Giry algebras

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For a measurable map $f: \Omega_1 \to \Omega_2$, **pushing forward** along f defines a measurable map $Gf: G\Omega_1 \to G\Omega_2$.

This gives an **endofunctor** $G : \mathbf{Mble} \to \mathbf{Mble}$.

• There is a measurable map $\mu_{\Omega}: GG\Omega \to G\Omega$:

$$\mu_{\Omega}(\mathbf{P})(A) := \int_{\lambda \in G\Omega} \lambda(A) \mathbf{P}(\mathrm{d}\lambda),$$

for all $\mathbf{P} \in GG\Omega$ and measurable subsets $A \subseteq \Omega$.

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• These form natural transformation $\mu: GG \to G$ and $\eta: 1_{\mathbf{Mble}} \to G$ and (G, μ, η) forms a monad, the Giry monad.

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We have that

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Therefore, $([0,1], \int)$ is a Giry algebra.

 \bullet For the set $\{0,\infty\}$ we can define a map $\alpha: {\it G}(\{0,\infty\}) \to \{0,\infty\}$ by

$$\alpha \left(\lambda \delta_0 + \overline{\lambda} \delta_\infty \right) := \begin{cases} 0 & \text{if } \lambda = 1, \\ \infty & \text{otherwise.} \end{cases}$$

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We have that $\alpha(\delta_0)=0$ and $\alpha(\delta_\infty)=\infty$. For a probability measure **P** on $G(\{0,\infty\})$, we have that

$$\mathbf{P}(\alpha^{-1}(0)) = 1 \Leftrightarrow \mathbf{P}(\delta_0) = 1 \Leftrightarrow \mu(\mathbf{P})(0) = 1.$$

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Other examples: [0,1), $[0,\infty]$, $[0,1]^{\mathbb{R}}$, lattices, . . .

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The algebras of this monad are **convex compact Hausdorff spaces**, i.e. compact convex subsets of (Hausdorff) locally convex topological vector spaces [3].

Given a convex compact Hausdorff space K, a structure map $R(K) \to K$ can be defined by sending every probability measure to its *barycenter*.

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Define the subspace

$$X := \left\{ \sum_{n=1}^N \lambda_n (p_n - \delta_{\gamma(p_n)}) \mid \lambda_n \in \mathbb{R}, p_n \in D(C) \text{ for all } n
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This defines a left adjoint functor $\mathbf{Set}^D \to \mathbf{Vect}$ to the forgetful functor $\mathbf{Vect} \to \mathbf{Set}^D$.

For a distribution monad algebra (C, γ) , there is a canonical convex map

$$\varphi_C:C\to W_C:c\mapsto [\delta_c].$$

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Theorem (Stone [4])

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- φ_C is injective.
- For $a, b, c \in C$, $\lambda \in (0, 1]$

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• Convex morphisms $C \rightarrow [0, 1]$ separate points.

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In this case we say that (C, γ) satisfies the (first) cancellation property (C1).

Associated LCTVS

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Consider the subspace

$$Y:=\{\mu\mid \mu(f)=0\quad \text{for all algebra morphisms } f:A o [0,1]\}\,.$$

Now define

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We will now give V_A a LCTVS structure.

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For an algebra morphism $f:A \rightarrow [0,1]$, the assignment

$$[\mu] \mapsto \left| \int f \mathrm{d}\mu \right|$$

forms a seminorm $\rho_f: V_A \to \mathbb{R}$.

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The family $(\rho_f)_f$ defines a Hausdorff LCTVS structure on V_A .

This defines a functor $\mathbf{Mble}^G \to \mathbf{LCTVS}$.

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If a Giry algebra satisfies (C2), then it satisfies (C1).

For a counterexample of a Giry algebra that is (C1), but not (C2), the locally convex topological vector space V_A should be *infinite*-dimensional.

We will now study the topological properties of the subset $\phi_A(A) \subseteq V_A$.

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Theorem

Let (A, α) be a Giry algebra, then $\phi_A(A) \subseteq V_A$ is a **convex, relatively compact, Hausdorff** subspace.

