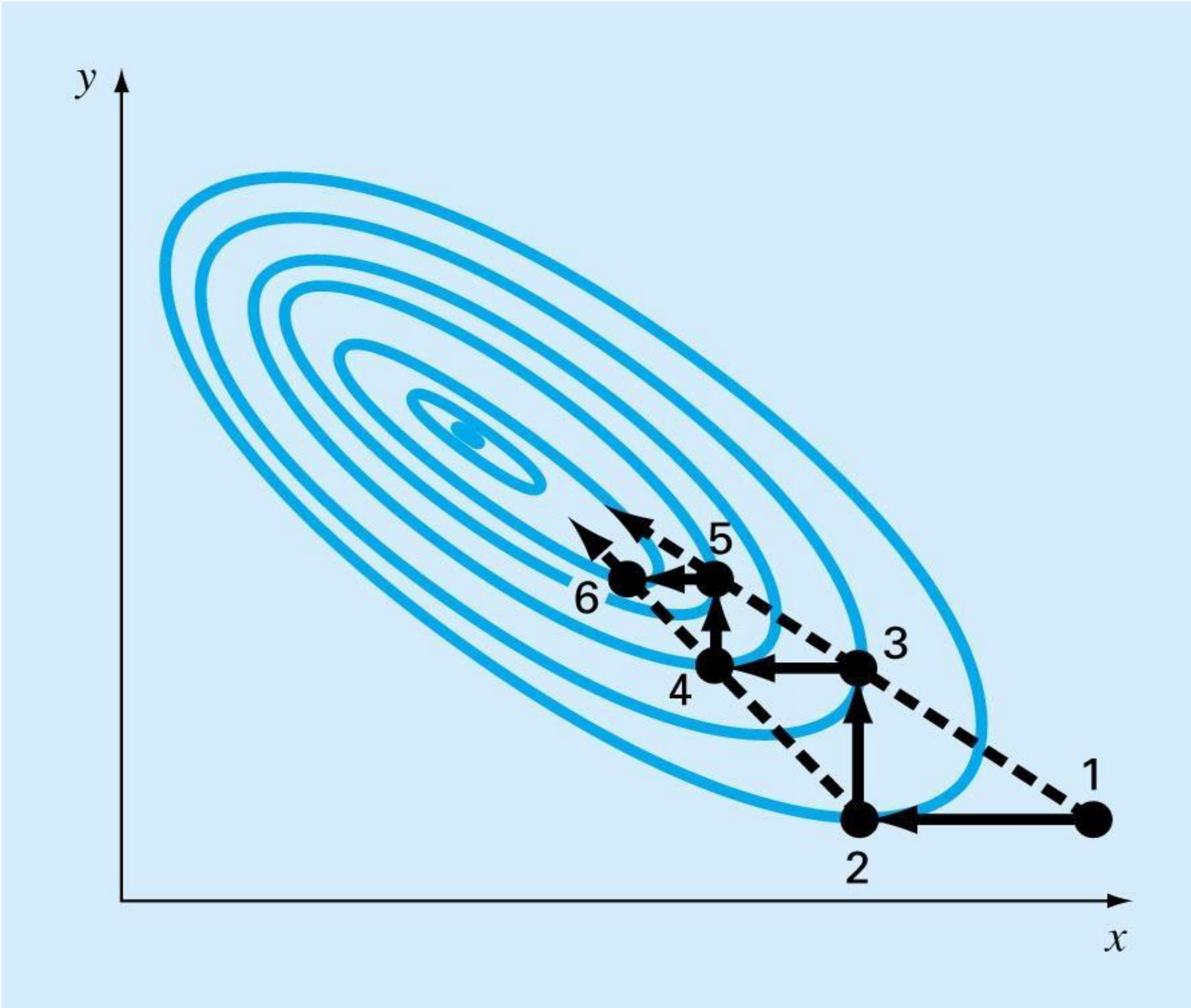
- Advantages
  - Works even for discontinuous and nondifferentiable functions
  - Always find global optimum rather than a local optimum

- Disadvantages
  - As the number of independent variables grows, the task can become onerous
  - Not efficient, it does not account for the behavior of underlying function

Direct Methods: Univariate and Pattern Searches

- More efficient than random search and still doesn’t require derivative evaluation.

