Quantum institutions

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October 4, 2005

Abstract

The exogenous approach to enriching any given base logic for probabilistic and quantum reasoning is brought into the realm of institutions. The theory of institutions helps in capturing the precise relationships between the logics that are obtained, and, furthermore, helps in analyzing some of the key design decisions and opens the way to make the approach more useful and, at the same time, more abstract.

1 Introduction

A new logic was proposed in [16, 18] for modeling and reasoning about quantum states, embodying the relevant postulates of quantum physics (as presented, for instance, in [22]). The logic was designed from the semantics upwards, starting with the key idea of adopting superpositions of classical models as the models of the quantum logic. In [19], other instances of the exogenous approach to enriching logics were presented in detail. In short, the exogenous approach is based on adopting as models of the new envisaged logic (enriched) sets of models of the given base logic without tempering with the models of the original logic.

This novel approach to quantum logic semantics is completely different from the traditional approach [11, 8] to the problem, as initially proposed by Birkhoff and von Neumann [3], that focuses on the lattice of closed subspaces of a Hilbert space. Our semantics has the advantage of closely guiding the design of the language around the underlying concepts of quantum physics while keeping the classical connectives and was inspired by the possible worlds approach originally proposed by Kripke [15] for modal logic. It is also akin to the society semantics introduced in [6] for many-valued logic and to the possible translations semantics proposed in [5] for paraconsistent logic. The possible worlds approach was also used in [23, 24, 2, 1, 10] for probabilistic logic. Our semantics to quantum logic, although inspired by modal logic, is also completely different from the alternative Kripke semantics given to traditional quantum logics (as first proposed in [9]) still closely related to the lattice-oriented operations. The resulting quantum logic also incorporates probabilistic reasoning (in the style of Nilsson's calculus [23, 24]) since the postulates of quantum physics impose uncertainty on the outcome of measurements. From a quantum state (superposition of classical valuations living in a suitable Hilbert space) it is

straightforward to generate a probability space of classical valuations in order to provide the semantics for reasoning about the probabilistic measurements made on that state.

Herein, we present within the theory of institutions the exogenous-style construction of a quantum logic from any given base logic in order to assess how general the construction is. To this end, a logic is identified with an institution, as originally proposed in [12, 13] with great success in both theoretical issues and applications. The construction is carried out in three main steps. Given an arbitrary institution we first build its global extension where each model is just a set of models of the original institution. Then, we proceed with the construction of its probabilistic extension where each model is a probability space where the outcomes are models of the original institution. Finally, we obtain the quantum extension of the given institution where each model is a unit vector in the Hilbert space freely generated from a set of models of the original institution. Obviously, in each step the language is enriched to take advantage and to express properties of the new models.

In Section 2, we briefly present the relevant notions and results of the theory of institutions. The globalization step is described in Section 3. The probabilization step is presented in Section 4. Finally, in Section 5 we carry out the quantization step of the enrichment. We conclude with an outline of further research directions.

2   Institutional preliminaries

In this paper, as a first step towards the full understanding of the proposed approach to enriching logics, we shall adopt an original variant of the notion of institution, without morphisms between models (c.f. [14]). For simplicity we shall just call it an institution, without any further qualifiers.

Definition 2.1. *Institution*
An *institution* is a tuple $I = \langle \mathbf{Sig}, \mathbf{Sen}, \mathbf{Mod}, \Vdash \rangle$ where:

- **Sig** is a category (of *signatures*);

- $\mathbf{Sen} : \mathbf{Sig} \rightarrow \mathbf{Set}$ is a (*formula*) functor;

- $\mathbf{Mod} : \mathbf{Sig} \rightarrow \mathbf{Cls^{op}}$ is a (*model*) functor; and

- $\Vdash = \{\Vdash_\Sigma\}_{\Sigma \in |\mathbf{Sig}|}$ is a family of (*satisfaction*) relations $\Vdash_\Sigma \subseteq \mathbf{Mod}(\Sigma) \times \mathbf{Sen}(\Sigma)$,

such that the following *satisfaction condition* holds, for every signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, every formula $\varphi \in \mathbf{Sen}(\Sigma)$, and every model $m' \in \mathbf{Mod}(\Sigma')$:

$$\mathbf{Mod}(\sigma)(m') \Vdash_\Sigma \varphi \text{ iff } m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi).$$

