Knowledge Representation and Reasoning

Exercises on Description Logics

1 Relationship with First-Order Logic

Translate the following $\mathcal{ALC}$ concepts into English and then into FOL:

1. $Father \sqcap \forall.child.(Doctor \sqcup Manager)$
2. $\exists manages.(Company \sqcap \exists employs.Doctor)$
3. $Father \sqcap \forall.child.(Doctor \sqcup \exists manages.(Company \sqcap \exists employs.Doctor))$

Answer:

1. Fathers whose children are either doctors or managers.
$Father(x) \wedge \forall y(child(x, y) \supset (Doctor(y) \vee Manager(y)))$
2. Those who manage a company that employs at least one doctor.
$\exists y(manages(x, y) \wedge (Company(y) \wedge \exists x(employs(x, y) \wedge Doctor(x)))$
3. Fathers whose children are either doctors or managers of companies that employ some doctor.
$Father(x) \wedge \forall y(child(x, y) \supset (Doctor(y) \vee \exists x(manages(y, x) \wedge$
$(Company(x) \wedge \exists y(employs(x, y) \wedge Doctor(y))))))$

2 Knowledge Representation in $\mathcal{ALC}$

Let Man, $Woman$, $Male$, $Female$, and $Human$ be concept names, and let *has-child*, *is-brother-of*, *is-sister-of*, and *is-married-to* be role names. Construct a TBox that contains definitions for $Mother$, $Grandfather$, $Niece$, $Father$, $Aunt$, $Nephew$, $Grandmother$, $Uncle$, and *Mother-of-at-least-one-male*.

Answer: Left as an exercise...

3 Knowledge Representation in $\mathcal{ALC}$

Express the following sentences in terms of the description logic $\mathcal{ALC}$.

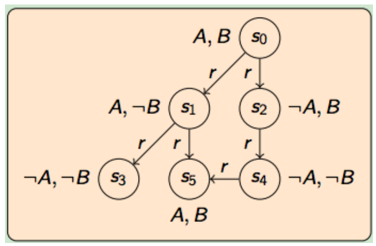
1. All employees are humans.
2. A mother is a female who has a child.
3. A parent is a mother or a father.
4. A grandmother is a mother who has a child who is a parent.
5. Only humans have children that are humans.

Answer:

1. $Employee \sqsubseteq Human$
2. $Mother \equiv Female \sqcap \exists hasChild.\top$
3. $Parent \equiv Mother \sqcup Father$
4. $Grandmother \equiv Mother \sqcap \exists hasChild.Parent$
5. $\exists hasChild.Human \sqsubseteq Human$

4 Semantics of $\mathcal{ALC}$

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, ..., s_5\}$.



Determine the interpretation of the following concepts:

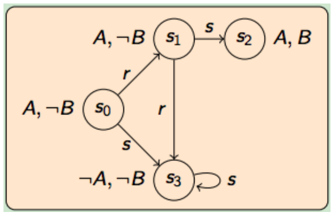
1. $\top^{\mathcal{I}}$.
2. $\bot^{\mathcal{I}}$.
3. $A^{\mathcal{I}}$.
4. $B^{\mathcal{I}}$.
5. $(A \sqcap B)^{\mathcal{I}}$.
6. $(A \sqcup B)^{\mathcal{I}}$.
7. $(\neg A)^{\mathcal{I}}$.
8. $(\exists r.A)^{\mathcal{I}}$.
9. $(\forall r.\neg B)^{\mathcal{I}}$.
10. $(\forall r.(A \sqcup B))^{\mathcal{I}}$.

Answer:

1. $\top^{\mathcal{I}} = \{s_0, s_1, \ldots, s_5\}$.
2. $\bot^{\mathcal{I}} = \emptyset$.
3. $A^{\mathcal{I}} = \{s_0, s_1, s_5\}$.
4. $B^{\mathcal{I}} = \{s_0, s_2, s_5\}$.
5. $(A \sqcap B)^{\mathcal{I}} = \{s_0, s_5\}$.
6. $(A \sqcup B)^{\mathcal{I}} = \{s_0, s_1, s_2, s_5\}$.
7. $(\neg A)^{\mathcal{I}} = \{s_2, s_3, s_4\}$.
8. $(\exists r.A)^{\mathcal{I}} = \{s_0, s_1, s_4\}$.
9. $(\forall r.\neg B)^{\mathcal{I}} = \{s_2, s_3, s_5\}$.
10. $(\forall r. (A \sqcup B))^{\mathcal{I}} = \{s_0, s_3, s_4, s_5\}$.

5 Semantics of $\mathcal{ALC}$

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, ..., s_3\}$.



