5/3/21 Lec 20

$$P(X_*|X) \propto \int_0^\infty \int_{\mathbb{R}} k(X_*|\theta,\sigma^2)\, k(\theta,\sigma^2|X)\, d\theta\, d\sigma^2$$

Assume Jeffrey's Prior $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$

$$= \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_*-\theta)^2} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\sigma^2)}\, d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\underbrace{\left((X_*-\theta)^2+n(\theta-\bar{X})^2\right)}_{X_*^2-2X_*\theta+\theta^2+n\theta^2-2n\theta\bar{X}+n\bar{X}^2}} d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2+X_*^2/2+n\bar{X}^2/2}{\sigma^2}} \int_{\mathbb{R}} \overbrace{e^{\frac{X_*+n\bar{X}}{\sigma^2}\theta-\frac{n+1}{2\sigma^2}\theta^2}}^{e^{a\theta+b\theta^2}}\, d\theta\, d\sigma^2$$

Sidenote:

$$\int_{\mathbb{R}} e^{a\theta+b\theta^2}\, d\theta = \sqrt{\frac{\pi}{b}}\, e^{a^2/4b}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2+X_*^2/2+n\bar{X}^2/2}{\sigma^2}} \sqrt{\frac{\pi}{\frac{n+1}{2\sigma^2}}}\, e^{\left(\frac{X_*+n\bar{X}}{\sigma^2}\right)^2 / 4\frac{n+1}{2\sigma^2}}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2+X_*^2/2+n\bar{X}^2/2}{\sigma^2}} \sqrt{\frac{\pi}{\frac{n+1}{2\sigma^2}}}\, e^{\frac{(X_*+n\bar{X})^2}{2\sigma^2(n+1)}}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{\overbrace{(n-1)s^2/2+X_*^2/2+n\bar{X}^2/2-(X_*+n\bar{X})^2/(2(n+1))}^{\beta}}{\sigma^2}} (\sigma^2)^{\frac{1}{2}} \sqrt{\frac{2\pi}{n+1}}\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{\overbrace{-\frac{n}{2}}^{\alpha}-1} e^{-\frac{\beta}{\sigma^2}}\, d\sigma^2 = \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\beta^{-\alpha} \propto \beta^{-\alpha}$$

$$= \left[\frac{(n-1)s^2}{2} + \frac{X_*^2}{2} + \frac{n\bar{X}^2}{2} - \frac{\overbrace{(X_*+n\bar{X})^2}^{X_*^2+2nX_*\bar{X}+n^2\bar{X}^2}}{2n+2}\right]^{-\frac{n}{2}} = (aX_*^2+bX_*+c)^{-\frac{n}{2}}$$

$$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1-\frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$$

$$b = \frac{2n\bar{X}}{2n+2} = -\frac{n\bar{X}}{n+1}$$

$$c = \frac{(n-1)s^2}{2} + \frac{n\bar{X}^2}{2} - \frac{n^2\bar{X}^2}{2n+2} = \frac{1}{2}\left((n-1)s^2+n\bar{X}^2-\frac{n^2\bar{X}^2}{n+1}\right)$$

$$= \left(\frac{1}{a}\right)^{\frac{n}{2}} \left(\frac{1}{a}\right)^{\frac{-n}{2}} (ax_*^2 + bx_* + c)^{\frac{-n}{2}} = \left(\frac{1}{a}\right)^{\frac{-n}{2}} \left(x_*^2 + \frac{b}{a}x_* + \frac{c}{a}\right)^{\frac{-n}{2}}$$

$$\propto \left(x_*^2 + \frac{b}{a}x_* + \frac{c}{a}\right)^{\frac{-n}{2}} \propto \left(\left(x_* + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)^{\frac{-n}{2}} \left(\frac{1}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-n/2}$$

$$\propto \left(1 + \frac{\left(x_* + \frac{b}{2a}\right)^2}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-\frac{\overset{\nu}{(n-1)}+1}{2}}$$

$$= \left(1 + \frac{1}{\underset{2S}{n-1}} \frac{\left(x_* - \overset{\mu}{\frac{-b}{2a}}\right)^2}{\underbrace{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)(n-1)}_{S_\circ^2}}\right)^{-\frac{(n-1)+1}{2}} \propto T_2(\mu, S^2)$$

$$= T_{n-1}\left(\bar{x}, \frac{n+1}{n}S^2\right) \overset{n\text{-large}}{\approx} N(\bar{x}, S^2)$$

$$\frac{-b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{\frac{n}{n+1}} = \bar{x}$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{c}{a} - \frac{b^2}{(2a)^2} = \frac{\frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\frac{1}{2}\frac{n}{n+1}} \rightarrow \frac{(n-1)(n+1)}{n} s^2$$

$$= \frac{(n-1)(n+1)}{n} s^2 + \underbrace{(n+1)\bar{x}^2}_{n\bar{x}^2 + \bar{x}^2} - n\bar{x}^2$$

$$-\left(\frac{b^2}{4a^2}\right) = \left(\frac{-b}{2a}\right)^2 = -\bar{x}^2 \Rightarrow \frac{c}{a} - \frac{b^2}{4a^2}$$

