Ben Criger

Ket Notation

Linear Algebra

column & row vectors: $\vec{x} = \mathbf{x} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}, \quad \vec{y}^\top = \mathbf{y}^\top = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

inner products: $\vec{y}^\top \vec{x} = \mathbf{y} \cdot \mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = 68$

matrices: $\mathbf{A}\vec{x} = \hat{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$

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column vectors $\mapsto$ *kets*: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$

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states can be expressed in different bases

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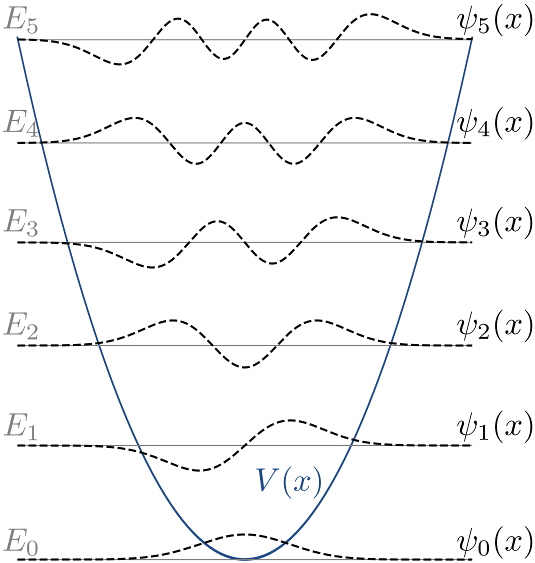
Quantum Mechanics



every device has a *Hamiltonian*

$$H = \sum_k E_k \left|\psi_k\right\rangle \left\langle\psi_k\right|$$

states $|\psi_k\rangle$ are *orthogonal*

$$\langle\psi_j|\psi_k\rangle = \delta_{j,k}$$

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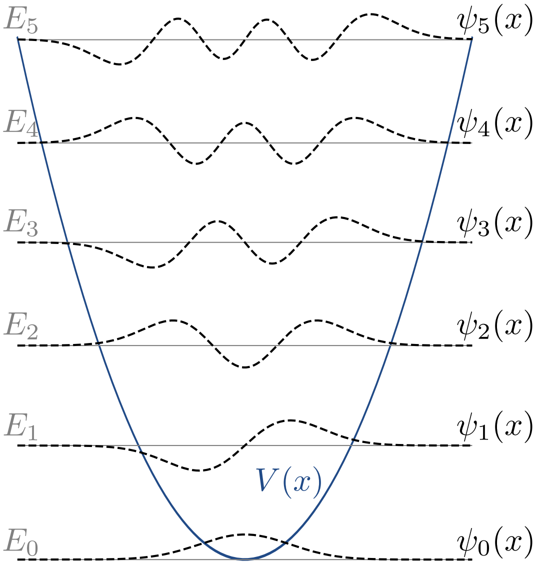

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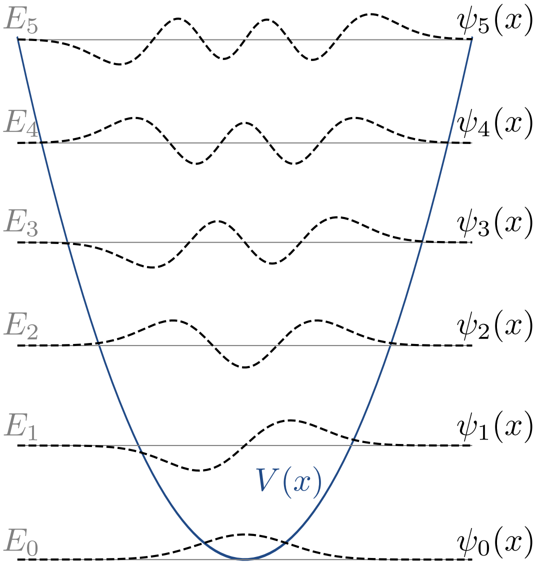

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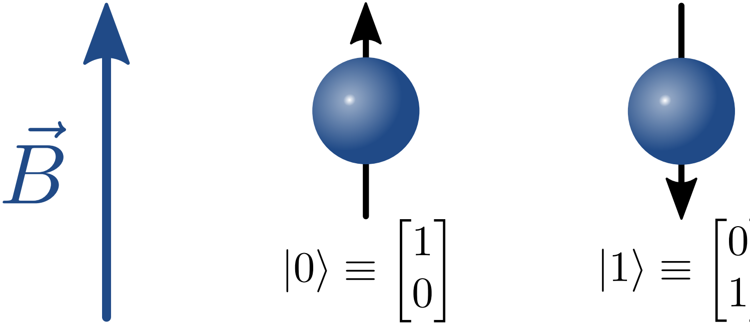
Quantum Mechanics

finite-dimensional system $\rightarrow$ finite-length vectors



$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Operations & Observables

logic gates $\equiv$ *unitary* matrices $\equiv$ changes of basis

$$U = \sum_k |\psi_k\rangle \langle k| \quad (\langle \psi_j | \psi_k \rangle = \delta_{jk})$$

readout $\equiv$ measurement operators $\equiv$ *hermitian* matrices

$$A = \sum_k r_k |\psi_k\rangle \langle \psi_k| \quad (r_k \text{ real})$$

average experimental outcomes $\equiv$ 'sandwich' products

$$\langle \phi | A | \phi \rangle = \sum_k r_k \langle \phi | \psi_k \rangle \langle \psi_k | \phi \rangle = \sum_k r_k \left| \langle \phi | \psi_k \rangle \right|^2$$

