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Characterizations of one-way general quantum finite automata[☆]

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ABSTRACT

Generally, unitary transformations limit the computational power of *quantum finite automata* (QFA). In this paper, we study a generalized model named *one-way general quantum finite automata* (1gQFA), in which each symbol in the input alphabet induces a trace-preserving quantum operation, instead of a unitary transformation. Two different kinds of 1gQFA will be studied: *measure-once one-way general quantum finite automata* (MO-1gQFA) where a measurement deciding to accept or reject is performed at the end of a computation, and *measure-many one-way general quantum finite automata* (MM-1gQFA) where a similar measurement is performed after each trace-preserving quantum operation on reading each input symbol.

We characterize the measure-once model from three aspects: the closure property, the language recognition power, and the equivalence problem. We prove that MO-1gQFA recognize, with bounded error, precisely the set of all regular languages. Our results imply that some models of quantum finite automata proposed in the literature, which were expected to be more powerful, still cannot recognize non-regular languages.

We prove that MM-1gQFA also recognize only regular languages with bounded error. Thus, MM-1gQFA and MO-1gQFA have the same language recognition power, in sharp contrast with traditional MO-1QFA and MM-1QFA, the former being strictly less powerful than the latter. Finally, we present a necessary and sufficient condition for two MM-1gQFA to be equivalent.

1. Introduction

Since Shor's quantum algorithm for factoring integers in polynomial time [42] and Grover's algorithm for searching in database of size n with only $O(\sqrt{n})$ accesses [13], quantum computation and information have attracted more and more attention in the community. As we know, these algorithms are implemented on *quantum Turing machines* which seem complicated to realize using current technology. Therefore, since building full quantum computers is still a long term goal, it is important to study "small-size" quantum processors (such as *quantum finite automata*).

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Indeed, quantum finite automata (QFA), as a theoretical model for quantum computers with finite memory, have interested many researchers (see, e.g., [1–9,11,12,14–16,19,20,22–24,26–31,33–39,44,45]). From a theoretical point of view, exploring QFA may provide meaningful insights into the power of quantum computation.

So far, several models of QFA have been studied. These models differ from each other mainly by two factors: the moving direction of the tape head and the measurement policy. Roughly speaking, we have two kinds of QFA: *one-way* QFA (1QFA) where the tape heads are allowed only to move right at each step and *two-way* QFA (2QFA) where the tape heads are allowed to move left or right, and even to stay stationary.

The model of 2QFA was firstly studied by Kondacs and Watrous [20]. It was proved that 2QFA not only recognize[1] all regular languages, but also some non-regular languages (such as $L_{eq} = \{a^n b^n | n > 0\}$) in linear time [20]. It is worth pointing out that any two-way probabilistic automaton needs average exponential time to recognize a non-regular language [21]. However, it seems to be difficult to implement a 2QFA, since the size of a 2QFA's quantum part depends on the length of the input.

Compared with 2QFA, 1QFA seem simpler to design and implement. In general, theoretical problems regarding 1QFA are easier to investigate than those regarding 2QFA. Thus, 1QFA may be an appropriate starting point for the study on quantum computing models. Two important types of 1QFA are *measure-once* 1QFA (MO-1QFA) proposed by Moore and Crutchfield [27] where a measurement is performed at the end of a computation, and *measure-many* 1QFA (MM-1QFA) defined by Kondacs and Watrous [20] where a measurement is performed at each step during a computation. It has been proved that both MO-1QFA and MM-1QFA can recognize only a proper subset of regular languages. More exactly, MO-1QFA recognize only group languages [7,11], while MM-1QFA recognize more languages than MO-1QFA but cannot recognize all regular languages [20].

Obviously, both MO-1QFA and MM-1QFA have a very limited computational power. Thus, some generalizations or modifications were made to the definition of 1QFA, with the aim of enhancing computational power. A usual modification to the definition of 1QFA is to allow an arbitrary projective measurement as an intermediate step in the computation.

For instance, Ambainis et al. [1] studied the so-called *Latvian* QFA (LQFA) in which the operation corresponding to an input symbol is the combination of a unitary transformation and a projective measurement. LQFA can be regraded as a generalized version of MO-1QFA. In fact, LQFA are closely related with the classical model PRA-C (*probabilistic reversible automata with classical acceptance*) [15], and from the results in [1] it follows that the two models recognize the same class of languages, i.e., the languages whose syntactic monoid is a block group [1]. More concretely, a language is recognized by LQFA if and only if it is a Boolean combination of languages of the form $L_0 a_1 L_1 \dots a_k L_k$ where the a_i's are letters and the L_i's are group languages. Hence, LQFA can recognize only a proper subset of regular languages; for example, LQFA cannot recognize regular languages $\Sigma^* a$ and $a\Sigma^*$ [1]. In fact, the class of languages recognized by LQFA is a proper subset of the languages recognized by MM-1QFA.

The measure-many version of LQFA was defined by Nayak [26], and we call this model GQFA in this paper. It can be seen that GQFA allow more general operations than models mentioned before. From the results about LQFA stated above, it follows that GQFA are strictly more powerful than LQFA. However, GQFA still cannot recognize all regular languages; for example, GQFA cannot recognize language $\{a, b\}^* a$ [26]. Also, it is still not known whether GQFA can recognize strictly more languages than MM-1QFA. An interesting result about GQFA is that there exist languages for which GQFA take exponentially more states than equivalent classical automata [5,26].

Bertoni et al. [8] defined a model called *one-way quantum finite automata with control language* (CL-1QFA). The accepting behavior of this model is controlled by the sequence of the measurement results obtained along the computation. If the result sequence is in a given language, then the input is accepted. In [28], it was proved that CL-1QFA recognize exactly regular languages with bounded error. Recently, Qiu et al. [39] studied *one-way quantum finite automata together with classical states* and showed that this model can also recognize any regular language with no error.

Besides the 1QFA mentioned above, there are some other models of 1QFA which go further in the direction of modifying the original definition of 1QFA. For instance, Ciamarra [12] thought that the reason for the computational power of 1QFA being weaker than that of their classical counterparts is that the definition of 1QFA neglects the concept of quantum reversibility. Thus, following the idea in [10], Ciamarra [12] proposed a new model of 1QFA that was believed to be strictly reversible, and whose computational power was proved to be at least equal to that of (one-way) classical automata. Paschen [31] introduced another model of 1QFA named *ancilla QFA*, where an ancilla quantum part is imported, and then the internal control states and the states of the ancilla part together evolve by a unitary transformation. Paschen [31] showed that ancilla QFA can recognize any regular language with certainty.

These two latter models of QFA can recognize at least regular languages. In a certain sense, the increased computational power comes from the generalization of the operations allowed by the models. In fact, the two models of QFA defined in [12] and in [31] share a common point that the state evolution of the internal state controller together with an auxiliary quantum part complies with a unitary transformation, and thus the evolution of the internal control states is generally not unitary. Therefore, a natural question is: how much computational power can no-unitary operations bring to quantum finite automata?

We will address this question here. We study the generalized version of 1QFA, called *one-way general quantum finite automata* (1gQFA), in which each symbol in the input alphabet induces a trace-preserving quantum operation instead

[1] Without additional explanation, in this paper, recognizing a language always means recognizing a language with bounded error.

of a unitary transformation. Two kinds of 1gQFA will be studied: *measure-once one-way general quantum finite automata* (MO-1gQFA) which can be seen as a generalized version of MO-1QFA, and *measure-many one-way general quantum finite automata* (MM-1gQFA), a generalized version of MM-1QFA.

We study MO-1gQFA from three aspects: the closure property, the language recognition power, and the equivalence problem. In fact, such a kind of QFA has already been considered by Hirvensalo [16], but with no further attention paid to this model. Hirvensalo [16] showed that MO-1gQFA can simulate MM-1QFA, LQFA and even probabilistic automata, and thus can recognize any regular language. In general, it is believed that the unitarity of evolution puts some limit on the computational power of QFA. Now, MO-1gQFA allow any physical admissible operation—the trace-preserving quantum operation. Then MO-1gQFA are expected to be more powerful. In particular, we investigate whether MO-1gQFA recognize some non-regular languages.

In this paper, we will prove that despite the most general operations allowed, MO-1gQFA can recognize only regular languages with bounded error. Moreover, the two types of QFA defined in [12] and in [31] are shown to be special cases of MO-1gQFA, and thus recognize only regular languages. Another problem worthwhile to be pursued is the equivalence between MO-1gQFA. We will give a necessary and sufficient condition for two MO-1gQFA to be equivalent. Also, we will present some closure properties of MO-1gQFA.

We study MM-1gQFA from two aspects: the language recognition power and the equivalence problem. Generally, the number of times the measurement is performed in the computation affects the computational power of 1QFA. For instance, MM-1QFA can recognize more languages than MO-1QFA, and GQFA also recognize more languages than LQFA. Therefore, it is expected that MM-1gQFA are more powerful than MO-1gQFA. However, we will prove that MM-1gQFA also recognize only regular languages with bounded error. Thus, MM-1gQFA and MO-1gQFA have the same computational power. This reveals an essential difference between 1QFA and their generalized versions. Finally, we discuss the equivalence problem of MM-1gQFA. Specifically, we give a sufficient and necessary condition to determine whether two MM-1gQFA are equivalent or not. This also offers a different solution to the equivalence problem of MM-1QFA discussed in [23].

It is worth pointing out that all the above discussions regarding MM-1gQFA are based on the following result proved by us: an MM-1gQFA can be simulated by a relaxed version of MO-1gQFA whose operation corresponding to the input symbol is a general linear super-operator, not necessarily a trace-preserving quantum operation.

2. Preliminaries

2.1. Notations and quantum operations

Some notations used in this paper are explained here. $|S|$ denotes the cardinality of set S. For non-empty set Σ, by Σ^* we mean the set of all strings over Σ with finite length. $|w|$ denotes the length of string w. Symbols $*$, $\dagger$, and $\top$ denote the conjugate operation, the conjugate-transpose operation, and the transpose operation, respectively. $Tr(A)$ denotes the trace of matrix (operator) A. $supp(A)$ denotes the support of operator A. For a positive operator A, $supp(A)$ is the space spanned by the eigenvectors of A corresponding to the non-zero eigenvalues. $\dim V$ denotes the dimension of finite-dimensional space V. Generally, we use $\mathcal{H}$ to denote a finite-dimensional Hilbert space. Let $L(\mathcal{H})$ denote the set of all linear operators from $\mathcal{H}$ to itself. A mapping $\Phi: L(\mathcal{H}) \to L(\mathcal{H})$ is called a super-operator on $\mathcal{H}$.

Let $\mathbb{C}$ and $\mathbb{R}$ denote the sets of complex numbers and real numbers, respectively. Let $\mathbb{C}^{n\times m}$ denote the set of all $n \times m$ complex matrices. For two matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$, their direct sum is defined as

$$A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix},$$

and their tensor product is

$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \dots & A_{nm}B \end{bmatrix}.$$

For vectors $v = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ and $u = (y_1, y_2, \dots, y_m) \in \mathbb{C}^m$, their direct sum is $v \oplus u = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$.

The detailed background on quantum information can be referred to [25], and here we just introduce some notions. According to the postulates of quantum mechanics, the state of a closed quantum system is represented by a unit vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, and the state evolution of a closed quantum system is described by a unitary transformation on $\mathcal{H}$. A more general tool to describe the state of a quantum system is the density operator. A density operator ρ on Hilbert space $\mathcal{H}$ is a linear operator satisfying the following conditions:

(1) (Trace condition) ρ has trace equal to 1, that is, $Tr(\rho) = 1$.
(2) (Positivity condition) $\rho \geq 0$, that is, for any $|\psi\rangle \in \mathcal{H}$, $\langle\psi|\rho|\psi\rangle \geq 0$.

