Chapter 9 Relations

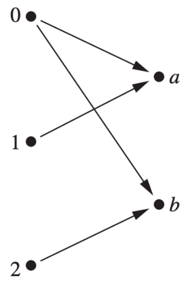
9.1 Relation and their properties

Definition 1

Let A and B be sets. A binary relation from A to B is a subset of A × B

In other words, a binary relation from A to B is **a set R of ordered pairs**, where the first element of each ordered pair comes from A and the second element comes from B. We use the notation aRb to denote that (a, b) ∈ R and a$\not R$b to denote that (a, b) ∉ R. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

EXAMPLE 3 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\,R\,a$, but that $1 \not R\, b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1. We will discuss representations of relations in more detail in Section 9.3. ◀



R	a	b
0	×	×
1	×	
2		×

Definition 2

A relation on a set A is a relation from A to A.

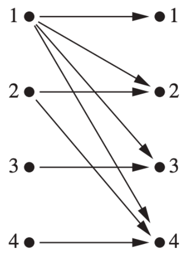
In other words, a relation on a set A is a subset of A × A.

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The pairs in this relation are displayed both graphically and in tabular form in Figure 2. ◀



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

EXAMPLE 6 How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ◀

9.1.4 Properties of relations

Definition 3

A relation R on a set A is called **reflexive** if (a, a) ∈ R for **every element a** ∈ **A**.

Remark: Using quantifiers we see that the relation R on the set A is reflexive if ∀a((a, a) ∈ R), where the universe of discourse is the set of all elements in A.

EXAMPLE 7 Consider the following relations on $\{1, 2, 3, 4\}$:

$$
\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},\\
R_2 &= \{(1, 1), (1, 2), (2, 1)\},\\
R_3 &= \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},\\
R_4 &= \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},\\
R_5 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},\\
R_6 &= \{(3, 4)\}.
\end{aligned}
$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. ◀

Supplement Irreflexive

properties-relations

Irreflexive Relation

A binary relation R on a set A is called irreflexive if aRa does not hold for any $a \in A$. This means that there is no element in R which is related to itself.

Examples of irreflexive relations:

1. The relation $<$ ("is less than") on the set of real numbers.
2. Relation of one person being son of another person.
3. The relation $R = \{(1,2), (2,1), (1,3), (2,3), (3,1)\}$ on the set $A = \{1,2,3\}$.

$$
\begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

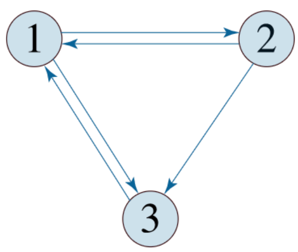


Figure 2.

The matrix of an irreflexive relation has all $0'$s on its main diagonal. The directed graph for the relation has no loops.

Definition 4

A relation R on a set A is called **symmetric** if (b, a) ∈ R **whenever (a, b) ∈ R, for all a, b ∈ A**. A relation R on a set A such that for all a, b ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called **antisymmetric**.

Remark: Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.

- antisymmetric
  1. if `(a,b)` in the set , `(b,a)` not in the set
  2. if `(a,b)` in the set and `(b,a)` in the set , then `a=b`
- symmetric
  1. if `(a,b)` in the set , then `(b,a)` must in the set

The terms symmetric and antisymmetric are not opposites, because a relation can have both of these properties or may lack both of them (see Exercise 10). A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) in which a ≠ b.

EXAMPLE 10 Which of the relations from Example 7 are symmetric and which are antisymmetric?

Extra Examples

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3, it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation, and $(1, 4)$ and $(4, 1)$ belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation. ◀

Definition 5

A relation R on a set A is called **transitive** if whenever (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R, for all a, b, c ∈ A.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have ∀a∀b∀c(((a, b) ∈ R ∧ (b, c) ∈ R) → (a, c) ∈ R).

- Once there exist a pair that `(a,b),(b,c)∈ R` , but , `(a,c) not ∈ R` , then R is not transitive

EXAMPLE 16 How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R. However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R. Each of the other $n(n-1)$ ordered pairs of the form (a, b), where $a \neq b$, may or may not be in R. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations [this is the number of ways to choose whether each element (a, b), with $a \neq b$, belongs to R]. ◀

9.1.5 Combining Relationships

Because relations from A to B are subsets of A × B, two relations from A to B can be combined in any way two sets can be combined.

Example 18

Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that R1 consists of all ordered pairs (a, b), where a is a student who has taken course b, and R2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate. What are the relations R1 ∪ R2, R1 ∩ R2, R1 ⊕ R2, R1 − R2, and R2 − R1?

Solution: The relation R1 ∪ R2 consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and R1 ∩ R2 is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, R1 ⊕ R2 consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. R1 − R2 is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. R2 − R1 is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken.

Definition 6

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by S ○R.

Definition 7

Let R be a relation on the set A. The powers Rn, n = 1, 2, 3, … , are defined recursively by

$$R^1 = R \quad , \quad R^{n+1} = R^n \circ R.$$

THEOREM 1 The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Proof: We first prove the "if" part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R. To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. ◁

Exercise

1. List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3}, where (a, b) ∈ R if and only if

   a) a = b.

   b) a + b = 4.

   c) a > b.

   d) a | b.

   e) gcd(a, b) = 1.

   f) lcm(a, b) = 2.

a) (0,0)(1,1)(2,2)(3,3)✓

b) (1,3)(2,2)(3,1)(4,0)✓

c) (1,0)(2,0)(2,1)(3,0)(3,1)(3,2)(4,0)(4,1)(4,2)(4,3)✓

d) (1,0)(1,1)(1,2)(1,3)(2,0)(2,2)(3,0)(3,3)(4,0)✓

~~e) (1,1)(1,2)(1,3)(2,1)(2,3)(3,1)(3,2)(4,1)(4,3)~~✗

gcd(0,a) = a

e) {(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)}

f) (1,2)(2,1)(2,2)✓

> 3. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
>
> a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
>
> b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
>
> c) {(2, 4), (4, 2)}
>
> d) {(1, 2), (2, 3), (3, 4)}
>
> e) {(1, 1), (2, 2), (3, 3),(4, 4)}
>
> f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

