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Abstract. The 'quantum cognition' paradigm was recently challenged by its proven impossibility to simultaneously model 'question order effects' and 'response replicability'. In the present article we describe sequential dichotomic measurements within an operational and realistic framework for human cognition, and represent them in a quantum-like 'extended Bloch representation', where the Born rule of quantum probability does not necessarily hold. We then apply this mathematical framework to successfully model question order effects, response replicability and unpacking effects, thus opening the way toward 'quantum cognition beyond Hilbert space'.

Keywords: Cognitive modeling; quantum structures; general tension-reduction model; order effects; response replicability; unpacking effects

1 Introduction

'Quantum cognition' is the name given to the approaches that apply the mathematics of quantum theory in Hilbert space to model cognitive phenomena. Conjunctive and disjunctive fallacies, over- and under-extension effects in membership judgments, unpacking effects and expected utility paradoxes are some of the situations where quantum probabilistic approaches show significant advantages over the approaches in cognitive psychology that use classical probability theory (see, e.g., [1–12]). Notwithstanding this success, two well known experimental situations, 'question order effects' and 'response replicability', seriously challenge the acceptance of 'Hilbert space quantum cognition' as a universally valid paradigm in cognitive psychology [13–17], as we will emphasize in Sec. 2.

The difficulties of quantum approaches to model the statistics of responses of sequential questions where these cognitive effects occur led us to investigate origins and range of applicability of the Born rule of quantum probability. To this end, we firstly developed an operational and realistic framework to describe cognitive entities, states and context-induced changes of states, individual and sequential measurements, measurement outcomes and their probabilities, etc. [18]. This general framework is applied in Sec. 3 to the operational and realistic description of a wide class of dichotomic measurements, including those exhibiting the above mentioned cognitive effects.

We also elaborated a 'general tension-reduction (GTR) model' [21, 22] that extends some previous results obtained on the 'hidden measurements interpretation of quantum mechanics' [19]. This GTR-model, together with the associated 'extended Bloch representation (EBR)' [20], puts forward an explanation for the concrete effectiveness of the mathematical formalism of quantum theory in cognition. Indeed, the Born rule of quantum probability can be characterized in it as uniform fluctuations of the measurement context, and emerges as a 'universal average' over all possible forms of non-uniform fluctuations of the said context [20–22]. In this way, the GTR-model is also able to explain the difficulties of the Hilbert space formalism in the simultaneous modeling of question order effects and response replicability within a quantum-like framework where the Born rule does not generally hold [17].

Accordingly, in Sec. 4, we apply the GTR-model to represent dichotomic measurements that are performed individually and sequentially, together with the corresponding probabilities, and show that the GTR-model exhibits quantum-like aspects, although the Born rule of quantum probability is not generally valid in it. Hence, the GTR-model is generally non-Hilbertian. However, we also observe that the model is compatible with an operational and realistic framework for cognitive entities, and we also provide an intuitive illustration of how it can be interpreted in cognition.

In Sec. 5, we apply the model to the experimental data collected by Moore [24] and exhibiting question order effects. We then show, in Sec. 6, how question order effects and response replicability can be modeled together within the GTR-model, which is not the case in the Hilbert space quantum modeling [13, 14]. Finally, in Sec. 7, we observe that another cognitive effect, the 'unpacking effect', also requires a non-Kolmogorovian probability framework, like the one provided by the GTR-model, when unpacking effects are interpreted in terms of the relationship between measurements and sub-measurements.

2 Challenges to quantum cognition in Hilbert space

Quantum cognition in Hilbert space was recently challenged by 'question order effects' and 'response replicability' (the latter effect, however, is still waiting for a clear experimental confirmation). In the former, the response probabilities of two sequential questions, in an opinion poll, depend on the order in which the questions are asked, whereas in the latter the response to a given question should give the same outcome if repeated, regardless of whether another question is asked and answered in between [13]. More precisely, let us consider two dichotomic questions that are asked sequentially, in whatever order, on a sample of participants, such that probabilities of 'yes' and 'no' responses are collected as large number limits of statistical frequencies. The two questions thus correspond to two 'yes-no measurements' A and B. Their possible outcomes are 'yes' and 'no', which we denote by A_y, A_n, and B_y, B_n, respectively. Hence, performing first A then B produces the possible outcomes A_iB_j, while performing first B then A produces the possible outcomes B_jA_i, $i, j \in \{y, n\}$. A question order effect occurs when, in a given cognitive situation, the probability distribution of measurement outcomes depends on the order in which the two sequential measurements are performed, i.e. $p(A_iB_j) \neq p(B_jA_i)$.

Response replicability may instead appear in two forms, 'adjacent replicability' and 'separated replicability' [14]. Suppose that a measurement A (B) is performed twice sequentially in a given cognitive situation. Then, adjacent replicability requires that, if the outcome A_i (B_j) is obtained in the first measurement, then the same outcome A_i (B_j) should be obtained in the second measurement with certainty, i.e. with probability 1. Suppose now that the sequence of three measurements ABA (BAB) are performed in a given cognitive situation. Then, separated replicability requires that, if the outcome A_i (B_j) is obtained in the first measurement, then the same outcome A_i (B_j) should be obtained in the final measurement with certainty, i.e. with probability 1. We thus formalize response replicability by setting the conditional probability $p(A_i|A_i) = 1$ in a AA sequence, $p(B_j|B_j) = 1$ in a BB sequence, $p(A_i|A_iB_j) = 1$ in a ABA sequence, and $p(B_j|B_jA_i) = 1$ in a BAB sequence, $i, j \in \{y, n\}$.

Let us now come to the way in which the above class of psychological measurements are modeled in Hilbert space. The cognitive situation is represented by a unit vector $|\psi\rangle$ of a suitable Hilbert space, the measurements A and B are represented by the spectral measures $\{P_i^A\}$ and $\{P_j^B\}$, $i, j \in \{y, n\}$, and the Born rule is assumed to hold in both individual and sequential measurements, that is, $p_\psi(A_i) = \langle\psi|P_i^A|\psi\rangle$, $p_\psi(B_j) = \langle\psi|P_j^B|\psi\rangle$, $p_\psi(A_iB_j) = \langle\psi|P_i^A P_j^B P_i^A|\psi\rangle$, and $p_\psi(B_jA_i) = \langle\psi|P_j^B P_i^A P_j^B|\psi\rangle$, $i, j \in \{y, n\}$. Finally, this class of psychological measurements are assumed to be 'ideal first kind' measurements in a standard quantum sense, hence the state transformations induced by the measurements A and B are $|\psi\rangle \rightarrow \frac{P_i^A|\psi\rangle}{\|P_i^A|\psi\rangle\|}$ and $|\psi\rangle \rightarrow \frac{P_j^B|\psi\rangle}{\|P_j^B|\psi\rangle\|}$, respectively, for every $i, j \in \{y, n\}$, according to the Lüders postulate.

