HatcherNotes

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Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps)**.** Let X, Y be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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Definition 2 (Path)**.** A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop)**.**

A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths)**.**

We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y.

Lemma 5 (All paths from x to y in $\mathbb{R}^n$ are Homotopic)**.**

Any two paths γ_1, γ_2 from x to y in $\mathbb{R}^n$ are homotopic.

Theorem 6 (Homotopy is equvialence relation)**.**

Relation $\simeq$ is an equvialence relation.

Theorem 7 (Homotopy of paths is equvialence relation)**.**

Relation $\simeq_p$ of paths is an equvialence relation.

Definition 8 (Composition of paths)**.**

Given to paths γ_1, γ_2 we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1 - 2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Lemma 9 (Composition of paths is a path)**.**
Composition of paths is a path (The map given by 8 is continoues)

Lemma 10 (Composition of paths depend on homotopy class)**.**
If $f_0 \simeq_p f_1$ *and* $g_0 \simeq_p g_1$ *then* $f_0 \cdot g_0 \simeq_p athf_1 \cdot g_1$

Theorem 11 (Homotopy of loops is equvialence relation)**.**
Relation $\simeq_l$ *of loops is an equvialence relation. (We use* $\simeq$ *to simplify notation)*

Lemma 12 (Composition of loops is a loop)**.**
Composition of loops is a loop.

Lemma 13 (Composition of loops depend on homotopy class)**.**
If $f_0 \simeq_p f_1$ *and* $g_0 \simeq_p g_1$ *then* $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$