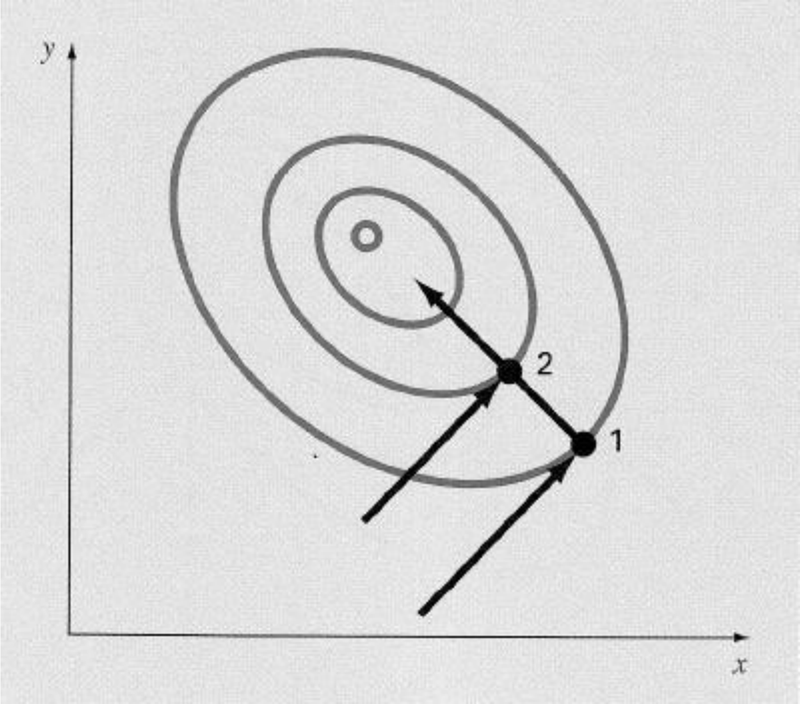
- The basic strategy is:
  - Change one variable at a time while the other variables are held constant.
  - Thus problem is reduced to a sequence of one-dimensional searches that can be solved by variety of methods.
  - The search becomes less efficient as you approach the maximum.



