Linear Algebra
Homework 3

Prof. Dr. André de Almeida (andre@gtel.ufc.br)
Kenneth B. dos A. Benício (kenneth@gtel.ufc.br)

2024.1

Singular value decomposition applications

Problem 1 (Image reconstruction) The SVD is a powerful tool that can be employed for data compression of images. Since the SVD orders the eigenvalues from the ones with the most energy to the ones with less energy, by analyzing the energy distribution of the eigenvalues of a matrix $\boldsymbol{X}$ with rank R, it is possible to discard the less prominent eigenvalues to eliminate redundancy. To verify this, load the image file inside **"image.zip"** on the IDE of your preference. Supposing that the matrix contained in this file has rank R find the best J-rank approximation, with $J <<< R$, that reconstructs the original image with minimal loss. To achieve this, you can employ the truncated SVD given by

$$\hat{\boldsymbol{A}} = \sum_{j=1}^{J} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}}. \tag{1}$$

To discard the correct number of eigenvalues, analyze the energy profile of the image to choose the best approximation that reproduces the image with minimal loss. After this, consider the following model

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z} \in \mathbb{C}^{M \times N} \tag{2}$$

where $\boldsymbol{Y} \in \mathbb{C}^{M \times N}$ is the original image degraded by noise, $\boldsymbol{X} \in \mathbb{C}^{M \times N}$ is the original image, and $\boldsymbol{Z} \in \mathbb{C}^{M \times N}$ is the AWGN component with zero mean and $\sigma_{\boldsymbol{Z}}^2$ variance. To control the noise level, consider the following formulation for the signal-to-noise-ratio (SNR)

$$\mathrm{SNR} = \frac{||\boldsymbol{Y}||^2}{\sigma_{\boldsymbol{Z}}^2 ||\boldsymbol{Z}||^2}, \tag{3}$$

where $||\boldsymbol{Y}||^2$ is the energy of the signal defined by Equation (2) and $\sigma_{\boldsymbol{z}}^2$ is the variance of the AWGN matrix, $\boldsymbol{Z}$. Analyze the impact of the AWGN component on the reconstruction performance of the SVD. Does the SVD minimize the degradation introduced by the AWGN component?

Problem 2 (Rank estimation) Several signal processing techniques in the context of communications depend on a precise estimation of the rank of the channel matrix. As an example of this class of techniques, the alternating least squares (ALS) algorithm can be employed to solve multidimensional parameter estimation problems. To verify that, let us consider the following signal model

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{I}}_{3,R} \times_1 \boldsymbol{A}^{(1)} \times_2 \boldsymbol{A}^{(2)} \times_3 \boldsymbol{A}^{(3)} \in \mathbb{C}^{I_1 \times I_2 \times I_3} + \boldsymbol{\mathcal{Z}}, \tag{4}$$

where $\boldsymbol{\mathcal{H}}$ is the tensor containing information on all channel parameters. The objective of this exercise is to employ the SVD to correctly guess the rank of the tensor $\boldsymbol{\mathcal{H}}$ and to observe how an incorrect estimation of the rank can potentially degrade the parameter estimation performance of the ALS algorithm. To that end, on **"tensor.zip"**, the necessary files to execute the ALS algorithm on a given tensor $\boldsymbol{\mathcal{H}}$ are provided. The ALS function is defined as

$$[\boldsymbol{y}, x] = \text{ALS\_estimation}(\boldsymbol{\mathcal{H}}, \hat{R}, \text{SNR}) \tag{5}$$

where the inputs are the tensor $\boldsymbol{\mathcal{H}}$, the estimated rank of the reconstruction $\hat{R}$, and the noise level SNR. The outputs of this function are the error vector $\boldsymbol{y}$ containing the error of reconstruction at each iteration of the ALS algorithm, and x represents the error of reconstruction at the final iteration of the ALS algorithm. In light of this, estimate the correct rank of $\boldsymbol{\mathcal{H}}$ by employing the ALS function for a varying number of $\hat{R}(< 10)$ considering 1000 Monte Carlo experiments to acquire the mean performance of the ALS algorithm for each $\hat{R}$. What can you formulate about the impact on the performance of the ALS algorithm when the incorrect rank is chosen?