ECS230

Summary 2

Vector and matrix norms

Vector norm: a function mapping $x \in \mathbb{R}^n$ to $||x|| \in \mathbb{R}$ such that

- $||x|| > 0$ if $x \neq 0$
- $||\alpha x|| = |\alpha|||x||$, $\alpha \in \mathbb{R}$
- $||x + y|| \leq ||x|| + ||y||$

Vector p-norms on $\mathbb{R}^n$

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

$$||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

$$||x||_\infty = \max_{i \in [1,n]} |x_i|$$

Norm inequalities:

$$x^T y \leq ||x||_p ||y||_q \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad \text{Hölder}$$

$$x^T y \leq ||x||_2 ||y||_2 \qquad \text{Cauchy-Schwarz}$$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n} ||x||_\infty$$

Matrix norms: $A \in \mathbb{R}^{m \times n}$

$$||A||_F = \left(\sum_{i \in [1,m]} \sum_{j \in [1,n]} |a_{ij}|^2 \right)^{\frac{1}{2}} \qquad \text{Frobenius norm}$$

$$||A||_p = \sup_{x \neq 0} \frac{||Ax||_p}{||x||_p}$$

$$||A||_2 \leq ||A||_F \leq \sqrt{n} ||A||_2$$

$$||A||_1 = \max_{j \in [1,n]} \sum_{i \in [1,m]} |a_{ij}|$$

$$||A||_\infty = \max_{i \in [1,m]} \sum_{j \in [1,n]} |a_{ij}|$$

$$||AB||_p \leq ||A||_p ||B||_p \quad B \in \mathbb{R}^{n \times k}$$

$$||A||_2 = \sqrt{\mu} \qquad \text{where} \quad \mu \quad \text{is the largest eigenvalue of} \quad A^T A$$

Linear transformations $A : V \mapsto V$
Range: $R(A) = \{w \in V : w = Av, v \in V\}$
Nullspace: $N(A) = \{v \in V : Av = 0\}$
Rank: $\text{rank}(A) = \dim(R(A))$

$$A \in \mathbb{R}^{m \times n} \quad \text{is } \underline{\text{onto}} \quad \Leftrightarrow \quad \text{rank}(A) = m$$

$$A \in \mathbb{R}^{m \times n} \quad \text{is } \underline{\text{one-to-one}} \quad \Leftrightarrow \quad \text{rank}(A) = n$$

$\text{rank}(A) \leq \min(m, n)$
Moore-Penrose pseudoinverse:

$$A^+ = A^T (AA^T)^{-1} \quad m < n$$

$$A^+ = (A^T A)^{-1} A^T \quad m > n$$