Assignment 14

Rubeena Aafreen

Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment14

1 Problem

Let $\mathbf{A}$ be the 4×4 real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \tag{1.0.1}$$

Show that the characteristic polynomial for $\mathbf{A}$ is $x^2(x-1)^2$ and that it is also the minimal polynomial

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left|x\mathbf{I} - \mathbf{A}\right|$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is he smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Algebraic Multiplicity (A_M)	No. of times an eigen value appears in a characteristic equation.

TABLE 1: Definitions

3 Solution

Characteristic polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$ $= \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix}$ $= ((x-1)(x+1)+1)((x-2)x+1)$ $= x^2\left(x^2-2x+1\right)$ $= x^2(x-1)^2$
A_M	For $\lambda = 0, \quad A_M = 2$ For $\lambda = 1, \quad A_M = 2$
Minimal Polynomial	$p(x) = x^a(x-1)^b, \quad a \le 2, b \le 2$
$a = 1, b = 1$	$m(x) = x(x-1)$ $\implies m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $= \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{pmatrix} \neq \mathbf{0}$ $\implies x(x-1)$ is not a minimal polynomial
$a = 2, b = 1$	$m(x) = x^2(x-1)$ $\implies m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \neq \mathbf{0}$ $\implies x^2(x-1)$ is not a minimal polynomial

$a = 1, b = 2$	$m(x) = x(x-1)^2$ $\implies m(\mathbf{A}) = \mathbf{A}(\mathbf{A}-\mathbf{I})^2$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$ $\implies x^2(x-1)$ is not a minimal polynomial
$a = 2, b = 2$	$m(x) = x^2(x-1)^2$ $\implies m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A}-\mathbf{I})^2$ $= p(\mathbf{A})$ $= \mathbf{0}$ (Cayley-Hamilton Theorem) $\implies x^2(x-1)^2$ is a minimal polynomial
Conclusion	For the given matrix $\mathbf{A}$, $x^2(x-1)^2$ is the characteristic polynomial as well as minimal polynomial.

TABLE 2: Checking for minimal polynomial