By $D(\mathcal{H})$ we mean the set of all density operators on Hilbert space $\mathcal{H}$.

In practice, an absolutely closed system does not exist, because a system interacts more or less with its outer environment, and thus it is open. Then the state evolution of an open quantum system is characterized by a *quantum operation* [25]. A quantum operation, denoted by $\mathcal{E}$, has an *operator-sum representation* as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^{\dagger}, \tag{1}$$

where ρ is a density operator on the input space $\mathcal{H}_{in}$, $\mathcal{E}(\rho)$ is a linear operator on the output space $\mathcal{H}_{out}$, and the set of $\{E_k\}$ known as *operation elements* are linear operators from $\mathcal{H}_{in}$ to $\mathcal{H}_{out}$. Furthermore, $\mathcal{E}$ is said to be trace-preserving if the following holds:

$$\sum_k E_k^{\dagger} E_k = I,$$

where I is the identity operator on $\mathcal{H}_{in}$.

Any physical admissible operation is a trace-preserving quantum operation (also called a completely positive trace-preserving mapping), which has another representation—*Stinespring representation*:

$$\mathcal{E}(\rho) = Tr_{\mathcal{H}_a}(V \rho V^{\dagger}), \tag{2}$$

where V is a linear isometry operator from $\mathcal{H}_{in}$ to $\mathcal{H}_{out} \otimes \mathcal{H}_a$, and $Tr_{\mathcal{H}_a}$ is the operation of partial trace that discards the subsystem a.

When the input space and the output space of quantum operation $\mathcal{E}$ are the same, say $\mathcal{H}$, we say $\mathcal{E}$ is a quantum operation acting on $\mathcal{H}$. In fact, the quantum operations used in the subsequent sections are all in this case.

2.2. A brief review on MO-1QFA and MM-1QFA

In this paper, we are interested in quantum finite automata with a one-way tape head. Two important models of 1QFA are MO-1QFA firstly defined by Moore and Crutchfield [27] and MM-1QFA proposed by Kondacs and Watrous [20].

An MO-1QFA is defined as a quintuple $\mathcal{A} = (Q, \Sigma, |\psi_0\rangle, \{U(\sigma)\}_{\sigma\in\Sigma}, Q_{acc})$, where Q is a set of finite states, $|\psi_0\rangle$ is the initial state that is a superposition of the states in Q, Σ is a finite input alphabet, $U(\sigma)$ is a unitary transformation for each $\sigma \in \Sigma$, and $Q_{acc} \subseteq Q$ is the set of accepting states. The computing process of MO-1QFA $\mathcal{A}$ on input string $x = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^*$ is as follows: the unitary transformations $U(\sigma_1), U(\sigma_2), \ldots, U(\sigma_n)$ are performed in succession on the initial state $|\psi_0\rangle$, and finally a measurement is performed on the final state, deciding to accept the input or not. The languages recognized by MO-1QFA with bounded error are group languages [11], a proper subset of regular languages.

An MM-1QFA is defined as a 6-tuple $\mathcal{M} = (Q, \Sigma, |\psi_0\rangle, \{U(\sigma)\}_{\sigma\in\Sigma\cup\{¢,\$\}}, Q_{acc}, Q_{rej})$, where $Q, Q_{acc} \subseteq Q, |\psi_0\rangle, \Sigma, \{U(\sigma)\}$ are the same as those in the MO-1QFA defined above, $Q_{rej} \subseteq Q$ represents the set of rejecting states, and ¢, $\$ \notin \Sigma$ are respectively the left end-marker and the right end-marker. For any input string ¢x\$ with $x \in \Sigma^*$, the computing process is similar to that of MO-1QFA except that after every transition, $\mathcal{M}$ measures its state with respect to the three subspaces that are spanned by the three subsets Q_{acc}, Q_{rej}, and Q_{non}, respectively, where $Q_{non} = Q \setminus (Q_{acc} \cup Q_{rej})$. The languages recognized by MM-1QFA with bounded error are more than those recognized by MO-1QFA, but still a proper subset of regular languages.

From the study on MO-1QFA and MM-1QFA, we make two observations: (i) the number of times the measurement is performed in the computation affects the computational power of 1QFA; (ii) by considering just unitary transformations one limits the computation power of 1QFA such that the two typical models of 1QFA (MO-1QFA and MM-1QFA) are less powerful than their classical counterparts.

Inspired by these observations, in this paper we are going to study the generalized versions of MO-1QFA and MM-1QFA, in which the most general operations—trace-preserving quantum operations—are allowed upon reading each input symbol. By studying these models, we hope to address the following question: what are the limitations imposed by unitary transformations on the computation power of 1QFA? Or, in other words, what extra computational power can non-unitary transformations bring to 1QFA?

3. Measure-once one-way general quantum finite automata (MO-1gQFA)

In this section, we consider the model of MO-1gQFA which has a one-way tape head, and in which each symbol in the input alphabet induces a trace-preserving quantum operation. In the subsequent sections, after giving the definition of MO-1gQFA, we will discuss the closure property, the language recognition power, and the equivalence problem for MO-1gQFA.

3.1. Closure properties of MO-1gQFA

First, we give the definition of MO-1gQFA as follows.

Definition 1. An MO-1gQFA $\mathcal{M}$ is a five-tuple $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$, where $\mathcal{H}$ is a finite-dimensional Hilbert space, Σ is a finite input alphabet, ρ_0, the initial state of $\mathcal{M}$, is a density operator on $\mathcal{H}$, $\mathcal{E}_\sigma$ corresponding to $\sigma \in \Sigma$ is a trace-preserving quantum operation acting on $\mathcal{H}$, P_{acc} is a projector on the subspace called accepting subspace of $\mathcal{H}$. Denote $P_{rej} = I - P_{acc}$, then $\{P_{acc}, P_{rej}\}$ form a projective measurement on $\mathcal{H}$.

On input word $\sigma_1\sigma_2 \dots \sigma_n \in \Sigma^*$, the above MO-1gQFA $\mathcal{M}$ proceeds as follows: the quantum operations $\mathcal{E}_{\sigma_1}, \mathcal{E}_{\sigma_2}, \dots, \mathcal{E}_{\sigma_n}$ are performed on ρ_0 in succession, and then the projective measurement $\{P_{acc}, P_{rej}\}$ is performed on the final state, obtaining the accepting result with a certain probability. Thus, MO-1gQFA $\mathcal{M}$ defined above induces a function $f_{\mathcal{M}} : \Sigma^* \to [0, 1]$ as

$$f_{\mathcal{M}}(\sigma_1\sigma_2 \dots \sigma_n) = Tr(P_{acc}\mathcal{E}_{\sigma_n} \circ \dots \circ \mathcal{E}_{\sigma_2} \circ \mathcal{E}_{\sigma_1}(\rho_0)),$$

where $\circ$ denotes the composition. In fact, for every $x \in \Sigma^*$, $f_{\mathcal{M}}(x)$ represents the probability that $\mathcal{M}$ accepts x.

In the following, we present some closure properties of MO-1gQFA.

Theorem 1. *The class of MO-1gQFA are closed under the following operations:*

(i) *If f is a function induced by an MO-1gQFA, then $1 - f$ is also induced by an MO-1gQFA.*

(ii) *If $f_1, f_2, \dots, f_k$ are functions induced by MO-1gQFA, then $\sum_i^k c_i f_i$ is also induced by an MO-1gQFA for any real constants $c_i > 0$ such that $\sum_i^k c_i = 1$.*

(iii) *If $f_1, f_2, \dots, f_k$ are functions induced by MO-1gQFA, then $\prod_{i=1}^k f_i$ is also induced by an MO-1gQFA.*

Proof. (a) If f is induced by an MO-1gQFA $\mathcal{M}$ with projector P_{acc}, then $1-f$ can be induced by MO-1gQFA $\mathcal{M}'$ that is almost the same as $\mathcal{M}$, but with $P'_{acc} = I - P_{acc}$.

(b) We prove item (ii) for $k = 2$ in detail. Assume that f_i is induced by MO-1gQFA $\mathcal{M}_i = \{\mathcal{H}_i, \Sigma, \rho_0^{(i)}, \{\mathcal{E}_\sigma^{(i)}\}_{\sigma\in\Sigma}, P_{acc}^{(i)}\}$ with $i = 1, 2$, respectively. Then for c_1, c_2 satisfying $c_1 > 0$, $c_2 > 0$ and $c_1 + c_2 = 1$, we construct $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and $\rho_0 = c_1\rho_0^{(1)} \oplus c_2\rho_0^{(2)}$. ρ_0 is obviously a density operator on $\mathcal{H}$. Moreover, for every $\sigma \in \Sigma$, we construct $\mathcal{E}_\sigma = \mathcal{E}_\sigma^{(1)} \oplus \mathcal{E}_\sigma^{(2)}$; more specifically, if $\mathcal{E}_\sigma^{(1)}$ and $\mathcal{E}_\sigma^{(2)}$ have operator element sets $\{E_i\}_{i\in N}$ and $\{F_j\}_{j\in M}$, respectively, then $\mathcal{E}_\sigma$ is constructed such that it has operator element set $\{\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\}_{i\in N, j\in M}$. Then we have

$$\begin{aligned}\sum_{i\in N, j\in M}\left(\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\right)^\dagger\left(\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\right) &= \sum_{i\in N, j\in M}\frac{1}{M}E_i^\dagger E_i \oplus \frac{1}{N}F_j^\dagger F_j \\ &= \sum_{i\in N}E_i^\dagger E_i \oplus \sum_{j\in M}F_j^\dagger F_j \\ &= I_{\mathcal{H}_1} \oplus I_{\mathcal{H}_2}.\end{aligned}$$

Hence, for any $\sigma \in \Sigma$, $\mathcal{E}_\sigma$ constructed above is a trace-preserving quantum operation acting on $\mathcal{H}$. Therefore, by letting $P_{acc} = P_{acc}^{(1)} \oplus P_{acc}^{(2)}$, we get an MO-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$.

Furthermore, for any $\rho = \rho_1 \oplus \rho_2 \in D(\mathcal{H})$ where ρ_1, ρ_2 are density operators up to some coefficients, we have

$$\begin{aligned}\mathcal{E}_\sigma(\rho) &= \sum_{i\in N, j\in M}\left(\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\right)(\rho_1 \oplus \rho_2)\left(\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\right)^\dagger && (3)\\ &= \sum_{i\in N, j\in M}\frac{1}{M}E_i\rho_1E_i^\dagger \oplus \frac{1}{N}F_j\rho_2F_j^\dagger \\ &= \sum_{i\in N}E_i\rho_1E_i^\dagger \oplus \sum_{j\in M}F_j\rho_2F_j^\dagger \\ &= \mathcal{E}_\sigma^{(1)}(\rho_1) \oplus \mathcal{E}_\sigma^{(2)}(\rho_2) && (4)\\ &\in D(\mathcal{H}).\end{aligned}$$

Then it is not difficult to see that for any $x \in \Sigma^*$, we have

$$f_{\mathcal{M}}(x) = c_1f_1(x) + c_2f_2(x).$$

Thus, we have proved item (ii) for $k = 2$. It is easy to generalize this proof for the general case $k > 2$.

(c) Similarly, we prove item (iii) for $k = 2$. Assume that f_i is induced by MO-1gQFA $\mathcal{M}_i = \{H_i, \Sigma, \rho_0^{(i)}, \{\mathcal{E}_\sigma^{(i)}\}_{\sigma\in\Sigma}, P_{acc}^{(i)}\}$ for $i = 1, 2$, respectively. Then we construct $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$ as follows:

- $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$;
- $\rho_0 = \rho_0^{(1)} \otimes \rho_0^{(2)}$;
- $\mathcal{E}_\sigma = \mathcal{E}_\sigma^{(1)} \otimes \mathcal{E}_\sigma^{(2)}$; more specifically, if $\mathcal{E}_\sigma^{(1)}$ and $\mathcal{E}_\sigma^{(2)}$ have operator element sets $\{E_i\}_{i\in N}$ and $\{F_j\}_{j\in M}$, then $\mathcal{E}_\sigma$ is constructed such that it has operator element set $\{E_i \otimes F_j\}_{i\in N, j\in M}$;
- $P_{acc} = P_{acc}^{(1)} \otimes P_{acc}^{(2)}$.

