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Exercise 1.1

(a) At first we write down our different sets of the Domain. Then we combine them to form our desired result.
Whe have: $\Sigma_2 \setminus \Sigma_1 = \{ d, e \}$, and $Q^2 = \{ (q_1, q_1), (q_1, q_2), (q_2, q_1), (q_2, q_2) \}$.
If we make the scalar product of this two sets we obtain the following result:

$$R = \{ ((q_1, q_1), d), ((q_1, q_2), e), ((q_2, q_1), d), ((q_2, q_2), e), ((q_1, q_1), e) \}$$

(b) First of all we write down the domain, that is also the power set of $\Sigma_1 \cap \Sigma_2$:

$$\mathcal{P}(\Sigma_1 \cap \Sigma_2) = \{ \emptyset, \{ b \}, \{ c \}, \{ b, c \} \}$$

and the codomain:

$$Q \times V = \{ (q_1, X), (q_1, Y), (q_1, Z), (q_2, X), (q_2, Y), (q_2, Z) \}$$

An example for a total function could be:

$$f : \{ \emptyset \mapsto (q_1, X), \{ b \} \mapsto (q_2, X), \{ c \} \mapsto (q_2, Z), \{ b, c \} \mapsto (q_1, Y) \}$$

(c) We want to calculate how many partial functions $f : \Sigma_1 \cup \Sigma_2 \to_p V$ there are.
The domain of the function f is:

$$\Sigma_1 \cup \Sigma_2 = \{ a, b, c, d, e \}$$

and the codomain:

$$V = \{ X, Y, Z \}$$

Since we have 4 possible mappings for every element of the set (the 3 variables and the emptyset) $\Sigma_1 \cup \Sigma_2$, with cardinality $|\Sigma_1 \cup \Sigma_2| = 5$, we have 4^5 possible partial functions.

Exercise 1.2

(a) Just for clarity we write the set of all cells: $P = \{ (1,1), (1,2), (2,1), \ldots, (n,m) \}$
In the given example we have: $n = 3$, $m = 2$, $C = \{ \text{White, Gray, Black} \}$ and the following function f:

$$f(P_{3\times 2}) := \begin{cases} Grey & P = (1,1) \\ Black & P = (3,2) \\ White & else \end{cases}$$

(b) The cartesian product of $P \times P$ results in:

$$P^2 = \{ ((1,1),(1,1)), ((1,1),(1,2)), \ldots, ((n,m),(n,m)) \}$$

and therefore the relation R:

$$R = \{ (p, p') = ((x_p, y_p), (x_{p'}, y_{p'})) \mid f(p) \neq f(p'),\ x_p = x_{p'} \text{ and } y_p = y_{p'} + 1 \}$$

R for concrete example in (a):

$$R = \{ ((1,2),(1,1)), ((3,2),(3,1)) \}$$

Exercise 1.3

The statement to prove is:

$$(A \cup B) \subseteq (A \cap B), \text{ then } A \subseteq B$$

(a) *Direct proof.* In a first step we take an $x \in (A \cup B)$. Because $A \cup B$ is a subset of $A \cap B$, any $x \in A$ is always also element of B and vice versa, and therefore $A \subseteq B$.

$\square$

(b) *Indirect proof.* Assumption: if $(A \cup B) \subseteq (A \cap B)$, then $A \nsubseteq B$
There must be an $x \in A \setminus B$, that is also element of $A \cup B$. Because $(A \cup B) \subseteq (A \cap B)$ and $x \notin (A \cap B)$, following the initial condition of our x, x cannot be in A which is a contraddiction to our assumption.

$\square$

(c) *Contrapositive.* We assume: if $A \nsubseteq B$, then $(A \cup B) \nsubseteq (A \cap B)$
Because of $A \nsubseteq B$, there must be an $x \in A \setminus B$ which contradicts with $(A \cup B) \subseteq (A \cap B)$ because if $x \in A \setminus B$, so also $x \in A \cup B$ but x is not element of $A \cap B$.

$\square$

Exercise 1.4

Proof. For $a \in \mathbb{R}$ and $a \neq 1$

Basis n = 0:

$$\sum_{i=0}^{0} a^0 = a^0 = 1 \stackrel{!}{=} \frac{1-a}{1-a} = 1$$

IH:

$$\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a} \qquad \forall n \in \mathbb{N}_0$$

Inductive step n → n + 1:

$$\begin{aligned}
\sum_{i=0}^{n+1} a^i = \sum_{i=0}^{n} a^i + a^{n+1} &\stackrel{IH}{=} \frac{1-a^{n+1}}{1-a} + a^{n+1} \\
&= \frac{1-a^{n+1} + (1-a)a^{n+1}}{1-a} \\
&= \frac{1-a^{n+1} + a^{n+1} - a^{n+2}}{1-a} \\
&= \frac{1-a^{n+2}}{1-a}
\end{aligned}$$

□