Noncooperative Game Theory

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Section 1

Imperfect-Information Extensive-Form Games

Imperfect-Information Extensive-Form Games

- Poker
  - Sequential play in betting/calling/folding
  - See some cards, but not all
  - See bets and react to them
  - Have beliefs about rationality and motivations of other players

Intro

- Up to this point, in our discussion of extensive-form games we have allowed players to specify the action that they would take at every choice node of the game.
  - This implies that players know the node they are in and all the prior choices, including those of other agents.
  - We may want to model agents needing to act with partial or no knowledge of the actions taken by others, or even themselves.
- This is possible using imperfect information extensive-form games.
  - each player's choice nodes are partitioned into information sets
  - if two choice nodes are in the same information set then the agent cannot distinguish between them.

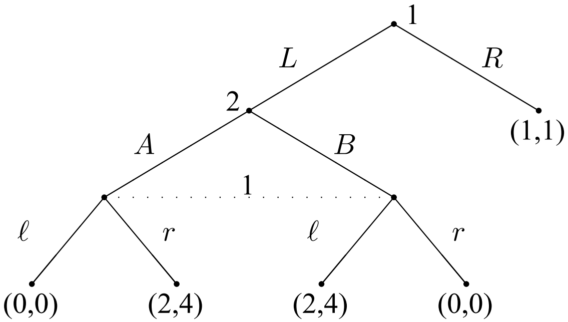
Formal definition

Definition

An imperfect-information game (in extensive form) is a tuple $(N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect-information extensive-form game, and
- $I = (I_1, \ldots, I_n)$, where $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an equivalence relation on (that is, a partition of) $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a j for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.

Example

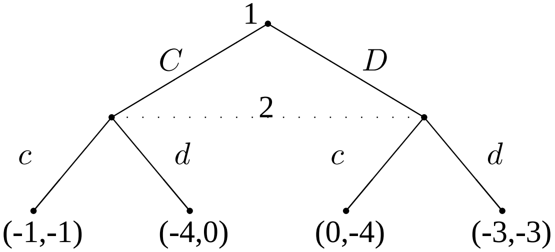


- What are the equivalence classes for each player?
- What are the pure strategies for each player?
  - choice of an action in each equivalence class.
- Formally, the pure strategies of player i consist of the cross product

$$\bigtimes_{I_{i,j} \in I_i} \chi(I_{i,j}).$$

Normal-form games

- We can represent any normal form game as an imperfect information extensive-form game.
- Prisoners' Dilemma:



  - Note that it would also be the same if we put player 2 at the root node.

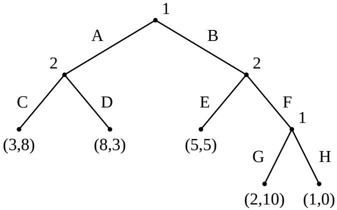
Induced Normal Form

- Same as before: enumerate pure strategies for all agents
- Mixed strategies are just mixtures over the pure strategies as before.
- Nash equilibria are also preserved.
- We've now defined two mappings: $NF \mapsto IIEF$ and $IIEF \mapsto NF$.
  - what happens if we apply each mapping in turn?
  - we might not end up with the same game, but we do get one with the same strategy spaces and equilibria.

Randomized Strategies

- It turns out there are two meaningfully different kinds of randomized strategies in imperfect information extensive form games
  - mixed strategies
  - behavioral strategies
- Mixed strategy: randomize over pure strategies
- Behavioral strategy: independent coin toss every time an information set is encountered

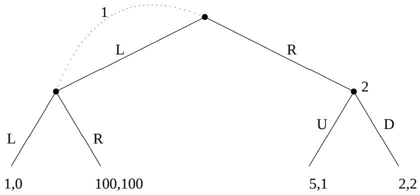
Randomized strategies example



- Give an example of a behavioral strategy:
  - A with probability $.5$ and G with probability $.3$
- Give an example of a mixed strategy that is not a behavioral strategy:
  - $(.6(A, G), .4(B, H))$ (why isn't this a good answer?)
  - Because it is equivalent to the behavioral strategy A with probability $.6$ and G with probability $.4$
- In fact, in this game, every behavioral strategy corresponds to a mixed strategy...

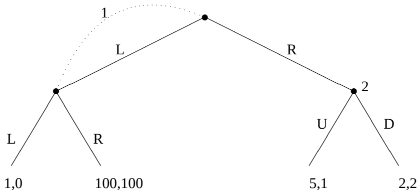
Games of imperfect recall

Imagine that player 1 sends two proxies to the game with the same strategies. When one arrives, he doesn't know if the other has arrived before him, or if he's the first one.



- What is the space of pure strategies in this game?
  - 1: (L, R); 2: (U, D)
- What is the mixed strategy equilibrium?
  - Observe that D is dominant for 2.
  - R, D is better for 1 than L, D, so R, D is an equilibrium.