Then, it is easy to see that for any $\sigma \in \Sigma$ and $\rho = \rho_1 \otimes \rho_2 \in D(\mathcal{H})$, we have

$$\mathcal{E}_\sigma(\rho) = \mathcal{E}_\sigma^{(1)}(\rho_1) \otimes \mathcal{E}_\sigma^{(2)}(\rho_2).$$

Furthermore, for any $x \in \Sigma^*$, we have $f_{\mathcal{M}}(x) = f_1(x)f_2(x)$. Thus, we have proved item (iii) for $k = 2$, and it is easy to generalize this proof for the general case $k > 2$.

Therefore, we have completed the proof for Theorem 1. □

Remark 1. The techniques used in Theorem 1 are generalizations of techniques previously introduced and used in the literature [9,27,29,30]

3.2. The computational power of MO-1gQFA

In this subsection, we investigate the computational power of MO-1gQFA. In Ref. [16], Hirvensalo showed that MO-1gQFA can simulate any probabilistic automaton. Thus, the class of languages recognized by MO-1gQFA with bounded error contains the set of all regular languages. Furthermore, with the most general operations allowed, MO-1gQFA are expected to be more powerful. However, we will prove that the languages recognized by MO-1gQFA with bounded error are exactly regular languages.

In the following, we first give the formal definition of an MO-1gQFA recognizing a language with bounded error.

Definition 2. A language L is said to be recognized by MO-1gQFA $\mathcal{M}$ with bounded ϵ ($\epsilon > 0$), if for some $\lambda \in (0, 1]$, $f_{\mathcal{M}}(x) \geq \lambda + \epsilon$ holds for any $x \in L$, and $f_{\mathcal{M}}(y) \leq \lambda - \epsilon$ holds for any $y \notin L$.

Before we start to prove the regularity of languages recognized by MO-1gQFA, we first recall some useful concepts and related results in [25]. The trace distance between density operators ρ and σ is[2]

$$D(\rho, \sigma) = ||\rho - \sigma||_{tr}$$

where $||A||_{tr} = Tr\sqrt{A^\dagger A}$ is the trace norm of operator A. (For a positive operator $A = \sum_i \lambda_i |i\rangle\langle i|$, its square is $\sqrt{A} = \sum_i \sqrt{\lambda_i}|i\rangle\langle i|$.) The trace distance between two probability distributions $\{p_x\}$ and $\{q_x\}$ is

$$D(p_x, q_x) = \sum_x |p_x - q_x|.$$

In the following, we recall two results regarding the trace distance that will be used later on.

Lemma 2 (*[25]*). *Let ρ and σ be two density operators. Then we have*

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma)$$

for any trace-preserving quantum operation $\mathcal{E}$.

Lemma 3 (*[25]*). *Let ρ and σ be two density operators. Then we have*

$$D(\rho, \sigma) = \max_{\{E_m\}} D(p_m, q_m)$$

where $p_m = Tr(\rho E_m)$, $q_m = Tr(\sigma E_m)$ and the maximization is over all POVMs $\{E_m\}$.

For the sake of readability, we also recall the Myhill–Nerode theorem [18]:

Theorem 4 (*Myhill–Nerode Theorem [18]*). *The following three statements are equivalent:*

1. *The set $L \subseteq \Sigma^*$ is accepted by some finite automata.*
2. *L is the union of some equivalence classes of a right invariant equivalence relation of finite index.*
3. *Let equivalence relation R_L be defined by: xR_Ly if and only if for all $z \in \Sigma^*$, xz is in L exactly when yz is in L. Then R_L is of finite index.*

Now, we present the main result of this subsection.

Theorem 5. *The languages recognized by MO-1gQFA with bounded error are regular.*

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[2] The original definition of trace distance between operators is $D(\rho, \sigma) = \frac{1}{2}||\rho - \sigma||_{tr}$, and here we ignore the factor $\frac{1}{2}$ for our purposes. Later, we do similar on the definition of trace distance between two probability distributions.

Proof. Assume that L is recognized by MO-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$ with bounded error ϵ. We define an equivalence relation "$\equiv_L$" on $x, y \in \Sigma^*$ such that $x \equiv_L y$ if for any $z \in \Sigma^*$, $xz \in L$ iff $yz \in L$. Then in terms of Theorem 4, if we prove that the number of equivalence classes induced by "$\equiv_L$" is finite, then L is regular.

Let $S = \{A : \|A\|_{tr} \le 1$, and A is a linear operator on $\mathcal{H}\}$. Then S is a bounded subset from a finite-dimensional space. Let $\rho_x = \mathcal{E}_{x_n} \circ \cdots \circ \mathcal{E}_{x_2} \circ \mathcal{E}_{x_1}(\rho_0)$, i.e., the state of $\mathcal{M}$ after having been fed with word x. Then for every $x \in \Sigma^*$, it can be seen that $\rho_x \in S$, since we have $||\rho_x||_{tr} = Tr(\rho_x) = Tr(\rho_0) = 1$, where the second equality holds, because every operation used is trace-preserving. Now, suppose that $x \not\equiv_L y$, that is, there exists a string $z \in \Sigma^*$ such that $xz \in L$ and $yz \notin L$. Then we have

$$Tr(P_{acc}\mathcal{E}_z(\rho_x)) \ge \lambda + \epsilon \quad \text{and} \quad Tr(P_{acc}\mathcal{E}_z(\rho_y)) \le \lambda - \epsilon$$

for some $\lambda \in (0, 1]$, where $\mathcal{E}_z$ stands for $\mathcal{E}_{z_n} \circ \cdots \circ \mathcal{E}_{z_2} \circ \mathcal{E}_{z_1}$. Denote $p_{acc} = Tr(P_{acc}\mathcal{E}_z(\rho_x))$, $p_{rej} = Tr(P_{rej}\mathcal{E}_z(\rho_x))$, $q_{acc} = Tr(P_{acc}\mathcal{E}_z(\rho_y))$, and $q_{rej} = Tr(P_{rej}\mathcal{E}_z(\rho_y))$. Then by Lemma 3, we have

$$||\mathcal{E}_z(\rho_x) - \mathcal{E}_z(\rho_y)||_{tr} \ge |p_{acc} - q_{acc}| + |p_{rej} - q_{rej}| \ge 2\epsilon.$$

On the other hand, by Lemma 2, we have

$$||\rho_x - \rho_y||_{tr} \ge ||\mathcal{E}_z(\rho_x) - \mathcal{E}_z(\rho_y)||_{tr}.$$

Consequently, for any two strings $x, y \in \Sigma^*$ satisfying $x \not\equiv_L y$, we always have

$$||\rho_x - \rho_y||_{tr} \ge 2\epsilon. \tag{5}$$

Now, suppose that $\equiv_L$ has infinite equivalence classes, say $[x^{(1)}], [x^{(2)}], [x^{(3)}], \ldots$. Then by the boundedness of S from a finite-dimensional space, from the sequence $\{\rho_{x^{(n)}}\}_{n\in N}$, we can extract a Cauchy sequence $\{\rho_{x^{(n_k)}}, \}_{k\in N}$, i.e., a convergent subsequence. Thus, there exist x and y satisfying $x \not\equiv_L y$ such that

$$||\rho_x - \rho_y||_{tr} < 2\epsilon,$$

which contradicts Inequality (5). Therefore, the number of the equivalence classes in Σ^* induced by the equivalence relation "$\equiv_L$" must be finite, which implies that L is a regular language. □

Remark 2. The idea of the above proof is essentially the same as the one in Rabin's seminal paper [41] where it was proved that probabilistic automata recognize only regular languages with bounded error. However, some technical treatment is required to adjust it to the case of MO-1gQFA. We also note that from the standpoint of topological space, Jeandel [19] offered some more general and abstract conditions for the regularity of the languages recognized by an automaton.

Remark 3. LQFA introduced by Ambainis et al. [1] are special cases of MO-1gQFA. Thus, from Theorem 5 it follows straightforwardly that the languages recognized by LQFA with bounded error are in the class of regular languages. Note that Ambainis et al. [1] characterized the languages recognized by LQFA using an algebraic approach.

The model of MO-1gQFA can also simulate classical automata—DFA and even probabilistic automata. Therefore, we have the following result which is owed to Hirvensalo [16].

Theorem 6. *MO-1gQFA recognize all regular languages with certainty.*

Proof. The proof is to simulate any probabilistic automaton by an MO-1gQFA. Indeed, Hirvensalo [16] has already presented such a simulating process. For the sake of readability, we reproduce the simulating process in more detail here.

First recall that an n-state probabilistic automaton $\mathcal{A}$ can be represented as

$$\mathcal{A} = (\pi, \Sigma, \{A(\sigma) : \sigma \in \Sigma\}, \eta),$$

where π is a stochastic n-dimensional row vector, η is an n-dimensional column vector whose entries are 0s or 1s, and for each $\sigma \in \Sigma$, $A(\sigma) = [A(\sigma)_{ij}]$ is a stochastic $n \times n$ matrix (i.e., each row of it is a stochastic vector), where $A(\sigma)_{ij}$ is the probability of $\mathcal{A}$ going from the state q_i to the state q_j upon reading σ. The probability of accepting a string $x_1x_2 \ldots x_m \in \Sigma^*$ is defined as

$$P_{\mathcal{A}}(x_1x_2 \ldots x_m) = \pi A(x_1)A(x_2) \ldots A(x_m)\eta.$$

Now to simulate the above probabilistic automaton $\mathcal{A}$, we construct an MO-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$ such that $\mathcal{H} = span\{|q_1\rangle, |q_2\rangle, \ldots, |q_n\rangle\}$, $\rho_0 = \sum_i \pi_i|q_i\rangle\langle q_i|$, and $P_{acc} = \sum_{i:\eta_i=1} |q_i\rangle\langle q_i|$. For each stochastic matrix $A(\sigma)$, define a set of operators $\{E_{ij} = \sqrt{A(\sigma)_{ij}}|q_j\rangle\langle q_i| : i, j = 1, 2, \ldots, n\}$. Then a direct calculation shows that $\sum_{i,j=1}^n E_{ij}^\dagger E_{ij} = I$. Thus, a trace-preserving quantum operation is defined as

$$\mathcal{E}_\sigma(\rho) = \sum_{i,j}^{n} E_{ij}\rho E_{ij}^\dagger.$$

The action of $\mathcal{E}_\sigma$ on a pure state $|q_i\rangle\langle q_i|$ is as follows:

$$\mathcal{E}_\sigma(|q_i\rangle\langle q_i|) = \sum_{j=1}^{n} A(\sigma)_{ij}|q_j\rangle\langle q_j|,$$

which means that under the operation $\mathcal{E}_\sigma$, state $|q_i\rangle$ evolves into $|q_j\rangle$ with probability $A(\sigma)_{ij}$. This is consistent with the action of $A(\sigma)$ in probabilistic automata $\mathcal{A}$. Assume that after reading input x, the states of $\mathcal{A}$ and $\mathcal{M}$ are $\pi_x = (\pi_1, \pi_2, \ldots, \pi_n)$ and $\rho_x = \sum p_i|q_i\rangle\langle q_i|$, respectively. Then by induction on the length of x, it is easy to verify that $\pi_i = p_i$ holds for $i = 1, 2, \ldots, n$. Therefore, MO-1gQFA $\mathcal{M}$ and probabilistic automaton $\mathcal{A}$ defined above have the same accepting probability for each string $x \in \Sigma^*$.