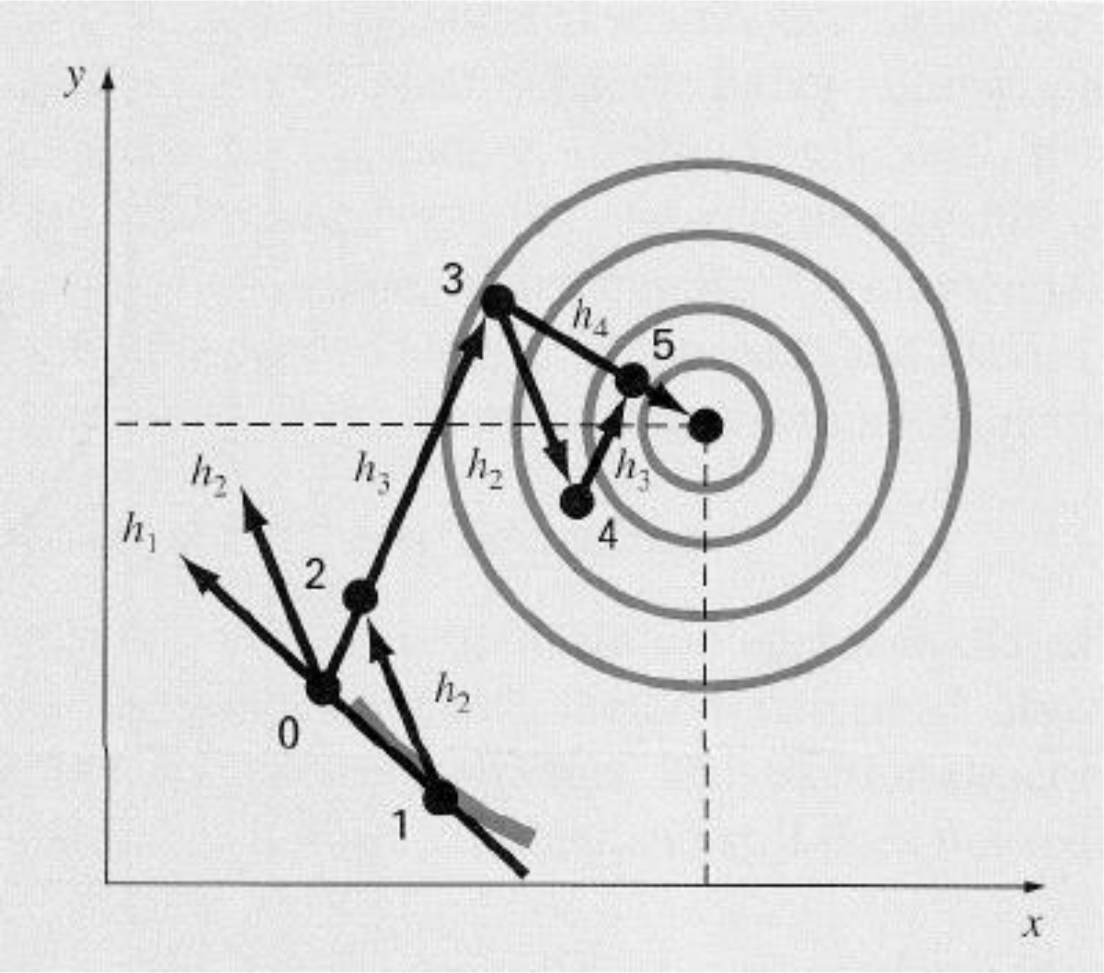
y
x
1
2
3
4
5
6

Direct Methods: Powell's Method

- If points 1 and 2 are obtained by one dimensional searches in the same direction but from different starting points
  - The line formed by 1 and 2 will be directed toward the maximum
  - Conjugated Direction



Direct Methods: Powell's Method



Gradient Methods

Gradient Methods: Gradients and Hessians

The Gradient

- If f(x,y) is a two dimensional function, the gradient vector tells us
  - What direction is the steepest ascend?
  - How much we will gain by taking that step?

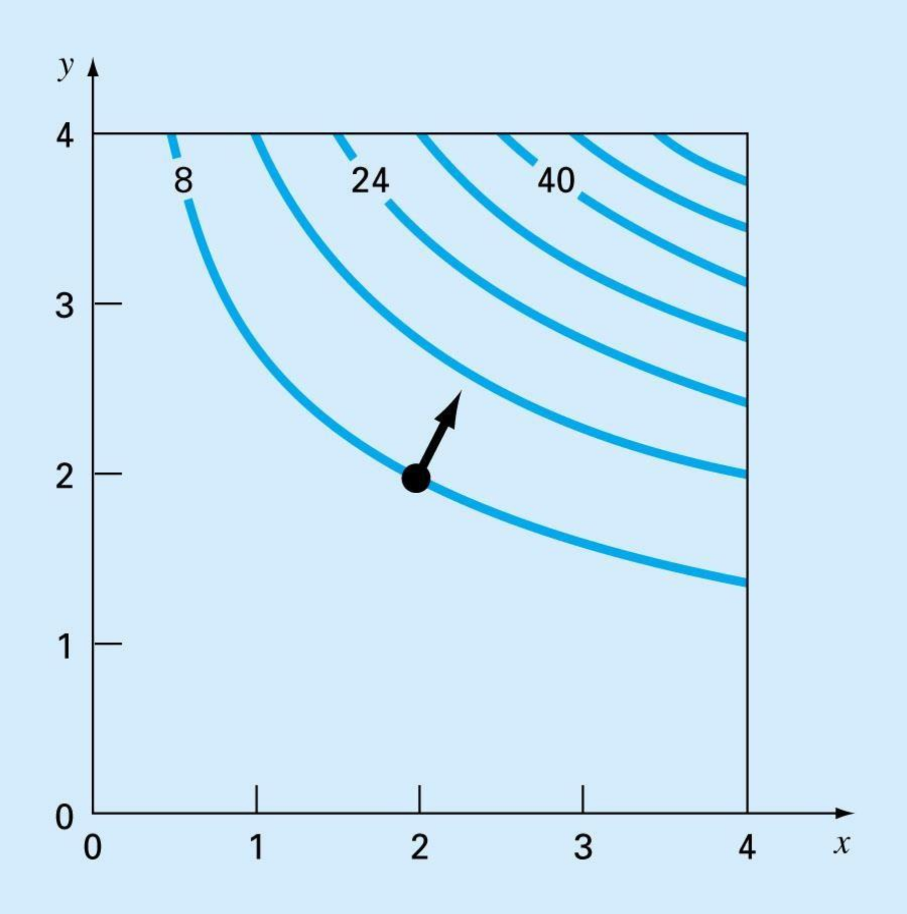
$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \quad \text{or } del\ f$$

