

Assignment 14

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Download the latex-tikz codes from

<https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment14>

1 PROBLEM

Let \mathbf{A} be the 4×4 real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \quad (1.0.1)$$

Show that the characteristic polynomial for \mathbf{A} is $x^2(x-1)^2$ and that it is also the minimal polynomial

2 DEFINITIONS

| | |
|----------------------------------|---|
| Characteristic Polynomial | For an $n \times n$ matrix \mathbf{A} , characteristic polynomial is defined by, $p(x) = x\mathbf{I} - \mathbf{A} $ |
| Minimal Polynomial | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = 0$ |
| Algebraic Multiplicity (A_M) | No. of times an eigen value appears in a characteristic equation. |

TABLE 1: Definitions

3 SOLUTION

| | |
|---------------------------|---|
| Characteristic polynomial | $ \begin{aligned} p(x) &= x\mathbf{I} - \mathbf{A} \\ &= \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix} \\ &= ((x-1)(x+1) + 1)((x-2)x + 1) \\ &= x^2(x^2 - 2x + 1) \\ &= x^2(x-1)^2 \end{aligned} $ |
| A_M | <p>For $\lambda = 0$, $A_M = 2$ For $\lambda = 1$, $A_M = 2$</p> |
| Minimal Polynomial | $p(x) = x^a(x-1)^b, \quad a \leq 2, b \leq 2$ |
| $a = 1, b = 1$ | $m(x) = x(x-1)$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})$ $ = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} $ $ = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{pmatrix} \neq \mathbf{0} $ $\Rightarrow x(x-1) \text{ is not a minimal polynomial}$ |
| $a = 2, b = 1$ | $m(x) = x^2(x-1)$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})$ $ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} $ $ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \neq \mathbf{0} $ $\Rightarrow x^2(x-1) \text{ is not a minimal polynomial}$ |

| | |
|-------------------|---|
| $a = 1, b = 2$ | $m(x) = x(x-1)^2$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})^2$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$ $\Rightarrow x^2(x-1) \text{ is not a minimal polynomial}$ |
| $a = 2, b = 2$ | $m(x) = x^2(x-1)^2$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})^2$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ $\Rightarrow x^2(x-1)^2 \text{ is a minimal polynomial}$ |
| <p>Conclusion</p> | <p>For the given matrix \mathbf{A}, $x^2(x-1)^2$ is the characteristic polynomial as well as minimal polynomial.</p> |

TABLE 2: Proving that the given characteristic polynomial is also minimal polynomial