Linear Programming
Iterated Dominance

Computational Game Theory – 2018/2019

(partially adapted from Kevin Leyton-Brown)

Linear Programming

- Maximize:

x_j produced quantity of product j

total profit $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

c_j profit per unit of product j

- Subject to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$

b_i available quantity of resource i

a_{ij} quantity of resource i consumed in the production of one unit of product j

Linear Programming

- Example:

Resource	Product: Regular	Product: Premium	Resource Availability
Raw gas	7 m^3/tonne	11 m^3/tonne	77 m^3/week
Production time	10 hr/tonne	8 hr/tonne	80 hr/week
Storage	9 tonnes	6 tonnes	
Profit	150/tonne	175/tonne	

$x_1 \rightarrow$ Regular $\quad x_2 \rightarrow$ Premium $\quad x_1 \geq 0 \quad x_2 \geq 0$

- Total profit: $150x_1 + 175x_2 \longrightarrow \max Z = 150x_1 + 175x_2$
- Gas: $7x_1 + 11x_2 \longrightarrow 7x_1 + 11x_2 \leq 77$
- Time: $10x_1 + 8x_2 \longrightarrow 10x_1 + 8x_2 \leq 80$
- Storage: $\longrightarrow x_1 \leq 9 \quad x_2 \leq 6$

Linear Programming

- Example:

Maximize: $Z = 150x_1 + 175x_2$

Subject to: $7x_1 + 11x_2 \leq 77$

$$10x_1 + 8x_2 \leq 80$$

$$x_1 \leq 9$$

$$x_2 \leq 6$$

$$x_1 \geq 0$$

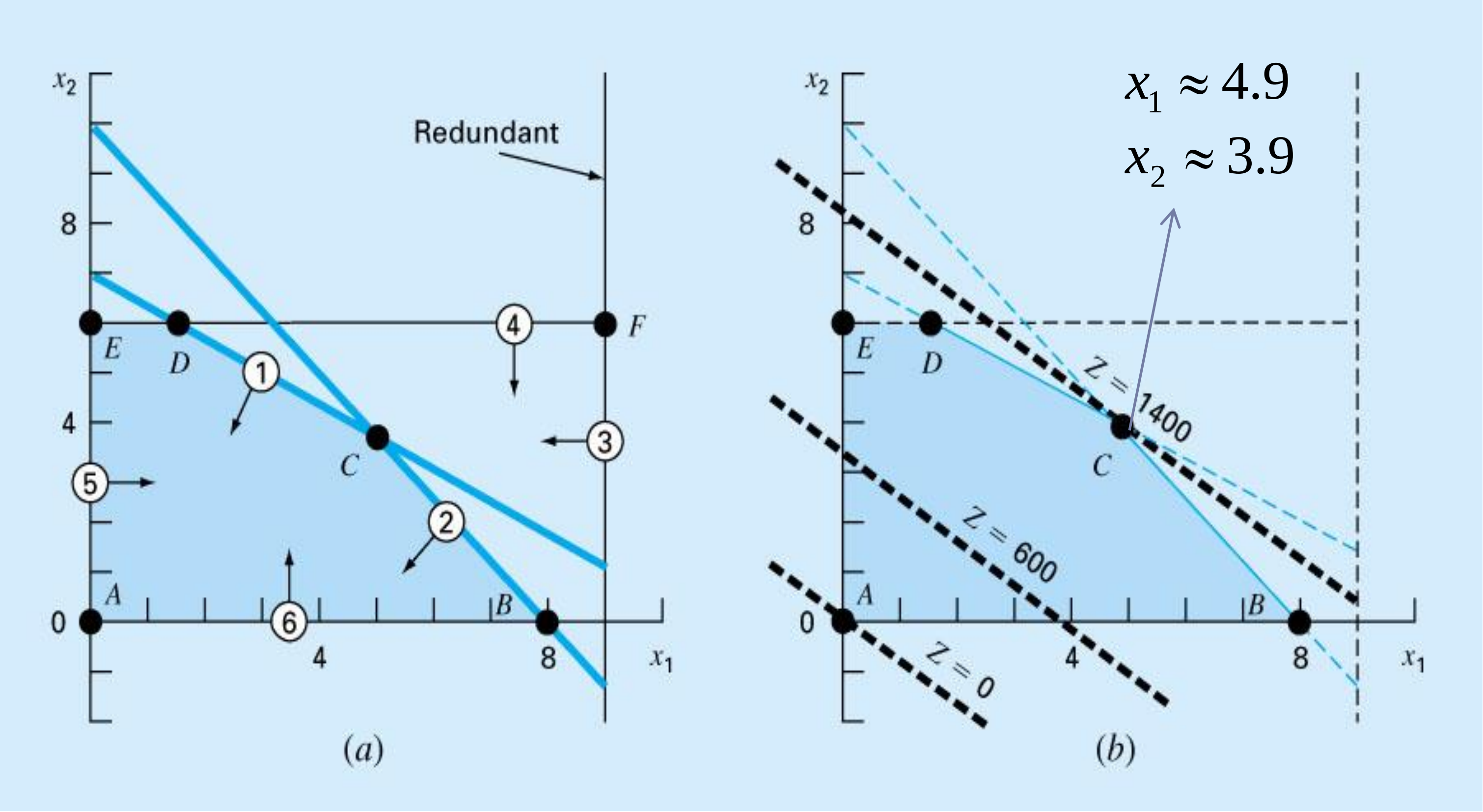
$$x_2 \geq 0$$

Linear Programming

- **Ex:** Maximize: $Z = 150x_1 + 175x_2$

  Subject to: $7x_1 + 11x_2 \le 77$ (1) $\quad x_1 \le 9$ (3) $\quad x_1 \ge 0$ (5)

  $10x_1 + 8x_2 \le 80$ (2) $\quad x_2 \le 6$ (4) $\quad x_2 \ge 0$ (6)



Linear Programming

- **Ex:** Maximize: $Z = 150(4.9) + 175(3.9) \approx 1400$

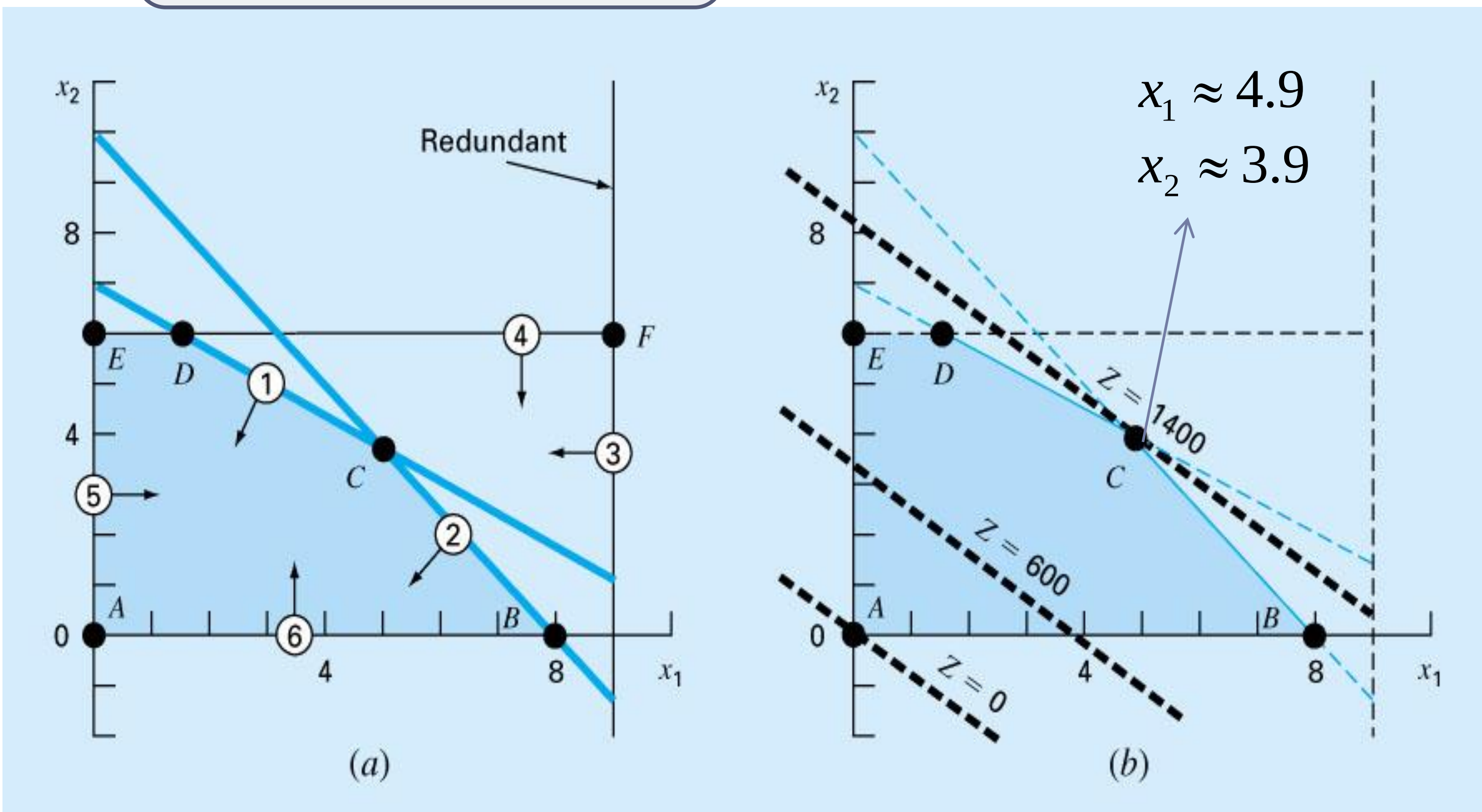
Subject to:

$$\boxed{\begin{aligned} 7(4.9) + 11(3.9) &\approx 77 \quad \color{red}{(1)} \\ 10(4.9) + 8(3.9) &\approx 80 \quad \color{red}{(2)} \end{aligned}}$$

$4.9 \leq 9$ (3) $\quad$ $4.9 \geq 0$ (5)

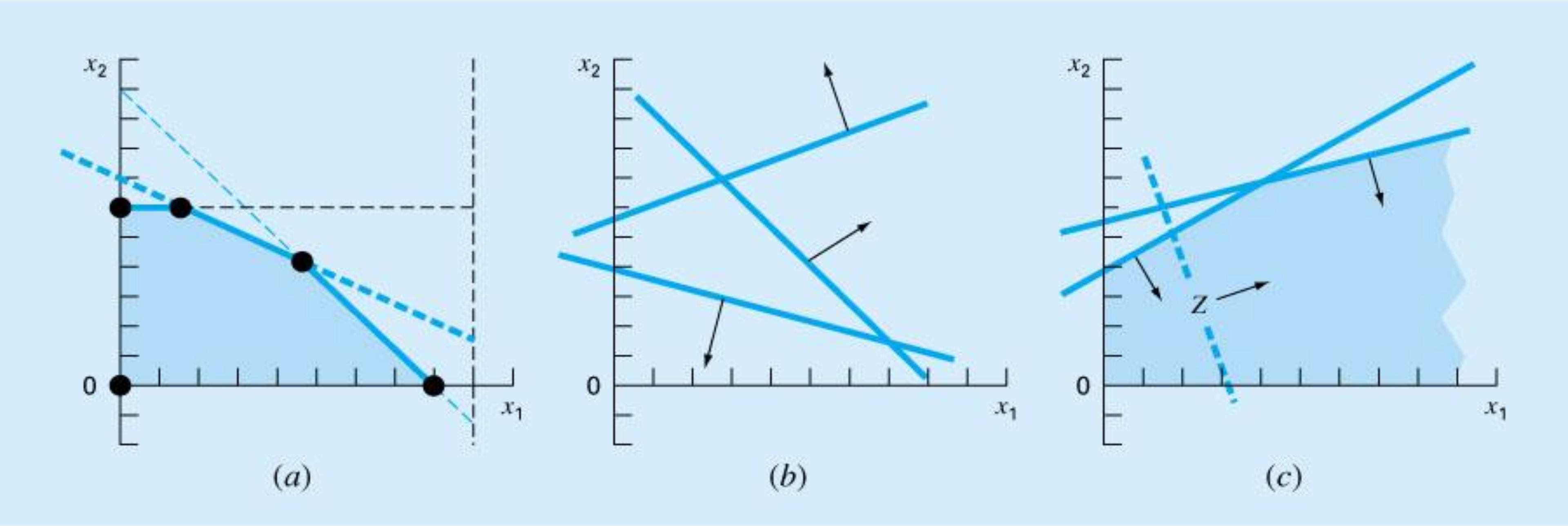
$3.9 \leq 6$ (4) $\quad$ $3.9 \geq 0$ (6)