Directional derivative of *f(x,y)* at point x=a and y=b

$$\nabla f(x) = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \dfrac{\partial f}{\partial x_2}(x) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) \end{Bmatrix}$$

Gradient Methods: Gradients and Hessians

- Example $f(x,y) = xy^2$



Direction of steepest ascent is perpendicular, or orthogonal to the elevation contour at the coordinate (2,2)

Gradient Methods: Gradients and Hessians

The Hessian

- For one dimensional functions both first and second derivatives are valuable information for searching out optima
- First derivative provides (a) the steepest trajectory of the function and (b) tells us that we have reached the maximum
- Second derivative tells us that whether we are a maximum or minimum (or saddle)
- For two dimensional functions whether a maximum or a minimum occurs involves not only the partial derivatives w.r.t. x and y but also the second partials w.r.t. x and y

Gradient Methods: Gradients and Hessians

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$ ← Hessian of f

$$|H| = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$ ← Determinant of Hessian

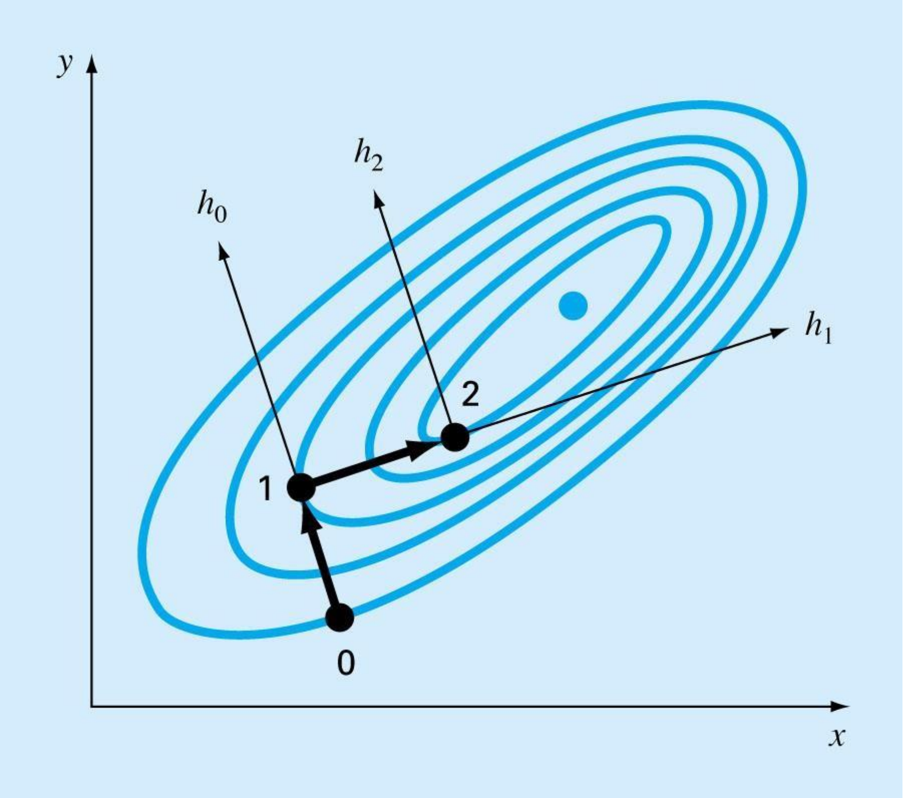
If $|H| > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, then f(x, y) has a local minimum

If $|H| > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, then f(x, y) has a local maximum

If $|H| < 0$, then f(x, y) has a saddle point

Gradient Methods: Steepest Ascent Method

- Start at an initial point (x_o, y_o), determine the direction of steepest ascend (the gradient).
- Search along the direction of the gradient, h_o, until we find maximum.
- Process is then repeated.



