Auctions

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Section 1

Auctions

Motivation

- Auctions are any mechanisms for allocating resources among self-interested agents
- Very widely used
  - government sale of resources
  - privatization
  - stock market
  - request for quote
  - FCC spectrum
  - real estate sales
  - eBay

CS Motivation

- resource allocation is a fundamental problem in CS
- increasing importance of studying distributed systems with heterogeneous agents
- markets for:
  - computational resources (JINI, etc.)
  - P2P systems
  - network bandwidth
- currency needn't be real money, just something scarce
  - that said, real money trading agents are also an important motivation

Section 2

Canonical Single-Good Auctions

Some Canonical Auctions

- English
- Japanese
- Dutch
- First-Price
- Second-Price
- All-Pay

English Auction

English Auction

- auctioneer starts the bidding at some “reservation price”
- bidders then shout out ascending prices
- once bidders stop shouting, the high bidder gets the good at that price

Japanese Auction

Japanese Auction

- Same as an English auction except that the auctioneer calls out the prices
- all bidders start out standing
- when the price reaches a level that a bidder is not willing to pay, that bidder sits down
  - once a bidder sits down, they can't get back up
- the last person standing gets the good

- analytically more tractable than English because jump bidding can't occur
  - consider the branching factor of the extensive form game...

Dutch Auction

Dutch Auction

- the auctioneer starts a clock at some high value; it descends
- at some point, a bidder shouts “mine!” and gets the good at the price shown on the clock

First-, Second-Price Auctions

First-Price Auction

- bidders write down bids on pieces of paper
- auctioneer awards the good to the bidder with the highest bid
- that bidder pays the amount of his bid

Second-Price Auction

- bidders write down bids on pieces of paper
- auctioneer awards the good to the bidder with the highest bid
- that bidder pays the amount bid by the second-highest bidder

All-Pay auction

All-Pay Auction

- bidders write down bids on pieces of paper
- auctioneer awards the good to the bidder with the highest bid
- everyone pays the amount of their bid regardless of whether or not they win

Auctions as Structured Negotiations

Any negotiation mechanism that is:

- market-based (determines an exchange in terms of currency)
- mediated (auctioneer)
- well-specified (follows rules)

Defined by three kinds of rules:

- rules for bidding
- rules for what information is revealed
- rules for clearing

Auctions as Structured Negotiations

Defined by three kinds of rules:

- rules for bidding
  - who can bid, when
  - what is the form of a bid
  - restrictions on offers, as a function of:
    - bidder's own previous bid
    - auction state (others' bids)
    - eligibility (e.g., budget constraints)
    - expiration, withdrawal, replacement
- rules for what information is revealed
- rules for clearing

Auctions as Structured Negotiations

Defined by three kinds of rules:

- rules for bidding
- rules for what information is revealed
  - when to reveal what information to whom
- rules for clearing

Auctions as Structured Negotiations

Defined by three kinds of rules:

- rules for bidding
- rules for what information is revealed
- rules for clearing
  - when to clear
    - at intervals
    - on each bid
    - after a period of inactivity
  - allocation (who gets what)
  - payment (who pays what)

Section 3

Second-price auctions

Second-Price

Theorem

Truth-telling is a dominant strategy in a second-price auction.

- In fact, we know this already (do you see why?)
- However, we'll look at a simple, direct proof.

Second-Price proof

Theorem

Truth-telling is a dominant strategy in a second-price auction.

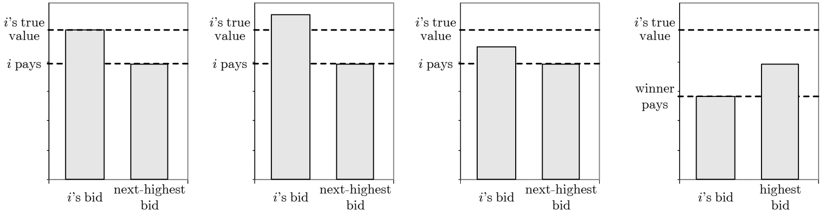
Proof.

Assume that the other bidders bid in some arbitrary way. We must show that i's best response is always to bid truthfully. We'll break the proof into two cases:

1. Bidding honestly, i would win the auction
2. Bidding honestly, i would lose the auction

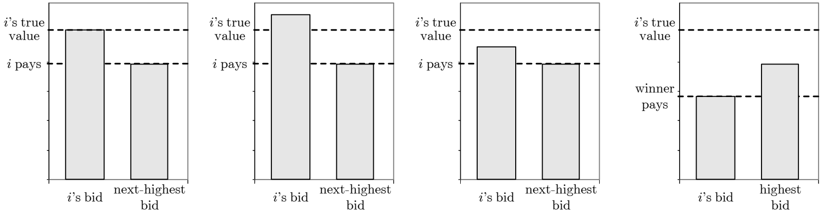
□

Second-Price proof (2)



- Bidding honestly, i is the winner
- If i bids higher, he will still win and still pay the same amount
- If i bids lower, he will either still win and still pay the same amount. . .

