Linear Algebra
Homework 2

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Least Squares optimization problem

Problem 1 (Data fitting with least squares) For a given set of data points, find the curve that minimizes the sum of the squared vertical distances between the points and the line. Use the file "Homework_02_data.mat"to validate your function. In this file, there are 3 data sets. Solve the problem for each.

Problem 2 (Projection error analysis) Let us consider the following signal model

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{z}, \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^{10\times 1}$ is the received vector at some equipment, $\boldsymbol{A} \in \mathbb{R}^{10\times 5}$ is the known parameter, $\boldsymbol{x} \in \mathbb{R}^{5\times 1}$ is the unknown parameter, and $\boldsymbol{z} \in \mathbb{R}^{10\times 1} \sim \mathcal{N}(0, \sigma_{\boldsymbol{z}}^2)$ is the additive white Gaussian noise (AWGN) component [1]. From Equation (1) we can estimate $\boldsymbol{x}$ by defining the least squares optimization problem as

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}||_2^2, \tag{2}$$

where the knowledge of $\boldsymbol{A}$ is assumed to be known *a priori*. First, derive the theoretical solution that minimizes the quadratic error of the problem defined by Equation (2) in the absence of noise. In the sequence, analyze the performance of the least squares in the presence of AWGN noise. To this end, vary the noise power according to the following formulation for the signal-to-noise-ratio (SNR)

$$\text{SNR} = \frac{||\boldsymbol{y}||^2}{\sigma_{\boldsymbol{z}}^2 ||\boldsymbol{z}||^2}, \tag{3}$$

where $||\boldsymbol{y}||^2$ is the energy of the signal defined by Equation (1) and $\sigma_{\boldsymbol{z}}^2$ is the variance of the AWGN vector, $\boldsymbol{z}$. Regarding the simulation, consider at least $M = 100$ individual

[1] $\mathcal{N}(0, \sigma_{\boldsymbol{z}}^2)$ means that the elements of the vector $\boldsymbol{z}$ are taken from a normal distribution that has zero mean and $\sigma_{\boldsymbol{z}}^2$ variance

experiments to obtain the average behavior of the solution. As the performance metric, let us consider the normalized mean square error (NMSE) given by

$$\text{NMSE}(\boldsymbol{y}) = \mathbb{E}\left\{\frac{||\boldsymbol{y}^{(m)} - \boldsymbol{A}\hat{\boldsymbol{x}}^{(m)}||_2^2}{||\boldsymbol{y}^{(m)}||_2^2}\right\} = \frac{1}{M}\sum_{m=1}^{M}\frac{||\boldsymbol{y}^{(m)} - \boldsymbol{A}\hat{\boldsymbol{x}}^{(m)}||_2^2}{||\boldsymbol{y}^{(m)}||_2^2}, \tag{4}$$

where $\boldsymbol{y}^{(m)}$ is the signal generated following Equation (1) at the mth experiment, and $\hat{\boldsymbol{x}}^{(m)}$ is the estimate solution of the problem defined in Equation (2) at the mth experiment.

Problem 3 (Computational complexity analysis) One of the biggest problems regarding the estimation done by the least squares algorithm is its considerably high complexity due to the computation of the pseudoinverse of $\boldsymbol{A} \in \mathbb{R}^{M\times N}$. The approximate complexity in FLOPS of the computation of the pseudoinverse of a matrix $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ is given by $\max(M, N)\min(M, N)^2$. Plot the complexity varying the number of columns of $\boldsymbol{A} \in \mathbb{R}^{M\times N}$. Consider $M = 100$ and $N = [10, 100, 200, 300, 400, 500]$.