

Hatcher Algebraic Topology

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Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps). Let X, Y be topological spaces. We say that maps $f, g : X \rightarrow Y$ are homotopic ($f \simeq g$) iff there exists a continuous map $H : X \times I \rightarrow Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

Definition 2 (Path). A path between points $x, y \in X$ is a continuous function $\gamma : I \rightarrow X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop).

A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths).

We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \rightarrow X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y .

Lemma 5 (All paths from x to y in \mathbb{R}^n are Homotopic).

Any two paths γ_1, γ_2 from x to y in \mathbb{R}^n are homotopic.

Proof. See here! □

Theorem 6 (Homotopy is equivalence relation).

Relation \simeq is an equivalence relation.

Theorem 7 (Homotopy of paths is equivalence relation).

Relation \simeq_p of paths is an equivalence relation.

Proof. See here! □

Definition 8 (Composition of paths).

Given to paths γ_1, γ_2 we define $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1 - 2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Definition 9 (Inverse of paths).

Given to paths γ we define $\bar{\gamma}$ (inverse of γ) by the formula:

$$\bar{\gamma}(t) = \gamma(1 - t)$$

Lemma 10 (Composition of paths is a path).

Composition of paths is a path (The map given by 8 is continuous)

Lemma 11 (Inverse of paths is a path).

Inverse of path is a path (The map given by 9 is continuous)

Lemma 12 (Composition of paths depend on homotopy class).

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Lemma 13 (Inverse of paths depend on homotopy class).

If f_0, f_1 are two homotopic paths then \bar{f}_0, \bar{f}_1 are also homotopic.

Theorem 14 (Homotopy of loops is equivalence relation).

Relation \simeq_l of loops is an equivalence relation. (We use \simeq to simplify notation)

Lemma 15 (Composition of loops is a loop).

Composition of loops is a loop.

Lemma 16 (Inverse of loop is a loop).

Inverse of loop is a loop (The map given by 9 is continuous)

Lemma 17 (Composition of loops depend on homotopy class).

If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Lemma 18 (Inverse of loops depend on homotopy class).

If f_0, f_1 are two homotopic loops then \bar{f}_0, \bar{f}_1 are also homotopic.

Definition 19 (Fundamental Group).

We define the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equivalence classes of relation \simeq with the operation \cdot - composition of loops

Lemma 20 (Composition is associative).

The operation \cdot is associative.

Lemma 21 (Composition has natural element).

There is a neutral element of \cdot , which is $[const_{x_0}]_\simeq$

Lemma 22 (Composition has inverse).

For every element of $\pi_1(X, x_0)$ there exists an inverse such that:

$$[f] \cdot [g] = [const_{x_0}]$$

Theorem 23 (Fundamental Group is a Group).

The fundamental group is a group

Proof. See here! □

Theorem 24 (Fundamental Group of \mathbb{R}^n).
The fundamental group of \mathbb{R}^n is trivial

Proof. See here! □

1.2 Fundamental Group properties

Definition 25 (β_h map).

Given a path h from x_0 to x_1 we define a map $\beta_h : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ as

$$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$$

Lemma 26 (β_h map is well defined).

Map β_h is well defined.

Proof. See here! □

Lemma 27 (β_h map is isomorphism of groups).

Map β_h is an isomorphism.

Proof. See here! □

Lemma 28 (Fundamental Group doesn't depend on base point).

For path connected space fundamental groups based on x_0, x_1 are isomorphic.

Proof. See here! □

Definition 29 (Loops in S^1). For $n \in \mathbb{Z}$ let us define:

$$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$$

Note: ω_n is the loop running n -times around the circle clockwise or counterclockwise depending on the sign of n .

Lemma 30 (ω_n is a loop).

For each $n \in \mathbb{Z}$: ω_n is a loop based in $(1, 0)$.

Definition 31 (Evenly covered). Let $f : X \rightarrow Y$ be a map and $U \subset Y$ be an open set. We say that U is evenly covered by f when $f^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by f .

Definition 32 (Covering space).

Given a space X , a covering space is a space \tilde{X} together with a map $p : \tilde{X} \rightarrow X$ such that:
For each point $x \in X$, x has a evenly covered neighbourhood U by p .

Lemma 33 (Homotopy lifting property).

Given a map $F : Y \times I \rightarrow X$ and a map $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$ lifting $F|Y \times \{0\}$, then there is a unique map $\tilde{F} : Y \times I \rightarrow \tilde{X}$ lifting F and restricting to the given \tilde{F} on $Y \times \{0\}$.

Proof. See here! (Part "c)" in this proof)

Or here! Here it is done for S^1 but it can be swapped to X □

Lemma 34 (Path lifting property).

For each path $f : I \rightarrow X$ starting at a point $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \rightarrow \tilde{X}$ starting at \tilde{x}_0 .

Proof. See here! □

Lemma 35 (Homotopy path lifting property).

For each homotopy $H : X \times I \rightarrow X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{H} : \tilde{X} \times I \rightarrow \tilde{X}$ of paths starting at \tilde{x}_0 .

Proof. See here! □

Lemma 36 (ω_n are not homotopic).

Let $n \neq m$ be integers. Then $\omega_n \not\sim \omega_m$.

Proof. See here! □

Lemma 37 (All S^1 loops are ω_n).

Let γ be a loop in S^1 based at $(1, 0)$. Then there exists $n \in \mathbb{Z}$ such that $\gamma \sim \omega_n$.

Proof. See here! □

Lemma 38 (Additive structure on ω_n). Let $n, m \in \mathbb{Z}$. Then

$$[\omega_n] \cdot [\omega_m] = [\omega_{n+m}]$$

Theorem 39 (Fundamental group of S^1).

$$\pi_1(S^1) \cong \mathbb{Z}$$

Proof. See here! □