Determine the interpretation of the following concepts:

1. $(A \sqcup B)^{\mathcal{I}}$.
2. $(\exists s.\neg A)^{\mathcal{I}}$.
3. $(\forall s.A)^{\mathcal{I}}$.
4. $(\exists s.\exists s.\exists s.\exists s.A)^{\mathcal{I}}$.
5. $(\neg\exists r. (\neg A \sqcup \neg B))^{\mathcal{I}}$.
6. $(\exists s. (A \sqcup \forall s.\neg B) \sqcup \neg\forall r.\exists r. (A \sqcup \neg A))^{\mathcal{I}}$.

Answer:

1. $(A \sqcup B)^{\mathcal{I}} = \{s_0, s_1, s_2\}$.
2. $(\exists s.\neg A)^{\mathcal{I}} = \{s_0, s_3\}$.
3. $(\forall s.A)^{\mathcal{I}} = \{s_1, s_2\}$.
4. $(\exists s.\exists s.\exists s.\exists s.A)^{\mathcal{I}} = \emptyset$.
5. $(\neg\exists r. (\neg A \sqcup \neg B))^{\mathcal{I}} = \{s_2, s_3\}$.
6. $(\exists s. (A \sqcup \forall s.\neg B) \sqcup \neg\forall r.\exists r. (A \sqcup \neg A))^{\mathcal{I}} = \{s_0, s_1, s_3\}$.

6 (Un)Satisfiability and Validity of $\mathcal{ALC}$

For each of the following formulas, indicate if it is valid, satisfiable or unsatisfiable. If it is not valid, provide a model that falsifies it:

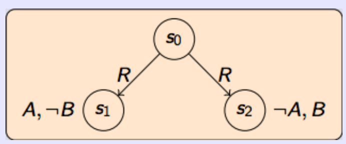
1. $\forall r. (A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$.
2. $\forall r. (A \sqcup B) \equiv \forall r.A \sqcup \forall r.B$.
3. $\exists r. (A \sqcap B) \equiv \exists r.A \sqcap \exists r.B$.
4. $\exists r. (A \sqcup B) \equiv \exists r.A \sqcup \exists r.B$.

Answer:

1. $\forall r. (A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$ is valid. We can prove that $(\forall r. (A \sqcap B))^{\mathcal{I}} = (\forall r.A \sqcap \forall r.B)^{\mathcal{I}}$ for all interpretations $\mathcal{I}$.

$$\begin{aligned}
(\forall r. (A \sqcap B))^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x, y) \in r^{\mathcal{I}} \rightarrow y \in (A \sqcap B)^{\mathcal{I}}\} \\
&= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x, y) \in r^{\mathcal{I}} \rightarrow y \in (A^{\mathcal{I}} \cap B^{\mathcal{I}})\} \\
&= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x, y) \in r^{\mathcal{I}} \rightarrow y \in A^{\mathcal{I}}\} \cap \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x, y) \in r^{\mathcal{I}} \rightarrow y \in B^{\mathcal{I}}\} \\
&= (\forall r.A)^{\mathcal{I}} \cap (\forall r.B)^{\mathcal{I}} \\
&= (\forall r.A \sqcap \forall r.B)^{\mathcal{I}}
\end{aligned}$$

2. $\forall r. (A \sqcup B) \equiv \forall r.A \sqcup \forall r.B$ is not valid. The following model is such that $(\forall r.(A \sqcup B))^{\mathcal{I}} \neq (\forall r.A \sqcup \forall r.B)^{\mathcal{I}}$.



- $s_0 \in (\forall r.(A \sqcup B))^{\mathcal{I}}$ but
- $s_0 \notin (\forall r.A)^{\mathcal{I}}$ and
- $s_0 \notin (\forall r.B)^{\mathcal{I}}$.

However, notice that $\forall r.A \sqcup \forall r.B \sqsubseteq \forall r. (A \sqcup B)$ is valid.

3. $\exists r. (A \sqcap B) \equiv \exists r.A \sqcap \exists r.B$ is not valid. The previous model is such that $(\exists r.(A \sqcap B))^{\mathcal{I}} \neq (\exists r.A \sqcap \exists r.B)^{\mathcal{I}}$.

- $s_0 \in (\exists r.A)^{\mathcal{I}}$ and
- $s_0 \in (\exists r.B)^{\mathcal{I}}$ but
- $s_0 \notin (\exists r.(A \sqcap B))^{\mathcal{I}}$.

However, notice that $\exists r. (A \sqcap B) \sqsubseteq \exists r.A \sqcap \exists r.B$ is valid.

