Social Choice

J. Leite (adapted from Kevin Leyton-Brown)

Voting seems very simple

How hard can this be?

But sometimes things go wrong

In 2000, the US presidential election came down to Florida. George Bush won by 537 votes. But Ralph Nader got 97,421 votes.
Twice as many Nader voters would have chosen Gore over Bush.

Is there a better way?

British and Canadian protesters demanding new voting systems, May 2010 and April 2011.

Section 1

Social Choice

Introduction

Our setting now:

- a set of outcomes or alternatives
- agents have preferences over them
- the goal: a social choice function: a mapping from profiles of preferences to a particular outcome
  - which such functions have desirable properties?

Preferences

- Given is a finite set of outcomes or alternatives O.
- agents have (strict) preferences, $\succ$, over the outcomes: linear orders (or total orders).
- Linear orders L: binary relations $\succ$ that are total and transitive
  - total: for every pair of outcomes $a \neq b$ either $a \succ b$ or $b \succ a$ (but not both: so it is complete and antisymmetric)
  - transitive: $a \succ b$ and $b \succ c$ implies $a \succ c$.
- Non-strict preferences L_{NS}: binary relations $\succ$ that are complete and transitive:
  - complete: for every pair of outcomes a, b either $a \succeq b$ or $b \succeq a$ (both indicates indifference)
  - transitive: $a \succeq b$ and $b \succeq c$ implies $a \succeq c$.

Formal model

Given a set of agents $N = \{1, 2, \dots, n\}$, a finite set of outcomes (or alternatives, or candidates) O, and the linear orders on the outcomes, L.

Definition (Social choice function)

A social choice function (over N and O) is a function $C : L_{NS}^n \mapsto O$.

Definition (Social welfare function)

A social welfare function (over N and O) is a function $C : L_{NS}^n \mapsto L_{NS}$.

Non-Ranking Voting Schemes

- Plurality
  - pick the outcome which is most-preferred by the most people
- Cumulative voting
  - distribute e.g., 5 votes each
  - possible to vote for the same outcome multiple times
- Approval voting
  - accept as many outcomes as you “like”

Voting Schemes based on Ranking

- Plurality with elimination (“instant runoff”, “transferrable voting”)
  - everyone selects their favorite outcome
  - if some outcome has a majority, it is the winner
  - otherwise, the outcome with the fewest votes is eliminated (may need some tie-breaking procedure)
  - repeat until there is a winner
- Borda Rule, Borda Count
  - assign each outcome a number.
  - The most preferred outcome gets a score of $n-1$, the next most preferred gets $n-2$, down to the n^{th} outcome which gets 0.
  - Then sum the numbers for each outcome, and choose the one that has the highest score
- Pairwise elimination
  - in advance, decide a schedule for the order in which pairs will be compared.
  - given two outcomes, have everyone determine the one that they prefer
  - eliminate the outcome that was not preferred, and continue with the schedule

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Condorcet Condition

- If there is a candidate or outcome that is preferred to every other candidate in pairwise majority-rule comparisons, that candidate should be the winner.
  - A social choice function is Condorcet consistent if it picks a Condorcet winner whenever one exists.
  - There is not always a Condorcet winner
  - sometimes, there is a cycle where A defeats B, B defeats C, and C defeats A in their pairwise runoffs, known as a Condorcet Cycle.

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Fun Game

- Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:
  - (B) Beijing, China
  - (R) Rio de Janeiro, Brazil
  - (S) Sydney, Australia
  - (L) Los Angeles, USA
- Construct your preference ordering (write it down)
- Vote (truthfully) using each of the following schemes:
  - plurality (raise hands)
  - plurality with elimination (raise hands)
  - pairwise elimination (raise hands, I'll pick a schedule)
  - Borda (volunteer to tabulate)

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Section 2

Voting Paradoxes

Condorcet example

$$
\begin{aligned}
\text{499 agents:} &\quad A \succ B \succ C \\
\text{3 agents:} &\quad B \succ C \succ A \\
\text{498 agents:} &\quad C \succ B \succ A
\end{aligned}
$$

- What is the Condorcet winner?
- What would win under plurality voting?
- What would win under plurality with elimination?

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Sensitivity to Losing Candidate

35 agents:	$A \succ C \succ B$
33 agents:	$B \succ A \succ C$
32 agents:	$C \succ B \succ A$

- What candidate wins under plurality voting?
- What candidate wins under Borda voting?
- Now consider dropping C. Now what happens under both Borda and plurality?

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Sensitivity to Agenda Setter

35 agents:	$A \succ C \succ B$
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- Who wins pairwise elimination, with the ordering A, B, C?
- Who wins with the ordering A, C, B?
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- Who wins with the ordering B, C, A? A

Another Pairwise Elimination Problem

1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
1 agent: $C \succ A \succ B \succ D$

- Who wins under pairwise elimination with the ordering A, B, C, D?
- What is the problem with this?

