Noncooperative Game Theory

J. Leite (adapted from Kevin Leyton-Brown)

Cortes and The Burning of the Boats

Section 1

Perfect-Information Extensive-Form Games

Introduction

- The normal form game representation does not incorporate any notion of sequence, or time, of the actions of the players
- The extensive form is an alternative representation that makes the temporal structure explicit.
- Two variants:
  - perfect information extensive-form games
  - imperfect-information extensive-form games

Definition

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- Players: N is a set of n players

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- Actions: A is a (single) set of actions

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- Players: N
- Actions: A
- Choice nodes and labels for these nodes:
  - Choice nodes: H is a set of non-terminal choice nodes

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  - Player function: $\rho : H \rightarrow N$ assigns to each non-terminal node h a player $i \in N$ who chooses an action at h

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  - Action function: $\chi : H \to 2^A$
  - Player function: $\rho : H \to N$
- Terminal nodes: Z is a set of terminal nodes, disjoint from H

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- Players: N
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  - Action function: $\chi : H \to 2^A$
  - Player function: $\rho : H \to N$
- Terminal nodes: Z
- Successor function: $\sigma : H \times A \to H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
  - The choice nodes form a tree, so we can identify a node with its history.

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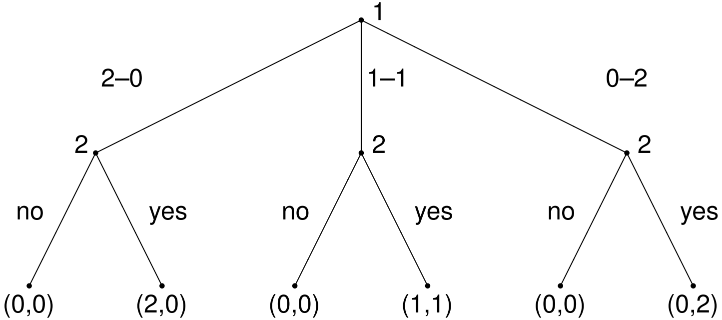
  - Action function: $\chi : H \rightarrow 2^A$
  - Player function: $\rho : H \rightarrow N$
- Terminal nodes: Z
- Successor function: $\sigma : H \times A \rightarrow H \cup Z$
- Utility function: $u = (u_1, \ldots, u_n)$; $u_i : Z \rightarrow \mathbb{R}$ is a utility function for player i on the terminal nodes Z

Example: the sharing game



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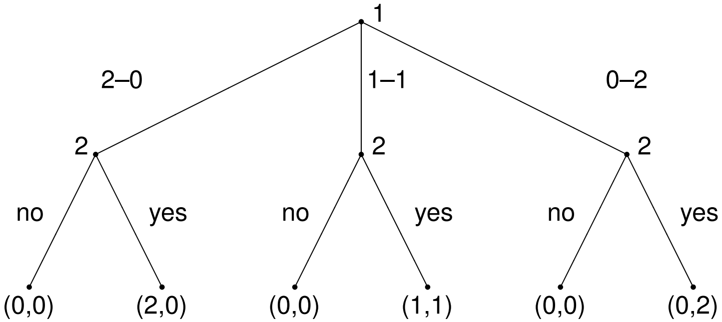
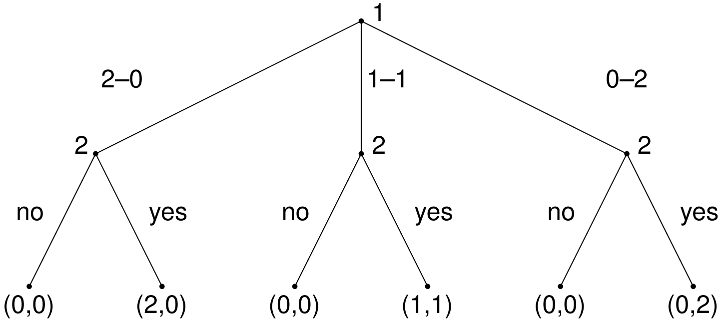


- Play as a fun game, dividing 100 euros in coins. (Play each partner only once.)

Example: the sharing game



- In the sharing game (splitting 2 coins) how many pure strategies does each player have?

