The Exponential Function

Ruben Olmos

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1 Introduction

In this essay we will prove the following:

- There is a number:
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$
- There exists a function
$$exp : \mathbb{Q} \to \mathbb{R}, exp(1) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$$
- There exists a continous function
$$exp : \mathbb{R} \to \mathbb{R}$$
- e^x is differentiable and it is equal to its own derivative
- $cos^2 z + sin^2 z = 1$ for all $z \in \mathbb{C}$ by deducing Eulers form.
- e^x is a homomorphism, that is $e^{x+y} = e^x \cdot e^y$

2 e

We define a sequence
$$a_n = (1 + \frac{1}{n})^n$$

using the binomial expansion we have that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \frac{n-1}{n} ... \frac{n-(k-1)}{n} = \sum_{k=0}^{n} \frac{1}{k!}(1-\frac{1}{n})...(1-\frac{k-1}{n})$$

We make a note of an observation here, we see that for all $l \in \{1, 2, 3, .., (k-1)\}$ it holds that $(1-\frac{l}{n}) < (1-\frac{l}{n+1})$ and it is easy to notice that the sum of a_{n+1} has one more term than that of a_n and can therefore conclude that we have a monotonic sequence, that is it is increasing. We return to the sequence a_n and expanding by multiplying out the $(1+\frac{1}{n})$ term by itself n times :

$$a_n = 1 + \frac{n}{n} + \frac{1}{2!} \frac{n(n-1)}{n^2} + \frac{1}{3!} \frac{n(n-1)(n-2)}{n^3} + ... + \frac{1}{n!} \frac{n(n-1)...(n-n+1)}{n^n}$$

Here we observe that the for all the $k \in \{2, 3, ..n\}$ it holds that

$$\frac{1}{k!} < \frac{1}{2^{k-1}}$$

and

$$\frac{n(n-1)...(n-(k-1))}{n^k} < 1$$

then it must hold that using the formula for geometric progressions with a ratio of $\frac{1}{2}$ that

$$a_n < 1 + (1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + ... + \frac{1}{2^{n-1}}) < 1 + (\frac{1-\frac{1}{2^n}}{1-\frac{1}{1/2}}) < 3 - \frac{1}{2^{n-1}} < 3$$

Then given this fact we have that the a_n is bounded above by 3 and it is monotonic, therefore the sequence a_n must converge and the limit must exist and be the list upper bound of the set $\{a_n\}$ where we have that $a_n \subset [2, 3)$ Now taking the sequence a_n by bimomial exapnsion we have:

$$a_n = (1 + \frac{1}{n})^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2}(\frac{1}{n})^2 + ... + (\frac{1}{n})^n$$

$$= \frac{1}{0!} + \frac{1}{1!} + (1-\frac{1}{n})\frac{1}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{1}{3!} + ... + (1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{n-1}{n})\frac{1}{n!}$$

then taking the limit of a_n and knowing that it exists we label it the constant e and arrive at:

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = \sum_{k=0}^{\infty} \frac{1}{k!} = e$$

3 There exists exp:$\mathbb{Q} \to \mathbb{R}$

We begin by defining a sequence $\{a_n\}$ letting $a_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$. By definition of uniform convergence we have that a sequence A sequence $f_n(x)$ is Uniformly Cauchy if $\lim_{n\to\infty} ||f_{n+m}(x) - f_n(x)|| = 0$, such that $\forall m \geq 1$

$$|a_{n+m} - a_n| = \Big|\sum_{k=0}^{n+m} \frac{x^k}{k!} - \sum_{k=0}^{n} \frac{x^k}{k!}\Big| = \Big|\sum_{k=n+1}^{n+m} \frac{x^k}{k!}\Big| \leq \sum_{k=n+1}^{n+m} \frac{x^k}{k!} \leq \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

Taking a look at $\sum_{k=n+1}^{\infty} \frac{x^k}{k!}$ and applying a limit the ratio of its kth and (k+1)th terms we have:

$$\lim_{k\to\infty} \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = \lim_{k\to\infty} \frac{x}{k+1} = 0$$

