Knowledge Representation and Reasoning

Solutions to Review Exercises on First Order Logic

1 Alpine Club

Formulate the following pieces of knowledge as sentences of first-order logic:

Tony, Mike and John belong to the Alpine Club. Every member of the Alpine Club who is not a skier is a mountain climber. Mountain climbers do not like rain, and anyone who does not like snow is not a skier. Mike dislikes whatever Tony likes, and likes whatever Tony dislikes. Tony likes rain and snow.

Answer:
The answer uses the following predicates and constants:

- *Member*: unary predicate meaning a member of the Alpine Club;
- *Skier*: unary predicate meaning a skier;
- *Climber*: unary predicate meaning a climber;
- *Likes*: binary predicate where $Likes\,(x, y)$ means that x likes y;
- constants *tony*, *mike*, *john*, *rain*, *snow*.

In the translation, we name sentences so that it is easy to refer to them later.

- Tony, Mike and John belong to the Alpine Club.

$$S1 : Member\,(tony) \qquad S2 : Member\,(mike) \qquad S3 : Member\,(john)$$

  Alternatively, we may use one conjunction $Member\,(tony) \land Member\,(mike) \land Member\,(john)$.

- Every member of the Alpine Club who is not a skier is a mountain climber.

$$S4 : \forall x\,(Member\,(x) \land \neg Skier\,(x) \supset Climber\,(x))$$

- Mountain climbers do not like rain

$$S5 : \forall x\,(Climber\,(x) \supset \neg Likes\,(x, rain))$$

- and anyone who does not like snow is not a skier.

$$S6 : \forall x\,(\neg Likes\,(x, snow) \supset \neg Skier\,(x))$$

- Mike dislikes whatever Tony likes

$$S7 : \forall x\,(Likes\,(tony, x) \supset \neg Likes\,(mike, x))$$

- and likes whatever Tony dislikes.

$$S8 : \forall x\,(\neg Likes\,(tony, x) \supset Likes\,(mike, x))$$

- Tony likes rain and snow.

$$S9 : Likes\,(tony, rain)$$
$$S10 : Likes\,(tony, snow)$$

2 Reduction to CNF

Rewrite all sentences in $KB = \{(p \vee q) \supset r, r \supset s, p\}$ in conjunctive normal form, and present KB in clausal form.

Answer:

- $(p \vee q) \supset r$ is, by definition of $\supset$ equivalent to $\neg(p \vee q) \vee r$
  $\neg(p \vee q) \vee r$ is by de Morgan's law equivalent to $(\neg p \wedge \neg q) \vee r$
  By distributivity, $(\neg p \wedge \neg q) \vee r$ is equivalent to $(\neg p \vee r) \wedge (\neg q \vee r)$
  $(\neg p \vee r) \wedge (\neg q \vee r)$ is in CNF and corresponds to two clauses $[\neg p, r]\,[\neg q, r]$.
- $r \supset s$ is, by definition of $\supset$ equivalent to $\neg r \vee s$
  $\neg r \vee s$ is in CNF and corresponds to the clause $[\neg r, s]$.

The KB written in clausal form is $KB = \{[\neg p, r], [\neg q, r], [\neg r, s], [p]\}$.

3 Propositional Resolution

a) Show by resolution that the following set of clauses is inconsistent (derive empty clause from it):

$$[A, B, C], [A, B, \neg C], [A, \neg B, C], [A, \neg B, \neg C]$$
$$[\neg A, B, C], [\neg A, B, \neg C], [\neg A, \neg B, C], [\neg A, \neg B, \neg C]$$

b) Show by resolution that the following sentence is inconsistent:

$$\neg\neg A \wedge (\neg A \vee ((\neg B \vee C) \wedge B)) \wedge \neg C$$

Answer:
a) We can apply resolution as follows:

1. $[A, B]$ from $[A, B, C], [A, B, \neg C]$.
2. $[A, \neg B]$ from $[A, \neg B, C], [A, \neg B, \neg C]$.
3. $[A]$ from 1. and 2.
4. $[\neg A, B]$ from $[\neg A, B, C], [\neg A, B, \neg C]$.
5. $[\neg A, \neg B]$ from $[\neg A, \neg B, C], [\neg A, \neg B, \neg C]$.
6. $[\neg A]$ from 4. and 5.
7. $[]$ from 3. and 6.

b) We first need to transform into conjunctive normal form to obtain the clauses:

- $\neg\neg A \wedge (\neg A \vee ((\neg B \vee C) \wedge B)) \wedge \neg C$ is equivalent to
- $A \wedge (\neg A \vee \neg B \vee C) \wedge (\neg A \vee B) \wedge \neg C$ which corresponds to the clauses
- $[A], [\neg A, \neg B, C], [\neg A, B], [\neg C]$; Then:

1. $[\neg B, C]$ from $[A], [\neg A, \neg B, C]$
2. $[B]$ from $[A], [\neg A, B]$
3. $[C]$ from 1. and 2.
4. $[]$ from 3. and $[\neg C]$

4 First-Order Resolution

Determine whether the following sentences are valid using resolution:

a) $\exists x \forall y \forall z \left(\left(P\left(y\right) \supset Q\left(z\right)\right) \supset \left(P\left(x\right) \supset Q\left(x\right)\right)\right)$

b) $\exists x \left(P\left(x\right) \supset \forall y \left(P\left(y\right)\right)\right)$

c) $\neg \exists x \forall y \left(E\left(x, y\right) \leftrightarrow \neg E\left(y, y\right)\right)$