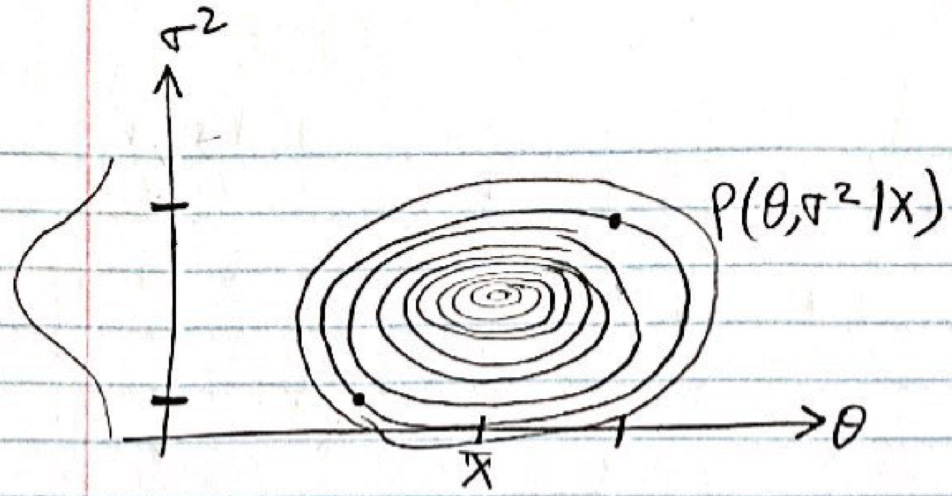
$$S_\circ^2 = \frac{\frac{(n-1)(n+1)s^2}{n}}{n-1} = \frac{n+1}{n} s^2$$

under Jeffreys Prior

$$\underbrace{P(\theta, \sigma^2 | X)}_{} = \underbrace{P(\theta | X, \sigma^2)}_{} \underbrace{P(\sigma^2 | X)}_{}$$ By def of Cond. Probability

$$\text{NormInvGamma}(\ldots) = N\left(\bar{x}, \frac{\sigma^2}{n}\right) \cdot \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

You can think of a normal inverse gamma as first sampling from an Inverse Gamma$\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$ to get a σ^2 value and then you use that value of σ^2 to draw a θ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ and return the two-dimensional point $[\theta \sigma^2]$.



This can also be done another way....

$$P(\theta,\sigma^2|X) = \underbrace{P(\sigma^2|X,\theta)}_{\text{InvGamma}\left(\frac{n}{2},\frac{n\hat{\sigma}_{MLE}}{2}\right)} \underbrace{P(\theta|X)}_{T_{n-1}\left(\bar{X},\frac{s^2}{2}\right)}$$

If we decompose the first way, we draw θ from $N(\bar{X},\frac{\sigma^2}{n})$ and thus σ^2 must be known. What if we break this by instead of using **Jeffrey's** prior, using

$$P(\theta) = N(\mu_0,\tau^2) \text{ and } P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2},\frac{n_0\sigma_0^2}{2}\right)$$

These were the two priors we began with when we started investigating the normal likelyhood model. However, it's important to note we are <u>not</u> allowing $\tau^2 = \sigma^2/n_0$

What happens? The two priors are disconnected completly.

$$P(\theta,\sigma^2) = P(\theta)P(\sigma^2) \text{ not } P(\theta|\sigma^2)P(\sigma^2)$$

Let's derive the posterior under this two-Dimen. Prior.

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P(\theta,\sigma^2) = P(X|\theta,\sigma^2)P(\theta)P(\sigma^2)$$

$$\propto k(X|\theta,\sigma^2)\,k(\theta)\,k(\sigma^2)$$

$$= (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + \underbrace{n(\bar{x}-\theta)^2}_{n\bar{x}-2\bar{x}n\theta+n\theta^2}\right)} e^{-\frac{1}{2\tau^2}\overbrace{(\theta-\mu_0)^2}^{\theta^2-2\theta\mu_0+\mu_0^2}} (\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma^2/2}{\sigma^2}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\frac{n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n_0\sigma_0^2 + n\bar{x}^2\right)} e^{\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)\theta - \left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)\theta^2}$$