Noncooperative Game Theory

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Section 1

Self-interested agents

Self-interested agents

- What does it mean to say that an agent is self-interested?
  - not that they want to harm other agents
  - not that they only care about things that benefit them
  - that the agent has its own description of states of the world that it likes, and that its actions are motivated by this description
- Utility theory:
  - quantifies degree of preference across alternatives
  - understand the impact of uncertainty on these preferences
  - utility function: a mapping from states of the world to real numbers, indicating the agent's level of happiness with that state of the world
    - expected utility: when the agent is uncertain about which state of the world he faces, his utility is defined as the expected value of his utility function with respect to the appropriate probability distribution over states
  - Decision-theoretic rationality: take actions to maximize expected utility.

Example: friends and enemies

- Alice has three options: club (c), movie (m), watching a video at home (h)
- On her own, her utility for these three outcomes is 100 for c, 50 for m and 50 for h
- However, Alice also cares about Bob (who she hates) and Carol (who she likes)
  - Bob is at the club 60% of the time, and at the movies otherwise
  - Carol is at the movies 75% of the time, and at the club otherwise
- If Alice runs into Bob at the movies, she suffers disutility of 40; if she sees him at the club she suffers disutility of 90.
- If Alice sees Carol, she enjoys whatever activity she's doing 1.5 times as much as she would have enjoyed it otherwise (taking into account the possible disutility caused by Bob)
- What should Alice do (show of hands)?

What activity should Alice choose?

$A = c$

	$B = c$	$B = m$
$C = c$	15	150
$C = m$	10	100

$A = m$

	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

- Alice's expected utility for c:

$$0.25(0.6 \cdot 15 + 0.4 \cdot 150) + 0.75(0.6 \cdot 10 + 0.4 \cdot 100) = 51.75.$$

- Alice's expected utility for m:

$$0.25(0.6 \cdot 50 + 0.4 \cdot 10) + 0.75(0.6(75) + 0.4(15)) = 46.75.$$

- Alice's expected utility for h: 50.

Alice prefers to go to the club (though Bob is often there and Carol rarely is), and prefers staying home to going to the movies (though Bob is usually not at the movies and Carol almost always is).

Section 2

Utility Theory

Why utility?

- Why would anyone argue with the idea that an agent's preferences could be described using a utility function as we just did?
  - why should a single-dimensional function be enough to explain preferences over an arbitrarily complicated set of alternatives?
  - Why should an agent's response to uncertainty be captured purely by the <u>expected value</u> of his utility function?
- It turns out that the claim that an agent has a utility function is substantive.

Preferences Over Outcomes

If o_1 and o_2 are outcomes

- $o_1 \succeq o_2$ means o_1 is at least as desirable as o_2.
  - read this as "the agent weakly prefers o_1 to o_2"
- $o_1 \sim o_2$ means $o_1 \succeq o_2$ and $o_2 \succeq o_1$.
  - read this as "the agent is indifferent between o_1 and o_2."
- $o_1 \succ o_2$ means $o_1 \succeq o_2$ and $o_2 \not\succeq o_1$
  - read this as "the agent strictly prefers o_1 to o_2"

Lotteries

- An agent may not know the outcomes of his actions, but may instead only have a probability distribution over the outcomes.

Definition (lottery)

A lottery is a probability distribution over outcomes. It is written

$$[p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]$$

where the o_i are outcomes and each $p_i \geq 0$ and

$$\sum_i p_i = 1$$

- The lottery specifies that outcome o_i occurs with probability p_i.
- We will consider lotteries to be outcomes.

Preference Axioms: Completeness

Definition (Completeness)

A preference relationship must be defined between every pair of outcomes:

$$\forall o_1 \forall o_2 \; o_1 \succ o_2 \text{ or } o_2 \succ o_1 \text{ or } o_1 \sim o_2$$

Preference Axioms: Transitivity

Definition (Transitivity)

Preferences must be transitive:

$$\text{if } o_1 \succeq o_2 \text{ and } o_2 \succeq o_3 \text{ then } o_1 \succeq o_3$$

- This makes good sense: otherwise $o_1 \succeq o_2$ and $o_2 \succeq o_3$ and $o_3 \succ o_1$.
- An agent should be prepared to pay some amount to swap between an outcome they prefer less and an outcome they prefer more
- Intransitive preferences mean we can construct a “money pump”!

