# Noncooperative Game Theory

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Section 1

# More about mixed-strategy Nash Equilibria

# Fun Game!

| | $L$ | $R$ |
|---|---|---|
| $T$ | $80, 40$ $320, 4044, 40$ | $40, 80$ |
| $B$ | $40, 80$ | $80, 40$ |

- Play once as each player, recording the strategy you follow.

# Football Penalty Kicks

- Mixed strategies in sports and competitive games
- Be unpredictable
- How do equilibrium strategies adjust to skills?

| $Kicker \backslash Goalie$ | $Left$ | $Right$ |
|---|---|---|
| $Left$ | $0, 1$ | $1, 0$ |
| $Right$ | $1, 0$ $.75, .25$ | $0, 1$ |

# Football Penalty Kicks

| *Kicker\Goalie* | *Left* | *Right* |
|---|---|---|
| *Left* | $0, 1$ | $1, 0$ |
| *Right* | $.75, .25$ | $0, 1$ |

- Let the Goalie play *Left* with $p$, *Right* with $1 - p$.
- If the Kicker best-responds with a mixed strategy, the Goalie must make him indifferent between *Left* and *Right*.
- $u_{Kicker}(Left) = u_{Kicker}(Right) \iff (1 - p) = .75p \iff p = \frac{4}{7}$
- Likewise, the Kicker must randomize to make the Goalie indifferent.
- Let the Kicker play *Left* with $q$, *Right* with $1 - q$.
- $u_{Goalie}(Left) = u_{Goalie}(Right) \iff q + .25(1 - q) = 0q + (1 - q) \iff q = \frac{3}{7}$
- Thus the mixed strategies $(\frac{3}{7}, \frac{4}{7})$, $(\frac{4}{7}, \frac{3}{7})$ are a Nash equilibrium.

# Fun Game!

| | $L$ | $R$ |
|---|---|---|
| $T$ | $80, 40$;  $320, 40$; $44, 40$ | $40, 80$ |
| $B$ | $40, 80$ | $80, 40$ |

- What does row player do in equilibrium of this game?
  - row player randomizes 50-50 all the time
  - that's what it takes to make column player indifferent
- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time

# Professional Football Penalty Kicks

- Some counter-intuitive features...
- Do people really play equilibria?
- Ignacio Palacios-Heurta (2003)
  - 1417 Penalty kicks from FIFA games: Spain, England, Italy...

| *Kicker\Goalie* | *Left* | *Right* |
|---|---|---|
| *Left* | .58, .42 | .95, .05 |
| *Right* | .93, .07 | .70, .30 |

- The mixed strategies $(.38, .62)$, $(.42, .58)$ are a Nash equilibrium.
- Real data...

| | *Goalie Left* | *Goalie Right* | *Kicker Left* | *Kicker Right* |
|---|---|---|---|---|
| *Nash Freq.* | .42 | .58 | .38 | .62 |
| *Actual Freq.* | .42 | .58 | .40 | .60 |

Section 2

## Beyond Nash Equlibrium

# Dominated Strategies

Should Grace celebrate her 90th birthday by jumping out of a plane strapped to this guy?

# Zero-Sum Games

Is he really solving for the Nash equilibrium?

# Coordination

Battle of the Sexes: either unfairness or miscoordination?

Section 3

## Domination

# Domination

- Let $s_i$ and $s_i'$ be two strategies for player $i$, and let $S_{-i}$ be is the set of all possible strategy profiles for the other players

**Definition**

$s_i$ strictly dominates $s_i'$ if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

**Definition**

$s_i$ weakly dominates $s_i'$ if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $\exists s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

**Definition**

$s_i$ very weakly dominates $s_i'$ if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

# Equilibria and dominance

- If one strategy dominates all others, we say it is dominant.
- A strategy profile consisting of dominant strategies for every player must be a Nash equilibrium.
  - An equilibrium in strictly dominant strategies must be unique.
- Consider Prisoner's Dilemma again
  - not only is the only equilibrium the only non-Pareto-optimal outcome, but it's also an equilibrium in strictly dominant strategies!