~~a) antisymmetric,transitive~~✗ (2,3)(3,2)

a) transitive

b) reflexisve, symmetric, transitive✓

c)symmetric✓

d)antisymmetric✓

e)reflexive, symmetric, antisymmetric, transitive✓

f)none✓

9.2 N-ray relations and their application

We will study relationships among elements from more than two sets in this section. These relationships are called n-ary relations. These relations are used to represent computer databases

Definition 1

Let A1, A2, … , An be sets. An n-ary relation on these sets is a subset of A1 × A2 × ⋯ × An. The sets A1, A2, … , An are called the **domains** of the relation, and n is called its **degree**.

EXAMPLE 3 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m), where a, b, and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$, and $(14, 0, 7)$ all belong to R, but $(7, 2, 3)$, $(-2, -8, 5)$, and $(11, 0, 6)$ do not belong to R because $8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$, but $7 \not\equiv 2 \pmod{3}$, $-2 \not\equiv -8 \pmod{5}$, and $11 \not\equiv 0 \pmod{6}$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers. ◀

Relational Datebase

A database consists of **records**, which are n-tuples, made up of fields. The fields are the entries of the n-tuples. For instance, a database of student records may be made up of **fields** containing the name, student number, major, and grade point average of the student.

Relations used to represent databases are also called **tables**, because these relations are often displayed as tables. Each column of the table corresponds to an attribute of the database. For instance, the same database of students is displayed in Table 1. The attributes of this database are Student Name, ID Number, Major, and GPA

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

A domain of an n-ary relation is called a primary key when **the value of the n-tuple from this domain determines the n-tuple**. That is, a domain is a primary key when **no two n-tuples in the relation have the same value from this domain.**

EXAMPLE 5 Which domains are primary keys for the n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?

Extra Examples

Solution: Because there is only one 4-tuple in this table for each student name, the domain of student names is a primary key. Similarly, the ID numbers in this table are unique, so the domain of ID numbers is also a primary key. However, the domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study. The domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA. ◀

Combinations of domains can also uniquely identify n-tuples in an n-ary relation. When the **values of a set of domains determine an n-tuple in a relation**, the Cartesian product of these domains is called a composite key.

EXAMPLE 6 Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation from Table 1, assuming that no n-tuples are ever added?

Solution: Because no two 4-tuples from this table have both the same major and the same GPA, this Cartesian product is a composite key. ◀

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database. Hence, checks should be made to ensure that every new record has values that are different in the appropriate field, or fields, from all other records in this table. For instance, it makes sense to use the student identification number as a key for student records if no two students ever have the same student identification number. A university should not use the name field as a key, because two students may have the same name (such as John Smith).

9.2.4 Operations on n-ray relations

The most basic operation on an n-ary relation is determining all n-tuples in the n-ary relation that satisfy certain conditions. For example, we may want to find all the records of all computer science majors in a database of student records. We may want to find all students who have a grade point average above 3.5. We may want to find the records of all computer science majors who have a grade point average above 3.5. To perform such tasks we use the selection operator

> ## Definition 2
>
> Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the **selection operator** s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C

Definition 3

The projection $P_{i_1 i_2,\ldots,i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple (a1, a2, … , an) to the m-tuple (ai1 , ai2 , … , aim), where m ≤ n.

Projections are used to form new n-ary relations by deleting the same fields in every record of the relation.

In other words, the projection Pi1,i2,…,im deletes n − m of the components of an n-tuple, leaving the i1th, i2th, … , and imth components.

EXAMPLE 8 What results when the projection $P_{1,3}$ is applied to the 4-tuples (2, 3, 0, 4), (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

Solution: The projection $P_{1,3}$ sends these 4-tuples to (2, 0), (Jane Doe, Geography), and (a_1, a_3), respectively. ◀

EXAMPLE 9 What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. Table 2 displays the results of this projection. ◀

Definition 4

Let R be a relation of degree m and S a relation of degree n. The join $J_p(R, S)$, where p ≤ m and p ≤ n, is a relation of degree m + n − p that consists of all (m + n − p)-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$belongs to S.

The **join** operation is used to combine two tables into one when these tables share some identical fields. For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.

In other words, the join operator J_p produces a new relation from two relations by combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples.

9.2.5 SQL(Structured Query Language)

```
SELECT keyname
FROM tablename
Where anotherkeyname = 'TheWantedValue'
```

9.2.6 Associating Rules from Data Mining

An important problem in data mining is to find strong association rules, which have support greater than or equal to a minimum support level and confidence greater than or equal to a minimum confidence level.

Exercise

9.3 Representing Relations

Ways to represent a finite relation

- list all its ordered pairs
- usa a table
- zero-one Matrix(computer-friendly)
- directed graph(mankind-friendly)

Matricx

The relation R can be represented by the matrix MR = $[m_{ij}]$, where

$$m_{ij} = \begin{cases} 0, & (a_i, b_j) \notin R, \\ 1, & (a_i, b_j) \in R \end{cases}$$

