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# Hatcher Algebraic Topology

the Scientific Circle of Theoretical Mathematicians

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# Chapter 1

## Fundamental Group of Circle

### 1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

**Definition 1** (Homotopy of maps). Let  $X, Y$  be topological spaces. We say that maps  $f, g : X \rightarrow Y$  are homotopic ( $f \simeq g$ ) iff there exists a continuous map  $H : X \times I \rightarrow Y$  such that for any  $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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**Definition 2** (Path). A path between points  $x, y \in X$  is a continuous function  $\gamma : I \rightarrow X$  such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

**Definition 3** (Loop). *def: path*

A loop is a path where  $x = y$ .

**Definition 4** (Homotopy of Paths). *def: path, def: homotopy*

We say that two paths  $\gamma_1, \gamma_2$  ( $\gamma_1 \simeq_p \gamma_2$ ) from  $x$  to  $y$  are homotopic iff there exists homotopy map  $H : I \times I \rightarrow X$  such that  $H$  is homotopy of  $\gamma_1, \gamma_2$  and for all  $t \in I$  function  $H(\cdot, t)$  is a path from  $x$  to  $y$ .

**Lemma 5** (All paths from  $x$  to  $y$  in  $\mathbb{R}^n$  are Homotopic). *def: path, homotopy*

Any two paths  $\gamma_1, \gamma_2$  from  $x$  to  $y$  in  $\mathbb{R}^n$  are homotopic.

*Proof.* See here!

□

**Theorem 6** (Homotopy is equivalence relation). *def:homotopy*  
*Relation  $\simeq$  is an equivalence relation.*

**Theorem 7** (Homotopy of paths is equivalence relation). *thm:homotopyequiv, def : path\_homotopy*  
*Relation  $\simeq_p$  of paths is an equivalence relation.*

*Proof.* See here! □

**Definition 8** (Composition of paths). *def:path*  
 Given to paths  $\gamma_1, \gamma_2$  we define  $\gamma_1 \cdot \gamma_2$  by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1 - 2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

**Definition 9** (Inverse of paths). *def:path*  
 Given to paths  $\gamma$  we define  $\bar{\gamma}$  (inverse of  $\gamma$ ) by the formula:

$$\bar{\gamma}(t) = \gamma(1 - t)$$

**Lemma 10** (Composition of paths is a path). *def:path, def:path\_composition*  
*Composition of paths is a path (The map given by ?? is continuous)*

**Lemma 11** (Inverse of paths is a path). *def:path, def:path\_inverse*  
*Inverse of path is a path (The map given by ?? is continuous)*

**Lemma 12** (Composition of paths depend on homotopy class). *def:path, def:path\_composition, def : path\_homotopy, thm : path\_homotopyequiv*  
*If  $f_0 \simeq_p f_1$  and  $g_0 \simeq_p g_1$  then  $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$*

**Lemma 13** (Inverse of paths depend on homotopy class). *def:path, def:path\_inverse, def : path\_homotopy, thm : path\_homotopyequiv*  
*If  $f_0, f_1$  are homotopic paths then  $\bar{f}_0, \bar{f}_1$  are also homotopic.*

**Theorem 14** (Homotopy of loops is equivalence relation). *thm:path\_homotopyequiv, def : path\_homotopy*  
*Relation  $\simeq_l$  of loops is an equivalence relation. (We use  $\simeq$  to simplify notation)*

**Lemma 15** (Composition of loops is a loop). *def:loop, lem:path\_comppath*  
*Composition of loops is a loop.*

**Lemma 16** (Inverse of loop is a loop). *def:loop, def:path\_inverse\_path*  
*Inverse of loop is a loop (The map given by ?? is continuous)*

**Lemma 17** (Composition of loops depend on homotopy class). *lem:loop\_comploop, lem : path\_comphomoclass, thm : loop\_homotopyequiv*  
*If  $f_0 \simeq_p f_1$  and  $g_0 \simeq_p g_1$  then  $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$*

**Lemma 18** (Inverse of loops depend on homotopy class). *lem:loop<sub>i</sub>nverse<sub>l</sub>oop, lem : path<sub>i</sub>nverse<sub>h</sub>omoclass, thm : loop<sub>h</sub>omotopy<sub>e</sub>quiv*

*If  $f_0, f_1$  are to homotopic loops then  $\bar{f}_0, \bar{f}_1$  are also homotopic.*

**Definition 19** (Fundamental Group). *lem:loop<sub>c</sub>omp<sub>h</sub>omoclass, lem : loop<sub>i</sub>nverse<sub>h</sub>omoclass*

We define the fundamental group of  $(\pi_1(X, x_0), \cdot)$  as the set of equivalence classes of relation  $\simeq$  with the operation  $\cdot$  - composition of loops

**Lemma 20** (Composition is associative). *def:fundamental<sub>g</sub>roup*

*The operation  $\cdot$  is associative.*

**Lemma 21** (Composition has natural element). *def:fundamental<sub>g</sub>roup*

*There is an neutral element of  $\cdot$ , which is  $[const_{x_0}] \simeq$*

**Lemma 22** (Composition has inverse). *def:fundamental<sub>g</sub>roup*

*For every element of  $\pi_1(X, x_0)$  there exists an inverse such that:*

$$[f] \cdot [g] = [const_{x_0}]$$

**Theorem 23** (Fundamental Group is a Group). *lem:loop<sub>c</sub>omp<sub>i</sub>nv, lem : loop<sub>c</sub>omp<sub>n</sub>eutral, lem : loop<sub>c</sub>omp<sub>a</sub>ssoc*

