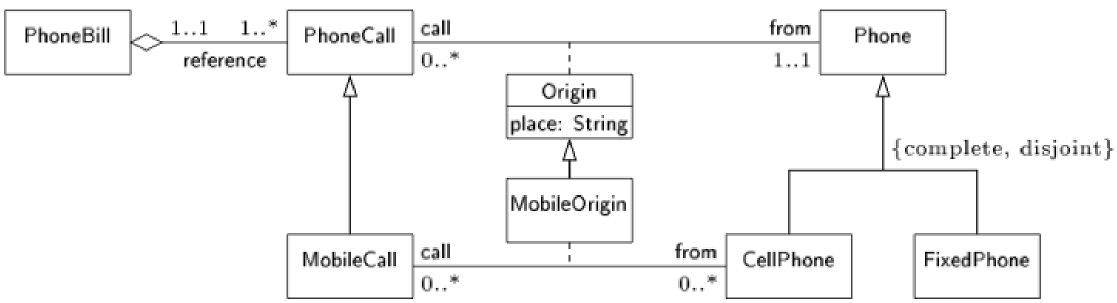
Knowledge Representation and Reasoning

Solutions to Exercises on Ontologies

1 Converting from UML to First-Order Logic

Consider the following UML class diagram about different kinds of phones, and phone bills they belong to.



The diagram shows that a MobileCall is a particular kind of PhoneCall and that the Origin of each PhoneCall is one and only one Phone. Additionally, a Phone can be only of two different kinds: a Fixed Phone or a Cell Phone. Mobile calls originate (through the association MobileOrigin) from cell phones. The association MobileOrigin is contained in the binary association Origin: hence MobileOrigin inherits the attribute place of association class Origin. Finally, a PhoneCall is referenced in one and only one PhoneBill, whereas a PhoneBill contains at least one PhoneCall.

1. Convert the UML diagram into Description Logics.

Answer:

$$\begin{aligned}
\exists place &\sqsubseteq Origin \\
\exists place^- &\sqsubseteq String \\
Origin &\sqsubseteq \exists place \sqcap (\leq 1 place) \\
\exists reference &\sqsubseteq PhoneBill \\
\exists reference^- &\sqsubseteq PhoneCall \\
PhoneBill &\sqsubseteq (\geq 1 reference) \\
PhoneCall &\sqsubseteq (\geq 1 reference^-) \sqcap \\
&\quad (\leq 1 reference^-) \\
\exists callO &\sqsubseteq Origin \\
\exists callO^- &\sqsubseteq PhoneCall \\
\exists fromO &\sqsubseteq Origin \\
\exists fromO^- &\sqsubseteq Phone \\
Origin &\sqsubseteq \exists callO \sqcap (\leq 1 callO) \sqcap \\
&\quad \exists fromO \sqcap (\leq 1 fromO)
\end{aligned}$$

$$\begin{aligned}
PhoneCall &\sqsubseteq (\geq 1 callO^-) \sqcap (\leq 1 callO^-) \\
\exists callMO &\sqsubseteq MobileOrigin \\
\exists callMO^- &\sqsubseteq MobileCall \\
\exists fromMO &\sqsubseteq MobileOrigin \\
\exists fromMO^- &\sqsubseteq CellPhone \\
MobileOrigin &\sqsubseteq \exists callMO \sqcap (\leq 1 callMO) \sqcap \\
&\quad \exists fromMO \sqcap (\leq 1 fromMO) \\
MobileOrigin &\sqsubseteq Origin \\
callMO &\sqsubseteq callO \\
fromMO &\sqsubseteq fromO \\
MobileCall &\sqsubseteq PhoneCall \\
CellPhone &\sqsubseteq Phone \\
FixedPhone &\sqsubseteq Phone \sqcap \neg CellPhone \\
Phone &\sqsubseteq CellPhone \sqcup FixedPhone
\end{aligned}$$

2. Convert the Description Logic result into first-order logic.

Answer:

$\forall x.\,(\exists y.place\,(x,y) \rightarrow Origin\,(x))$
$\forall x.\,(\exists y.place\,(y,x) \rightarrow String\,(x))$
$\forall x.\,(Origin\,(x) \rightarrow (\exists y.place\,(x,y) \wedge \forall y,z.\,((place\,(x,y) \wedge place\,(x,z)) \rightarrow y = z)))$
$\forall x.\,(\exists y.reference\,(x,y) \rightarrow PhoneBill\,(x))$
$\forall x.\,(\exists y.reference\,(y,x) \rightarrow PhoneCall\,(x))$
$\forall x.\,(PhoneBill\,(x) \rightarrow (\exists y.reference\,(x,y)))$
$\forall x.\,(PhoneCall\,(x) \rightarrow (\exists y.reference\,(y,x) \wedge \forall y,z.\,((reference\,(y,x) \wedge reference\,(z,x)) \rightarrow y = z)))$
$\forall x.\,(\exists y.callO\,(x,y) \rightarrow Origin\,(x))$
$\forall x.\,(\exists y.callO\,(y,x) \rightarrow PhoneCall\,(x))$
$\forall x.\,(\exists y.fromO\,(x,y) \rightarrow Origin\,(x))$
$\forall x.\,(\exists y.fromO\,(y,x) \rightarrow Phone\,(x))$
$\forall x.(Origin\,(x) \rightarrow (\exists y.callO\,(x,y) \wedge \exists y.fromO\,(x,y) \wedge \forall y,z.\,((callO\,(x,y) \wedge callO\,(x,z)) \rightarrow y = z) \wedge$
$\quad \forall y,z.\,((fromO\,(x,y) \wedge fromO\,(x,z)) \rightarrow y = z)))$
$\forall x.\,(PhoneCall\,(x) \rightarrow (\exists y.callO\,(y,x) \wedge \forall y,z.\,((callO\,(y,x) \wedge callO\,(z,x)) \rightarrow y = z)))$
$\forall x.\,(\exists y.callMO\,(x,y) \rightarrow MobileOrigin\,(x))$
$\forall x.\,(\exists y.callMO\,(y,x) \rightarrow MobileCall\,(x))$
$\forall x.\,(\exists y.fromMO\,(x,y) \rightarrow MobileOrigin\,(x))$
$\forall x.\,(\exists y.fromMO\,(y,x) \rightarrow CellPhone\,(x))$
$\forall x.(MobileOrigin\,(x) \rightarrow (\exists y.callMO\,(x,y) \wedge \exists y.fromMO\,(x,y) \wedge$
$\quad \forall y,z.\,((callMO\,(x,y) \wedge callMO\,(x,z)) \rightarrow y = z) \wedge$
$\quad \forall y,z.\,((fromMO\,(x,y) \wedge fromMO\,(x,z)) \rightarrow y = z)))$
$\forall x.\,(MobileOrigin\,(x) \rightarrow Origin\,(x))$
$\forall x,y.\,(callMO\,(x,y) \rightarrow callO\,(x,y))$
$\forall x,y.\,(fromMO\,(x,y) \rightarrow fromO\,(x,y))$
$\forall x.\,(MobileCall\,(x) \rightarrow PhoneCall\,(x))$
$\forall x.\,(CellPhone\,(x) \rightarrow Phone\,(x))$
$\forall x.\,(FixedPhone\,(x) \rightarrow (Phone\,(x) \wedge \neg CellPhone\,(x)))$
$\forall x.\,(Phone\,(x) \rightarrow (CellPhone\,(x) \vee FixedPhone\,(x)))$

3. Suppose you add a generalization to the diagram asserting that each CellPhone is a FixedPhone. Which classes become inconsistent (i.e. they cannot be populated) and which pairs of classes become equivalent?

Answer:
First, the class CellPhone is inconsistent, i.e., it has no instances. Indeed, the disjointness constraint asserts that there are no cell phones that are also fixed phones, and since the empty set is the only set that can be at the same time disjoint from and contained in the class FixedPhone, the class CellPhone must have it as extension. Second, since the class Phone is made up by the union of classes CellPhone and FixedPhone, and since CellPhone is inconsistent, the classes Phone and FixedPhone are equivalent, hence one of them is redundant. Finally, since there are no cell phones, there are no pairs in the association MobileOrigin, and so it is inconsistent too. The class MobileCall is not inconsistent since it can be populated by instances that do not participate to association MobileOrigin.

2 Constructing Models of Ontologies

Consider the following **TBox**:

$$Cow \sqsubseteq Vegetarian$$
$$MadCow \sqsubseteq Cow \sqcap \exists\ eat.BrainOfSheep$$
$$Sheep \sqsubseteq Animal$$
$$Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)$$
$$BrainOfSheep \sqsubseteq PartOfAnimal$$

1. Translate the TBox into natural language, and compare with the translation into first-order logic.

$Cow \sqsubseteq Vegetarian$**:** All cows are vegetarians.

$$\forall x.\,(Cow(x) \rightarrow Vegetarian(x))$$

$MadCow \sqsubseteq Cow \sqcap \exists\ eat.BrainOfSheep$**:** All mad cows are cows that eat (some) brain of sheep.

$$\forall x.\,[MadCow(x) \rightarrow (Cow(x) \wedge \exists y.\,(eat(x,y) \wedge BrainOfSheep(y)))]$$

