Computing Nash Equilibria

Computational Game Theory – 2018/2019

Two Players Game

		Player 2	
		4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

- Two players, each one with their own strategies
- Each strategy is labeled with numbers:
  - $M \rightarrow$ set of the m pure strategies of player 1
  - $N \rightarrow$ set of the n pure strategies of player 2

$$M=\{1, \ldots, m\},\ N=\{m+1, \ldots, m+n\}$$

Mixed Strategies

		Player 2: 4	Player 2: 5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} y_4 \\ y_5 \end{bmatrix}$$

- Let A and B be $|M|$ x $|N|$ matrices.
- Mixed strategies x, y : Probability distributions over M and N
- If players 1 and 2 play x and y, the payoffs are $x^T A y$ and $y^T B^T x$

$$x^T A y = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} y_{m+1} \\ \vdots \\ y_{m+n} \end{bmatrix}$$

Mixed Strategies

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$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Let A and B be $|M|$ x $|N|$ matrices.
- Mixed strategies x, y : Probability distributions over M and N
- If players 1 and 2 play x and y, the payoffs are $x^T A y$ and $y^T B^T x$

$$x^T A y = [x_1 \quad \cdots \quad x_m] \begin{bmatrix} (Ay)_1 \\ \vdots \\ (Ay)_m \end{bmatrix} \qquad y^T B^T x = [y_{m+1} \quad \cdots \quad y_{m+n}] \begin{bmatrix} (B^T x)_1 \\ \vdots \\ (B^T x)_n \end{bmatrix}$$

Best Response

		Player 2	
		4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
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$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

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$$y^T B^T x = \begin{bmatrix} y_4 & y_5 \end{bmatrix} \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Given y, player 1's best response maximizes its payoff $x^T A y$
- x is a best response to y if and only if:

$$\forall_{i \in M} x_i > 0 \Rightarrow (Ay)_i = u = \max\{(Ay)_k \mid k \in M\}$$

$$x^T A y = \begin{bmatrix} x_1 & \cdots & x_k & \cdots & x_m \end{bmatrix} \begin{bmatrix} (Ay)_1 \\ \vdots \\ (Ay)_k \\ \vdots \\ (Ay)_m \end{bmatrix}$$

If $x_k \neq 0$, $(Ay)_k = u$

Best Response

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- Proof: $\quad x^T A y = \sum_{i \in M} x_i (Ay)_i = \sum_{i \in M} x_i [u - (u - (Ay)_i)]$

Best Response

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- x is a best response to y if and only if:

$$\forall_{i \in M} x_i > 0 \Rightarrow (Ay)_i = u = \max\{(Ay)_k \mid k \in M\}$$

- Proof: $x^T A y = \sum_{i \in M} x_i (Ay)_i = u - \sum_{i \in M} x_i (u - (Ay)_i)$

  Because $x_i \geq 0, (u - (Ay)_i) \geq 0 \rightarrow x^T A y \leq u$

  $x^T A y = u$ happens only when $x_i > 0, (Ay)_i = u$

Nash Equilibrium

		Player 2: 4	Player 2: 5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

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$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Given y, player 1's best response maximizes its payoff $x^T A y$
- Given x, player 2's best response maximizes its payoff $y^T B^T x$
- (x,y) is a Nash equilibrium iff x and y are best responses to each other
- The support of a mixed strategy is the set of actions that are played with positive probability

Zero-Sum Games

		Player 2: 4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- In zero-sum games $A = -B$, the Nash Equilibrium can be expressed as a LP

minimize u

subject to $\forall_{j \in M}\ (Ay)_j \leq u$

Player 2 plays the mixed strategy y that minimizes the utility of player 1 by playing his best response

$\forall_{i \in N}\ y_i \geq 0$

$\sum_{i \in N} y_i = 1$

The values of variables y_i are consistent with their interpretation as probabilities

- A similar LP dual problem expresses the player 1's mixed strategies
- In general-sum games the problem of finding Nash equilibria cannot be formulated as an LP

