Fundamentals of MIMO Wireless Communication

Tutorial

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Entropy:-

$$x_1, x_2, \ldots, x_N$$

$$\boxed{H(X) = \sum_{i=1}^{N} p(x_i) \log_2 \frac{1}{p(x_i)}}$$

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x) \quad \text{bits.}$$

$$= -\sum_{x \in \mathcal{X}} p(x) \ln p(x) \quad \text{nats.}$$

$$0 \le p(x) \le 1 \rightarrow \ln p(x) \le 0$$

$$\rightarrow \boxed{H(X) \ge 0}$$

Joint Entropy: $X, Y \to$ pair of r.v.

$$H(X,Y) = -\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x,y)\log_2 p(x,y)$$

Conditional Entropy:

$$H(Y|X) = -\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} \underbrace{p(x)\, p(y|x)} \log_2 p(y|x)$$

$$= -\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x,y)\log_2 p(y|x)$$

$$= -E_{p(x,y)} \log_2 (y|x)$$

Chain Rule :-

$$H(X,Y) = H(X) + H(Y|X)$$

LHS $\quad H(X,Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 p(x,y)$

$$= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 \left[p(x)\, p(y|x) \right]$$

$$= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 p(x) - \sum \sum p(x,y) \log_2 p(y|x)$$

$$= H(X) + H(Y|X)$$

$$= H(Y) + H(X|Y)$$

Relative Entropy: Kullback Leibler Distance.

→ measure of distance b/w two distribution.

$$D(p(x) \| q(x)) \Rightarrow D(p\|q)$$

$$= \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

$$= E_{p(x)} \log \frac{p(x)}{q(x)}$$

$$\boxed{D(p\|q) \neq D(q\|p).}$$

Mutual Information: Relative entropy b/w joint distribution & product distribution.

2 r.v. X & Y $\underbrace{p(x,y)}_{\text{joint}}$, $p(x)$, $p(y)$.

$$I(X,Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$

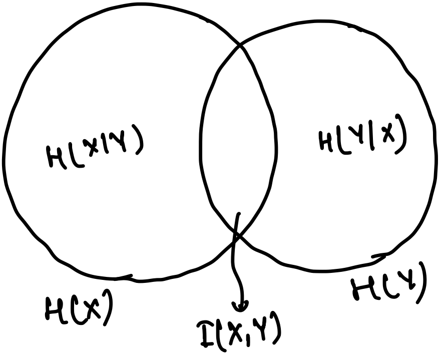
$$= \sum \sum p(x,y) \log_2 \frac{\cancel{p(x)} \cdot p(y/x)}{\cancel{p(x)}\, p(y)}$$

$$= \sum \sum p(x,y) \left[\log_2 p(y/x) - \log_2 p(y) \right]$$

$$\boxed{= H(Y) - H(Y/X)}$$

$$= H(X) - H(X|Y)$$

$$= H(X) + H(Y) - H(X,Y)$$



$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$

$$= H(Y) - H(Y|X)$$

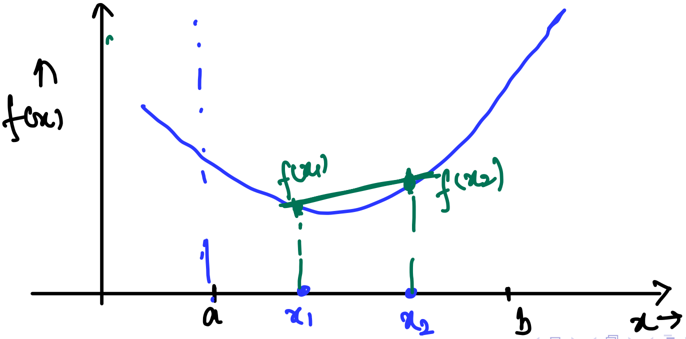
$$= H(X) - H(X|Y)$$

Jensen's inequality:

$f(x)$ is convex over interval (a,b),
if for each $(x_1, x_2) \in (a,b)$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$0 \leq \lambda \leq 1$$



If f is convex and X is a r.v.

$$\boxed{E[f(X)] \geq f[EX]}$$ Jensens inequality

$X = EX \rightarrow X$ is const

- $p(x)$, $q(x)$ for $x \in \mathcal{X}$

$D(p||q) \geq 0$, equality will hold iff $p(x) = q(x)$ $\forall x$.