Limiting Constraints!



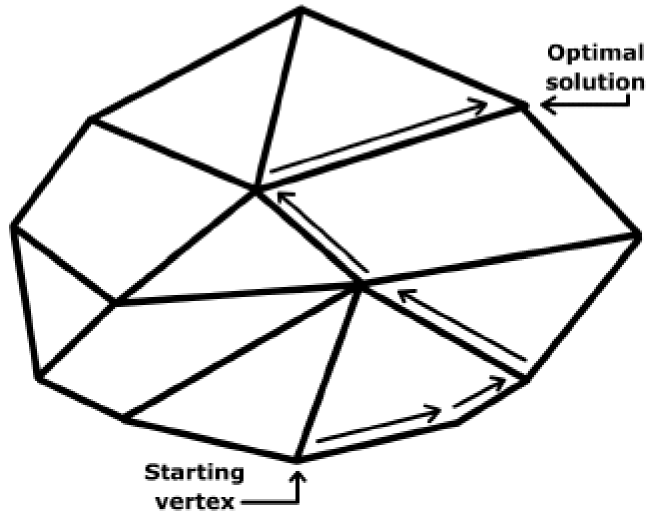
Linear Programming

- Special cases where there is not a unique solution:
  a) Infinite number of solutions
  b) Infeasible problem
  c) Unbounded problem



Linear Programming

- In general, when there is a bounded feasible region:



- Verify all vertices of the feasible region:
  - Too inefficient!
- Simplex method:
  - Explore a sequence of vertices to find the optimal solution

Simplex Method

- Convert the problem to the augmented form
- Ex: Maximize: $Z = 150x_1 + 175x_2$

  Subject to: $7x_1 + 11x_2 \le 77$ $\quad x_1 \le 9 \quad x_1 \ge 0$

  $10x_1 + 8x_2 \le 80 \quad x_2 \le 6 \quad x_2 \ge 0$

- augmented form:

  Maximize: $Z = 150x_1 + 175x_2$

  Subject to:

$$
\begin{aligned}
7x_1 + 11x_2 + S_1 &= 77 \\
10x_1 + 8x_2 + S_2 &= 80 \\
x_1 + S_3 &= 9 \\
x_2 + S_4 &= 6 \\
x_1,\ x_2,\ S_1,\ S_2,\ S_3,\ S_4 &\ge 0
\end{aligned}
$$

Simplex Method

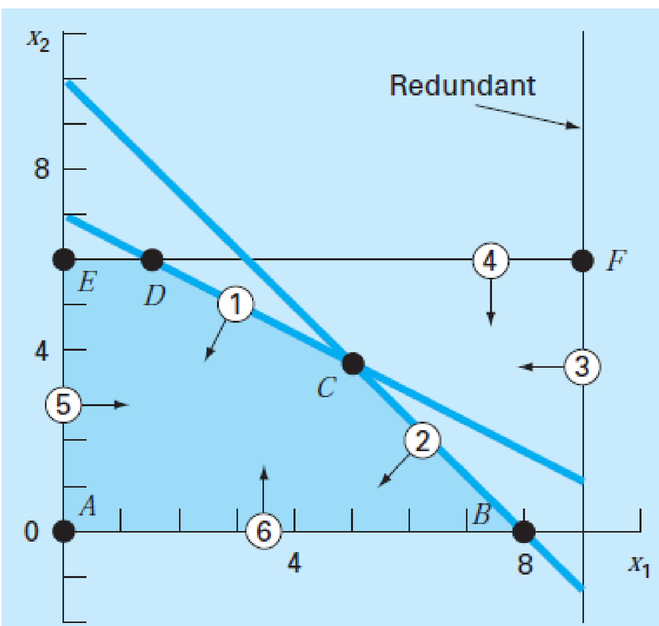
- augmented form:
  - 6 variables
  - 4 equations
  - 1 solution per each pair of variables equal to 0

