Giry algebras

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Giry monad

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  $G\Omega$ becomes a measurable spaces by endowing it with the σ-algbra generated by the evaluation maps.

  For a measurable map $f : \Omega_1 \to \Omega_2$, **pushing forward** along f defines a measurable map $Gf : G\Omega_1 \to G\Omega_2$.

  This gives an **endofunctor** G : **Mble** $\to$ **Mble**.

- There is a measurable map $\mu_\Omega : G G\Omega \to G\Omega$:

$$\mu_\Omega(\mathbf{P})(A) := \int_{\lambda \in G\Omega} \lambda(A)\mathbf{P}(\mathrm{d}\lambda),$$

for all $\mathbf{P} \in GG\Omega$ and measurable subsets $A \subseteq \Omega$.

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- These form natural transformation $\mu : GG \to G$ and $\eta : 1_{\mathbf{Mble}} \to G$ and (G, μ, η) forms a monad, *the Giry monad*.

Examples of Giry algebras

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We have that
$$\int x\delta_{x_0}(\mathrm{d}x) = x_0$$
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  Therefore, $([0,1], \int)$ is a Giry algebra.

Examples of Giry algebras

- For the set $\{0, \infty\}$ we can define a map $\alpha : G(\{0, \infty\}) \to \{0, \infty\}$ by

$$\alpha\left(\lambda\delta_0 + \overline{\lambda}\delta_\infty\right) := \begin{cases} 0 & \text{if} \quad \lambda = 1, \\ \infty & \text{otherwise.} \end{cases}$$

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Other examples: $[0, 1)$, $[0, \infty]$, $[0, 1]^{\mathbb{R}}$, lattices, ...

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Examples: $\mathbb{R}$, vector spaces, lattices, ...

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Given a convex compact Hausdorff space K, a structure map $R(K) \to K$ can be defined by sending every probability measure to its *barycenter*.

Associated vector space

Let (C, γ) be an algebra of the distribution monad and let $M_f(C)$ be the vector space of finitely supported finite signed measures on X.

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Define the subspace

$$X := \left\{ \sum_{n=1}^{N} \lambda_n (p_n - \delta_{\gamma(p_n)}) \mid \lambda_n \in \mathbb{R}, p_n \in D(C) \text{ for all } n \right\}$$

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This defines a left adjoint functor $\mathbf{Set}^D \to \mathbf{Vect}$ to the forgetful functor $\mathbf{Vect} \to \mathbf{Set}^D$.

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$$\varphi_C : C \to W_C : c \mapsto [\delta_c].$$

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- For $a, b, c \in C$, $\lambda \in (0, 1]$

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- Convex morphisms $C \to [0, 1]$ separate points.

In this case we say that (C, γ) satisfies the (first) cancellation property (C1).

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Consider the subspace

$$Y := \{\mu \mid \mu(f) = 0 \quad \text{for all algebra morphisms } f : A \to [0,1]\}.$$

Now define

$$V_A := M(A)/Y.$$

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The family $(\rho_f)_f$ defines a Hausdorff LCTVS structure on V_A.

This defines a functor $\mathbf{Mble}^G \to \mathbf{LCTVS}$.

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In this case we say that (A, α) satisfies the second cancellation property (C2).

Cancellation properties

Proposition

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If a Giry algebra satisfies (C2), then it satisfies (C1).

For a counterexample of a Giry algebra that is (C1), but not (C2), the locally convex topological vector space V_A should be *infinite*-dimensional.

Topologizing Giry algebras

We will now study the topological properties of the subset $\phi_A(A) \subseteq V_A$.

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Theorem

Let (A, α) be a Giry algebra, then $\phi_A(A) \subseteq V_A$ is a **convex, relatively compact, Hausdorff** subspace.

Topologizing Giry algebras

Proof: The vector space $M(A)$ of finite signed measures becomes topological, by endowing it with the topology generated by the maps

$$\rho_f : M(A) \to \mathbb{R} : \mu \mapsto |\mu(f)|$$

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By the Banach-Alaoglu theorem, it follows that the subset

$$\left\{ \mu \in M(A) \mid \sup_{\|f\|_\infty < 1} |\mu(f)| \leq 1 \right\}$$

is a compact subset of $M(A)$.

