Assignment 10

Rubeena Aafreen

***Abstract*—This assignment deals with annihilator of vector space.**

Download all solutions from

https://github.com/rubeenaafreen20/EE5609/tree/master/Assignment10

1 Problem

Let $\mathbf{W}$ be the subspace of $\mathbb{R}^5$ which is spanned by the vectors

$$\begin{aligned}\alpha_1 &= \epsilon_1 + 2\epsilon_2 + \epsilon_3,\\ \alpha_2 &= \epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + \epsilon_5,\\ \alpha_3 &= \epsilon_1 + 4\epsilon_2 + 6\epsilon_3 + 4\epsilon_4 + \epsilon_5\end{aligned} \tag{1.0.1}$$

Find a basis for $\mathbf{W^0}$

2 Annihilator

2.1 Definition

If $\mathbf{V}$ is a vector space over the field $\mathbb{F}$ and $\mathbf{W}$ is a subset of $\mathbf{V}$, the annihilator of $\mathbf{W}$ is the set $\mathbf{W^0}$ of linear functionals $\mathbf{f}$ on $\mathbf{V}$ such that $\mathbf{f}(\alpha) = 0$ for every α in $\mathbf{W}$.

2.2 Properties of Annihilator

If f is a linear functional on R^n:

$$f(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} c_j x_j \tag{2.2.1}$$

Then f is in W^0 if and only if

$$\forall \alpha \in W\colon f(\alpha) = 0 \iff f = 0 \tag{2.2.2}$$

3 Solution

equation (2.2.1) can be expressed as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T\mathbf{c} \tag{3.0.1}$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\\ c_5\end{pmatrix} \tag{3.0.2}$$

Given three vectors

$$\alpha_1 = \begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}0\\1\\3\\3\\1\end{pmatrix}, \alpha_3 = \begin{pmatrix}1\\4\\6\\4\\1\end{pmatrix} \tag{3.0.3}$$

Let matrix A with column vectors $\alpha_1, \alpha_2, \alpha_3$:

$$\mathbf{A} = \begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 4\\ 1 & 3 & 6\\ 0 & 3 & 4\\ 0 & 1 & 1\end{pmatrix} \tag{3.0.4}$$

Given that $\mathbf{f}$ is a linear functional on $\mathbb{R}^5$, then $\mathbf{f}$ is in $\mathbf{W^0}$ if and only if,

$$f(\alpha_i) = 0, i = 1, 2, 3 \tag{3.0.5}$$

$$\implies \mathbf{A}^T\mathbf{c} = \mathbf{0} \tag{3.0.6}$$

Converting the equation (3.0.6) into system of equations, we have,

$$\begin{pmatrix}1 & 2 & 1 & 0 & 0\\ 0 & 1 & 3 & 3 & 1\\ 1 & 4 & 6 & 4 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{pmatrix} = 0 \tag{3.0.7}$$

Converting equation (3.0.7) into row reduced echelon form,

$$\begin{pmatrix}1 & 2 & 1 & 0 & 0\\ 0 & 1 & 3 & 3 & 1\\ 1 & 4 & 6 & 4 & 1\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}1 & 0 & 0 & 4 & 3\\ 0 & 1 & 0 & -3 & -2\\ 0 & 0 & 1 & 2 & 1\end{pmatrix} \tag{3.0.8}$$

From equation (3.0.8), we have,

$$c_1 = -(4c_4 + 3c_5) \tag{3.0.9}$$

$$c_2 = (3c_4 + 2c_5) \tag{3.0.10}$$

$$c_3 = -(2c_4 + c_5) \tag{3.0.11}$$

Therefore, $\mathbf{c}$ can be expressed as,

$$\mathbf{c} = \begin{pmatrix} -4c_4 - 3c_5 \\ 3c_4 + 2c_5 \\ -2c_4 - c_5 \\ c_4 \\ c_5 \end{pmatrix} \tag{3.0.12}$$

$$\implies \mathbf{c} = \begin{pmatrix} -4 \\ 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} c_4 + \begin{pmatrix} -3 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} c_5 \tag{3.0.13}$$

Therefore,

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \tag{3.0.14}$$

is the general element of $\mathbf{W^0}$. Dimension of $\mathbf{W^0}$ is 2 and a basis $\{\mathbf{f_1}, \mathbf{f_2}\}$ can be obtained by putting $c_4 = 1, c_5 = 0$ and $c_4 = 0, c_5 = 1$ in equation (3.0.13)

$$\mathbf{f_1}(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} -4 \\ 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} \tag{3.0.15}$$

$$\mathbf{f_2}(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} -3 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{3.0.16}$$