$Sheep \sqsubseteq Animal$**:** All sheep are animals.

$$\forall x.\,(Sheep(x) \vee Cow(x) \supset Animal(x))$$

$Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)$**:** All vegetarians eat something, but never anything which is an animal or part of an animal.

$$\forall x.\,[Vegetarian(x) \rightarrow \exists y.\,(eat(x,y)) \wedge \forall y.\,(eat(x,y) \rightarrow \neg(Animal(y) \vee PartOfAnimal(y)))]$$

$BrainOfSheep \sqsubseteq PartOfAnimal$**:** All brains of sheep are parts of animals.

$$\forall x.\,(BrainOfSheep(x) \rightarrow PartOfAnimal(x))$$

2. Construct a model for the ontology $\mathcal{O}_1 = (\mathbf{TBox}, Cow(mimosa))$.

Answer: A model is $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ (others exist), where the domain is $\Delta^{\mathcal{I}} = \{m, e\}$ and the interpretation mapping is:

$$\begin{array}{lcl} mimosa^{\mathcal{I}} & = & m \\ Cow^{\mathcal{I}} & = & \{m\} \\ MadCow^{\mathcal{I}} & = & \{\} \\ Sheep^{\mathcal{I}} & = & \{\} \\ BrainOfSheep^{\mathcal{I}} & = & \{\} \\ Animal^{\mathcal{I}} & = & \{\} \\ PartOfAnimal^{\mathcal{I}} & = & \{\} \\ Vegetarian^{\mathcal{I}} & = & \{m\} \\ eat^{\mathcal{I}} & = & \{(m,e)\} \end{array}$$

All assertions must be satisfied, i.e. $\mathcal{I} \models \mathcal{O}_1$ iff $\mathcal{I} \models \mathbf{TBox}$ and $\mathcal{I} \models Cow(mimosa)$:

$\mathcal{I} \models Cow \sqsubseteq Vegetarian$ iff $Cow^{\mathcal{I}} \subseteq Vegetarian^{\mathcal{I}}$ iff $\{m\} \subseteq \{m\}$
$\mathcal{I} \models MadCow \sqsubseteq Cow \sqcap \exists\ eat.BrainOfSheep$ iff $MadCow^{\mathcal{I}} \subseteq Cow^{\mathcal{I}} \cap (\exists\ eat.BrainOfSheep)^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$
$\mathcal{I} \models Sheep \sqsubseteq Animal$ iff $Sheep^{\mathcal{I}} \subseteq Animal^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$
$\mathcal{I} \models Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)$
$\mathcal{I} \models BrainOfSheep \sqsubseteq PartOfAnimal$ iff $BrainOfSheep^{\mathcal{I}} \subseteq PartOfAnimal^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$

$\mathcal{I} \models Cow(mimosa)$ iff $mimosa^{\mathcal{I}} \in Cow^{\mathcal{I}}$ iff $m \in \{m\}$

3. Show that there is no model for the ontology $\mathcal{O}_2 = (\mathbf{TBox}, MadCow(mimosa))$.

We will show that it is impossible to construct an interpretation $\mathcal{I}$ that satisfies $\mathcal{O}_2$.
So suppose there is an interpretation that models $\mathcal{O}_2$.
Since we have to satisfy assertion $MadCow(mimosa)$, there is an individual m in the domain of $\mathcal{I}$ such that $mimosa^{\mathcal{I}} = m$ and $mimosa^{\mathcal{I}} \in MadCow^{\mathcal{I}}$, i.e., $m \in MadCow^{\mathcal{I}}$.
Since every $MadCow$ is a Cow, $m \in Cow^{\mathcal{I}}$ holds, and furthermore $m \in Vegetarian^{\mathcal{I}}$. Moreover, every $MadCow$ eats at least some brain of sheep (let's denote this brain by b, and thus $b \in BrainOfSheep^{\mathcal{I}}$ and $(m, b) \in eat^{\mathcal{I}}$. In addition, $b \in PartOfAnimal^{\mathcal{I}}$. But then, since $m \in Vegetarian^{\mathcal{I}}$, we also require that $m \in (\forall eat.\neg(Animal \sqcup PartOfAnimal))^{\mathcal{I}}$. Since $(m, b) \in eat^{\mathcal{I}}$, $b \in (\neg(Animal \sqcup PartOfAnimal))^{\mathcal{I}}$, i.e., $b \notin (Animal \sqcup PartOfAnimal)^{\mathcal{I}}$, and in particular $b \notin PartOfAnimal^{\mathcal{I}}$. We derive a contradiction.