Assignment 9

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***Abstract*—This assignment deals with linear transformation.**

Download all solutions from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment9

1 Problem

Let V be a vector space over the field of complex numbers , and suppose there is an isomorphism T of V onto C^3. Let α_1 , α_2 , α_3 , α_4 be vectors in V such that

$$\begin{aligned} T(\alpha_1) = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \\ T(\alpha_3) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix} \end{aligned} \tag{1.0.1}$$

Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2.

2 Theory

2.1 Properties of Isomorphism

T : V → W is an isomorphism if
(1) T is one one.
(2) T is onto.

2.2 Theorem 6

If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and

$$dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2) \tag{2.2.1}$$

3 Solution

We have,

$$T = \begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \tag{3.0.1}$$

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.2}$$

T is one one over C^3 if

$$T(\alpha) = 0 \implies \alpha = 0 \tag{3.0.3}$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \tag{3.0.4}$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xleftrightarrow[R_3 \to iR_3]{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2}+3i \end{pmatrix} \tag{3.0.5}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2}+4i+1 \end{pmatrix} \tag{3.0.6}$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.7}$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns).Thus the vectors are linearly independent hence it is onto . Hence, T is an isomorphism onto C^3.
Also, from equation (3.0.2), we observe,

$$T(\alpha_3) = -iT(\alpha_1) + \frac{i-1}{2} T(\alpha_2) \tag{3.0.8}$$

Hence $T(\alpha_3)$ belongs to the subspace spanned by $T(\alpha_1)$ and $T(\alpha_2)$.
Therefore, α_3 is in subspace spanned by α_1 and α_2.
Therefore,

$$\alpha_3 \in W_1 \tag{3.0.9}$$
$$\implies \alpha_3 \in W_1 \cap W_2 \tag{3.0.10}$$

Since $T(\alpha_1)$ and $T(\alpha_2)$ are linearly independent, and $T(\alpha_3)$ and $T(\alpha_4)$ are linearly independent, we have,

$$dim(W_1) = dim(W_2) = 2 \tag{3.0.11}$$

From equation (2.2.1),

$$dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2) \tag{3.0.12}$$
$$dim(W_1 + W_2) = 3 \tag{3.0.13}$$
$$\implies dim(W_1 \cap W_2) = 2 + 2 - 3 \tag{3.0.14}$$
$$\implies dim(W_1 \cap W_2) = 1 \tag{3.0.15}$$

Therefore,

$$W_1 \cap W_2 = c\alpha_3 \tag{3.0.16}$$