Assignment 14

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Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment14

1 Problem

Let $\mathbf{A}$ be the 4×4 real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \tag{1.0.1}$$

Show that the characteristic polynomial for $\mathbf{A}$ is $x^2(x-1)^2$ and that it is also the minimal polynomial

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $p(x) = \left|x\mathbf{I} - \mathbf{A}\right|$
Minimal Polynomial	Minimal polynomial $m(x)$ is he smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Algebraic Multiplicity (A_M)	No. Of times an Eigen value appears in a characteristic equation.

TABLE 1: Definitions

3 Solution

Characteristic polynomial	$\begin{aligned} p(x) &= \left|x\mathbf{I} - \mathbf{A}\right| \\ &= \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix} \\ &= ((x-1)(x+1)+1)((x-2)x+1) \\ &= x^2\left(x^2-2x+1\right) \\ &= x^2(x-1)^2 \end{aligned}$

<table>
<tr><td>A_M</td><td>For $\lambda = 0, \quad A_M = 2$
For $\lambda = 1, \quad A_M = 2$</td></tr>
<tr><td>Minimal Polynomial</td><td>$p(x) = x^a (x-1)^b, \quad a \leq 2, b \leq 2$</td></tr>
<tr><td>$a = 1, b = 1$</td><td>$m(x) = x(x-1)$
$$\begin{aligned} \implies m(\mathbf{A}) &= \mathbf{A}(\mathbf{A}-\mathbf{I}) \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{pmatrix} \neq \mathbf{0} \end{aligned}$$
$\implies x(x-1)$ is not a minimal polynomial</td></tr>
<tr><td>$a = 2, b = 1$</td><td>$m(x) = x^2(x-1)$
$$\begin{aligned} \implies m(\mathbf{A}) &= \mathbf{A}^2(\mathbf{A}-\mathbf{I}) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \neq \mathbf{0} \end{aligned}$$
$\implies x^2(x-1)$ is not a minimal polynomial</td></tr>
<tr><td>$a = 1, b = 2$</td><td>$m(x) = x(x-1)^2$
$$\begin{aligned} \implies m(\mathbf{A}) &= \mathbf{A}(\mathbf{A}-\mathbf{I})^2 \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0} \end{aligned}$$
$\implies x^2(x-1)$ is not a minimal polynomial</td></tr>
</table>

$a = 2, b = 2$	$m(x) = x^2 (x-1)^2$ $\implies m(\mathbf{A}) = \mathbf{A}^2 (\mathbf{A} - \mathbf{I})^2$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ $\implies x^2 (x-1)^2$ is a minimal polynomial
Conclusion	For the given matrix $\mathbf{A}$, $x^2 (x-1)^2$ is the characteristic polynomial as well as minimal polynomial.

TABLE 2: Proving that the given characteristic polynomial is also minimal polynomial