3/24/21 — Lec 13

Poisson λ

$\lambda \in (0, \infty)$, $Y \sim Poisson(\lambda)$, $Supp[Y] = \{0, 1, 2, \dots\} = \mathbb{N}_0$

$E[Y] = \lambda$.

$\mathcal{F}$: $n = 1$ Poisson(θ). θ is our unknown parameter of interest. It is not the same θ in the binomial. Let's try to find the conjugate prior for this parametric model (likelyhood)

$$P(\theta|x) \propto P(x|\theta)P(\theta) = \frac{e^{-\theta}\theta^x}{x!}P(\theta) \propto e^{-\theta}\theta^x k(\theta)$$

we want to find $P(\theta)$ of the same distribution as $P(\theta|x)$

Pattern matching

$$= e^{-\theta}\theta^x \underbrace{e^{-\theta b}\theta^a} = \underbrace{e^{-\theta(b+1)}\theta^{x+a}}_{\text{kernal for same rv}}$$

Let's figure out $P(\theta)$ from $k(\theta)$

$$\int_0^\infty k(\theta)\,d\theta = \int e^{-\theta b}\theta^a d\theta = \int_0^\infty \theta^{a+1-1}e^{-b\theta}d\theta \overset{\text{u-subst}}{=} \dots = \frac{\Gamma(a+1)}{\beta^{a+1}}$$

$$= \Gamma(a+1) = \frac{1}{c}$$

$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\theta^{a+1-1}e^{-b\theta}$$

$$= Gamma(a+1, b)$$

Back to probability class......

$$Y \sim Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} Y^{\alpha-1}e^{-\beta Y}$$

$Supp[Y] = (0, \infty)$. Parameter space: $\alpha > 0$, $\beta > 0$

$$E[Y] = \int_0^\infty Y f(y)\,dy \overset{\text{u-subst}}{=} \dots\dots = \frac{\alpha}{\beta}$$

$$Mode[Y] \overset{\text{calc}}{=} \dots\dots = \frac{\alpha - 1}{\beta} \text{ if } \alpha > 1 \approx \hat{\theta}_{MAP}$$

$Med[Y] = 2$ s.t. $\int_0^{2} f(y)\,dy = \frac{1}{2}$ ... no closed form expression is possible.

so we use a computer to do numerical integration.

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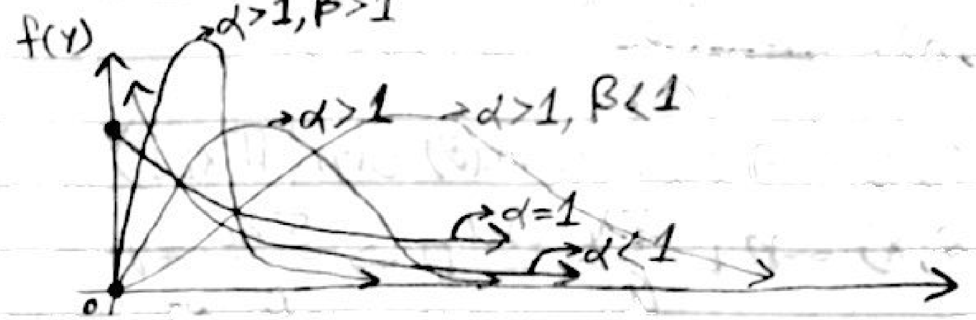
$Med[Y] = qgamma(0.5, \alpha, \beta)$



Let's go back to inference for θ in the Poisson model, Now we consider $\mathcal{F}$: iid Poisson(θ)

$$P(\theta|X) \propto P(X|\theta)P(\theta) = P(\theta)\prod_{i=1}^{n}\frac{e^{-\theta}\theta^{x_i}}{x!}$$

$$= P(\theta)\frac{e^{-n\theta}\theta^{\Sigma x_i}}{\Pi x_i!} \propto e^{-n\theta}\theta^{\Sigma x_i} K(\theta)$$

Pattern matching $= e^{-n\theta}\theta^{\Sigma x_i}\underbrace{\theta^{\alpha-1}e^{-\beta\theta}}_{\text{gomma kernal}}$

$$= \underbrace{\theta^{\alpha+\Sigma x_i-1}e^{-(\beta+n)\theta}}_{\text{gamma kernal}} \leftarrow \text{due to conjugacy}$$

$$\propto Gamma(\alpha+\Sigma x_i, \beta+n)$$

n: # of observation in data

n_0, # of pseudoobservation

# of events in the data

x_0, # of pseudoevents

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{\alpha+\Sigma x_i}{\beta+n}$$

$$\hat{\theta}_{MAP} = \frac{\alpha+\Sigma x_i-1}{\beta+n} \text{ only if } \alpha+\Sigma x_i \geq 1$$

$$\hat{\theta}_{MMAE} = qgamma(0.5, \alpha+\Sigma x_i, \beta+n)$$

$$CR_{\theta,1-\alpha_0} = \left[qgamma\left(\frac{\alpha_0}{2}, \alpha+\Sigma x_i, \beta+n\right), qgamma\left(1-\frac{\alpha_0}{2}, \alpha+\Sigma x_i, \beta+n\right)\right]$$

$H_a: \theta > \theta_0$ vs. $H_0: \theta \leq \theta_0$, $P\text{-val} = P(H_0|X) = P(\theta \leq \theta_0|X)$

$$= \int_0^{\theta_0} P(\theta|X)d\theta$$

$$= Pgamma(\theta_0, \alpha+\Sigma x_i, \beta+n)$$

Let's derive the MLE:

$$\mathcal{L}(\theta; x) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod x_i!}$$

$$\Rightarrow l(\theta; x) = -n\theta + \sum x_i \ln(\theta) - \ln\left(\prod x_i!\right)$$

$$\Rightarrow l'(\theta; x) = -n + \frac{\sum x_i}{\theta} \overset{set}{=} 0 \Rightarrow \frac{\sum x_i}{\theta} = n$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum x_i}{n} = \bar{X}$$

Let's prove that $\hat{\theta}_{MMSE}$ is a shrinkage estimator and let's find the value of ρ.

$$\hat{\theta}_{MMSE} = \frac{\alpha + \sum x_i}{\beta + n} = \frac{\alpha}{\beta+n} \cdot \frac{\beta}{\beta} + \frac{\sum x_i}{\beta+n} \cdot \frac{n}{n}$$

$$= \underbrace{\overbrace{\frac{\beta}{\beta+n}}^{\rho}}_{=\,\frac{n_0}{n+n_0}} \overbrace{\frac{\alpha}{\beta}}^{E[\theta]} + \overbrace{\frac{n}{\beta+n}}^{1-\rho} \overbrace{\frac{\sum x_i}{n}}^{\hat{\theta}_{MLE}}$$

$$\lim_{n\to\infty} \rho = 0$$