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

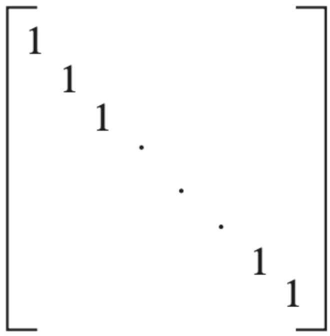
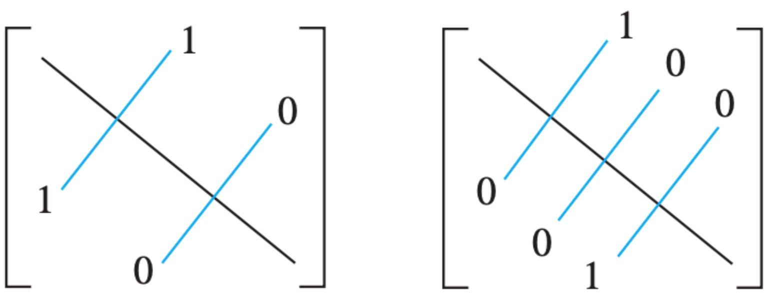


FIGURE 1 **The zero–one matrix for a reflexive relation. (Off diagonal elements can be 0 or 1.)**



(a) Symmetric (b) Antisymmetric

FIGURE 2 **The zero–one matrices for symmetric and antisymmetric relations.**

Let the zero– one matrices for S ○R, R, and S be $M_{S \circ R} = [t_{ij}]$, $M_R = [r_{ij}]$, and $M_S = [s_{ij}]$, respectively (these matrices have sizes m × p, m × n, and n × p, respectively). The ordered pair (a_i, c_j) belongs to S ○R if and only if **there is**[*Once there exist one is ok, only need there to be a way to get through*] an element b_k such that (a_i, b_k) belongs to R and(b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for **some k**. From the definition of the Boolean product, this means that

$$M_{R\circ S} = M_R \odot M_s (\odot \, means \times)$$

9.4 Closures of Relations

> ### Definition 1
>
> ---
>
> If R is a relation on a set A, then the closure of R with respect to P, if it exists, is the relation S on A with property P that contains R and is a subset of every subset of A × A containing R with property P.

NOTE : If there is a relation S that is a subset of every relation containing R with property P, it must be unique. To see this, suppose that relations S and T both have property P and are subsets of every relation with property P that contains R. Then, S and T are subsets of each other, and so are equal.

1. Reflexive Closure

The relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive. How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding (2, 2) and (3, 3) to R, because these are the only pairs of the form (a, a) that are not in R. This new relation contains R. Furthermore, any reflexive relation that contains R must also contain (2, 2) and (3, 3). Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta=\{(a, a) \mid a \in A\}$ is the diagonal relation on A. (The reader should verify this.)

2. Syemmtric Closure

The relation {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only add (2, 1) and (1, 3), because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R. This new relation is symmetric and contains R. Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain (2, 1) and (1, 3). Consequently, this new relation is called the symmetric closure of R.

As this example illustrates, the symmetric closure of a relation R can be constructed by adding all ordered pairs of the form (b, a), where (a, b) is in the relation, that are not already present in R. Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R. The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse (defined in the preamble of Exercise 26 in

Section 9.1); that is, R ∪ R−1 is the symmetric closure of R, where R−1 = {(b, a) ∣ (a, b) ∈ R}. The reader should verify this statement.

EXAMPLE 2 What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

Extra Examples

Solution: The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}.$$

This last equality follows because R contains all ordered pairs of positive integers, where the first element is greater than the second element, and R^{-1} contains all ordered pairs of positive integers, where the first element is less than the second. ◀

3.Transitive Closure

Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation? Consider the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R. The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1). **Adding these pairs does not produce a transitive relation**, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4). *(NOTE : because new uncompleted pair is generated after adding)*

This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure. The transitive closure of a relation can be found by adding new ordered pairs that must be present and then **repeating this process until no new ordered pairs are needed.**

9.4.3 Paths in Directed Graphs

Definition 2

A path from a to b in the directed graph G is a sequence of **edges** *(NOTE : edge, which means the two vertices are connected, (a,b) is in relation R)* , $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and x_0 = a and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. We view the empty set of edges as a path of length zero from a to a. A path of length n ≥ 1 that begins and ends at the same vertex is called a **circuit** or **cycle**.

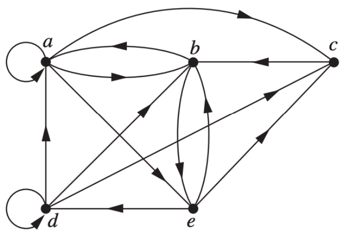


FIGURE 1 **A directed graph.**

A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

EXAMPLE 3 Which of the following are paths in the directed graph shown in Figure 1: *a, b, e, d*; *a, e, c, d, b*; *b, a, c, b, a, a, b*; *d, c*; *c, b, a*; *e, b, a, b, a, b, e*? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Solution: Because each of (*a, b*), (*b, e*), and (*e, d*) is an edge, *a, b, e, d* is a path of length three. Because (*c, d*) is not an edge, *a, e, c, d, b* is not a path. Also, *b, a, c, b, a, a, b* is a path of length six because (*b, a*), (*a, c*), (*c, b*), (*b, a*), (*a, a*), and (*a, b*) are all edges. We see that *d, c* is a path of length one, because (*d, c*) is an edge. Also *c, b, a* is a path of length two, because (*c, b*) and (*b, a*) are edges. All of (*e, b*), (*b, a*), (*a, b*), (*b, a*), (*a, b*), and (*b, e*) are edges, so *e, b, a, b, a, b, e* is a path of length six.

The two paths *b, a, c, b, a, a, b* and *e, b, a, b, a, b, e* are circuits because they begin and end at the same vertex. The paths *a, b, e, d*; *c, b, a*; and *d, c* are not circuits. ◀

Theorem 1

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

9.4.4 Transitive Closures

Definition 3

Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

The connectivity relation is useful in many models.

