Assignment 5

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QR Decomposition

***Abstract*—This document solves problem based on QR decomposition.**

Download all solutions from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment5

1 Problem

Perform QR decomposition on the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{1.0.1}$$

2 Solution

The columns of the matrix $\mathbf{A}$ can be represented as:

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.1}$$

$$\beta = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.0.2}$$

For QR decomposition, matrix A is represented in the form:

$$\mathbf{A} = \mathbf{QR} \tag{2.0.3}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.5}$$

Here $\mathbf{R}$ is a upper triangular matrix and Q is a orthogonal matrix such that,

$$\mathbf{Q^T Q} = \mathbf{I} \tag{2.0.6}$$

Now we calculate the above values,

$$k_1 = \|\alpha\| \tag{2.0.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{2.0.8}$$

$$r_1 = \frac{\mathbf{u_1^T}\beta}{\|\mathbf{u_1}\|^2} \tag{2.0.9}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{2.0.10}$$

$$k_2 = \mathbf{u_2^T}\beta \tag{2.0.11}$$

Substituting (2.0.1) and (2.0.2) in the above equations, we get

$$k_1 = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{2.0.12}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.13}$$

$$r_1 = \frac{1}{\left(\sqrt{\frac{1}{5} + \frac{4}{5}}\right)^2}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.0.14}$$

$$\implies r_1 = \frac{11}{\sqrt{5}} \tag{2.0.15}$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{11}{5} \\ \frac{22}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{u_2} = \frac{\begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix}}{\sqrt{\frac{4}{5}^2 + \frac{-2}{5}^2}} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{2.0.17}$$

$$k_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{2.0.18}$$

Therefore, from (2.0.4) and (2.0.5) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.20}$$

where

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (2.0.21)$$

Therefore matrix **A** in QR decomposed form is,

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.22)$$