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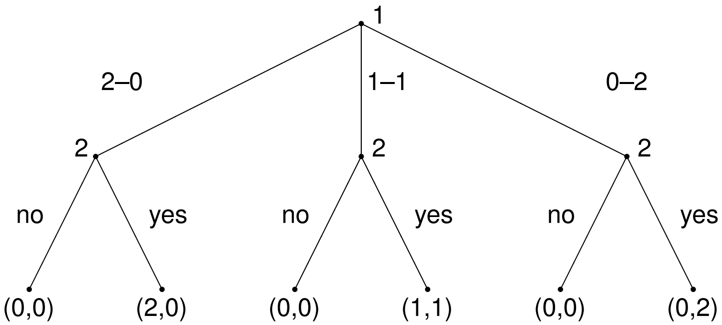

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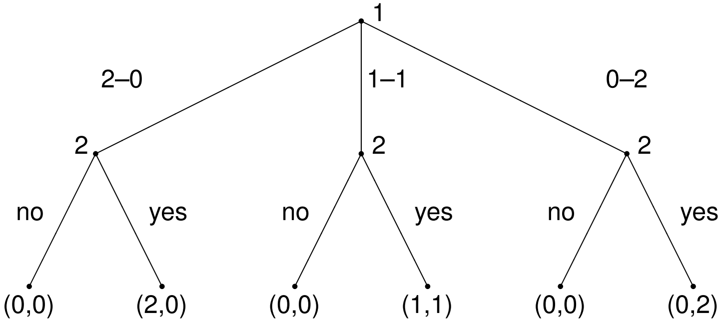


- In the sharing game (splitting 2 coins) how many pure strategies does each player have?
  - player 1: 3
  - player 2: 8

Pure Strategies

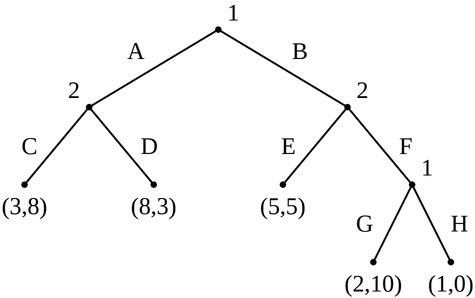
- Overall, a pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player.

Definition (pure strategies)

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consist of the cross product

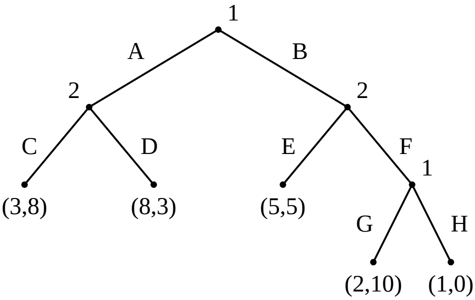
$$\bigtimes_{h \in H, \rho(h)=i} \chi(h)$$

Pure Strategies Example



What are the pure strategies for player 2?

Pure Strategies Example



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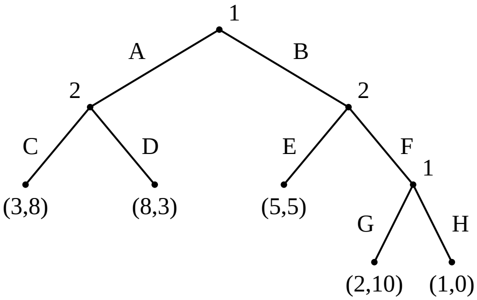
- $S_2 = \{(C, E); (C, F); (D, E); (D, F)\}$

Pure Strategies Example



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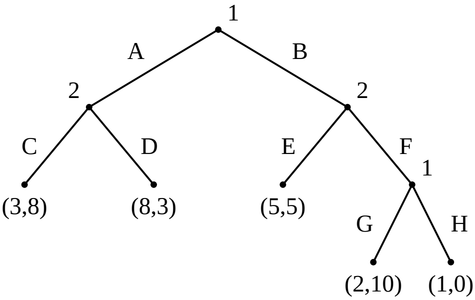


What are the pure strategies for player 2?

- $S_2 = \{(C,E);(C,F);(D,E);(D,F)\}$

What are the pure strategies for player 1?