Second-Price proof (2)



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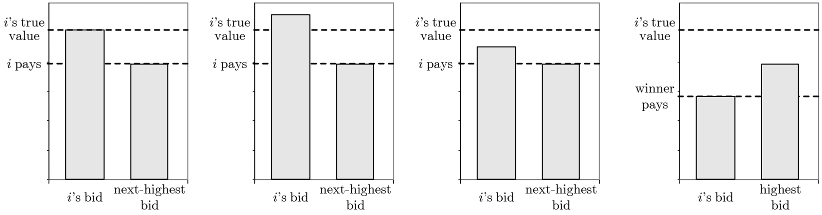

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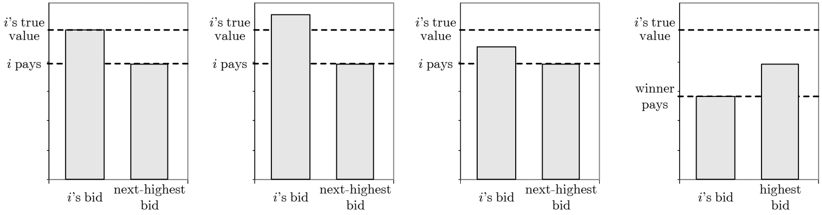

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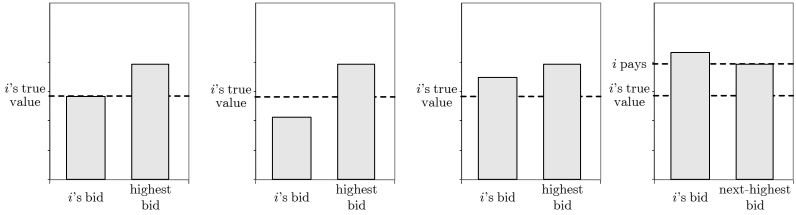

- Bidding honestly, i is not the winner
- If i bids lower, he will still lose and still pay nothing
- If i bids higher, he will either still lose and still pay nothing… or win and pay more than his valuation.

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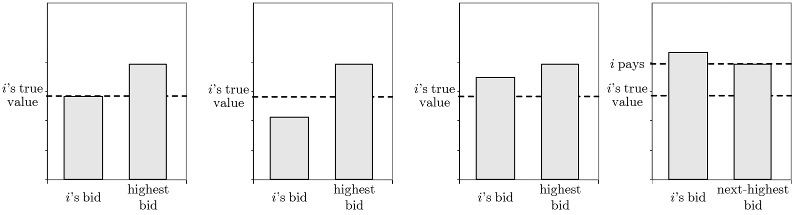

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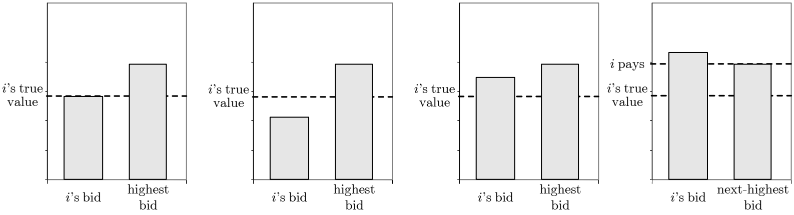

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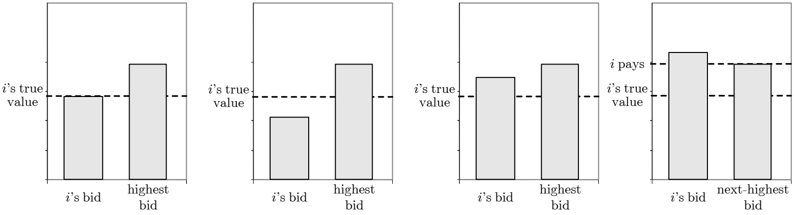

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English and Japanese auctions

- A much more complicated strategy space
  - extensive form game
  - bidders are able to condition their bids on information revealed by others
  - in the case of English auctions, the ability to place jump bids
- intuitively, though, the revealed information doesn't make any difference in the IPV setting.

Theorem

Under the independent private values model (IPV), it is a dominant strategy for bidders to bid up to (and not beyond) their valuations in both Japanese and English auctions.