Recent studies confirm that, while the standard quantum formalism in Hilbert space is able to separately model question order effects and response replicability [3, 7, 9, 12], the same formalism does not work in cognitive situations where both effects are simultaneously present [13, 14, 17]. Roughly speaking, while the latter effect requires the spectral measures representing measurements to commute, the former can only be reproduced by non-commuting spectral families – the possibility of solving this problem by using more general positive operator values measurements is still under investigation. One may then wonder whether cognitive experiments exist where question order effects and response replicability are effectively observed. In this respect, one typically accepts the latter effect as a natural requirement for a wide class of psychological measurements. On the other hand, order effects in sequential measurements have been thoroughly studied since the seventies [23]. In particular, Moore reviewed a Gallup poll conducted in 1997, in which he reported interesting results of different experiments on question order effects [24].

Hilbert space models of question order effects predict that a so-called 'QQ equality' should be satisfied by the experimental data, for every initial state $|\psi\rangle$ [9]. The QQ equality is important, as it provides a 'parameter-free test of quantum models for question order effects', valid for projection operators of arbitrary dimension. Interestingly enough, this equality is approximately satisfied by some of the data collected by Moore, like those of the 'Clinton/Gore experiment', while it is significantly violated by others, like the 'Rose/Jackson experiment'. Furthermore, some authors [15, 16], including ourselves [17], lately observed that a special version of the quantum model, the 'non-degenerate model', should satisfy further parameter-free conditions, which are instead generally violated by the data. As we will analyse more specifically in Sec. 5, we can thus already draw a major result from the preceding discussion: at the level of question order effects, not only when they are simultaneously present with response replicability, quantum modeling in Hilbert space is problematical, and a more general probabilistic framework becomes necessary.

3 An operational and realistic framework for cognitive entities and measurements

In this section, we apply an operational and realistic framework to describe cognitive situations of the type mentioned in Sec. 2 [18]. This framework rests on the operational and realistic foundations of quantum physics and quantum probability that were formalized by the SCoP formalism [25]. Here, the terms 'operational' and 'realistic' have a precise meaning. Our approach to cognition is 'operational', in the sense that the basic notions (states, measurements, outcomes and their probabilities, etc.) are defined in terms of the concrete operations that are performed in the laboratory of experimentation. Furthermore, our approach to cognition is 'realistic', in the sense that the state of the cognitive entity is interpreted as a 'state of affairs', hence it expresses a reality of the cognitive entity, albeit a reality not of a physical but of a conceptual nature.

In experimental psychology, we can introduce 'psychological laboratories', that is, spatio-temporal domains where cognitive experiments are performed. Let us focus ourselves on opinion polls, where a large number of human participants are asked questions in the form of structured questionnaires, and let the questions involve a 'cognitive entity' S (a concept, a combination of concepts, or a more complex conceptual situation). The experimental design, the questionnaire and the cognitive effect under study define a 'preparation' of the cognitive entity S, which is thus assumed to be in an 'initial state' p_S, and all participants interact with the cognitive entity in the state p_S. Suppose that the question, or 'yes-no measurement', A is asked to a participant as part of the opinion poll. The measurement has the possible outcomes A_y and A_n, depending on whether the response of the participant was 'yes' or 'no'. The interaction of the participant with the cognitive entity S, when the dichotomic measurement A is performed, thus leads to one of the two possible outcomes, and generally also gives rise to a change of the state of the entity from p_S to either p_{A_y} or p_{A_n}, depending on whether the response is 'yes' or 'no'. Hence, the participant acts as a measurement context for the cognitive entity in the state p_S. If the same measurement A is performed by making use of a large sample of participants, a statistics of responses is collected, which determines in the large number limit a 'transition probability' $\mu(p_{A_i}, e_A, p_S)$ that the initial state p_S of the cognitive entity S changes to the state p_{A_i}, $i \in \{y, n\}$, under the effect of the context e_A determined by the measurement A.

The above framework formalizes the situation of the 'Clinton/Gore experiment' mentioned in Sec. 2, where the participant is asked to answer 'yes' or 'no' to the question: "Is Gore honest and trustworthy?". If, for a given participant, the response is 'yes', the initial state $p_{Honesty}$ of the conceptual entity *Honesty and Trustworthiness* (which we will simply denote *Honesty*, for the sake of simplicity) changes to a new state p_{A_y}, which is the state the entity is in when the choice 'Gore is honest' is added to its original content. Let us now suppose that a second question B is asked to the participants as part of the opinion poll. This defines a measurement B, with possible outcomes B_y and B_n, on the cognitive entity in the state p_S. Also in this case, the response determines a change of the state of S from p_S to either p_{B_y} or p_{B_n}, depending on whether the response is 'yes' or 'no'. In the large number limit, we get a transition probability $\mu(p_{B_j}, e_B, p_S)$ that the initial state p_S of S changes to the state p_{B_j}, $j \in \{y, n\}$, under the effect of the context e_B determined by the measurement B.

The measurement B formalizes the situation where the participant is asked to answer 'yes' or 'no' to the question: "Is Clinton honest and trustworthy?". If, for a given participant, the response is 'yes', the initial state $p_{Honesty}$ of the conceptual entity *Honesty and Trustworthiness* changes to a new state p_{B_y}, which is the state the entity is in when the choice 'Clinton is honest' is added to its original content. Then, let us suppose that each participant is first asked question A and then question B. This defines a new measurement AB, with possible outcomes A_iB_j, $i, j \in \{y, n\}$, on the cognitive entity S in the state p_S. The probability $p_S(A_iB_j)$ of obtaining the outcome A_iB_j in the measurement AB, i.e. the outcome A_i when performing A, and then B_j when performing B, $i, j \in \{y, n\}$, on S in the state p_S, is given by the product $p_S(A_iB_j) = \mu(p_{A_i}, e_A, p_S)\mu(p_{B_j}, e_B, p_{A_i})$. Finally, let us suppose that each participant is first asked question B and then question A. This defines a new measurement BA, with possible outcomes B_jA_i, $i, j \in \{y, n\}$, on the cognitive entity S in the state p_S. The probability $p_S(B_jA_i)$ of obtaining the outcome B_jA_i in the measurement BA, i.e. the outcome B_j when performing B, and then A_i when performing A, $i, j \in \{y, n\}$, on S in the state p_S, is given by the product $p_S(B_jA_i) = \mu(p_{B_j}, e_B, p_S)\mu(p_{A_i}, e_A, p_{B_j})$.