<u>Proof</u>: The vector space M(A) of finite signed measures becomes topological, by endowing it with the topology generated by the maps

$$\rho_f: \mathcal{M}(A) \to \mathbb{R}: \mu \mapsto |\mu(f)|$$

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By the Banach-Alaoglu theorem, it follows that the subset

$$\left\{\mu \in M(A) \mid \sup_{\|f\|_{\infty} < 1} |\mu(f)| \le 1\right\}$$

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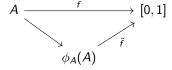
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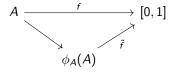
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Furthermore, we have that $\phi_A(A) \subseteq p(U)$ and therefore it is relatively compact.

Every algebra morphism $f:A \to [0,1]$ factors uniquely through $A \to \phi_A(A)$.



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Proposition

A map $f:A\to [0,1]$ is an algebra morphism if and only if $\tilde f:\phi_A(A)\to [0,1]$ is convex and continuous.

<u>Proof</u>: Let \mathbb{P} be a probability measure on A. There exist a net $(p_i)_i$ such that

$$p_i(g) \to \mathbb{P}(g)$$

for all convex morphisms $g: A \rightarrow [0,1]$.

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for all convex morphisms $g:A \rightarrow [0,1].$ Then,

$$\mathbb{P}(f) = \lim_{i} p_{i}(f) = \lim_{i} f(\alpha(p_{i})) = \tilde{f}(\lim_{i} \phi_{A}(\alpha(p_{i}))) = \tilde{f}\phi_{A}\alpha(\mathbb{P}) = f(\alpha(\mathbb{P})).$$

Corollary

Let (A, α) be a Giry algebra such that V_A is *finite*-dimensional.

- (A, α) satisfies (C1) if and only if (A, α) satisfies (C2).
- Every convex map $f: A \rightarrow [0,1]$ is an algebra morhpsism.

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Let (A, α) be a Giry algebra. For a probability measure \mathbb{P} on A,

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<u>Proof</u>: Because $\phi_A(A)$ is a convex, relatively compact subset of V_A , we know that the Pettis integral exists. Let $g: V_A \to \mathbb{R}$ be a linear continuous functional.

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$$g(\phi_{\mathcal{A}}(\alpha(\mathbb{P}))) = \int g\phi_{\mathcal{A}} \mathsf{d}\mathbb{P}.$$

By the defining property of the Pettis integral, the statement follows.



¹Note that it is bounded because $\overline{\phi_A(A)}$ is compact.

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We obtain a **full and faithful** functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$.

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Theorem

The functor $R: \mathbf{CH}^R \to \mathbf{Mble}^G$ is right adjoint to $L: \mathbf{Mble}^G \to \mathbf{CH}^R$. Moreover, the unit is a natural isomorphism and $\eta_{(A,\alpha)}$ is monic if and only if (A,α) satisfies (C2).

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For the unit η of the first adjunction, we have for every Giry algebra (A, α) that

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For the unit η of the first adjunction, we have for every Giry algebra (A, α) that $\eta_{(A,\alpha)}$ is an isomorphism \Leftrightarrow (A,α) satisfies (C2).

Theorem

The categories \mathbf{Mble}_{C2}^G and $\mathbf{RelCompConv}$ are equivalent.

Consider the Giry algebra $([0,\infty],\alpha)$, where $\alpha(\mathbb{P})=\int_0^\infty x\mathbb{P}(\mathrm{d}x)$.

Consider the Giry algebra ([0, ∞], α), where $\alpha(\mathbb{P}) = \int_0^\infty x \mathbb{P}(dx)$. This algebra is not (C2).

