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Exercise 7.1

(a)

$$x_1 = \texttt{dec}(101) = 5, \qquad x_2 = \texttt{dec}(0) = 0, \qquad x_3 = \texttt{dec}(11) = 3, \qquad y = \texttt{dec}(1001) = 9$$

(b)

$$x_1 = \texttt{dec}(11) = 3, \qquad x_2 = \texttt{dec}(100) = 4, \qquad x_3 = \texttt{dec}(1) = 1, \qquad y = \textit{undefined}$$

Exercise 7.2

Step 1. Check if $n_2 = 0$: if **true** write $n_1 = 1$, remove n_2, n_3 from tape and terminate.

Step 2. Check if $n_2 = 1$: if **true** remove n_2, n_3 from tape and terminate.

Step 3. Copy n_1 to n_3.

Step 4. $M_{pred_1}(n_2)$.

Step 5. $M_{mul}(n_1, n_3)$ and write result to n_1.

Step 6. Check if $n_2 = 1$: if **true** remove n_2, n_3 from tape and terminate.

Step 7. Jump to Step 4.

Exercise 7.3

To show that the composition $(f \circ g) : \Sigma_1^* \to \Sigma_2^*$ is Turing-computable, we can build a TM that simulates the computations of f and g sequentially. This TM would work as follows:

1. Start the TM with the input x.
2. Use a Turing machine that simulates the computation of g on x. If $g(x)$ is undefined, then halt and output *undefined*.
3. Use a Turing machine that simulates the computation of f on $g(x)$. If $f(g(x))$ is undefined, then halt and output *undefined*.
4. If $f(g(x))$ is defined, output it as the result of the composition.

Since we assume that f and g are Turing-computable, we know that there exist Turing machines that can compute f and g respectively. Therefore, we can construct a Turing machine that combines the computations of these two machines, and this new machine can compute the composition of f and g for any input. Therefore, we have shown that the composition of two Turing-computable partial functions is also Turing-computable.

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Exercise 7.4

Let w be an arbitrary string in A. Since A is a subset of B, we know that w is also in B. Similarly, since B is a subset of C, we know that w is also in C. Therefore, w is in A and in C, which implies that A is a subset of C, or $A \leq C$.
Thus, we have shown that if $A \leq B$ and $B \leq C$, then $A \leq C$ for any languages A, B and C. This is a result of the transitive property of set inclusion.

Exercise 7.5

(a) If $R \leq L$, and R is not decidable, then L is also not decidable. However, since R is Turing-recognizable, L must also be Turing-recognizable.

(b) If $L \leq U$, and U is not Turing-recognizable, then L is also not Turing-recognizable and cannot be decidable either.

(c) If $D \leq L$, and D is decidable, then we cannot tell from the given information whether L is decidable and/or Turing-recognizable.

(d) If $L \leq D$, and D is decidable, then L is also decidable. However, since D is decidable, it is also Turing-recognizable. Therefore, L is both decidable and Turing-recognizable.