## Transcript Synthesising rotations - Ben Criger

I'd like to show you an example of something that's a little bit difficult to do when looking at the matrix based picture of quantum mechanics, but if we put everything on the Bloch sphere it makes perfect sense.

In many experiments, experimentalists can only rotate along two axes of the Bloch sphere. The x-axis, which goes in this direction and the y-axis, which is going in this way.

Many algorithms rely on rotations around the z-axis, which is independent from these other two. And we would like to see how to synthesize the z-rotations out of the tools that we have available: x and y rotations.

Now, I happen to know the answer to this question and I can begin by writing down the matrices that describe the operations that we would like to do. So, here is z and I'll say this is z(theta). For those of you who are really in the know, this is e^{i*z*theta}. But if this doesn't make sense to you, then this is just a diagonal matrix with e^{i*theta/2} and e^{-i*theta/2} on the diagonal, zeros off the diagonal. And we can see immediately that x and y rotations on their own aren't going to give us this z-rotation.

So we can write down x theta and y theta, which are just rotations around these axes by angles theta. E^{ix*theta/2}, e^{i*y*theta/2}. And this guy is equal to cosine of theta over two minus i sine of theta over two, minus i sine theta over two cosine theta over two. E^{i*y*theta/2} is quite similar. Cosine theta over two, sine theta over two, minus sine theta over two, cosine theta over two. Now, there's a product of these rotations. A y an x and a y, that will excecute one of these sets.

Lets look at those matrix rotations first. What I am going to do is set theta, well, equal to pi over two for a y rotation. And that's going to give me one over root two, one one, minus one, one. Y theta equal to minus pi over 2 will give me operations that look like one, minus one, one, one. You might recognize these to be quite similar to the Hadamard basis that changes from the computational basis to the plus minus basis, and that is by design.

Now lets take a look at what happens if we sandwich an x rotations around theta with these two matrices, which we lovingly call y 90 and y -90. Because they are 90 degree rotations. So, we're going to get a, well, y pi by 2 x theta y minus pi 2. Is equal to 1 over root two one, minus one, one, one, cosine theta over two, minus i sine theta over two, minus i sine theta over two, cosine of theta over two. Then we have another one over root two, one, minus one, one, one. We multiply these through. So, first off I can take these two factors of one over square root of two, make them a factor of a half. One minus one one one. Then if I multiply these here, I get cos theta over two plus I sine theta over two, because these two minus signs cancel. So that's cos theta over two plus I sine theta over two. Here I'll get cosine theta over two minus I sine theta over two. Here I'll get minus cosine, minus I sine. So that's minus I cosine theta plus I sine theta over two. And then here I'll get cosine minus I sine. So, cosine theta over two minus I sine theta over two.

Continuing on with all this math I can notice that these two elements are equal and opposite. So If I take one and minus one, I'll get cosine plus I sine, minus negative cosine plus I sine. Which is simply two cosine theta over two plus I sine theta over two. Here I take a number and it's opposite, I get zero. Here I can take these two numbers which are equal and add them. That gives me theta over two minus I sine theta over two. And if I take one and minus one and these two I get zero as well. That's a lot of matrix multiplication.

Now, let's also use Eulers identity. We know that this is equal to e^{i*theta/2} and that this is equal to e^{-i*Theta/2}. From arithmetic using complex numbers. And so we have managed to synthesize this z rotation around theta just by using x rotations and y rotations. So by using a little bit of ket notation and some trigonometric identities, we can show that we can obtain universal control using only the tools that are available to the experimentalists. This is nice because it means we don't have to design new hardware. We can just put together operations that we already have in order to obtain universal control. But this is not that simple as it could be. There is a far easier way to see this, which we can do using the Bloch sphere.

The action of this sequence of unitary operations. Y 90, x around some angle theta and then y minus 90 can be expressed in terms of matrices. But it's much easier to see using the Bloch sphere. If we write down the Bloch sphere with the x axis pointing out of the board, we can see the effect of a y 90, a rotation around 90 degrees of the y axis, is just turning the sphere sort of like a steering wheel, so that the x axis now faces down and the z axis is where the x axis used to be. If we now apply a rotation around the x axis, that is giving us a rotation around our old z axis. And once we have that done, all we have to do is reverse the y 90 with a y minus 90 to restore the orientation of our original axes, having picked up a phase on the z. And that's how we can synthesize a diagonal unitary. e^{itheta/2}, e^{-i*theta/2} out of the operations to which the experimentalists have access. A little bit of notations saves you a lot of hardware design.