As usual, given a set $\Gamma \subseteq \mathbf{Sen}(\Sigma)$ of formulas and a model $m \in \mathbf{Mod}(\Sigma)$, we will write $m \Vdash_\Sigma \Gamma$ to denote the fact that $m \Vdash_\Sigma \varphi$ for every $\varphi \in \Gamma$. Mutatis

mutandis, given a set $M \subseteq \mathbf{Mod}(\Sigma)$ of models and a formula $\varphi \in \mathbf{Sen}(\Sigma)$, we will write $M \Vdash_\Sigma \varphi$ to denote the fact that $m \Vdash_\Sigma \varphi$ for every $m \in M$.

Recall that I induces a family $\vDash = \{\vDash_\Sigma\}_{\Sigma \in |\mathbf{Sig}|}$ of (*entailment*) relations $\vDash_\Sigma \subseteq \mathbf{Pw}(\mathbf{Sen}(\Sigma)) \times \mathbf{Sen}(\Sigma)$ defined by

$$\Gamma \vDash_\Sigma \varphi \text{ if, for every } m \in \mathbf{Mod}(\Sigma), \text{ if } m \Vdash_\Sigma \Gamma \text{ then } m \Vdash_\Sigma \varphi.$$

The notions of arrow between institutions are at least as important as the notion of institution itself. There is a rather extense and prolific bibliography on this subject, where various meaningful notions of arrow between institutions are proposed, used, examplified, and related with each other. A recent systematization of the field can be found in [14]. The notion of arrow that we will be using in this paper can be classified as a *comorphism* (or a *plain map* as originally named in [20], or also a *representation* as rebaptized in [25]). It is however a modified comorphism that maps models to sets of models, which can be explained as an instance of the general monad construction of [21].

The definition will take advantage of the usual covariant powerset endofunctor $\mathbf{Pw}$, in this case extended to classes, that is, $\mathbf{Pw} : \mathbf{Cls} \to \mathbf{Cls}$ is such that $\mathbf{Pw}(X) = 2^X$, and $\mathbf{Pw}(f : X \to X')$ maps each $Y \subseteq X$ to $f[Y] = \{f(x) : x \in Y\}$.

Definition 2.2. *Power-model comorphism*
Let $I = \langle \mathbf{Sig}, \mathbf{Sen}, \mathbf{Mod}, \Vdash \rangle$ and $I' = \langle \mathbf{Sig}', \mathbf{Sen}', \mathbf{Mod}', \Vdash' \rangle$ be institutions. A *power-model comorphism* from I to I' is a tuple $\langle \Phi, \alpha, \beta \rangle$ where:

- $\Phi : \mathbf{Sig} \to \mathbf{Sig}'$ is a (*signature translation*) functor;

- $\alpha : \mathbf{Sen} \to \mathbf{Sen}' \circ \Phi$ is a (*formula translation*) natural transformation; and

- $\beta : \mathbf{Mod}' \circ \Phi \to \mathbf{Pw} \circ \mathbf{Mod}$ is a (*power-model translation*) natural transformation,

such that the following *coherence condition* holds, for every signature $\Sigma \in |\mathbf{Sig}|$, every formula $\varphi \in \mathbf{Sen}(\Sigma)$, and every model $m' \in \mathbf{Mod}'(\Phi(\Sigma))$:

$$\beta_\Sigma(m') \Vdash_\Sigma \varphi \text{ iff } m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma(\varphi).$$

In the definition above, $\beta_\Sigma(m')$ is a set of models. Hence, the coherence condition states that $m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$ iff, for every $m \in \beta_\Sigma(m')$, $m \Vdash_\Sigma \varphi$. Clearly, the possibility that $\beta_\Sigma(m') = \emptyset$ is not excluded. In that case, m' must satisfy the translation via α of any formula whatsoever. A particularly interesting case corresponds to the situation when $\beta_\Sigma(m')$ is a singleton. If this happens for every model then we can recast the power-model natural transformation simply to $\beta : \mathbf{Mod}' \circ \Phi \to \mathbf{Mod}$, thus obtaining the usual notion of comorphism between institutions.

