Assignment 18

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Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment18

1 Problem

(UGC June 2018, 76)
Let $\mathbf{V}$ be an inner product space and $\mathcal{S}$ be a subset of $\mathbf{V}$. Let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in $\mathbf{V}$ with respect to the topology induced by the metric given by the inner product. Which of the following statements are true?

1) $\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$
2) $\overline{\mathcal{S}} = (\mathcal{S}^{\perp})^{\perp}$
3) $\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$
4) $\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$

2 Outline

Orthogonal Complement	Let $\mathcal{S}$ be a subset of an inner product space $\mathbf{V}$. The space of all vectors orthogonal to $\mathcal{S}$ is called the **orthogonal complement** of $\mathcal{S}$: $$\mathcal{S}^{\perp} = \{\mathbf{x} \in \mathbf{V} : \ < \mathbf{x}, \mathbf{y} >= 0, \quad \forall \mathbf{y} \in \mathcal{S}\}$$
Closure of subset	closure of a set $\mathcal{S}$ is the set of all limits of points from $\mathcal{S}$ Let $\mathcal{S}$ be a subset of an inner product space $\mathbf{V}$. Then closure of $\mathcal{S}$ satisfies, $$\overline{\mathcal{S}} = \{\mathbf{y} \in \mathbf{V} : \ \textit{there exist } \mathbf{x_n} \in \mathcal{S} \textit{ such that } \mathbf{x_n} \to \mathbf{y}\}$$
Projection Theorem	Let $\mathcal{S}$ be a closed subspace of a finite dimensional vector space $\mathbf{V}$, then, Every $\mathbf{x} \in \mathcal{S}$ can be expressed as, $$\mathbf{x} = \mathbf{u} + \mathbf{v}, \text{ where,}$$ $$\mathbf{u} \in \mathcal{S}, \quad \mathbf{v} \in \mathcal{S}^{\perp}$$
Theorem	If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbf{V}$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $$\mathcal{S}_2^{\perp} \subseteq \mathcal{S}_1^{\perp} .$$

TABLE 1: Definitions and results used

3 Solution

Given	Let $\mathcal{S}$ be any set, then $\mathcal{S}^{\perp}$ is the set of all vectors that are perpendicular to all elements of $\mathcal{S}$ We will check if $\mathcal{S}^{\perp}$ is a subspace (1) Closed on Addition Let $\mathbf{u}, \mathbf{v} \in \mathcal{S}^{\perp}$, then, for $\mathbf{x} \in \mathbf{V}$, $< \mathbf{x}, \mathbf{u} + \mathbf{v} >=< \mathbf{x}, \mathbf{u} > + < \mathbf{x}, \mathbf{v} >= 0$ $\implies \mathbf{u} + \mathbf{v} \in \mathcal{S}^{\perp}$ (2) Closed on Multiplication Let $\mathbf{u} \in \mathcal{S}^{\perp}$, then, for $\mathbf{x} \in \mathbf{V}$ and scalar $\alpha \in \mathbb{F}$, $< \mathbf{x}, \alpha\mathbf{u} >= \alpha^* < \mathbf{x}, \mathbf{u} >= 0$ $\implies \alpha\mathbf{u} \in \mathcal{S}^{\perp}$ Therefore, $\mathcal{S}^{\perp}$ is a subspace Therefore, $(\mathcal{S}^{\perp})^{\perp}$ is also a subspace
	Checking the options
$\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$	We have, $\mathcal{S}^{\perp} = \{x \in \mathbf{V} : \ < x, y >= 0, \quad \forall y \in \mathcal{S}\}$ $\implies \quad (\mathcal{S}^{\perp})^{\perp} = \{x \in \mathbf{V} : \ < x, y >= 0, \quad \forall y \in \mathcal{S}\}$ Let $\mathbf{s} \in \mathcal{S}$, then $< \mathbf{s}, \mathbf{v} >= 0, \quad \forall \mathbf{v} \in \mathcal{S}^{\perp}$ $\implies \mathbf{s} \in (\mathcal{S}^{\perp})^{\perp}$ Therefore, $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp} \qquad \ldots(1)$ We have proved that $(\mathcal{S}^{\perp})^{\perp}$ is a subspace But, $\mathcal{S}$ is a subset of $\mathbf{V}$ and is not necessarily a subspace. Therefore, this option is **false**.
$\overline{\mathcal{S}} = (\mathcal{S}^{\perp})^{\perp}$	Similarly, $\overline{\mathcal{S}}$ is a subset of $\mathbf{V}$ and is not necessarily a subspace. Therefore, this option is **false**.
$\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$	Let $\mathbf{v}$ is a limit of some $\mathbf{v_i}$ such that $\mathbf{v_i} \in span\,(\mathcal{S})$ $\implies \mathbf{v} \in \overline{span\,(\mathcal{S})}$ Now,