To conclude this section, we stress that the state of a cognitive entity describes an element of a conceptual reality that is independent of the subjective beliefs of the persons questioning about that entity. Such subjective beliefs are rather incorporated in the measurement context, which describes the cognitive interaction between the entity and the persons deciding on it. As such, our operational and realistic approach to cognition departs from other approaches that apply the quantum formalism to model cognitive phenomena [3, 7, 8].

4 The GTR-model for dichotomic measurements

In this section, we present a geometric representation, in the 3-dimensional Euclidean space $\mathbb{R}^3$, of the operational and realistic entities we have introduced in Sec. 3, focusing on the representation of the sequential measurements AB and BA. Our results rest on [17], to which we refer for technical details and calculations. The model presented here is an application of the 'general tension-reduction (GTR) model, where quantum probabilities are recovered as 'universal averages' over all possible forms of non-uniform fluctuations [21, 22]. When the state space is Hilbertian, as in quantum physics, the GTR-model reduces to the so-called 'extended Bloch representation' (EBR) of quantum theory [20].

Let us firstly consider individual measurements with two outcomes on a cognitive entity, and study how they are represented in the EBR representation. The cognitive entity S is represented by an abstract point particle that can move on the surface of a 3-dimensional unit sphere, called the 'Bloch sphere'. The initial state p_S of S is represented by a state of the point particle on the sphere corresponding to a given position $\mathbf{x}_\psi$ on the sphere. Dichotomic measurements on S are then represented by 1-dimensional breakable and elastic structures, anchored at two antipodal points, corresponding to the two possible outcome states. More precisely, the measurement A is represented by a breakable elastic band stretched between two points $\mathbf{a}_y$ and $\mathbf{a}_n = -\mathbf{a}_y$, $\|\mathbf{a}_y\| = \|\mathbf{a}_n\| = 1$, corresponding to the two outcomes A_y and A_n, respectively. Analogously, the measurement B is represented by a breakable elastic band stretched between two points $\mathbf{b}_y$ and $\mathbf{b}_n = -\mathbf{b}_y$, $\|\mathbf{b}_y\| = \|\mathbf{b}_n\| = 1$, corresponding to the two outcomes B_y and B_n, respectively. Accordingly, the outcome states p_{A_i} and p_{B_j} are represented by the positions $\mathbf{a}_i$ and $\mathbf{b}_j$, respectively, $i, j \in \{y, n\}$.

We assume that the points x of the two breakable elastics are parameterized in such a way that the end points coordinate $x = 1$ and $x = -1$ correspond to the outcome 'yes' and 'no', respectively, with $x = 0$ describing the center of the elastics, also coinciding with the center of the Bloch sphere. Each elastic represents a possible dichotomic measurement, and is described not only by its orientation within the sphere, but also by 'the way' it can break. More concretely, breakability of the elastic representing the measurement A is formalized by a probability distribution $\rho_A(x|\psi)$ such that $\int_{x_1}^{x_2} \rho_A(x|\psi)dx$ is the probability that the elastic breaks in the interval $[x_1, x_2]$, $-1 \leq x_1 \leq x_2 \leq 1$, when the measurement A is performed and the point particle is in the initial position $\mathbf{x}_\psi$. The condition $\int_{-1}^{1} \rho_A(x|\psi)dx = 1$ guarantees that the elastic will break in one of its points, with certainty, i.e. that the measurement will produce an outcome.

Let us now describe the measurement A on the cognitive entity S in the state p_S as represented in the Bloch sphere. When the measurement A is performed and the point particle is in the initial position $\mathbf{x}_\psi$, a certain probability distribution $\rho_A(x|\psi)$ is actualized, which describes the way the A-elastic band will break, in accordance with the fluctuations that are present in the measurement context e_A. Then, the point particle "falls" from its original position $\mathbf{x}_\psi$, orthogonally onto the A-elastic band, and sticks to it. Next, the elastic breaks in some point, and its two broken fragments contract toward the corresponding anchor points, bringing with them the point particle (see Fig. 1). If x_A is the position of the

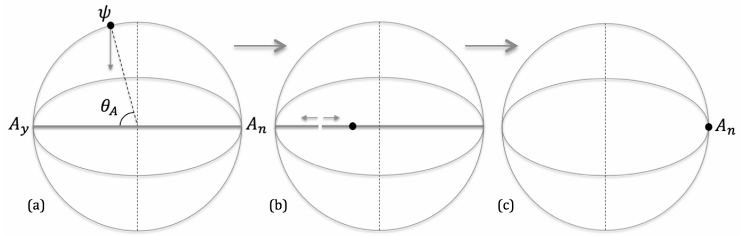

Fig. 1. The unfolding of the A-measurement, here producing outcome A_n.

point particle onto the elastic, i.e. $x_A = \mathbf{x}_\psi \cdot \mathbf{a}_y = \cos\theta_A$, and the elastic breaks in a point λ, with $x_A < \lambda$, then the particle attached to the elastic fragment $[-1, \lambda]$ is drawn toward the position $\mathbf{a}_y$. In this case, we say that the measurement A gives the outcome 'yes'. If instead $x_A > \lambda$, then the particle attached to the elastic fragment $[\lambda, 1]$ is drawn toward the position $\mathbf{a}_n$. In this case, we say that the measurement A gives the outcome 'no'. The transition probability $p_\psi(A_y)$ that the initial position $\mathbf{x}_\psi$ collapses to $\mathbf{a}_y$, and $p_\psi(A_n)$ that the initial position $\mathbf{x}_\psi$ collapses to $\mathbf{a}_n$, are given by:

$$p_\psi(A_y) = \int_{-1}^{\cos\theta_A} \rho_A(x|\psi)dx \qquad p_\psi(A_n) = \int_{\cos\theta_A}^{1} \rho_A(x|\psi)dx \tag{1}$$

and represent the transition probabilities $\mu(p_{A_y}, e_A, p_S)$ and $\mu(p_{A_n}, e_A, p_S)$, respectively, which we have introduced in Sec. 3.

It is worth noticing that: (i) the probabilities in (1) formalize a lack of knowledge about the measurement process, i.e. the breaking point λ corresponds to a 'hidden measurement-interaction'; (ii) the Born rule of quantum probability is recovered when $\rho_A(x|\psi) = \frac{1}{2}$, i.e. when the probability distribution is globally uniform, in which case (1) becomes: $p_\psi(A_y) = \frac{1}{2}(1 + \cos\theta_A)$ and $p_\psi(A_n) = \frac{1}{2}(1 - \cos\theta_A)$. This result is not limited to dichotomic measurements, but has a general validity, i.e. it can be naturally generalized to degenerate and non-degenerate measurements having an arbitrary number of outcomes [20–22]. For the transition probabilities $p_\psi(B_y)$ and $p_\psi(B_n)$, associated with measurement B, one has the same formulae, simply replacing θ_A by θ_B and $\rho_A(x|\psi)$ by $\rho_B(x|\psi)$, with θ_B now defining the landing point $x_B = \mathbf{x}_\psi \cdot \mathbf{b}_y = \cos\theta_B$ of the point particle onto the B-elastic band, and $\rho_B(x|\psi)$ being the probability distribution associated with the latter (generally different from $\rho_A(x|\psi)$).