# Section 4

## Fun Game

# Traveler's Dilemma

*Two travelers purchase identical African masks while on a tropical vacation. Their luggage is lost on the return trip, and the airline asks them to make independent claims for compensation. In anticipation of excessive claims, the airline representative announces: "We know that the bags have identical contents, and we will entertain any claim between \$180 and \$300, but you will each be reimbursed at an amount that equals the minimum of the two claims submitted. If the two claims differ, we will also pay a reward $R$ to the person making the smaller claim and we will deduct a penalty $R$ from the reimbursement to the person making the larger claim."*

# Traveler's Dilemma

- Action: choose an integer between 180 and 300
- If both players pick the same number, they both get that amount as payoff
- If players pick a different number:
  - the low player gets his number ($L$) plus some constant $R$
  - the high player gets $L - R$.
- Set $R = 5$ and play this game once with a partner; play with as many different partners as you like.
- Now set $R = 180$, and again play with as many partners as you like.

# Traveler’s Dilemma

- What is the equilibrium?
  - $(180, 180)$ is the only equilibrium, for all $R \geq 2$.
- What (usually) happens?
  - with $R = 5$ most people choose 295–300
  - with $R = 180$ most people choose 180

Section 5

# Iterated Removal of Dominated Strategies

# “Rationality”

- A basic premise: players maximize their payoffs
- What if all players know this?
- And they know that other players know it?
- And they know that other players know that they know it?
- ...

# Dominated strategies

- No equilibrium can involve a strictly dominated strategy
  - Thus we can remove it, and end up with a strategically equivalent game
  - This might allow us to remove another strategy that wasn't dominated before
  - Running this process to termination is called iterated removal of dominated strategies.

# Iterated Removal of Dominated Strategies: Example

| | L | C | R |
|---|---|---|---|
| U | $3, 0$ | $2, 1$ | $0, 0$ |
| M | $1, 1$ | $1, 1$ | $5, 0$ |
| D | $0, 1$ | $4, 2$ | $0, 1$ |

- $R$ is dominated by $C$.

# Iterated Removal of Dominated Strategies: Example

| | L | C |
|---|---|---|
| U | $3, 0$ | $2, 1$ |
| M | $1, 1$ | $1, 1$ |
| D | $0, 1$ | $4, 2$ |

- $M$ is dominated by $U$.

# Iterated Removal of Dominated Strategies: Example

| | L | C |
|---|---|---|
| U | $3, 0$ | $2, 1$ |
| D | $0, 1$ | $4, 2$ |

- $L$ is dominated by $C$.

# Iterated Removal of Dominated Strategies: Example

| | C |
|---|---|
| U | $2, 1$ |
| D | $4, 2$ |

- $U$ is dominated by $D$.

# Iterated Removal of Dominated Strategies: Example

| | C |
|---|---|
| D | $4, 2$ |

# Iterated Removal of Dominated Strategies: Example

| | L | C | R |
|---|---|---|---|
| U | $3, 0$ | $2, 1$ | $0, 0$ |
| M | $1, 1$ | $1, 1$ | $5, 0$ |
| D | $0, 1$ | $4, 2$ | $0, 1$ |

- A unique Nash equilibrium $D, C$.

# Iterated Removal of Dominated Strategies: Another Example

| | L | C | R |
|---|---|---|---|
| U | $3, 1$ | $0, 1$ | $0, 0$ |
| M | $1, 1$ | $1, 1$ | $5, 0$ |
| D | $0, 1$ | $4, 1$ | $0, 0$ |

- $R$ is dominated by $L$ or $C$.

# Iterated Removal of Dominated Strategies: Another Example

| | L | C |
|---|---|---|
| U | $3, 1$ | $0, 1$ |
| M | $1, 1$ | $1, 1$ |
| D | $0, 1$ | $4, 1$ |

- $M$ is dominated by the mixed strategy that selects $U$ and $D$ with equal probability.

# Iterated Removal of Dominated Strategies: Another Example

| | L | C |
|---|---|---|
| U | 3, 1 | 0, 1 |
| D | 0, 1 | 4, 1 |

- No other strategies are dominated.

# Iterated Removal of Dominated Strategies

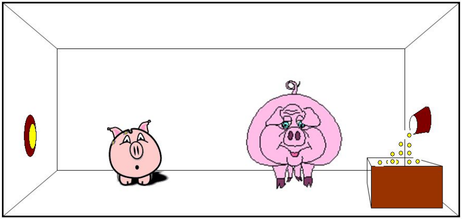
- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique: dominance solvable
  - Example: Traveler's Dilemma!
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

Section 6

## Are pigs rational?

# Feeding Behaviour among Pigs and Iterative Strict Dominance

- Experiment by B. A. Baldwin and G. B. Meese (1979) “Social Behaviour in Pigs Studied by Means of Operant Conditioning”, Animal Behaviour, Vol 27, pp 947-957.
- Two pigs in a cage, one is larger.
- need to press a button to get food to arrive
- food and button are at opposite sides of cage
- run to press and the other pig gets the food...



