# Hatcher Algebraic Topology

the Scientific Circle of Theoretical Mathematicians

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# Chapter 1

# Fundamental Group of Circle

## 1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

**Definition 1** (Homotopy of maps)**.** Let $X, Y$ be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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**Definition 2** (Path)**.** A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

**Definition 3** (Loop)**.**

A loop is a path where $x = y$.

**Definition 4** (Homotopy of Paths)**.**

We say that two paths $\gamma_1, \gamma_2$ ($\gamma_1 \simeq_p \gamma_2$) from $x$ to $y$ are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of $\gamma_1, \gamma_2$ and for all $t \in I$ function $H(\cdot, t)$ is a path from $x$ to $y$.

**Definition 5** (Homotopy of Loops)**.**

We say that two loops $\gamma_1, \gamma_2$ ($\gamma_1 \simeq_l \gamma_2$) based in $x$ are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of $\gamma_1, \gamma_2$ and for all $t \in I$ function $H(\cdot, t)$ is a loop based in $x$.

**Lemma 6** (All paths from $x$ to $y$ in $\mathbb{R}^n$ are Homotopic)**.**

*Any two paths $\gamma_1, \gamma_2$ from $x$ to $y$ in $\mathbb{R}^n$ are homotopic.*

*Proof.* See here! □

**Theorem 7** (Homotopy is eqvivalence relation)**.**

*Relation $\simeq$ is an eqvivalence relation.*

**Theorem 8** (Homotopy of paths is equvialence relation)**.**
*Relation $\simeq_p$ on paths is an equvialence relation.*

*Proof.* See here! ◻

**Theorem 9** (Homotopy of loops is equvialence relation)**.**
*Relation $\simeq_l$ on loops is an equvialence relation.*

*Proof.* See here! ◻

**Definition 10** (Composition of paths)**.**
Given to paths $\gamma_1, \gamma_2$ we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1-2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

**Definition 11** (Inverse of paths)**.**
Given to paths $\gamma$ we definte $\bar{\gamma}$ (inverse of $\gamma$) by the formula:

$$\bar{\gamma}(t) = \gamma(1-t)$$

**Lemma 12** (Composition of paths is a path)**.**
*Composition of paths is a path (The map given by 10 is continoues)*

**Lemma 13** (Inverse of paths is a path)**.**
*Inverse of path is a path (The map given by 11 is continoues)*

**Lemma 14** (Composition of paths depend on homotopy class)**.**
*If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p ath f_1 \cdot g_1$*

**Lemma 15** (Inverse of paths depend on homotopy class)**.**
*If $f_0, f_1$ are to homotopic paths then $\bar{f}_0, \bar{f}_1$ are also homotopic.*

**Theorem 16** (Homotopy of loops is equvialence relation)**.**
*Relation $\simeq_l$ of loops is an equvialence relation. (We use $\simeq$ to simplify notation)*

**Lemma 17** (Composition of loops is a loop)**.**
*Composition of loops is a loop.*

**Lemma 18** (Inverse of loop is a loop)**.**
*Inverse of loop is a loop (The map given by 11 is continoues)*

**Lemma 19** (Composition of loops depend on homotopy class)**.**
*If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$*

**Lemma 20** (Inverse of loops depend on homotopy class)**.**
*If $f_0, f_1$ are to homotopic loops then $\bar{f}_0, \bar{f}_1$ are also homotopic.*

**Definition 21** (Fundamental Group)**.**
We definie the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equvialence classes of relation $\simeq$ with the operation $\cdot$ - composition of loops

**Lemma 22** (Composition is associative)**.**
*The operation $\cdot$ is associative.*

**Lemma 23** (Composition has natural element)**.**
*There is an neutral element of* $\cdot$*, which is* $[const_{x_0}]_{\simeq}$

**Lemma 24** (Composition has inverse)**.**
*For every element of* $\pi_1(X, x_0)$ *there exists an inverse such that:*
$[f] \cdot [g] = [const_{x_0}]$

**Theorem 25** (Fundamental Group is a Group)**.**
*The fundamental group is a group*

*Proof.* See here! □

**Theorem 26** (Fundamental Group of $\mathbb{R}^n$)**.**
*The fundamental group of* $\mathbb{R}^n$ *is trivial*

*Proof.* See here! □

## 1.2 Fundamental Group properties

**Definition 27** ($\beta_h$ map)**.**
Given a path $h$ from $x_0$ to $x_1$ we definie a map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as

$$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$$

**Lemma 28** ($\beta_h$ map is well definied)**.**
*Map* $\beta_h$ *is well definied.*

*Proof.* See here! □

**Lemma 29** ($\beta_h$ map is isomorphism of groups)**.**
*Map* $\beta_h$ *is an isomorphism.*

*Proof.* See here! □

**Lemma 30** (Fundamental Group doesnt depend on base point)**.**
*For path connected space fundamental groups based on* $x_0, x_1$ *are isomorphic.*

*Proof.* See here! □

**Definition 31** (Loops in $S^1$)**.** For $n \in \mathbb{Z}$ let us define:

$$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$$

Note: $\omega_n$ is the loop running $n$-times around the circle clockwise or counterclockwise depending on the sign of $n$.

**Lemma 32** ($\omega_n$ is a loop)**.**
*For each* $n \in \mathbb{Z}$*:* $\omega_n$ *is a loop based in* $(1, 0)$*.*

**Definition 33** (Evenly covered)**.** Let $f : X \to Y$ be a map and $U \subset Y$ be an open set. We say that $U$ is evenly covered by $f$ when $f^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto $U$ by $f$.

**Definition 34** (Covering space).

Given a space $X$, a covering space is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that: For each point $x \in X$, $x$ has a evenly covered neighbourhood $U$ by $p$.

**Lemma 35** (Homotopy lifting property).

*Given a map $F : Y \times I \to X$ and a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ lifting $F|Y \times \{0\}$, then there is a unique map $\tilde{F} : Y \times I \to \tilde{X}$ lifting $F$ and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.*

*Proof.* See here! (Part "c)" in this proof )
Or here! Here it is done for $S^1$ but it can be swaped to $X$ ☐

**Lemma 36** (Path lifting property).

*For each path $f : I \to X$ starting at a point $x_0 \in X$ and each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x_0}$.*

*Proof.* See here! ☐

**Lemma 37** (Homotopy path lifting property).

*For each homotopy $H : X \times I \to X$ of paths starting at $x_0$ and each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{H} : X \times I \to \tilde{X}$ of paths starting at $\tilde{x_0}$.*

*Proof.* See here! ☐

**Lemma 38** ($\omega_n$ are not homotopic).

*Let $n \neq m$ be integers. Then $\omega_n \nsim \omega_m$.*

*Proof.* See here! ☐

**Lemma 39** (All $S^1$ loops are $\omega_n$).

*Let $\gamma$ be a loop in $S^1$ based at $(1, 0)$. Then there exists $n \in \mathbb{Z}$ such that $\gamma \sim \omega_n$.*

*Proof.* See here! ☐

**Lemma 40** (Additive structure on $\omega_n$). *Let $n, m \in \mathbb{Z}$. Then*

$$[\omega_n] \cdot [\omega_m] = [\omega_{n+m}]$$

**Theorem 41** (Fundamental group of S1).

$\pi_1(S^1) \cong \mathbb{Z}$

*Proof.* See here! ☐