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There is a continuous linear map $p : M(A) \to V_A$, hence $p(U)$ is a compact subset of V_A.

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There is a continuous linear map $p : M(A) \to V_A$, hence $p(U)$ is a compact subset of V_A.

Furthermore, we have that $\phi_A(A) \subseteq p(U)$ and therefore it is relatively compact.

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Every algebra morphism $f : A \to [0,1]$ factors uniquely through $A \to \phi_A(A)$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & {[0,1]} \\
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Proposition

A map $f : A \to [0,1]$ is an algebra morphism if and only if $\tilde{f} : \phi_A(A) \to [0,1]$ is convex and continuous.

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Proof: Let $\mathbb{P}$ be a probability measure on A. There exist a net $(p_i)_i$ such that

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$$\mathbb{P}(f) = \lim_i p_i(f) = \lim_i f(\alpha(p_i)) = \tilde{f}(\lim_i \phi_A(\alpha(p_i)) = \tilde{f}\phi_A\alpha(\mathbb{P}) = f(\alpha(\mathbb{P})).$$

Topologizing Giry algebras

Corollary

Let (A, α) be a Giry algebra such that V_A is *finite*-dimensional.

- (A, α) satisfies (C1) if and only if (A, α) satisfies (C2).
- Every convex map $f : A \to [0, 1]$ is an algebra morhpsism.

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Theorem

Let (A, α) be a Giry algebra. For a probability measure $\mathbb{P}$ on A,

$$\phi_A(\alpha(\mathbb{P})) = \int \phi_A \mathrm{d}\mathbb{P}.$$

[1] Note that it is bounded because $\overline{\phi_A(A)}$ is compact.

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Proof: Because $\phi_A(A)$ is a convex, relatively compact subset of V_A, we know that the Pettis integral exists. Let $g : V_A \to \mathbb{R}$ be a linear continuous functional.

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<u>Proof</u>: Because $\phi_A(A)$ is a convex, relatively compact subset of V_A, we know that the Pettis integral exists. Let $g : V_A \to \mathbb{R}$ be a linear continuous functional. Then $g \circ \phi_A$ is an algebra morphism[1]. Therefore,

$$g(\phi_A(\alpha(\mathbb{P}))) = \int g\phi_A \mathrm{d}\mathbb{P}.$$

By the defining property of the Pettis integral, the statement follows.

[1] Note that it is bounded because $\overline{\phi_A(A)}$ is compact.

Radon-Giry algebra adjunction

We will now construct an adjunction between Giry algebras and Radon algebras,

$$\begin{tikzcd} \mathbf{Mble}^G \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{CH}^R \arrow[l, shift left] \end{tikzcd}$$

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Given a Radon algebra (K, κ), we consider the measurable space K, whose σ-algebra is generated by the continuous convex maps $K \to [0, 1]$.

Using Pettis integration, we can define a structure map $GK \to K$, extending κ. This makes K into a Giry algebra.

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We obtain a **full and faithful** functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$.

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The functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$ is right adjoint to $L : \mathbf{Mble}^G \to \mathbf{CH}^R$.

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Theorem

The functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$ is right adjoint to $L : \mathbf{Mble}^G \to \mathbf{CH}^R$. Moreover, the unit is a natural isomorphism and $\eta_{(A,\alpha)}$ is monic if and only if (A, α) satisfies (C2).

Characterization of Giry algebras satisfying (C2)

Let **RelCompConv** be the category of relatively compact, convex subsets of locally convex topological vector spaces.

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We can decompose the previous adjunctions as follows:

$$\begin{tikzcd} \mathbf{Mble}^G \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{RelCompConv} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{CH}^R \arrow[l, shift left] \end{tikzcd}$$

Characterization of Giry algebras satisfying (C2)

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Theorem

The categories $\mathbf{Mble}^G_{C2}$ and **RelCompConv** are equivalent.