Let, p(x,y) be given as,

X \ Y	0	1
0	$\frac{1}{3}$	0
1	$\frac{1}{3}$	$\frac{1}{3}$

What is the value of H(X), H(Y)?

a. H(X)=H(Y)=0.918 bits

b. H(X)=H(Y)=1.585 bits

c. H(X)=0.918 bits, H(Y)=1.585 bits.

d. None of the above

$$H(X) = -\sum p(x) \log_2 p(x).$$

$$= -\left[p(x=0)\log_2 p(x=0)\right]$$

$$- \left[p(x=1)\log_2 (p(x=1))\right]$$

$p(x=0) = 1/3$, $p(x=1) = 2/3$

$p(y=0) = 2/3$, $p(y=1) = 1/3$.

$$\Rightarrow -\frac{1}{3}\log_2 1/3 - \frac{2}{3}\log_2 2/3$$

$$= 0.918$$

$$H(Y) = -p(y=0)\log_2 p(y=0)$$

$$- p(y=1)\log_2 p(y=1)$$

$$= 0.918$$

The value of I(X;Y) in the previous problem is

a. I(X;Y) = 0.918 bits.

b. I(X;Y) = 0.421 bits.

c. I(X;Y) = 0.251 bits.

d. None of the above

$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X)$$
$$= H(X) + H(Y) - H(X,Y).$$

$$H(X|Y) = p(y=0)H(X|y=0) + p(y=1)\cdot H(X|y=1)$$
$$= \frac{2}{3} H(X|y=0) + \frac{1}{3} H(X|y=1)$$

$$H(X|y=0) = -\Pr(x=0|y=0)\log_2 p(x=0|y=0)$$
$$-\Pr(x=1|y=0)\log_2 p(x=1|y=0)$$

$$\Pr\left(x(0)/y(0)\right) = \frac{\Pr(x=0, y=0)}{P(y=0)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$$\Pr\left(x=1/y=0\right) = \frac{\Pr(x=1, y=0)}{P(y=0)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$$H(x|y=0) = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = 1.$$

$$H(x|y=1) = 0.$$

$$H(x|y) = \frac{2}{3} H(x|y=0) + \cancel{\frac{1}{3} H(x|y=1)} \rightarrow 0$$

$$= \frac{2}{3} \times 1 = \frac{2}{3} = 0.667$$

$$I(X,Y) = H(X) - H(X|Y)$$

$$= 0.918 - 0.667 = 0.251 \text{ bits}$$

X is a r.v.

$F(X) = \text{Prob}(X \leq x)$.

$f(x) = F'(x)$.

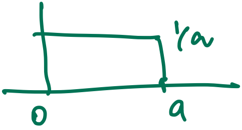
$\int_{-\infty}^{\infty} f(x) = 1 \Rightarrow f(x)$ is pdf of x.

Support set: Set of values where $f(x) > 0$.

Differential Entropy:

$$\boxed{h(x) = -\int_S f(x)\log f(x)\,dx}$$

h(t)



$$h(x) = -\int_0^a \frac{1}{a}\log \frac{1}{a}\,dx$$

$$= \log a$$

$a < 1. \Rightarrow h(x) \leq 0$

Differential entropy can be -ve.