From the above process, a DFA as a special probabilistic automaton can be simulated exactly by an MO-1gQFA, and thus, for every regular language, there is an MO-1gQFA recognizing it with certainty. □

Furthermore, we can show that the computational power of MO-1gQFA equals that of the QFA defined in [12] and in [31]. To see that, we first show that the two kinds of QFA defined in [12,31] are special cases of MO-1gQFA. We explain this point in detail for the model in [31].

Paschen [31] proposed a new QFA by adding some ancilla qubits to avoid the restriction of unitarity. This is done by adding an output alphabet. Formally, we have

Definition 3 (*[31]*). An ancilla QFA is a 6-tuple $\mathcal{M} = (Q, \Sigma, \Omega, \delta, q_0, F)$, where Q is a finite state set, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of accepting states, Ω is an output alphabet, and the transition function $\delta : Q \times \Sigma \times Q \times \Omega \longrightarrow \mathbb{C}$ satisfies

$$\sum_{p\in Q,\omega\in\Omega} \delta(q_1, \sigma, p, \omega)^*\delta(q_2, \sigma, p, \omega) = \begin{cases} 1, & q_1 = q_2 \\ 0, & q_1 \neq q_2 \end{cases}$$

for all states $q_1, q_2 \in Q$ and $\sigma \in \Sigma$.

The transition function δ corresponding to the input symbol $\sigma \in \Sigma$ can be described by an isometry mapping V_σ from Q to $Q \times \Omega$. Suppose the current state of $\mathcal{M}$ defined above is ρ. Then after reading σ, the state of $\mathcal{M}$ evolves to

$$\rho' = Tr_\Omega(V_\sigma \rho V_\sigma^\dagger).$$

Recalling the Stinespring representation (Eq. (2)) of quantum operations, it is easy to see that the state of $\mathcal{M}$ evolves by a trace-preserving quantum operation. Thus, an ancilla QFA is just a special MO-1gQFA.

Therefore, the language recognized by an ancilla QFA with bounded error is a regular language. On the other hand, in Ref. [31], it was proved that ancilla QFA can recognize any regular language with certainty. Hence, the languages recognized by ancilla QFA with bounded error are exactly regular languages.

Following the idea in [10], Ciamarra [12] proposed a new model of 1QFA whose computational power was shown to be at least equal to that of classical automata. For convenience, we call the QFA defined in [12] as *Ciamarra QFA* named after the author. Similar to the above process, it is not difficult to see that the internal state of a Ciamarra QFA evolves by a trace-preserving quantum operation, and thus, a Ciamarra QFA is also a special MO-1gQFA.

In summary, we have the following result.

Corollary 7. *The ancilla QFA in [31] and the Ciamarra QFA in [12] are both special cases of MO-1gQFA. Furthermore, the three kinds of QFA have the same computational power, and recognize exactly regular languages with bounded error.*

3.3. The equivalence problem of MO-1gQFA

In this subsection, we discuss the equivalence problem of MO-1gQFA. As we know, determining the equivalence between computing models is of importance in the theory of classical computation. For example, determining whether two DFA are equivalent is an important problem in the theory of classical automata [18], and determining whether two probabilistic automata are equivalent has also been deeply studied [32,43]. Similarly, the equivalence problem for quantum computing models is also worth studying, which may redounds to clarifying the essential difference between quantum and classical computing models. Indeed, there has already been some work done on the equivalence problem for quantum automata [22–24,37,38].

In the following, we first give the formal definition of the equivalence between two MO-1gQFA.

Definition 4. Two MO-1gQFA $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same input alphabet Σ are said to be equivalent (k-equivalent, resp.), if $f_{\mathcal{M}_1}(w) = f_{\mathcal{M}_2}(w)$ holds for any $w \in \Sigma^*$ (for any $w \in \Sigma^*$ with $|w| \leq k$, resp.).

Lemma 8. *For an MO-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$, denote*

$$\varphi(k) = span\{\rho_x : \rho_x = \mathcal{E}_x(\rho_0), |x| \leq k\}.$$

Then there exists an integer $i_0 \leq n^2$ such that $\varphi(i_0) = \varphi(i_0 + j)$ for $j = 1, 2, \ldots$, where $n = \dim \mathcal{H}$.

Proof. First, from the definition of $\varphi(k)$, it is readily seen that

$$1 \leq \dim\ \varphi(1) \leq \dim\ \varphi(2) \leq \cdots \leq \dim\ \varphi(i) \leq \cdots \leq n^2.$$

Thus, there exists an integer $i_0 \leq n^2$ such that $\varphi(i_0) = \varphi(i_0+1)$. Next we prove that $\varphi(i_0) = \varphi(i_0+j)$ holds for $j = 2, 3, \ldots$. Without loss of generality, we prove that $\varphi(i_0) = \varphi(i_0+2)$. Firstly, from the fact that $\varphi(i_0) = \varphi(i_0+1)$, for any $\rho \in \varphi(i_0+1)$, we have

$$\rho = \sum_i \alpha_i \rho_{x_i}, \quad \forall x_i : |x_i| \leq i_0.$$

Then for any $\rho' \in \varphi(i_0 + 2)$, we have

$$\begin{aligned}
\rho' &= \sum_j \beta_j \rho_{x_j} && |x_j| \leq i_0 + 2 && (6)\\
&= \sum_j \beta_j \mathcal{E}_{\sigma_j}(\rho_{x'_j}) && x_j = \sigma_j x'_j,\ |x'_j| \leq i_0 + 1 && (7)\\
&= \sum_j \beta_j \mathcal{E}_{\sigma_j}\left(\sum_i \alpha_i \rho_{x''_i}\right) && |x''_j| \leq i_0 && (8)\\
&= \sum_{i,j} \alpha_i \beta_j \mathcal{E}_{\sigma_j}(\rho_{x''_i}) && |x''_j| \leq i_0 && (9)\\
&= \sum_{i,j} \alpha_i \beta_j \rho_{x''_{ij}} && |x''_{ij}| \leq i_0 + 1 && (10)\\
&\in \varphi(i_0 + 1). && && (11)
\end{aligned}$$

Hence, we have $\varphi(i_0) = \varphi(i_0 + 2)$. Similarly, we can show that $\varphi(i_0) = \varphi(i_0 + j)$ for $j \geq 3$. This ends the proof. □

Note that in the above proof, we used only the linearity but no more properties of quantum operations.

Based on the above lemma, we have the following theorem.

Theorem 9. *Two MO-1gQFA $\mathcal{M}_i = \{\mathcal{H}_i, \Sigma, \rho_0^{(i)}, \{\mathcal{E}_\sigma^{(i)}\}_{\sigma\in\Sigma}, P_{acc}^{(i)}\}$ $(i = 1, 2)$ on the same input alphabet Σ are equivalent if and only if they are $(n_1 + n_2)^2$-equivalent, where $n_i = \dim \mathcal{H}_i$ for $i = 1, 2$.*

Proof. The necessity is obvious. So we verify the sufficiency. For the two MO-1gQFA

$$\mathcal{M}_i = \{\mathcal{H}_i, \Sigma, \rho_0^{(i)}, \{\mathcal{E}_\sigma^{(i)}\}_{\sigma\in\Sigma}, P_{acc}^{(i)}\} \quad (i = 1, 2),$$

denote that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\rho_0 = \frac{1}{2}\left(\rho_0^{(1)} \oplus \rho_0^{(2)}\right)$, and $\mathcal{E}_\sigma = \mathcal{E}_\sigma^{(1)} \oplus \mathcal{E}_\sigma^{(2)}$ for any $\sigma \in \Sigma$. More specifically, similar to the construction process in Section 3.1, if $\mathcal{E}_\sigma^{(1)}$ and $\mathcal{E}_\sigma^{(2)}$ have operator element sets $\{E_i\}_{i\in N}$ and $\{F_j\}_{j\in M}$, respectively, then $\mathcal{E}_\sigma$ is constructed such that it has operator element set $\{\frac{1}{\sqrt{M}}E_i \oplus \frac{1}{\sqrt{N}}F_j\}_{i\in N, j\in M}$. Then $\mathcal{E}_\sigma$ is a trace-preserving quantum operation for any $\sigma \in \Sigma$, and from Eqs. (3)–(4), we have

$$\mathcal{E}_x(\rho_0) = \frac{1}{2}\mathcal{E}_x^{(1)}(\rho_0^{(1)}) \oplus \frac{1}{2}\mathcal{E}_x^{(2)}(\rho_0^{(2)}).$$

Let $P = -P_{acc}^{(1)} \oplus P_{acc}^{(2)}$. Then for any $x \in \Sigma$, we have

$$\begin{aligned}
Tr\ (P\mathcal{E}_x(\rho_0)) &= \frac{1}{2}Tr\left(P_{acc}^{(2)}\mathcal{E}_x^{(2)}(\rho_0^{(2)})\right) - \frac{1}{2}Tr\left(P_{acc}^{(1)}\mathcal{E}_x^{(1)}(\rho_0^{(1)})\right)\\
&= \frac{1}{2}f_{\mathcal{M}_2}(w) - \frac{1}{2}f_{\mathcal{M}_1}(w).
\end{aligned}$$

Hence, $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent if and only if $Tr\ (P\mathcal{E}_x(\rho_0)) = 0$ holds for any $x \in \Sigma^*$.

From Lemma 8, there exists an integer $i_0 \leq (n_1 + n_2)^2$ ($n_i = \dim \mathcal{H}_i$ for $i = 1, 2$) such that $\varphi(i_0) = \varphi(i_0 + j)$ holds for $j = 1, 2, \ldots$. Thus, for any $|x| > (n_1+n_2)^2$, $\mathcal{E}_x(\rho_0)$ can be linearly represented by some elements in $\{\mathcal{E}_y(\rho_0) : |y| \leq (n_1+n_2)^2\}$. Therefore, if $Tr\ (P\mathcal{E}_x(\rho_0)) = 0$ holds for $|x| \leq (n_1 + n_2)^2$, then so does it for any $x \in \Sigma^*$. □

Remark 4. The above result can be seen as a generalized version of the one on the equivalence problem of *quantum sequential machines* given in [22]. Thus, the result without loss of generality can be applied to more models. For instance, MO-1QFA [27], LQFA [1], ancilla QFA [31], and Ciamarra QFA [12] can all be seen as special cases of MO-1gQFA. Thus, the equivalence criterion given in Theorem 9 also holds for these models. Note that the equivalence problem for these models had not been addressed before the result given here, except for MO-1QFA [24].

4. Measure-many one-way general quantum finite automata (MM-1gQFA)

In this section, we study another kind of general quantum finite automata, called MM-1gQFA. Similar to the case of MO-1gQFA, each input symbol of MM-1gQFA also induces a trace-preserving quantum operation. The difference is that in an MM-1gQFA, a measurement deciding to accept or reject is performed after a trace-preserving quantum operation on reading each symbol, while in an MO-1gQFA, a similar measurement is allowed only after all the input symbols have been scanned.

It is known that MM-1QFA recognize with bounded error more languages than MO-1QFA, and even more than LQFA [1], which implies that the measurement policy affects the computational power of one-way QFA. In the foregoing section, we have proved that MO-1gQFA can recognize any regular language with bounded error. Hence, if measurements also affect the computational power of 1gQFA, then the model of MM-1gQFA should recognize some non-regular languages with bounded error.

Our aim in this section is to characterize the languages recognized by MM-1gQFA. Also, we will discuss the equivalence problem of MM-1gQFA. To address these problems, in Section 4.1 we first develop some techniques to simulate an MM-1gQFA by a relaxed version of MO-1gQFA in which each symbol induces a linear super-operator instead of a trace-preserving quantum operation. Based on these techniques, in Section 4.2 we will prove that the languages recognized by MM-1gQFA with bounded error are exactly regular languages, which are the same as those recognized by MO-1gQFA. Therefore, the number of times measurements are performed has no effect on the computational power of 1gQFA; this in sharp contrast with the conventional case in 1QFA. In Section 4.3, we will discuss the equivalence problem of MM-1gQFA.

