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HatcherNotes

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October 21, 2025

Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps)**.** Let X, Y be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x)$$

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Definition 2 (Path)**.** A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop)**.** def:path

A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths)**.** def:path, def:homotopy

We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y.

Lemma 5 (All paths from x to y in $\mathbb{R}^n$ are Homotopic)**.** *def:path$_h$omotopy*

Any two paths γ_1, γ_2 from x to y in $\mathbb{R}^n$ are homotopic.

Theorem 6 (Homotopy is equvialence relation)**.** *def:homotopy*

Relation $\simeq$ is an equvialence relation.

Theorem 7 (Homotopy of paths is equvialence relation)**.** *thm:homotopy*$_e$*quiv*$, def:$ $path_homotopy$

Relation $\simeq_p$ of paths is an equvialence relation.

Definition 8 (Composition of paths)**.** def:path

Given to paths γ_1, γ_2 we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1-2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Definition 9 (Inverse of paths)**.** def:path

Given to paths γ we definte $\bar{\gamma}$ (inverse of γ) by the formula:

$$\bar{\gamma}(t) = \gamma(1-t)$$

Lemma 10 (Composition of paths is a path)**.** *def:path,def:path*$_c$*omposition*

Composition of paths is a path (The map given by **??** *is continoues)*

Lemma 11 (Inverse of paths is a path)**.** *def:path,def:path*$_i$*nverse*

Inverse of path is a path (The map given by **??** *is continoues)*

Lemma 12 (Composition of paths depend on homotopy class)**.** *def:path, def:path*$_c$*omposition*$, def:$ $path_homotopy, thm : path_homotopy_equiv$

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p athf_1 \cdot g_1$

Lemma 13 (Inverse of paths depend on homotopy class)**.** *def:path, def:path*$_i$*nverse*$, def:$ $path_homotopy, thm : path_homotopy_equiv$

If f_0, f_1 are to homotopic paths then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Theorem 14 (Homotopy of loops is equvialence relation)**.** *thm:path*$_h$*omotopy*$_e$*quiv*$, def:$ $path_homotopy$

Relation $\simeq_l$ of loops is an equvialence relation. (We use $\simeq$ to simplify notation)

Lemma 15 (Composition of loops is a loop)**.** *def:loop, lem:path*$_c$*omp*$_p$*ath*

Composition of loops is a loop.

Lemma 16 (Inverse of loop is a loop)**.** *def:loop,def:path*$_i$*nverse*$_p$*ath*

Inverse of loop is a loop (The map given by **??** *is continoues)*

Lemma 17 (Composition of loops depend on homotopy class)**.** *lem:loop*$_c$*omp*$_l$*oop*$, lem :$ $path_comp_homoclass, thm : loop_homotopy_equiv$

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Lemma 18 (Inverse of loops depend on homotopy class)**.** *lem:loop*$_i$*nverse*$_l$*oop*$, lem :$ $path_inverse_homoclass, thm : loop_homotopy_equiv$

If f_0, f_1 are to homotopic loops then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Definition 19 (Fundamental Group)**.** lem:loop$_comp_homoclass, lem : loop_inverse_homoclass$

We definie the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equvialence classes of relation $\simeq$ with the operation $\cdot$ - composition of loops

Lemma 20 (Composition is associative). *def:fundamental$_g$roup*

The operation $\cdot$ is associative.

Lemma 21 (Composition has natural element). *def:fundamental$_g$roup*

There is an neutral element of $\cdot$, which is $[const_{x_0}]_\simeq$

Lemma 22 (Composition has inverse). *def:fundamental$_g$roup*

For every element of $\pi_1(X, x_0)$ there exists an inverse such that:
$[f] \cdot [g] = [const_{x_0}]$

Theorem 23 (Fundamental Group is a Group). *lem:loop$_c$omp$_i$nv*$, lem : loop_comp_neutral, lem : loop_comp_assoc$

The fundamental group is a group

Theorem 24 (Fundamental Group of $\mathbb{R}^n$). *thm:fundamental$_g$roup$_i$s$_g$roup*$, lem : Rn_path_equiv_lemma$

The fundamental group of $\mathbb{R}^n$ is trivial

1.2 Fundamental Group properties

Definition 25 (β_h map). def:fundamental$_{group}$

Given a path h from x_0 to x_1 we definie a map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as

$$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$$

Lemma 26 (β_h map is well definied). *def:beta$_{hm}$ap*

Map β_h is well definied.

Lemma 27 (β_h map is isomorphism of groups). *def:beta$_{hm}$ap*

Map β_h is an isomorphism.

Lemma 28 (Fundamental Group doesnt depend on base point). *lem:beta$_{hm}$ap$_i$so*

For path connected space fundamental groups based on x_0, x_1 are isomorphic.