Infinite elements

Consider the Giry algebra $([0, \infty], \alpha)$, where $\alpha(\mathbb{P}) = \int_0^\infty x\mathbb{P}(\mathrm{d}x)$.

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\begin{aligned}
\alpha(\mathbb{P}) &= \int_0^\infty x\mathbb{P}(\mathrm{d}x) \\
&= \lim_n \int_0^n x\mathbb{P}(\mathrm{d}x) \\
&= \lim_n \frac{1}{\mathbb{P}([0,n])} \int_0^n x\mathbb{P}(\mathrm{d}x) \\
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Note that $[0,n]$ is an algebra that does satisfy (C2).

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Let (A, α) be a Giry algebra.

Definition

An element $c \in A$ is an **infinite element** if there *exists* distinct $a, b \in A$ and $\lambda \in (0, 1)$ such that

$$\lambda a + \overline{\lambda} c = \lambda b + \overline{\lambda} c.$$

Example: ∞ in $[0, \infty]$.

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Definition

For a (C2) subalgebra B of A and an infinite element $c \in A$ such that *for all* distinct $a, b \in B$ and $\lambda \in (0, 1)$,

$$\lambda a + \overline{\lambda} c = \lambda b + \overline{\lambda} c,$$

we write $\mathbf{B} \leq \mathbf{c}$.

Example: $[0, n] \times \{0\} \leq (\infty, 0)$ in $[0, \infty]^2$ for all $n \geq 0$.

Infinite elements

Definition

For infinite elements $c_1, c_2 \in A$, we write $\mathbf{c_1} \leq \mathbf{c_2}$ if for all $\lambda \in (0, 1]$,

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<u>Example</u>: $(\infty, 0) \leq (\infty, \infty)$ in $[0,\infty]^2$.

This defines a **poset** structure on the set $\mathrm{Inf}(A)$ of infinite elements in A.

Infinite elements

For an infinite element c, consider the collection

$$S_c := \{B \subset A \mid B \leq c \text{ and } B \text{ is a (C2) subalgebra.}\}$$

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For an infinite element c, consider the collection

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Define $B_c := \bigcup S_c$. For $c_1 \leq c_2$, we have that $B_{c_1} \subseteq B_{c_2}$.

Goal: Can we make sense of the following?

- For a probability measure $\mathbb{P}$ on B_c,

$$\text{"} \quad \alpha(\mathbb{P}) = \lim_{B \in S_c} \int \phi_B \mathrm{d}\mathbb{P} \quad \text{"}.$$

- For a probability measure $\mathbb{P}$ on A,

$$\text{"} \quad \alpha(\mathbb{P}) = \lim_{c \in \mathrm{Inf}(A)} \alpha(\mathbb{P}_{B_c}).$$

Giry algebra-valued random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (A, α) be a Giry algebra.

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Example: For a random variable f taking values in the Giry algebra $([0,1], \int)$, we have

$$\mathbb{E}[f] = \int(\mathbb{P} \circ f^{-1}) = \int x \mathbb{P} \circ f^{-1}(\mathrm{d}x) = \int f \mathrm{d}\mathbb{P}.$$

Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

For a random variable f taking values in a Giry algebra (A, α), a **conditional expectation of f with respect to $\mathcal{G}$** is a $\mathcal{G}$-measurable random variable $g : \Omega \to A$ such that

$$\alpha(\mathbb{P}_E \circ f^{-1}) = \alpha(\mathbb{P}_E \circ g^{-1})$$

for all $E \in \mathcal{G}$ such that $\mathbb{P}(E) \neq 0$.

Here $\mathbb{P}_E$ is defined as $\frac{\mathbb{P}(- \cap E)}{\mathbb{P}(E)}$.

Conditional expectation

Proposition

Let (A, α) be a σ-algebra such that V_A satisfies the *Radon-Nikodym property*.

Conditional expectation

Proposition

Let (A, α) be a σ-algebra such that V_A satisfies the *Radon-Nikodym property*. Then conditional expectation of random variables valued in A *exist* and are *almost surely unique*.

References

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