Normal Distribution:

$$X \sim \phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

$$h(\phi) = -\int \phi \ln \phi$$

$$= -\int \phi(x) \cdot \left[\frac{-x^2}{2\sigma^2} - \frac{1}{2} \ln (2\pi\sigma^2) \right] dx$$

$$h(\phi) = \int \frac{x^2 \phi(x)}{2\sigma^2} dx + \frac{1}{2} \int \phi(x) \ln(2\pi\sigma^2) dx.$$

$$= \frac{1}{2\sigma^2} \cdot E[X^2] + \frac{1}{2} \ln (2\pi\sigma^2)$$

σ^2

$$= \frac{1}{2} \ln e + \frac{1}{2} \ln(2\pi\sigma^2)$$

$$= \frac{1}{2} \ln(2\pi e \sigma^2) \quad \text{nats.}$$

$$= \frac{1}{2} \log_2 (2\pi e \sigma^2) \quad \text{bits.}$$

Joint differential entropy :-

Set $X_1, X_2, \ldots X_n$ with pdf $f(x_1, x_2 \ldots x_n)$

$$h(X_1, X_2, \ldots X_n) = -\int f(x_1, \ldots x_n) \log_2 f(x_1, x_2 \ldots x_n)\, dx_1 \ldots dx_n.$$

Joint Conditional Differential Entropy:

$$h(X|Y) = -\int f(x,y) \log f(x|y)\, dx\, dy.$$

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$$\boxed{h(X|Y) = h(X,Y) - H(Y)}$$

Entropy of multivariate normal distribution:-

$(X_1, X_2, \dots X_n)$ mean $\underline{\mu}$ covariance matrix $\underline{\underline{K}}$

$$\mathcal{N}(\underline{\mu}, \underline{\underline{K}})$$

$$f(x) = \frac{1}{\underbrace{(\sqrt{2\pi})^n}|K|^{1/2}} e^{-\frac{1}{2}(\underline{X}-\underline{\mu})^T K^{-1} (\underline{X}-\underline{\mu})}$$

$$h(X) = -\int f(x) \ln f(x)$$

$$= -\int f(x) \left[-\underbrace{\ln \left((2\pi)^n |K|\right)^{1/2}} - \frac{1}{2} \underset{1\times n}{(\underline{X}-\underline{\mu})^T} \underset{n\times n}{K^{-1}} \underset{n\times 1}{(\underline{X}-\underline{\mu})} \right] dx.$$

$$= \frac{1}{2}\ln\left((2\pi)^n |K|\right) + \frac{1}{2}\int f(x)(x-\mu)^T K^{-1}(x-\mu)\, dx$$

$$= \left(\qquad \right) + \frac{1}{2} E \sum_{i,j} (x_i - \mu_i) K_{ij}^{-1} (x_j - \mu_j)$$

$$= \left(\qquad \right) + \frac{1}{2} \sum_{i,j} E\left[\underbrace{(x_i - \mu_i)(x_j - \mu_j)}_{K_{ij}}\right](K_{ij})^{-1}$$

$$= \frac{1}{2}\ln\left((2\pi)^n |K|\right) + \frac{1}{2}\sum_{i,j} E\left[K_{ij} K_{ij}^{-1}\right]$$

$$= \frac{1}{2}\ln\left((2\pi)^n |K|\right) + \frac{1}{2} n$$

$$= \frac{1}{2} \ln \left((2\pi e)^n |K| \right) \quad \text{nats}.$$

$$\boxed{h(x) = \frac{1}{2} \log_2 \left((2\pi e)^n |K| \right) \quad \text{bits}.}$$

for multivariate complex gaussian.

$$\boxed{h(x) = \log_2 \left((\pi e)^n |K| \right) \quad \text{bits}.}$$

Relative entropy:

$$D(f\|g) = \int f \log f/g.$$

Mutual Information:

$$I(X;Y) = \int f(x,y) \log_2 \frac{f(x,y)}{f(x) f(y)} \, dx\, dy.$$

$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X).$$

$$D(f\|g) \geq 0 \Rightarrow I(X;Y) \geq 0$$

* $X \in \mathbb{R}^n$ zero mean & Covariance matrix. $K = E[XX^T]$

$$\boxed{h(X) \leq \frac{1}{2}\left(\log_2 (2\pi e)^n |K|\right)}$$

→ entropy is max. iff $X \sim \mathcal{N}(0, K)$.