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1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
1 agent: $C \succ A \succ B \succ D$

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Another Pairwise Elimination Problem

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- Who wins under pairwise elimination with the ordering A, B, C, D? D
- What is the problem with this?
  - all of the agents prefer B to D—the selected candidate is Pareto-dominated!

Section 3

Properties

Pareto Efficiency

Definition (Pareto Efficiency (PE))

A social welfare function W is Pareto efficient if whenever all agents agree on the ordering of two outcomes, the social welfare function selects that ordering.

Independence of Irrelevant Alternatives

Definition (Independence of Irrelevant Alternatives (IIA))

A social welfare function W is independent of irrelevant alternatives if the selected ordering between two outcomes depends only on the relative orderings they are given by the agents.

Dictatorship

Definition (Dictatorial)

A social welfare function W is dictatorial (i.e. has a dictator) if there is a single agent whose preferences always determine the social ordering.

Arrow's Theorem

Theorem (Arrow, 1951)

Any social welfare function W over three or more outcome that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

Notation

- N is the set of agents
- O is a finite set of outcomes with $|O| \geq 3$
- L is the set of all possible strict preference orderings over O.
  - for ease of exposition we switch to strict orderings
  - we will end up showing that desirable SWFs cannot be found even if preferences are restricted to strict orderings
- $[\succ]$ is an element of the set L^n (a preference ordering for every agent; the input to our social welfare function)
- $\succ_W$ is the preference ordering selected by the social welfare function W.
  - When the input to W is ambiguous we write it in the subscript; thus, the social order selected by W given the input $[\succ']$ is denoted as $\succ_{W([\succ'])}$.

Pareto Efficiency

Definition (Pareto Efficiency (PE))

W is Pareto efficient if for any $o_1, o_2 \in O$, $\forall i\, o_1 \succ_i o_2$ implies that $o_1 \succ_W o_2$.

- W is Pareto efficient if whenever all agents agree on the ordering of two outcomes, the social welfare function selects that ordering.

Independence of Irrelevant Alternatives

Definition (Independence of Irrelevant Alternatives (IIA))

W is independent of irrelevant alternatives if, for any $o_1, o_2 \in O$ and any two preference profiles $[\succ'], [\succ''] \in L^n$, $\forall i\,(o_1 \succ'_i o_2$ if and only if $o_1 \succ''_i o_2)$ implies that $(o_1 \succ_{W([\succ'])} o_2$ if and only if $o_1 \succ_{W([\succ''])} o_2)$.

- W is independent of irrelevant alternatives if the selected ordering between two outcomes depends only on the relative orderings they are given by the agents.

Nondictatorship

Definition (Non-dictatorship)

W does not have a dictator if $\neg\exists i\, \forall o_1, o_2 (o_1 \succ_i o_2 \Rightarrow o_1 \succ_W o_2)$.

- W is non-dictatorial if there does not exist a single agent whose preferences always determine the social ordering.
- We say that W is dictatorial if it fails to satisfy this property.

Section 4

Arrow's Theorem

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Theorem (Arrow, 1951)

Any social welfare function W that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

We will assume that W is both PE and IIA, and show that W must be dictatorial. Our assumption that $|O| \geq 3$ is necessary for this proof. The argument proceeds in four steps.

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Arrow's Theorem, Step 1

Step 1: If every voter puts an outcome b at either the very top or the very bottom of his preference list, b must be at either the very top or very bottom of $\succ_W$ as well.

Consider an arbitrary preference profile $[\succ]$ in which every voter ranks some $b \in O$ at either the very bottom or very top, and assume for contradiction that the above claim is not true. Then, there must exist some pair of distinct outcomes $a, c \in O$ for which $a \succ_W b$ and $b \succ_W c$.

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Now let's modify $[\succ]$ so that every voter moves c just above a in his preference ranking, and otherwise leaves the ranking unchanged; let's call this new preference profile $[\succ']$. We know from IIA that for $a \succ_W b$ or $b \succ_W c$ to change, the pairwise relationship between a and b and/or the pairwise relationship between b and c would have to change. However, since b occupies an extremal position for all voters, c can be moved above a without changing either of these pairwise relationships. Thus in profile $[\succ']$ it is also the case that $a \succ_W b$ and $b \succ_W c$. From this fact and from transitivity, we have that $a \succ_W c$. However, in $[\succ']$ every voter ranks c above a and so PE requires that $c \succ_W a$. We have a contradiction.