Definition (Monotonicity)

An agent prefers a larger chance of getting a better outcome to a smaller chance:

- If $o_1 \succ o_2$ and $p > q$ then

$$[p : o_1, 1 - p : o_2] \succ [q : o_1, 1 - q : o_2]$$

Preference Axioms

Let $P_\ell(o_i)$ denote the probability that outcome o_i is selected by lottery ℓ. For example, if $\ell = [0.3 : o_1; 0.7 : [0.8 : o_2; 0.2 : o_1]]$ then $P_\ell(o_1) = 0.44$ and $P_\ell(o_3) = 0$.

Definition (Decomposability ("no fun in gambling"))

If $\forall o_i \in O$, $P_{\ell_1}(o_i) = P_{\ell_2}(o_i)$ then $\ell_1 \sim \ell_2$.

Preference Axioms

Definition (Substitutability)

If $o_1 \sim o_2$ then for all sequences of one or more outcomes $o_3, \ldots, o_k$ and sets of probabilities $p, p_3, \ldots, p_k$ for which $p + \sum_{i=3}^{k} p_i = 1$,
$[p : o_1, p_3 : o_3, \ldots, p_k : o_k] \sim [p : o_2, p_3 : o_3, \ldots, p_k : o_k]$.

Definition (Continuity)

Suppose $o_1 \succ o_2$ and $o_2 \succ o_3$, then there exists a $p \in [0, 1]$ such that $o_2 \sim [p : o_1, 1 - p : o_3]$.

Preferences and utility functions

Theorem (von Neumann and Morgenstern, 1944)

If an agent's preference relation satisfies the axioms Completeness, Transitivity, Decomposability, Substitutability, Monotonicity and Continuity then there exists a function $u : O \rightarrow [0, 1]$ with the properties that:

1. $u(o_1) \geq u(o_2)$ *iff the agent prefers* o_1 *to* o_2*; and*
2. *when faced about uncertainty about which outcomes he will receive, the agent prefers outcomes that maximize the expected value of* u*.*

Proof idea:

- define the utility of the best outcome $u(\overline{o}) = 1$ and of the worst $u(\underline{o}) = 0$
- now define the utility of each other outcome o as the p for which $o \sim [p : \overline{o}; (1 - p) : \underline{o}]$.

Section 3

What is Game Theory?

Non-Cooperative Game Theory

- What is it?
  - mathematical study of interaction between rational, self-interested agents
- Why is it called non-cooperative?
  - while it's most interested in situations where agents' interests conflict, it's not restricted to these settings
  - the key is that the individual is the basic modeling unit, and that individuals pursue their own interests
    - cooperative/coalitional game theory has teams as the central unit, rather than agents

TCP Backoff Game

- Consider this situation as a two-player game:
  - both use a correct implementation: both get 1 ms delay
  - one correct, one defective: 4 ms delay for correct, 0 ms for defective
  - both defective: both get a 3 ms delay.

- Questions:
  - What action should a player of the game take?
  - Would all users behave the same in this scenario?
  - What global patterns of behaviour should the system designer expect?
  - Under what changes to the delay numbers would behavior be the same?
  - What effect would communication have?
  - Repetitions? (finite? infinite?)
  - Does it matter if I believe that my opponent is rational?

Defining Games – Key Ingredients

- Players: who are the decision makers?
  - People? Governments? Companies? Somebody employed by a company? Computational Agent?...
- Actions: what can the players do?
  - Enter a bid in an auction? Decide whether to end a strike? Decide when to sell a stock? Decide how to vote? Decide how to play a game?...
- Payoffs: what motivate players?
  - Do they care about some profit? Do they care about other players?...