EXAMPLE 4 Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b. Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1, x_1 has met x_2, …, and x_{n-1} has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence. (There are many interesting conjectures about R^*. Do you think that this connectivity relation includes the pair with you as the first element and the president of Mongolia as the second element? We will use graphs to model this application in Chapter 10.) ◀

EXAMPLE 5 Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^*?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most $n - 1$ changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary. (The reader should verify these statements.) ◀

EXAMPLE 6 Let R be the relation on the set of all states in the United States that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

Solution: The relation R^n consists of the pairs (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. R^* consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary. (The reader should verify these statements.) The only ordered pairs not in R^* are those containing states that are not connected to the continental United States (that is, those pairs containing Alaska or Hawaii). ◀

Theorem 2

The transitive closure of a relation R equals the connectivity relation R^*

Lemma 1

Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when a ≠ b, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n − 1.

Suppose that xi = xj with 0 ≤ i < j ≤ m − 1. Then the path contains a circuit from xi to itself. This circuit can be deleted from the path from a to b, leaving a path, namely, x0, x1, … , xi , xj+1, … , xm−1, xm, from a to b of shorter length. Hence, the path of shortest length must have length less than or equal to n. The case where a ≠ b is left as an exercise for the reader.

From Lemma 1, we see that the transitive closure of R is the union of $R, R^2, R^3, \ldots$, and R^n. This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer i with $i \leq n$. Because

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

and the zero–one matrix representing a union of relations is the join of the zero–one matrices of these relations, the zero–one matrix for the transitive closure is the join of the zero–one matrices of the first n powers of the zero–one matrix of R.

THEOREM 3 Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
  $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
  $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}$ {$\mathbf{B}$ is the zero–one matrix for R^*}

Complexity Analysis

We can easily find the number of bit operations used by Algorithm 1 to determine the transitive closure of a relation. Computing the Boolean powers $\mathbf{M}_R, \mathbf{M}_R^{[2]}, \ldots, \mathbf{M}_R^{[n]}$ requires that $n-1$ Boolean products of $n \times n$ zero–one matrices be found. Each of these Boolean products can be found using $n^2(2n-1)$ bit operations. Hence, these products can be computed using $n^2(2n-1)(n-1)$ bit operations.

To find $\mathbf{M}_{R^*}$ from the n Boolean powers of $\mathbf{M}_R$, $n-1$ joins of zero–one matrices need to be found. Computing each of these joins uses n^2 bit operations. Hence, $(n-1)n^2$ bit operations are used in this part of the computation. Therefore, when Algorithm 1 is used, the matrix of the transitive closure of a relation on a set with n elements can be found using $n^2(2n-1)(n-1) + (n-1)n^2 = 2n^3(n-1)$, which is $O(n^4)$ bit operations. The remainder of this section describes a more efficient algorithm for finding transitive closures.

[if you don't know the (2n-1)]

[https://math.stackexchange.com/questions/1080574/number-of-bit-operations-in-nxn-zero-one-matrix-boolean-product]

9.4.5 Warshall's Algorithm

[Here is my understanding]

Warshall Algorithm use a concept called `interior vertices`.

> If $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, and b. The interior vertices of a, c, d, a, f, b are c, d, a, and f .

Warshall get the transitive closure step by step, add one vertice per step

First, at the beginning, only direct paths without interior vertices are allowed.

Say we have this directed graph

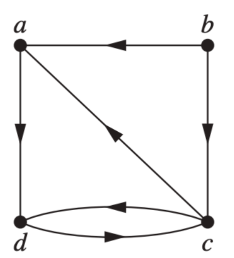


FIGURE 3 **The directed graph of the relation *R*.**

Solution: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, and $v_4 = d$. $\mathbf{W}_0$ is the matrix of the relation. Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $W_{ij} = 1$ when `i` and `j` is connected directly, otherwise 0

Secondly, add v_1, `a` in this case, to be an interior vertice.

$\mathbf{W}_1$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. Also, there is now an allowable path from b to d, namely, b, a, d. Hence,

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

....

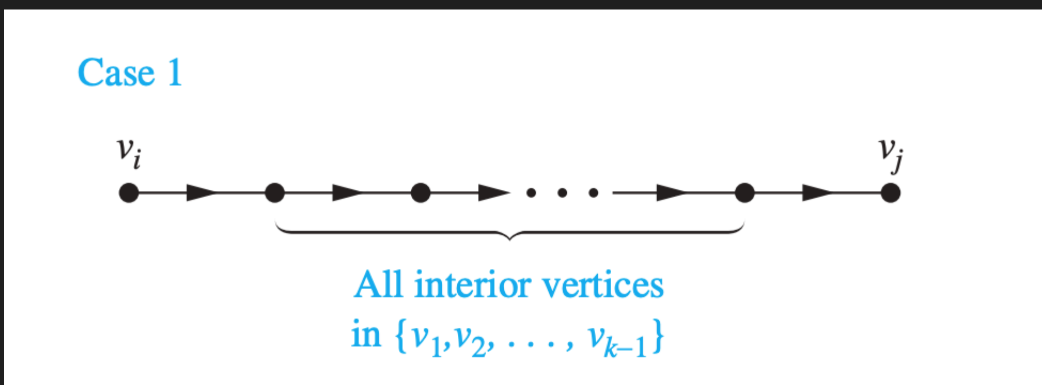
Do this until all vertices are added, that is `k == n`

But how can we know whether there is a new path been created between two vertices under the circumstance of v_k been added?