- $S_1 = \{(B,G);(B,H),(A,G),(A,H)\}$
- This is true even though, conditional on taking A, the choice between G and H will never have to be made

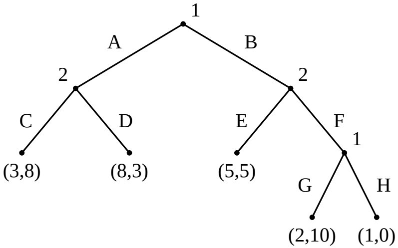
Nash Equilibria

Given our new definition of pure strategy, we are able to reuse our old definitions of:

- mixed strategies
- best response
- Nash equilibrium

Induced Normal Form

- In fact, the connection to the normal form is even tighter
  - we can "convert" an extensive-form game into normal form



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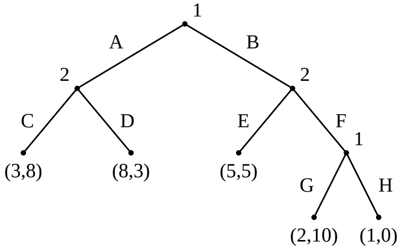

	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
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BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

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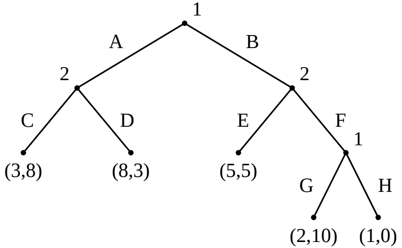

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- this illustrates the lack of compactness of the normal form
  - games aren't always this small
  - even here, we write down 16 payoff pairs instead of 5
  - exponential growth

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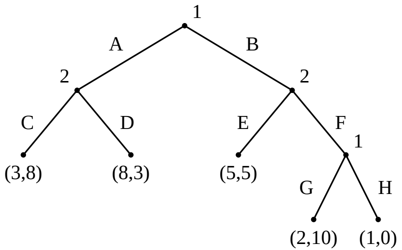

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- while we can write any extensive-form game as a NF, we can’t do the reverse.
  - e.g., matching pennies cannot be written as a perfect-information extensive form game

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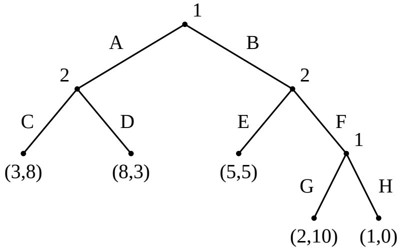

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Theorem

Every perfect information game in extensive form has a PSNE

- This is easy to see, since the players move sequentially.

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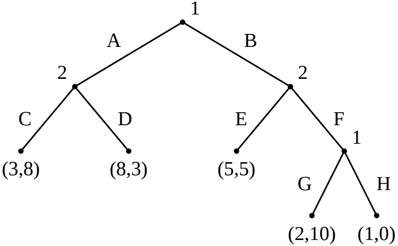

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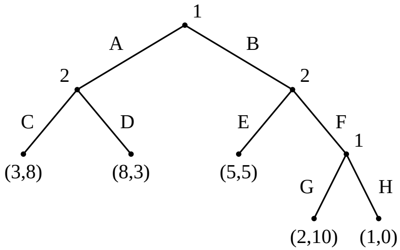

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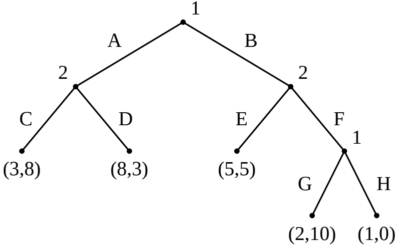

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- What are the (three) pure-strategy equilibria?
  - $(A,G),(C,F)$
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- One of these equilibria is preferable—which one?

Section 2

Subgame Perfection

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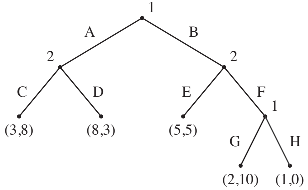
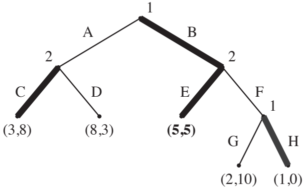


- Consider the equilibrium $(B, H), (C, E)$

Subgame Perfection



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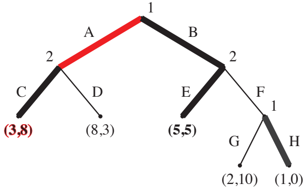