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Since, $\mathbf{v_i} \in span\,(\mathcal{S})$,
$\implies \mathbf{v_i} = \sum \beta_j \mathbf{s_j}, \quad \mathbf{s_j} \in \mathcal{S}$

Let $\mathbf{w} \in \mathcal{S}^{\perp}$,
$\implies <\mathbf{w}, \mathbf{s_j}>= 0$
Now,
$<\mathbf{w}, \mathbf{v_i}>= \sum \beta_j <\mathbf{w}, \mathbf{s_j}> = 0$
Therefore,
$\mathbf{w} \perp \mathbf{v_i}$, hence,
$\mathbf{w} \perp \mathbf{v}$
$\implies \mathbf{v} \in (\mathcal{S}^{\perp})^{\perp}$
$\implies \overline{span\,(\mathcal{S})} \subseteq (\mathcal{S}^{\perp})^{\perp} \qquad \ldots (2)$

Therefore, this option is **false**.

However, if we assume that $\mathbf{V}$ is a finite dimensional space, which implies, $\mathbf{V}$ is a hilbert space, then we have,

for $\mathbf{x} \in (\mathcal{S}^{\perp})^{\perp}$,
$\mathbf{x} = \mathbf{u} + \mathbf{v}, \quad \mathbf{u} \in \overline{span(\mathcal{S})}, \mathbf{v} \perp \overline{span(\mathcal{S})}$
Now,
$<\mathbf{x}, \mathbf{u}>= 0$
$\implies <\mathbf{u} + \mathbf{v}, \mathbf{v}>= 0$
$\implies <\mathbf{u}, \mathbf{v}> + <\mathbf{v}, \mathbf{v}>= 0$
$\implies \|\mathbf{v}\| = 0$
$\implies \mathbf{v} = 0$
$\implies \mathbf{x} = \mathbf{u} \in \overline{span(\mathcal{S})}$
$\implies (\mathcal{S}^{\perp})^{\perp} \subseteq \overline{span(\mathcal{S})} \qquad \ldots (3)$

From (2) and (3),
$\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$ if $\mathbf{V}$ is a hilbert space.
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<td>$\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$</td>
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From (1), we have,
$\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$
$\implies \mathcal{S}^{\perp} \subseteq \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} \qquad \ldots (4)$

We know that,
$\mathcal{S}_2^{\perp} \subseteq \mathcal{S}_1^{\perp}$
Therefore,
$\left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} \subseteq \mathcal{S}^{\perp} \qquad \ldots (5)$

From (4) and (5), we have,
$\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$

Therefore, this option is **True**.
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Example:	Let $\mathbf{V} = \mathbb{R}^2$ We want a subset $\mathcal{S}$ of $\mathbf{V}$ which is not a subspace. Let $\mathcal{S} = \left\{ \begin{pmatrix} x \\ 3x+1 \end{pmatrix} \right\}, x \in \mathbb{R},$ Then, $\mathcal{S}^{\perp} = \left\{ \begin{pmatrix} x \\ -\frac{1}{3}x + c \end{pmatrix} \right\} \qquad \ldots(1)$ $\implies (\mathcal{S}^{\perp})^{\perp} = \left\{ \begin{pmatrix} x \\ 3x + c \end{pmatrix} \right\}$ Therefore, $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$ $\implies \boxed{\mathcal{S} \neq (\mathcal{S}^{\perp})^{\perp}}$ Similarly, $\implies \boxed{\overline{\mathcal{S}} \neq (\mathcal{S}^{\perp})^{\perp}}$ Also, $\left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} = \left\{ \begin{pmatrix} x \\ -\frac{1}{3}x + c \end{pmatrix} \right\} \qquad \ldots(2)$ From (1) and (2), we get, $\boxed{\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}}$

TABLE 2: Solution

4 Conclusion

$\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$	**false**.
$\overline{\mathcal{S}} = (\mathcal{S}^{\perp})^{\perp}$	**false**.
$\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$	**false**
$\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$	**True**.

TABLE 2: Conclusion