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Arrow's Theorem, Step 2

Step 2: There is some voter n^* who is extremely pivotal in the sense that by changing his vote at some profile, he can move a given outcome b from the bottom of the social ranking to the top.

Consider a preference profile $[\succ]$ in which every voter ranks b last, and in which preferences are otherwise arbitrary. By PE, W must also rank b last. Now let voters from 1 to n successively modify $[\succ]$ by moving b from the bottom of their rankings to the top, preserving all other relative rankings. Denote as n^* the first voter whose change causes the social ranking of b to change. There clearly must be some such voter: when the voter n moves b to the top of his ranking, PE will require that b be ranked at the top of the social ranking.

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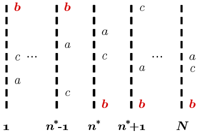
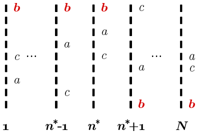
Denote by $[\succ^1]$ the preference profile just before n^* moves b, and denote by $[\succ^2]$ the preference profile just after n^* has moved b to the top of his ranking. In $[\succ^1]$, b is at the bottom in $\succ_W$. In $[\succ^2]$, b has changed its position in $\succ_W$, and every voter ranks b at either the top or the bottom. By the argument from Step 1, in $[\succ^2]$ b must be ranked at the top of $\succ_W$.

Profile $[\succ^1]$:



Profile $[\succ^2]$:



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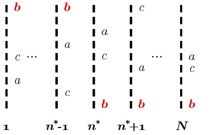

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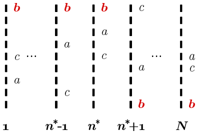

Arrow's Theorem, Step 3

Step 3: n^* (the agent who is extremely pivotal on outcome b) is a dictator over any pair ac not involving b.

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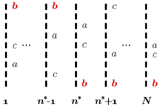

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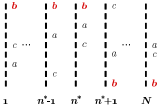

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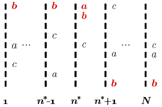

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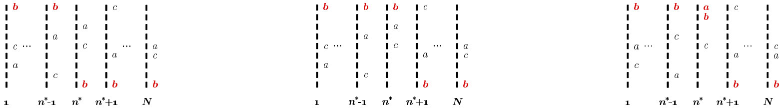

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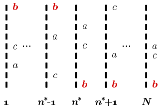

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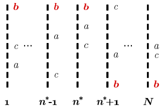

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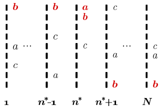

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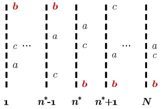

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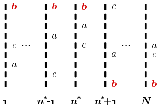

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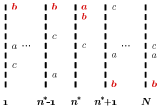

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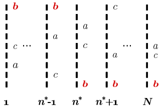

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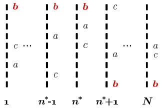

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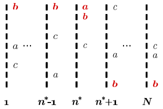

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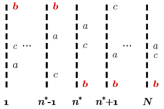

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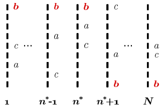

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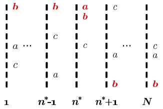

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Now construct one more preference profile, $[\succ^4]$, by changing $[\succ^3]$ in two ways. First, arbitrarily change the position of b in each voter's ordering while keeping all other relative preferences the same. Second, move a to an arbitrary position in n^*'s preference ordering, with the constraint that a remains ranked higher than c. Observe that all voters other than n^* have entirely arbitrary preferences in $[\succ^4]$, while n^*'s preferences are arbitrary except that $a \succ_{n^*} c$.

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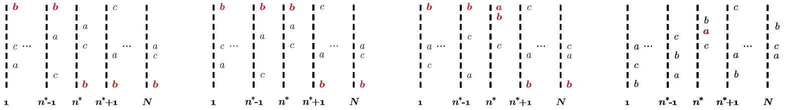

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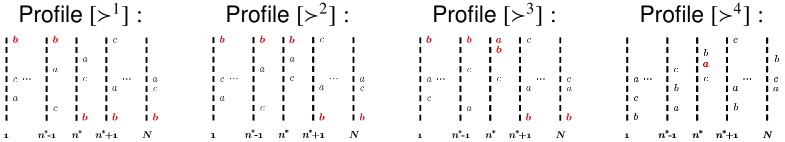

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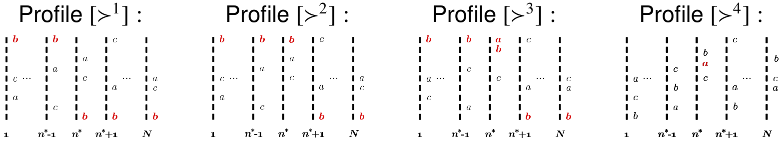