4.1. Preprocessing an MM-1gQFA

In this subsection, we first give the definitions related to MM-1gQFA. Afterward, we develop some techniques to transform an MM-1gQFA into a relaxed version of MO-1gQFA.

Definition 5. An MM-1gQFA $\mathcal{M}$ is a six-tuple $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, \mathcal{H}_{acc}, \mathcal{H}_{rej}\}$, where $\mathcal{H}$ is a finite-dimensional Hilbert space, Σ is a finite input alphabet, ¢ and \$ are respectively the left end-marker and the right end-marker, ρ_0, the initial state of $\mathcal{M}$, is a density operator on $\mathcal{H}$, $\mathcal{E}_\sigma$ corresponding to symbol σ is a trace-preserving quantum operation acting on $\mathcal{H}$, $\mathcal{H}_{acc}$ and $\mathcal{H}_{rej}$ are the "accepting" and "rejecting" subspaces of $\mathcal{H}$, respectively, and they together with another subspace $\mathcal{H}_{non}$ span the full space $\mathcal{H}$. There is a measurement $\{P_{non}, P_{acc}, P_{rej}\}$, of which the elements in turn are the projectors onto subspace $\mathcal{H}_{non}$, $\mathcal{H}_{acc}$, and $\mathcal{H}_{rej}$, respectively.

In the above definition, it is assumed that the initial state ρ_0 is a density operator from the subspace $\mathcal{H}_{non}$, and has no common part with the other two subspaces. That is, $supp(\rho_0) \subseteq \mathcal{H}_{non}$ and $supp(\rho_0) \cap \mathcal{H}_l = \emptyset$ for $l \in \{acc, rej\}$. This assumption does not affect the computational power of MM-1gQFA, since we can produce arbitrary density operator from ρ_0 by adjusting operation $\mathcal{E}_{¢}$. In fact, a similar assumption was also made in the definition of 2QFA [20].

The input string of MM-1gQFA $\mathcal{M}$ has this form: ¢x\$ with $x \in \Sigma^*$ and ¢, \$ the left end-maker and the right end-marker, respectively. The behavior of MM-1gQFA is similar to that of MM-1QFA. Reading each symbol σ in the input string, the machine has two actions: (i) first $\mathcal{E}_\sigma$ is performed such that the current state ρ evolves into $\mathcal{E}_\sigma(\rho)$; (ii) the measurement $\{P_{non}, P_{acc}, P_{rej}\}$ is performed on the state $\mathcal{E}_\sigma(\rho)$. If the result "acc" (or "rej") is observed, the machine halts in an accepting (or rejecting) state with a certain probability. With probability $Tr(P_{non}\mathcal{E}_\sigma(\rho))$ the machine continues to read the next symbol.

Define $\mathcal{V} = L(\mathcal{H}) \times \mathbb{R} \times \mathbb{R}$. Elements of $\mathcal{V}$ will represent the total states of $\mathcal{M}$ as follows: a machine described by $(\rho, p_{acc}, p_{rej}) \in \mathcal{V}$ has accepted with probability p_{acc}, rejected with probability p_{rej}, and neither with probability $tr(\rho)$. In this latter case, the current density operator is $\frac{1}{tr(\rho)}\rho$. The evolution of $\mathcal{M}$ reading symbol $\sigma \in \Sigma \cup \{¢, \$\}$ can be described by an operator $\mathcal{T}_\sigma$ on $\mathcal{V}$ as follows:

$$\mathcal{T}_\sigma(\rho, p_{acc}, p_{rej}) = (P_{non}\mathcal{E}_\sigma(\rho)P_{non}, Tr(P_{acc}\mathcal{E}_\sigma(\rho)) + p_{acc}, Tr(P_{rej}\mathcal{E}_\sigma(\rho)) + p_{rej}).$$

We use $f_{\mathcal{M}}(x)$ to denote the probability that MM-1gQFA $\mathcal{M}$ accepts $x \in \Sigma^*$. Then $f_{\mathcal{M}}(x)$ accumulates all the accepting probabilities produced on reading each symbol in the input string ¢x\$.

Obviously, an MM-1QFA [6] is a special MM-1gQFA, and the model, named GQFA, defined by Nayak [26] is also a special case of MM-1gQFA. Thus, all the results obtained later for MM-1gQFA also hold for the two models.

It is easy to see that an MO-1gQFA can be simulated by an MM-1gQFA. Here we ask a question in the opposite direction: can an MM-1gQFA be simulated by an MO-1gQFA? If we relax the definition of MO-1gQFA, we find the answer is "yes". We first define a model named *Measure-Once Linear Machine* (MO-LM) as follows.

Definition 6. An MO-LM, represented by $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\Theta_\sigma\}_{\sigma\in\Sigma}, P_{acc}\}$, is similar to an MO-1gQFA, where all the elements except Θ_σ are the same as those in MO-1gQFA, and $\Theta_\sigma : L(\mathcal{H}) \to L(\mathcal{H})$ is a linear super-operator, not necessarily a trace-preserving quantum operation.

An MO-LM $\mathcal{M}$ induces a function $f_{\mathcal{M}} : \Sigma^* \to \mathbb{C}$ as follows:

$$f_{\mathcal{M}}(x_1x_2\ldots x_m) = Tr(\Theta_{x_m} \circ \cdots \circ \Theta_{x_2} \circ \Theta_{x_1}(\rho_0)P_{acc}).$$

In the following, we decompose each trace-preserving quantum operation in an MM-1gQFA into three parts, which will be useful when we construct an MO-LM simulating an MM-1gQFA.

Lemma 10. *Given a trace-preserving quantum operation $\mathcal{E}(\rho) = \sum_m E_m \rho E_m^\dagger$ acting on the finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{non} \oplus \mathcal{H}_{acc} \oplus \mathcal{H}_{rej}$, there is a decomposition $E_m = E_m^{(non)} + E_m^{(acc)} + E_m^{(rej)}$ for every E_m, such that for any $l \in \{non, acc, rej\}$, there is*

$$\sum_m E_m^{(l)\dagger} E_m^{(l)} = I_l$$

where I_l is the identity on subspace $\mathcal{H}_l$, and for any positive operator ρ_l on $\mathcal{H}$ satisfying $supp(\rho_l) \subseteq \mathcal{H}_l$ with $l \in \{non, acc, rej\}$, there is

$$\mathcal{E}(\rho_l) = \sum_m E_m^{(l)} \rho_l E_m^{(l)\dagger}. \tag{12}$$

Proof. Let $\{|n_i\rangle\}$, $\{|a_i\rangle\}$, and $\{|r_i\rangle\}$ be the orthonormal bases of $\mathcal{H}_{non}$, $\mathcal{H}_{acc}$, and $\mathcal{H}_{rej}$, respectively. Then $\{|n_i\rangle\} \cup \{|a_i\rangle\} \cup \{|r_i\rangle\}$ form an orthonormal base of $\mathcal{H}$, and for simplicity, we refer this base as $\{|l\rangle\}$. Then each element E_m in the operator-sum representation of $\mathcal{E}$ can be represented by the *outer product representation* [25, pp. 67] as

$$E_m = \sum_{ll'} e_{ll'} |l\rangle\langle l'|,$$

where $e_{ll'} = \langle l|E_m|l'\rangle$. More specifically, E_m can be decomposed into three parts $E_m = E_m^{(non)} + E_m^{(acc)} + E_m^{(rej)}$ where

$$E_m^{(non)} = \sum e_{ln_i} |l\rangle\langle n_i|,$$
$$E_m^{(acc)} = \sum e_{la_i} |l\rangle\langle a_i|,$$
$$E_m^{(rej)} = \sum e_{lr_i} |l\rangle\langle r_i|.$$

Since $\mathcal{E}$ is trace-preserving, there is

$$\sum_m E_m^\dagger E_m = I$$
$$\Leftrightarrow \sum_m \left(E_m^{(non)} + E_m^{(acc)} + E_m^{(rej)}\right)^\dagger \left(E_m^{(non)} + E_m^{(acc)} + E_m^{(rej)}\right) = I$$
$$\Leftrightarrow \sum_m \left(\sum_l E_m^{(l)\dagger} E_m^{(l)} + \sum_{l \neq l'} E_m^{(l)\dagger} E_m^{(l')} \right) = I.$$

In the above, we should note that for each $l \in \{non, acc, rej\}$, $E_m^{(l)\dagger} E_m^{(l)}$ includes both diagonal and non-diagonal elements of $\sum_m E_m^\dagger E_m$, and $E_m^{(l)\dagger} E_m^{(l')}$ with $l \neq l'$ includes only non-diagonal elements of that. Furthermore, it should be noticed that $E_m^{(l)\dagger} E_m^{(l)}$ with $l \in \{non, acc, rej\}$ and $E_m^{(l)\dagger} E_m^{(l')}$ with $l \neq l'$ do not simultaneously have no-zero elements in the same position. For example, it is easy to see that $E_m^{(non)\dagger} E_m^{(non)}$ and $E_m^{(non)\dagger} E_m^{(acc)}$ are in the following forms:

$$E_m^{(non)\dagger} E_m^{(non)} = \sum e'_{ij} |n_i\rangle\langle n_j|,$$
$$E_m^{(non)\dagger} E_m^{(acc)} = \sum e'_{ij} |n_i\rangle\langle a_j|.$$

Obviously, they do not simultaneously have no-zero elements in the same position. Similarly, we can verify the other cases.

Therefore, from the equality $\sum_m E_m^\dagger E_m = I$ we conclude that

$$\sum_m \sum_{l \neq l'} E_m^{(l)\dagger} E_m^{(l')} = 0,$$

and

$$\sum_m E_m^{(non)\dagger} E_m^{(non)} + \sum_m E_m^{(acc)\dagger} E_m^{(acc)} + \sum_m E_m^{(rej)\dagger} E_m^{(rej)} = I.$$

At the same time, we note that

$$supp(E_m^{(l)\dagger} E_m^{(l)}) \subseteq \mathcal{H}_l$$

holds for each $l \in \{non, acc, rej\}$. Thus we have

$$\sum_m E_m^{(l)\dagger} E_m^{(l)} = I_l$$

for each $l \in \{non, acc, rej\}$, where I_l is the identity on subspace $\mathcal{H}_l$.

Next, we prove Eq. (12) for the case $l = non$. Given a positive operator ρ_{non} satisfying $supp(\rho_{non}) \subseteq \mathcal{H}_{non}$, it is easy to verify that

$$E_m^{(l)}\rho_{non} = \rho_{non}E_m^{(l)\dagger} = 0 \quad \text{for } l \in \{acc, rej\}.$$

Thus we have

$$\mathcal{E}(\rho_{non}) = \sum_m E_m^{(non)}\rho_{non}E_m^{(non)\dagger}.$$

Similarly, we can also prove Eq. (12) for the other cases where $l \in \{acc, rej\}$. Hence, we have completed the proof of Lemma 10. □

Now we are ready to simulate an MM-1gQFA by an MO-LM.