We notice that by the ratio test $\sum_{k=n+1}^{\infty} \frac{x^k}{k!}$ is convergent this implies that:

$$|a_{n+m} - a_n| = \Big|\sum_{k=0}^{n+m} \frac{x^k}{k!} - \sum_{k=0}^{n} \frac{x^k}{k!}\Big| = \Big|\sum_{k=n+1}^{n+m} \frac{x^k}{k!}\Big| \leq \sum_{k=n+1}^{n+m} \frac{x^k}{k!} \leq \sum_{k=n+1}^{\infty} \frac{x^k}{k!} < \epsilon$$

therefore the sequence is cauchy. And taking an arbitrary element $x \in \mathbb{Q}$ we see that the above holds and have convergence for all rational numbers, further we observe that $\forall x \in \mathbb{Q}$ we have convergence to a real number since the sum of rational numbers will always be real. Therefore there must exist a function $exp : \mathbb{Q} \to \mathbb{R}$ and provided we defined $exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ then $exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$.

4 There exists exp:$\mathbb{R} \to \mathbb{R}$

We begin by defining the sequence $e_n^x = \sum_{k=0}^{n} \frac{x^k}{k!}$ we note that this is a polynomial. We let $b \in \mathbb{R}$,$b > 0$ We assume the sequence $\{e_i^b\}_{i\in\mathbb{N}}$, we note that $\{e_i^b\}_{i\in\mathbb{N}} \subset \mathbb{R}$. In general we recall that for a sequence if the sequence limit $\lim_{k\to\infty} |\frac{a_k}{a_{k+1}}|$ converges then the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is r such that $r := \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$. and it follows that given a b such that , $0 \leq b < r$, the sequence of partial sums $x \mapsto \sum_{k=0}^{n} a_k x^k$ is uniformly convergent on and uniform cauchy $[-b, b]$. Then taking our given sequence we let $a_k = 1/k!$ and let $a_{k+1} = 1/(k+1)!$ and so taking the limit we have, $\lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} |k+1| = \infty$, so the radius of convergence is $r = \infty$. Hence for any $b \in [0, \infty)$, e_n^x is uniformly convergent on $[-b, b]$. This implies that there must exist a continuous function $exp : [-b, b] \to \mathbb{R}$ letting $b \in [0, \infty]$ and therefore it must follow that there must exist a continuous function $exp : \mathbb{R} \to \mathbb{R}$. Since $\forall r \in \mathbb{R}$, it must be that $r \in [-b, b]$ for $b \in [0, \infty)$.

5 e^x is equal to its derivative

We define the exponential function e^x to be a function $f : \mathbb{R} \to \mathbb{R}$ We begin with the binomial series expansion:

$$\left(1 + \frac{x}{n}\right)^n = 1 + \frac{n}{1!}\left(\frac{x}{n}\right)^1 + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3 + \cdots$$

$$\left(1 + \frac{x}{n}\right)^n = 1 + \frac{n}{n}x + \frac{n(n-1)}{n^2}\frac{x^2}{2!} + \frac{n(n-1)(n-2)}{n^3}\frac{x^3}{3!} + \cdots$$

As $n \to \infty$ the coefficients in n all tend to 1. Hence:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Here we can recognize the Taylor series expansion of e^x

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and consequently we have that:

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

We recall that by definition of continuity: e^x is continuous on $\mathbb{R}$ if and only if

$$(\forall x_0 \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(0 < |x - x_0| < \delta \implies |e^x - e^{x_0}| < \varepsilon).$$

Taking the limit of the exponential in the form that we derived above:

$$\lim_{x\to x_0}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{x\to x_0} e^x = e^{x_0}$$

And by definition $\lim_{x\to x_0} e^x = L$ if and only if

$$(\forall\varepsilon>0)(\exists\delta>0)(0<|x-x_0|<\delta \implies |e^x-L|<\varepsilon).$$

And given arbitrary x_0 then:

$$(\forall x_0\in\mathbb{R})(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})(0<|x-x_0|<\delta \implies |e^x-e^{x_0}|<\varepsilon).$$

Thus, e^x is continuous for all $x\in\mathbb{R}$.