General-Sum Games

		Player 2	
		4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^TAy = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^TB^Tx = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- If there is a Nash equilibrium (x,y) with a support $X \subseteq M, Y \subseteq N$ then:

$$\forall_{i \in X} \ x_i > 0 \longrightarrow (Ay)_i = u$$

$$\forall_{j \in M \setminus X} \ x_j = 0$$

$$\sum_{i \in X} \ x_i = 1$$

General-Sum Games

		Player 2: 4	Player 2: 5
Player 1	1	3, 3	3, 2
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- If there is a Nash equilibrium (x,y) with a support $X \subseteq M, Y \subseteq N$ then:

$\forall_{i \in X}\ x_i > 0$ $\qquad (Ay)_i = u \qquad \max\{(Ay)_k \mid k \in M\}$

$\forall_{j \in M \setminus X}\ x_j = 0$ $\qquad (Ay)_j \leq u$

$\sum_{i \in X}\ x_i = 1$

General-Sum Games

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- If there is a Nash equilibrium (x,y) with a support $X \subseteq M, Y \subseteq N$ then:

$$\forall_{i \in X} \;\; x_i > 0 \qquad (Ay)_i = u \qquad (B^T x)_k = v \qquad \forall_{k \in Y} \;\; y_k > 0$$

$$\forall_{j \in M \setminus X} \;\; x_j = 0 \qquad (Ay)_j \leq u \qquad (B^T x)_l \leq v \qquad \forall_{l \in N \setminus Y} \;\; y_l = 0$$

$$\sum_{i \in X} \;\; x_i = 1 \qquad \qquad \sum_{k \in Y} \;\; y_k = 1$$

General-Sum Games

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- With $X = \{1,2,3\}, Y = \{4,5\}$ then:

$$x_1 > 0, \quad x_2 > 0, \quad x_3 > 0,$$

$$\boxed{\begin{aligned} 3y_4 + 3y_5 &= u, \\ 2y_4 + 5y_5 &= u, \\ 0y_4 + 6y_5 &= u, \end{aligned}}$$

$$3x_1 + 2x_2 + 3x_3 = v, \quad 2x_1 + 6x_2 + 1x_3 = v,$$

$$y_4 > 0, \quad y_5 > 0,$$

$$\boxed{y_4 + y_5 = 1}$$

$$x_1 + x_2 + x_3 = 1$$

Infeasible!

General-Sum Games

		Player 2: 4	5
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- With $X = \{1,2\}, Y = \{4,5\}$ then:

$$x_1 > 0, \quad 3y_4 + 3y_5 = u, \quad 3x_1 + 2x_2 + 3x_3 = v, \quad y_4 > 0,$$

$$x_2 > 0, \quad 2y_4 + 5y_5 = u, \quad 2x_1 + 6x_2 + 1x_3 = v, \quad y_5 > 0,$$

$$x_3 = 0, \quad 0y_4 + 6y_5 \leq u, \quad y_4 + y_5 = 1$$

$$x_1 + x_2 = 1$$

Feasible!

$$\boxed{x_1 = \frac{4}{5}, \quad x_2 = \frac{1}{5}, \quad x_3 = 0, \quad y_4 = \frac{2}{3}, \quad y_5 = \frac{1}{3},} \rightarrow v = \frac{14}{5}, u = 3,$$

General-Sum Games

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$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- With $X = \{1,3\}, Y = \{4,5\}$ then:

$$\begin{array}{llll} x_1 > 0, & 3y_4 + 3y_5 = u, & 3x_1 + 2x_2 + 3x_3 = v, & y_4 > 0, \\ x_2 = 0, & 2y_4 + 5y_5 \leq u, & 2x_1 + 6x_2 + 1x_3 = v, & y_5 > 0, \\ x_3 > 0, & 0y_4 + 6y_5 = u, & & y_4 + y_5 = 1 \end{array}$$

$$x_1 + x_3 = 1$$

Infeasible!

Support Enumeration Methods

		Player 2: 4	Player 2: 5
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- A two-player game is called nondegenerate if no mixed strategy of support size k has more than k pure best responses.
- In any Nash equilibrium (x,y) of a nondegenerate game, x and y have supports of equal size.
- Enumeration method for nondegenerate games:

For each $k \in 1, \dots, \min\{m, n\}$

test all possible k-sized support mixed strategy pairs $(X \subseteq M, Y \subseteq N)$

Support Enumeration Methods

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			4	5
Player 1	1		3, 3	3, 2
	2		2, 2	5, 6
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$$x^T A y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

