Bayesian Games

J. Leite (adapted from Kevin Leyton-Brown)

Auctions

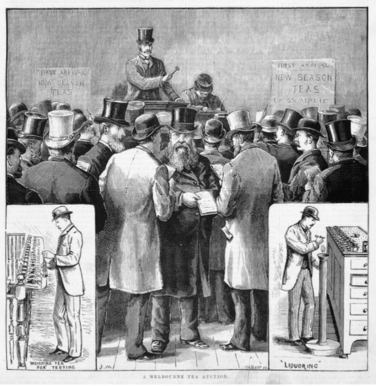


Figure: Tea Auction, Melbourne, Australia, 1885

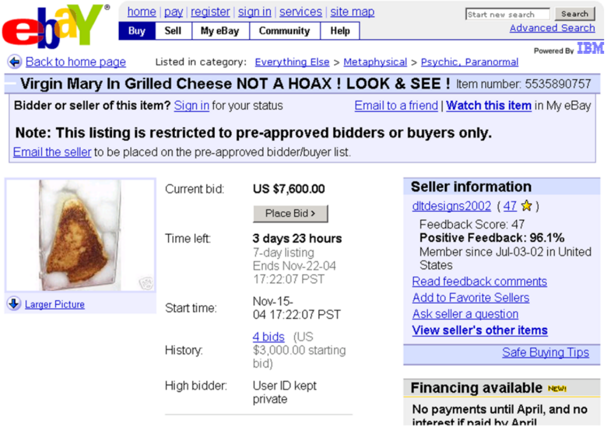
Auctions

Figure: Bluefin Tuna Auction, Tokyo, Japan, 2008

Auctions

Figure: Auction of seized horses, Dixon, IL

Auctions



Back to home page   Listed in category: Everything Else > Metaphysical > Psychic, Paranormal

Virgin Mary In Grilled Cheese NOT A HOAX ! LOOK & SEE ! Item number: 5535890757

Bidder or seller of this item? Sign in for your status   Email to a friend | **Watch this item** in My eBay

Note: This listing is restricted to pre-approved bidders or buyers only.

Email the seller to be placed on the pre-approved bidder/buyer list.

Larger Picture

Current bid:	**US $7,600.00**
	Place Bid >
Time left:	**3 days 23 hours** 7-day listing Ends Nov-22-04 17:22:07 PST
Start time:	Nov-15-04 17:22:07 PST
History:	4 bids (US $3,000.00 starting bid)
High bidder:	User ID kept private

Seller information

dltdesigns2002 (47 ★)

Feedback Score: 47
Positive Feedback: 96.1%
Member since Jul-03-02 in United States

Read feedback comments
Add to Favorite Sellers
Ask seller a question
View seller's other items

Safe Buying Tips

Financing available NEW!

No payments until April, and no interest if paid by April

Auctions

Figure: A silent auction – looks suspiciously like a game

Section 1

Bayesian Games

Fun Game

- Choose a phone number none of your neighbours knows; consider it to be ABCDEFGHI
    - take "DE" as your valuation
    - play a first-price auction with three neighbours, where your utility is your valuation minus the amount you pay
    - now play the auction again, same neighbours, same valuation
    - now play again, with "FG" as your valuation
- Questions:
    - what is the role of uncertainty here?
    - can we model this uncertainty using an imperfect information extensive form game?
        - imperfect info means not knowing what node you're in in the info set
        - here we're not sure what game is being played (though if we allow a move by nature, we can do it)

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Introduction

- So far, we've assumed that all players know what game is being played. Everyone knows:
  - the number of players
  - the actions available to each player
  - the payoff associated with each action vector
- Why is this true in imperfect information games?
- We'll now consider games of incomplete (not imperfect) information: players are uncertain about the game being played.

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- Why is this true in imperfect information games?
- We'll now consider games of incomplete (not imperfect) information: players are uncertain about the game being played.

