Performing arbitrary measurements - Ben Criger

I'd like to talk today about how a little bit of ket-notation can save us a big experimental headache, when it comes time to try measure a qubit in an arbitrary basis. First, we have to review a few of the important facts.

The conjugate transpose of a matrix or a vector is given by first taking the transpose of this 2-by-2 matrix that leaves the diagonal elements in the same place, and exchanges the elements being c (and b) here, but also taking the complex conjugate of each element in the matrix. A second important fact, which you can prove yourself using this definition, and I encourage you to do so, is that the dagger or complex conjugate transpose of U- psi (ket) is just the bra for psi times U-dagger. Now, that follows from linear algebra, but it is easy to check for yourself.

There's also another construction that I'd like to use, which is that for every state psi, there is another state which I call psi-perp(endicular), which consists of these coefficients. So, if psi is alpha-beta, psi-perp is the conjugate of beta and minus the conjugate of alpha. And these states are by construction orthogonal, which you can also see just by taking the inner product and seeing that it's always zero regardless of what alpha and beta are.

And with those three ingredients, we can solve this question. So, experimentalists usually measure a single operator: the Pauli-Z operator, that was perhaps discussed earlier. But we'd like to be able to measure in whatever basis we want. So that if we have a question: "Is this state this or that?", we can answer it without having to restrict ourselves to the 0-1 basis.

So, we have a two-step plan to solve this. First, we're going to apply some operator U, and then we're going to measure in the 0-1 basis. Now, if you've got a controllable qubit, you can apply an operator U as you see fit, and as discussed in the statement of the question, we can measure in the 0-1 basis. We can measure the Z-operator. So, the expectation value of this measurement, if we first apply U on psi, that replaces psi with psi-U-dagger in the bra and U-psi in the ket, and then we take the familiar sandwich product to determine the expectation value of the operator, we end up with psi-U-dagger-Z-U-psi.

So, it's as if we had our original state psi, and instead of measuring Z, we measured U-dagger-Z-U. Now, U-dagger-Z-U we can write out like so. And if we label the state U-dagger-0 as psi, then U-dagger-Z-U is equal to psi-psi minus psi-perp-psi-perp. This is a measurement that will return 1 if the state is psi, and -1 if the state is psi-perp, giving us an arbitrary basis to measure in.