Gradient Methods: Steepest Ascent Method

- The problem has two parts
  - Determining the "best direction" and
  - Determining the "best value" along that search direction.
- Steepest ascent method uses the gradient approach as its choice for the "best" direction.
- To transform a function of x and y into a function of h along the gradient section:

$$x = x_0 + \frac{\partial f}{\partial x} h$$

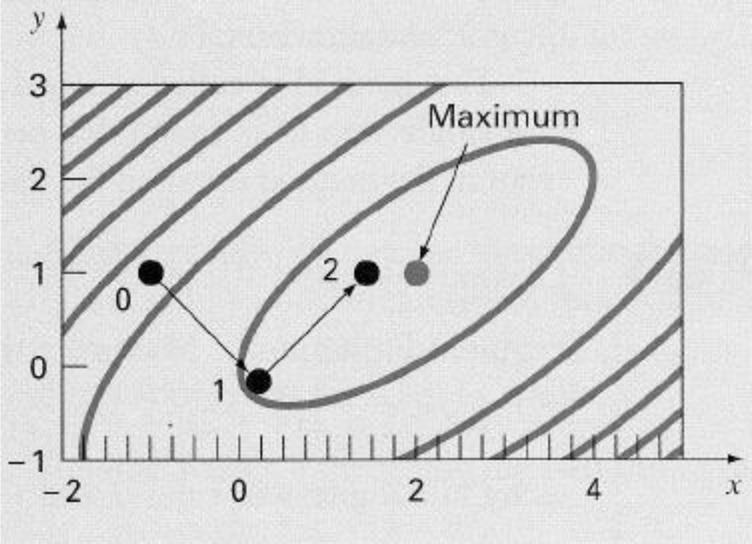
$$y = y_0 + \frac{\partial f}{\partial y} h$$

h is distance along the h axis

Gradient Methods: Steepest Ascent Method

- Example $f(x) = 2xy + 2x - x^2 - 2y^2$

Gradient Methods:
Steepest Ascent Method



Constrained Optimization

Constrained Optimization

- Optimization problem that deals with meeting a desired objective in the presence of constraints
  - Desired objective such as maximizing profit, minimizing cost
  - Constraints such as limited resources
- Linear Constrained Optimization
  - If both objective function and constraints are linear
- Nonlinear Constrained Optimization
  - Otherwise
- Linear Programming (LP)
  - Doesn't mean "computer programming"
  - Means "Scheduling" or "Setting an agenda"

Linear Programming

- Standard Form

$$\text{Maximizing } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

  - c_j = payoff of each unit of the j^{th} activity
  - x_j = magnitude of the j^{th} activity
  - Z = the total payoff due to the total number of activities n

$$\text{Constraint s} \Rightarrow \quad a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i$$

  - a_{ij} =amount of the i^{th} resource that is consumed for each unit of j^{th} activity
  - b_i = amount of the i^{th} resource that is available
  - All activities must have a positive value $x_1 \ge 0$

