Lesson 2: Simulation of stochastic dynamic models

Qianying (Ruby) Lin   Spencer J. Fox   Zian (Larry) Zhuang

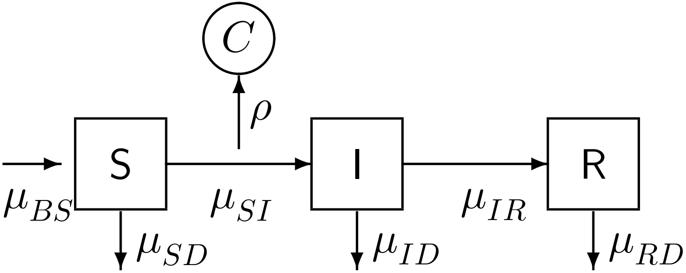
Objectives

This tutorial develops some classes of dynamic models relevant to biological systems, especially for epidemiology.

1. Dynamic systems can often be represented in terms of *flows* between *compartments*.
2. We develop the concept of a *compartment model* for which we specify *rates* for the flows between compartments.
3. We show how deterministic and stochastic versions of a compartment model are derived and related.
4. We introduce Euler's method to simulate from dynamic models.
5. We specify deterministic and stochastic compartment models in pomp using Euler method simulation.

A basic compartment model: The SIR model I

- We develop deterministic and stochastic representations of a susceptible-infected-recovered (SIR) system, a fundamental class of models for disease transmission dynamics.
- We set up notation applicable to general compartment models (Bretó et al. 2009).



S : susceptible I : infected and infectious
R : recovered and/or removed C : reported cases

A basic compartment model: The SIR model II

- We suppose that each arrow has an associated rate, so here there is a rate $\mu_{SI}(t)$ at which individuals in S transition to I, and μ_{IR} at which individuals in I transition to R.
- To account for demography (births/deaths/migration) we allow the possibility of a source and sink compartment, which is not usually represented on the flow diagram. We write μ_{BS} for a rate of births into S, and denote mortality rates by μ_{SD}, μ_{ID}, μ_{RD}.
- The rates may be either constant or time-varying.
- For the simplest SIR model, ignoring demography, we set

$$\mu_{BS} = \mu_{SD} = \mu_{ID} = \mu_{RD} = 0.$$

General notation for compartment models I

To develop a systematic notation, it turns out to be convenient to keep track of the flows between compartments as well as the number of individuals in each compartment:

- $N_{SI}(t)$: the number of individuals who have transitioned from S to I **by** time t. We say that $N_{SI}(t)$ is a *counting process*.
- $N_{IR}(t)$: the number of individuals transitioning from I to R **by** time t.

To include demography, we could keep track of birth and death events by the counting processes:

- $N_{BS}(t)$: the number of newborns into S **by** time t.
- $N_{SD}(t)$, $N_{ID}(t)$, $N_{RD}(t)$: the number of deaths from S, I, and R compartments **by** time t, respectively.

General notation for compartment models II

- For discrete population compartment models, the flow counting processes are non-decreasing and integer valued.
- For continuous population compartment models, the flow counting processes are non-decreasing and real valued.

Compartment model from counting processes

- The numbers of people in each compartment can be computed via these counting processes. Ignoring demography, we have:

$$
\begin{aligned}
S(t) &= S(0) - N_{SI}(t) & & \\
I(t) &= I(0) + N_{SI}(t) & &- N_{IR}(t) \\
R(t) &= R(0) & &+ N_{IR}(t)
\end{aligned}
$$

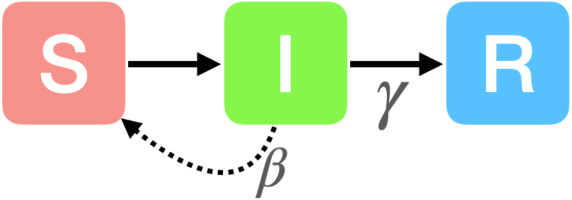
- These equations represent *conservation of individuals* or *what goes in must come out*.