Answer:
To do this we need to check if from the negation of the sentence we can derive an empty clause (a contradiction).
a) First transform the negation into clausal form:

$$\begin{aligned}
&\neg \exists x \forall y \forall z \left(\left(P\left(y\right) \supset Q\left(z\right)\right) \supset \left(P\left(x\right) \supset Q\left(x\right)\right)\right)\\
&\neg \exists x \forall y \forall z \left(\neg \left(\neg P\left(y\right) \vee Q\left(z\right)\right) \vee \left(\neg P\left(x\right) \vee Q\left(x\right)\right)\right)\\
&\forall x \exists y \exists z \neg \left(\neg \left(\neg P\left(y\right) \vee Q\left(z\right)\right) \vee \left(\neg P\left(x\right) \vee Q\left(x\right)\right)\right)\\
&\forall x \exists y \exists z \left(\neg\neg \left(\neg P\left(y\right) \vee Q\left(z\right)\right) \wedge \neg \left(\neg P\left(x\right) \vee Q\left(x\right)\right)\right)\\
&\forall x \exists y \exists z \left(\left(\neg P\left(y\right) \vee Q\left(z\right)\right) \wedge \left(\neg\neg P\left(x\right) \wedge \neg Q\left(x\right)\right)\right)\\
&\forall x \exists y \exists z \left(\left(\neg P\left(y\right) \vee Q\left(z\right)\right) \wedge P\left(x\right) \wedge \neg Q\left(x\right)\right)\\
&\forall x \left(\left(\neg P\left(f\left(x\right)\right) \vee Q\left(g\left(x\right)\right)\right) \wedge P\left(x\right) \wedge \neg Q\left(x\right)\right)
\end{aligned}$$

Clauses:

$$\begin{aligned}
&C1 : \left[\neg P\left(f\left(x\right)\right) \vee Q\left(g\left(x\right)\right)\right]\\
&C2 : \left[P\left(x_1\right)\right]\\
&C3 : \left[\neg Q\left(x_2\right)\right]
\end{aligned}$$

Proof:

1. $\left[Q\left(g\left(x\right)\right)\right]$ from $C1$ and $C2$, $x_1/f\left(x\right)$.
2. $[]$ from (1) and $C3$, $x_2/g\left(x\right)$.

b) Transform the negation into clausal form:

$$\begin{aligned}
&\neg \exists x \left(P\left(x\right) \supset \forall y \left(P\left(y\right)\right)\right)\\
&\forall x \neg \left(\neg P\left(x\right) \vee \forall y \left(P\left(y\right)\right)\right)\\
&\forall x \left(P\left(x\right) \wedge \exists y \, \neg P\left(y\right)\right)\\
&\forall x \left(P\left(x\right) \wedge \neg P\left(f\left(x\right)\right)\right)
\end{aligned}$$

Clauses:

$$\begin{aligned}
&C1 : \left[P\left(x\right)\right]\\
&C2 : \left[\neg P\left(f\left(x_1\right)\right)\right]
\end{aligned}$$

Proof:

1. $[]$ from $C1$ and $C2$, $x/f\left(x_1\right)$

c) Transform the negation into clausal form:

$$\begin{aligned}
&\neg\neg \exists x \forall y \left(E\left(x, y\right) \leftrightarrow \neg E\left(y, y\right)\right)\\
&\exists x \forall y \left(\left(\neg E\left(x, y\right) \vee \neg E\left(y, y\right)\right) \wedge \left(E\left(x, y\right) \vee E\left(y, y\right)\right)\right)\\
&\forall y \left(\left(\neg E\left(a, y\right) \vee \neg E\left(y, y\right)\right) \wedge \left(E\left(a, y\right) \vee E\left(y, y\right)\right)\right)
\end{aligned}$$

Clauses:

$$C1 : [\neg E(a, y), \neg E(y, y)]$$
$$C2 : [E(a, y_1), E(y_1, y_1)]$$

Proof:

1. $[\neg E(a, a)]$ factorization $C1$, y/a
2. $[E(a, a)]$ factorization $C2$, y_1/a
3. $[]$ from 1. and 2.

5 Alpine Club and First-Order Resolution

As a follow-up to the Alpine Club Exercise, use resolution to prove that there exists a member of the Alpine club who is a climber but not a skier.

Answer:
Translation into first-order logic as given in the solution for Exercise 1 (S1 – S10) together with:

$$S11 : \exists x\,(Member(x) \land Climber(x) \land \neg Skier(x))$$

Now in clausal form (with S11 negated):

$$C1 : [Member(tony)]$$
$$C2 : [Member(mike)]$$
$$C3 : [Member(john)]$$
$$C4 : [\neg Member(x), Skier(x), Climber(x)]$$
$$C5 : [\neg Climber(x_1), \neg Likes(x_1, rain)]$$
$$C6 : [Likes(x_2, snow), \neg Skier(x_2)]$$
$$C7 : [\neg Likes(tony, x_3), \neg Likes(mike, x_3)]$$
$$C8 : [Likes(tony, x_4), Likes(mike, x_4)]$$
$$C9 : [Likes(tony, rain)]$$
$$C10 : [Likes(tony, snow)]$$
$$C11 : [\neg Member(x_5), \neg Climber(x_5), Skier(x_5)]$$

Prove that, together, $C1 - C11$ are inconsistent:

1. $[\neg Likes(mike, snow)]$ from $C10$ and $C7$, $x_3/snow$
2. $[\neg Skier(mike)]$ from 1. and $C6$, $x_2/mike$
3. $[\neg Member(mike), Climber(mike)]$ from 2. and $C4$, $x/mike$
4. $[Climber(mike)]$ from 3. and $C2$
5. $[\neg Member(mike), Skier(mike)]$ from (4) and $C11$, $x_5/mike$
6. $[Skier(mike)]$ from 5. and $C2$
7. $[]$ from 6. and 2.

Can you find an even shorter proof?