Let us then consider sequential measurements on a cognitive entity and study how they are represented in the GTR-model. Suppose that we firstly perform the measurement A and then the measurement B. We thus have the four transition probabilities $p_\psi(A_iB_j)$ that the point particle position $\mathbf{x}_\psi$, representing the initial state, first changes to the position $\mathbf{a}_i$ and then to the position $\mathbf{b}_j$ (sequential outcome A_i and then B_j), $i,j \in \{y,n\}$. If we set $\cos\theta = \mathbf{a}_y \cdot \mathbf{b}_y$, we can first write the conditional probabilities $p_{A_i}(B_j)$ that the position $\mathbf{a}_i$ changes to the position $\mathbf{b}_j$, $i,j \in \{y,n\}$, as

$$\begin{aligned} p_{A_y}(B_y) &= \int_{-1}^{\cos\theta} \rho_B(x|A_y)dx \qquad p_{A_y}(B_n) = \int_{\cos\theta}^{1} \rho_B(x|A_y)dx \\ p_{A_n}(B_y) &= \int_{-1}^{-\cos\theta} \rho_B(x|A_n)dx \qquad p_{A_n}(B_n) = \int_{-\cos\theta}^{1} \rho_B(x|A_n)dx \end{aligned} \tag{2}$$

where $\rho_B(x|A_y)$ (respectively $\rho_B(x|A_n)$) is the probability distribution actualized during the measurement B, knowing that the measurement A produced the transition from $\mathbf{x}_\psi$ to $\mathbf{a}_y$ (respectively to $\mathbf{a}_n$). Now, for every $i,j \in \{y,n\}$, we have $p_\psi(A_iB_j) = p_\psi(A_i)p_{A_i}(B_j)$ for the transition probabilities in the sequential measurement AB. More explicitly, using (2) and (1), we can write:

$$\begin{aligned} p_\psi(A_yB_y) &= \int_{-1}^{\cos\theta} \rho_B(x|A_y)dx \int_{-1}^{\cos\theta_A} \rho_A(x|\psi)dx \\ p_\psi(A_yB_n) &= \int_{\cos\theta}^{1} \rho_B(x|A_y)dx \int_{-1}^{\cos\theta_A} \rho_A(x|\psi)dx \\ p_\psi(A_nB_y) &= \int_{-1}^{-\cos\theta} \rho_B(x|A_n)dx \int_{\cos\theta_A}^{1} \rho_A(x|\psi)dx \\ p_\psi(A_nB_n) &= \int_{-\cos\theta}^{1} \rho_B(x|A_n)dx \int_{\cos\theta_A}^{1} \rho_A(x|\psi)dx \end{aligned} \tag{3}$$

and by exchanging the role of A and B in (3), we get similar expressions for the probabilities $p_\psi(B_yA_y)$, $p_\psi(B_yA_n)$, $p_\psi(B_nA_y)$ and $p_\psi(B_nA_n)$ of the sequential measurement BA. Clearly, these sequential probabilities coincide by construction with the probabilities $p_S(A_iB_j)$ and $p_S(B_jA_i)$, $i,j \in \{y,n\}$, given in Sec. 3.

Our general modeling of cognitive entities, states, dichotomic measurements and sequential measurement processes is thus completed. One realizes at once that it incorporates quantum aspects, as context induced changes of state, pure potentiality, unavoidable and uncontrollable uncertainty. In this sense, one can say that the model that we have presented is 'quantum-like'. However, it is more general than the standard Hilbert space representation, as the Born rule of quantum probability is only recovered in the specific case in which ρ_A and ρ_B are both globally uniform probability distributions (describing uniform elastic structures, having the same probability to break in all their points).

In order to find explicit solutions, to be used in specific applications, one needs to add some reasonable constraints to the measurements A and B, in particular for what concerns the probability densities ρ_A and ρ_B. Before doing so, let us observe that the elastic mechanism we have described also provides a possible representation of what we intuitively feel when confronted with decision contexts, and a neural/mental equilibrium is progressively built, resulting from the balancing of the different tensions between the initial state and the available mutually excluding answers. Indeed, an elastics stretched between two antipodal points in the Bloch sphere can be seen as an abstract representation of such equilibrium, which at some moment will be altered in a non-predictable way (when the elastic breaks), causing a sudden and irreversible process during which the initial conceptual state is drawn to one of the possible answers.

The compatibility of the GTR-model with our intuitive understanding of the human cognitive processes remains such also when psychological measurements with an arbitrary number N of outcomes are considered [20–22]. The elastics are then replaced by disintegrable hyper-membranes having the shape of $(N-1)$-dimensional simplexes. Similarly to the $N=2$ situation, the latter can still be viewed not only as mathematical objects naturally representing the measurements' probabilities, and their relations, but also as a way to 'give shape' to the different mental states of equilibrium, characterized by the existence of different competing 'tension lines' going from the on-membrane position of the point particle to the N vertices of the simplex, representing the different answers.

These 'tension-reduction processes' can also describe situations where the conflicts between the competing answers cannot be fully resolved, so that the system is brought into another state of equilibrium, between a reduced set of possibilities, which in the GTR-model correspond to lower-dimensional sub-simplexes [20–22]. These are situations describing sub-measurements of a given mesurement, called degenerate measurements in quantum mechanics. As we see in Sec. 7, they may have some relevance in the description of unpacking effects.

Let us now provide an exact solution to the modeling of data about sequential measurements. For this, we will assume in the following that $\rho_A(x|\psi)$ does not depend on the initial state and that it is 'locally uniform', i.e. only characterized by two parameters $\epsilon_A \in [0,1]$ and $d_A \in [-1+\epsilon_A, 1-\epsilon_A]$, such that: $\rho_A(x)=0$ if $x \in [-1, d_A-\epsilon_A) \cup (d_A+\epsilon_A, 1]$, and $\rho_A(x) = 1/2\epsilon_A$ if $x \in [d_A-\epsilon_A, d_A+\epsilon_A]$. To obtain compact expressions, we also assume that $\cos\theta_A \in [d_A-\epsilon_A, d_A+\epsilon_A]$. If we describe in a similar way a second dichotomic measurement B, then in addition to the three parameters ϵ_A, d_A and θ_A, characterizing A, we have three more parameters ϵ_B, d_B and θ_B, characterizing B, and a supplementary parameter θ, defined by $\cos\theta = \mathbf{a}_y \cdot \mathbf{b}_y$, characterizing the relative orientation of the two measurements within the Bloch sphere. In the following, we also assume that $\cos\theta \in [d_A-\epsilon_A, d_A+\epsilon_A]$ and $\cos\theta \in [d_B-\epsilon_B, d_B+\epsilon_B]$. Then, if we perform in sequence the measurement A followed by the measurement B (which we denote AB), the sequential measurement has the 4 outcomes A_iB_j, $i,j \in \{y,n\}$, and the associated probabilities are given by the products $p_\psi(A_iB_j) = p_\psi(A_i)p_{A_i}(B_j)$. Performing the integrals (1), one obtains:

