1 Asymmetric Random Walk

$$X_n = X_{n-1} + \Delta x B_n \qquad \text{with initial } X_0 = x_0 \tag{1}$$

$$B_n := \begin{cases} 1 & \text{with probability} \quad p \\ -1 & \text{with probability} \quad 1-p \end{cases} \tag{2}$$

$$X_n = x_0 + \Delta x \sum_{i=1}^{n} B_i \;=\; x_0 + \Delta x D_n, \quad \text{where} \quad D_n = 2m - n,\; m \in \{0, 1, 2, \ldots, n\} \tag{3}$$

$$D_n \sim \mathrm{Bino}(n, m) \qquad \text{and} \qquad \Pr(D_n = 2m - n) \;=\; \binom{n}{m} p^m (1-p)^{n-m} \tag{4}$$

$$f_{X_n}(x) = \binom{n}{m} p^m (1-p)^{n-m} \mathbb{1}_{\{m \in \mathbb{N}_0\}} \quad \text{where} \quad m = \frac{x - (x_0 - n\Delta x)}{2\Delta x} \tag{5}$$

$$M_{X_n}(\alpha) = \mathrm{E}[e^{\alpha X_n}] \;=\; (1 - p + pe^{2\alpha\Delta x})^n e^{\alpha(x_0 - n\Delta x)} \tag{6}$$

$$\mathrm{E}[X_n] = x_0 + n\Delta x(2p - 1) \tag{7}$$

$$\mathrm{Var}(X_n) = 4(\Delta x)^2 np(1-p) \tag{8}$$

$$\mathrm{Skew}(X_n) = \frac{8(\Delta x)^3 np(1-p)(1-2p)}{\left(\sqrt{\mathrm{Var}(X_n)}\right)^3} \tag{9}$$

$$\mathrm{Kurt}(X_n) = \frac{16(\Delta x)^4 np(p-1)\big(3(n-2)p^2 - 3(n-2)p + 1\big)}{\big(\mathrm{Var}(X_n)\big)^2} \tag{10}$$

$$\mathrm{E}[(X_n - K)^+] = \sum_{m=\max\left(0, \left\lceil \frac{K - (x_0 - n\Delta x)}{2\Delta x} \right\rceil\right)}^{n} \binom{n}{m} p^m (1-p)^{n-m} \big((2m - n)\Delta x + x_0 - K\big) \tag{11}$$

$$\mathrm{E}[(K - X_n)^+] = \sum_{m=0}^{\min\left(n, \left\lfloor \frac{K - (x_0 - n\Delta x)}{2\Delta x} \right\rfloor\right)} \binom{n}{m} p^m (1-p)^{n-m} \big(K - x_0 - (2m - n)\Delta x\big) \tag{12}$$

2 Brownian Motion with Drift

$$
\begin{aligned}
dX_t &= \mu dt + \sigma dW_t, \quad \text{where } W_t \sim \mathcal{N}(0,t) && (13)\\
X_{t+\Delta t} &= X_t + \mu \Delta t + \sigma \sqrt{\Delta t} Z, \quad \text{where } Z \sim \mathcal{N}(0,1) && (14)\\
X_t &\sim \mathcal{N}(x_0 + \mu t, \sigma^2 t) && (15)\\
f_{X_t}(x) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} && (16)\\
\mathrm{E}[X_t] &= x_0 + \mu t && (17)\\
\mathrm{Var}(X_t) &= \sigma^2 t && (18)\\
\mathrm{Skew}(X_t) &= 0 && (19)\\
\mathrm{Kurt}(X_t) &= 3 && (20)\\
\mathrm{E}[(X_t - K)^+] &= \int_K^\infty (x-K) f_{X_t}(x) dx && (21)\\
&= \sigma^2 t \; \varphi\left(\frac{x_0 + \mu t - K}{\sigma\sqrt{t}}\right) + (x_0 + \mu t - K) \; \Phi\left(\frac{x_0 + \mu t - K}{\sigma\sqrt{t}}\right) && (22)\\
\mathrm{E}[(K - X_t)^+] &= \int_{-\infty}^K (K-x) f_{X_t}(x) dx && (23)\\
&= \sigma^2 t \; \varphi\left(\frac{K - x_0 - \mu t}{\sigma\sqrt{t}}\right) + (K - x_0 - \mu t) \; \Phi\left(\frac{K - x_0 - \mu t}{\sigma\sqrt{t}}\right) && (24)
\end{aligned}
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from A. RW to BM with D.

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3 Geometric Random Walk

Let $Y_n = \exp(X_n)$, where X_n is Asymmetric Random Walk.