Example

- Consider the payoff matrix shown here
  - ϵ is a small positive constant
  - Agent 1 knows its value

	L	R
T	$100, a$	$1 - \epsilon, b$
B	$2, c$	$1, d$

- Agent 1 doesn't know the values of a, b, c, d
  - Thus the matrix represents a set of games
  - Agent 1 doesn't know which of these games is the one being played
- Agent 1 wants a strategy that makes sense despite this lack of knowledge

- If Agent 1 thinks Agent 2 is malicious, then Agent 1 might want to play a maxmin, or "safety level" strategy
  - minimum payoff of T is $1 - \epsilon$
  - minimum payoff of B is 1

- So, agent 1's maxmin strategy is B

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Bayesian Games

- Suppose we know the set G of all possible games and we have enough information to put a probability distribution over the games in G
- A Bayesian Game is a class of games G that satisfies two fundamental conditions:

Condition 1	All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
Condition 2	Agent's beliefs are posteriors, obtained by conditioning a common prior on individual private signals.

Bayesian Games: Condition 1

- All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
- This condition isn't very restrictive. Other types of uncertainty can be reduced to the above, by reformulating the problem
- Suppose we don't know whether player 2 only has strategies L and R, or also an additional strategy C:

Game G_1:

	L	R
T	$1, 1$	$1, 3$
B	$0, 5$	$1, 13$

Game G_2:

	L	C	R
T	$1, 1$	$0, 2$	$1, 3$
B	$0, 5$	$2, 8$	$1, 13$

- If player 2 doesn't have strategy C, this is equivalent to having a strategy C that's strictly dominated by other strategies:

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Game G'_1:

	L	C	R
T	$1, 1$	$0, -100$	$1, 3$
B	$0, 5$	$2, -100$	$1, 13$

- The Nash equilibria for G'_1 are the same as those for G_1.

Bayesian Games: Condition 2

- Agent's beliefs are posteriors, obtained by conditioning a common prior on individual private signals.
- The probability distribution over the games in G is common knowledge (i.e., known to all the agents).
- The beliefs of the different agents are posterior probabilities
    - Combine the common prior distribution with individual "private signals" (what's "revealed" to the individual players)

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Bayesian Games

- So a Bayesian game defines
  - the uncertainties of agents about the game being played,
  - what each agent believes the other agents believe about the game being played
- We'll discuss three, essentially equivalent, different ways to define Bayesian Games.
  - based on Information Sets
  - based on Extensive Form with Chance Moves
  - based on Epistemic Types

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Definition 1: Information Sets

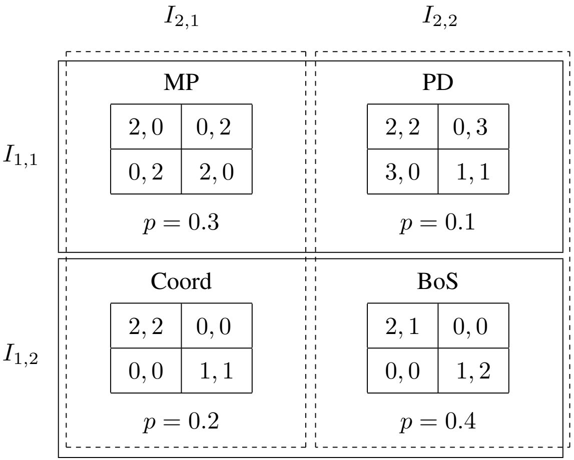
- Bayesian game: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

Definition (Bayesian Game: Information Sets)

A Bayesian game is a tuple (N, G, P, I) where

- N is a set of agents,
- G is a set of games with N agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g',
- $P \in \Pi(G)$ is a common prior over games, where $\Pi(G)$ is the set of all probability distributions over G, and
    - common: common knowledge (known to all the agents)
    - prior: probability before learning any additional information
- $I = (I_1, ..., I_N)$ is tuple of information sets i.e. a tuple of partitions of G, one for each agent.