In fact there are only two cases:

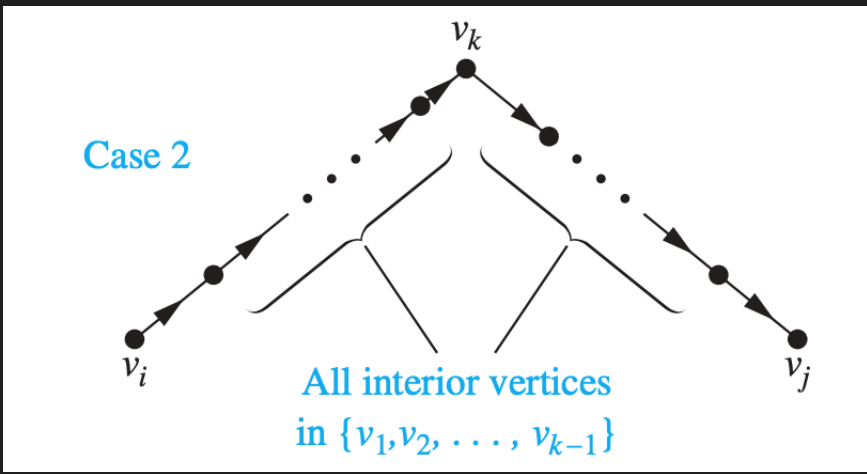
- Case 1:

  The path already existed before.



- Case 2:

  There exist a path from `start vertice` to `the vertice to be added` `&&` there exist a path from `the vertice to be added` to `destination vertice`



So we have this lemma

The first type of path exists if and only if $w_{ij}^{[k-1]} = 1$, and the second type of path exists if and only if both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. Hence, $w_{ij}^{[k]}$ is 1 if and only if either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. This gives us Lemma 2.

LEMMA 2 Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

And finally have codes of this algorithm

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
    for $i := 1$ **to** n
        for $j := 1$ **to** n
            $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

The complexity

The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry $w_{ij}^{[k]}$ from the entries $w_{ij}^{[k-1]}$, $w_{ik}^{[k-1]}$, and $w_{kj}^{[k-1]}$ using Lemma 2 requires two bit operations. To find all n^2 entries of $\mathbf{W}_k$ from those of $\mathbf{W}_{k-1}$ requires $2n^2$ bit operations. Because Warshall's algorithm begins with $\mathbf{W}_0 = \mathbf{M}_R$ and computes the sequence of n zero–one matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n = \mathbf{M}_{R^*}$, the total number of bit operations used is $n \cdot 2n^2 = 2n^3$.

[See the origin at page 657]

9.5 Equivalence Relations

Definition 1

A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2

Two elements a and b that are related by an equivalence relation are called equivalent. The notation `a ~ b` is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation

9.5.3 Equivalence Classes

Definition 3

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s|(a, s) \in R\}$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

9.5.4 Equivalence Classes and Partition

THEOREM 2 Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

EXAMPLE 13 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 12.

Solution: The subsets of S in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the S in the partition. The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair (6, 6) belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R. ◀

9.6 Partial Ordering

Definition 1

A relation R on a set S is called a partial ordering or partial order if it is **reflexive, antisymmetric, and transitive.** A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by(S, R). Members of S are called elements of the poset. (poset stands for `Partial Ordering set`)

EXAMPLE 1 Show that the greater than or equal to relation ($\geq$) is a partial ordering on the set of integers.

Extra Examples

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

Example 4 illustrates a relation that is not a partial ordering.

EXAMPLE 4 Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Extra Examples

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if xRy, then $y \not R x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy and yRz, then xRz. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R x$ for all people x. It follows that R is not a partial ordering. ◀

Definition 2

The elements a and b of a poset (S, $\preceq$) are called **comparable** if either a $\preceq$ b or b $\preceq$ a. When a and b are elements of S such that neither a $\preceq$ b nor b $\preceq$ a, a and b are called incomparable.

WHY we call things `partial ordering` *rather than* `total ordering` *:The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a total ordering.*

Definition 3

If (S, $\preccurlyeq$) is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and is called a total order or a linear order. A totally ordered set is also called a **chain.**

EXAMPLE 6 The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers. ◀

EXAMPLE 7 The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7. ◀

Definition 4

(S,$\preccurlyeq$) is a **well-ordered set** if it is a poset such that is a total ordering and every nonempty subset of S has a **least** element.

EXAMPLE 8 The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set. The verification of this is left as Exercise 53. The set $\mathbf{Z}$, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element. ◀

THEOREM 1 **THE PRINCIPLE OF WELL-ORDERED INDUCTION** Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

MY Understanding : for every x,y in well-ordered set S, if $P(x)$ is true and $x \prec y$,then $P(y)$ is true

9.6.2 Lexicographic Order

EXAMPLE 10 Note that $(1, 2, 3, 5) \prec (1, 2, 4, 3)$, because the entries in the first two positions of these 4-tuples agree, but in the third position the entry in the first 4-tuple, 3, is less than that in the second 4-tuple, 4. (Here the ordering on 4-tuples is the lexicographic ordering that comes from the usual less than or equal to relation on the set of integers.) ◀

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set S. Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if

$(a_1, a_2, \ldots, a_t) \prec (b_1, b_2, \ldots, b_t)$, or

$(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,

where $\prec$ in this inequality represents the lexicographic ordering of S^t. In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms. Then the t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on S^t. One string is less than another string if the t-tuple corresponding to the first string is less than the t-tuple of the second string, or if these two t-tuples are the same, but the second string is longer. The verification that this is a partial ordering is left as Exercise 38 for the reader.

```
/*question:
given an array of strings, how to sort them quickly
in lexicographic order?
(there is an answer that use STL sort with
costomized compare function, the time complexity is
O(n*log(n)), but can you come up with something new
or faster?)
*/
```

