

1 Asymmetric Random Walk

$$X_n = X_{n-1} + \Delta x B_n \quad \text{with initial } X_0 = x_0 \quad (1)$$

$$B_n := \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases} \quad (2)$$

$$X_n = x_0 + \Delta x \sum_{i=1}^n B_i = x_0 + \Delta x D_n, \quad \text{where } D_n = 2m - n, \quad m \in \{0, 1, 2, \dots, n\} \quad (3)$$

$$D_n \sim \text{Bino}(n, m) \quad \text{and} \quad \Pr(D_n = 2m - n) = \binom{n}{m} p^m (1-p)^{n-m} \quad (4)$$

$$f_{X_n}(x) = \binom{n}{m} p^m (1-p)^{n-m} \mathbb{1}_{\{m \in \mathbb{N}_0\}} \quad \text{where } m = \frac{x - (x_0 - n\Delta x)}{2\Delta x} \quad (5)$$

$$M_{X_n}(\alpha) = \mathbb{E}[e^{\alpha X_n}] = (1-p + pe^{2\alpha\Delta x})^n e^{\alpha(x_0 - n\Delta x)} \quad (6)$$

$$\mathbb{E}[X_n] = x_0 + n\Delta x(2p-1) \quad (7)$$

$$\text{Var}(X_n) = 4(\Delta x)^2 np(1-p) \quad (8)$$

$$\text{Skew}(X_n) = \frac{8(\Delta x)^3 np(1-p)(1-2p)}{(\sqrt{\text{Var}(X_n)})^3} \quad (9)$$

$$\text{Kurt}(X_n) = \frac{16(\Delta x)^4 np(1-p)(1+3(n-2)p-3(n-2)p^2)}{(\text{Var}(X_n))^2} \quad (10)$$

$$\mathbb{E}[(X_n - K)^+] = \sum_{m=\max\left(0, \left\lceil \frac{K - (x_0 - n\Delta x)}{2\Delta x} \right\rceil\right)}^n \binom{n}{m} p^m (1-p)^{n-m} ((2m-n)\Delta x + x_0 - K) \quad (11)$$

$$\mathbb{E}[(K - X_n)^+] = \sum_{m=0}^{\min\left(n, \left\lfloor \frac{K - (x_0 - n\Delta x)}{2\Delta x} \right\rfloor\right)} \binom{n}{m} p^m (1-p)^{n-m} (K - x_0 - (2m-n)\Delta x) \quad (12)$$

2 Brownian Motion with Drift

$$dX_t = \mu dt + \sigma dW_t, \quad \text{where } W_t \sim \mathcal{N}(0, t) \quad (13)$$

$$X_{t+\Delta t} = X_t + \mu\Delta t + \sigma\sqrt{\Delta t}Z, \quad \text{where } Z \sim \mathcal{N}(0, 1) \quad (14)$$

$$X_t \sim \mathcal{N}(x_0 + \mu t, \sigma^2 t) \quad (15)$$

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} \quad (16)$$

$$\mathbb{E}[X_t] = x_0 + \mu t \quad (17)$$

$$\text{Var}(X_t) = \sigma^2 t \quad (18)$$

$$\text{Skew}(X_t) = 0 \quad (19)$$

$$\text{Kurt}(X_t) = 3 \quad (20)$$

$$\mathbb{E}[(X_t - K)^+] = \int_K^\infty (x - K) f_{X_t}(x) dx \quad (21)$$

$$= \sigma^2 t \varphi\left(\frac{x_0 + \mu t - K}{\sigma\sqrt{t}}\right) + (x_0 + \mu t - K) \Phi\left(\frac{x_0 + \mu t - K}{\sigma\sqrt{t}}\right) \quad (22)$$

$$\mathbb{E}[(K - X_t)^+] = \int_{-\infty}^K (K - x) f_{X_t}(x) dx \quad (23)$$

$$= \sigma^2 t \varphi\left(\frac{K - x_0 - \mu t}{\sigma\sqrt{t}}\right) + (K - x_0 - \mu t) \Phi\left(\frac{K - x_0 - \mu t}{\sigma\sqrt{t}}\right) \quad (24)$$

from A. RW to BM with D.

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3 Geometric Random Walk

Let $Y_n = \exp(X_n)$, where X_n is Asymmetric Random Walk.