$$
\begin{aligned}
p_\psi(A_yB_y) &= \tfrac{1}{4}(1+\tfrac{\cos\theta-d_B}{\epsilon_B})(1+\tfrac{\cos\theta_A-d_A}{\epsilon_A})\\
p_\psi(A_yB_n) &= \tfrac{1}{4}(1-\tfrac{\cos\theta-d_B}{\epsilon_B})(1+\tfrac{\cos\theta_A-d_A}{\epsilon_A})\\
p_\psi(A_nB_y) &= \tfrac{1}{4}(1-\tfrac{\cos\theta+d_B}{\epsilon_B})(1-\tfrac{\cos\theta_A-d_A}{\epsilon_A})\\
p_\psi(A_nB_n) &= \tfrac{1}{4}(1+\tfrac{\cos\theta+d_B}{\epsilon_B})(1-\tfrac{\cos\theta_A-d_A}{\epsilon_A})
\end{aligned}
\tag{4}
$$

and by exchanging the role of A and B in (4), we get similar expressions for the probabilities $p_\psi(B_yA_y)$, $p_\psi(B_yA_n)$, $p_\psi(B_nA_y)$ and $p_\psi(B_nA_n)$ of the sequential measurement BA. These systems of equations are underdetermined, as the 8 outcome probabilities can determine all the parameters but one. Thus, we are free to choose one of the parameters, for instance ϵ_A, and by doing so all the others will be fixed. Since we must have $\epsilon_A(1+\frac{d_A}{\epsilon_A})\leq 1$, i.e. $\epsilon_A\leq 1/(1+\frac{d_A}{\epsilon_A})$, this means that if $\frac{d_A}{\epsilon_A}$ is different from zero, it is not be possible to model the data by means of the standard quantum formalism (in a 2-dimensional Hilbert space), as the Born rule corresponds to the choice $d_A=0$ and $\epsilon_A=1$.

5 Modeling Moore's data

We now use the system of equations (4), for the sequential measurement AB, and its reversed order version, for the sequential measurement BA, to 'exactly' model the data obtained in a Gallup poll conducted in 1997, as presented in a review of question order effects by Moore [24]. More precisely, we consider the probabilities given by [26] (see also [9]), where the participants who did not provided a 'yes' or 'no' answer have been excluded from the statistics. In one of the experiments, a thousand participants were subjected to a pair of questions, asked in a sequence. The first question, which we associate with measurement A, is Clinton's question, and the second question, which we associate with measurement B, is Gore's question, as we described them in Sec. 3. Half of the participants were submitted to the two questions in the order AB (first 'Clinton' then 'Gore') and the other half in the reversed order BA, and the collected response probabilities are: $p(A_yB_y)=0.4899$, $p(A_yB_n)=0.0447$, $p(A_nB_y)=0.1767$, $p(A_nB_n)=0.2887$, $p(B_yA_y)=0.5625$, $p(B_yA_n)=0.1991$, $p(B_nA_y)=0.0255$, $p(B_nA_n)=0.2129$.

These probabilities show a significant question order effect. Inserting them in (4), and in its reversed order version, one obtains, after some calculations, the following explicit values for the model's parameters (we refer to [17] for a detailed analysis): $\frac{d_A}{\epsilon_A}=0.1545$, $\frac{\cos\theta_A}{\epsilon_A}=0.2237$, $\frac{\cos\theta}{\epsilon_A}=0.6316$, $\frac{d_B}{\epsilon_B}=-0.2961$, $\frac{\cos\theta_B}{\epsilon_B}=0.2271$ and $\frac{\cos\theta}{\epsilon_B}=0.5367$.

We see at once that the solution does not admit a representation by means of the Born rule, considering that $\frac{d_A}{\epsilon_A},\frac{d_B}{\epsilon_B}\neq 0$. Furthermore, we see that we cannot have $\epsilon_A=\epsilon_B$ and $d_A=d_B$, i.e. the solution requires the two measurements to be characterized by different rules of probabilistic assignment ($\rho_A\neq\rho_B$). The structure of the probabilistic data is thus irreducibly non-Hilbertian. If we choose $\epsilon_A=1/2$, we obtain for the other parameters (writing them in approximate form, to facilitate their reading): $\epsilon_A=0.5$, $\epsilon_B\approx 0.59$, $d_A\approx 0.08$, $d_B\approx -0.17$, $\cos\theta\approx 0.32$, $\cos\theta_A\approx 0.11$, and $\cos\theta_B\approx 0.13$ – hence, $\theta\approx 72°$, $\theta_A\approx 84°$, and $\theta_B\approx 74°$.

In another experiment reported by Moore, always performed on a thousand participants, the opinion poll consisted in a pair of questions about the baseball players Pete Rose and Shoeless Joe Jackson. More precisely, question A was: "Do you think Rose should or should not be eligible for admission to the Hall of Fame?". Similarly, question B was: "Do you think Jackson should or should not be eligible for admission to the Hall of Fame?". The collected response probabilities are (also in this case we use the data given in [9, 26]): $p(A_yB_y) = 0.3379$, $p(A_yB_n) = 0.3241$, $p(A_nB_y) = 0.0178$, $p(A_nB_n) = 0.3202$, $p(B_yA_y) = 0.4156$, $p(B_yA_n) = 0.0671$, $p(B_nA_y) = 0.1234$, $p(B_nA_n) = 0.3939$, and the modeling now gives [17]: $\frac{d_A}{\epsilon_A} = -0.0995$, $\frac{\cos\theta_A}{\epsilon_A} = 0.2245$, $\frac{\cos\theta}{\epsilon_A} = 0.6224$, $\frac{d_B}{\epsilon_B} = 0.4369$, $\frac{\cos\theta_B}{\epsilon_B} = 0.4023$ and $\frac{\cos\theta}{\epsilon_B} = 0.4578$. Again, we can observe that these values are irreducibly non-Hilbertian. For $\epsilon_A = 1/2$, we obtain: $\epsilon_A = 0.5$, $\epsilon_B \approx 0.68$, $d_A \approx -0.05$, $d_B \approx 0.30$, $\cos\theta \approx 0.31$, $\cos\theta_A \approx 0.11$, $\cos\theta_B \approx 0.27$.

