Theory of Computer Science

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Exercise Sheet 1

Due: Wednesday, March 8, 2023

Note: The goal of this exercise sheet is to learn how to correctly express formal proofs. A formally correct proof consists of single steps where each step follows *immediately* from the previous steps or from the assumptions (for example when replacing a value by its definition). Please write down your proofs in detail and in a formal fashion. Examples can be found in the lecture slides.

Exercise 1.1 (Sets, Functions and Relations; 0.5 + 0.5 + 1 points)
Consider the sets $V = \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$, $\Sigma_1 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, $\Sigma_2 = \{\mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $Q = \{q_1, q_2\}$.

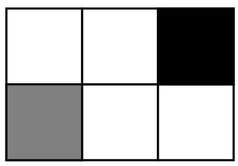
(a) Specify a non-trivial example for a binary relation $R \subseteq (Q^2) \times (\Sigma_2 \setminus \Sigma_1)$ with $|R| = 5$.

(b) Specify a example for a (total) function $f : \mathcal{P}(\Sigma_1 \cap \Sigma_2) \to Q \times V$.

(c) How many partial functions $f : \Sigma_1 \cup \Sigma_2 \to_{\mathrm{p}} V$ are there? Justify your answer (a proof is not necessary).

Exercise 1.2 (Mathematical Modeling; 0.5 + 1.5 points)
Consider a two dimensional grid with width n and height m, where positions are denoted by $\langle x, y\rangle$ with $x \in \{1, \dots, n\}$, $y \in \{1, \dots, m\}$, and $\langle 1, 1\rangle$ being the bottom left cell. We define P to be the set of all cells. Additionally there is a set of colors C, and function $f : P \to C$ maps each cell to a color.

(a) Explicitly specify n, m, C and f for the following concrete grid:



(b) Consider relation R over $P \times P$ which contains $\langle p, p'\rangle$ iff p and p' have different colors and p is directly above p'. Specify a general definition of R for arbitrary P and f, and additionally specify R for the concrete example in (a).

Exercise 1.3 (Proofs, 1 + 1 + 1 points)
Consider the following statement: If $(A \cup B) \subseteq (A \cap B)$, then $A \subseteq B$.

(a) Show the statement with a direct proof.

(b) Show the statement with an indirect proof.

(c) Show the statement by contrapositive.

Exercise 1.4 (Mathematical Induction, 3 points)
Prove using mathematical induction that for $a \in \mathbb{R}$ and $a \neq 1$ the following statement holds.

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a} \quad \text{for all } n \in \mathbb{N}_0$$