9.6.3 Hasse Diagrams

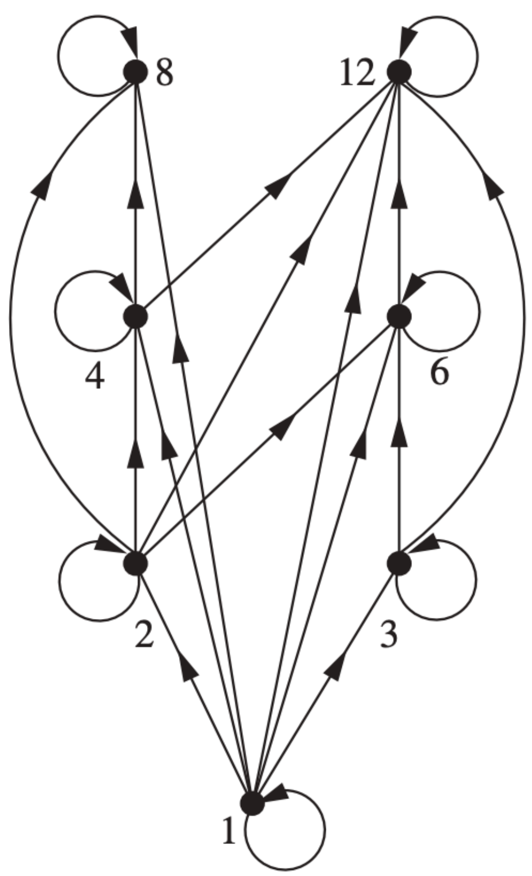
Attention! : An error on page 655 !!!(you could report this to the publisher)

of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$. Finally, arrange each edge so that its initial vertex is

It should be : remove all edges (x,y) for which there is an element z $\in$ S such that $x \prec z$ and $z \prec y$.

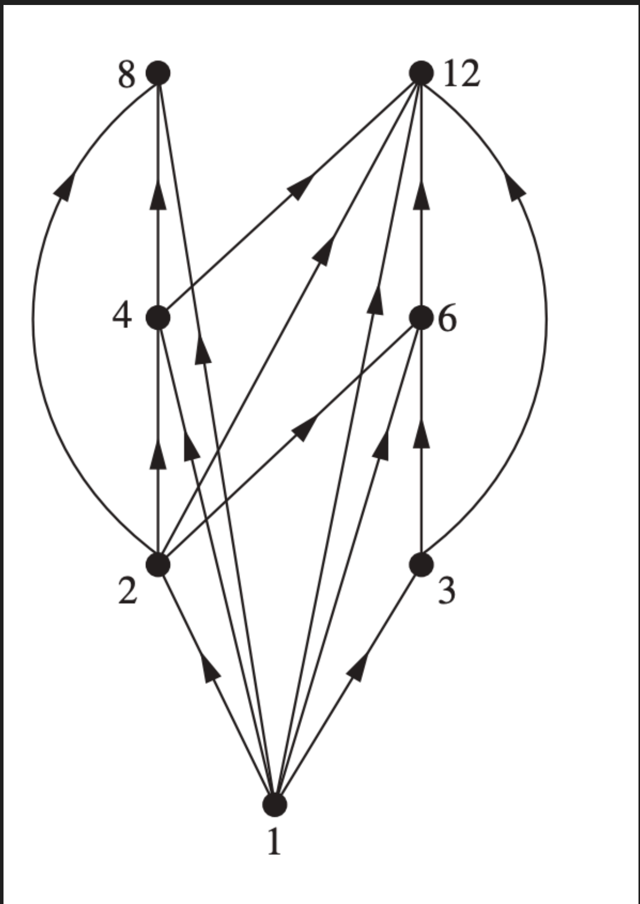
Procedures of drawing a Hasse Digram

1. draw a `directed graph` of the relation



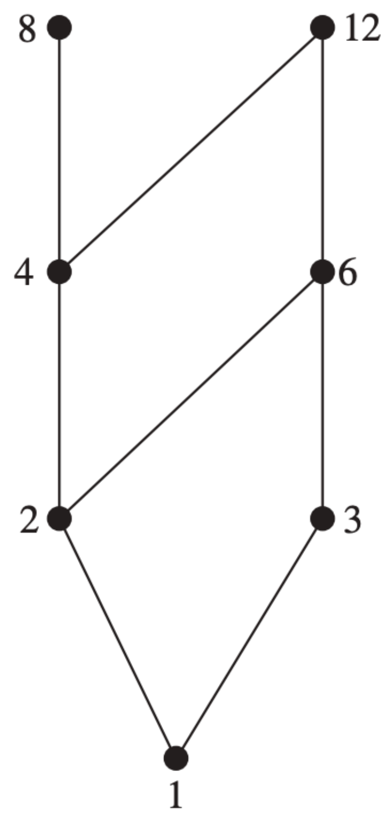
(a)

2. remove all `self-loop`s which represent `reflexive`



(b)

3. remove all `“indirected” path`s which represent `transitive`
4. remove the arrows, because the graph is always "upwards"



(c)

5. finish

> ## Covering Relationship
>
> Let $(S, \preceq)$ be a poset. We say that an element $y \in S$ covers an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the covering relation of $(S, \preceq)$.

From the description of the Hasse diagram of a poset, we see that the edges in the Hasse diagram of $(S, \preceq)$ are upwardly pointing edges corresponding to the pairs in the covering relation of $(S, \preceq)$. Furthermore, we can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation.

This tells us that we can construct a partial ordering from its Hasse diagram.

9.6.4 Maximal and Minimal Elements

> # Definition
>
> An element of a poset is called **maximal** if it is not less than any element of the poset. That is, a is maximal in the poset (S, $\preceq$) if there is no b $\in$ S such that a $\prec$ b. Similarly, an element of a poset is called **minimal** if it is not greater than any element of the poset. That is, a is minimal if there is no element b $\in$ S such that b $\prec$ a. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

EXAMPLE 14 Which elements of the poset ($\{2, 4, 5, 10, 12, 20, 25\}$, $|$) are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

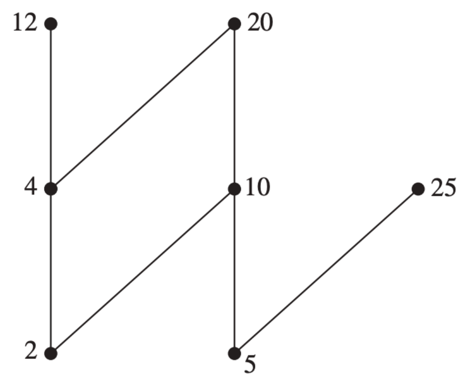


FIGURE 5 **The Hasse diagram of a poset.**

Definition

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the **greatest element**. That is, a is the greatest element of the poset (S,$\preceq$) if b a for all b $\in$ S. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the **least element** if it is less than all the other elements in the poset. That is, a is the least element of (S, $\preceq$) if a $\preceq$ b for all b $\in$ S. The least element is unique when it exists [see Exercise 40(b)].