* for complex distributions

$$\boxed{h(f) \leq \log_2\left((\pi e)^n |K|\right)}$$

ZMCSCG

Asymptotic Equipartition Property :-

iid r.v

$X_1, X_2, \ldots X_n$ pdf $\sim p(x)$.

then,

$$-\frac{1}{n}\log p(x_1, x_2 \ldots x_n) \longrightarrow H(X)$$ if n is large.

$$\Rightarrow \left(-\frac{1}{n}\sum_{i=1}^{n}\right)\log_2 p(x_i) \longrightarrow -E\log p(x)$$

$$-E\log_2 p(x) = -\sum_{i=1}^{N} p(x)\log_2 p(x) = H(X)$$

$$-\frac{1}{n}\log p(x_1, x_2 \ldots x_n) \longrightarrow H(X)$$

$$\Rightarrow \log_2 p(x_1, x_2 \ldots x_n) = -nH(X)$$

$$\Rightarrow \boxed{p(x_1, x_2 \ldots x_n) = 2^{-nH(X)}}$$

0 1 0 0 1 0 0 1 0 ...

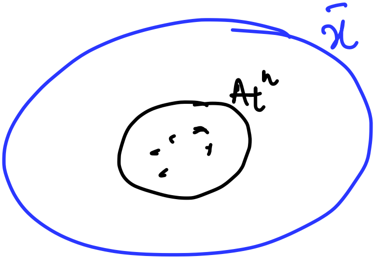
$p(1) = p_1$, $p(0) = p_0$.

No. of 1s $\to np_1$. no. of 0s $\to np_0$.

$$H(X) = -p_1 \log p_1 - p_0 \log p_0$$

$$\Rightarrow nH(X) = -\log \underbrace{p_1^{np_1} \cdot p_0^{np_0}} \Rightarrow p_1^{np_1} p_0^{np_0} = 2^{-nH(X)}.$$

Typical Sequences: A_t^n



All seq. in typical set are equiprobable with prob. $2^{-nH(x)}$

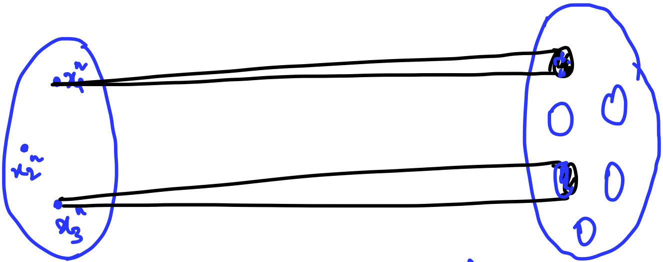
No. of sequences: $2^{nH(X)}$

$$|\hat{A}_\epsilon| \leq 2^{nH(x)+\epsilon}$$

$$|\hat{A}_\epsilon| \geq (1-\epsilon)\, e^{nH(x)-\epsilon}$$

Tx   Channel.   Rx

$(x_1, x_2, \ldots x_n)$ $(y_1, y_2, \ldots y_n)$



For each seq., no. of seq. generated in y = $2^{nH(Y|X)}$

Total no. of such seq = $2^{nH(Y)}$

No. of diff sequences in y

$$= \frac{2^{nH(Y)}}{2^{nH(Y|X)}}$$

$$= 2^{n(H(Y) - H(Y|X))}$$

$$\max[H(Y) - H(Y|X)] = \max I(X, Y)$$

Channel Capacity:

$$\boxed{C = \max_{f(x)} I(X, Y)}$$

ZMCG.

Suppose, a transmitter generates a sequence of N number of symbols $(X_1, X_2, X_3, ..., X_N)$. Then, which of the following statement(s) is/are true:

(i) Typical sequence can be formed if the symbols are equiprobable.

(ii) Each of the symbols will occure with the probability of $2^{-NH(X)}$.

(iii) Each of the symbols will occur with the probability of $NH(X)$.

(iv) Typical sequence can not be formed if the symbols are equiprobable.

a. (i) and (ii) are correct.

b. only (i) is correct.

c. (i) and (iii) are correct.

d. None of these above.

For a discrete random variable X, the Entropy of X will be maximized if X follows :-

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x)$$

a. Exponential distribution.

b. Uniform distribution.

c. Chi-square distribution.

d. Gaussian distribution.

NPTEL

Which of the following statement(s) is/are true.