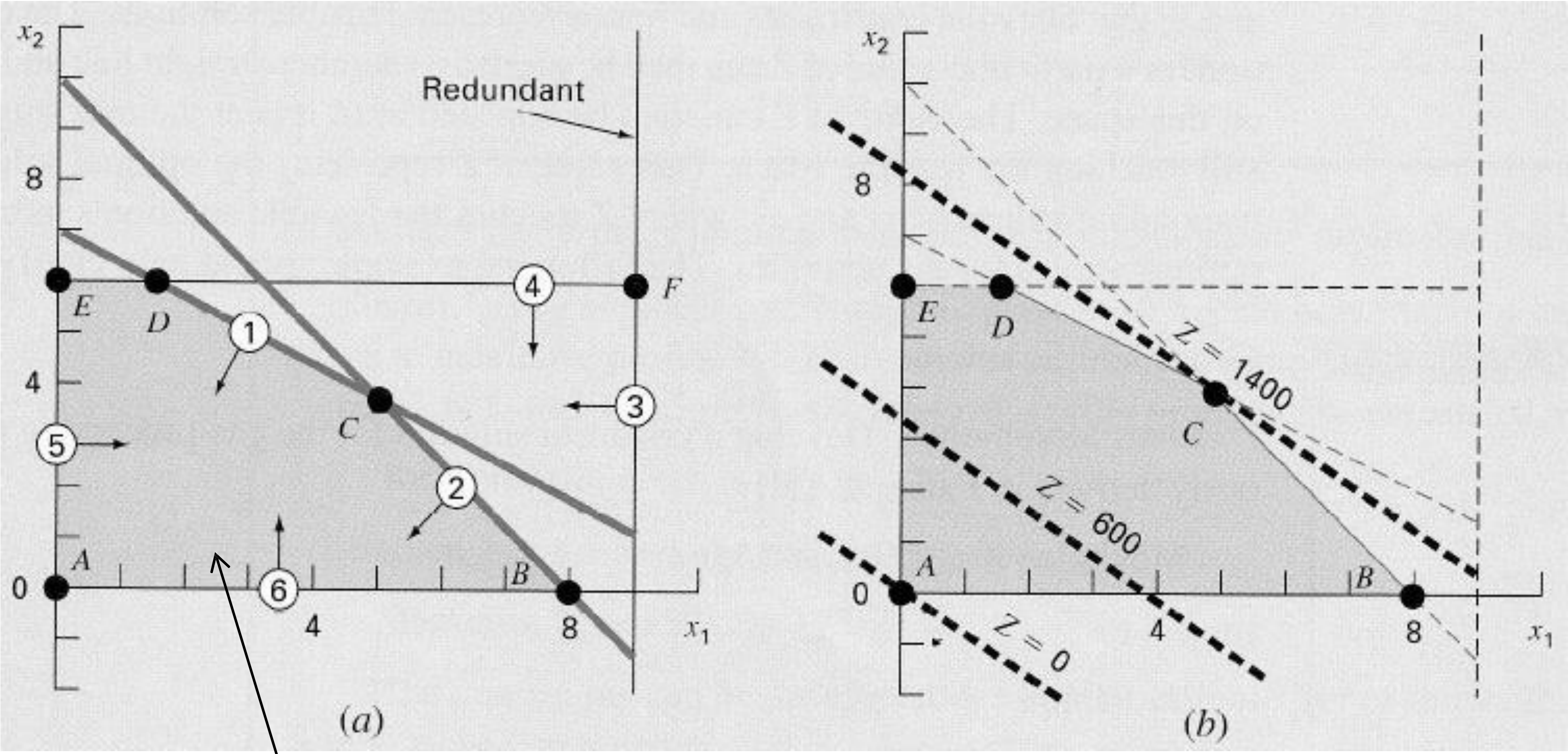
Linear Programming

- Example

Resource	Product		Resource Availability
	Regular	Premium	
Raw gas	7 m^3/tonne	11 m^3/tonne	77 m^3/week
Production time	10 hr/tonne	8 hr/tonne	80 hr/week
Storage	9 tonnes	6 tonnes	
Profit	150/tonne	175/tonne	