Maximize: $Z = 150x_1 + 175x_2$

Subject to:

$$
\begin{aligned}
7x_1 + 11x_2 + S_1 \quad\quad\quad\quad &= 77 \\
10x_1 + 8x_2 \quad + S_2 \quad\quad &= 80 \\
x_1 \quad\quad\quad\quad + S_3 \quad &= 9 \\
x_2 \quad\quad\quad\quad + S_4 &= 6
\end{aligned}
$$

$$x_1,\ x_2,\ S_1,\ S_2,\ S_3,\ S_4\ \geq\ 0$$

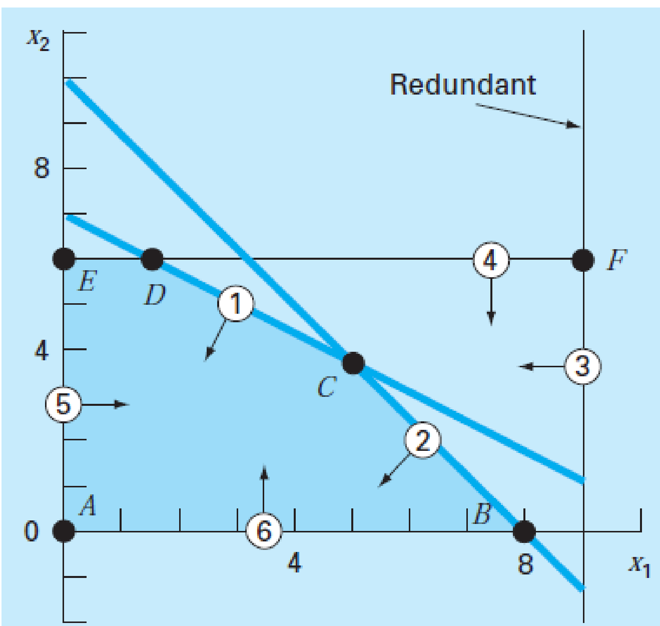


Ex: to obtain point E do x_1=S_4=0

Extreme Point	Zero Variables
A	x_1, x_2
B	x_2, S_2
C	S_1, S_2
D	S_1, S_4
E	x_1, S_4

Simplex Method

- augmented form:
  - 6 variables
  - 4 equations
  - 1 solution per each pair of variables equal to 0



Maximize: $Z = 150x_1 + 175x_2$

Subject to:

$$
\begin{aligned}
7x_1 + 11x_2 + S_1 &= 77 \\
10x_1 + 8x_2 + S_2 &= 80 \\
x_1 + S_3 &= 9 \\
x_2 + S_4 &= 6 \\
x_1,\ x_2,\ S_1,\ S_2,\ S_3,\ S_4 &\geq 0
\end{aligned}
$$

Ex: to obtain point E do x_1=S_4=0

$$
\begin{aligned}
11x_2 + S_1 &= 77 \\
8x_2 + S_2 &= 80 \\
+ S_3 &= 9 \\
x_2 &= 6
\end{aligned}
$$