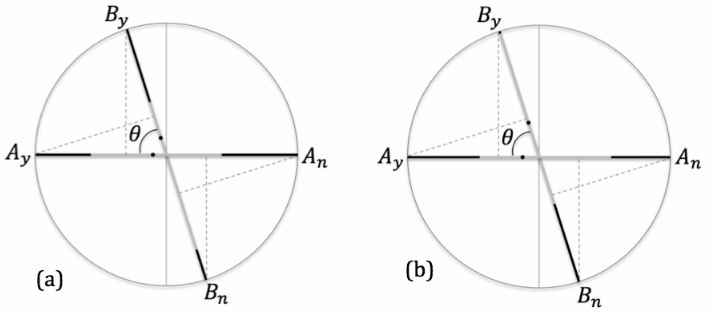

Fig. 2. The (a) Clinton/Gore and (b) Rose/Jackson probability distributions.

The two solutions are graphically represented in Fig. 2. The two black dots denote the values of $\cos\theta_A$ and $\cos\theta_B$, and the black regions are those where the probability distributions are zero (corresponding to the unbreakable elastic regions). What strikes the eye is that the Clinton/Gore and Rose/Jackson solutions are structurally very similar, despite the fact that only the former (almost) obey the 'QQ equality' [9, 26]. This is because the latter is insufficient to fully characterize a Hilbertian structure and that both solutions are actually intrinsically non-Hilbertian [17].

It is worth mentioning that the QQ equality, i.e. the equality

$$p_\psi(A_yB_y) - p_\psi(B_yA_y) + p_\psi(A_nB_n) - p_\psi(B_nA_n) = 0 \tag{5}$$

simply follows by taking the average $\langle\psi|Q|\psi\rangle$ of the operatorial identity $Q = 0$, where $Q \equiv P_y^A P_y^B P_y^A - P_y^B P_y^A P_y^B + P_n^A P_n^B P_n^A - P_n^B P_n^A P_n^B = 0$ [17, 18]. Now, it has been pointed out that the Clinton/Gore data are different from the Rose/Jackson, as for the latter, participants also received some sequential background information before answering the two questions, and this would explain why, contrary to the Clinton/Gore data, they violate the QQ equality. Indeed, if this supply of information is modeled by using two unitary operators U (for the information given before A) and V (for that given before B), we now have to write $p_\psi(A_iB_j) = \langle\psi|U^\dagger P_i^A V^\dagger P_j^B V P_i^A U|\psi\rangle$, and similarly for $p_\psi(B_jA_i)$. Thus, the relevant operator becomes:

$$
\begin{aligned}
Q' &= U^\dagger P_y^A P'^B_y P_y^A U - V^\dagger P_y^B P'^A_y P_y^B V + U^\dagger P_n^A P'^B_n P_n^A U - V^\dagger P_n^B P'^A_n P_n^B V \\
&= [P'^B_y - U^\dagger P'^B_y U] + [V^\dagger P'^A_y V - P'^A_y] + [U^\dagger P'^B_y U P'^A_y - P'^B_y V^\dagger P'^A_y V] \\
&\quad + [P'^A_y U^\dagger P'^B_y U - V^\dagger P'^A_y V P'^B_y]
\end{aligned}
\tag{6}
$$

where we have defined $P'^A_i \equiv U^\dagger P_i^A U$, $P'^B_j \equiv V^\dagger P_j^B V$, $i,j \in \{y,n\}$. Since the average $\langle\psi|Q'|\psi\rangle$ can now in principle take any value within the interval $[-1,1]$ (unless $U = V = \mathbb{I}$), this could explain why the QQ equality is violated in the Rose/Jackson situation.

The above argument, however, is weakened by the observation that there are other quantum equalities that are strongly disobeyed both by the Clinton/Gore and Rose/Jackson data, like for instance, in the situation of non-degenerate measurements [17, 18]: $q' \equiv p_\psi(A_yB_n)p_\psi(A_nB_n) - p_\psi(A_nB_y)p_\psi(A_yB_y) = 0$, which must be obeyed also when participants receive some background information. Indeed, we have in this case $p_\psi(A_iB_j) = |\langle A_i|U|\psi\rangle|^2|\langle B_j|V|A_i\rangle|^2$, where $P_i^A = |A_i\rangle\langle A_i|$ and $P_j^B = |B_j\rangle\langle B_j|$, $i,j \in \{y,n\}$, so that we can write:

$$
\begin{aligned}
q' = |\langle A_y|U|\psi\rangle|^2|\langle A_n|U|\psi\rangle|^2 \times& \\
\times \left[|\langle B_n|V|A_y\rangle|^2|\langle B_n|V|A_n\rangle|^2 - |\langle B_y|V|A_n\rangle|^2|\langle B_y|V|A_y\rangle|^2\right]&
\end{aligned}
\tag{7}
$$

Using $|\langle B_y|V|A_n\rangle|^2 = 1 - |\langle B_n|V|A_n\rangle|^2$ and $|\langle B_y|V|A_y\rangle|^2 = 1 - |\langle B_n|V|A_y\rangle|^2$, it is easy to check that the terms in the above bracket cancel, so that $q' = 0$. Thus, we have a genuine quantum equality which must be satisfied also when some information is sequentially provided to the participants. However, it is strongly violated by the experimental data [17].

6 Response replicability

As emphasized in [13], the standard quantum formalism is unable to jointly model question order effects and response replicability. The reason is simple to understand: response replicability, the situation where a question, if asked a second time, receives the same answer, even if other questions have been answered in between, requires commuting observables to be modeled. Indeed, since we have the operatorial identity $P_n^B P_y^A P_n^B - P_y^A P_n^B P_y^A = (P_y^B - P_y^A)[P_y^B, P_y^A]$, it follows that the difference $p_\psi(B_nA_y) - p_\psi(A_yB_n)$ can generally be non-zero only if $[P_y^B, P_y^A] \neq 0$, i.e. the spectral families associated with the A and B measurements do not commute. Thus, not only an exact description of question order effects requires to go beyond-quantum, but the combination of the latter with response replicability also creates a contradiction, which persists even when measurements are represented by positive-operator valued measures [13, 14].

The reason why the above contradiction cannot be eliminated is that in quantum theory an observable automatically determines, via the Born rule, the outcome probabilities. This means that, once the initial state is given, and the possible outcomes are also given, there is only one way to choose them: that prescribed by the Born rule. This means that, if a specific participant were able to interact with a cognitive entity by employing different 'ways of choosing', at least one of them has to be non-Bornian. In our opinion, such a situation precisely occurs when considering the effect of response replicability. Indeed, in this case there are at least two possible ways of choosing an outcome from the memory of the previous interaction. In the standard formalism there is no place to describe such a memory effect, hence the impossibility to model it in a consistent way, beyond the so-called 'adjacent replicability' [14], which is built-in in all first-kind measurements. On the other hand, in the richer structure of the GTR-model, changes in the way outcomes are selected can be easily modeled as changes in the measurements' probability distributions [17]. In other words, the reason why 'separated replicability' can be taken into account in the GTR-model, jointly with possible question order effects, is that it allows not only to describe how the action of contexts can produce state transitions, but also how state transitions can determine a change of future contexts, via a change of the associated probability distributions.

