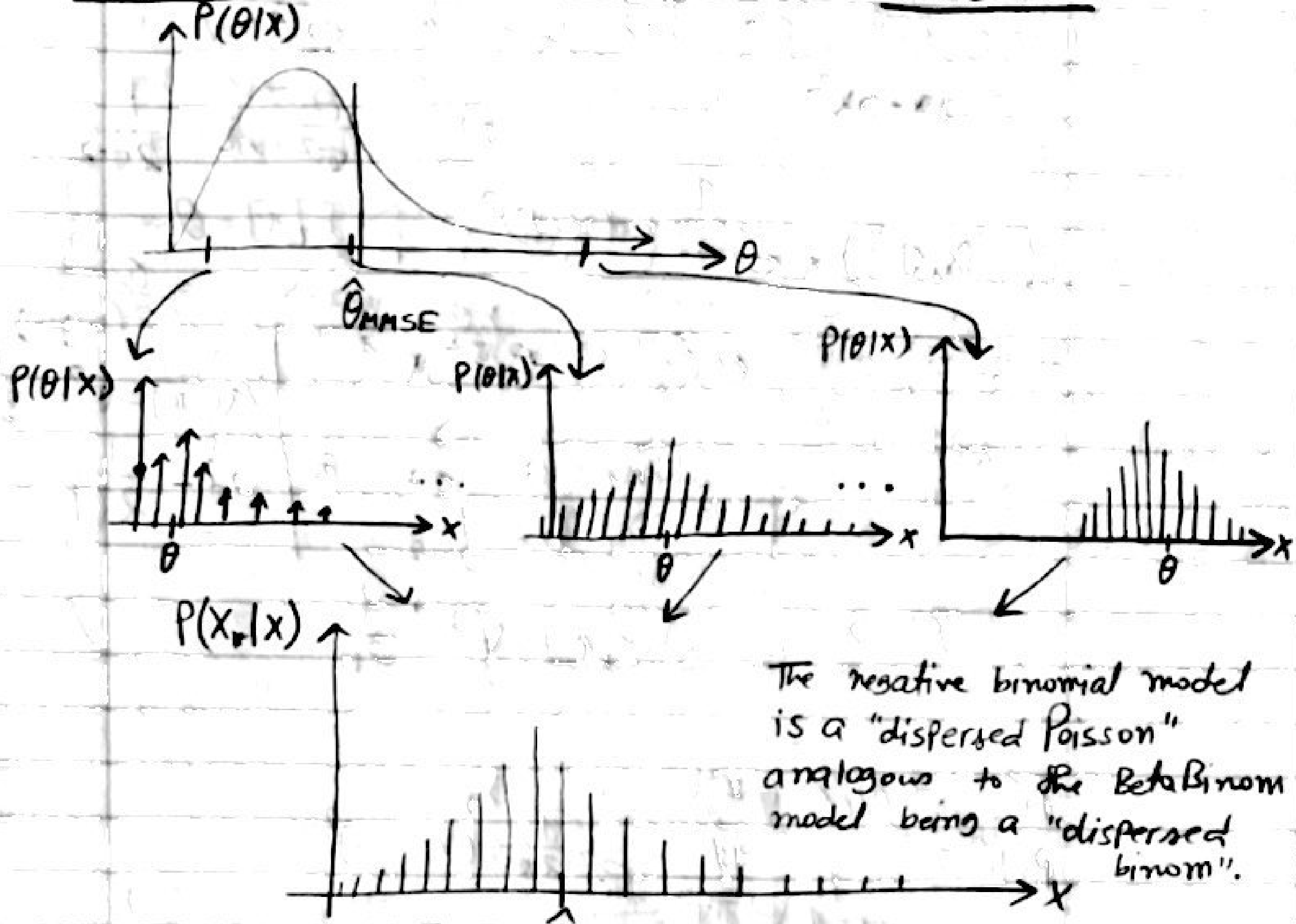
MATH 341/650

4/7/21 Lec 15



The negative binomial model is a "dispersed Poisson" analogous to the BetaBinom model being a "dispersed binom".

