ECS230

Summary 3

Sensitivity of linear systems $Ax = b$

Sensitivity of x to changes in b: $A \in \mathbb{R}^{n\times n}, x, \delta x, b, \delta b \in \mathbb{R}^n$
Assume A non-singular, $b \neq 0$. Approximate solution: $\hat{x} = x + \delta x$

$$A\hat{x} = b + \delta b$$

Norm-wise relative error:

$$\frac{||\delta x||}{||x||} \leq ||A||||A^{-1}||\frac{||\delta b||}{||b||}$$

Definition: $\kappa(A)$ is the condition number of A

$$\kappa(A) = ||A||||A^{-1}||$$

$$\frac{||\delta x||}{||x||} \leq \kappa(A)\frac{||\delta b||}{||b||}$$

Notes:
$\kappa(A)$ can be defined using any p-norm. Notation: $\kappa_p(A)$. The above inequalities are then valid using the same p-norm.
$\kappa(A) = \kappa(A^{-1})$
$\kappa(\alpha A) = \kappa(A), \alpha \neq 0$
Lower bound: $\kappa(A) \geq 1$

The problem "solving $Ax = b$" is well-conditioned if $\kappa(A) \simeq 1$. The problem is ill-conditioned if $\kappa(A)$ is large.
Note: A set of linear equations can be ill-conditioned due to bad scaling of one or more of the equations. This can occur because of a bad choice of units.

Sensitivity of b to changes in x when computing $b = Ax$:

$$\frac{||\delta b||}{||b||} \leq \kappa(A)\frac{||\delta x||}{||x||}$$

Sensitivity of solutions to uncertainty in A:
$A \to A + \delta A$
Theorem: If A is non-singular and

$$\frac{||\delta A||}{||A||} < \frac{1}{\kappa(A)}$$

then $A + \delta A$ is non-singular.
Theorem:

$$\frac{||\delta x||}{||x||} \leq \frac{\kappa(A)\frac{||\delta A||}{||A||}}{1 - \kappa(A)\frac{||\delta A||}{||A||}}$$

Perturbation of both A and b:

$$\frac{||\delta x||}{||x||} \leq \frac{\kappa(A)\left[\frac{||\delta A||}{||A||} + \frac{||\delta b||}{||b||}\right]}{1 - \kappa(A)\frac{||\delta A||}{||A||}}$$

A posteriori error analysis: Given $\hat{x}$ obtained by solving numerically $Ax = b$.
How good is $\hat{x}$?
Definition: residual $\hat{r} = b - A\hat{x}$

$$\frac{||\delta x||}{||x||} \leq \kappa(A)\frac{||\hat{r}||}{||b||}$$