Assignment 17

Rubeena Aafreen

Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment17

1 Problem

Let $\mathbf{T}$ be the diagonalizable linear operator on $\mathbb{R}^3$ which we discussed in example 3 of section 6.2. Use the Lagrange polynomials to write the representing matrix $\mathbf{A}$ in the form

$$\mathbf{A} = \mathbf{E_1} + 2\mathbf{E_2}, \quad \mathbf{E_1} + \mathbf{E_2} = \mathbf{I}, \mathbf{E_1}\mathbf{E_2} = \mathbf{0} \tag{1.0.1}$$

2 Outline

Diagonalizable Operator	For a linear operator $\mathbf{T}\colon \mathbf{V} \longrightarrow \mathbf{V}$, $\mathbf{T}$ is a diagonalizable operator if $\exists$ some basis for $\mathbf{V}$ such that the matrix representing $\mathbf{T}$ is a diagonal matrix i.e. $$\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$$ $\implies$ $\mathbf{A}$ is a diagonalizable matrix
Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A}\right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial
Lagrange Polynomials	For a set of scalars $c_0, c_1, \ldots, c_n \in \mathbb{F}$, Lagrange Polynomial is defined as: $$p_j = \prod_{i \neq j} \frac{(x - c_i)}{\left(c_j - c_i\right)}$$

TABLE 1: Definitions

3 Solution

Given	$$\mathbf{A} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$
Characteristic polynomial	$$\begin{aligned} p(x) &= \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert \\ &= \begin{vmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{vmatrix} \\ &= x^3 - 5x^2 + 8x - 4 \\ &= (x-1)(x-2)^2 \end{aligned}$$ $\implies \lambda = 1, 2$
Minimal Polynomial	$p(x) = (x-1)(x-2)^b, \quad b \leq 2$ $$(\mathbf{A} - \mathbf{I})(\mathbf{A} - 2\mathbf{I}) = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} = \mathbf{0}$$ Therefore, $(x-1)(x-2)$ is the minimal polynomial.
Lagrange Polynomial	$$p_j = \prod_{i \neq j} \frac{(x - c_i)}{\left(c_j - c_i\right)}$$ For characteristic values $c_1 = 1, \quad c_2 = 2,$ $\implies p_1 = \frac{(x-1)}{2-1}, \qquad p_2 = \frac{(x-2)}{1-2}$ $\implies p_1 = (x-1)$, and $p_2 = (2-x)$
Projection Maps	We know that, $$\mathbf{E_j} = p_j(\mathbf{T})$$ $\implies \mathbf{E_1} = \mathbf{A} - \mathbf{I}$ and $\mathbf{E_2} = 2\mathbf{I} - \mathbf{A}$ $\implies \mathbf{E_1} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$, and $\mathbf{E_2} = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$

Verification	We have, $\mathbf{E_1} = \mathbf{A} - \mathbf{I}$ $\implies \mathbf{A} - \mathbf{E_1} = \mathbf{I} \qquad \ldots (1)$ $\mathbf{E_2} = 2\mathbf{I} - \mathbf{A}$ From (1), $\implies \mathbf{E_2} = 2(\mathbf{A} - \mathbf{E_1}) - \mathbf{A}$ $\implies \boxed{\mathbf{A} = 2\mathbf{E_1} + \mathbf{E_2}}$ Also, $\mathbf{E_1} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}, \qquad \mathbf{E_2} = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $\implies \mathbf{E_1} + \mathbf{E_2} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} + \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\implies \boxed{\mathbf{E_1} + \mathbf{E_2} = \mathbf{I}}$ $\mathbf{E_1}\mathbf{E_2} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\implies \boxed{\mathbf{E_1}\mathbf{E_2} = \mathbf{0}}$

TABLE 2: Using Lagrange Polynomials to represent $\mathbf{A}$