Definition 1: Example



$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP

$2,0$	$0,2$
$0,2$	$2,0$

$p = 0.3$

PD

$2,2$	$0,3$
$3,0$	$1,1$

$p = 0.1$

$I_{1,2}$

Coord

$2,2$	$0,0$
$0,0$	$1,1$

$p = 0.2$

BoS

$2,1$	$0,0$
$0,0$	$1,2$

$p = 0.4$

Definition 1: Example

- Suppose the randomly chosen game is MP
- Agent 1's information set is $I_{1,1}$
  - 1 knows it's MP or PD
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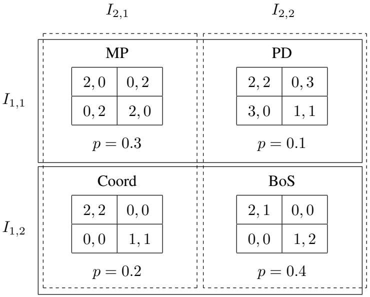

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$I_{1,2}$

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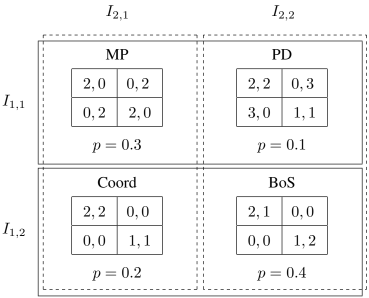

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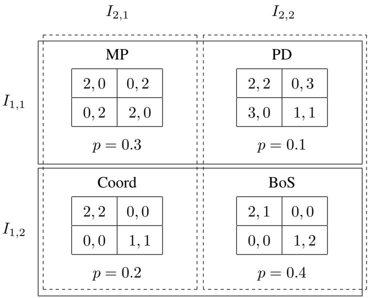

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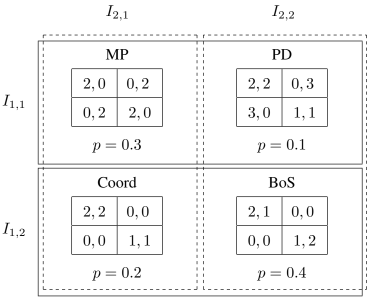

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- Agent 1's information set is $I_{1,1}$
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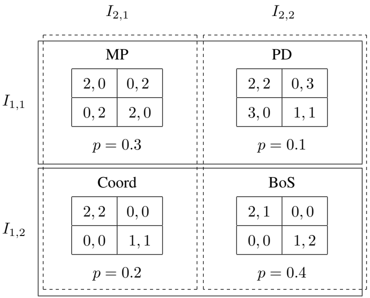
- 1 can infer posterior probabilities for each



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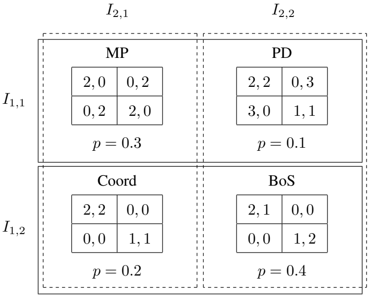

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- 1 can infer posterior probabilities for each

$$Pr(MP|I_{1,1}) = \frac{Pr(MP)}{Pr(MP) + Pr(PD)} = \frac{3}{4}$$

$$Pr(PD|I_{1,1}) = \frac{Pr(PD)}{Pr(MP) + Pr(PD)} = \frac{1}{4}$$

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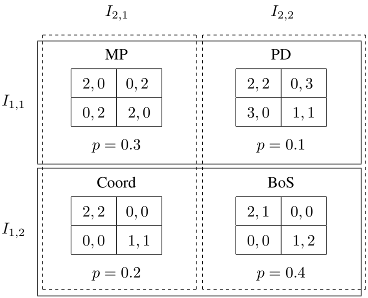

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$$Pr(PD|I_{1,1}) = \frac{Pr(PD)}{Pr(MP) + Pr(PD)} = \frac{1}{4}$$

- 2 can infer posterior probabilities for each

$$Pr(MP|I_{2,1}) = \frac{Pr(MP)}{Pr(MP) + Pr(Coord)} = \frac{3}{5}$$

$$Pr(Coord|I_{2,1}) = \frac{Pr(Coord)}{Pr(MP) + Pr(Coord)} = \frac{2}{5}$$

Definition 2: Extensive Form with Chance Moves

- Add an agent, "Nature," who follows a commonly known mixed strategy, according to the common prior, and has no utility function.
- At the start of the game, Nature makes its move.
- The agents receive individual signals about Nature's choice
    - Some of Natures choices are revealed to some players, others to other players
- Thus, reduce Bayesian games to extensive form games of imperfect information.
- This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner's Dilemma
    - however, it makes sense when the agents really do move sequentially, and at least occasionally observe each other's actions.
    - extensions exist where Nature makes choices and sends signals throughout the game.
    - This allows to model e.g. Backgammon and Bridge