Linear Programming

- Graphical Solution

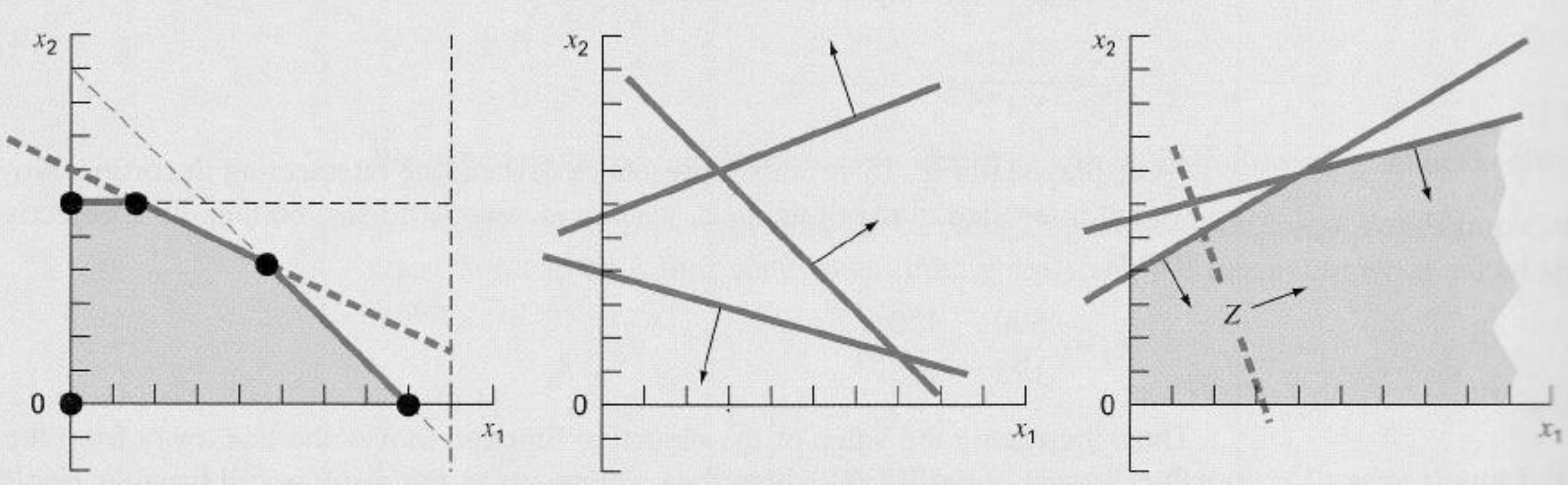


Feasible solution space

Linear Programming

- Four possible outcomes generally obtained in a linear programming problem
  - *Unique Solution.* The maximum objective function intersects a single point
  - *Alternate Solution.* The problem has an infinite number of optima
  - *No feasible Solution.* The problem is overconstrained to the point that no solution can satisfy all constraints
  - *Unbounded Problems.* The problem is underconstrained and therefore open-ended

Linear Programming



(a) (b) (c)

The Simplex Method

- Not every extreme point is feasible
  - Limiting ourselves to feasible extreme points narrows the problem
- Simplex method
  - offers a preferable strategy that charts a selective course through a sequence of feasible extreme points
  - Arrive at the optimum in an extremely efficient manner VS. exhaustively evaluating the value

Linear Programming in Standard Form

$$\begin{array}{rrcl}
 & \text{Max } c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\
 & \ldots \quad \ldots & & \ldots \\
 & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \\
 & x_1 \geq 0, \ldots x_n \geq 0 & &
\end{array}$$

where the objective is maximized, the constraints are equalities and the variables are all nonnegative.

This is done as follows:

- If the problem is $min\ z$, convert it to $max\ -z$.
- If a constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$, convert it into an equality constraint by adding a nonnegative *slack* variable s_i. The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + s_i = b_i$, where $s_i \geq 0$.
- If a constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$, convert it into an equality constraint by subtracting a nonnegative *surplus* variable s_i. The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - s_i = b_i$, where $s_i \geq 0$.

Example 7.2.1 *Solve the linear program*

$$\begin{array}{lrrcl}
max & x_1 & +x_2 & & \\
 & 2x_1 & +x_2 & \leq & 4 \\
 & x_1 & +2x_2 & \leq & 3 \\
 & x_1 \geq 0, & x_2 \geq 0 & &
\end{array}$$

First, we convert the problem into standard form by adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$.

$$\begin{array}{lrrrrcl}
\max & x_1 & +x_2 & & & & \\
 & 2x_1 & +x_2 & +x_3 & & = & 4 \\
 & x_1 & +2x_2 & & +x_4 & = & 3 \\
 & x_1 \geq 0, & x_2 \geq 0 & x_3 \geq 0, & x_4 \geq 0 & &
\end{array}$$

Let z denote the objective function value. Here, $z = x_1 + x_2$ or, equivalently,

$$z - x_1 - x_2 = 0.$$

Putting this equation together with the constraints, we get the following system of linear equations.

$$\begin{array}{rrrrrclr}
z & -x_1 & -x_2 & & & = & 0 & \text{Row 0} \\
 & 2x_1 & +x_2 & +x_3 & & = & 4 & \text{Row 1} \\
 & x_1 & +2x_2 & & +x_4 & = & 3 & \text{Row 2}
\end{array} \tag{7.1}$$