Theory of Computer Science

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Exercise Sheet 9

Due: Wednesday, May 10, 2023

Due to the Monday lecture on May 1 not taking place, this exercise sheet covers only one lecture and is worth 5 points.

Exercise 9.1 (Proving NP-completeness, 2 points)

Consider the following two problems:

HittingSet:
Given: finite non-empty set U,
finite set of sets $S = \{S_1, \dots, S_n\}$ with $S_i \subseteq U$ for all $S_i \in S$,
natural number $k \in \mathbb{N}_0$
Question: Does a set $H \subseteq U$ with $|H| \leq k$ exist such that $S_i \cap H \neq \emptyset$ for all $S_i \in S$?

SAT':
Given: finite set of variables V,
finite set of clauses $C = \{C_1, \dots, C_m\}$ with $C_i \subseteq V \cup \{\neg v \mid v \in V\}$ for all $C_i \in C$.
Question: Is there an assigment $I : V \to \{\top, \bot\}$ such that for each $C_i \in C$ there is a v with either $v \in C$ and $I(v) = \top$, or $\neg v \in C$ and $I(v) = \bot$?

Find and briefly explain all the flaws in the following attempt of proving that SAT' is NP-complete based on the knowledge that HittingSet is NP-complete.
Hint: There are more than two flaws.

We first show SAT' $\leq$ HittingSet. Consider an arbitrary instance $\mathbf{F} = \langle V, C\rangle$ of SAT'. We define a HittingSet instance $\mathbf{H} = \langle U, S, k\rangle$ by interpreting the variables V of $\mathbf{F}$ combined with their negation as the universe U of $\mathbf{H}$, the set of clauses C of $\mathbf{F}$ as the set S in $\mathbf{H}$, and the amount of variables of $\mathbf{F}$ as the size k of $\mathbf{H}$.
Formally, the reduction is defined by the function $f : \text{SAT'} \to \text{HittingSet}$ with

$$f(\langle V, C\rangle) = \langle V \cup \{\neg v \mid v \in V\}, C, |V|\rangle$$

If I is an assignment for $\langle V, C\rangle$ with the required properties, we can build a set $H = \{v \mid I(v) = \top\} \cup \{\neg v \mid I(v) = \bot\}$. This set is a hitting set for $f(\langle V, C\rangle) = \langle U, S, k\rangle$: We know that for each $C_i \in C$ $(= S_i \in S)$ we have some v with either $I(v) = \top$ and $v \in C$, or $I(v) = \bot$ and $\neg v \in C$. In the former case we add v to the hitting set, and in the latter $\neg v$, meaning we hit S_i. Furthermore, since I is a full assignment over V we have $|H| = |V| = k$. Putting everything together we have that H is a solution for $f(\langle V, C\rangle)$, and thus SAT' can be reduced to HittingSet.
We know that HittingSet is NP-complete, from which follows that all problems in NP can be reduced to HittingSet. Together with the above reduction this implies that all problems in NP can be reduced to SAT', meaning SAT' is NP-complete.

Exercise 9.2 (NP; 0.5+0.5 points)

Are the following statements correct? Briefly justify your answer.

(a) A language cannot be in both P and NP.

(b) Let X be an NP-complete problem and Y a problem with $X \leq_\text{p} Y$. Then Y is NP-complete.

Exercise 9.3 (Decidability and NP; 2 points)

Prove that all languages in NP are decidable.