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- A generic Support Enumeration Method (SEM)
  - Search the space of supports
  - Considers every support size separately
  - Favors support sizes that are balanced and small
  - Uses strict dominance to prune the search space

Support Enumeration Methods

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- A generic Support Enumeration Method (SEM)

  for all (X, Y) sorted in increasing order of, first $\big||X| - |Y|\big|$ and second $|X| + |Y|$ **do**

      if $(X, Y) == pruneStrictlyDominated(X, Y)$

          test the support mixed strategy pair (X, Y)

Polytopes

		4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
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(Columns: Player 2)

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

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- An alternative method via polytopes (geometric objects with flat sides)
- Transform games to polytopes:

$$\bar{P} \rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \leq v \\ 2x_1 + 6x_2 + 1x_3 \leq v \\ x_1 + x_2 + x_3 = 1 \end{cases} \qquad \bar{Q} \rightarrow \begin{cases} 3y_4 + 3y_5 \leq u \\ 2y_4 + 5y_5 \leq u \\ 0y_4 + 6y_5 \leq u \\ y_4 + y_5 = 1 \end{cases}$$

$$\bar{P} = \{(x, v) \in R^M \times R \mid x \geq 0, \mathbf{1}^T x = 1, B^T x \leq \mathbf{1} v\}$$
$$\bar{Q} = \{(y, u) \in R^N \times R \mid y \geq 0, \mathbf{1}^T y = 1, Ay \leq \mathbf{1} u\}$$

Polytopes

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- An alternative method via polytopes (geometric objects with flat sides)
- Simplify by eliminating payoff variables:

$$\bar{P} \rightarrow \begin{cases} 3\frac{x_1}{v} + 2\frac{x_2}{v} + 3\frac{x_3}{v} \leq 1 \\ 2\frac{x_1}{v} + 6\frac{x_2}{v} + 1\frac{x_3}{v} \leq 1 \\ \frac{x_1}{v} + \frac{x_2}{v} + \frac{x_3}{v} = \frac{1}{v} \end{cases} \qquad \bar{Q} \rightarrow \begin{cases} 3\frac{y_4}{u} + 3\frac{y_5}{u} \leq 1 \\ 2\frac{y_4}{u} + 5\frac{y_5}{u} \leq 1 \\ 0\frac{y_4}{u} + 6\frac{y_5}{u} \leq 1 \\ \frac{y_4}{u} + \frac{y_5}{u} = \frac{1}{u} \end{cases}$$

$$\bar{P} = \{(x, v) \in R^M \times R \mid x \geq 0, \mathbf{1}^T x = 1, B^T x \leq \mathbf{1} v\}$$
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Polytopes

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$$y^T B^T x = \begin{bmatrix} y_4 & y_5 \end{bmatrix} \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- An alternative method via polytopes (geometric objects with flat sides)
- Simplify by eliminating payoff variables:

$$P \longrightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \leq 1 \\ 2x_1 + 6x_2 + 1x_3 \leq 1 \\ x_1 + x_2 + x_3 = \frac{1}{v} \end{cases} \qquad Q \longrightarrow \begin{cases} 3y_4 + 3y_5 \leq 1 \\ 2y_4 + 5y_5 \leq 1 \\ 0y_4 + 6y_5 \leq 1 \\ y_4 + y_5 = \frac{1}{u} \end{cases}$$

$$P = \{x \in R^m \mid x \geq 0, B^T x \leq \mathbf{1}\}$$
$$Q = \{y \in R^n \mid y \geq 0, Ay \leq \mathbf{1}\}$$

Polytopes

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- An alternative method via polytopes (geometric objects with flat sides)
- Simplify by eliminating payoff variables:

$$P \rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \leq 1 \\ 2x_1 + 6x_2 + 1x_3 \leq 1 \\ v = \frac{1}{x_1+x_2+x_3} \end{cases} \qquad Q \rightarrow \begin{cases} 3y_4 + 3y_5 \leq 1 \\ 2y_4 + 5y_5 \leq 1 \\ 0y_4 + 6y_5 \leq 1 \\ u = \frac{1}{y_4+y_5} \end{cases}$$

$$P = \{x \in R^m \mid x \geq 0, B^T x \leq \mathbf{1}\}$$
$$Q = \{y \in R^n \mid y \geq 0, Ay \leq \mathbf{1}\}$$

