Theory of Computer Science

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Exercise Sheet 10

Due: Wednesday, May 17, 2023

Exercise 10.1 (Polynomial Reduction, 0.5 + 3 + 0.5 points)

A k-coloring of an undirected graph $G = \langle V, E\rangle$ is a function $c : V \to \{1, \dots, k\}$ such that for all $\{x, y\} \in E$, $c(x) \neq c(y)$. Intuitively, a k-coloring assigns every vertex one of k colors (represented by numbers) such that no two adjacent vertices have the same color.

We consider the problem KB-Coloring:

- *Given:* Tuple $\langle G, k, b\rangle$, where G is an undirected graph, and k and $b \in \mathbb{N}_0$ are natural numbers.
- *Question:* Does G have a k-coloring c such that there is an $n \in \{1, \dots, k\}$ such that at least b elements of V are mapped to n?

Intuitively, KB-Coloring asks whether the graph has a k-coloring, where one of the colors is used for at least b vertices.

(a) Consider graph $G = \langle \{v_1, v_2, \dots, v_5\}, \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_4, v_5\}\}\rangle$ and specify a coloring that shows that $\langle G, 3, 3\rangle \in$ KB-Coloring.

(b) Provide a polynomial reduction from IndSet (lecture slides D4) to KB-Coloring and prove that it has the required properties.

(c) What do you know about IndSet from the lecture? What can you conclude for KB-Coloring from the existence of a polynomial reduction as required in part (b)?

Exercise 10.2 (Proving NP-Completeness; 6 points)

Consider the following decision problem:

SetPacking:

- *Given:* finite set M, set of sets $\mathcal{S} = \{S_1, \dots, S_n\}$ with $S_i \subseteq M$ for all $i \in \{1, \dots, n\}$, natural number $k \in \mathbb{N}_0$
- *Question:* Is there a set $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \geq k$, such that all sets in $\mathcal{S}'$ are pairwise disjoint, i.e., for all $S_i, S_j \in \mathcal{S}'$ with $S_i \neq S_j$ it holds that $S_i \cap S_j = \emptyset$?

Prove that SetPacking is NP-complete. You may use that IndSet (from slides D4) is NP-complete.