Now given that e^x is a function from $\mathbb{R}$ to $\mathbb{R}$ where $x\in\mathbb{R}$ and $e^x\in\mathbb{R}$ and it is continuous we can say that e^x is differentiable at x_0, where $x_0\in\mathbb{R}$ if there exists a $t_0\in\mathbb{R}$ such that:

$$\lim_{h\to 0}\frac{e^{x_0+h}-e^{x_0}}{h}$$

We let $h=x-x_0$. Further if e^x is differentiable at x_0 we say that t_0 is the derivative of e^x at x_0. We notice that:

$$\lim_{h\to 0}\frac{e^{x_0+h}-e^{x_0}}{h} = \lim_{h\to 0}\frac{e^{x_0}(e^h-1)}{h} = e^{x_0}\cdot\lim_{h\to 0}\frac{e^h-1}{h}$$

Taking $\frac{e^h-1}{h}$ and expanding it using the fact that $e=\lim_{n\to\infty}(1+\frac{1}{n})^n$ from the general taylor expansion of e^x for some $x\in\mathbb{R}$ we have the following:

$$\frac{e^h-1}{h} = \frac{\lim_{n\to\infty}\left(1+\frac{h}{n}\right)^n-1}{h} = \frac{\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{h}{n}\right)^k-1}{h}$$

$$= \lim_{n\to\infty}\left(\binom{n}{0}1-1+\binom{n}{1}\left(\frac{h}{n}\right)\frac{1}{h}+\sum_{k=2}^{n}\binom{n}{k}\left(\frac{h}{n}\right)^k\frac{1}{h}\right)$$

$$= \lim_{n\to\infty}1+\lim_{n\to\infty}\sum_{k=2}^{n}\binom{n}{k}\frac{h^{k-1}}{n^k} = 1+h\lim_{n\to\infty}\sum_{k=2}^{n}\binom{n}{k}\frac{h^{k-2}}{n^k}$$

Using the above expansion for $\frac{e^h-1}{h}$ and taking $\lim_{h\to 0}\frac{e^h-1}{h}$, we have the following:

$$\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{h\to 0}(1+h\lim_{n\to\infty}\sum_{k=2}^{n}\binom{n}{k}\frac{h^{k-2}}{n^k})=1.$$

The above holds since the term added to 1 goes to zero as $h\to 0$.Thus we have that:

$$\lim_{h\to 0}\frac{e^{x_0+h}-e^{x_0}}{h}=e^{x_0}\cdot\lim_{h\to 0}\frac{e^h-1}{h}=e^{x_0}\cdot(1)=e^{x_0}$$

Therefore,we have that e^x is differentiable at $x_0\in\mathbb{R}$ with a derivative of e^{x_0}. And since x_0 is arbitrary we can say that e^x is differentiable $\forall x_0\in\mathbb{R}$ and for each x_0 the derivative of e^x at x_0 is e^{x_0}.

6 $cos^2 z+sin^2 z=1$

By familiarity of the exponential extended to the complex numbers we have that $e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$ letting $z\in\mathbb{C}$. We let $z=a+bi$, letting $a,b\in\mathbb{R}$ and i be the imaginary unit. Then:

$$e^{zi}=\sum_{k=0}^{\infty}\frac{(zi)^k}{k!}=\sum_{k=0}^{\infty}(\frac{(zi)^{2k}}{(2k)!}+\frac{(zi)^{2k+1}}{(2k+1)!})=$$

$$\sum_{k=0}^{\infty}((-1)^k\frac{(z)^{2k}}{(2k)!}+i(-1)^k\frac{(z)^{2k+1}}{(2k+1)!})=\sum_{k=0}^{\infty}(-1)^k\frac{(z)^{2k}}{(2k)!}+i\sum_{k=0}^{\infty}(-1)^k\frac{(z)^{2k+1}}{(2k+1)!}=$$

$$Cos(z)+iSin(z)$$

since:

$$Cos(z)=\sum_{k=0}^{\infty}(-1)^k\frac{(z)^{2k}}{(2k)!}$$

and

$$Sin(z)=\sum_{k=0}^{\infty}(-1)^k\frac{(z)^{2k+1}}{(2k+1)!}$$

Therefore

$$e^{zi}=Cos(z)+iSin(z)$$

Now supposing:

$$e^{-zi} = \sum_{k=0}^{\infty} \frac{(-zi)^k}{k!} = \sum_{k=0}^{\infty} (\frac{(-zi)^{2k}}{(2k)!} + \frac{(-zi)^{2k+1}}{(2k+1)!}) =$$

$$\sum_{k=0}^{\infty}((-1)^k \frac{(z)^{2k}}{(2k)!} + i(-1)^{2k+1}(-1)^k \frac{(z)^{2k+1}}{(2k+1)!}) = \sum_{k=0}^{\infty}(-1)^k \frac{(z)^{2k}}{(2k)!} - i\sum_{k=0}^{\infty}(-1)^k \frac{(z)^{2k+1}}{(2k+1)!} =$$

$$Cos(z) - iSin(z)$$

given that we are subtracting the Sine series. Therfore it holds that:

$$e^{-zi} = Cos(z) - iSin(z)$$

Now given that:

$$e^{-zi} = Cos(z) - iSin(z)$$

and

$$e^{zi} = Cos(z) + iSin(z)$$

for $z \in \mathbb{C}$

$$\frac{e^{zi} + e^{-zi}}{2} = \frac{Cos(z) + iSin(z) + Cos(z) - iSin(z)}{2} = \frac{2Cos(z)}{2} = Cos(z)$$

$$\frac{e^{zi} - e^{-zi}}{2i} = \frac{Cos(z) + iSin(z) - Cos(z) - iSin(z)}{2i} = \frac{2iSin(z)}{2i} = Sin(z)$$

Therefore we have that:

$$Cos(z) = \frac{e^{zi} + e^{-zi}}{2}$$

$$Sin(z) = \frac{e^{zi} - e^{-zi}}{2i}$$

Then we have that

$$(Cos(z))^2 + (Sin(z))^2 = (\frac{e^{zi} + e^{-zi}}{2})^2 + (\frac{e^{zi} - e^{-zi}}{2i})^2 =$$

$$\frac{(e^{zi} + e^{-zi})(e^{zi} + e^{-zi})}{4} + \frac{(e^{zi} - e^{-zi})(e^{zi} - e^{-zi})}{4i^2} =$$

$$\frac{e^{2zi} + 2e^0 + e^{-2zi}}{4} - (\frac{e^{2zi} - 2e^0 + e^{-2zi}}{4}) = \frac{4e^0}{4} = e^0 = 1$$

Therefore we have that $\forall z \in \mathbb{C}$:

$$(Cos(z))^2 + (Sin(z))^2 = 1$$

7 Exponential is a homomorphism

We have that the exponential function extends to complex numbers such that for $z \in \mathbb{C}$ it is the case that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, we let $x, y \in \mathbb{C}$. We iterate the fact that $\mathbb{R} \subset \mathbb{C}$. Then by properties of complex numbers $(x+y) \in \mathbb{C}$ and:

$$e^{x+y} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

We recall that the binomial theorem states that:

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$$

Then applying the binomial theorem to the above form of e^{x+y}:

$$e^{x+y} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\binom{k}{j} x^{k-j} y^j}{k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\frac{k!}{(k-j)!(j!)} x^{k-j} y^j}{k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{x^{k-j} y^j}{(k-j)!(j!)}$$

Here we recall the Cauchy Product which implies that for the infinite series $\sum_{i=0}^{\infty} a_i$ and $\sum_{l=0}^{\infty} b_l$:

$$(\sum_{i=0}^{\infty} a_i)(\sum_{l=0}^{\infty} b_l) = \sum_{k=0}^{\infty} c_k$$

Where $c_k = \sum_{j=0}^{k} a_j \cdot b_{k-j}$ we let $a_j = \frac{y^j}{(j!)}$ and $b_{k-j} = \frac{x^{k-j}}{(k-j)!}$ then:

$$e^{x+y} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{x^{k-j} y^j}{(k-j)!(j!)} = (\sum_{i=0}^{\infty} \frac{y^i}{(i!)})(\sum_{l=0}^{\infty} \frac{x^l}{l!}) = (e^y)(e^x)$$

Therefore we can conclude that:

$$e^{x+y} = e^x \cdot e^y$$

and therefore the exponential function is a homomorphism.