$E[X_*|x] = \hat{\theta}_{MMSE}$

$$Var[X_*|X] = r\frac{1-p}{p^2} = \hat{\theta}_{MMSE}\cdot\frac{1}{p} = \underbrace{\frac{\beta+n+1}{\beta+n}}_{\in[1,2]}\hat{\theta}_{MMSE}$$

Thus, the Posterior Predictive distribution has Varience up to 2x the Poisson (i.e. more spread out or less sure of where the relalization will be).

$$\mathcal{F}: \text{one } N(\theta,\sigma^2) =: P(X|\underbrace{\theta,\sigma^2}_{\vec{\theta}=\begin{bmatrix}\theta\\ \sigma^2\end{bmatrix}}) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

Let σ^2 be fixed/known in advanced.

$$P(x|\theta,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{1}{2\sigma^2}(x^2-2x\theta+\theta^2)}$$

$$= e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \propto e^{\frac{-x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}}$$

$$= e^{ax-bx^2} \quad \text{where } a = \frac{\theta}{\sigma^2}, b = \frac{1}{2\sigma^2}$$

$$P(\theta|X,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$= e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}} \propto e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}}$$

$$= e^{a\theta-b\theta^2} \quad \text{where } a = \frac{x}{\sigma^2}, \; b = \frac{1}{2\sigma^2}$$

$$E[\theta] = X = \frac{a}{2b}, \; Var[\theta] = \sigma^2 = \frac{1}{2b}$$

$$E[x] = \theta = \frac{a}{2b} = \frac{\theta/\sigma^2}{2\left(\frac{1}{2\sigma^2}\right)} = \theta$$

$$Var[x] = \sigma^2 = \frac{1}{2b} = \frac{1}{2\left(\frac{1}{2\sigma^2}\right)} = \sigma^2$$

F: iid $N(\theta,\sigma^2)$ $X_1, \ldots, X_n$

$$P(X|\theta) = \prod_{i=1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(X_i-\theta)^2}$$

$$\sum(X_i-\theta)^2 = \sum X_i^2 - 2X_i\theta + \theta^2 = \sum X_i^2 - 2n\bar{x}\theta + n\theta^2$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum X_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}}$$

$$= \mathcal{L}(\theta; x, \sigma^2) \propto e^{\frac{-\sum X_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}}$$

$$P(\theta|X,\sigma^2) \propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}}$$

$$= e^{a\theta-b\theta^2} \quad \text{where } a = \frac{n\bar{x}}{\sigma^2}, \; b = \frac{n}{2\sigma^2} \Rightarrow 2b = \frac{n}{\sigma^2}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N\left(\frac{\frac{n\bar{x}}{\sigma^2}}{\frac{n}{\sigma^2}}, \frac{1}{\frac{n}{\sigma^2}}\right) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

All we have done thus far is Probability theory and we seemingly just made random computions for fun. Now we'll do Bayesian .....

F: $X_1, \ldots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ with σ^2 = known. Let's find Posterior.

$$P(\theta | x, \sigma^2) \propto P(X | \theta, \sigma^2)\, P(\theta | \sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} P(\theta | \sigma^{2})$$

$$\propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta | \sigma^2) \Big\} {}^{*} \Rightarrow P(\theta | \sigma^2) = N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right)$$

$$\propto e^{-a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2}$$

$$= e^{(a+\alpha)\theta - (b+\beta)\sigma^2} \propto N\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right)$$

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \alpha}{\frac{n}{\sigma^2} + \beta}, \frac{1}{\frac{n}{\sigma^2} + 2\beta}\right)$$

* Traditionally... $\alpha = \frac{\mu_0}{\tau^2}$, $\beta = \frac{1}{2\tau^2}$

$$P(\theta | \sigma^2) = N(\mu_0, \tau^2)$$

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$$

The normal-normal Conjugate model (where σ^2 is assumed to be fixed).

$$P(\theta | \sigma^2) = N(\mu_0, \tau^2) \Rightarrow P(\theta | x, \sigma^2) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$$

Point Estimation:

$$\hat{\theta}_{MMSE} = E[\theta | X, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MMAE} = Med[\theta | X, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAP} = Mode[\theta | X, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\Rightarrow \hat{\theta}_{MMSE} = \hat{\theta}_{MMAE} = \hat{\theta}_{MAP}$$

<u>Credible Regions</u>

$$CR_{\theta, 1-\alpha_0} = \left[qnorm\left(\frac{\alpha_0}{2}, \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \right), qnorm\left(1 - \frac{\alpha_0}{2}, mean, Var\right) \right]$$

<u>Hypothesis Tests:</u>

$H_a : \theta < \theta_0 \Rightarrow H_0 : \theta \geq \theta_0$

$$P_{val} = P(H_0 | X, \sigma^2) = \int_{\theta}^{\infty} P(\theta | x, \sigma^2) \, d\theta$$

$$= 1 - Pnorm(\theta_0, mean, Var)$$

Let's calculate the MLE (was on HW1)

$$\mathcal{L}(\theta; x, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{\Sigma x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}}$$

$$l(\theta; x, \sigma^2) = ln(2\pi\sigma^2)^{-n/2} - \frac{\Sigma x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$l'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \stackrel{set}{=} 0$$

$$\Rightarrow \cancel{n}\bar{x} = \cancel{n}\theta$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}.$$