Ordinary differential equation interpretation

Together with initial conditions specifying $S(0)$, $I(0)$ and $R(0)$, we just need to write down ordinary differential equations (ODEs) for the flow counting processes. These are:

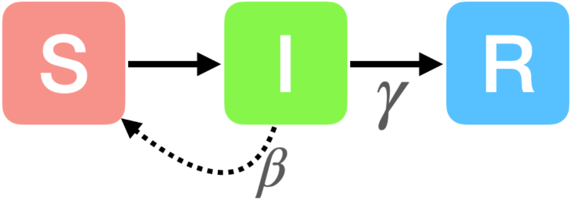
$$\frac{\mathrm{d}N_{SI}}{\mathrm{d}t} = \mu_{SI}(t)\, S(t)$$
$$\frac{\mathrm{d}N_{IR}}{\mathrm{d}t} = \mu_{IR}\, I(t)$$

Common notation for a deterministic SIR model



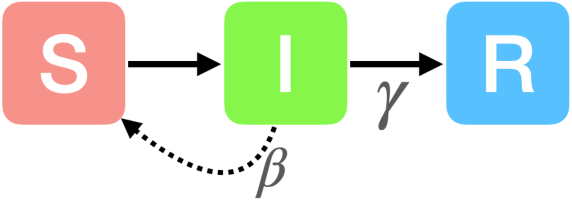
- β: transmission rate, encompasses the frequency of contacts and transmission probability between individuals
- γ: recovery rate, rate that infected individuals become "uninfectious"
  - Duration of infectiousness on average is $\frac{1}{\gamma}$
- $S + I + R = N$

Common notation for a deterministic SIR model - equations



$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S \frac{I}{N}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta S \frac{I}{N} - \gamma * I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I$$

Common notation for a deterministic SIR model - Skeleton code



$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S \frac{I}{N}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta S \frac{I}{N} - \gamma * I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I$$

```
library(pomp)
Csnippet("
        DS = -Beta*S*I/N;
        DI = Beta*S*I/N - I*Gamma;
        DR = I*Gamma;
        ") -> sir_det_skel
```

Spencer component done...

Stochastic Differential Equations (SDEs)

- By including randomness in the ODE system, we can have the stochastic differential equation (SDE) system.
- For example, for the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = h(x)$, a natural way to add stochastic variation is

  $$\frac{\mathrm{d}X}{\mathrm{d}t} = h(X) + \sigma \frac{\mathrm{d}B}{\mathrm{d}t}$$

  where $\{B(t)\}$ is Brownian motion and so dB/dt is Brownian noise.

The simple counting process and the reactions I

- A deterministic SIR model has a fixed trajectory, indicating that the number of each compartment at any time is fixed with given parameters and intial states; thus the transitions between compartments are fixed at any time.
- A stochastic SIR model, in the contrary, the trajectory and the transitions between compartments at any time are stochastic.
- Recall $N_{SI}(t)$ and $N_{IR}(t)$ are counting processes, indicating the number of total individuals transitioned from S to I and I to R by time t, respectively.
- A *simple counting process* is one which cannot count more than one event at a time.

The simple counting process and the reactions II

- We then can relate the counting process to the common SIR reactions with the corresponding probabilities.
- Note that we are using *little o notation* and we write $h(\delta) = o(\delta)$ to mean $\lim_{\delta \to 0} \frac{h(\delta)}{\delta} = 0$.

Table 1: Relationship between the counting processes, the reactions, and the probabilities.

Counting	Reaction	Probability
$N_{SI}(t+\delta) = N_{SI}(t) + 1$	$S \to S - 1$ $I \to I + 1$	$\beta S(t) I(t) \delta / N + o(\delta)$
$N_{SI}(t+\delta) = N_{SI}(t)$		$1 - \beta S(t) I(t) \delta / N + o(\delta)$
$N_{IR}(t+\delta) = N_{IR}(t) + 1$	$I \to I - 1$ $R \to R + 1$	$\gamma I(t) \delta + o(\delta)$
$N_{IR}(t+\delta) = N_{IR}(t)$		$1 - \gamma I(t) \delta + o(\delta)$