(i) H(X,Y)= H(X) + H(Y/X)   Chain rule.

(ii) Relative entropy measures the distance between two distributions.

(iii) H(X,Y)= H(X) - H(Y/X)

(iv) Kullback Leibler Distance measures the distance between two points.

a. (i) and (ii) are correct

b. (ii) and (iii) are correct

c. (i),(ii) and (iv) are correct.

d. None of these are correct.

Calculate the Entropy of a discrete random variable X whose probability distribution is given below:-

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1 \\ \frac{1}{4}, & \text{if } x = 2 \\ \frac{1}{8}, & \text{if } x = 4 \\ \frac{1}{8}, & \text{if } x = 7 \end{cases}$$

a. 1.75 bits.

b. 1.90 bits.

c. 1.37 bits.

d. None of these.

$$H(X) = -p(x=1)\log_2 p(x=1) - p(x=2)\log_2 p(x=2) - p(x=4)\log_2 p(x=4) - p(X=7)\log_2 p(x=7)$$

$$= -\frac{1}{2}\log \frac{1}{2} - \frac{1}{4}\log_2 \frac{1}{4} - \frac{1}{8}\log_2 \frac{1}{8} - \frac{1}{8}\log_2 \frac{1}{8}$$

$$= 1.75 \text{ bits}.$$

Which of the following statement(s) is/are true:-

(i) $H(X) \geq 0$. ✓

(ii) For any two random variable, X & Y we have,

$$H(X|Y) > H(X)$$ ✗

With equality iff X and Y are independent.

(iii) $I(X;Y) \geq 0$, with equality iff

$$p(x, y) < p(x)p(y)$$

.

(iv) $D(p||q) \geq 0$ with equality iff p(x)=q(x).

a. (i) and (ii) are correct.

b. (ii) and (iv) are correct.

c. (i) and (iv) are correct. ✓

d. None of these above.

$$I(X,Y) = H(X) - H(X|Y)$$

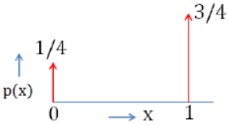
$$I(X,Y) \geqslant 0$$

$$\Rightarrow H(X) - H(X|Y) \geqslant 0$$

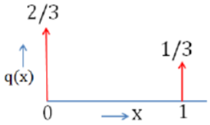
$$\Rightarrow H(X) \geqslant H(X|Y).$$

Let, $\chi = 0, 1$, consider the distributions p and q on χ

Let,



Distribution of p



Distribution of q

What are the values of $D(p||q)$ & $D(q||p)$, respectively?

a. 0.5237 bits & 0.5237 bits.

b. 0.5237 bits & 0.5534 bits.

c. 0.5534 bits & 0.5534 bits.

d. 0.5534 bits & 0.5237 bits.

$$D(p||q) = p(x=0)\log_2 \frac{p(x=0)}{q(x=0)} + p(x=1)\log_2 \frac{p(x=1)}{q(x=1)}$$

$$= \frac{1}{4}\log_2 \frac{1/4}{2/3} + \frac{3}{4}\log_2 \frac{3/4}{1/3}$$

$$= 0.5237 \text{ bits.}$$

D

$$D(q\|p) = q(0)\log_2\frac{q_0}{p_0} + q_1\log_2\frac{q_1}{p_1}$$

$$= \frac{2}{3}\log_2\frac{2/3}{1/4} + \frac{1}{3}\log_2\frac{1/3}{3/4}$$

$$= 0.5534$$

A function $f(x)$ said to be convex over an interval (a,b) if for every $(x_1, x_2) \in (a,b)$ & $0 \leq \lambda \leq 1$

a. $f\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

b. $f\left(\lambda x_1 + (1-\lambda)x_2\right) > \lambda f(x_1) + (1-\lambda) f(x_2)$

c. $f\left(x_1 + x_2\right) \leq f(x_1) + (1-\lambda) f(x_2)$

d. $f\left(x_1 + x_2\right) > f(x_1) + (1-\lambda) f(x_2)$

Evaluate differential entropy of following function,

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$h(f) = -\int_0^\infty f(x) \log_2 f(x)\, dx$$

$$= -\int_0^\infty \lambda e^{-\lambda x} \ln(\lambda e^{-\lambda x})\, dx \quad \text{nats.}$$

$$= -\int_0^\infty \lambda e^{-\lambda x} [\ln \lambda - \lambda x]\, dx.$$

$$= (1 - \ln \lambda) \text{ nats.}$$

$$= \ln(e/\lambda) \text{ nats.}$$

$$= \log_2(e/\lambda) \text{ bits.}$$

a. $log_2(\frac{2e}{\lambda})$ bits.

