4/19/21 — Lec 18

Jeffrey's $P_J(\sigma^2|\theta) \propto \sqrt{I(\sigma^2;\theta)}$

$$\ell'(\sigma^2;X,\theta) = -\frac{n}{2\sigma^2} - \frac{\Sigma(X_i-\theta)^2}{2(\sigma^2)^2}$$

$$\ell''(\sigma^2;X,\theta) = \frac{1}{2(\sigma^2)^2} + \frac{\Sigma(X_i-\theta)^2}{(\sigma^2)^3}$$

$$I(\sigma^2;\theta) = E_X[-\ell''] = +E_X\left[\frac{-n}{2(\sigma^2)^2} + \frac{\Sigma(X_i-\theta)^2}{(\sigma^2)^3}\right]$$

Defn of $Var[X] = \sigma^2$, $X \sim N(\theta,\sigma^2)$

$$= +\left(\frac{-n}{2(\sigma^2)^2} + \frac{\Sigma E[(X_i-\theta)^2]}{(\sigma^2)^3}\right)$$

$$= +\left(\frac{-n}{2(\sigma^2)^2} + \frac{\Sigma\sigma^2}{(\sigma^2)^2}\right)$$

$$= -\frac{n}{2(\sigma^2)^2} + \frac{n}{(\sigma^2)^2} = \frac{n}{(\sigma^2)^2}\left(-\frac{1}{2}+1\right) = \frac{n}{2(\sigma^2)^2}$$

$$P_J(\sigma^2|\theta) \propto \sqrt{\frac{n}{2(\sigma^2)^2}} \propto \sqrt{\frac{1}{(\sigma^2)^2}} = (\sigma^2)^{-1} \propto \text{InvGamma}(0,0)$$

$= \text{Haldane}$

Jeffrey's and Haldane are equivalent principled objective priors and they're the default for this model.

Shrinkage $P(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right) \Rightarrow E[\sigma^2|\theta] = \frac{n_0\sigma_0^2}{n_0-2}$ (valid for $n_0>2$)

$$\hat{\sigma}^2_{MMSE} = \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{n+n_0-2} = \frac{n\hat{\sigma}^2_{MLE}}{n+n_0-2} + \frac{n_0\sigma_0^2}{n+n_0-2}\cdot\frac{n_0-2}{n_0-2}$$

$$= \underbrace{\frac{n}{n+n_0-2}}_{1-\rho}\hat{\sigma}^2_{MLE} + \underbrace{\frac{n_0-2}{n+n_0-2}}_{\rho} E[\sigma^2|\theta]$$

Posterior Predictive distribution

$$P(X_*|X,\theta) = \int_0^\infty \underbrace{P(X_*|\theta,\sigma^2)}_{N(\theta,\sigma^2)} \underbrace{P(\sigma^2|X,\theta)}_{\text{InvGamma}\left(\underbrace{\frac{n+n_0}{2}}_{\alpha}, \underbrace{\frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}}_{\beta}\right)}$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}} \, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}} e^{-\frac{(x_*-\theta)^2/2}{\sigma^2}} (\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}} \, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\overbrace{(\alpha+\frac{1}{2})}^{\alpha'}-1} e^{-\frac{\overbrace{(x_*-\theta)^2/2+\beta}^{\beta'}}{\sigma^2}} \, d\sigma^2$$

$$\overset{\text{u-Subst}}{=} \frac{\Gamma(\alpha')}{\beta'^{\alpha'}}$$

$$\parallel$$

$$P(x_*|X,\theta) \propto \Gamma\left(\frac{n+n_0+1}{2}\right) \left(\frac{(x_*-\theta)^2 + \overbrace{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}^{a}}{2}\right)^{-\frac{\overbrace{n+n_0}^{\nu}+1}{2}}$$

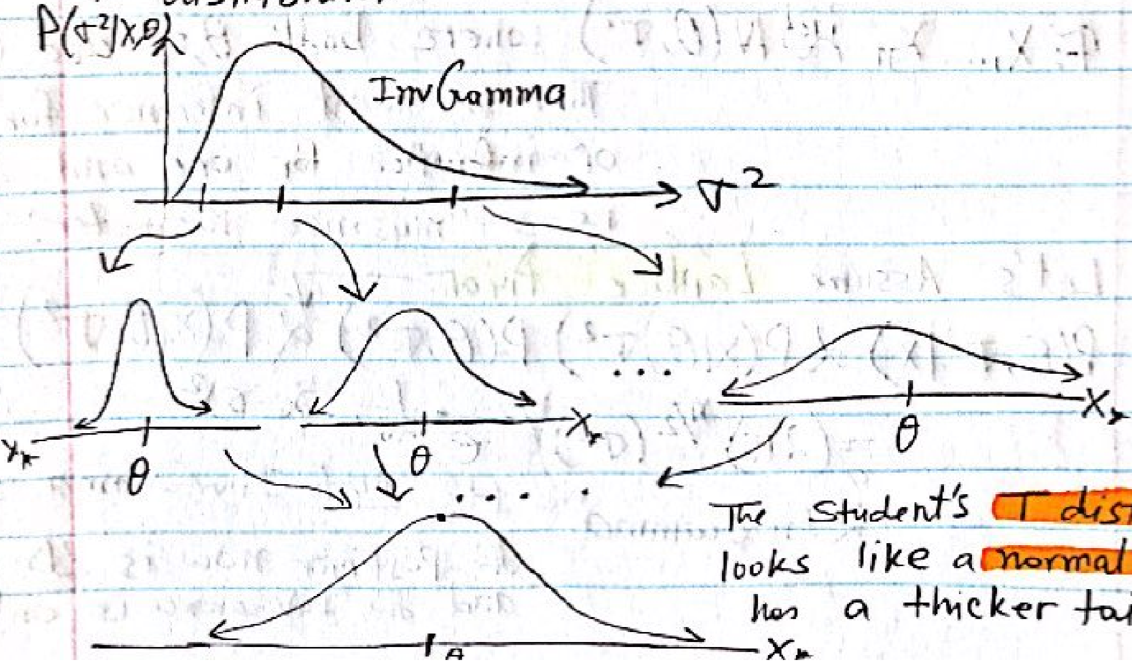
$$\propto \left(\frac{(x_*-\theta)^2+a}{2}\right)^{-\frac{\nu+1}{2}} \cdot \left(\frac{2}{a}\right)^{-\frac{\nu+1}{2}} \cdot \left(\frac{2}{a}\right)^{\frac{\nu+1}{2}}$$