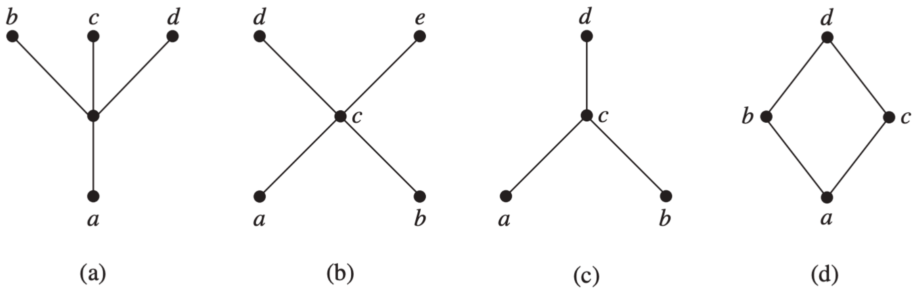


FIGURE 6 **Hasse diagrams of four posets.**

Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d. ◀

EXAMPLE 16 Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S. ◀

Definition

> Sometimes it is possible to find an element that is greater than or equal to all the elements in a **subset A** of a poset $(S, \preceq)$.If u is an element of S such that $a \preceq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preceq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

You may wonder are there any differences between `the greatest element` and the `upper bound`, here's the answer. see it yourself

Definition [edit]

$S = \{1,2,3,4\}$ has two maximal elements, viz. 3 and 4, and one minimal element, viz. 1, which is also its least element.

Throughout, let $(P, \leq)$ be a partially ordered set and let $S \subseteq P$.

> **Definition**: An element g of a subset S of P is said to be a **greatest element of S** if it satisfies
>
> $s \leq g$, for all $s \in S$.
>
> If S has a greatest element then it is necessarily unique so we may speak of ***the* greatest element of S**.

By using $\geq$ instead of $\leq$ in the above definition, one defines the least element of S.

Contrast to maximal elements, upper bounds, and local/absolute maximums [edit]

The greatest element of a partially ordered subset must not be confused with **maximal elements** of the set, which are elements that are not smaller than any other element in the set. A set can have several maximal elements without having a greatest element. Like upper bounds and maximal elements, greatest elements may fail to exist.

> **Definitions**:
>
> 1. An element $m \in S$ is said to be a **maximal element of S** if there does *not* exist any $s \in S$ such that $m \leq s$ and $s \neq m$.
> 2. An **upper bound of S *in* P** is an element u such that $u \in P$ and $s \leq u$ for all $s \in S$.

In the particular case where $P = S$, the definition of "u is an upper bound of S *in* S" becomes: u is an element such that $u \in S$ and $s \leq u$ for all $s \in S$, which is **completely identical** to the definition of a greatest element given before. Thus g is a greatest element of S if and only if g is an upper bound of S *in* S.

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a. ◀

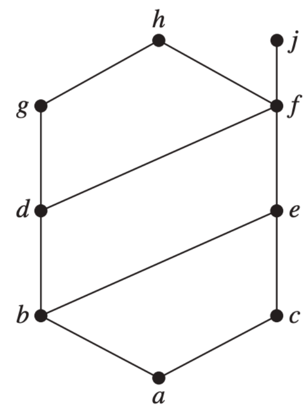


FIGURE 7 The Hasse diagram of a poset.

Attention!!!

- *It is useful to **reveal all the upwards arrows** when finding upperor lower bound*
- Don't forget that upper/lower bound can be the element in subset A rather than in set S($\preceq$ means it can be "equal")

Definition

The element x is called the **least upper bound** of the **subset A** if x is an upper bound that is less than every other upper bound of A. Because there is **only one** such element, if it exists, it makes sense to call this element the least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A, if $a \preceq x$ whenever $a \in A$, and $x \preceq z$ whenever z is an upper bound of A. Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preceq y$ whenever z is a lower bound of A. The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively

