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HatcherNotes

ru6by

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Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps)**.** Let X, Y be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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Definition 2 (Path)**.** A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop)**.** def:path
A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths)**.** def:path, def:homotopy
We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y.

Lemma 5 (All paths from x to y in $\mathbb{R}^n$ are Homotopic)**.** *def:path$_h$omotopy*
Any two paths γ_1, γ_2 from x to y in $\mathbb{R}^n$ are homotopic.

Theorem 6 (Homotopy is equvialence relation)**.** *def:homotopy*
Relation $\simeq$ is an equvialence relation.

Theorem 7 (Homotopy of paths is equvialence relation)**.** *thm:homotopy*$_e$*quiv*$, def:$ $path_homotopy$

Relation $\simeq_p$ of paths is an equvialence relation.

Definition 8 (Composition of paths)**.** def:path

Given to paths γ_1, γ_2 we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1-2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Lemma 9 (Composition of paths is a path)**.** *def:path,def:path*$_c$*omposition*

Composition of paths is a path (The map given by **??** *is continoues)*

Lemma 10 (Composition of paths depend on homotopy class)**.** *def:path, def:path*$_c$*omposition*$, def:$ $path_homotopy, thm: path_homotopy_equiv$

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p ath f_1 \cdot g_1$

Theorem 11 (Homotopy of loops is equvialence relation)**.** *thm:path*$_h$*omotopy*$_e$*quiv*$, def:$ $path_homotopy$

Relation $\simeq_l$ of loops is an equvialence relation. (We use $\simeq$ to simplify notation)

Lemma 12 (Composition of loops is a loop)**.** *def:loop, lem:path*$_c$*omp*$_p$*ath*

Composition of loops is a loop.

Lemma 13 (Composition of loops depend on homotopy class)**.** *lem:loop*$_c$*omp*$_l$*oop*$, lem:$ $path_comp_homoclass, thm: loop_homotopy_equiv$

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Definition 14 (Fundamental Group)**.** lem:loop$_c omp_h omoclass$

We definie the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equvialence classes of relation $\simeq$ with the operation $\cdot$ - composition of loops

Lemma 15 (Composition is associative)**.** *def:fundamental*$_g$*roup*

The operation $\cdot$ is associative.

Lemma 16 (Composition has natural element)**.** *def:fundamental*$_g$*roup*

There is an neutral element of $\cdot$, which is $[const_{x_0}]_\simeq$

Lemma 17 (Composition has inverse)**.** *def:fundamental*$_g$*roup*

For every element of $\pi_1(X, x_0)$ there exists an inverse such that:
$[f] \cdot [g] = [const_{x_0}]$

Theorem 18 (Fundamental Group is a Group)**.** *lem:loop*$_c$*omp*$_i$*nv*$, lem: loop_comp_neutral, lem:$ $loop_comp_assoc$

The fundamental group is a group

Theorem 19 (Fundamental Group of $\mathbb{R}^n$)**.** *thm:fundamental*$_g$*roup*$_i$*s*$_g$*roup*$, lem:$ $Rn_path_equiv_lemma$

The fundamental group of $\mathbb{R}^n$ is trivial