Hatcher Algebraic Topology

the Scientific Circle of Theoretical Mathematicians

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Chapter 1

Fundamental Group of Circle

1.1 Homotopy Definitions

In this section we provide all the definition , lemmas and theorems regarding homotopies. At the end we provide the definition of Fundamental Group of Topological Space and proof that it has a group structure.

Definition 1 (Homotopy of maps). Let X, Y be topological spaces. We say that maps $f, g : X \to Y$ are homotopic ($f \simeq g$) iff there exists a continoues map $H : X \times I \to Y$ such that for any $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

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Definition 2 (Path). A path between points $x, y \in X$ is a continoues function $\gamma : I \to X$ such that

$$\gamma(0) = x \text{ and } \gamma(1) = y$$

Definition 3 (Loop).
    A loop is a path where $x = y$.

Definition 4 (Homotopy of Paths).
    We say that two paths γ_1, γ_2 ($\gamma_1 \simeq_p \gamma_2$) from x to y are homotopic iff there exists homotopy map $H : I \times I \to X$ such that H is homotopy of γ_1, γ_2 and for all $t \in I$ function $H(\cdot, t)$ is a path from x to y.

Lemma 5 (All paths from x to y in $\mathbb{R}^n$ are Homotopic).
    Any two paths γ_1, γ_2 from x to y in $\mathbb{R}^n$ are homotopic.

Proof. See here! □

Theorem 6 (Homotopy is equvialence relation).
    Relation $\simeq$ is an equvialence relation.

Theorem 7 (Homotopy of paths is equvialence relation).
    Relation $\simeq_p$ of paths is an equvialence relation.

Proof. See here! □

Definition 8 (Composition of paths)**.**
    Given to paths γ_1, γ_2 we definte $\gamma_1 \cdot \gamma_2$ by the formula:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2s), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(1-2s), & \text{if } t \geq \frac{1}{2} \end{cases}$$

Definition 9 (Inverse of paths)**.**
    Given to paths γ we definte $\bar{\gamma}$ (inverse of γ) by the formula:

$$\bar{\gamma}(t) = \gamma(1-t)$$

Lemma 10 (Composition of paths is a path)**.**
    Composition of paths is a path (The map given by 8 is continoues)

Lemma 11 (Inverse of paths is a path)**.**
    Inverse of path is a path (The map given by 9 is continoues)

Lemma 12 (Composition of paths depend on homotopy class)**.**
    If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p ath f_1 \cdot g_1$

Lemma 13 (Inverse of paths depend on homotopy class)**.**
    If f_0, f_1 are to homotopic paths then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Theorem 14 (Homotopy of loops is equvialence relation)**.**
    Relation $\simeq_l$ of loops is an equvialence relation. (We use $\simeq$ to simplify notation)

Lemma 15 (Composition of loops is a loop)**.**
    Composition of loops is a loop.

Lemma 16 (Inverse of loop is a loop)**.**
    Inverse of loop is a loop (The map given by 9 is continoues)

Lemma 17 (Composition of loops depend on homotopy class)**.**
    If $f_0 \simeq_p f_1$ and $g_0 \simeq_p g_1$ then $f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$

Lemma 18 (Inverse of loops depend on homotopy class)**.**
    If f_0, f_1 are to homotopic loops then $\bar{f}_0, \bar{f}_1$ are also homotopic.

Definition 19 (Fundamental Group)**.**
    We definie the fundamental group of $(\pi_1(X, x_0), \cdot)$ as the set of equvialence classes of relation $\simeq$ with the operation $\cdot$ - composition of loops

Lemma 20 (Composition is associative)**.**
    The operation $\cdot$ is associative.

Lemma 21 (Composition has natural element)**.**
    There is an neutral element of $\cdot$, which is $[const_{x_0}]_{\simeq}$

Lemma 22 (Composition has inverse)**.**
    For every element of $\pi_1(X, x_0)$ there exists an inverse such that:
    $[f] \cdot [g] = [const_{x_0}]$

Theorem 23 (Fundamental Group is a Group)**.**
    The fundamental group is a group

Proof. □

Theorem 24 (Fundamental Group of $\mathbb{R}^n$)**.**
    The fundamental group of $\mathbb{R}^n$ is trivial

Proof. □

1.2   Fundamental Group properties

Definition 25 (β_h map)**.**
    Given a path h from x_0 to x_1 we definie a map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as

$$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$$

Lemma 26 (β_h map is well definied)**.**
    Map β_h is well definied.

Proof. □

Lemma 27 (β_h map is isomorphism of groups)**.**
    Map β_h is an isomorphism.

Proof. □

Lemma 28 (Fundamental Group doesnt depend on base point)**.**
    For path connected space fundamental groups based on x_0, x_1 are isomorphic.

Proof. □

Definition 29 (Loops in S^1)**.** For $n \in \mathbb{Z}$ let us define:

$$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$$

    Note: ω_n is the loop running n-times around the circle clockwise or counterclockwise depending on the sign of n.

Lemma 30 (ω_n is a loop)**.**
    For each $n \in \mathbb{Z}$: ω_n is a loop based in $(1, 0)$.

Definition 31 (Evenly covered)**.** Let $f : X \to Y$ be a map and $U \subset Y$ be an open set. We say that U is evenly covered by f when $f^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by f.

Definition 32 (Covering space)**.**
    Given a space X, a covering space is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that: For each point $x \in X$, x has a evenly covered neighbourhood U by p.

Lemma 33 (Homotopy lifting property)**.**
    Given a map $F : Y \times I \to X$ and a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ lifting $F|Y \times \{0\}$, then there is a unique map $\tilde{F} : Y \times I \to \tilde{X}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

Proof. (Part "c)" in this proof)
Here it is done for S^1 but it can be swaped to X □

Lemma 34 (Path lifting property)**.**

For each path $f : I \to X$ starting at a point $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x}_0$.

Proof. See here! □

Lemma 35 (Homotopy path lifting property)**.**

For each homotopy $H : X \times I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{H} : X \times I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

Proof. See here! □

Lemma 36 (ω_n are not homotopic)**.**

Let $n \neq m$ be integers. Then $\omega_n \nsim \omega_m$.

Proof. See here! □

Lemma 37 (All S^1 loops are ω_n)**.**

Let γ be a loop in S^1 based at $(1,0)$. Then there exists $n \in \mathbb{Z}$ such that $\gamma \sim \omega_n$.

Proof. See here! □

Lemma 38 (Additive structure on ω_n)**.** *Let $n, m \in \mathbb{Z}$. Then*

$$[\omega_n] \cdot [\omega_m] = [\omega_{n+m}]$$

Theorem 39 (Fundamental group of S1)**.**

$\pi_1(S^1) \cong \mathbb{Z}$

Proof. See here! □