Games of imperfect recall



- What is an equilibrium in behavioral strategies?
  - again, D strongly dominant for 2
  - if 1 uses the behavioural strategy $(p, 1-p)$, his expected utility is $1 * p2 + 100 * p(1-p) + 2 * (1-p)$
  - simplifies to $-99p^2 + 98p + 2$
  - maximum at $p = 98/198$
  - thus equilibrium is $(98/198, 100/198), (0, 1)$
  - corresponding to an expected utility for payer 1 of 26.25
- Thus, we can have behavioral strategies that are different from mixed strategies.

Section 2

Perfect Recall

Perfect Recall: mixed and behavioral strategies coincide

No player forgets anything he knew about moves made so far.

Definition

Player i has perfect recall in an imperfect-information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, h_2, \ldots, h_n, a_n, h$ from the root of the game to h (where the h_j are decision nodes and the a_j are actions) and any path $h_0, a'_0, h'_1, a'_1, h'_2, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$
2. For all $0 \leq j \leq n$, if $\rho(h_j) = i$ (that is, h_j is a decision node of player i), then h_j and h'_j are in the same equivalence class for player i.
3. For all $0 \leq j \leq n$, if $\rho(h_j) = i$ (that is, h_j is a decision node of player i), then $a_j = a'_j$.

G is a game of perfect recall if every player has perfect recall in it.

Perfect Recall

Clearly, every perfect-information game is a game of perfect recall.

Theorem (Kuhn, 1953)

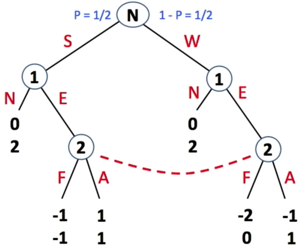
In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy. Here two strategies are equivalent in the sense that they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining agents.

Corollary

In games of perfect recall the set of Nash equilibria does not change if we restrict ourselves to behavioral strategies.

Beyond Sub-game Perfection

With incomplete information, there may not be many proper sub-games...



- Equilibrium concepts that explicitly model players' beliefs about where they are in the tree for every information set (what other players have done)
- Sequential Equilibrium and Perfect Bayesian Equilibrium – key features:
  - Beliefs are not contradicted by the actual play of the game (on the equilibrium path)
  - Players best respond to their beliefs

Section 3

Repeated Games

Repeated Games

- Many (most?) interactions occur more than once:
  - Firms in a marketplace
  - Political Alliances
  - Friends (favour exchange...)
  - Workers (team production...)

Introduction

- OPEC: Oil Prices
  - 20$/bbl or less, from 1930-1973
  - 50$/bbl by 1976
  - 90$/bbl by 1982
  - 40$/bbl or less, from 1986 to 2002
  - 100$/bbl by late 2008...
- Cooperative Behaviour: Cartel is much like a repeated Prisoner's Dilemma
  - Need to easily observe each other's plays and react (quickly) to punish undesired behaviour
  - Need patient players who value the long run (wars don't help!)
  - Need a stable set of players and some stationarity helps
    - constantly changing sources of production can hurt, but growing demand can help...

Introduction

- Play the same normal-form game over and over
  - each round is called a “stage game”
- Questions we’ll need to answer:
  - what will agents be able to observe about others’ play?
  - how much will agents be able to remember about what has happened?
  - what is an agent’s utility for the whole game?
- Some of these questions will have different answers for finitely- and infinitely-repeated games.

Finitely Repeated Games

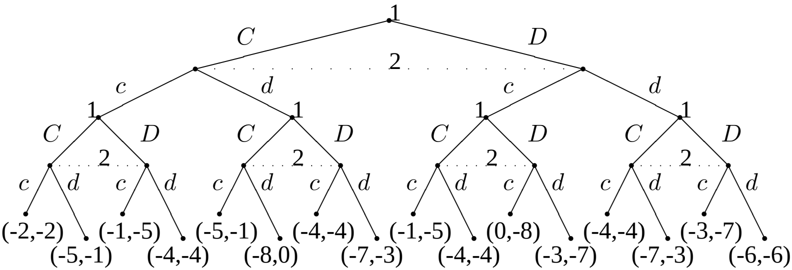
- Everything is straightforward if we repeat a game a finite number of times
- We can write the whole thing as an extensive-form game with imperfect information
  - at each round players don't know what the others have done; afterwards they do
  - overall payoff function is additive: sum of payoffs in stage games

Example

	C	D
C	$-1,-1$	$-4,0$
D	$0,-4$	$-3,-3$

$\Rightarrow$

	C	D
C	$-1,-1$	$-4,0$
D	$0,-4$	$-3,-3$



Play repeated prisoner's dilemma with one or more partners. Repeat the game five times.

Notes

- Observe that the strategy space is much richer than it was in the NF setting
- Repeating a Nash strategy in each stage game will be an equilibrium in behavioral strategies (called a stationary strategy)
- In general strategies adopted can depend on actions played so far
- We can apply backward induction in these games when the normal form game has a dominant strategy.