Assignment 16

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Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment16

1 Problem

True or False? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection.

2 Definitions

Diagonalizable Operator	For a linear operator $\mathbf{T}\colon \mathbf{V} \longrightarrow \mathbf{V}$, $\mathbf{T}$ is a diagonalizable operator if $\exists$ some basis for $\mathbf{V}$ such that the matrix representing $\mathbf{T}$ is a diagonal matrix i.e. $$\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$$ $\implies$ $\mathbf{A}$ is a diagonalizable matrix
Properties of Projection	If $n \times n$ matrix $\mathbf{A}$ is projecion matrix, then $$\mathbf{A}^2 = \mathbf{A}$$

TABLE 1: Definitions

3 Solution

Diagonalizability	Let $\mathbf{A}$ be $n \times n$ matrix. Given that $\mathbf{A}$ is diagonalizable, it can be expressed as, $$\begin{aligned} &\mathbf{A} = \mathbf{P\Lambda P}^{-1} \\ \implies &\mathbf{AP} = \mathbf{P\Lambda} \end{aligned} \qquad \ldots(1)$$ where, $\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$
Eigen values	Given that $\mathbf{A}$ has eigen values 0 and 1 $\implies$ $\mathbf{\Lambda}$ has diagonal entries of $0s$ and $1s$ only $\implies \lambda_i = 0$ or $1, \qquad i = 0, 1, \ldots, n$ $\implies \lambda_i^2 = 0$ or 1 $\implies \lambda_i^2 = \lambda_i$ $$\implies \mathbf{\Lambda}^2 = \mathbf{\Lambda\Lambda} = \begin{pmatrix} \lambda_1^2 & 0 & \ldots & 0 \\ 0 & \lambda_2^2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n^2 \end{pmatrix} = \mathbf{\Lambda} \qquad \ldots(2)$$
Projection	$$\begin{aligned} &\mathbf{A} = \mathbf{P\Lambda P}^{-1} \\ \implies &\mathbf{AA} = \mathbf{AP\Lambda P}^{-1} \\ \implies &\mathbf{A}^2 = (\mathbf{AP})\,\mathbf{\Lambda P}^{-1} \end{aligned}$$ From (1), $$\begin{aligned} \implies \mathbf{A}^2 &= \mathbf{P\Lambda\Lambda P}^{-1} \\ &= \mathbf{P\Lambda}^2\mathbf{P}^{-1} \end{aligned}$$ From (2), $$\begin{aligned} \implies \mathbf{A}^2 &= \mathbf{P\Lambda P}^{-1} \\ &= \mathbf{A} \end{aligned}$$ Therefore, $$\mathbf{A}^2 = \mathbf{A}$$ Hence, $\mathbf{A}$ is a projection matrix
Example	Consider a 2×2 matrix $$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

	Characteristic polynomial, $p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert = \begin{vmatrix} x & -1 \\ 0 & x-1 \end{vmatrix} = x(x-1)$ $\implies \lambda_1 = 0, \lambda_2 = 1$ Also, $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{PJP}^{-1}$ where $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ Now we check if $\mathbf{A}$ is projection matrix. $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \mathbf{A}$ Therefore, $\mathbf{A}^2 = \mathbf{A}$ Hence, if $\mathbf{A}$ is diagonalizable and has eigen values 0 and 1, then $\mathbf{A}$ is a projection matrix.
Conclusion	Given statement is True

TABLE 2: Checking for projection matrix