4. $\exists r. (A \sqcup B) \equiv \exists r.A \sqcup \exists r.B$ is valid. We could provide a similar proof to the case $\forall r. (A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$, but we show here an alternative proof which is based on other equivalences.

$$\begin{aligned}
\exists r.(A \sqcup B) &\equiv \neg\forall r.(\neg(A \sqcup B)) \\
&\equiv \neg\forall r.(\neg A \sqcap \neg B)) \\
&\equiv \neg(\forall r.(\neg A) \sqcap \forall r.(\neg B)) \\
&\equiv \neg\forall r.(\neg A) \sqcup \neg\forall r.(\neg B)) \\
&\equiv \exists r.A \sqcup \exists r.B
\end{aligned}$$

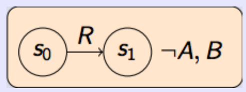
7 (Un)Satisfiability and Validity of $\mathcal{ALC}$

For each of the following concepts, indicate if it is valid, satisfiable or unsatisfiable. If it is valid, or unsatisfiable, provide a proof. If it is satisfiable (and not valid), then exhibit a model that interprets the concept in a non-empty set:

1. $\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))$.
2. $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$.
3. $(\exists s.C \sqcap \exists s.D) \sqcap \forall s.(\neg C \sqcup \neg D)$.
4. $\exists s.(C \sqcap D) \sqcap (\forall s.\neg C \sqcup \forall s.\neg D)$.
5. $C \sqcap \exists r.A \sqcap \exists r.B \sqcap \neg\exists r.(A \sqcap B)$.

Answer:

1. $\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))$. Satisfiable.

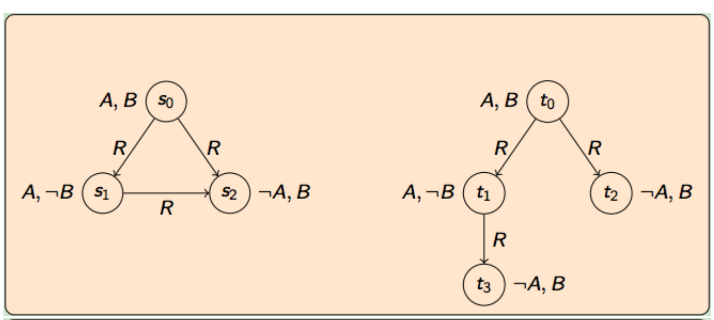


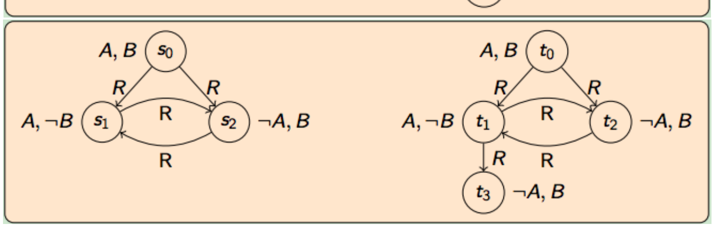
   - $s_0 \in (\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))^{\mathcal{I}}$
   - $s_1 \notin (\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))^{\mathcal{I}}$

2. $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$. Unsatisfiable.
   Since $\exists r.(\forall s.C) \equiv \neg\forall r.(\neg\forall s.C) \equiv \neg\forall r.(\exists s.(\neg C))$, this implies that $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$ is equivalent to $\neg\forall r.(\exists s.(\neg C)) \sqcap \forall r.(\exists s.(\neg C))$, which is a concept of the form $\neg B \sqcap B$ which is always unsatisfiable.
3. $(\exists s.C \sqcap \exists s.D) \sqcap \forall s.(\neg C \sqcup \neg D)$. Satisfiable.
4. $\exists s.(C \sqcap D) \sqcap (\forall s.\neg C \sqcup \forall s.\neg D)$. Unatisfiable.
5. $C \sqcap \exists r.A \sqcap \exists r.B \sqcap \neg\exists r.(A \sqcap B)$. Satisfiable.

8 Bissimulation

For each of the following pairs of models, check if they are bisimular. If yes, find the bisimulation relation, if not, find a formula that is true in the first model and false in the second.





Answer:

- The first pair of models is bisimilar and the bisimulation is $\{(s_0,t_0),(s_1,t_1),(s_2,t_2),(s_2,t_3)\}$.
- The second pair of models is not bisimular on s_0 and t_0. Note that (s_0,t_0) would have to belong to the bisimulation. However, we have that $s_0 \in (\forall r.(\forall r.(\exists r.\top)))^{\mathcal{I}_1}$ and $t_0 \notin (\forall r.(\forall r.(\exists r.\top)))^{\mathcal{I}_2}$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ and the interpretations shown above.