*The fundamental group is a group*

*Proof.* See here! □

**Theorem 24** (Fundamental Group of  $\mathbb{R}^n$ ). *thm:fundamental<sub>g</sub>roup<sub>i</sub>s<sub>g</sub>roup, lem : Rn<sub>path<sub>e</sub>quiv<sub>i</sub>emma</sub>*

*The fundamental group of  $\mathbb{R}^n$  is trivial*

*Proof.* See here! □

## 1.2 Fundamental Group properties

**Definition 25** ( $\beta_h$  map). *def:fundamental<sub>g</sub>roup*

Given a path  $h$  from  $x_0$  to  $x_1$  we define a map  $\beta_h : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$  as

$$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$$

**Lemma 26** ( $\beta_h$  map is well defined). *def:beta<sub>h</sub>m<sub>a</sub>p*

*Map  $\beta_h$  is well defined.*

*Proof.* See here! □

**Lemma 27** ( $\beta_h$  map is isomorphism of groups). *def:beta<sub>h</sub>m<sub>a</sub>p*

*Map  $\beta_h$  is an isomorphism.*

*Proof.* See here! □

**Lemma 28** (Fundamental Group doesn't depend on base point). *lem:beta<sub>h</sub>m<sub>a</sub>p<sub>i</sub>so*

*For path connected space fundamental groups based on  $x_0, x_1$  are isomorphic.*

*Proof.* See here! □

**Definition 29** (Loops in  $S^1$ ). For  $n \in \mathbb{Z}$  let us define:

$$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$$

Note:  $\omega_n$  is the loop running  $n$ -times around the circle clockwise or counterclockwise depending on the sign of  $n$ .

**Lemma 30** ( $\omega_n$  is a loop). *def:wnloop, def:loop*

For each  $n \in \mathbb{Z}$ :  $\omega_n$  is a loop based in  $(1, 0)$ .

**Definition 31** (Evenly covered). Let  $f : X \rightarrow Y$  be a map and  $U \subset Y$  be an open set. We say that  $U$  is evenly covered by  $f$  when  $f^{-1}(U)$  is a union of disjoint open sets each of which is mapped homeomorphically onto  $U$  by  $f$ .

**Definition 32** (Covering space). *def:evenlycovered*

Given a space  $X$ , a covering space is a space  $\tilde{X}$  together with a map  $p : \tilde{X} \rightarrow X$  such that: For each point  $x \in X$ ,  $x$  has a evenly covered neighbourhood  $U$  by  $p$ .

**Lemma 33** (Homotopy lifting property). *def:coveringspace*

Given a map  $F : Y \times I \rightarrow X$  and a map  $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$  lifting  $F|_{Y \times \{0\}}$ , then there is a unique map  $\tilde{F} : Y \times I \rightarrow \tilde{X}$  lifting  $F$  and restricting to the given  $\tilde{F}$  on  $Y \times \{0\}$ .

*Proof.* See here! (Part "c)" in this proof )

Or here! Here it is done for  $S^1$  but it can be swaped to  $X$  □

**Lemma 34** (Path lifting property). *lem:homotoplifting*

For each path  $f : I \rightarrow X$  starting at a point  $x_0 \in X$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$  there is a unique lift  $\tilde{f} : I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .

*Proof.* See here! □

**Lemma 35** (Homotopy path lifting property). *lem:homotoplifting*

For each homotopy  $H : X \times I \rightarrow X$  of paths starting at  $x_0$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$  there is a unique lifted homotopy  $\tilde{H} : X \times I \rightarrow \tilde{X}$  of paths starting at  $\tilde{x}_0$ .

*Proof.* See here! □

**Lemma 36** ( $\omega_n$  are not homotopic). *lem:homotoppathlifting, lem: pathlifting, lem: Rnpathequivlemma, lem: wnloop, sloop*

Let  $n \neq m$  be integers. Then  $\omega_n \not\sim \omega_m$ .

*Proof.* See here! □

**Lemma 37** (All  $S^1$  loops are  $\omega_n$ ). *lem:homotoppathlifting, lem: pathlifting, lem: Rnpathequivlemma, lem: wnloop, sloop*

Let  $\gamma$  be a loop in  $S^1$  based at  $(1, 0)$ . Then there exists  $n \in \mathbb{Z}$  such that  $\gamma \sim \omega_n$ .

*Proof.* See here!

□

**Lemma 38** (Additive structure on  $\omega_n$ ). *Let  $n, m \in \mathbb{Z}$ . Then*

$$[\omega_n] \cdot [\omega_m] = [\omega_{n+m}]$$

**Theorem 39** (Fundamental group of  $S^1$ ). *def:fundamental\_group, lem : loops1\_homotopic\_w n, lem : wn\_def\_homoclass, lem : wn\_comp\_is\_sum*  
 $\pi_1(S^1) \cong \mathbb{Z}$

*Proof.* See here!

□