Polytopes

		Player 2 4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- An alternative method via polytopes (geometric objects with flat sides)
- Simplify by eliminating payoff variables:

$$P \rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \le 1 \\ 2x_1 + 6x_2 + 1x_3 \le 1 \\ \times \end{cases} \qquad Q \rightarrow \begin{cases} 3y_4 + 3y_5 \le 1 \\ 2y_4 + 5y_5 \le 1 \\ 0y_4 + 6y_5 \le 1 \\ \times \end{cases}$$

$$v = \frac{1}{x_1 + x_2 + x_3} \qquad u = \frac{1}{y_4 + y_5}$$

$$x_1^* = vx_1, x_2^* = vx_2, x_3^* = vx_3$$

$$P = \{x \in R^m \mid x \ \ge 0, B^T x \ \le \mathbf{1}\}$$
$$Q = \{y \in R^n \mid y \ \ge 0, Ay \le \mathbf{1}\}$$

$$y_4^* = uy_4, y_5^* = uy_5$$

Labeling Vertices

		Player 2: 4	Player 2: 5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- An alternative method via polytopes (geometric objects with flat sides)
- Label the vertices:

$$y_5 \geq 0 \quad 5$$

$$Q \rightarrow \begin{cases} 3y_4 + 3y_5 \leq 1 & 1 \\ 2y_4 + 5y_5 \leq 1 & 2 \\ 0y_4 + 6y_5 \leq 1 & 3 \end{cases}$$

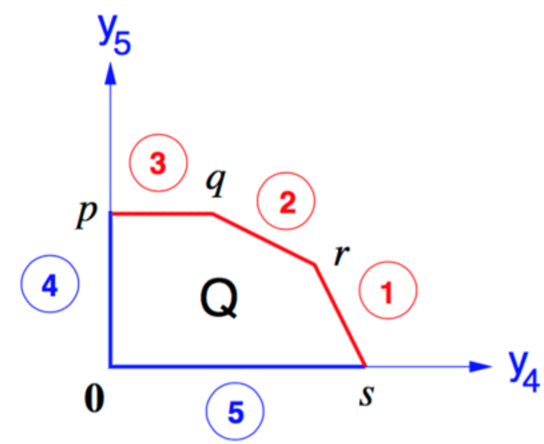
$$y_4 \geq 0 \quad 4$$

$s = (1/3, 0)$

$r = (2/9, 1/9)$

$q = (1/12, 1/6)$

$p = (0, 1/6)$



Labeling Vertices

Player 1 \ Player 2	4	5
1	3, 3	3, 2
2	2, 2	5, 6
3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

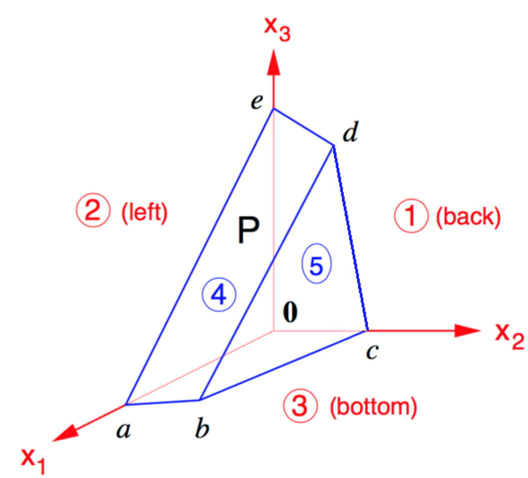
$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- An alternative method via polytopes (geometric objects with flat sides)
- Label the vertices:

$$P \rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \leq 1 & 4 \\ 2x_1 + 6x_2 + 1x_3 \leq 1 & 5 \end{cases}$$

$$x_1 \geq 0 \quad 1$$
$$x_2 \geq 0 \quad 2$$
$$x_3 \geq 0 \quad 3$$

$$a = (1/3, 0, 0)$$
$$b = (2/7, 1/14, 0)$$
$$c = (0, 1/6, 0)$$
$$d = (0, 1/8, 1/4)$$
$$e = (0, 0, 1/3)$$



Nash Equilibria from completed labeled pairs

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & 4 & 5 \\ \hline & 1 & 3,\,3 & 3,\,2 \\ \hline \text{Player 1} & 2 & 2,\,2 & 5,\,6 \\ \hline & 3 & 0,\,3 & 6,\,1 \\ \hline \end{array}$$

