HW1 ECS 230

Heqiao Ruan
email:hruan@ucdavis.edu

October 10, 2018

1 Problem 1:

Here as we assume that both A^{-1} and B^{-1} exist, we only need to validate that the left and right multiplication by the matrix the problem describe map the AB to the identity matrix I_n. Then we first derive: $B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n$ then we derive: $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$. So it satisfies the property of the inverse of matrix. So we have proved that the inverse of AB is $B^{-1}A^{-1}$.

2 Problem 2:

Here the matrix $A \in \mathbb{R}^{n\times n}$ and it satisfy $Ax = 0$ for each $x \in \mathbb{R}^n$ if and only if $A == 0$. Here if $A == 0$, then we can see that for all $x \in \mathbb{R}^n$ the Ax absolutely should be 0 which is pretty trivial.
Then on the other side of the problem we can prove it by contradiction. WLOG we suppose that anyone of the element $A_{ij} \neq 0$, then we can see that there exist a $x \in \mathbb{R}^n$ which makes $A_{ij}x_j \neq 0$. Note that this is the case of $A_{ij} \neq 0$ with other elements 0. So we conclude that for any matrix A with any non-zero element we can find a subspace of $\mathbb{R}^n$ which is orthogonal to $span\{A\}$. Here we can express it as $span\{A\} \oplus span\{x\} = \mathbb{R}^n$. Thus we finish the other side of the problem.

3 Problem 3

Here $A \in \mathbb{R}^{n\times n}$ and we suppose that A is strictly lower triangular. Then we can see that the first row of A itself is 0 vector. Then we can see that for A^2, the first row: $A^2_{1,:} = [0, ..., 0]^T A$ are 0 vector obviously. What's more the second row will be $A^2_{2,:} = [a_{2,1}, 0, ..., 0]^T A = 0$ as the first row of A is all 0.($A^2_{2,j} = [a_{2,1}, 0, ..., 0]^T [0, a_{1,j}, ..., a_{n,j}]$ for each j) Then iteratively we know that the first 3 row of A^3 are 0 vector. Then we can see that for A^n, the first n row are all 0 vector which indicate that A^n is zero matrix.

4 Problem 4

To compute the inverse of matrix $A = I + uv^T$ we try the matrix with form $\alpha(I - uv^T)$. Then we just need to check it by left and right multiply the matrix. We start by checking the right multiplication: $(I+uv^T)(I-\alpha uv^T) = I+uv^T-\alpha uv^T-\alpha uv^T uv^T = I+(1-\alpha-\alpha u^T v)uv^T$. Note that $\alpha uv^T uv^T = \alpha uu^T vv^T = u^T vuv^T$ which is the product of two inner products. Here wlog, we denote $u^T v$ as β. Firstly if $uv^T = 0$, no need to solve α(works for all $\alpha \in \mathbb{R}$)and then the inverse should be Identity matrix. Otherwise we can see that $\alpha = \frac{1}{1+\beta}$ which is a number $\in \mathbb{R}$. Then we validate it by the left multiplication: $(I-\alpha uv^T)(I+uv^T) = I-\alpha uv^T+uv^T-\alpha\beta uv^T = I + (1 - \alpha(1+\beta))uv^T = I$ which indicate that $\alpha = \frac{1}{\beta+1}$, and $A^{-1} = I - \alpha uv^T$. Note that here we need to ensure that $u^T v \neq 0$(indeed the inner product) or there the α we get doesn't make sense.

5 Problem 5

Here we need to prove that $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A-1}{1+v^T A^{-1}u}$. From the previous problem we know that $A^{-1} - \frac{A^{-1}uv^T A-1}{1+v^T A^{-1}u} = A^{-1}(I - \beta uv^T A^{-1}) = (I - \beta uv^T A^{-1})A^{-1}$.(the similar trick) Here we can calculate by left and right multiply.Then we start from right multiplication $(A + uv^T)A^{-1}(I - \beta uv^T A^{-1}) = (I + uv^T A^{-1})(I - \beta uv^T A^{-1}) = I + uv^T A^{-1} - \beta uv^T A^{-1} - \beta uv^T A^{-1}uv^T A^{-1} = I$ which indicate that $uv^T A^{-1} - \beta uv^T A^{-1} - \beta uv^T A^{-1}uv^T A^{-1} = 0$. So here we must have $\hat{\beta} = \frac{1}{1+v^T A^{-1}u}$ note that only $v^T A^{-1}u$ is a number. Similarly we can see that this is based on the assumption that $(1 + v^T A^{-1}u) \neq 0$ or it doesn't make sense.
Then we only need to validate it by the left multiplication which help us to understand. $(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1}u})(A + uv^T) = I + (1 - \beta - \beta u^T A^{-1}v)A^{-1}uv^T$ and this equals to Identity matrix as we plug in the proposed $\hat{\beta}$.
So in conclusion we prove that $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A-1}{1+v^T A^{-1}u}$.