Theorem 11. *An MM-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, \mathcal{H}_{acc}, \mathcal{H}_{rej}\}$ can be simulated by an MO-LM $\mathcal{M}' = \{\mathcal{H}, \Sigma, \rho_0, \{\Theta_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, P_{acc}\}$, such that $f_{\mathcal{M}}(x) = f_{\mathcal{M}'}(¢x\$)$ holds for each $x \in \Sigma^*$.*

Proof. Given an MM-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, \mathcal{H}_{acc}, \mathcal{H}_{rej}\}$, we construct an MO-LM $\mathcal{M}'$ such that all the elements except Θ are the same as those in MM-1gQFA $\mathcal{M}$. Then the key step is to construct a linear super-operator Θ to simulate the quantum operation $\mathcal{E}$ and the measurement $\{P_{non}, P_{acc}, P_{rej}\}$ performed by $\mathcal{M}$. We complete this with two steps: (i) first construct a linear super-operator $\mathcal{F}: L(\mathcal{H}) \to L(\mathcal{H})$ to simulate the quantum operation $\mathcal{E}$; (ii) next construct another linear super-operator $\mathcal{F}'$ to simulate the measurement $\{P_{non}, P_{acc}, P_{rej}\}$.

For the trace-preserving quantum operation in $\mathcal{M}$: $\mathcal{E}(\rho) = \sum_{m=1}^{M} E_m\rho E_m^\dagger$, in terms of Lemma 10, each E_m can be decomposed as $E_m = E_m^{(non)} + E_m^{(acc)} + E_m^{(rej)}$. Then we construct a linear operator on $\mathcal{H}$ as

$$F_m = E_m^{(non)} + \frac{1}{\sqrt{M}}P_{acc} + \frac{1}{\sqrt{M}}P_{rej},$$

where M is the number of operators in the operator-sum representation of $\mathcal{E}$, and P_{acc} and P_{rej} are the projectors onto subspaces $\mathcal{H}_{acc}$ and $\mathcal{H}_{rej}$, respectively. Furthermore, construct a linear super-operator $\mathcal{F} : L(\mathcal{H}) \to L(\mathcal{H})$ as

$$\mathcal{F}(\rho) = \sum_{m=1}^{M} F_m\rho F_m^\dagger.$$

Then for any $\rho = \rho_{non} + \rho_{acc} + \rho_{rej}$ satisfying $supp(\rho_l) \subseteq \mathcal{H}_l$ with $l \in \{non, acc, rej\}$, we have

$$\begin{aligned}\mathcal{F}(\rho) = \mathcal{F}(\rho_{non}) + \mathcal{F}(\rho_{acc}) + \mathcal{F}(\rho_{rej}) &= \sum_{m=1}^{M} E_m^{(non)}\rho_{non}E_m^{(non)\dagger} + \frac{1}{M}\sum_{m=1}^{M} P_{acc}\rho_{acc}P_{acc} + \frac{1}{M}\sum_{m=1}^{M} P_{rej}\rho_{rej}P_{rej} \\ &= \mathcal{E}(\rho_{non}) + \rho_{acc} + \rho_{rej}.\end{aligned} \tag{13}$$

In the above process, we used Lemma 10 and these properties:

$$\begin{aligned}&P_l\rho_{non} = 0, \qquad \rho_{non}P_l = 0 \quad \text{for } l \in \{acc, rej\},\\ &E_m^{(non)}\rho_l = 0, \qquad \rho_l E_m^{(non)\dagger} = 0 \quad \text{for } l \in \{acc, rej\} \text{ and any } m,\\ &P_l\rho_l P_l = \rho_l \quad \text{for } l \in \{acc, rej\}.\end{aligned}$$

The next step is to simulate the measurement $\{P_{non}, P_{acc}, P_{rej}\}$ performed by MM-1gQFA $\mathcal{M}$. To do this, construct a trace-preserving quantum operation $\mathcal{F}'$ as follows:

$$\mathcal{F}'(\rho) = P_{non}\rho P_{non} + P_{acc}\rho P_{acc} + P_{rej}\rho P_{rej}.$$

For $\mathcal{F}(\rho)$ given in Eq. (13), we have

$$\begin{aligned}\mathcal{F}'(\mathcal{F}(\rho)) &= \mathcal{F}'(\mathcal{E}(\rho_{non})) + \mathcal{F}'(\rho_{acc}) + \mathcal{F}'(\rho_{rej})\\ &= \mathcal{F}'(\mathcal{E}(\rho_{non})) + \rho_{acc} + \rho_{rej}\\ &= P_{non}\mathcal{E}(\rho_{non})P_{non} + (\rho_{acc} + P_{acc}\mathcal{E}(\rho_{non})P_{acc}) + (\rho_{rej} + P_{rej}\mathcal{E}(\rho_{non})P_{rej}).\end{aligned}$$

In summary, corresponding to the quantum operation $\mathcal{E}$ and the measurement $\{P_{non}, P_{acc}, P_{rej}\}$ performed by MM-1gQFA $\mathcal{M}$ when reading a symbol, we construct a linear super-operator $\Theta : L(\mathcal{H}) \to L(\mathcal{H})$ for MO-LM $\mathcal{M}'$ by letting $\Theta = \mathcal{F}' \circ \mathcal{F}$. Then for any $\rho = \rho_{non} + \rho_{acc} + \rho_{rej}$ satisfying $supp(\rho_l) \subseteq \mathcal{H}_l$ with $l \in \{non, acc, rej\}$, we have

$$\Theta : \rho_{non} + \rho_{acc} + \rho_{rej} \to \rho'_{non} + (\rho_{acc} + \rho'_{acc}) + (\rho_{rej} + \rho'_{rej}) \tag{14}$$

such that $supp(\rho'_l) \subseteq \mathcal{H}_l$ for $l \in \{non, acc, rej\}$, and more specifically $\rho'_{non} = P_{non}\mathcal{E}(\rho_{non})P_{non}$, $\rho'_{acc} = P_{acc}\mathcal{E}(\rho_{non})P_{acc}$, and $\rho'_{rej} = P_{rej}\mathcal{E}(\rho_{non})P_{rej}$.

Next we should prove that $\mathcal{M}$ and $\mathcal{M}'$ have the same accepting probability for each input string. First we mention that the state $\bar{\rho} \in L(\mathcal{H})$ of MO-LM $\mathcal{M}'$ after reading some input string can always be written in this form:

$$\bar{\rho} = \rho_{non} + \rho_{acc} + \rho_{rej}$$

with $supp(\rho_l) \subseteq \mathcal{H}_l$ for $l \in \{non, acc, rej\}$. To see that, first we note that the initial state ρ_0 is trivially in the form, and from Eq. (14), we see that the linear super-operator Θ maintains states in the form.

Also recall that the state of MM-1gQFA $\mathcal{M}$ can be described by an element in $\mathcal{V} = L(\mathcal{H}) \times \mathbb{R} \times \mathbb{R}$ as

$$(\rho, p_{acc}, p_{rej}).$$

To prove that $\mathcal{M}$ and $\mathcal{M}'$ have the same accepting probability for each input string, we prove the following proposition.

Proposition 12. *After reading any string, the state (ρ, p_{acc}, p_{rej}) of MM-1gQFA $\mathcal{M}$ and the state of MO-LM $\mathcal{M}'$ $\bar{\rho} = \rho_{non} + \rho_{acc} + \rho_{rej}$ where $supp(\rho_l) \subseteq \mathcal{H}_l$ for $l \in \{non, acc, rej\}$ satisfy the following equalities:*

$$\rho = \rho_{non}, \tag{15}$$

$$p_{acc} = Tr(\bar{\rho}P_{acc}). \tag{16}$$

Proof. We prove this proposition by induction on the length of input string y.

Base: When $|y| = 0$, the result holds trivially if only we note that $supp(\rho_0) \subseteq \mathcal{H}_{non}$. When $|y| = 1$, the state of $\mathcal{M}$ evolves as

$$\mathcal{T}_y : (\rho_0, 0, 0) \to (P_{non}\mathcal{E}_y(\rho_0)P_{non}, Tr(P_{acc}\mathcal{E}_y(\rho_0)), Tr(P_{rej}\mathcal{E}_y(\rho_0))),$$

and the state of $\mathcal{M}'$ evolves as

$$\Theta_y : \rho_0 \to \bar{\rho} = P_{non}\mathcal{E}_y(\rho_0)P_{non} + P_{acc}\mathcal{E}_y(\rho_0)P_{acc} + P_{rej}\mathcal{E}_y(\rho_0)P_{rej}.$$

Then it is readily seen that Eqs. (15) and (16) hold.

Induction: Assume that after reading y with $|y| = k$, the states of $\mathcal{M}$ and $\mathcal{M}'$ are (ρ, p_{acc}, p_{rej}) and $\bar{\rho} = \rho_{non} + \rho_{acc} + \rho_{rej}$, respectively, and they satisfy $\rho = \rho_{non}$ and $p_{acc} = Tr(\bar{\rho}P_{acc})$. For $|y| = k+1$, let $y = y'\sigma$ satisfying $|y'| = k$ and $\sigma \in \Sigma \cup \{¢, \$\}$. Then the state of $\mathcal{M}$ evolves as:

$$\mathcal{T}_\sigma : (\rho, p_{acc}, p_{rej}) \to (\rho', p'_{acc}, p'_{rej}),$$

where $\rho' = P_{non}\mathcal{E}_\sigma(\rho)P_{non}$, $p'_{acc} = Tr(P_{acc}\mathcal{E}_\sigma(\rho)) + p_{acc}$, and $p'_{rej} = Tr(P_{rej}\mathcal{E}_\sigma(\rho)) + p_{rej}$. The state of $\mathcal{M}'$ evolves as:

$$\Theta_\sigma : \bar{\rho} = \rho_{non} + \rho_{acc} + \rho_{rej} \to \bar{\rho}' = \rho'_{non} + (\rho_{acc} + \rho'_{acc}) + (\rho_{rej} + \rho'_{rej}),$$

where $\rho'_{non} = P_{non}\mathcal{E}_\sigma(\rho_{non})P_{non}$, $\rho'_{acc} = P_{acc}\mathcal{E}_\sigma(\rho_{non})P_{acc}$, and $\rho'_{rej} = P_{rej}\mathcal{E}_\sigma(\rho_{non})P_{rej}$.

Then from the assumption $\rho = \rho_{non}$, it is easily seen that $\rho'_{non} = \rho'$, i.e., Eq. (15) holds. Also, we have

$$\begin{aligned} Tr(\bar{\rho}'P_{acc}) &= Tr((\rho_{acc} + \rho'_{acc})P_{acc}) \\ &= Tr(\rho_{acc}P_{acc}) + Tr(\rho'_{acc}P_{acc}) \\ &= Tr(\bar{\rho}P_{acc}) + Tr(P_{acc}\mathcal{E}_\sigma(\rho_{non})) \\ &= p_{acc} + Tr(P_{acc}\mathcal{E}_\sigma(\rho)) \quad \text{(by the assumption)} \\ &= p'_{acc}. \end{aligned}$$

Thus, we have completed the proof of Proposition 12. □

From Proposition 12, we know that MM-1gQFA $\mathcal{M}$ and MO-LM $\mathcal{M}'$ have the same accepting probability for any input string. Therefore, we have completed the proof of Theorem 11. □

Remark 5. In the above proof, we should observe the following two points, which will be useful in the proof of the regularity of languages recognized by MM-1gQFA in the next subsection:

(i) The linear super-operator $\mathcal{F} : L(\mathcal{H}) \to L(\mathcal{H})$ defined as $\mathcal{F}(\rho) = \sum_{m=1}^{M} F_m \rho F_m^\dagger$ is generally not a trace-preserving quantum operation, since direct calculation shows that

$$\sum_{m=1}^{M} F_m^\dagger F_m = \sum_{m=1}^{M} E_m^{(non)\dagger} E_m^{(non)} + P_{acc} + P_{rej} + \frac{1}{\sqrt{M}} \sum_{m=1}^{M} \left(E_m^{(non)\dagger} P_{acc} + E_m^{(non)\dagger} P_{rej} + P_{acc} E_m^{(non)} + P_{rej} E_m^{(non)} \right)$$

where $\sum_{m=1}^{M} E_m^{(non)\dagger} E_m^{(non)} + P_{acc} + P_{rej} = I_{\mathcal{H}}$. However, it is easy to see that for any $\rho = \rho_{non} + \rho_{acc} + \rho_{rej}$ satisfying $supp(\rho_l) \subseteq \mathcal{H}_l$ with $l \in \{non, acc, rej\}$, $\mathcal{F}$ is trace-preserving, i.e., $Tr(\mathcal{F}(\rho)) = Tr(\rho)$.