$$= \left(\frac{(x_*-\theta)^2}{a} \cdot \frac{\frac{1}{\nu}}{\frac{1}{\nu}} + 1\right)^{-\frac{\nu+1}{2}} \left(\frac{2}{a}\right)^{\frac{\nu+1}{2}}$$

$$= \left(1 + \frac{1}{\nu} \frac{(x_*-\theta)^2}{\frac{a}{\nu}}\right)^{-\frac{\nu+1}{2}} \left(\frac{2}{a}\right)^{\frac{\nu+1}{2}}$$

$$\propto \left(1 + \frac{1}{\nu} \frac{(x_*-\theta)^2}{\frac{a}{\nu}}\right)^{-\frac{\nu+1}{2}} \propto T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{n+n_0}\right)$$

This distribution is "non-standard Student's T distribution" or "shifted and scaled Student's T distribution".



The Student's T distribution looks like a normal dist, but has a thicker tails.

$$T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{n+n_0}\right) \overset{n+n_0>20}{\approx} N\left(\theta, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{n+n_0}\right)$$

If n is large
$\Rightarrow \hat{\sigma}^2_{MLE} \to \sigma^2$

$$\approx N(\theta, \sigma^2)$$

$\theta = 5$

$n=12$, $\hat{\sigma}^2_{MLE} = 0.387$, Jeffry's Prior $\Rightarrow n_0 = \sigma_0^2 = 0$

$$P(X_* > 8 | X, \theta) = 1 - P(X_* < 8 | X, \theta)$$

$$= 1 - P.(t.\ scaled)\Big[\underset{x_*}{8}, \underset{\nu}{12}, \underset{\theta}{5}, \underbrace{\sqrt{\frac{12(.387)}{12}}}_{\sqrt{scale}}\Big]$$

Predictive Intervals (PI)

$$PI_{X_*, 1-\alpha_0} = \left[Q[X_*|X, \tfrac{\alpha_0}{2}], Q[X_*|X, 1-\tfrac{\alpha_0}{2}]\right]$$

$$P(X_* \in PI_{X_*, 1-\alpha_0} | X) = 1 - \alpha_0$$

$\sigma^2 = 1.1$, $\bar{x} = 1.89$, $n=13$, Jeffrey's Prior $\Rightarrow P(X_*|X, \sigma^2) = N(\bar{x}, \sigma^2)$

$$PI_{X_*, 95\%} = \left[qnorm(0.025, 1.89, \sqrt{1.1}), qnorm(0.975, 1.89, \sqrt{1.1})\right]$$

*** End of Mid II Matenilas ↑ ***

*** Start of final Exam Matrials ↓ ***

F: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ where both θ, σ^2 are Unknown.
Thus we want inference for both, or inference for one and the other is a "nuisance Parameter."

Let's Assume Laplace Prior

$$P(\theta, \sigma^2 | X) \propto P(X|\theta, \sigma^2)\, P(\theta, \sigma^2) \propto P(X|\theta, \sigma^2)$$

$$= (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$$

$\overset{?}{\propto}$ InvGamma

It's Not InvGamma since the Posterior now is 2D dist. and the InvGamma is only 1D.

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What we have is a known distribution but to get it into Cononical form, we need to do algebra.

$$\sum(x_i-\theta)^2 = \sum((x_i-\bar{x})+(\bar{x}-\theta))^2 = \sum(x_i-\bar{x})^2 + 2\sum(x_i-\bar{x})(\bar{x}-\theta) + \sum(\bar{x}-\theta)^2$$

$S^2 := \frac{1}{n-1}\sum(x_i-\bar{x})^2$ the sample varience formula from MATH 241

$\bar{x} = \frac{1}{n}\sum x_i \Rightarrow \sum x_i = n\bar{x}$

$$= (n-1)S^2 + 2\sum(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x}-\theta)^2$$

$$= (n-1)S^2 + n(\bar{x}-\theta)^2 + 2(\cancel{n\bar{x}^2} - \cancel{n\bar{x}^2} - \cancel{n\bar{x}\theta} + \cancel{n\bar{x}\theta})$$

$$P(\theta,\sigma^2|x) \propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}((n-1)S^2 + n(\bar{x}-\theta)^2)}$$

$$= (\sigma^2)^{-(\frac{n}{2}+1)-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}} e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2}$$

$$= \underbrace{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}_{\propto N(\bar{x}, \frac{\sigma^2}{n})} \underbrace{(\sigma^2)^{-(\frac{n}{2}+1)-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}}}_{\propto \text{InvGamma}\left(\frac{n+2}{2}, \frac{(n-1)S^2}{2}\right)}$$

$$\propto \text{NormalInvGamma}\left(\mu=\bar{x}, \lambda=n, \alpha=\frac{n+2}{2}, \beta=\frac{(n-1)S^2}{2}\right)$$

This is the "normal-inverse-gamma" distribution with four parameters!