Defining Games – Two Standard Representations

- Normal Form (a.k.a. Matrix Form, Strategic Form) List what payoffs players get as a function of their actions
  - It is as if players moved simultaneously
  - but strategies encode many things...
- Extensive Form Includes timing of moves (later in the course)
  - Players move sequentially, represented as a tree
    - Chess: white player moves, then black player can see white's move and react...
  - Keeps track of what each player knows when they make each decision
    - Poker: bet sequentially – what can a given player see when they bet?

Defining Games - Normal Form Games

- A finite, n-person, normal form game is a triple $\langle N, A, u \rangle$:
  - Players: $N = \{1, ..., n\}$ is a finite set of n players, typically indexed by i
  - Space of Action profiles: $A = A_1 \times \ldots \times A_n$, where A_i is the action set for player i
    - $a = (a_1, \ldots, a_n) \in A$ is an action profile.
  - Utility function or Payoff function for each player i: $u_i : A \mapsto \mathbb{R}$
    - $u = (u_1, \ldots, u_n)$, is a profile of utility functions

- Standard Matrix Representation – writing a 2-player game as a matrix:
  - row player is player 1, column player is player 2
  - rows are actions $a \in A_1$, columns are $a' \in A_2$
  - cells are outcomes, written as a tuple of utility values for each player

Games in Matrix Form

TCP Backoff Game written as a matrix ("normal form").

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

A Large Collective Action Game

Some normal form games cannot be conveniently written as a matrix, so we can write them abstractly:

- Players: $N = \{1, \ldots, 10.000.000\}$
- Action Set for player i: $A_i = \{Revolt, Not\}$
- Utility Function for player i:
  - $u_i(a) = 1$ if $\#\{j : a_j = Revolt\} \geq 2.000.000$
  - $u_i(a) = -1$ if $\#\{j : a_j = Revolt\} < 2.000.000$ and $a_i = Revolt$
  - $u_i(a) = 0$ if $\#\{j : a_j = Revolt\} < 2.000.000$ and $a_i = Not$

Section 4

Example Matrix Games

More General Form

Prisoner's dilemma is any game

	C	D
C	a, a	b, c
D	c, b	d, d

with $c > a > d > b$.

Games of Pure Competition

Players have exactly opposed interests

- There must be precisely two players (otherwise they can't have exactly opposed interests)
- For all action profiles $a \in A$, $u_1(a) + u_2(a) = c$ for some constant c
  - Special case: zero sum
- Thus, we only need to store a utility function for one player
  - in a sense, it's a one-player game

Matching Pennies

One player wants to match; the other wants to mismatch.

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

Play this game with someone near you, repeating five times.

Rock-Paper-Scissors

Generalized matching pennies.

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

...Believe it or not, there’s an annual international competition for this game!

Games of Cooperation

Players have exactly the same interests.

- no conflict: all players want the same things
- $\forall a \in A, \forall i, j,\ u_i(a) = u_j(a)$
- we often write such games with a single payoff per cell
- why are such games "noncooperative"?

Coordination Game

Which side of the road should you drive on?

	Left	Right
Left	1,1	0,0
Right	0,0	1,1

Play this game with someone near you. Then find a new partner and play again. Play five times in total.

General Games: Battle of the Sexes

The most interesting games combine elements of cooperation and competition.

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Play this game with someone near you. Then find a new partner and play again. Play five times in total.

Section 5

Pareto Optimality

Analyzing Games

- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - we have no way of saying that one agent's interests are more important than another's
  - intuition: imagine trying to find the revenue-maximizing outcome when you don't know what currency has been used to express each agent's payoff
- Are there situations where we can still prefer one outcome to another?

Pareto Optimality

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
  - in this case, it seems reasonable to say that o is better than o'
  - we say that o Pareto-dominates o'.

Definition (Pareto Optimality)

An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.

- can a game have more than one Pareto-optimal outcome?
- does every game have at least one Pareto-optimal outcome?