$$Y_n = \exp(X_{n-1} + \Delta x B_n) = Y_{n-1} \times G_n \quad \text{with initial } Y_0 = y_0 \quad (25)$$

$$G_n := \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases} \quad \text{where } d = \frac{1}{u} \quad (26)$$

$$Y_n = y_0 \prod_{i=1}^n G_i = y_0 \times H_n, \quad \text{where } H_n = u^{2m-n}, \quad m \in \{0, 1, 2, \dots, n\} \quad (27)$$

$$G_n \sim \text{Bino}(n, m) \quad \text{and} \quad \Pr(G_n = u^{2m-n}) = \binom{n}{m} p^m (1-p)^{n-m} \quad (28)$$

$$f_{Y_n}(y) = \binom{n}{m} p^m (1-p)^{n-m} \mathbf{1}_{\{m \in \mathbb{N}_0\}} \quad \text{where } m = \frac{\ln y - (\ln y_0 - n \ln u)}{2 \ln u} \quad (29)$$

$$\mathbb{E}[e^{\alpha \ln Y_n}] = (1-p + pe^{2\alpha \ln u})^n e^{\alpha \ln(\frac{y_0}{u^n})} \quad (30)$$

$$\mathbb{E}[Y_n] = \frac{y_0}{u^n} (1 + (u^2 - 1)p)^n \quad (31)$$

$$\text{Var}(Y_n) = \left(\frac{y_0}{u^n}\right)^2 \left[(1 + (u^4 - 1)p)^n - (1 + (u^2 - 1)p)^{2n} \right] \quad (32)$$

$$\text{Skew}(Y_n) = \frac{\left(\frac{y_0}{u^n}\right)^3 \left[(1 + (u^6 - 1)p)^n - 3(1 + (u^4 - 1)p)^n (1 + (u^2 - 1)p)^n + 2(1 + (u^2 - 1)p)^{3n} \right]}{\left(\sqrt{\text{Var}(Y_n)}\right)^3} \quad (33)$$

$$\begin{aligned} \text{Kurt}(Y_n) &= \frac{\left(\frac{y_0}{u^n}\right)^4 \left[(1 + (u^8 - 1)p)^n - 4(1 + (u^6 - 1)p)^n (1 + (u^2 - 1)p)^n \right]}{(\text{Var}(Y_n))^2} \\ &+ \frac{\left(\frac{y_0}{u^n}\right)^4 \left[6(1 + (u^4 - 1)p)^n (1 + (u^2 - 1)p)^{2n} - 3(1 + (u^2 - 1)p)^{4n} \right]}{(\text{Var}(Y_n))^2} \end{aligned} \quad (34)$$

$$\mathbb{E}[(Y_n - K)^+] = \sum_{m=\max\left(0, \left\lceil \frac{\ln K - (\ln y_0 - n \ln u)}{2 \ln u} \right\rceil\right)}^n \binom{n}{m} p^m (1-p)^{n-m} (y_0 u^{2m-n} - K) \quad (35)$$

$$\mathbb{E}[(K - Y_n)^+] = \sum_{m=0}^{\min\left(n, \left\lfloor \frac{\ln K - (\ln y_0 - n \ln u)}{2 \ln u} \right\rfloor\right)} \binom{n}{m} p^m (1-p)^{n-m} (K - y_0 u^{2m-n}) \quad (36)$$

4 Geometric Brownian Motion

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t, \quad \text{where } W_t \sim \mathcal{N}(0, t) \quad (37)$$

$$Y_{t+\Delta t} = Y_t \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \sqrt{\Delta t} Z \right), \quad \text{where } Z \sim \mathcal{N}(0, 1) \quad (38)$$

$$Y_t \sim \text{LN}(\ln y_0 + \nu_t, \sigma^2 t), \quad \text{where } \nu_t = \left(\mu - \frac{\sigma^2}{2} \right) t \quad (39)$$

$$f_{Y_t}(y) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} e^{-\frac{(\ln(\frac{y}{y_0}) - \nu_t)^2}{2\sigma^2 t}} \quad (40)$$

$$\mathbb{E}[Y_t] = y_0 e^{\mu t} \quad (41)$$

$$\text{Var}(Y_t) = y_0^2 (e^{\sigma^2 t} - 1) e^{2\mu t} \quad (42)$$

$$\text{Skew}(Y_t) = (e^{\sigma^2 t} + 2) \sqrt{e^{\sigma^2 t} - 1} \quad (43)$$

$$\text{Kurt}(Y_t) = e^{4\sigma^2 t} + 2e^{3\sigma^2 t} + 3e^{2\sigma^2 t} - 3 \quad (44)$$

$$\mathbb{E}[(Y_t - K)^+] = \int_K^\infty (y - K) f_{Y_t}(y) dy \quad (45)$$

$$= y_0 e^{\mu t} \Phi \left(-\frac{\ln \left(\frac{K}{y_0} \right) - \nu_t}{\sigma \sqrt{t}} + \sigma \sqrt{t} \right) - K \Phi \left(-\frac{\ln \left(\frac{K}{y_0} \right) - \nu_t}{\sigma \sqrt{t}} \right) \quad (46)$$

$$\mathbb{E}[(K - Y_t)^+] = \int_{-\infty}^K (K - y) f_{Y_t}(y) dy \quad (47)$$

$$= K \Phi \left(\frac{\ln \left(\frac{K}{y_0} \right) - \nu_t}{\sigma \sqrt{t}} \right) - y_0 e^{\mu t} \Phi \left(\frac{\ln \left(\frac{K}{y_0} \right) - \nu_t}{\sigma \sqrt{t}} - \sigma \sqrt{t} \right) \quad (48)$$