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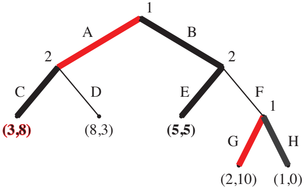

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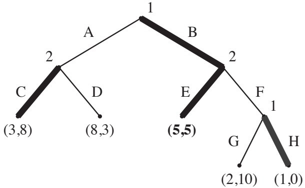

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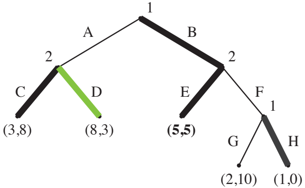

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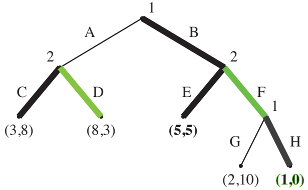

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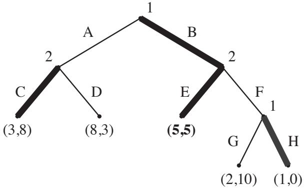

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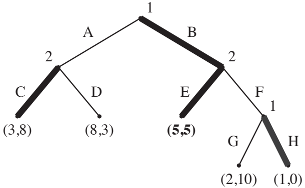

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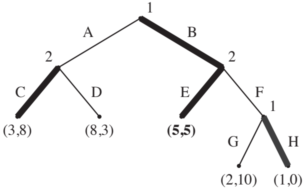

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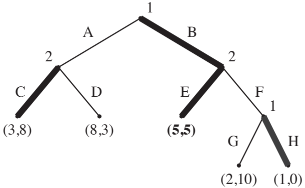

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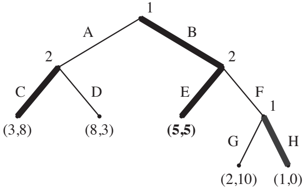

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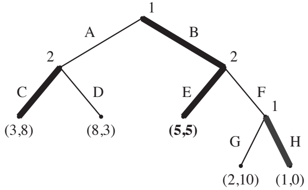

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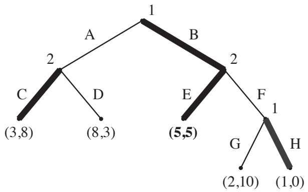

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  - He does it to threaten player 2, to prevent him from choosing F, and so gets 5
    - However, this seems like a non-credible threat
    - If player 1 reached his second decision node, would he really follow through and play H?

Formal Definition

Definition (subgame of G rooted at h)

The subgame of G rooted at h is the restriction of G to the descendents of H.

Definition (subgames of G)

The set of subgames of G is defined by the subgames of G rooted at each of the nodes in G.

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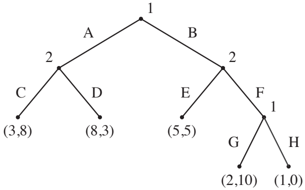
Definition (Subgame perfect equilibrium)

s is a subgame perfect equilibrium of G iff for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'

- Notes:
  - since G is its own subgame, every SPE is a NE.
  - this definition rules out “non-credible threats”

Which equilibria are subgame perfect?



- Which equilibria from the example are subgame perfect?
  - $(A,G),(C,F)$:
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  - $(A,H),(C,F)$:

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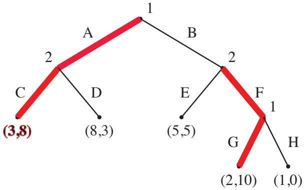

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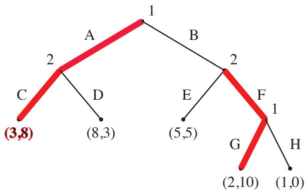

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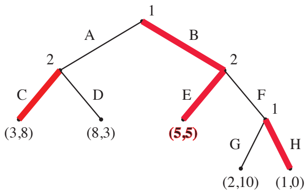

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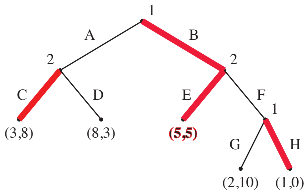

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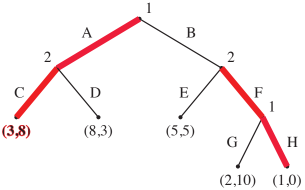

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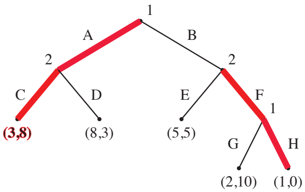
  - $(A, H), (C, F)$:

Which equilibria are subgame perfect?