To see how the above works, let us consider the sequence of three measurements ABA on a given cognitive entity (see also [17] for a more general discussion). Let ρ_A be the probability distribution describing the measurement A, and let us suppose that the outcome is A_y. We do not need to associate any change of the probability distribution ρ_A to this transition, as measurements are already first kind measurements in the GTR-model, as in quantum theory. Then, let us suppose that, when the measurement B is performed, with the entity now in the state associated with outcome A_y, the outcome B_y is obtained. Again, we do not need to associate any change of the probability distribution ρ_B to this second transition, but we now have to update the probability distribution describing the measurement A, to guarantee that, if we repeat the latter, the outcome A_y is certain in advance. In other words, we now associate a probability distribution transition $\rho_A \rightarrow \rho'_A$, to ensure response replicability. Similarly, when the measurement A is performed, giving A_y with probability 1, there will be a transition $\rho_B \rightarrow \rho'_B$, to ensure that a subsequent measurement B will give B_y with certainty, and from that point on subsequent A or B measurements can only deterministically reproduce the same outcomes, with no further changes of contexts. More precisely, the probability distributions ρ'_A and ρ'_B can be obtained by simple truncation and renormalization [17]:

$$\rho'_A(x) = \frac{\rho_A}{\int_{-1}^{\cos\theta} \rho_A dx} \chi_{[-1,\cos\theta)}(x) \qquad \rho'_B(x) = \frac{\rho_B}{\int_{-1}^{\cos\theta} \rho_B dx} \chi_{[-1,\cos\theta)}(x) \quad (8)$$

where $\chi_I(x)$ is the characteristic function of the interval I. Fig. 3 illustrates this 'double transition process', where not only states but also probability distributions can change. Fig. 3 (a) represents the situation following the first measurement A, the outcome being A_y. Fig. 3 (b) describes the subsequent measurement B, the outcome being B_y, also producing the transition from ρ_A to ρ'_A. Fig. 3 (c) describes the second measurement A, giving again outcome A_y, with certainty, which is also accompanied by the transition from ρ_B to ρ'_B.

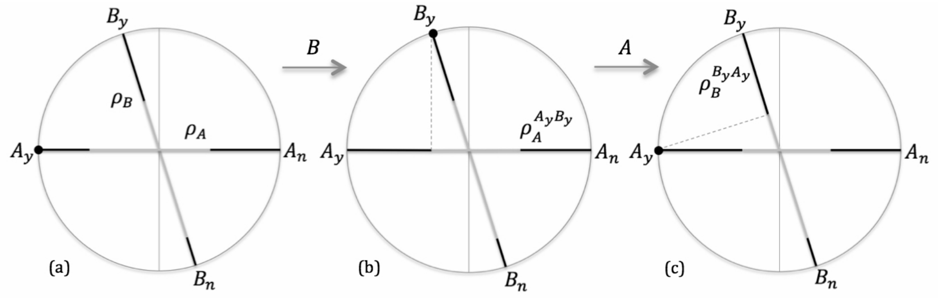

Fig. 3. The measurement sequence BA, in the GTR-model.

7 Unpacking effects

In this final section, we analyze the so-called 'unpacking effects', usually modeled in the quantum formalism by assuming that the participants actually perform non-compatible sequential measurements, in a predetermined order [27]. Our thesis is that, if we consider these effects in relation to the notion of sub-measurement, they point to an inadequacy of the quantum formalism in Hilbert space, as they describe situations that are incompatible with the quantum representation of degenerate measurements.

Two kinds of unpacking are usually considered, 'implicit' and 'explicit'. The implicit unpacking is when a question is addressed in two different ways, a 'packed way' and an 'unpacked way'. More precisely, if A and B are two dichotomic measurements with outcomes A_y and A_n, and B_y and B_n, respectively, we can define a measurement A', with outcomes A'_y and A'_n, where A'_n is the same as A_n, and A'_y describes a possibility that is logically equivalent to A_y, expressed as an alternative over two mutually exclusive and exhaustive possibilities, defined by the outcomes of B. In other words, $A'_y = (A_y \wedge B_y) \oplus (A_y \wedge B_n)$, where the symbol $\oplus$ denotes the logical exclusive conjunction.

An example adapted from a list of experiments performed by Rottenstreich and Tversky [28] is the following. Measurement A is the question: "Is the winner of next US presidential election a non-Democrat?", with outcome A_y and A_n corresponding to the answers "Yes, is a non-Democrat," and "No, is a Democrat," respectively. Measurement B is the question: "If the winner of next US presidential election is a non-Democrat, will be an Independent?", with outcome B_y and B_n corresponding to the answers "Yes, an Independent," and "No, not an Independent," respectively. On the other hand, the implicitly unpacked measurement A' is defined by the question: "Is the winner of the next presidential election an Independent or Republican rather than a Democrat?", with outcome A'_y corresponding to the (unpacked) answer "Yes, is an Independent or a Republican rather than a Democrat" and outcome A'_n to the answer "No, is a Democrat," which is the same as A_n.

Following Sec. 3, we denote by p_S the initial state of the conceptual entity S, which in our case is: *The winner of next US presidential election.* Moreover, we denote by p_{A_i} and $p_{A'_i}$ the final states of S associated with the outcomes A_i and A'_i, $i \in \{y, n\}$, respectively. Then, we can write the corresponding probabilities as $p_S(A_i) = \mu(p_{A_i}, e_A, p_S)$ and $p_S(A'_i) = \mu(p_{A'_i}, e_{A'}, p_S)$, $i \in \{y, n\}$, where e_A and $e_{A'}$ are the contexts associated with A and A', respectively, causing the transitions from the initial state p_S to the observed outcome states p_{A_i} and $p_{A'_i}$, respectively. If $p_S(A'_y)$ is found to be sensibly different from $p_S(A_y)$, one says that there is an unpacking effect, i.e. an effect where logically equivalent descriptions of a same possibility can produce different probabilities, thus violating the so-called principle of 'description invariance'. More precisely, one speaks of 'superadditivity' if $p_S(A_y) > p_S(A'_y)$ and 'subadditivity' if $p_S(A_y) < p_S(A'_y)$.

