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Exercise 1.1

- (a) At first we write down our different sets of the Domain. Then we combine them to form our desired result.

We have: $\Sigma_2 \setminus \Sigma_1 = \{d, e\}$, and $Q^2 = \{(q_1, q_1), (q_1, q_2), (q_2, q_1), (q_2, q_2)\}$.

If we make the scalar product of this two sets we obtain the following result:

$$R = \{((q_1, q_1), d), ((q_1, q_2), e), ((q_2, q_1), d), ((q_2, q_2), e), ((q_1, q_1), e)\}$$



- (b) First of all we write down the domain, that is also the power set of $\Sigma_1 \cap \Sigma_2$:

$$\mathcal{P}(\Sigma_1 \cap \Sigma_2) = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$$

and the codomain:

$$Q \times V = \{(q_1, X), (q_1, Y), (q_1, Z), (q_2, X), (q_2, Y), (q_2, Z)\}$$

An example for a total function could be:

$$f : \{\emptyset \mapsto (q_1, X), \{b\} \mapsto (q_2, X), \{c\} \mapsto (q_2, Z), \{b, c\} \mapsto (q_1, Y)\}$$



- (c) We want to calculate how many partial functions $f : \Sigma_1 \cup \Sigma_2 \rightarrow_p V$ there are.
The domain of the function f is:

$$\Sigma_1 \cup \Sigma_2 = \{a, b, c, d, e\}$$

and the codomain:

$$V = \{X, Y, Z\}$$

Since we have 4 possible mappings for every element of the set (the 3 variables and the emptyset) $\Sigma_1 \cup \Sigma_2$, with cardinality $|\Sigma_1 \cup \Sigma_2| = 5$, we have 4^5 possible partial functions.



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Exercise 1.2

- (a) Just for clarity we write the set of all cells: $P = \{ (1, 1), (1, 2), (2, 1), \dots, (n, m) \}$
In the given example we have: $n = 3$, $m = 2$, $C = \{ \text{White, Gray, Black} \}$ and the following function f :

$$f(P_{\text{3x2}}) := \begin{cases} \text{Grey} & \text{if } P = (1, 1) \\ \text{Black} & \text{if } P = (3, 2) \\ \text{White} & \text{else} \end{cases}$$

- (b) The cartesian product of $P \times P$ results in:

$$P^2 = \{ ((1, 1), (1, 1)), ((1, 1), (1, 2)), \dots, ((n, m), (n, m)) \}$$

and therefore the relation R :

$$R = \{ (p, p') = ((x_p, y_p), (x_{p'}, y_{p'})) \mid f(p) \neq f(p'), x_p = x_{p'} \text{ and } y_p = y_{p'} + 1 \}$$

R for concrete example in (a):

$$R = \{ ((1, 2), (1, 1)), ((3, 2), (3, 1)) \}$$

2.5/3 Exercise 1.3

The statement to prove is:

$$(A \cup B) \subseteq (A \cap B), \text{ then } A \subseteq B$$

- (a) *Direct proof.* In a first step we take an $x \in (A \cup B)$. Because $A \cup B$ is a subset of $A \cap B$, any $x \in A$ is always also element of B and vice versa, and therefore $A \subseteq B$. ✓

□

- (b) *Indirect proof.* Assumption: if $(A \cup B) \subseteq (A \cap B)$, then $A \not\subseteq B$. ✓
There must be an $x \in A \setminus B$, that is also element of $A \cup B$. Because $(A \cup B) \subseteq (A \cap B)$ and $x \notin (A \cap B)$, following the initial condition of our x , x cannot be in A which is a contradiction to our assumption. ✓

□

- (c) *Contrapositive.* We assume: if $A \not\subseteq B$, then $(A \cup B) \not\subseteq (A \cap B)$. ✓
Because of $A \not\subseteq B$, there must be an $x \in A \setminus B$ which contradicts with $(A \cup B) \subseteq (A \cap B)$ because if $x \in A \setminus B$, so also $x \in A \cup B$ but x is not element of $A \cap B$.
but we want to show here that this does not hold so its not a contradiction but proving our statement -0.5

3/3 Exercise 1.4

Proof. For $a \in \mathbb{R}$ and $a \neq 1$

Basis $n = 0$:

$$\sum_{i=0}^0 a^i = a^0 = 1 \stackrel{?}{=} \frac{1-a}{1-a} = 1 \quad \checkmark$$

IH:

$$\sum_{i=0}^n a^i = \frac{1-a^{n+1}}{1-a} \quad \forall n \in \mathbb{N}_0 \quad \checkmark$$

Inductive step $n \rightarrow n + 1$:

$$\begin{aligned} \sum_{i=0}^{n+1} a^i &= \sum_{i=0}^n a^i + a^{n+1} \stackrel{IH}{=} \frac{1-a^{n+1}}{1-a} + a^{n+1} \\ &= \frac{1-a^{n+1} + (1-a)a^{n+1}}{1-a} \\ &= \frac{1-a^{n+1} + a^{n+1} - a^{n+2}}{1-a} \\ &= \frac{1-a^{n+2}}{1-a} \quad \checkmark \end{aligned}$$

□