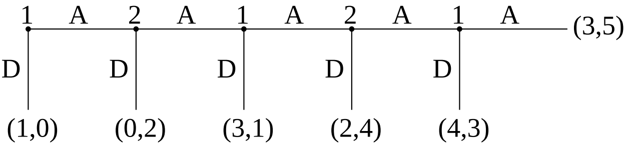


- Which equilibria from the example are subgame perfect?
  - $(A,G),(C,F)$: is subgame perfect
  - $(B,H),(C,E)$: (B,H) is an non-credible threat; not subgame perfect
  - $(A,H),(C,F)$: (A,H) is also non-credible, even though H is "off-path"

Section 3

Backward Induction

Centipede Game



- Play this as a fun game...

Computing Subgame Perfect Equilibria

Idea: Identify the equilibria in the bottom-most trees, and adopt these as one moves up the tree

function BACKWARDINDUCTION (node h) **returns** $u(h)$
if $h \in Z$ **then**
  return $u(h)$
$best\_util \leftarrow -\infty$
forall $a \in \chi(h)$ **do**
  $util_at_child \leftarrow$BACKWARDINDUCTION$(\sigma(h,a))$
  if $util\_at\_child_{\rho(h)} > best\_util_{\rho(h)}$ **then**
    $best\_util \leftarrow util\_at\_child$
return $best\_util$

- *util_at_child* is a vector denoting the utility for each player
- the procedure doesn't return an equilibrium strategy, but rather labels each node with a vector of real numbers.
  - This labeling can be seen as an extension of the game's utility function to the non-terminal nodes
  - The equilibrium strategies: take the best action at each node.

Computing Subgame Perfect Equilibria

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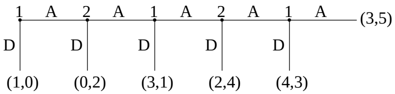
return $best\_util$

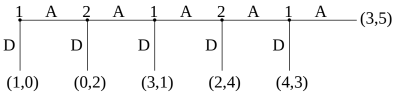
- For zero-sum games, BACKWARDINDUCTION has another name: the minimax algorithm.
  - Here it's enough to store one number per node.
  - It's possible to speed things up by pruning nodes that will never be reached in play: "alpha-beta pruning".

Backward Induction



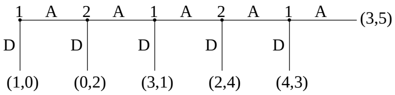
- What happens when we use this procedure on Centipede?

Backward Induction



- What happens when we use this procedure on Centipede?
  - In the only equilibrium, player 1 goes down in the first move.
  - However, this outcome is Pareto-dominated by all but one other outcome.

Backward Induction



- What happens when we use this procedure on Centipede?
  - In the only equilibrium, player 1 goes down in the first move.
  - However, this outcome is Pareto-dominated by all but one other outcome.
- Two considerations:
  - practical: human subjects don't go down right away
  - theoretical: what should you do as player 2 if player 1 doesn't go down?
    - SPE analysis says to go down. However, that same analysis says that P1 would already have gone down. How do you update your beliefs upon observation of a measure zero event?
    - but if player 1 knows that you'll do something else, it is rational for him not to go down anymore... a paradox
    - there's a whole literature on this question

Centipede Game

- Some Experimental Results (with 6 nodes):
  - 1% stop at 1st node;
  - 6% stop at 2nd node;
  - 21% stop at 3rd node;
  - 53% stop at 4th node;
  - 73% stop at 5th node;
  - 85% stop at 6th node;

Centipede Game

- Some Experimental Results (with 6 nodes):
  - 1% stop at 1st node;
  - 6% stop at 2nd node;
  - 21% stop at 3rd node;
  - 53% stop at 4th node;
  - 73% stop at 5th node;
  - 85% stop at 6th node;
- How to explain?
  - Bounded ability to reason
    - Player's own limitations
    - Or unsure of other player's reasoning
  - Altruism
    - Player's own altruism
    - Or belief that other player is altruistic

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players

Centipede Game

- Are Chess players playing centipede more rational?
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    - 69% end at first node

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    -

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    - 100% end at first node

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    - 100% end at first node
  - Students vs. Students
    -