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- Thus, reduce Bayesian games to extensive form games of imperfect information.
- This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner’s Dilemma
  - however, it makes sense when the agents really do move sequentially, and at least occasionally observe each other’s actions.
  - extensions exist where Nature makes choices and sends signals throughout the game.
  - This allows to model e.g. Backgammon and Bridge

Definition 2: Extensive Form with Chance Moves

- Add an agent, “Nature,” who follows a commonly known mixed strategy, according to the common prior, and has no utility function.
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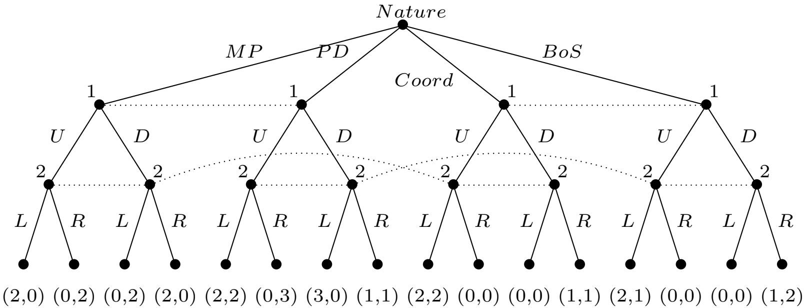
Definition 2: Example

- Same example as before, but translated into extensive form



Definition 3: Epistemic Types

- Recall that we can assume the only thing players are uncertain about is the game's utility function.
- We can directly represent uncertainty over utility function, using the notion of epistemic type.
- An agent's epistemic type consists of all the information it has that isn't common knowledge, e.g.,
    - The agent's actual payoff function
    - The agent's beliefs about other agents' payoffs,
    - The agent's beliefs about their beliefs about his own payoff
    - Any other higher-order beliefs

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Definition 3: Epistemic Types

Definition

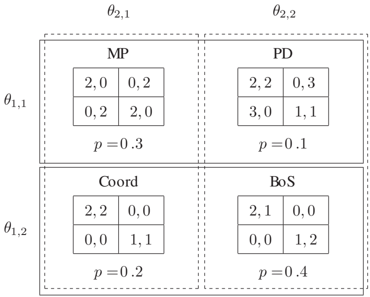
A Bayesian game is a tuple (N, A, Θ, p, u) where

- N is a set of agents,
- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to player i,
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$, where Θ_i is the set of possible types of player i,
- $p : \Theta \rightarrow [0, 1]$ is the common prior over types,
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \rightarrow \mathbb{R}$ is the utility function for player i.

- All this is common knowledge among the players, and each agent knows its own type.

Definition 3: Example

- Agent 1's possible types: $\theta_{1,1}$ and $\theta_{1,2}$
    - 1's type is $\theta_{1,j} \Leftrightarrow$ 1's info set is $I_{1,j}$
- Agent 2's possible types: $\theta_{2,1}$ and $\theta_{2,2}$
    - 2's type is $\theta_{2,j} \Leftrightarrow$ 2's info set is $I_{2,j}$
- Joint distribution on the types:
  $Pr(\theta_{1,1}, \theta_{2,1}) = 0.3$; $Pr(\theta_{1,1}, \theta_{2,2}) = 0.1$
  $Pr(\theta_{1,2}, \theta_{2,1}) = 0.2$; $Pr(\theta_{1,2}, \theta_{2,2}) = 0.4$