(ii) It is not difficult to check that the states of MO-LM $\mathcal{M}'$ constructed in the above proof are always positive operators, and for a positive operator ρ, we have $\|\rho\|_{tr} = Tr(\rho)$.

4.2. The computational power of MM-1gQFA

In this subsection, we are going to investigate the language recognition power of MM-1gQFA.

We first recall some notions and results that will be used later. In the following theorem, $rank(A)$ denotes the rank of A.

Theorem 13 (*Singular-Value Theorem [17,25]*). *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator and let $rank(A) = r$. Then there exist positive real numbers $s_1, s_2, \ldots, s_r$ and orthonormal sets $\{|v_1\rangle, |v_2\rangle, \ldots, |v_r\rangle\} \subset \mathcal{H}_1$ and $\{|u_1\rangle, |u_2\rangle, \ldots, |u_r\rangle\} \subset \mathcal{H}_2$ such that*

$$A = \sum_{i=1}^{r} s_i |u_i\rangle\langle v_i|.$$

We can characterize several important norms of linear operators using their singular values. There are different norms for linear operators, and here we present two norms: the Frobenius norm and the trace norm.

The Frobenius norm of $A \in L(\mathcal{H})$ is defined as

$$||A||_F = \sqrt{\langle A, A\rangle}$$

where $\langle A, B\rangle = Tr(A^\dagger B)$ is the Hilbert–Schmidt inner product between A and B. Then the Cauchy–Schwarz inequality implies

$$|\langle A, B\rangle| \le ||A||_F ||B||_F.$$

Equivalently, the Frobenius norm $||A||_F$ can be characterized by the singular values of A as follows:

$$||A||_F = \left(\sum s_i^2\right)^{\frac{1}{2}}.$$

The trace norm of $A \in L(\mathcal{H})$, defined as $||A||_{tr} = Tr\sqrt{A^\dagger A}$, will often be used in the foregoing sections. Note that if A is a positive operator, then $||A||_{tr} = Tr(A)$. Similar to the Frobenius norm, the trace norm can also be characterized by singular values as

$$||A||_{tr} = \sum_i s_i.$$

In terms of the singular values of $A \in L(\mathcal{H})$, it is not difficult to see

$$||A||_F \le ||A||_{tr}. \tag{17}$$

In fact, different norms defined for $A \in L(\mathcal{H})$ are equivalent in the following sense.

Lemma 14 ([17]). *Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two norms on a finite dimensional vector space V. Then there exist two finite positive constants c_1 and c_2 such that $c_1\|x\|_\alpha \le \|x\|_\beta \le c_2\|x\|_\alpha$ for all $x \in V$.*

Obviously, $L(\mathcal{H})$ is a finite dimensional vector space given $\mathcal{H}$ is finite. Thus, the norms on $L(\mathcal{H})$ defined above satisfy the property given in the above lemma.

Now for the MO-LM $\mathcal{M}' = \{\mathcal{H}, \Sigma, \rho_0, \{\Theta_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, P_{acc}\}$ which was constructed to simulate an MM-1gQFA, let $\mathcal{S} = span\{\Theta_{¢w}(\rho_0) : w \in \Sigma^*\}$. Then we have the following result.

Lemma 15. *There exists a constant c such that $\left\|\Theta_{y\$}(\rho)\right\|_{tr} \le c\|\rho\|_{tr}$ for any $\rho \in \mathcal{S}$ and $y \in \Sigma^*$.*

Proof. First find a base for $\mathcal{S}$ as: $\rho_1 = \Theta_{¢w_1}(\rho_0), \rho_2 = \Theta_{¢w_2}(\rho_0), \ldots, \rho_m = \Theta_{¢w_m}(\rho_0)$. Note that for $A, B \in L(\mathcal{H})$, $A \perp B$ means $\langle A, B\rangle = tr(A^\dagger B) = 0$. For each $1 \le i \le m$, let $e_i \in L(\mathcal{H})$ satisfy $||e_i||_F = 1$, $e_i \perp \{\rho_j : j \ne i\}$ and $e_i \not\perp \rho_i$. Then $\rho \in \mathcal{S}$ can be linearly represented as $\rho = \sum_{i=1}^{m} \alpha_i\rho_i$, and it holds that

$$||\rho||_F \ge |\langle e_i, \rho\rangle| = |\alpha_i| \cdot |\langle e_i, \rho_i\rangle|. \tag{18}$$

Therefore, we have

$$\begin{aligned}
\|\Theta_{y\$}(\rho)\|_F &= \left\|\sum_{i=1}^{m}\alpha_i\Theta_{y\$}(\rho_i)\right\|_F = \left\|\sum_{i=1}^{m}\alpha_i\Theta_{¢w_ix\$}(\rho_0)\right\|_F \\
&\le \sum_{i=1}^{m}|\alpha_i|\cdot\|\Theta_{¢w_ix\$}(\rho_0)\|_F \le \sum_{i=1}^{m}|\alpha_i|\cdot\left\|\Theta_{¢w_ix\$}(\rho_0)\right\|_{tr} && \text{(by Inequality (17))} \\
&= \sum_{i=1}^{m}|\alpha_i|Tr(\rho_0) = \sum_{i=1}^{m}|\alpha_i| \\
&\le ||\rho||_F\sum_{i=1}^{m}1/|\langle e_i,\rho_i\rangle| && \text{(by Inequality (18))} \\
&= K||\rho||_F,
\end{aligned}$$

where $K = \sum_{i=1}^{m} 1/|\langle e_i, \rho_i\rangle|$ is a constant not depending on ρ, and the third equality follows from the observations (i) and (ii) made at the end of Section 4.1. Furthermore, by Lemma 14 and Inequality (17), we have

$$\|\Theta_{y\$}(\rho)\|_{tr} \le c_1\|\Theta_{y\$}(\rho)\|_F \le c_1K||\rho||_F \le c_1K||\rho||_{tr}.$$

Thus, by letting $c = c_1K$, we have completed the proof of Lemma 15. □

The definition of MM-1gQFA recognizing a language with bounded error is similar to the one for MO-1gQFA given in Definition 2. In the following, we present a complete characterization of the languages recognized by MM-1gQFA with bounded error.

Theorem 16. *The languages recognized by MM-1gQFA with bounded error are regular.*

Proof. Assume that L is recognized by MM-1gQFA $\mathcal{M} = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, \mathcal{H}_{acc}, \mathcal{H}_{rej}\}$ with bounded error ϵ. Then in terms of Theorem 11, there exists an MO-LM $\mathcal{M}' = \{\mathcal{H}, \Sigma, \rho_0, \{\Theta_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, P_{acc}\}$ such that for some $\lambda \in (0, 1]$, $f_{\mathcal{M}'}(¢x\$) \ge \lambda + \epsilon$ holds for any $x \in L$, and $f_{\mathcal{M}'}(¢y\$) \le \lambda - \epsilon$ holds for any $y \notin L$.

We define an equivalence relation "$\equiv_L$" on $x, y \in \Sigma^*$ such that $x \equiv_L y$ if for any $z \in \Sigma^*$, $xz \in L$ iff $yz \in L$. Then in terms of the Myhill–Nerode theorem (Theorem 4), it is sufficient to prove that the number of equivalence classes induced by "$\equiv_L$" is finite.

Let $S = \{A : \|A\|_{tr} \le 1$, and A is a linear operator on $\mathcal{H}\}$. Then S is a bounded subset from a finite-dimensional space. Let $\rho_x = \Theta_{x_n}\circ\cdots\circ\Theta_{x_2}\circ\Theta_{x_1}\circ\Theta_¢(\rho_0)$, i.e., the state of $\mathcal{M}'$ after having been fed with input string $¢x$ with $x \in \Sigma^*$. Then for every x, it can be seen that $\rho_x \in S$, since we have $||\rho_x||_{tr} = \mathrm{Tr}(\rho_x) = Tr(\rho_0) = 1$ which follows from the observations (i) and (ii) made at the end of Section 4.1. Now, suppose that $x \not\equiv_L y$, that is, there exists a string $z \in \Sigma^*$ such that $xz \in L$ and $yz \notin L$. Then we have

$$Tr(P_{acc}\Theta_{z\$}(\rho_x)) \ge \lambda + \epsilon \quad\text{and}\quad Tr(P_{acc}\Theta_{z\$}(\rho_y)) \le \lambda - \epsilon$$

for some $\lambda \in (0, 1]$.

Denote $\overline{P}_{acc} = I - P_{acc}$. Then $\{P_{acc}, \overline{P}_{acc}\}$ is a POVM measurement (a projective measurement) on space $\mathcal{H}$. Note that Lemma 3 also holds for any two positive operators. That is, for any two positive operators A, B, it holds that

$$||A - B||_{tr} = \max_{\{E_m\}}\sum_m |Tr(E_mA) - Tr(E_mB)|,$$

where the maximization is over all POVMs $\{E_m\}$. Indeed, this property has already been observed in [40]. Therefore, we have

$$\begin{aligned}
||\mathcal{E}_{z\$}(\rho_x) - \mathcal{E}_{z\$}(\rho_y)||_{tr} &\ge \left|Tr(P_{acc}\mathcal{E}_{z\$}(\rho_x)) - Tr(P_{acc}\mathcal{E}_{z\$}(\rho_y))\right| + \left|Tr(\overline{P}_{acc}\mathcal{E}_{z\$}(\rho_x)) - Tr(\overline{P}_{acc}\mathcal{E}_{z\$}(\rho_y))\right| \\
&\ge 2\epsilon.
\end{aligned}$$

On the other hand, by Lemma 15, we have

$$||\rho_x - \rho_y||_{tr} \ge \frac{1}{c}||\mathcal{E}_{z\$}(\rho_x) - \mathcal{E}_{z\$}(\rho_y)||_{tr},$$

where c is a constant. Consequently, for any two strings $x, y \in \Sigma^*$ satisfying $x \not\equiv_L y$, we always have

$$||\rho_x - \rho_y||_{tr} \ge \frac{1}{c}2\epsilon. \tag{19}$$

Now, suppose that Σ^* consists of infinite equivalence classes, say $[x^{(1)}], [x^{(2)}], [x^{(3)}], \ldots$. Then by the boundedness of S from a finite-dimensional space, from the sequence $\{\rho_{x^{(n)}}\}_{n\in N}$, we can extract a Cauchy sequence $\{\rho_{x^{(n_k)}},\}_{k\in N}$, i.e., a convergent subsequence. Thus, there exist x and y satisfying $x \not\equiv_L y$ such that

$$||\rho_x - \rho_y||_{tr} < \frac{1}{c}2\epsilon,$$

which contradicts Inequality (19). Therefore, the number of the equivalence classes in Σ^* induced by the equivalence relation "$\equiv_L$" must be finite, which implies that L is a regular language. □

Now we have proved the languages recognized by MM-1gQFA with bounded error are in the set of regular languages. On the other hand, it is easy to see that an MO-1gQFA can be simulated by an MM-1gQFA. Hereby, MM-1QFA can recognize any regular language with bounded error. Therefore, we have the following result.

Theorem 17. *The languages recognized by MM-1gQFA with bounded error are exactly regular languages.*

Remark 6. As we know, so far no QFA with a one-way tape head can recognize non-regular languages. Although we allow the most general operations—trace-preserving quantum operations, QFA with one-way tape heads still recognize only regular languages. On the other hand, the two-way QFA defined in [20] can recognize some non-regular languages. Thus, the uppermost factor affecting the computational power of a QFA should be the moving direction of its tape head, but not the operations induced by the input alphabet.