9 $\mathcal{ALC}$ Tableaux

Check by means of tableaux whether the following subsumption is valid:

1. $\neg\forall r.A \sqcap \forall r.\left((\forall r.B) \sqcup A\right) \sqsubseteq \forall r.\neg\left(\exists r.A\right) \sqcup \exists r.\left(\exists r.B\right)$

Answer: To check whether the given subsumption is valid we can use tableaux to verify whether the following concept is insatisfiable:

$$\neg\forall r.A \sqcap \forall r.\left(\forall r.B \sqcup A\right) \sqcap \neg\left(\forall r.\neg\left(\exists r.A\right) \sqcup \exists r.\left(\exists r.B\right)\right)$$

We have to transform into negation normal form first:

$$C_0 = \exists r.\neg A \sqcap \forall r.\left(\forall r.B \sqcup A\right) \sqcap \exists r.\exists r.A \sqcap \forall r.\forall r.\neg B$$

We apply the tableaux algorithm starting with C_0:

$$
\begin{array}{llll}
 & \mathcal{A}_0 = \{C_0(x_0)\} & & \\
\rightarrow_\sqcap & \mathcal{A}_1 = \mathcal{A}_0 \cup \{\exists r.\neg A \sqcap \forall r.\left(\forall r.B \sqcup A\right)(x_0), \exists r.\exists r.A \sqcap \forall r.\forall r.\neg B(x_0)\} & & \\
\rightarrow_\sqcap & \mathcal{A}_2 = \mathcal{A}_1 \cup \{\exists r.\neg A(x_0), \forall r.\left(\forall r.B \sqcup A\right)(x_0)\} & & \\
\rightarrow_\sqcap & \mathcal{A}_3 = \mathcal{A}_2 \cup \{\exists r.\exists r.A(x_0), \forall r.\forall r.\neg B(x_0)\} & & \\
\rightarrow_\exists & \mathcal{A}_4 = \mathcal{A}_3 \cup \{r(x_0,x_1), \exists r.A(x_1)\} & & \\
\rightarrow_\exists & \mathcal{A}_5 = \mathcal{A}_4 \cup \{r(x_1,x_2), A(x_2)\} & & \\
\rightarrow_\forall & \mathcal{A}_6 = \mathcal{A}_5 \cup \{\forall r.\neg B(x_1)\} & & \\
\rightarrow_\forall & \mathcal{A}_7 = \mathcal{A}_6 \cup \{\neg B(x_2)\} & & \\
\rightarrow_\forall & \mathcal{A}_8 = \mathcal{A}_7 \cup \{\forall r.B \sqcup A(x_1)\} & & \\
\rightarrow_\sqcup & \mathcal{A}_9 = \mathcal{A}_8 \cup \{A(x_1)\} & & \mathcal{A}_{9'} = \mathcal{A}_8 \cup \{\forall r.B(x_1)\} \\
\rightarrow_\exists & \mathcal{A}_{10} = \mathcal{A}_9 \cup \{r(x_0,x_3), \neg A(x_3)\} & \rightarrow_\forall & \mathcal{A}_{10'} = \mathcal{A}_{9'} \cup \{B(x_2)\} \times \\
\rightarrow_\forall & \mathcal{A}_{11} = \mathcal{A}_{10} \cup \{\forall r.B \sqcup A(x_3)\} & & \\
\rightarrow_\sqcup & \mathcal{A}_{12} = \mathcal{A}_{11} \cup \{\forall r.B(x_3)\} & & \mathcal{A}_{12''} = \mathcal{A}_{11} \cup \{A(x_3)\} \times \\
\rightarrow_\forall & \mathcal{A}_{13} = \mathcal{A}_{12} \cup \{\forall r.\neg B(x_3)\} \checkmark & &
\end{array}
$$

Since $\mathcal{A}_{13}$ is complete and clash-free, C_0 is satisfiable, and the initial subsumption inclusion is not valid.

10 $\mathcal{ALC}$ Tableaux

Which of the following statements are true? Explain your answer.

1. $\forall r.\left(A \sqcap B\right) \sqsubseteq \forall r.A \sqcap \forall r.B$
2. $\forall r.A \sqcap \forall r.B \sqsubseteq \forall r.\left(A \sqcap B\right)$
3. $\forall r.A \sqcup \forall r.B \sqsubseteq \forall r.\left(A \sqcup B\right)$
4. $\forall r.\left(A \sqcup B\right) \sqsubseteq \forall r.A \sqcup \forall r.B$

5. $\exists r.(A \sqcap B) \sqsubseteq \exists r.A \sqcap \forall r.B$
6. $\exists r.(A \sqcup B) \sqsubseteq \exists r.A \sqcup \forall r.B$
7. $\exists r.A \sqcup \forall r.B \sqsubseteq \exists r.(A \sqcup B)$
8. $\exists r.A \sqcap \forall r.B \sqsubseteq \exists r.(A \sqcap B)$

Answer: Left as an exercise...