whose solution is: x_2=6, S_1=11, S_2=32 e S_3=9

Simplex Method

- Start with a initial feasible solution: A $\quad x_1 = x_2 = 0$

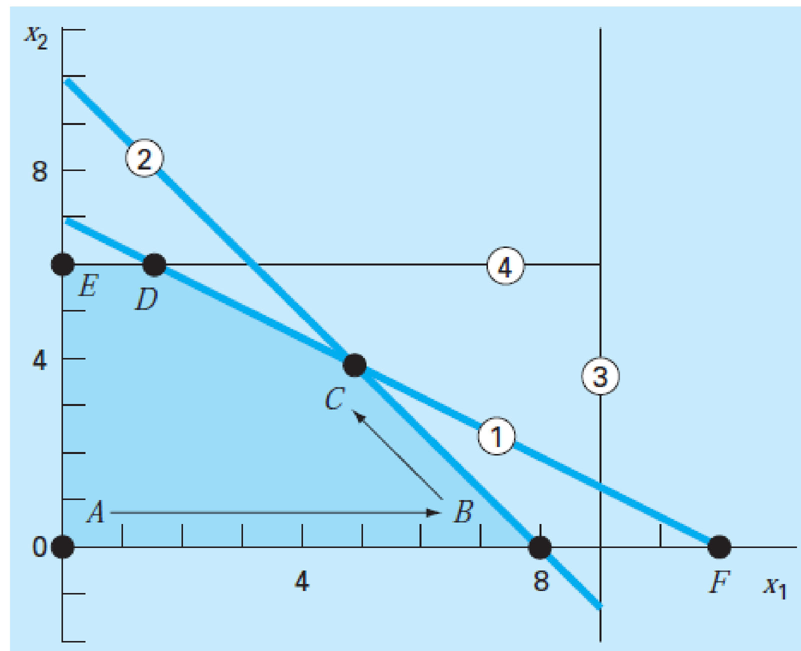
$$\begin{aligned} 7x_1 + 11x_2 + S_1 \qquad\qquad &= 77 \\ 10x_1 + 8x_2 \quad + S_2 \qquad &= 80 \\ x_1 \qquad\qquad + S_3 \quad &= 9 \\ x_2 \qquad\qquad + S_4 &= 6 \end{aligned} \longrightarrow \begin{aligned} S_1 \qquad\quad &= 77 \\ S_2 \qquad &= 80 \\ S_3 \quad &= 9 \\ S_4 &= 6 \end{aligned}$$

$$x_1,\ x_2,\ S_1,\ S_2,\ S_3,\ S_4\ \geq\ 0$$

- Next feasible solution: B

$$x_2 = 0, S_2 = 0$$

$$\begin{aligned} 7x_1 + S_1 \qquad\quad &= 77 \\ 10x_1 \qquad\qquad &= 80 \\ x_1 \qquad + S_3 \quad &= 9 \\ S_4 &= 6 \end{aligned}$$



- Stop optimal solution: C

Domination

Let s_i and s_i' be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for the other players

- s_i strictly dominates s_i' if $\forall_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- s_i weakly dominates s_i' if $\forall_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$
  and $\exists_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- s_i very weakly dominates s_i' if $\forall_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

Dominated Strategies

No equilibrium can involve a strictly dominated strategy

- Thus we can remove it, and end up with a strategically equivalent game

- This might allow us to remove another strategy that wasn't dominated before

- Running this process to termination is called iterated removal of dominated strategies.

Iterated Removal of Dominated Strategies

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

- *R* is strictly dominated by *L*

Iterated Removal of Dominated Strategies

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

- *M* is dominated by the mixed strategy that selects *U* and *D* with equal probability.

Iterated Removal of Dominated Strategies

	L	C
U	3, 1	0, 1
D	0, 1	4, 1

- No other strategies are dominated.

Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria
  - strict dominance: all equilibria preserved
  - weak/very weak dominance: at least one equilibrium preserved
- Can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique
  - Example: Traveler's Dilemma
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter
  - weak/very weak dominance: affect which equilibria are preserved

Is s_i strictly dominated by any pure strategy?

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
    $dom \leftarrow true$
    for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
        if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
            $dom \leftarrow false$
            break
        end if
    end for
    if $dom = true$ **then return** $true$
end for
return $false$

- What about mixed strategies?

Is s_i strictly dominated by any mixed strategy?

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?
  - strict inequality in the first constraint means we don't have an LP

Is s_i strictly dominated by any mixed strategy?

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in A_i} p_j \\ \text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i} \end{aligned}$$

- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add a positive amount to each p_j and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)

Iterated Removal of Dominated Strategies

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in P:
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
  - Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N} (|A_i| - 1)$ steps.
  - Thus we need to solve $O\left((n \times a^*)^2\right)$ linear programs, where $a^* = max_i |A_i|$