Assignment 17

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Download the latex-tikz codes from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment17

1 Problem

Let $\mathbf{T}$ be the diagonalizable linear operator on $\mathbb{R}^3$ which we discussed in example 3 of section 6.2. Use the Lagrange polynomials to write the representing matrix $\mathbf{A}$ in the form

$$\mathbf{A} = \mathbf{E_1} + 2\mathbf{E_2}, \quad \mathbf{E_1} + \mathbf{E_2} = \mathbf{I}, \mathbf{E_1}\mathbf{E_2} = \mathbf{0} \tag{1.0.1}$$

2 Outline

Diagonalizable Operator	For a linear operator $\mathbf{T}\colon \mathbf{V} \longrightarrow \mathbf{V}$, $\mathbf{T}$ is a diagonalizable operator if $\exists$ some basis for $\mathbf{V}$ such that the matrix representing $\mathbf{T}$ is a diagonal matrix i.e. $$\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$$ $\implies$ $\mathbf{A}$ is a diagonalizable matrix
Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial
Lagrange Polynomials	For a set of scalars $c_0, c_1, \ldots, c_n \in \mathbb{F}$, Lagrange Polynomial is defined as: $$p_j = \prod_{i \neq j} \frac{(x - c_i)}{\left(c_j - c_i\right)}$$
Theorem	If $\mathbf{T}$ is a diagonalizable linear operator on a finite dimensional space $\mathbf{V}$, and if $c_1, c_2, \ldots, c_k$ are distinct characterictic values of $\mathbf{T}$, then there exist

	linear operators $\mathbf{E_1}, \mathbf{E_2}, \ldots, \mathbf{E_k}$ such that: (1) $\mathbf{T} = c_1\mathbf{E_1} + \cdots + c_k\mathbf{E_k}$ (2) $\mathbf{E_1} + \cdots + \mathbf{E_k} = \mathbf{I}$ (3) $\mathbf{E_i}\mathbf{E_j} = \mathbf{0}, \quad i \neq j$ (4) $\mathbf{E_i} = \mathbf{E_i}^2, \quad (\mathbf{E_i}$ is a projection$)$ (5) $\alpha = \mathbf{E_i}\alpha, \forall \alpha \in \mathbf{V}$
Relation between Lagrange Polynomials and Projection	We have: $\mathbf{T} = c_1\mathbf{E_1} + \cdots + c_k\mathbf{E_k}$ If g is any polynomial over field $\mathbb{F}$, $g(\mathbf{T}) = g(c_1)\mathbf{E_1} + \cdots + g(c_k)\mathbf{E_k} \quad \ldots(1)$ Now, $p_j = \prod_{i \neq j} \frac{(x - c_i)}{\left(c_j - c_i\right)}$ $\implies p_j(c_i) = \delta_{ij}$ (Kronecker Delta) $\quad \ldots(2)$ From (1) and (2), $\implies p_j(\mathbf{T}) = \sum_{i=1}^{k} \delta_{ij}\mathbf{E}_i = \mathbf{E_j}$ $\implies \boxed{p_j(\mathbf{T}) = \mathbf{E_j}}$ $\implies$ Projections $\mathbf{E_j}$ are polynomials in $\mathbf{T}$

TABLE 1: Definitions and results used

3 Solution

Given	Matrix of $\mathbf{T}$ in the standard basis of $\mathbb{R}^3$: $\mathbf{A} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$
Characteristic polynomial	$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$ $= \begin{vmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{vmatrix}$ $= x^3 - 5x^2 + 8x - 4$ $= (x-1)(x-2)^2$ $\implies \lambda = 1, 2$

Minimal Polynomial	$p(x) = (x-1)(x-2)^b, \quad b \leq 2$ $(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I}) = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} = \mathbf{0}$ Therefore, $(x-1)(x-2)$ is the minimal polynomial.
Lagrange Polynomial	$p_j = \prod_{i \neq j} \frac{(x-c_i)}{(c_j - c_i)}$ For characteristic values $c_1 = 1, \quad c_2 = 2,$ $\implies p_1 = \frac{(x-1)}{2-1}, \qquad p_2 = \frac{(x-2)}{1-2}$ $\implies p_1 = (x-1)$, and $p_2 = (2-x)$
Projection Maps	We know that, $\mathbf{E_j} = p_j(\mathbf{T})$ $\implies \mathbf{E_1} = \mathbf{A} - \mathbf{I}$ and $\mathbf{E_2} = 2\mathbf{I} - \mathbf{A}$ $\implies \mathbf{E_1} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$, and $\mathbf{E_2} = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$
Verification	We have, $\mathbf{E_1} = \mathbf{A} - \mathbf{I}$ $\implies \mathbf{A} - \mathbf{E_1} = \mathbf{I} \qquad \ldots(1)$ $\mathbf{E_2} = 2\mathbf{I} - \mathbf{A}$ From (1), $\implies \mathbf{E_2} = 2(\mathbf{A} - \mathbf{E_1}) - \mathbf{A}$ $\implies \boxed{\mathbf{A} = 2\mathbf{E_1} + \mathbf{E_2}}$ Also, $\mathbf{E_1} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}, \qquad \mathbf{E_2} = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$

	$$\implies \mathbf{E_1} + \mathbf{E_2} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} + \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $$\implies \boxed{\mathbf{E_1} + \mathbf{E_2} = \mathbf{I}}$$ $$\mathbf{E_1}\mathbf{E_2} = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ $$\implies \boxed{\mathbf{E_1}\mathbf{E_2} = \mathbf{0}}$$

TABLE 2: Using Lagrange Polynomials to represent **A**