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

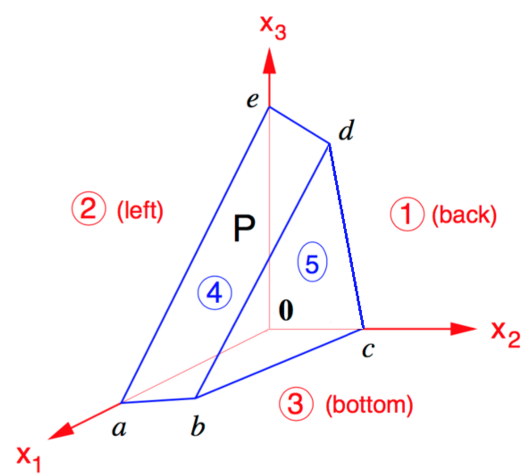
$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

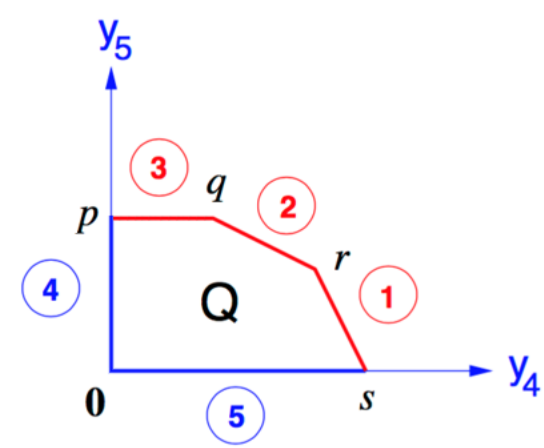
$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Nash equilibria are obtained from completed labeled pair of vertices

$$Q \rightarrow \begin{cases} 3y_4 + 3y_5 \le 1 & 1 \\ 2y_4 + 5y_5 \le 1 & 2 \\ 0y_4 + 6y_5 \le 1 & 3 \end{cases}$$

$$P \rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 \le 1 & 4 \\ 2x_1 + 6x_2 + 1x_3 \le 1 & 5 \end{cases}$$





Nash Equilibria from completed labeled pairs

Player 2

Player 1		4	5
	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

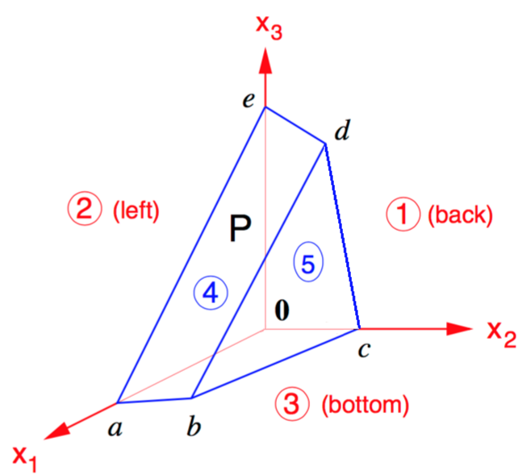
$$x^T A y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

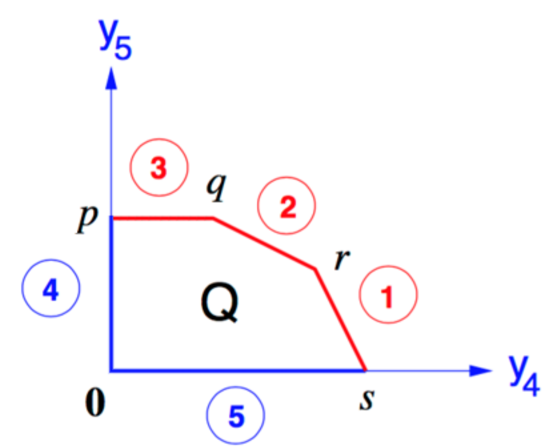
$$y^T B^T x = \begin{bmatrix} y_4 & y_5 \end{bmatrix} \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Nash equilibria are obtained from completed labeled pair of vertices

$a \rightarrow \{2,3,4\}$
$b \rightarrow \{3,4,5\}$
$c \rightarrow \{1,3,5\}$
$d \rightarrow \{1,4,5\}$
$e \rightarrow \{1,2,4\}$