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    - 100% end at first node
  - Students vs. Students
    - 3% end at first node

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    - 100% end at first node
  - Students vs. Students
    - 3% end at first node
  - Students vs. Chess players
    -

Centipede Game

- Are Chess players playing centipede more rational?
  - Chess players vs. Chess players
    - 69% end at first node
  - Grandmasters (player 1) vs. Chess players
    - 100% end at first node
  - Students vs. Students
    - 3% end at first node
  - Students vs. Chess players
    - 30% end at first node

Ultimatum Bargaining

- Player 1 makes an offer $x \in \{0, 1, ..., 10\}$ to player 2
- Player 2 can accept or reject
- 1 gets $10 - x$ and 2 gets x if accepted
- Both get 0 if rejected

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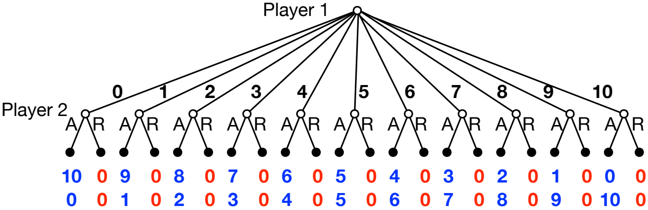

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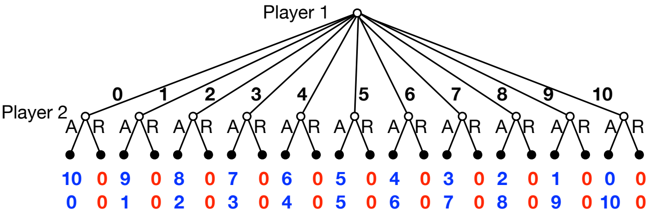
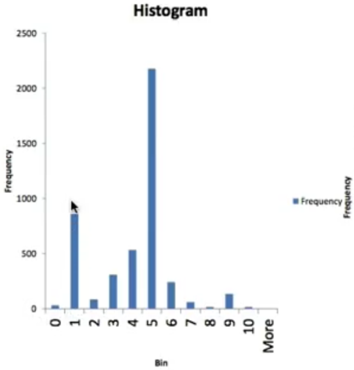
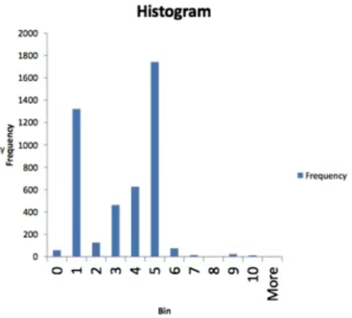


- Subgame Perfect Equlibria
  - Player 2 accepts every positive x.
  - If offered 0, Player 2 is indifferent could accept or reject (or even mix).
  - Player 1 offers either 0 or 1 depending on 2's decision at 0.

Ultimatum Bargaining

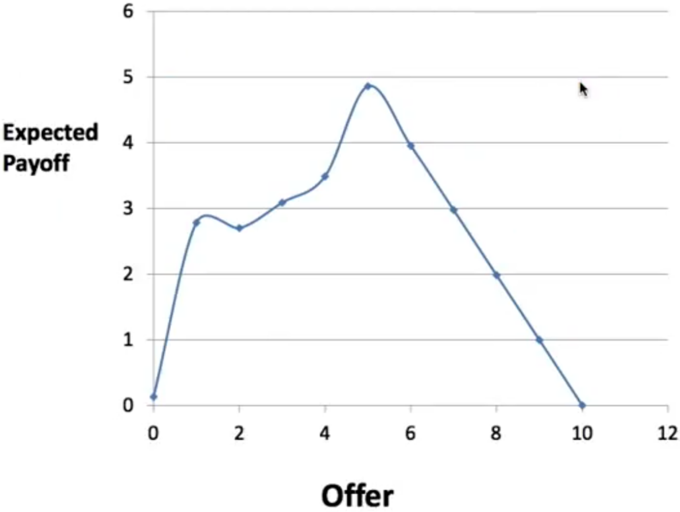


Offers



Min Accept

Ultimatum Bargaining



Ultimatum Bargaining

- Subgame perfection doesn't always match data.
- Rejections violate "rationality"?
- ... or do we have the payoffs incorrect: people value equity, or feel emotions: Behavioural Game Theory.