$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP

2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

2, 2	0, 3
3, 0	1, 1

$p = 0.1$

$\theta_{1,2}$

Coord

2, 2	0, 0
0, 0	1, 1

$p = 0.2$

BoS

2, 1	0, 0
0, 0	1, 2

$p = 0.4$

- Conditional probabilities for agent 1:
  $Pr(\theta_{2,1}|\theta_{1,1}) = 0.3/(0.3+0.1) = 3/4$; $Pr(\theta_{2,2}|\theta_{1,1}) = 0.1/(0.3+0.1) = 1/4$
  $Pr(\theta_{2,1}|\theta_{1,2}) = 0.2/(0.2+0.4) = 1/3$; $Pr(\theta_{2,2}|\theta_{1,2}) = 0.4/(0.2+0.4) = 2/3$
- Conditional probabilities for agent 2:
  $Pr(\theta_{1,1}|\theta_{2,1}) = 0.3/(0.3+0.2) = 3/5$; $Pr(\theta_{1,2}|\theta_{2,1}) = 0.2/(0.3+0.2) = 2/5$
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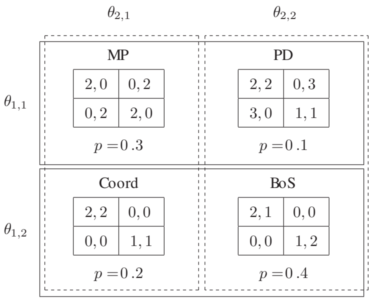

	$\theta_{2,1}$	$\theta_{2,2}$
$\theta_{1,1}$	MP 2, 0 | 0, 2 0, 2 | 2, 0 $p = 0.3$	PD 2, 2 | 0, 3 3, 0 | 1, 1 $p = 0.1$
$\theta_{1,2}$	Coord 2, 2 | 0, 0 0, 0 | 1, 1 $p = 0.2$	BoS 2, 1 | 0, 0 0, 0 | 1, 2 $p = 0.4$

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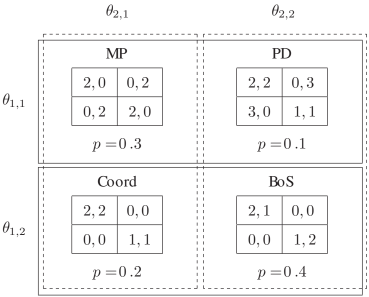

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0, 2	2, 0

PD

2, 2	0, 3
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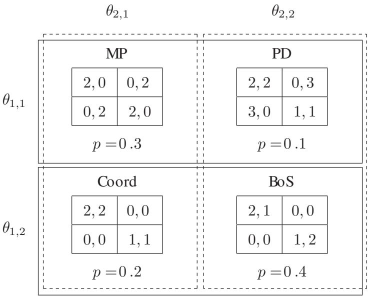

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$\theta_{1,2}$	Coord $2,2$ $0,0$ $0,0$ $1,1$ $p = 0.2$	BoS $2,1$ $0,0$ $0,0$ $1,2$ $p = 0.4$

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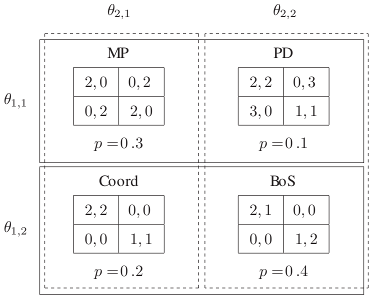

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2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

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3, 0	1, 1

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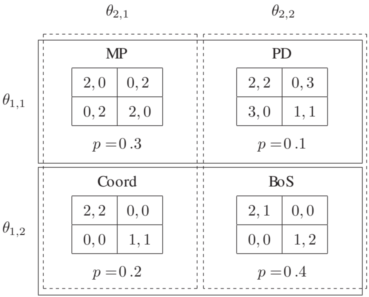

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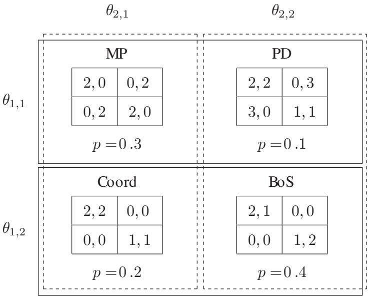
Definition 3: Example

- The players' payoffs depend on both their types and their actions
  - The types determine what game it is.
  - The actions determine the payoff within that game



	$\theta_{2,1}$	$\theta_{2,2}$
$\theta_{1,1}$	MP $p = 0.3$	PD $p = 0.1$
$\theta_{1,2}$	Coord $p = 0.2$	BoS $p = 0.4$