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Section 4

First-Price Auctions

First-Price and Dutch

Theorem

First-Price and Dutch auctions are strategically equivalent.

- In both first-price and Dutch, a bidder must decide on the amount he's willing to pay, conditional on having placed the highest bid.
  - despite the fact that Dutch auctions are extensive-form games, the only thing a winning bidder knows about the others is that all of them have decided on lower bids
    - e.g., he does not know what these bids are
    - this is exactly the thing that a bidder in a first-price auction assumes when placing his bid anyway.
- Note that this is a stronger result than the connection between second-price and English.

Discussion

- So, why are both auction types held in practice?
  - First-price auctions can be held asynchronously
  - Dutch auctions are fast, and require minimal communication: only one bit needs to be transmitted from the bidders to the auctioneer.
- How should bidders bid in these auctions?
  - They should clearly bid less than their valuations.
  - There's a tradeoff between:
    - probability of winning
    - amount paid upon winning
  - Bidders don't have a dominant strategy any more.

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Analysis

Theorem

In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from $[0, 1]$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ *is a Bayes-Nash equilibrium strategy profile.*

Proof.

Assume that bidder 2 bids $\frac{1}{2}v_2$, and bidder 1 bids s_1. From the fact that v_2 was drawn from a uniform distribution, all values of v_2 between 0 and 1 are equally likely. Bidder 1's expected utility is

$$E[u_1] = \int_0^1 u_1 dv_2. \tag{1}$$

Note that the integral in Equation (1) can be broken up into two smaller integrals that differ on whether or not player 1 wins the auction.

$$E[u_1] = \int_0^{2s_1} u_1 dv_2 + \int_{2s_1}^1 u_1 dv_2$$

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Proof (continued).

We can now substitute in values for u_1. In the first case, because 2 bids $\frac{1}{2}v_2$, 1 wins when $v_2 < 2s_1$, and gains utility $v_1 - s_1$. In the second case 1 loses and gains utility 0. Observe that we can ignore the case where the agents have the same valuation, because this occurs with probability zero.

$$\begin{aligned} E[u_1] &= \int_0^{2s_1} (v_1 - s_1) dv_2 + \int_{2s_1}^{1} (0) dv_2 \\ &= 2v_1 s_1 - 2s_1^2 \end{aligned} \tag{2}$$

□

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Proof (continued).

We can find bidder 1's best response to bidder 2's strategy by taking the derivative of Equation (2) and setting it equal to zero:

$$\begin{aligned} \frac{\partial}{\partial s_1}(2v_1 s_1 - 2s_1^2) &= 0 \\ 2v_1 - 4s_1 &= 0 \\ s_1 &= \frac{1}{2}v_1 \end{aligned}$$

Thus when player 2 is bidding half her valuation, player 1's best strategy is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game and the equilibrium. □

More than two bidders

- Very narrow result: two bidders, uniform valuations.
- Still, first-price auctions are not incentive compatible
  - hence, unsurprisingly, not equivalent to second-price auctions

Theorem

In a first-price sealed bid auction with n risk-neutral agents whose valuations are independently drawn from a uniform distribution on the same bounded interval of the real numbers, the unique symmetric equilibrium is given by the strategy profile $(\frac{n-1}{n}v_1, \ldots, \frac{n-1}{n}v_n)$.

- proven using a similar argument, but more involved calculus
- a broader problem: that proof only showed how to verify an equilibrium strategy.
  - How do we identify one in the first place?

Section 5

Revenue Equivalence

Revenue Equivalence

- Which auction should an auctioneer choose? To some extent, it doesn't matter...

Theorem (Revenue Equivalence Theorem)

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, each drawn from cumulative distribution F. Then any auction mechanism in which

- *in equilibrium, the good will be allocated in the same way (e.g. to the agent with the highest valuation); and*
- *any agent with valuation 0 has an expected utility of 0;*

yields the same expected revenue, and hence results in any bidder with valuation v making the same expected payment.

Section 6

Risk Attitudes

Risk Attitudes

What kind of auction would the auctioneer prefer?

- Buyer is not risk neutral:
  - no change under various risk attitudes for second price
  - in first-price, increasing bid amount increases probability of winning, decreases profit. This is good for risk-averse bidder, bad for risk-seeking bidder.
  - Risk averse, IPV: First $\succ$ [Japanese = English = Second]
  - Risk seeking, IPV: Second $\succ$ First
- Auctioneer is not risk neutral:
  - revenue is fixed in first-price auction (the expected amount of the second-highest bid)
  - revenue varies in second-price auction, with the same expected value
  - thus, a risk-averse seller prefers first-price to second-price.

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