Hatcher Algebraic Topology

the Scientific Circle of Theoretical Mathematicians

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Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps)**.** Let X, Y be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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Definition 2 (Path)**.** A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop)**.**
A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths)**.**
We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y.

Lemma 5 (All paths from x to y in $\mathbb{R}^n$ are Homotopic)**.**
Any two paths γ_1, γ_2 from x to y in $\mathbb{R}^n$ are homotopic.

Proof. See here! □

Theorem 6 (Homotopy is equvialence relation)**.**
Relation $\simeq$ is an equvialence relation.

Theorem 7 (Homotopy of paths is equvialence relation)**.**
Relation $\simeq_p$ of paths is an equvialence relation.

Proof. See here! □

Definition 8 (Composition of paths)**.**
Given to paths γ_1, γ_2 we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1-2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Definition 9 (Inverse of paths)**.**
Given to paths γ we definte $\bar{\gamma}$ (inverse of γ) by the formula:

$$\bar{\gamma}(t) = \gamma(1-t)$$

Lemma 10 (Composition of paths is a path)**.**
Composition of paths is a path (The map given by 8 is continoues)

Lemma 11 (Inverse of paths is a path)**.**
Inverse of path is a path (The map given by 9 is continoues)

Lemma 12 (Composition of paths depend on homotopy class)**.**
If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p athf_1 \cdot g_1$

Lemma 13 (Inverse of paths depend on homotopy class)**.**
If f_0, f_1 are to homotopic paths then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Theorem 14 (Homotopy of loops is equvialence relation)**.**
Relation $\simeq_l$ of loops is an equvialence relation. (We use $\simeq$ to simplify notation)

Lemma 15 (Composition of loops is a loop)**.**
Composition of loops is a loop.

Lemma 16 (Inverse of loop is a loop)**.**
Inverse of loop is a loop (The map given by 9 is continoues)

Lemma 17 (Composition of loops depend on homotopy class)**.**
If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Lemma 18 (Inverse of loops depend on homotopy class)**.**
If f_0, f_1 are to homotopic loops then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Definition 19 (Fundamental Group)**.**
We definie the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equvialence classes of relation $\simeq$ with the operation $\cdot$ - composition of loops

Lemma 20 (Composition is associative)**.**
The operation $\cdot$ is associative.

Lemma 21 (Composition has natural element)**.**
There is an neutral element of $\cdot$, which is $[const_{x_0}]_{\simeq}$

Lemma 22 (Composition has inverse)**.**
For every element of $\pi_1(X, x_0)$ there exists an inverse such that:
$[f] \cdot [g] = [const_{x_0}]$

Theorem 23 (Fundamental Group is a Group)**.**
The fundamental group is a group

Proof. See here! ∎

Theorem 24 (Fundamental Group of $\mathbb{R}^n$)**.**
The fundamental group of $\mathbb{R}^n$ is trivial

Proof. See here! ∎