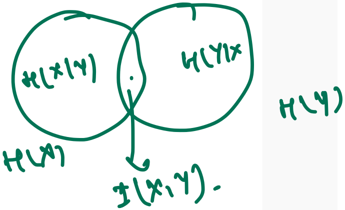
b. $log_2(\frac{\lambda}{e})$ bits.

c. $log_2(\frac{e}{\lambda})$ bits. ✓

d. $\frac{log_2(\frac{e}{\lambda})}{log_e(2)}$ bits.

Suppose, X & Y are two random variables and $I(X,Y)$ denotes the mutual information between X and Y. Which of the following statement(s) related to the mutual information is/are true:-

(i) $I(X,Y) = H(X) - H(X/Y)$.
(ii) $I(X,Y) = H(X) + H(X/Y)$.
(iii) $I(X,Y) = H(Y) - H(Y/X)$.
(iv) $I(X,Y) = H(X) + H(Y) - H(X,Y)$



a. Only (i) is correct.

b. (ii),(iii), and (iv) are correct.

c. (i),(iii), and (iv) are correct.

d. None of these above.

Gaussian Channel :

$(x_0, x_1 - x_n)$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \le P.$$

$$C = \max I(X;Y)$$
$$E[X^2] \le P.$$

$$I(X;Y) = h(Y) - h(\overset{X+Z.}{Y}|X)$$
$$= h(Y) - h(Z)$$

$$h(Z) = \frac{1}{2}\log_2(2\pi e N)$$

$$E Y^2 = E[(X+Z)^2] = E[X^2] + E[Z^2] + 2\,\overset{\to 0}{\cancel{E(X)}}E(Z)$$

$$= P + N.$$

$$h(Y) \le \frac{1}{2}\log_2\left(2\pi e\,(P+N)\right)$$

$$I(X,Y) = \frac{1}{2}\log_2\left(\frac{P+N}{N}\right) = \frac{1}{2}\log_2\left(1+\frac{P}{N}\right)$$

Capacity of gaussian channel.

Capacity of freq. flat deterministic MIMO channel.

M_T tx antennas

M_R rx antennas

$B = 1\,Hz$.

$$y_{M_R \times 1} = \sqrt{\frac{E_s}{M_T}}\, \underline{H}_{M_R \times M_T}\, \underline{s}_{M_T \times 1} + \underline{n}_{M_R \times 1}$$

R_{ss} = Covariance matrix of s $= E[s s^H]$

$E[s] = 0$.

Perfect CSI at Rx:

MIMO channel capacity,

$$C = \max_{f(s)} \underline{I(s,Y)}$$

$$I(s,Y) = h(Y) - h(Y|s)$$

$s+n$

$$= \underline{h(Y)} - \underbrace{h(n)}_{\text{fixed.}}$$

$$\max I(s,Y) \Longrightarrow \max h(Y)$$

$$y = S + n$$

Covariance matrix

$$R_{yy} = E[yy^H]$$

$$= E\left[\left(\sqrt{\frac{E_s}{M_T}} Hs + n\right)\left(\sqrt{\frac{E_s}{M_T}} s^H H^H + n^H\right)\right]$$

$$= E\left[\frac{E_s}{M_T} H s s^H H^H\right] + E\left[n n^H\right]$$

$N_0 I_{M_R}$. $\begin{bmatrix} n_1 \\ \vdots \\ n_{M_R} \end{bmatrix}[\quad]$

$$= \frac{E_s}{M_T} H \, E[ss^H] H^H + N_0 I_{M_R}$$

$\begin{bmatrix} N_0 & & \\ & N_0 & \\ & & \ddots \end{bmatrix}_{M_R}$

$$= \frac{E_s}{M_T} H R_{ss} H^H + N_0 I_{M_R}.$$

Theorem: $A \to n \times n$ matrix, $\alpha \to$ scalar.

