4/5/2021 Lec 14

Laplace's Prior: the prior of indifference/uniformity. In the Poisson model θ is $(0,\infty)$. we need a distribution that is uniform on that set. A distribution would look like:

$$P(\theta) = c > 0,\ \int_0^\infty P(\theta)d\theta = \int_0^\infty c\,d\theta = c\int_0^\infty d\theta = \infty$$

$\Rightarrow P(\theta) = c$ DNE.

There can't be a proper Laplace Prior. But there is an improper Laplace prior.

$$P(\theta|X) \propto P(X|\theta)\,P(\theta) = e^{-n\theta}\theta^{\Sigma X_i}\underbrace{P(\theta)}_{P(\theta)\propto 1 \Rightarrow \text{Laplace's idea}} \propto e^{-n\theta}\theta^{\Sigma X_i + 1 - 1}$$

$$\propto \text{Gamma}(\underbrace{1+\Sigma X_i}_{\alpha}, \underbrace{0+n}_{\beta})$$

$\Rightarrow P(\theta) = \text{Gamma}(1, 0)$, an improper Prior.

$X_0 = 1,\ n_0 = 0$ nonsense!

Is the posterior proper? Yes, Always! Since $\Sigma X_i \geq 0$, it's first parameter is always $\geq 1 > 0$ and since $n \geq 1$, it's second parameter is always $\geq 1 > 0$.

Haldan's Prior of complete ignorance. Setting all pseudodata to be zero. ie. $X_0 = 0,\ n_0 = 0$

$\Rightarrow$ Gamma$(0,0)$ improper!

$$\Rightarrow P(\theta|X) = \text{Gamma}(\Sigma X_i, n) \Rightarrow \hat{\theta}_{MMSE} = \frac{\Sigma X_i}{n} = \bar{X} = \hat{\theta}_{MLE}$$

Is this Posterior proper? only if $\Sigma X_i > 0$

Jeffrey's Prior. $P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \theta^{-\frac{1}{2}} \propto \text{Gamma}(\frac{1}{2}, 0)$ improper.

$$l'(\theta) = -n + \frac{\Sigma X_i}{\theta} \Rightarrow l''(\theta) = -\frac{\Sigma X_i}{\theta^2}$$

$P(\theta|X) = \text{Gamma}(\frac{1}{2} + \Sigma X_i, 0 + n)$

Is Jeffrey's Prior proper $\rightarrow$ Always Proper!

$$I(\theta) = E_X[-l''(\theta)] = \frac{E[\sum X_i]}{\theta^2} = \frac{n E[X_i]}{\theta^2} = \frac{n\theta}{\theta^2} = \frac{n}{\theta}$$

$X \sim \text{Poisson}(\theta)$

$$E[X] = \sum_{x \in Supp[X]} x\, P(x) = \sum_{x=0}^{\infty} x \frac{e^{-\theta}\theta^x}{x!}$$

If $x=0$, term $=0$

$$= e^{-\theta} \sum_{x=1}^{\infty} \frac{x\theta^x}{x!} = e^{-\theta} \sum_{x=1}^{\infty} \frac{\theta^x}{(x-1)!}$$

$y = x-1 \Rightarrow x = y+1$

$$= e^{-\theta} \sum_{y=0}^{\infty} \frac{\theta^{y+1}}{y!}$$

$$= \theta e^{-\theta} \underbrace{\sum_{y=0}^{\infty} \frac{\theta^y}{y!}}_{e^\theta} = \theta e^{-\theta} e^{\theta} = \theta$$

Posterior Predictive distribution. You see n observations and you want to know the distribution of n^* future observation(s). For our case here, let $n^* = 1$

$\underbrace{X_1, X_2 \ldots X_n}_{n} \overset{iid}{\sim} \text{Poisson}(\theta)$ | $X_* | X \sim ?$, $n_* = 1$

$\longleftarrow$ Past $\longrightarrow$ $\longleftarrow$ future $\longrightarrow$

$$P(X_*|X) = \int_{\Theta} \overbrace{P(X_*|\theta)}^{\text{Poisson}(\theta)}\, \overbrace{P(\theta|X)}^{\text{Gamma}(\alpha+\sum X_i,\, \beta+n)}\, d\theta$$

$$= \int_0^\infty \frac{e^{-\theta}\theta^{X_*}}{X_*!} \frac{(\beta+n)^{\alpha+\sum X_i}}{\Gamma(\alpha+\sum X_i)} \theta^{\alpha+\sum X_i - 1} e^{-(\beta+n)\theta}\, d\theta$$

$$= \frac{(\beta+n)^{\alpha+\sum X_i}}{X_*!\,\Gamma(\alpha+\sum X_i)} \underbrace{\int_0^\infty e^{-\theta}\theta^{\alpha_*}\theta^{\alpha+\sum X_i-1} e^{-(\beta+n)\theta}\, d\theta}$$

$$= \frac{(\beta+n)^{\alpha+\sum X_i}}{X_*!\,\Gamma(\alpha+\sum X_i)} \int_0^\infty \theta^{\alpha_*+\alpha+\sum X_i-1} e^{-(\beta+n+1)\theta}\, d\theta$$

let $t=(\beta+n+1)\theta \Rightarrow \theta=\frac{t}{\beta+n+1} \Rightarrow \frac{d\theta}{dt}=\frac{1}{\beta+n+1}$

$\Rightarrow d\theta=\frac{1}{\beta+n+1}dt$

If $\theta=0 \Rightarrow t=0$, if $\theta=\infty \Rightarrow t=\infty$

$$=\frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)}\int_0^\infty \frac{t^{(x_*+\alpha+\sum x_i-1)}}{(\beta+n+1)^{x_*+\alpha+\sum x_i-1}}e^{-t}\frac{1}{(\beta+n+1)^1}dt$$

$$=\frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}}\underbrace{\int_0^\infty t^{(x_*+\alpha+\sum x_i)-1}e^{-t}dt}_{\text{gamma Integral}}$$

$$=\frac{(\beta+n)^{\alpha+\sum x_i}\,\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}}$$

$$=\frac{(\beta+n)^{\alpha+\sum x_i}}{(\beta+n+1)^{\alpha+\sum x_i}}\frac{1}{(\beta+n+1)^{x_*}}\frac{\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)}$$

$$=\left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum x_i}\left(\frac{1}{\beta+n+1}\right)^{x_*}\frac{\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)}$$

let $P:=\frac{\beta+n}{\beta+n+1}\in(0,1)$, $1-P=\frac{1}{\beta+n+1}\in(0,1)$

$r:=\sum x_i+\alpha>0$

$$=P^r(1-P)^{x_*}\frac{\Gamma(x_*+r)}{x_*!\,\Gamma(r)}=\text{ExtNegBin}(r,P)$$

Extended negative binomial random variable model.

If $\alpha\in\{0,1,2\ldots\}$

$$=\binom{x_*+r-1}{r}P^r(1-P)^{x_*}=\text{NegBin}(r,P)$$

From 368, the NegBinom is the sum of iid Geometric rv. Since the expection of the geometric rv is $\frac{(1-p)}{p}$, the expection of the negative binomial by linearity is

$$P(X_*|x) = \text{Ext NegBin}(r, p) \Rightarrow E[X_*|X] \stackrel{\downarrow}{=} r\left(\frac{1-p}{p}\right)$$

$$= \left(\sum x_i + \alpha\right) \frac{\frac{1}{n+\beta+1}}{\frac{n+\beta}{n+(\beta+1)}}$$

$$= \frac{\sum x_i + \alpha}{n+\beta} = \hat{\theta}_{MMSE}$$