Assignment 8

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https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment8

1 Problem

Let $s<n$ and A an $s \times n$ matrix with entries in the field $\mathbb{F}$. Use Theorem 4 to show that there is a non-zero $\mathbf{x}$ in $\mathbb{F}^{n\times 1}$ such that $Ax = \mathbf{0}$.

2 Explanation

Theorem 4:*Let $\mathbb{V}$ be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ..., \beta_m$. Then any independent set of vectors in $\mathbb{V}$ is finite and contains no more than m elements.*

3 Solution

Let V be a vector space spanned by $A_1, A_2, \ldots, A_n$, where A_i, i=1,2,…,n are columns of matrix $A_{s\times n}$. let $\mathbf{x}$ be a vector in vector space $\mathbb{V}$ Let

$$S = \{x : Ax = \mathbf{0}\} \tag{3.0.1}$$
$$\text{If } x_1, x_2 \in S \tag{3.0.2}$$
$$\implies A(x_1 + x_2) = Ax_1 + Ax_2 = 0 \tag{3.0.3}$$
$$\implies (x_1 + x_2) \in S \tag{3.0.4}$$
$$\text{If } x \in S \tag{3.0.5}$$
$$\implies A(\alpha x) = \alpha(Ax) = 0 \tag{3.0.6}$$
$$\implies \alpha x \in S \tag{3.0.7}$$

From equations (3.0.4) and (3.0.7), S is a subspace of $\mathbb{V}$.
From equation (3.0.4), it is clear that,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.0.8}$$

is in S iff,

$$\sum_{i=1}^{n} A_i x_i = \mathbf{0} \tag{3.0.9}$$

Where A_i is the i^{th} column of matrix A.
Let $A_1, A_2, \ldots, A_s$ form a basis for column space of A, then,

$$Ax = \mathbf{0} \tag{3.0.10}$$
$$\implies A_1x_1 + \cdots + A_sx_s + A_{s+1}x_{s+1} + \cdots + A_nx_n = \mathbf{0} \tag{3.0.11}$$

The columns not in the basis can be expressed as linear combinations of the basis columns

$$A_{s+1} = \sum_{j=1}^{s} \beta_{1,j} x_i \tag{3.0.12}$$
$$A_{s+2} = \sum_{j=1}^{s} \beta_{2,j} x_i \tag{3.0.13}$$
$$\vdots \tag{3.0.14}$$
$$A_n = \sum_{j=1}^{s} \beta_{(n-s),j} x_i \tag{3.0.15}$$

Therefore, we can conclude that the columns of matrix A are not independent.
Equation (3.0.11) leads to a homogeneous system of linear equations with *s equations and n unknowns*.
Since, s rows can hold at most s pivots, there must be $(n-s)$ free variables.
These free variables can be assigned any value.
Hence, there are more solutions to equation (3.0.11) than the trivial $x = 0$.
$\implies$ Equation (3.0.11) will have at least one special solution.
Therefore, at least one $x_j \neq 0, j = 1, 2, \ldots, n$ exists such that $Ax = \mathbf{0}$.