ECS230

Summary 1

Vectors, matrices, basic facts

- A vector $x \in \mathbb{R}^n$ is a *column vector*. The corresponding row vector is x^T.
- $\mathbb{R}^{m\times n}$ is the set of real matrices with m rows and n columns. $\mathbb{C}^{m\times n}$ is the set of complex $m \times n$ matrices.
- Notation: A denotes the matrix, a_{ij} is an element of the matrix A, i.e. $a_{ij} = (A)_{ij}$. Use upper case for matrices and lower case for vectors.

<u>Block matrices:</u> A matrix can be represented in block form, for example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. The dimensions of the blocks must be compatible.

The matrix

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

is *upper block triangular* (it contains the zero matrix). Note that it is *not* upper triangular (it has elements below its main diagonal).

<u>Transpose:</u> $A^T \in \mathbb{R}^{n\times m}$ is the transpose of $A \in \mathbb{R}^{m\times n}$. $(A^T)_{ij} = a_{ji}$.

<u>Hermitian transpose:</u> $A^H \in \mathbb{C}^{n\times m}$ is the hermitian transpose of $A \in \mathbb{C}^{m\times n}$. $(A^H)_{ij} = \overline{a_{ji}}$.

<u>Symmetric matrix:</u> $A = A^T$

<u>Hermitian matrix:</u> $A = A^H$

<u>Matrix multiplication:</u> $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times p}, C = AB$. $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Note that in general $AB \neq BA$.

<u>Dot product of vectors:</u> $x, y \in \mathbb{R}^n$, the dot product of x and y is $x^T y = \sum_{i=1}^{n} x_i y_i$.

<u>Outer product of vectors:</u> $x, y \in \mathbb{R}^n$, the outer product of x and y is an $n\times n$ matrix of rank one. $A = xy^T$. $a_{ij} = x_i y_j$.

<u>Transpose of a product:</u> $(AB)^T = B^T A^T$.

Complex dot product: $x^H y = \sum_{i=1}^n \overline{x_i} y_i$. This is not symmetric: $x^H y = \overline{y^H x}$.
Orthogonal matrix: $A \in \mathbb{R}^{n \times n}$, $A^T A = A A^T = I$.
Unitary matrix: $A \in \mathbb{C}^{n \times n}$, $A^H A = A A^H = I$.

Matrix algebra, basic facts

In general, for matrices A, B, C, if $AB = AC$ for $A \neq 0$, this does not imply that $B = C$. If $AB = 0$, this does not imply that either A or B is zero.
Matrix inverse: $A \in \mathbb{R}^{n \times n}$, the inverse of A is A^{-1} such that $AA^{-1} = A^{-1}A = I$.
Trace: $\mathrm{Tr} A = \sum_i a_{ii}$. $\mathrm{Tr}(A + B) = \mathrm{Tr} A + \mathrm{Tr} B$. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
Determinant: $A \in \mathbb{R}^{n \times n}$

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma_i}$$

$\det A^T = \det A$, $\det A^H = \overline{\det A}$, $\det AB = \det A \det B$, $\det \alpha A = \alpha^n \det A$.
Schur complements:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B)$$

$D - CA^{-1}B$ is the Schur complement of A in $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C)$$

$A - BD^{-1}C$ is the Schur complement of D in $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.