4.3. The equivalence problem of MM-1gQFA

In this subsection, we discuss the equivalence problem of MM-1gQFA. In Section 3.3, we have dealt with the equivalence problem of MO-1gQFA. Apparently, the equivalence problem of MM-1gQFA is more difficult than that of MO-1gQFA. However, based on the techniques developed in Section 4.1, the equivalence problem of MM-1gQFA can be proved in the same way we did for MO-1gQFA.

The formal definitions related to the equivalence of MM-1gQFA are similar to those for MO-1gQFA given in Section 3.3, and we do not repeat them here. Our result is as follows.

Theorem 18. *Two MM-1gQFA $\mathcal{M}_i = \{\mathcal{H}_i, \Sigma, \rho_0^{(i)}, \{\mathcal{E}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\cent,\}}, \mathcal{H}_{acc}^{(i)}, \mathcal{H}_{rej}^{(i)}\}$ with $i = 1, 2$ are equivalent if and only if they are $(n_1 + n_2)^2$-equivalent, where $n_i = dim(\mathcal{H}_i)$ for $i = 1, 2$.*

Proof. In terms of Theorem 11, we know that two MM-1gQFA $\mathcal{M}_i$ with $i = 1, 2$ can be simulated by two MO-LM $\mathcal{M}_i' = \{\mathcal{H}_i, \Sigma, \rho_0^{(i)}, \{\Theta_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\cent,\}}, P_{acc}^{(i)}\}$, respectively. Then we need only to determine the equivalence between $\mathcal{M}_1'$ and $\mathcal{M}_2'$.
Note that the only difference between MO-1gQFA and MO-LM is that $\mathcal{E}_\sigma$ is a trace-preserving quantum operation while Θ_σ is a general linear super-operator. Also, note that we used only the linearity but no more properties of $\mathcal{E}_\sigma$ in the proof of Lemma 8. Therefore, Lemma 8 also holds for MO-LM. Furthermore, using similar techniques used in the proof of Theorem 9, we obtain the result stated in the above theorem. □

Remark 7. The above result also holds for the two special cases of MM-1gQFA: MM-1QFA [6] and GQFA [5]. Note that the equivalence problem of MM-1QFA has already been considered in [23]. In the above, viewing MM-1QFA as a special case of MM-1QFA, we have obtained an equivalence criterion slightly different from the one in [23]. In fact, here we have used a method different from the one in [23]. The equivalence problem of GQFA had not been discussed before the above result, and here we have addressed this problem.

5. Conclusion

In this paper, we have studied the model of one-way general quantum finite automata (1gQFA), in which each symbol in the input alphabet induces a trace-preserving quantum operation, instead of a unitary transformation. We have studied two typical models of 1gQFA: MO-1gQFA where a measurement deciding to accept or reject is allowed only at the end of a computation, and MM-1gQFA where a similar measurement is allowed at reading each symbol during a computation.

We have proved that the languages recognized by MO-1gQFA with bounded error are still in the scope of regular languages, despite the most general operations allowed by this model [Theorem 5]. More exactly, MO-1gQFA recognize exactly regular languages with bounded error [Theorems 5 and 6]. Also, two types of QFA defined in [12,31] which were expected to be more powerful than MO-1QFA, have been shown to be special cases of MO-1gQFA, and have the same computational power as MO-1gQFA. We have discussed the equivalence problem of MO-1gQFA, and it has been proved that two MO-1gQFA $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent if and only they are $(n_1 + n_2)^2$-equivalent, where n_1 and n_2 are the dimensions of the Hilbert spaces that $\mathcal{M}_1$ and $\mathcal{M}_2$ act on, respectively [Theorem 9]. In addition, some closure properties of MO-1gQFA have been presented.

The number of times the measurement is performed is generally thought to affect the computational power of 1QFA. With this belief, we have defined the model of MM-1gQFA, a measure-many version of MO-1gQFA. However, we have proved that MM-1gQFA recognize with bounded error the same class of languages as MO-1gQFA [Theorem 17]. Hence, the measurement times have no effect on the computational power of 1gQFA, which is greatly different from the conventional case where MM-1QFA recognize more languages than MO-1QFA [11]. Also, we have addressed the equivalence problem of MM-1gQFA.

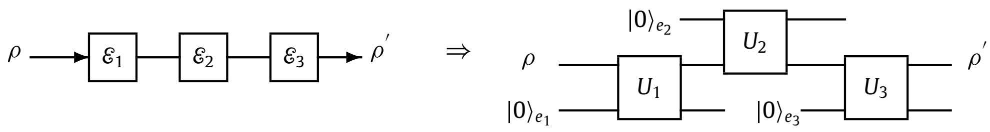

Fig. 1. The left-hand side denotes the state evolution of MO-1gQFA $\mathcal{M}$, and the right-hand side denotes the resulted machine $\mathcal{M}'$ that is to simulate $\mathcal{M}$. As shown, to simulate the quantum operation $\mathcal{E}_i$ in $\mathcal{M}$, an ancillary quantum system E_i should be added in $\mathcal{M}'$ to perform the unitary operation U_i. This leads to the size of $\mathcal{M}'$'s quantum part depending on the length of the input (i.e, the total running time of quantum operations), which implies that $\mathcal{M}'$ is no longer a QFA.

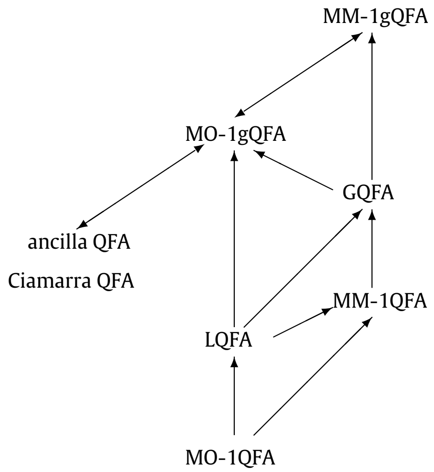

Fig. 2. A diagram illustrating known inclusions among the languages recognized with bounded error by most of the current known 1QFA. Directional lines indicate containments going from the tail to the head; for example, the languages recognized by MO-1QFA are contained in those recognized by MM-1QFA. Bidirectional lines between two models mean they are equivalent; for example, MO-1gQFA and MM-1gQFA recognize the same class of languages.

We have proved that the equivalence criterion for MO-1gQFA given above also holds for MM-1gQFA [Theorem 18]. The solution of all the above problems regarding MM-1gQFA is based on such a result proved by us that an MM-1gQFA can be simulated by a relaxed version of MO-1gQFA—MO-LM, in which each symbol in the input alphabet induces a general linear super-operator, not necessarily a trace-preserving quantum operation [Theorem 11].

From the study in this paper, we have seen that so far no quantum finite automaton with a one-way tape head can recognize with bounded error a language out of the scope of regular languages, even if the most general operations—trace-preserving quantum operations are allowed. On the other hand, we recall that 2QFA introduced by Kondacs and Watrous [20] can recognize the non-regular language $L_{eq} = \{a^nb^n|n > 0\}$ in linear time. Therefore, it may be asserted that the uppermost factor affecting the computational power of QFA is the moving direction of the tape head, neither the operation induced by the input symbol, nor the number of times the measurement is performed.

We note that, as proved by Aharonov et al. [4], quantum circuits with mixed states are equivalent to those with pure states [4]. However, such an equivalence relationship no longer holds for the restricted model—quantum finite automata, as we shown that one-way QFA with mixed states are more powerful than those with pure states. In fact, the equivalence between quantum circuits with mixed states and those with pure states is simply a corollary of the fact that every trace-preserving quantum operation $\mathcal{E}$ acting on $\mathcal{H}$ can be simulated by a unitary transformation U acting on a larger space $\mathcal{H}\otimes E$ in such a way $\mathcal{E}(\rho) = \mathrm{Tr}_E(U\rho\otimes|0^E\rangle\langle 0^E|U^\dagger)$ [4]. Unluckily, such a simulating process is not suitable for QFA. If we apply this simulating process to mixed-state QFA, for example MO-1gQFA $\mathcal{M}$, and denote the resulting machine by $\mathcal{M}'$, then as described in Fig. 1, at each running of a quantum operation in $\mathcal{M}$, a new ancillary quantum system E should be added in $\mathcal{M}'$. At the same time, we know that the total running time of quantum operations in QFA equals the length of the input (note that in a quantum circuit which consists of a finite number of quantum gates and some input ports, the total running time of quantum gates has no dependence on the input). Thus, the resulted machine $\mathcal{M}'$ has a quantum part whose size varies with the length of the input, which is clearly no longer a QFA.

Finally, we present Fig. 2 to depict the inclusion relations among the languages recognized by most of the current known 1QFA. Here we use the abbreviations of QFA to denote the classes of languages recognized by them; for example, "MM-1QFA" denotes the class of languages recognized by MM-1QFA with bounder error. Most of the inclusion relations depicted in Fig. 2 are proper inclusions, expect for the following two points: (i) it is still not known whether GQFA can recognize any language not recognized by MM-1QFA; (ii) MM-1gQFA, MO-1gQFA, ancilla QFA and Ciamarra QFA recognize the same class of languages (i.e., regular languages) as shown in this paper.

6. Further discussion

In this paper, we have addressed the equivalence problem of MO-1gQFA and MM-1gQFA, and obtained the same equivalence criterion ($(n_1 + n_2)^2$, see Theorems 9 and 18) for both of them. Recently, we noticed that Ref. [45] implied a different method to the equivalence problem of MO-1gQFA and MM-1gQFA, by which the equivalence criterion can be improved to $n_1^2 + n_1^2 - 1$. In fact, this is not an essential improvement. However, we would like to mention the different method here, since by comparing the two methods we may have a deeper understanding on QFA.

From Lemma 1 in [45], we know that an MO-1gQFA with an n-dimensional Hilbert space can be transformed to an equivalent n^2-state *Bilinear machine* (BLM) [23]. In this transformation, the mapping vec plays a key role. vec is defined as $vec(A)((i-1)n+j) = A(i,j)$ that maps an $n \times n$ matrix A to an n^2-dimensional vector. In other words, vec can be defined as $vec(|i\rangle\langle j|) = |i\rangle|j\rangle$. An n-state BLM has a form similar to that of probabilistic automata as shown in Eq. (6): $\mathcal{A} = (\pi, \Sigma, \{A(\sigma) : \sigma \in \Sigma\}, \eta)$, but for BLM, there is no more restriction than that π is an n-dimensional row vector, $A(\sigma)$ is an $n \times n$ matrix, η is an n-dimensional column vector, and all of them have entries in the set of complex numbers. In Ref. [24], it was shown that two BLMs with n_1 and n_2 states, respectively, are equivalent if and only if they are $(n_1 + n_2 - 1)$-equivalent. Therefore, combining the above results, we can obtain the equivalence criterion $n_1^2 + n_1^2 - 1$ for MO-1gQFA.

Similarly, we can also address the equivalence problem of MM-1gQFA. First, we have proved that an MM-1gQFA can be transformed to an equivalent MO-LM with the same Hilbert space (see Definition 6 and Lemma 11). Second, using the mapping vec we can also transform an MO-ML with an n-dimensional Hilbert space to an equivalent n^2-state BLM as did in [45], if only we note that the linear super-operator constructed in this paper also has an operator-sum representation as in Eq. (1).

By the way, in [45] it was proved that MO-1gQFA recognize only stochastic languages with cut-point. Based on the results in this paper, we can also prove that MM-1gQFA recognize only stochastic languages with cut-point. Thus, MO-1gQFA and MM-1gQFA have the same language recognition power as probabilistic automata in the sense of both bounded error and unbounded error.

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