Consider the Giry algebra ($[0,\infty],\alpha$), where $\alpha(\mathbb{P})=\int_0^\infty x\mathbb{P}(\mathrm{d}x)$. This algebra is not (C2). For a probability measure \mathbb{P} on $[0,\infty)$,

$$\alpha(\mathbb{P}) = \int_0^\infty x \mathbb{P}(dx)$$

$$= \lim_n \int_0^n x \mathbb{P}(dx)$$

$$= \lim_n \frac{1}{\mathbb{P}([0, n])} \int_0^n x \mathbb{P}(dx)$$

$$= \lim_n \alpha \left(\frac{\mathbb{P}(-\cap [0, n])}{\mathbb{P}([0, n])}\right).$$

Consider the Giry algebra ($[0,\infty],\alpha$), where $\alpha(\mathbb{P})=\int_0^\infty x\mathbb{P}(\mathrm{d}x)$. This algebra is not (C2). For a probability measure \mathbb{P} on $[0,\infty)$,

$$\alpha(\mathbb{P}) = \int_0^\infty x \mathbb{P}(dx)$$

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Note that [0, n] is an algebra that does satisfy (C2).

Let (A, α) be a Giry algebra.

Definition

An element $c \in A$ is an **infinite element** if there *exists* distinct $a, b \in A$ and $\lambda \in (0,1)$ such that

$$\lambda a + \overline{\lambda} c = \lambda b + \overline{\lambda} c.$$

Example: ∞ in $[0, \infty]$.

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Definition

For a (C2) subalgebra B of A and an infinite element $c \in A$ such that for all distinct $a, b \in B$ and $\lambda \in (0, 1)$,

$$\lambda a + \overline{\lambda} c = \lambda b + \overline{\lambda} c,$$

we write $\mathbf{B} < \mathbf{c}$.

Example: $[0, n] \times \{0\} \le (\infty, 0)$ in $[0, \infty]^2$ for all $n \ge 0$.

Definition

For infinite elements $c_1, c_2 \in A$, we write $\mathbf{c_1} \leq \mathbf{c_2}$ if for all $\lambda \in (0,1]$,

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Example: $(\infty, 0) \le (\infty, \infty)$ in $[0, \infty]^2$.

This defines a **poset** structure on the set Inf(A) of infinite elements in A.

For an infinite element c, consider the collection

$$S_c := \{B \subset A \mid B \leq c \text{ and } B \text{ is a (C2) subalgebra.}\}$$

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Define $B_c := \bigcup S_c$. For $c_1 \leq c_2$, we have that $B_{c_1} \subseteq B_{c_2}$.

Goal: Can we make sense of the following?

• For a probability measure \mathbb{P} on B_c ,

"
$$\alpha(\mathbb{P}) = \lim_{B \in S_c} \int \phi_B d\mathbb{P}$$
 ".

• For a probability measure \mathbb{P} on A,

"
$$\alpha(\mathbb{P}) = \lim_{c \in Inf(A)} \alpha(\mathbb{P}_{B_c}).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (A, α) be a Giry algebra.

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We can define the expectation of f as follows:

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Example: For a random variable f taking values in the Giry algebra $([0,1], \int)$, we have

$$\mathbb{E}[f] = \int (\mathbb{P} \circ f^{-1}) = \int x \mathbb{P} \circ f^{-1}(\mathsf{d}x) = \int f \mathsf{d}\mathbb{P}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let \mathcal{G} be a sub- σ -algebra of \mathcal{F} .

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For a random variable f taking values in a Giry algebra (A, α) , a **conditional expectation of** f **with respect to** \mathcal{G} is a \mathcal{G} -measurable random variable $g:\Omega\to A$ such that

$$\alpha(\mathbb{P}_{\mathsf{E}}\circ f^{-1})=\alpha(\mathbb{P}_{\mathsf{E}}\circ g^{-1})$$

for all $E \in \mathcal{G}$ such that $\mathbb{P}(E) \neq 0$.

Here \mathbb{P}_E is defined as $\frac{\mathbb{P}(-\cap E)}{\mathbb{P}(E)}$.

Proposition

Let (A, α) be a σ -algebra such that V_A satisfies the Radon-Nikodym property.

Proposition

Let (A, α) be a σ -algebra such that V_A satisfies the *Radon-Nikodym property*. Then conditional expectation of random variables valued in A exist and are almost surely unique.

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