$s \rightarrow \{1,5\}$
$r \rightarrow \{1,2\}$
$q \rightarrow \{2,3\}$
$p \rightarrow \{3,4\}$





Nash Equilibria from completed labeled pairs

Player 2

		4	5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Nash equilibria are obtained from completed labeled pair of vertices

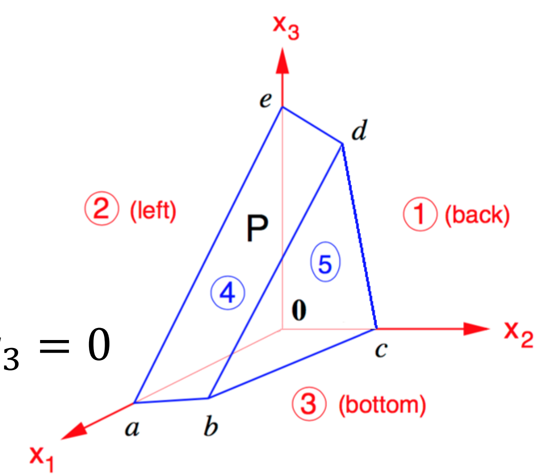
$$a = (1/3,0,0) \qquad s = (1/3,0)$$

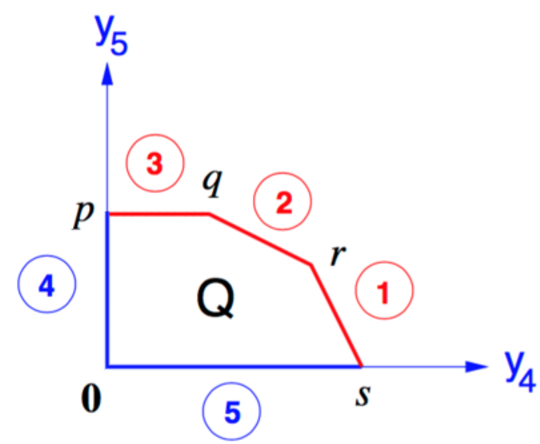
$$v = \frac{1}{x_1 + x_2 + x_3} = \frac{1}{1/3} = 3$$

$$u = \frac{1}{y_4 + y_5} = \frac{1}{1/3} = 3$$

$$x_1^* = vx_1 = 1, x_2^* = vx_2 = 0, x_3^* = vx_3 = 0$$

$$y_4^* = uy_4 = 1, y_5^* = uy_5 = 0$$





Nash Equilibria from completed labeled pairs

Player 2

Player 1	4	5
1	3, 3	3, 2
2	2, 2	5, 6
3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = [y_4 \quad y_5] \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Nash equilibria are obtained from completed labeled pair of vertices

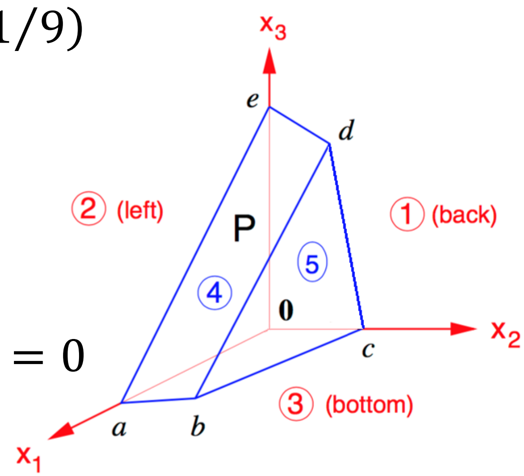
$b = (2/7\,, 1/14\,, 0) \qquad r = (2/9\,, 1/9)$

