ECS230

Summary 6

Eigenvalue Problems

The Power Method $A \in \mathbb{C}^{n \times n}$ non-defective (i.e. diagonalizable)
There are n eigenpairs (λ_i, x_i) $i = 1, \ldots, n$
We assume that eigenvalues are ordered by magnitude and

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$$

λ_1 is the dominant eigenvalue.
The eigenvectors x_i are linearly independent.
Any non-zero vector $u_0 \in \mathbb{C}^n$ can be written as a linear combination of the x_i:

$$u_0 = \gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n$$

The power method consists in computing $A^k u_0$.

$$A^k u_0 = \gamma_1 \lambda_1^k x_1 + \gamma_2 \lambda_2^k x_2 + \ldots + \gamma_n \lambda_n^k x_n$$

$\lim_{k \to \infty} A^k u_0 = \lambda_1^k \gamma_1 x_1$

Algorithm 1 Power Method

```
k = 0
while not converged do
    u_{k+1} = A u_k
    k := k + 1
end while
```

Algorithm 1 can lead to exponential growth (or decrease) of the norm of u_k, resulting in overflow (or underflow). Algorithm 2 solves this problem by normalizing u_k at each iteration.

Algorithm 2 Power Method

k = 0
while not converged **do**
  $u_{k+1} = Au_k$
  $u_{k+1} := u_{k+1}/||u_{k+1}||$
  $k := k + 1$
end while

Convergence:
The residual vector is $r_k = Au_k - \mu_k u_k$ where μ_k is an approximation of the eigenvalue λ_1. Convergence criterion: stop when $||r_k||$ is "small".
Rayleigh quotient For a given $u \neq 0$ and a given μ, define the residual as $r(\mu) = Au - \mu u$. The residual $r(\mu)$ is minimum if

$$\mu = \frac{u^H A u}{u^H u} \quad \text{(Rayleigh quotient)}$$

Algorithm 3 Power Method with convergence criterion

$x = u_0/||u_0||$
while not converged **do**
  $y = Ax$
  $\mu = x^H Ax$
  $r = y - \mu x$
  $x = y/||y||$
  if $||r|| < \epsilon$ **then**
    converged = true, done
  end if
end while

Convergence rate:
At iteration k, the error $u_k - x_1$ is dominated by a term proportional to $|\lambda_2/\lambda_1|^k$, i.e. at each iteration the error is reduced by a factor $|\lambda_2/\lambda_1|$. The convergence rate is $|\lambda_2/\lambda_1|$.
Shifted power method
Use $A - \sigma I$ instead of A.
$A - \sigma I$ has eigenpairs $(\lambda_1 - \sigma, x_1), (\lambda_2 - \sigma, x_2) \ldots$, i.e. eigenvectors of A are also eigenvectors of $A - \sigma I$. Choosing the shift σ carefully can improve the

convergence rate of the power iteration.

Example 1: A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.99$. The convergence rate of the power method (without shift) is $|0.99/1.0| = 0.99$, i.e. convergence is slow. Using a shift $\sigma = -1/2$, the eigenvalues of $A - \sigma I$ are $\lambda_1 - \sigma = 1.5$ and $\lambda_2 - \sigma = -0.49$. The convergence rate of the (shifted) power iteration is

$$\frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} = |-0.49/1.5| \simeq 0.33$$

and convergence is faster.

Example 2: A has eigenvalues $\lambda_1 = 1$ $\lambda_2 = -0.99$ and $\lambda_3 = 0$. The convergence rate of the power method (without shift) is $|0.99/1.0| = 0.99$, i.e. convergence is slow. Using a shift $\sigma = -1/2$, the eigenvalues of $A - \sigma I$ are $\lambda_1 - \sigma = 1.5$, $\lambda_2 - \sigma = -0.49$ and $\lambda_3 - \sigma = 0.50$. The convergence rate of the (shifted) power iteration is

$$\frac{|\lambda_3 - \sigma|}{|\lambda_1 - \sigma|} = |0.50/1.5| \simeq 0.33$$

and convergence is faster. Note that with this shift, the convergence rate is determined by λ_3 and not λ_2.