# Feeding Behaviour among Pigs and Iterative Strict Dominance

- 10 units of food – the typical split
  - if large gets to the food first then 1,9 split (1 for small, 9 for large),
  - if small gets to food first then 4,6 split,
  - if get to food at the same time then 3,7 split,
  - Pressing the button costs 2 units of food in energy.

| S\L | Press | Wait |
|---|---|---|
| Press | $1, 5$ | $-1, 9$ |
| Wait | $4, 4$ | $0, 0$ |

- What happens if we analyse the game through the iterative elimination of strictly dominated strategies?

# Pigs Behaviour: Frequency of pushing the button per 15 min, after 10 tests (learning...)

| | Alone | Together |
|---|---|---|
| Large | 75 | 105 |
| Small | 70 | 5 |

# Feeding Behaviour among Pigs and Iterative Strict Dominance

- Are pigs rational? Do they know Game Theory?
- They do seem to learn and respond to incentives
- Learn not to play strictly dominated strategy...
- Learn to not play strictly dominated strategies out of what remains
- Learning, evolution, and survival of the fittest: powerful game theory tools.

Section 7

## Maxmin and Minmax

# Maxmin Strategies

- Player $i$'s maxmin strategy is a strategy that maximizes $i$'s worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to $i$.
- The maxmin value (or safety level) of the game for player $i$ is that minimum amount of payoff guaranteed by a maxmin strategy.

## Definition (Maxmin)

The maxmin strategy for player $i$ is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and the maxmin value for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

- Why would $i$ want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

# Minmax Strategies

- Player $i$'s minmax strategy against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the minmax value for $i$ against $-i$ is payoff.
- Why would $i$ want to play a minmax strategy?
  - to punish the other agent as much as possible

## Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player $i$ against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

We can generalize to $n$ players.

## Definition (Minmax, $n$-player)

In an $n$-player game, the minmax strategy for player $i$ against player $j \neq i$ is $i$'s component of the mixed strategy profile $s_{-j}$ in the expression $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$, where $-j$ denotes the set of players other than $j$. As before, the minmax value for player $j$ is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$.

# Minmax Theorem

**Theorem (Minimax theorem (von Neumann, 1928))**

*In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.*

1. Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

Section 8

# Correlated Equilibrium

# Examples

- Consider again Battle of the Sexes.

| | $B$ | $F$ |
|---|---|---|
| $B$ | $2,1$ | $0,0$ |
| $F$ | $0,0$ | $1,2$ |

  - Intuitively, the best outcome seems a 50-50 split between $(F,F)$ and $(B,B)$.
  - But there's no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate

- Another classic example: traffic game

| | *go* | *wait* |
|---|---|---|
| *go* | $-10,-10$ | $1,0$ |
| *wait* | $0,1$ | $-1,-1$ |

# Intuition

- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium
- We could use the same idea to achieve the fair outcome in Battle of the Sexes.
- Correlated Equilibrium (informal): a randomised assignment of (potentially correlated) action recommendations to agents, such that nobody wants to deviate.

# Formal definition

## Definition (Correlated equilibrium)

Given an $n$-agent game $G = (N, A, u)$, a correlated equilibrium is a tuple $(v, \pi, \sigma)$, where $v$ is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, $\pi$ is a joint distribution over $v$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent $i$ and every mapping $\sigma'_i : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right) \geq \sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma'_i(d_i), \ldots, \sigma_n(d_n)\right).$$

# Existence

## Theorem

*For every Nash equilibrium $\sigma^*$ there exists a corresponding correlated equilibrium $\sigma$.*

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma_i^*(d_i)$
  - $\sigma_i$ maps each $d_i$ to the corresponding $a_i$.
- Thus, correlated equilibria always exist

# Additional Solution Concepts

There are more solution concepts defined in the literature. Examples:

- Trembling-hand perfect equilibrium: strategy profile that is the limit of an infinite sequence of fully-mixed-strategy profiles in which each player best-responds to the previous profile.
  - So: even if they make small mistakes, I'm responding rationally.
- $\epsilon$-Nash equilibrium: no player can gain more than $\epsilon$ in utility by unilaterally deviating from her assigned strategy.
  - How does the standard definition of NE relate to this?