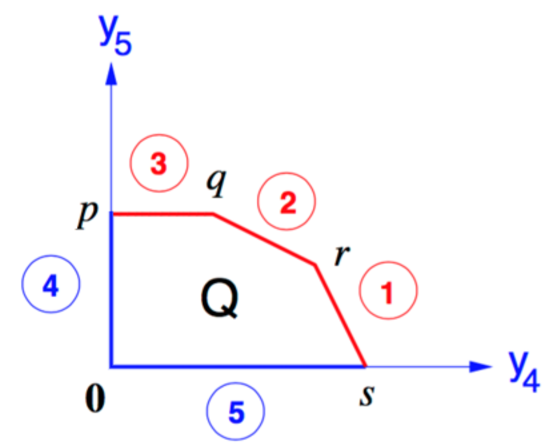
$$v = \frac{1}{2/7 + 1/14} = \frac{14}{5}$$

$$u = \frac{1}{2/9 + 1/9} = 3$$

$$x_1^* = vx_1 = \frac{4}{5}, x_2^* = vx_2 = \frac{1}{5}, x_3^* = vx_3 = 0$$

$$y_4^* = uy_4 = \frac{2}{3}, y_5^* = uy_5 = \frac{1}{3}$$





Nash Equilibria from completed labeled pairs

Player 1 \ Player 2	4	5
1	3, 3	3, 2
2	2, 2	5, 6
3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = \begin{bmatrix} y_4 & y_5 \end{bmatrix} \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Nash equilibria are obtained from completed labeled pair of vertices

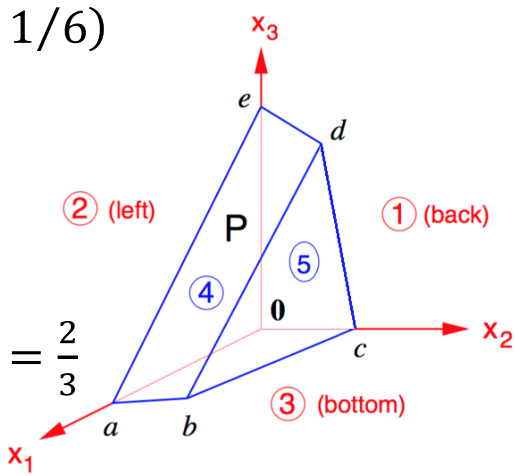
$d = (0, 1/8, 1/4) \qquad q = (1/12, 1/6)$

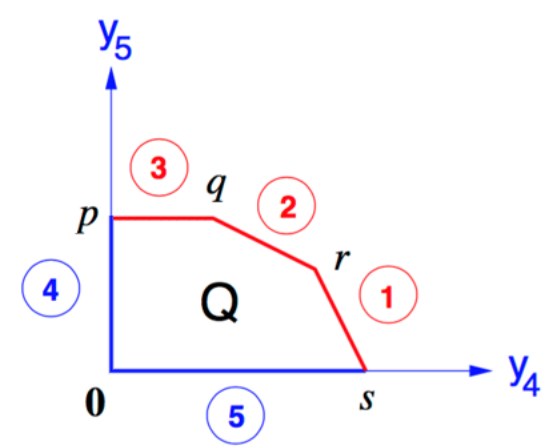
$$v = \frac{1}{1/8 + 1/4} = \frac{8}{3}$$

$$u = \frac{1}{1/12 + 1/6} = 4$$

$$x_1^* = vx_1 = 0, x_2^* = vx_2 = \frac{1}{3}, x_3^* = vx_3 = \frac{2}{3}$$

$$y_4^* = uy_4 = \frac{1}{3}, y_5^* = uy_5 = \frac{2}{3}$$





Lemke-Howson Algorithm

		Player 2: 4	Player 2: 5
Player 1	1	3, 3	3, 2
	2	2, 2	5, 6
	3	0, 3	6, 1

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}$$

$$x^T A y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3y_4 + 3y_5 \\ 2y_4 + 5y_5 \\ 0y_4 + 6y_5 \end{bmatrix}$$

$$y^T B^T x = \begin{bmatrix} y_4 & y_5 \end{bmatrix} \begin{bmatrix} 3x_1 + 2x_2 + 3x_3 \\ 2x_1 + 6x_2 + 1x_3 \end{bmatrix}$$

- Lemke-Howson algorithm to find one Nash equilibrium
  - Starts with $(0,0)$
  - Choose an initial label $k \in M \cup N$ $(k = 2)$
  - Drop label k $(c, 0)$
  - The duplicate label (5) is dropped in the other polytope (c, p)
  - Repeat dropping the duplicate label in the other polytope $(d, p) \rightarrow (d, q)$
  - Stop if there are no duplicate label (d, q)
  - Nash Eq: $x = \left(0, \frac{1}{3}, \frac{2}{3}\right)$, $y = \left(\frac{1}{3}, \frac{2}{3}\right)$