EXAMPLE 20 Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Extra Examples

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

9.6.5 Lattices

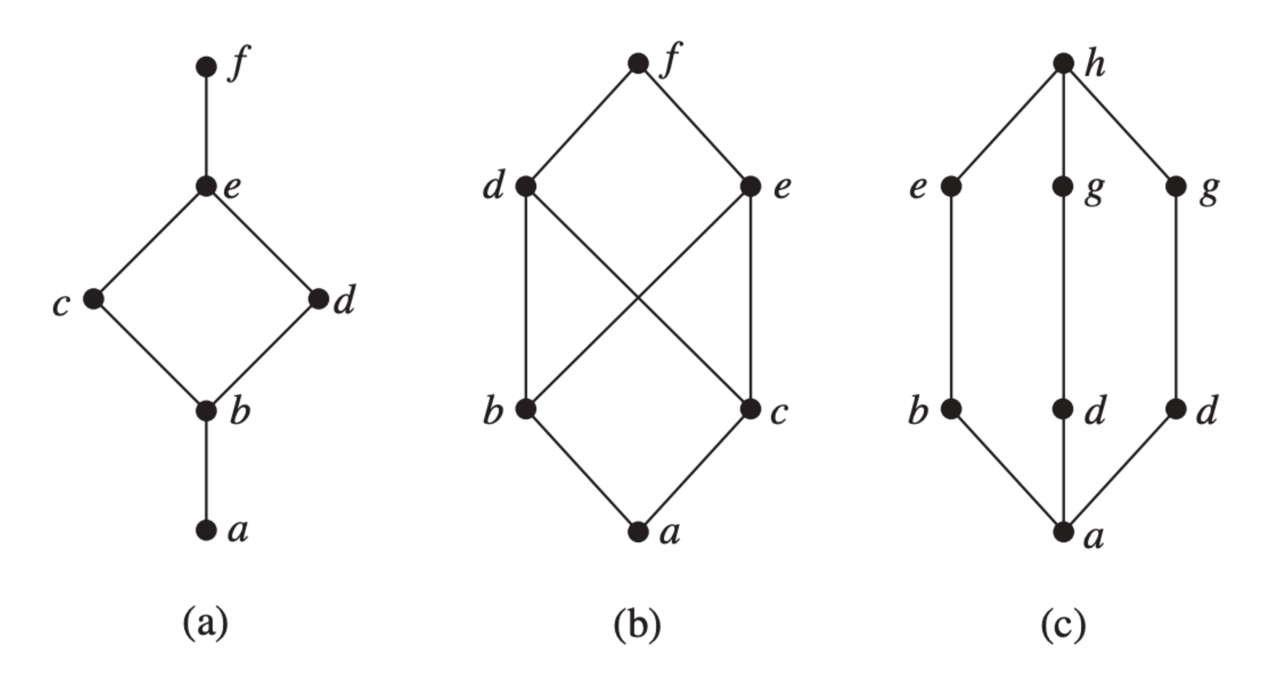
Definition

A partially ordered set in which every pair of elements has **both a least upper bound and a greatest lower bound** is called a lattice.

NOTE: every pair means the two elements do NOT need to be connected in Hasse Diagram

EXAMPLE 21 Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀



EXAMPLE 22 Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

EXAMPLE 24 Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

9.6.4 Topological Sort

> A total ordering is said to be compatible with the partial ordering R if a$\preceq$ b whenever aRb. Constructing a compatible total ordering from a partial ordering is called topological sorting.

> ## Lemma 1
>
> Every finite nonempty poset (S,$\preceq$) has at least one minimal element.

Proof: Choose an element a_0 of S. If a_0 is not minimal, then there is an element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, there is an element a_2 with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} \prec a_n$. Because there are only a finite number of elements in the poset, this process must end with a minimal element a_n. ◁

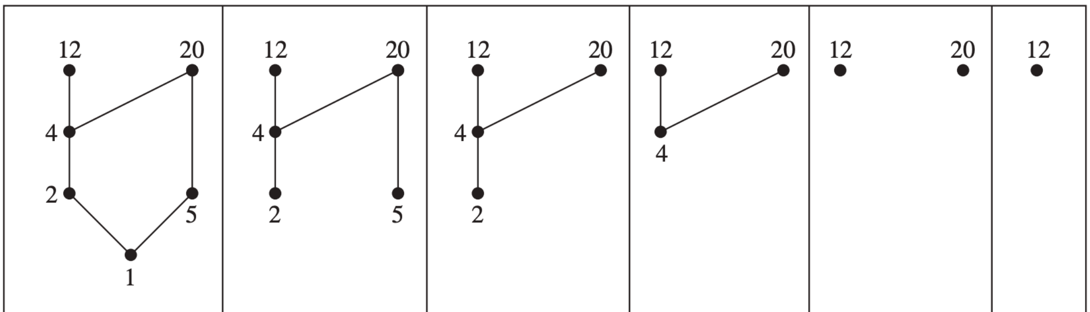
ALGORITHM 1 Topological Sorting.

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
  $S := S - \{a_k\}$
  $k := k + 1$
return $a_1, a_2, \ldots, a_n$ $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

EXAMPLE 26 Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution: The first step is to choose a minimal element. This must be 1, because it is the only minimal element. Next, select a minimal element of $(\{2, 4, 5, 12, 20\}, |)$. There are two minimal elements in this poset, namely, 2 and 5. We select 5. The remaining elements are $\{2, 4, 12, 20\}$. The only minimal element at this stage is 2. Next, 4 is chosen because it is the only minimal

*"Topological sorting" is terminology used by computer scientists; mathematicians use the terminology "linearization of a partial ordering" for the same thing. In mathematics, topology is the branch of geometry dealing with properties of geometric figures that hold for all figures that can be transformed into one another by continuous bijections. In computer science, a topology is any arrangement of objects that can be connected with edges.



Minimal element chosen	1	5	2	4	20	12

FIGURE 9 **A topological sort of $(\{1, 2, 4, 5, 12, 20\}, |)$.**

element of $(\{4, 12, 20\}, |)$. Because both 12 and 20 are minimal elements of $(\{12, 20\}, |)$, either can be chosen next. We select 20, which leaves 12 as the last element left. This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

Attention!!!

This algorithm only give an arbitary "order" of doing things, the $5 \prec 2$ *in the result do not mean* $5|2$ *or* $2|5$*, actually there is no relation between them*

```
/*question:
1. code the algorithm of finding the minimal
element out
2. Is there any way to also compose the edge(the
original relation)
3. I have an idea of `descend elements`
    3.1 list all N elements
    3.2 each of them have value N
    3.3 let each element traverse the set, N-1 when
comparing result is <
    3.4 sort the values
*/
```

Field 1	Field 2	Field 3	Field 4	Field 5	Field 6	Field 7	Field 8
8	8	8	8	8	8	8	8
ø	a	b	c	ac	ab	bc	abc
−1	−1	−1	−1	−1	−1	−1	8
−1	−1	−1	−1	7	7	7	
−1	−1	−1	−1				
−1	5	5	5				
−1							
−1							
−1							
1							