$$Y_n = \exp\left(X_{n-1} + \Delta x B_n\right) = Y_{n-1} \times G_n \qquad \text{with initial } Y_0 = y_0 \tag{25}$$

$$G_n := \begin{cases} u & \text{with probability} \quad p \\ d & \text{with probability} \quad 1-p \end{cases} \qquad \text{where } d = \frac{1}{u} \tag{26}$$

$$Y_n = y_0 \prod_{i=1}^{n} G_i = y_0 \times H_n, \quad \text{where} \quad H_n = u^{2m-n},\ m \in \{0, 1, 2, \ldots, n\} \tag{27}$$

$$G_n \sim \text{Bino}(n, m) \qquad \text{and} \qquad \Pr\left(G_n = u^{2m-n}\right) = \binom{n}{m} p^m (1-p)^{n-m} \tag{28}$$

$$f_{Y_n}(y) = \binom{n}{m} p^m (1-p)^{n-m} \mathbb{1}_{\{m \in \mathbb{N}_0\}} \quad \text{where} \quad m = \frac{\ln y - (\ln y_0 - n \ln u)}{2 \ln u} \tag{29}$$

$$\mathrm{E}[e^{\alpha \ln Y_n}] = (1 - p + p e^{2\alpha \ln u})^n e^{\alpha \ln\left(\frac{y_0}{u^n}\right)} \tag{30}$$

$$\mathrm{E}[Y_n] = \frac{y_0}{u^n} \left(1 + (u^2 - 1)p\right)^n \tag{31}$$

$$\mathrm{Var}(Y_n) = \left(\frac{y_0}{u^n}\right)^2 \left[\left(1 + (u^4 - 1)p\right)^n - \left(1 + (u^2 - 1)p\right)^{2n}\right] \tag{32}$$

$$\mathrm{Skew}(Y_n) = \frac{\left(\frac{y_0}{u^n}\right)^3 \left[\left(1 + (u^6 - 1)p\right)^n - 3\left(1 + (u^4 - 1)p\right)^n \left(1 + (u^2 - 1)p\right)^n + 2\left(1 + (u^2 - 1)p\right)^{3n}\right]}{\left(\sqrt{\mathrm{Var}(Y_n)}\right)^3} \tag{33}$$

$$\begin{aligned} \mathrm{Kurt}(Y_n) &= \frac{\left(\frac{y_0}{u^n}\right)^4 \left[\left(1 + (u^8 - 1)p\right)^n - 4\left(1 + (u^6 - 1)p\right)^n \left(1 + (u^2 - 1)p\right)^n\right]}{\left(\mathrm{Var}(Y_n)\right)^2} \\ &\quad + \frac{\left(\frac{y_0}{u^n}\right)^4 \left[6\left(1 + (u^4 - 1)p\right)^n \left(1 + (u^2 - 1)p\right)^{2n} - 3\left(1 + (u^2 - 1)p\right)^{4n}\right]}{\left(\mathrm{Var}(Y_n)\right)^2} \end{aligned} \tag{34}$$

$$\mathrm{E}[(Y_n - K)^+] = \sum_{m=\max\left(0, \left\lceil \frac{\ln K - (\ln y_0 - n \ln u)}{2 \ln u} \right\rceil\right)}^{n} \binom{n}{m} p^m (1-p)^{n-m} \left(y_0 u^{2m-n} - K\right) \tag{35}$$

$$\mathrm{E}[(K - Y_n)^+] = \sum_{m=0}^{\min\left(n, \left\lfloor \frac{\ln K - (\ln y_0 - n \ln u)}{2 \ln u} \right\rfloor\right)} \binom{n}{m} p^m (1-p)^{n-m} \left(K - y_0 u^{2m-n}\right) \tag{36}$$

4 Geometric Brownian Motion

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\begin{aligned}
dY_t &= \mu Y_t dt + \sigma Y_t dW_t, \quad \text{where } W_t \sim \mathcal{N}(0,t) && (37)\\
Y_{t+\Delta t} &= Y_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z\right), \quad \text{where } Z \sim \mathcal{N}(0,1) && (38)\\
Y_t &\sim \mathrm{LN}\left(\ln y_0 + \nu_t, \sigma^2 t\right), \qquad \text{where} \quad \nu_t = \left(\mu - \frac{\sigma^2}{2}\right)t && (39)\\
f_{Y_t}(y) &= \frac{1}{y\sqrt{2\pi\sigma^2 t}} e^{-\frac{\left(\ln\left(\frac{y}{y_0}\right) - \nu_t\right)^2}{2\sigma^2 t}} && (40)\\
\mathrm{E}[Y_t] &= y_0 e^{\mu t} && (41)\\
\mathrm{Var}(Y_t) &= y_0^2\left(e^{\sigma^2 t} - 1\right)e^{2\mu t} && (42)\\
\mathrm{Skew}(Y_t) &= \left(e^{\sigma^2 t} + 2\right)\sqrt{e^{\sigma^2 t} - 1} && (43)\\
\mathrm{Kurt}(Y_t) &= e^{4\sigma^2 t} + 2e^{3\sigma^2 t} + 3e^{2\sigma^2 t} - 3 && (44)\\
\mathrm{E}[(Y_t - K)^+] &= \int_K^\infty (y - K) f_{Y_t}(y)dx && (45)\\
&= y_0 e^{\mu t}\Phi\left(-\frac{\ln\left(\frac{K}{y_0}\right) - \nu_t}{\sigma\sqrt{t}} + \sigma\sqrt{t}\right) - K\Phi\left(-\frac{\ln\left(\frac{K}{y_0}\right) - \nu_t}{\sigma\sqrt{t}}\right) && (46)\\
\mathrm{E}[(K - Y_t)^+] &= \int_{-\infty}^K (K - y) f_{Y_t}(y)dx && (47)\\
&= K\Phi\left(\frac{\ln\left(\frac{K}{y_0}\right) - \nu_t}{\sigma\sqrt{t}}\right) - y_0 e^{\mu t}\Phi\left(\frac{\ln\left(\frac{K}{y_0}\right) - \nu_t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) && (48)
\end{aligned}
$$