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

$p = 0.1$

Coord

$2, 2$	$0, 0$
$0, 0$	$1, 1$

$p = 0.2$

BoS

$2, 1$	$0, 0$
$0, 0$	$1, 2$

$p = 0.4$

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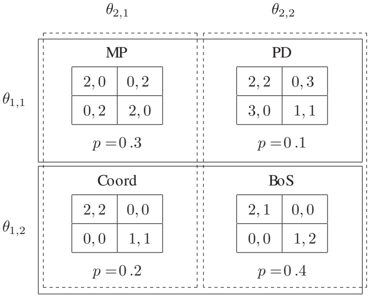

MP

2, 0	0, 2
0, 2	2, 0

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PD

2, 2	0, 3
3, 0	1, 1

$p = 0.1$

Coord

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0, 0	1, 1

$p = 0.2$

BoS

2, 1	0, 0
0, 0	1, 2

$p = 0.4$

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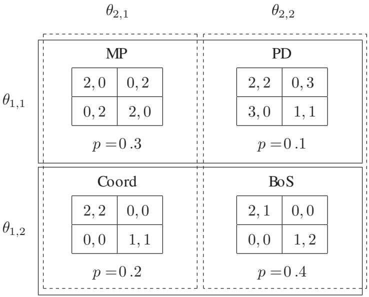

$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP

2, 0	0, 2
0, 2	2, 0

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PD

2, 2	0, 3
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$\theta_{1,2}$

Coord

2, 2	0, 0
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$p = 0.2$

BoS

2, 1	0, 0
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$p = 0.4$

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Section 2

Analyzing Bayesian games

Bayesian (Nash) Equilibrium

- A plan of action for each player as a function of types that maximize each type's expected utility:
  - expecting over the actions of other players
  - expecting over the types of other players

Strategies

Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, strategies are defined as follows:

- Pure strategy: $s_i : \Theta_i \rightarrow A_i$
  - a mapping from every type agent i could have to the action he would play if he had that type.
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- $s_j(a_j|\theta_j)$
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Strategies

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Expected Utility

Three meaningful notions of expected utility:

- *ex-post*
    - the agent knows all agents' types.
- *ex-interim*
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Ex-post expected utility

Definition (*Ex-post* expected utility)

Agent i's *ex-post* expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by s and the agents' types are given by θ, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents' mixed strategies, since i knows everyone's type.
- In a Bayesian game, no agent will know the others' types. So why is this notion useful?
  - Because it is used in defining the other notions of expected utility.
  - And also for defining a specialized notion of equilibrium.

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Ex-interim expected utility

Definition (*ex-interim* expected utility)

Agent i's *ex-interim* expected utility in a Bayesian game (N, A, Θ, p, u), where i's type is θ_i and where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) EU_i(s, (\theta_{-i}, \theta_i)).$$

or, equivalently:

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- i must consider every θ_{-i} and every a to evaluate $u_i(a, \theta_i, \theta_{-i})$.
- i must weight this utility value by:
    - the probability that a would be realized given all players' mixed strategies and types;
    - the probability that the other players' types would be θ_{-i} given that his own type is θ_i.

Ex-ante expected utility

Definition (*Ex-ante* expected utility)

Agent i's *ex-ante* expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$

Bayesian Equilibrium or Bayes-Nash Equilibrium

Definition (Bayesian Equilibrium)

A Bayes-Nash equilibrium is a mixed strategy profile s that satisfies

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i}|\theta_i).$$

for each i and $\theta_i \in \Theta_i$.

- The above is defined based on interim maximization. It is equivalent to an ex-ante formulation:
- If $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, then this is equivalent to requiring that

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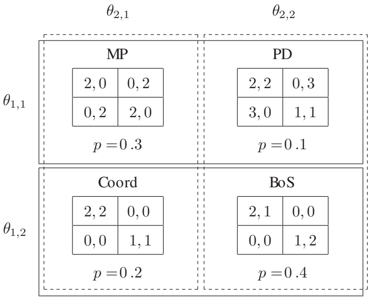
Computing Bayes-Nash Equilibria

- The idea is to construct a payoff matrix for the entire Bayesian game, and find equilibria on that matrix