Pareto Optimal Outcomes in Example Games

	Left	Right
Left	1,1	0,0
Right	0,0	1,1

	B	F
B	2, 1	0, 0
F	0, 0	1, 2

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Section 6

Strategic Reasoning

Keynes' Beauty Contest Game

- You hold a stock and the price is rising...
- You believe that the price is too high to be justified by the value of the company.
- You would like to sell it, but would like to wait until the price is almost at its peak.
- You would like to get out of the market just before other investors do.
- How will they act? What should you do in response?

Keynes' Beauty Contest Game

- Pick a number between 1 and 100.
- The player who names the integer closest to two thirds of the average integer wins a prize, the other players get nothing.
- Ties are broken uniformly at random.

Section 7

Best Response and Nash Equilibrium

Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.
  - now $a = (a_{-i}, a_i)$

- Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

Nash Equilibrium

- Now let's return to the setting where no agent knows anything about what the others will do
- What can we say about which actions will occur?

- Idea: look for stable action profiles.

Definition (Nash Equilibrium)

$a = \langle a_1, \ldots, a_n \rangle$ is a ("pure strategy") Nash equilibrium iff $\forall i,\ a_i \in BR(a_{-i})$.

Nash Equilibria of Example Games

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

	Left	Right
Left	1,1	0,0
Right	0,0	1,1

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

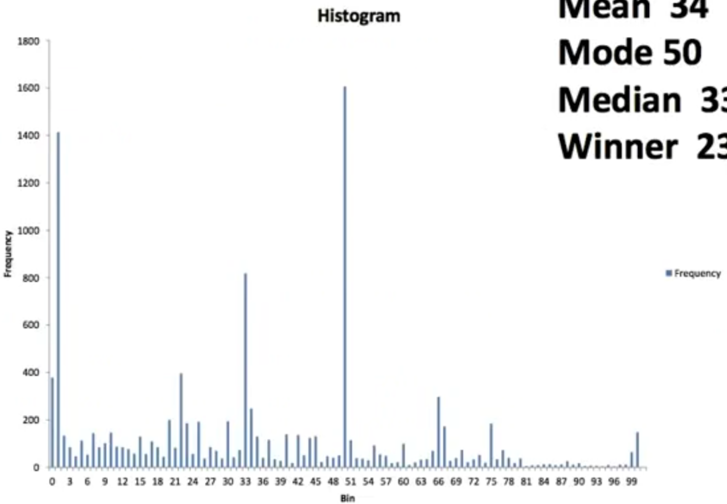
The paradox of Prisoner's dilemma: the Nash equilibrium is the only non-Pareto-optimal outcome!

Solving Keynes' Beauty Contest Game

- Suppose a player believes that the average play will be X (including his own integer)
- That player's optimal strategy is to say the closest integer to $\frac{2}{3}X$.
- X has to be less than 100, so the optimal strategy of any player has to be no more than 67.
- If X is no more than 67, then the optimal strategy of any player has to be no more than $\frac{2}{3}67$.
- If X is no more than $\frac{2}{3}67$, then the optimal strategy of any player has to be no more than $(\frac{2}{3})^2 67$.
- Iterating, everyone will end up announcing 1.

Keynes’ Beauty Contest Game

Online course: more than 10000 players:



2012 GTOC

Mean 34

Mode 50

Median 33

Winner 23

Section 8

Mixed Strategies

Mixed Strategies

UN soldiers inspecting a taxi at a security checkpoint in Kismayro Port, Somalia, 2006

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i.
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
    - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

Definition (Best response)

$s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i,\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

Definition (Nash equilibrium)

$s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i,\ s_i \in BR(s_{-i})$

Theorem (Nash, 1950)

Every finite game has a Nash equilibrium

e.g., matching pennies: both players play heads/tails 50%/50%

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support
- Let player 2 play B with p, F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

$$u_1(B) = u_1(F)$$
$$2p + 0(1 - p) = 0p + 1(1 - p)$$
$$p = \frac{1}{3}$$

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support
- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with $1 - q$.

$$u_2(B) = u_2(F)$$

$$q + 0(1-q) = 0q + 2(1-q)$$

$$q = \frac{2}{3}$$

- Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
  - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.