Let us also describe the situation corresponding to the 'explicit unpacking effect'. In this case the dichotomic measurement A' is further decomposed into a measurement having three distinct outcomes, transforming the implicit alternative into an explicit one. More precisely, this fully unpacked measurement, which we denote by A'', now has the three outcomes A''_{yy}, A''_{yn} and A''_n, and the associated states $p_{A''_{yy}}$, $p_{A''_{yn}}$ and $p_{A''_n}$, respectively, where $A''_n = A_n$, $A''_{yy} = A_y \wedge B_y$ and $A''_{yn} = A_y \wedge B_n$. Thus, participants can now choose among three distinct possibilities, with probabilities $p_S(A''_i) = \mu(p_{A''_i}, e_{A''}, p_S)$, $i \in \{yy, yn, n\}$. Again, one speaks of superadditivity if $p_S(A_y) > p_S(A''_{yy}) + p_S(A''_{yn})$ and of subadditivity if $p_S(A_y) < p_S(A''_{yy}) + p_S(A''_{yn})$.

Since superadditivity and subadditivity are in general both possible, the usual quantum analysis exploits the interference effects as a way to explain, by means of a single mechanism, both possibilities, as interference terms can take both positive and negative values [27]. The assumption behind this approach is that participants act in a sequential way, all with the same order for the sequence. Accordingly, one associates the non-commuting projection operators P_i^A and P_j^B to the outcomes A_i and B_j, respectively, $i, j \in \{y, n\}$, so that one can write, for every $i \in \{y, n\}$, $P_i^A = P_y^B P_i^A P_y^B + P_n^B P_i^A P_n^B + I_i$, for every $i \in \{y, n\}$, where $P_n^B = \mathbb{I} - P_y^B$ and $I_i = P_y^B P_i^A P_n^B + P_n^B P_i^A P_y^B$ is the interference contribution, responsible of the superadditivity or subadditivity effects.

The above analysis, however, has some weak points. Firstly, the above projection operators do not commute, hence the order of evaluation in the sequence becomes important, and one needs to assume that all participants always start by answering first the question B and only then the question A. However, since this sequentiality is not part of the experimental protocol, nothing guarantees that it will be carried out in practice, instead of considering A''_{yy} and A''_{yn} as outcomes of a single non-sequential measurement. Secondly, it is incompatible with the natural interpretation of the packed and explicitly unpacked outcomes as belonging to two measurements that are logically related, in the sense that A can be understood as the degenerate version of the non-degenerate measurement A'' or, to put it another way, as a sub-measurement of A''.

Considering the packed measurement A and the associated explicitly unpacked measurement A'', the question is: How should we use the quantum formalism to model these experimental situations? In both measurements we have a cognitive entity in the same initial state p_S. We also have outcomes that are the same for both measurements, A_n and A''_n, which therefore should be associated with the same state, describing the same intersubjective reality. Then, we have outcomes that are described in a packed way in one measurement and in an explicitly unpacked way in the other – in our example the outcome A_y that is decomposed into the two alternatives A''_{yy} and A''_{yn}.

If quantum theory is taken as a unitary and coherent framework, one should then be able to use the notion of 'degenerate measurement' (the quantum notion of sub-measurement) to model these two logically related experimental situations. Considering the previous example of the entity *The winner of next US presidential election*, it is clear that a 'non-Democrat' president is either a 'Republican' or an 'Independent,' and that 'Republican' and 'Independent' presidents are always 'non-Democrat' presidents. This means that the 'Republican' or the 'Independent' specification is an additional specification for the 'non-Democrat' state, and this means that when comparing an experimental situation where this specification is made, to a situation where it is not made, the latter should be considered as a sub-measurement of the former, i.e. a 'degenerate measurement' in the quantum jargon. Indeed, when the outcome is just 'Non-Democrat', the experimenter has no information about the 'Independent' or 'Republican' element, this being not specified in the outcome state. Also, since 'Republican' and 'Independent' are excluding possibilities, within the quantum formalism one should certainly describe them by two orthogonal subspaces, or two orthogonal states. Considering all this, one would thus expect to get: $p_S(A_y) = p_S(A''_{yy}) + p_S(A''_{yn})$ and $p_S(A_n) = p_S(A''_n)$.

However, since explicit unpacking effects are observed (which are generally stronger than the implicit ones), equalities like the above can be expected to be significantly violated, meaning that sub-measurements in psychology would not allow themselves to be consistently represented in the Hilbert space quantum formalism. Again, this can be attributed to the fact that the latter only admits a single 'way of choosing' the available outcomes, the 'Born way', whereas it is more natural to assume that the selection process can generally depend on the overall cognitive situation that is presented to the participants. Indeed, participants' propensity of choosing a given outcome certainly depends on the nature of the alternatives that are presented to them, and this is a contextuality effect that the quantum formalism is unable to describe. Yet, it can be represented in the GTR-model and its EBR implementation, by assuming that the probability distribution ρ_A characterizing the degenerate measurement A is not the same as the probability distribution $\rho_{A''}$ describing the corresponding non-degenerate versions A'', associated with an explicitly unpacked situation.

Concerning the implicitly unpacked case, one would also expect, if the standard quantum formalism applied, that $p_S(A_n) = p_S(A'_n)$, implying that $p_S(A_y) = p_S(A'_y)$. However, since the packed and implicitly unpacked measurements are dichotomic measurements, sharing the same state $p_{A_n} = p_{A'_n}$, if follows that the two states corresponding to the outcomes A_y and $A'_y = (A_y \wedge B_y) \oplus (A_y \wedge B_n)$ should also be equal, implying the equality of the associated transition probabilities. Thus, also in this case a Hilbert space quantum formalism cannot be used to model the data. In fact, even the EBR is too specific in this case, as it also relies on the Hilbert space structure for the representation of states (in the EBR, if two non-degenerate two-outcome measurements share an eigenstate, they necessarily also share the other one, as to each point on the 3-dimensional Bloch sphere there is only one corresponding antipodal point). This is a situation where the more general GTR-model is required, as it allows one to describe two dichotomic measurements by means of two probability distributions defined on line segments that share one of their vertex points (corresponding to the outcome A_n), but not the other.

To conclude, we observe that in [28] the protocol was such that respondents were partitioned in four groups, each group responding one of the four different 'yes/no' alternatives for *The winner of the next US presidential election*: 'Non-Democrat', 'Independent rather than Republican or Democrat', 'Republican rather than Independent or Democrat' and 'Independent or Republican rather than Democrat'. In other words, they were actually performing a single measurement with five distinct outcomes. The experimental situation we have discussed is different, although of course related, and to apply the data to our analysis one should repeat the experiment by splitting it into three measurements: (A) one with outcomes 'non-Democrat' and 'Democrat'; (A') another one with outcomes 'Independent or Republican rather than a Democrat' and 'Democrat'; (A'') and a last one with outcomes 'Independent', 'Republican' and 'Democrat'.

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