	$\theta_{2,1}$	$\theta_{2,2}$
$\theta_{1,1}$	MP $\begin{array}{|c|c|}\hline 2,0 & 0,2 \\ \hline 0,2 & 2,0 \\ \hline\end{array}$ $p = 0.3$	PD $\begin{array}{|c|c|}\hline 2,2 & 0,3 \\ \hline 3,0 & 1,1 \\ \hline\end{array}$ $p = 0.1$
$\theta_{1,2}$	Coord $\begin{array}{|c|c|}\hline 2,2 & 0,0 \\ \hline 0,0 & 1,1 \\ \hline\end{array}$ $p = 0.2$	BoS $\begin{array}{|c|c|}\hline 2,1 & 0,0 \\ \hline 0,0 & 1,2 \\ \hline\end{array}$ $p = 0.4$

- First, write each of the pure strategies as a list of actions, one for each type.
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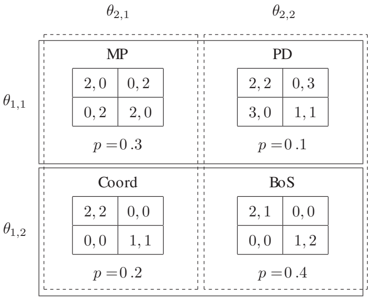

MP

2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

2, 2	0, 3
3, 0	1, 1

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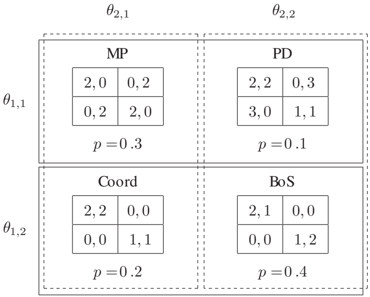

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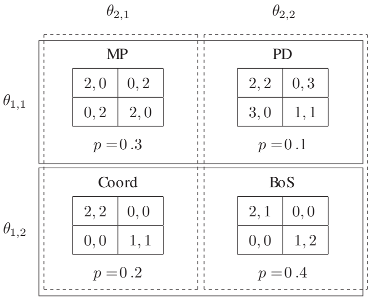

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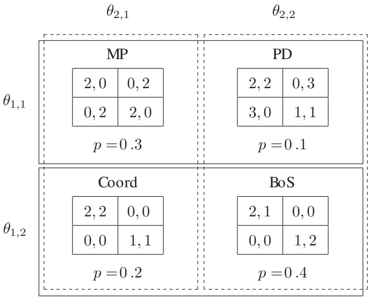

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$3,0$	$1,1$

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$\theta_{1,2}$

Coord

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\hline
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- Next, compute the ex ante expected utility for each pure-strategy profile

$$EU_i(s) = \sum_{\theta\in\Theta} p(\theta)u_i(s,\theta)$$

- For example:

$$EU_2(UD,LR) = \sum_{\theta\in\Theta} p(\theta)u_i(UD,LR,\theta)$$

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$$= 1.1$$

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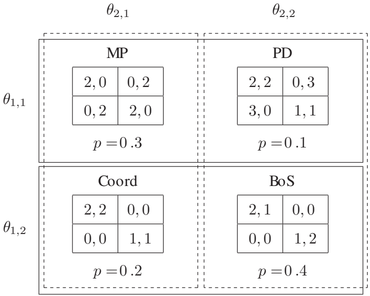
- The idea is to construct a payoff matrix for the entire Bayesian game, and find equilibria on that matrix



$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP

2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

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$\theta_{1,2}$

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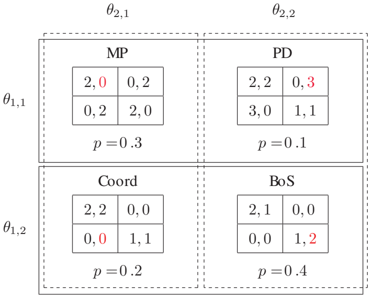