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Operations & Observables

unitary operations

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |1\rangle \langle 0| + |0\rangle \langle 1| \quad X|0\rangle = |1\rangle \;\; X|1\rangle = |0\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1| \quad H|0\rangle = |+\rangle \;\; H|1\rangle = |-\rangle$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = |0\rangle \langle 0| + i|1\rangle \langle 1| \quad P|0\rangle = |0\rangle \;\; P|1\rangle = i|1\rangle$$

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The Bloch Sphere

coefficients in polar co-ordinates

$$|\psi\rangle = r_0 e^{i\phi_0} |0\rangle + r_1 e^{i(\phi_0+\phi)} |1\rangle$$

global phases don't matter

$$|\psi\rangle \mapsto e^{i\alpha} |\psi\rangle$$

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normalization constrains qubits' states further

$$\langle\psi|\psi\rangle = r_0^2 + r_1^2 \equiv 1$$

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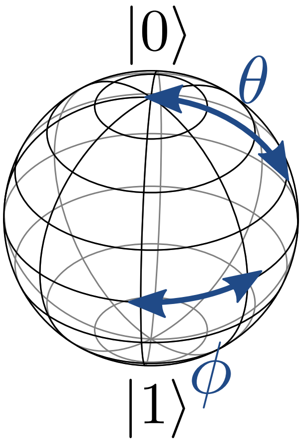

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unitary operations rotate the sphere

$\langle\psi|\psi'\rangle = 0 \leftrightarrow$ anti-parallel on the Bloch sphere

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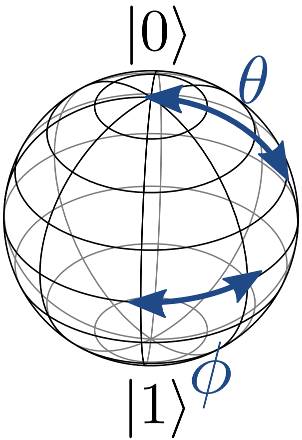

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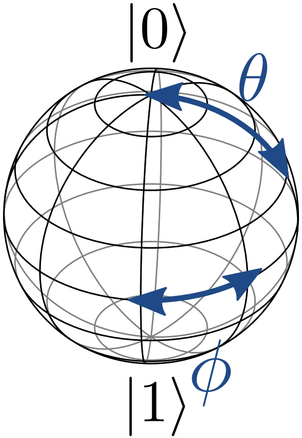

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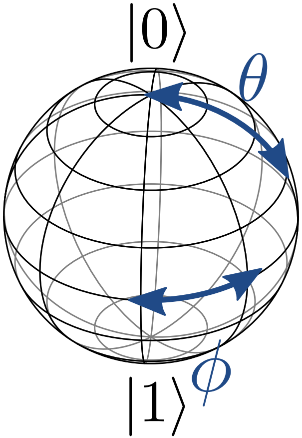

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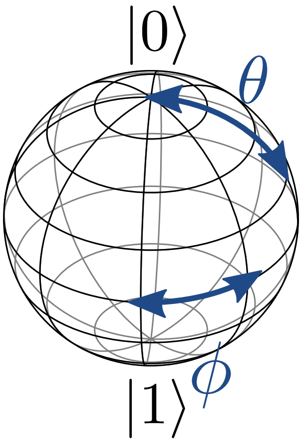

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Multi-Qubit States & Operations

multi-qubit states and operations: *tensor* (or *Kronecker*) products

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compatible with matrix-matrix product

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Example: Bell State Preparation

controlled-NOT: $\textsc{cnot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$

Bell state: $|\Omega\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Bell states are *entangled*: cannot be written as tensor product

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$$H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\text{CNOT} \times \big(H \otimes I\big)\big(|0\rangle \otimes |0\rangle\big)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

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Example: Bell State Preparation

$$H \otimes I(|0\rangle \otimes |0\rangle) = |+\rangle |0\rangle = |+0\rangle = \frac{1}{\sqrt{2}} |00\rangle + |10\rangle$$

$$|\Omega\rangle = \text{CNOT} \times \left(\tfrac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \right)$$

$$= \tfrac{1}{\sqrt{2}} (|0\rangle \langle 0|0\rangle \otimes I |0\rangle + |0\rangle \langle 0|1\rangle \otimes I |0\rangle$$
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