Estimating π With Polygons

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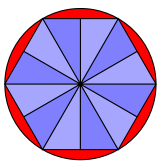
1 Introduction

To celebrate of March 14 (π-day), we seek to estimate π. As is well known, π is an important circle constant, which is approximately equal to 3.14, hence the date of π-day. However, it is defined as the ratio of the circumference to the diameter of a circle. Estimating π beyond those first three digits is both an academic exercise and an important task for physical sciences. In this paper we use the perimeter of regular polygons to develop our estimate, and we will develop an program in Common Lisp to calculate our estimate.

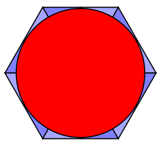
2 Developing the Estimate

Figure 1: Circumscribed and Inscribing Polygons

(a) Circumscribed Polygon

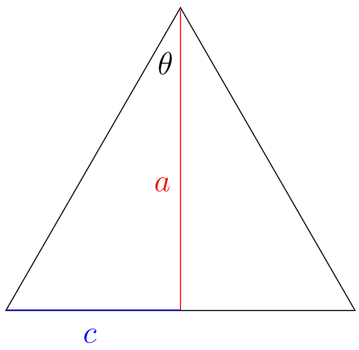


(b) Inscribing Polygon



The critical fact to this estimate is that $\pi = \frac{C}{d}$, that is circumference divided by diameter. If we keep diameter constant at 1, then $\pi = C$, so by estimating the circumference of the circle with diameter 1, we estimate π. We will use regular polygons for this task, since a regular polygon circumscribed by the circle will have a smaller perimeter than the circumference, and a regular polygon which the circle is inscribed within will have a larger perimeter than the circumference. See the figure. By developing a formula for these perimeter, as a function of n (where n is the number of sides, and is

Figure 2: One Triangle of the Circumscribed Polygon



an integer greater than or equal to 3), we can find an upper and lower bound for π where the precision increases as n increases.

2.1 Formula for Circumscribed Perimeter

First we will develop the formula for the Circumscribed Perimeter. The radius of the polygon is equal the radius of the circle, which is $\frac{1}{2}$. We will split the polygon into $2n$ triangles, where two identical but mirrored triangles makes up each side of the polygon. If we find the side one side length of the polygon, we can find the perimeter by multiplying by n. When we split the polygon into triangles, we are left with $2n$ equal angles that we will call θ. We also calculate that $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$, since the total sum is 2π and there is $2n$ angles. From the figure, we have the hypotenuse ($a = \frac{1}{2}$) and the angle of a right triangle (θ). We also see that we want to solve the side opposite the angle (c). Using the definition of of sine:

$$\sin\theta = \frac{c}{a} = \frac{c}{\frac{1}{2}}$$

$$\frac{\sin(\frac{\pi}{n})}{2} = c$$

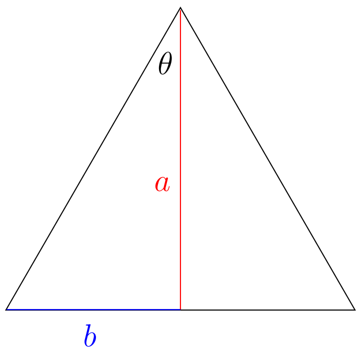
Finally, since c is half of one side, the perimeter is defined by $p(n) = 2n \cdot c = n\sin(\frac{\pi}{n})$. [1] Which is our lower bound.

2.2 Formula for Inscribing Perimeter

Next we will develop the formula for the Inscribing Perimeter. The altitude of each triangle equal to the radius of the circle. Again, we will split the

[1]To address the use of π in the estimate of π, the π used in the angles can be substituted with 180 if we use the degree version of trigonometric functions

Figure 3: One Triangle of the Circumscribed Polygon



polygon into $2n$ triangles, in the same way as before. So we find from the figure we have an angle θ, same as before, and the angle adjacent to the angle. We want to find the side opposite the triangle which we will call b. Using the definition of tangent:

$$\tan\theta = \frac{b}{a} = \frac{b}{\frac{1}{2}}$$

$$\frac{\tan(\frac{\pi}{n})}{2} = b$$

Finally, since b is half of one side, the perimeter is defied by $P(n) = 2n \cdot b = n\tan(\frac{\pi}{n})$

2.3 Combine Results

Recall that the circumference is equal to π, since $p(n)$ is less than the circumference and $P(n)$ is greater than the circumference then:

$$p(n) < \pi < P(n)$$

$$n\sin(\frac{\pi}{n}) < \pi < n\tan(\frac{\pi}{n})$$

When n is an integer which is greater than or equal to 3.