8 for all $z \in \mathbb{C}, exp(z) \neq 0$

Given that e^z is equal to $exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ for $z \in \mathbb{C}$, defining $exp(z) : \mathbb{C} \to \mathbb{C}$. Suppose the complex exponential e^z letting $z \in \mathbb{C}$, such that $z = a + bi$, we have already shown the property that

$$e^{bi} = cos(b) + isin(b)$$

then

$$e^z = e^{a+bi} = e^z \cdot e^{bi} = e^a(cos(b) + isin(b))$$

Noting the properties of the real exponential e^x we note that there does not exit $x \in \mathbb{R}$ such that $e^x = 0$. Then it must be that $e^a \neq 0$. Further suppose that

$$cos(b) + isin(b) = 0$$

then it must be that $cos(b) = sin(b) = 0$ since it is the only way that $cos(b) + isin(b) = 0$ and if $cos(b) = sin(b) = 0$ then

$$cos(b)^2 + sin(b)^2 = 0$$

, this is impossible since we have already proved, for $b \in \mathbb{R}$

$$cos(b)^2 + sin(b)^2 = 1$$

. Therefore since $e^a \neq 0$ and $cos(b) + isin(b) \neq 0$ then it must follow that $\forall z \in \mathbb{C}$:

$$e^z = e^{a+bi} = e^z \cdot e^{bi} = e^a(cos(b) + isin(b)) \neq 0$$

Therefore for all $z \in \mathbb{C}, exp(z) \neq 0$

9 $e^{\pi i} + 1 = 0$

we showed that $e^{bi} = cos(b) + isin(b)$ in section 6, we can now show that $e^{\pi i} + 1 = 0$:

$$e^{\pi i} = cos(\pi) + isin(\pi)$$

since $cos(\pi) = -1$ and since $sin(\pi) = 0$ then

$$e^{\pi i} = cos(\pi) + isin(\pi) = -1 + 0 = -1$$

and so

$$e^{\pi i} + 1 = cos(\pi) + isin(\pi) + 1 = -1 + 0 + 1 = -1 + 1 = 0$$

therefore it follows that

$$e^{\pi i} + 1 = 0$$

10 Conclusion

In this paper we proved some properties and relations of the exponential. We observed a couple of things, we observed that we can express $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = e$ letting e be a constant. Further we showed that $e \lim_{n\to\infty}(1+\frac{1}{n})^n$ showing that the constant e had an inherent relationship to a binomial form. From this point we showed that using the sequence $\sum_{k=0}^{n}\frac{x^k}{k!}$, essentially polynomials of real numbers, we could define continuous functions that extending from the rational numbers to the reals, and from the reals to the reals. This was done hinged on the fact that the sequence showed itself to converge for all reals and therefore all rationals. Here we reached a critical point we noted that we could define a function $exp : \mathbb{R} \to \mathbb{R}$ and state it as simply e^x this allowed for an attempt at showing differentiability and the form of differentiating of this function. We showed that e^x once more showed continuity on R and critically used the existence of the constant e to show that the function was differentiable. With some manipulation we found that for all reals, the derivative of e^x was itself, that is e^x. We have that the trigonometric functions cosine and sine are polynomials as well, via the taylor expansion. We expanded our exponential to the complex, and showed a very fundamental and important result: Eulers formula. We showed that applying the same sequence to a specific form of the complex is equivalent to a cosine term plus the imaginary unit scaled by a sine term. This result allowed us to extract properties of equivalence to cosine and sine, in complex terms. With these knew derived properties we showed that the sum of $cos(z)^2 + sin(z)^2 = 1$ for all complex numbers, Finally after having shown this property we showed that the exponential function was a homomorphism, by showing that the exponential of the sum of two numbers is equivalent to the product of the product of the exponentials of each particular number.

11 Works Cited by proof number

SECTION 3

- uniform convergence jerry shuman pdf

SECTION 4

- Rudin, Principles of Mathematical Analysis, Theorem 3.39 and Theorem 8.1.