$$\det\{\alpha A\} = \alpha^n \det(A).$$

$$h(Y) = \log_2\left(\det\left(\pi e R_{yy}\right)\right)$$

$$h(n) = \log_2\left(\det\left(\pi e N_0 I_{M_R}\right)\right)$$

$$I(S, Y) = h(Y) - h(n).$$

$$= \log_2 \frac{\det\left(R_{yy}\right)}{\det\left(N_0 I_{M_R}\right)}$$

$$= \log_2 \frac{\det\left(\frac{E_s}{M_T} H R_{ss} H^H + N_0 I_{M_R}\right)}{\det\left(N_0 I_{M_R}\right)}$$

$$I(S,Y) = \log_2\left(\det\left(I + \frac{E_s}{M_T N_0} H R_{ss} \cdot H^H\right)\right)$$

$$\boxed{Cap = \max_{Tr(R_{ss}) = M_T} \log_2\left(\det\left(I_{M_R} + \frac{E_s}{M_T N_0} H R_{ss} H^H\right)\right)}$$

Cap. of MIMO channel when CSI is known at receiver.

Consider a communication system where channel between transmitter and receiver is AWGN. The average received SNR is 5 dB. Maximum spectral efficiency that can be achieved in this channel is:

a. 1.4261 bps/Hz

b. 2.1817 bps/Hz

c. 2.8123 bps/Hz

d. 2.0574 bps/Hz

$$SNR = 5\,dB$$

$$= 10^{0.5} = 3.162.$$

$$\log_2(1 + SNR)$$

$$= \log_2(1 + 3.162)$$

$$= \log_2(4.162) = 2.057\,\text{bps/Hz}.$$

Consider a 2×2 MIMO channel with channel matrix $\mathbf{H} = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$. The average SNR $(\frac{E_S}{N_0}) = 4$ dB. The covariance matrix of transmit signal ($\mathbf{S}$) is given by $\mathbf{R_{SS}}$ which is a identity matrix of size 2×2. Calculate the mutual information between transmitted signal $\mathbf{S}$ and received signal $\mathbf{Y}$.

a. 1.3354 bits ✓

b. 1.8742 bits

c. 2.1271 bits

d. 2.2717 bits

$$SNR = 4\,dB = 10^{0.4} = 2.5119.$$

$$R_{SS} = I_2.$$

$$I(S,Y) = \log_2\left(\det\left(I + \frac{E_S}{M_T N_0} \cdot H \underbrace{R_{SS}}_{I} H^H\right)\right)$$

$$= \log_2\left(\det\left(I + \frac{2.5119}{2}\begin{bmatrix} 0.52 & 0.46 \\ 0.46 & 0.58 \end{bmatrix}\right)\right)$$

$$= \log_2(2.3235) = 1.3354 \text{ bits}.$$

A Bivariate normal distribution can be given as,

$$f_X(x) = \frac{1}{2\pi|K|^{\frac{1}{2}}} exp\Big[-\frac{1}{2}(X-\mu)^T K^{-1}(X-\mu)\Big]$$

Where,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$K = E[X^T X]$$

Suppose,

$$\mu = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0.2 \\ 0.4 & 5 \end{bmatrix}$$

Compute the Differential Entropy of $f_X(x)$.

a. 5.2435 bits.

b. 10.4870 bits.

c. 5.37 bits.

d. 6.37 bits.

$$h(X) = \frac{1}{2}\log_2\left((2\pi e)^2 |K|\right)$$

$$|K| = 5 - 0.08 = 4.92.$$

$$= \frac{1}{2}\log_2\left((2\pi e)^2 \times 4.92\right)$$

$$= \frac{1}{2} \times 10.487$$

$$= 5.243 \text{ bits.}$$