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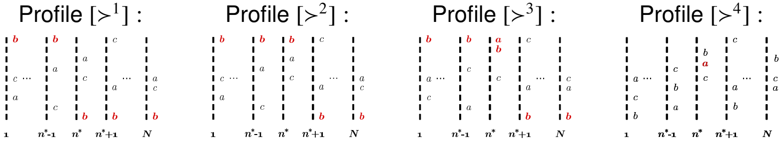

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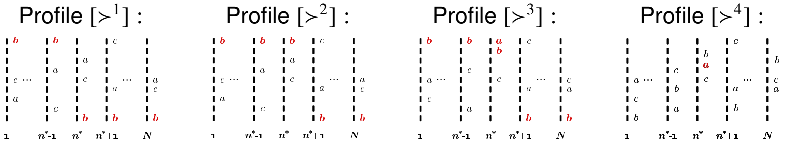
In $[\succ^3]$ and $[\succ^4]$ all agents have the same relative preferences between a and c; thus, since $a \succ_W c$ in $[\succ^3]$ and by IIA, $a \succ_W c$ in $[\succ^4]$. Thus we have determined the social preference between a and c without assuming anything except that $a \succ_{n^*} c$.



Arrow's Theorem, Step 4

Step 4: n^* is a dictator over all pairs ab.

Consider some third outcome c. By the argument in Step 2, there is a voter n^{**} who is extremely pivotal for c. By the argument in Step 3, n^{**} is a dictator over any pair $\alpha\beta$ not involving c. Of course, ab is such a pair $\alpha\beta$. We have already observed that n^* is able to affect W's ab ranking—for example, when n^* was able to change $a \succ_W b$ in profile $[\succ^1]$ into $b \succ_W a$ in profile $[\succ^2]$. Hence, n^{**} and n^* must be the same agent.

Section 5

Social Choice Functions

Social Choice Functions

- Maybe Arrow's theorem held because we required a whole preference ordering.
- Idea: social choice functions might be easier to find
- We'll need to redefine our criteria for the social choice function setting; PE and IIA discussed the ordering

Weak Pareto Efficiency

Definition (Weak Pareto Efficiency)

A social choice function C is weakly Pareto efficient if, for any preference profile $[\succ] \in L^n$, if there exist a pair of outcomes o_1 and o_2 such that $\forall i \in N$, $o_1 \succ_i o_2$, then $C([\succ]) \neq o_2$.

- A dominated outcome can't be chosen.

Monotonicity

Definition (Monotonicity)

C is monotonic if, for any $o \in O$ and any preference profile $[\succ] \in L^n$ with $C([\succ]) = o$, then for any other preference profile $[\succ']$ with the property that $\forall i \in N, \forall o' \in O$, $o \succ'_i o'$ if $o \succ_i o'$, it must be that $C([\succ']) = o$.

- an outcome o must remain the winner whenever the support for it is increased relative to a preference profile under which o was already winning

Non-dictatorship

Definition (Non-dictatorship)

C is non-dictatorial if there does not exist an agent j such that C always selects the top choice in j's preference ordering.

The bad news

Theorem (Muller-Satterthwaite, 1977)

Any social choice function that is weakly Pareto efficient and monotonic is dictatorial.

- Perhaps contrary to intuition, social choice functions are no simpler than social welfare functions after all.

But... Isn't Plurality Monotonic?

Plurality satisfies weak PE and ND, so it must not be monotonic.

Consider the following preferences:

$$\begin{array}{ll} \text{3 agents:} & a \succ b \succ c \\ \text{2 agents:} & b \succ c \succ a \\ \text{2 agents:} & c \succ b \succ a \end{array}$$

Plurality chooses a.

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Plurality chooses a.

Increase support for a by moving c to the bottom:

$$\begin{array}{ll} \text{3 agents:} & a \succ b \succ c \\ \text{2 agents:} & b \succ c \succ a \\ \text{2 agents:} & b \succ a \succ c \end{array}$$

Now plurality chooses b.

Single Peaked Preferences

- Sometimes voters' preferences have nicer properties
- Proeminent case: candidates can be ordered from left to right
- Voters: have a most-preferred candidate and then candidates who are more extreme are less preferred

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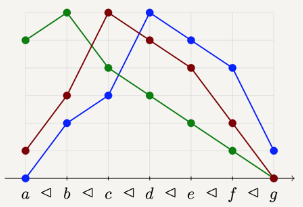

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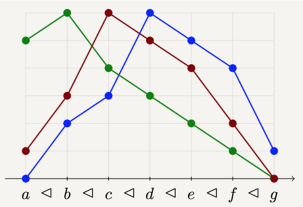

- A Condorcet winner always exists with an odd number of voters.
- Voters have an incentive to be truthful – dominant strategy