Assignment 13

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Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment13

1 Problem

Suppose that $\mathbf{A}$ is a 2×2 matrix with real entries which is symmetric ($\mathbf{A}^t = \mathbf{A}$). Prove that $\mathbf{A}$ is similar over $\mathbb{R}$ to a diagonal matrix.

2 Solution

Given	$\mathbf{A}$ is a 2×2 matrix with real entries and $\mathbf{A}$ is symmetric ($\mathbf{A}^t = \mathbf{A}$)
To Prove	$\mathbf{A}$ is similar to diagonal matrix over $\mathbb{R}$
Theory	$\mathbf{A}$ is similar to diagonal matrix $\mathbf{\Lambda}$ if $\exists$ an invertible matrix $\mathbf{P}$ such that: $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$
Proof	Let $\mathbf{A} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, a, b, c \in \mathbb{R}$ Characteristic polynomial: $p(t) = \left\lvert \mathbf{A} - \lambda \mathbf{I} \right\rvert$ $p\left(t\right) = \begin{vmatrix} a - t & c \\ c & b - t \end{vmatrix}$ $\implies p\left(t\right) = t^2 - (a + b)t + ab - c^2 = 0$ Roots of p(t) are eigenvalues of $\mathbf{A}$ Discriminant of $p\left(t\right)$ is given by $(a + b)^2 - 4(ab - c^2) = a^2 + b^2 - 2ab + c^2$ $= (a - b)^2 + 4c^2 > 0$ We observe that the above equation has positive discriminant, hence λ has real values

	Eigen vectors are obtained by: $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = 0$ Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the eigen vectors corresponding to eigen values λ_1 and λ_2 $\implies \mathbf{A}\mathbf{v_1} = \lambda_1\mathbf{v_1}$ and $\mathbf{A}\mathbf{v_2} = \lambda_2\mathbf{v_2}$ Let linear combination of the two eigen vectors be, $c_1\mathbf{v_1} + c_2\mathbf{v_2} = \mathbf{0}$ Multiplying both sides by λ_1, we have, $\implies c_1\lambda_1\mathbf{v_1} + c_2\lambda_1\mathbf{v_2} = 0 \qquad \ldots(1)$ Consider, $\mathbf{A}.\mathbf{0} = \mathbf{0}$ $\implies \mathbf{A}\left(c_1\mathbf{v_1} + c_2\mathbf{v_2}\right) = \mathbf{0}$ $\implies c_1\left(\mathbf{A}\mathbf{v_1}\right) + c_2\left(\mathbf{A}\mathbf{v_2}\right) = \mathbf{0}$ $\implies c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} = 0 \qquad \ldots(2)$ Subtracting equation (1) and (2), we have, $c_2\left(\lambda_1 - \lambda_2\right)\mathbf{v_2} = \mathbf{0}$ Since, λ_1 and λ_2 are real and distinct, $c_2 = 0$ Similarly, $c_1 = 0$ Therefore, eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. Let $\mathbf{P} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2}\end{pmatrix}$ $\implies \mathbf{AP} = \begin{pmatrix}\lambda_1\mathbf{v_1} & \lambda_2\mathbf{v_2}\end{pmatrix}$ $\implies \mathbf{AP} = \mathbf{P\Lambda}$, where $\mathbf{\Lambda} = \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}$ $\implies \mathbf{A} = \mathbf{P\Lambda P}^{-1}$ Therefore, $\mathbf{A}$ is similar to diagonal matrix $\mathbf{\Lambda}$ Hence, Proved.

TABLE 1: Proving that eigen vectors are linearly independent for real eigen values and symmetric matrix is similar to diagonal matrix