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

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$3, 0$	$1, 1$

$p = 0.1$

Coord

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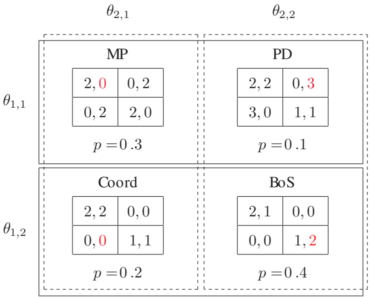
- The idea is to construct a payoff matrix for the entire Bayesian game, and find equilibria on that matrix



$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

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$\theta_{1,2}$

Coord

$2, 2$	$0, 0$
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BoS

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- Then, put all of the ex ante expected utilities into a payoff matrix
  - e.g. $EU_2(UD, LR) = 1.1$

- Now we can compute best responses and Nash equilibria

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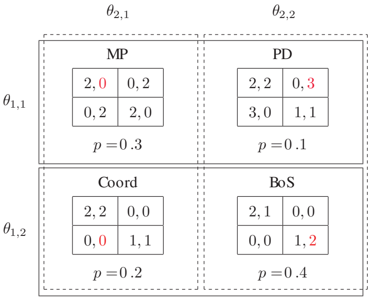

MP

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	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, **1.1**	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

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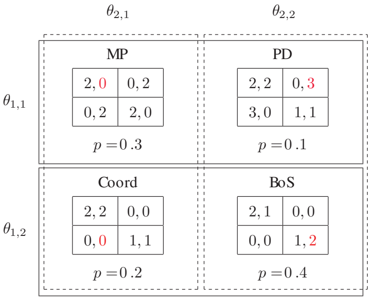

$\theta_{2,1}$ $\theta_{2,2}$

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MP

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$0, 2$	$2, 0$

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PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

$p = 0.1$

$\theta_{1,2}$

Coord

$2, 2$	$0, 0$
$0, 0$	$1, 1$

$p = 0.2$

BoS

$2, 1$	$0, 0$
$0, 0$	$1, 2$

$p = 0.4$

- Then, put all of the ex ante expected utilities into a payoff matrix
  - e.g. $EU_2(UD, LR) = 1.1$

	LL	LR	RL	RR
UU	$2, 1$	$1, 0.7$	$1, 1.2$	$0, 0.9$
UD	$0.8, 0.2$	$1,$ **1.1**	$0.4, 1$	$0.6, 1.9$
DU	$1.5, 1.4$	$0.5, 1.1$	$1.7, 0.4$	$0.7, 0.1$
DD	$0.3, 0.6$	$0.5, 1.5$	$1.1, 0.2$	$1.3, 1.1$

- Now we can compute best responses and Nash equilibria

ex-post Equilibrium

Ex-post utilities allows for a stronger equilibrium:

Definition (*ex-post* equilibrium)

A *ex-post* equilibrium is a mixed strategy profile s that satisfies $\forall\theta$, $\forall i$, $s_i \in \arg\max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta)$.

- Note that this notion does not presume that each agent actually does know the others' types.
- Instead, it says that no agent would ever want to deviate from his mixed strategy even if he knew the complete vector θ
- somewhat similar to dominant strategy, but not quite
    - EP: agents do not need to have accurate beliefs about the type distribution
    - DS: agents do not need to have accurate beliefs about others' strategies

Section 3

Bayesian Games: Example

A Sheriff's Dilemma

A sheriff faces an armed suspect and they each must (simultaneously) decide whether to shoot the other or not, and:

- the suspect is either a criminal with probability p or not with probability $1 - p$.
- the sheriff would rather shoot if the suspect shoots, but not if the suspect does not.
- the criminal would rather shoot even if the sheriff does not, as the criminal would be caught if does not shoot.
- the innocent suspect would rather not shoot even if the sheriff shoots.

$$Sheriff$$

$Innocent$	$Shoot$	Not
$Shoot$	$-3, -1$	$-1, -2$
Not	$-2, -1$	$0, 0$

$Criminal$	$Shoot$	Not
$Shoot$	$0, 0$	$2, -2$
Not	$-2, -1$	$-1, 1$

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Summary: Bayesian (Nash) Equilibrium

- Explicitly models behavior in an uncertain environment
- Players choose actions to maximize their payoffs in response to others accounting for:
    - strategic uncertainty about how others will play and
    - payoff uncertainty about the value to their actions