It is a well known fact that comorphisms preserve entailment. A further simple condition on the surjectivity of the translation of models can also guarantee the reflection of entailment. Such properties were studied in [7]. These

results can easily be lifted to the level of power-model comorphisms, as stated below. (Power-model) comorphisms compose in the usual way.

Proposition 2.3.
Let $I = \langle \boldsymbol{Sig}, \boldsymbol{Sen}, \boldsymbol{Mod}, \Vdash \rangle$ and $I' = \langle \boldsymbol{Sig'}, \boldsymbol{Sen'}, \boldsymbol{Mod'}, \Vdash' \rangle$ be institutions and $\langle \Phi, \alpha, \beta \rangle : I \to I'$ a power-model comorphism. Then

$$\Gamma \vDash_\Sigma \varphi \text{ implies } \alpha_\Sigma[\Gamma] \vDash_{\Phi(\Sigma)} \alpha_\Sigma(\varphi).$$

Additionally, if for each $m \in \boldsymbol{Mod}(\Sigma)$ there exists $m' \in \boldsymbol{Mod'}(\Phi(\Sigma))$ such that $\beta_\Sigma(m') = \{m\}$, then

$$\Gamma \vDash_\Sigma \varphi \text{ iff } \alpha_\Sigma[\Gamma] \vDash_{\Phi(\Sigma)} \alpha_\Sigma(\varphi).$$

Proof. Given the power-model comorphism, assume that $\Gamma \vDash_\Sigma \varphi$. If $m' \in \boldsymbol{Mod'}(\Phi(\Sigma))$ is such that $m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma[\Gamma]$ then, using the coherence condition of the power-model comorphism, we have that $\beta_\Sigma(m') \Vdash_\Sigma \Gamma$. Thus, by definition of entailment, it follows from $\Gamma \vDash_\Sigma \varphi$ that $\beta_\Sigma(m') \Vdash_\Sigma \varphi$. Using again the coherence condition, we now get $m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$. Therefore, $\alpha_\Sigma[\Gamma] \vDash_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$.

Assume now that the additional surjectivity condition holds and $\alpha_\Sigma[\Gamma] \vDash_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$. If $m \in \boldsymbol{Mod}(\Sigma)$ is such that $m \Vdash_\sigma \Gamma$ then $\{m\} \Vdash_\sigma \Gamma$. But we know that there exists $m' \in \boldsymbol{Mod'}(\Phi(\Sigma))$ such that $\beta_\Sigma(m') = \{m\}$. Thus, $\beta_\Sigma(m') \Vdash_\sigma \Gamma$ and it follows from the coherence condition of the power-model comorphism that $m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma[\Gamma]$. Hence, by definition of entailment, it follows that $m' \Vdash'_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$. Using again the coherence condition we obtain that $\beta_\Sigma(m') \Vdash_\sigma \varphi$, or equivalently, $m \Vdash_\sigma \varphi$. Therefore, $\Gamma \vDash_\Sigma \varphi$. $\qquad\square$

Hence, the existence of a power-model comorphism that fulfills the *surjectivity condition* stated in the second half of Proposition 2.3, for every signature, allows one to say that the target institution is a conservative extension of the source institution.

Note that, for comorphisms, the surjectivity condition stated above simply boils down to requiring that each map $\beta_\Sigma : \boldsymbol{Mod'}(\Phi(\Sigma)) \to \boldsymbol{Mod}(\Sigma)$ is surjective. It is also a trivial task to check that the the surjectivity condition is preserved by composing (power-model) comorphisms.

3 Globalization

3.1 Global institution

As a first step in our development, we aim at characterizing the exogenous enrichment of a given logic with a layer of global reasoning. For the purpose, let $I = \langle \boldsymbol{Sig}, \boldsymbol{Sen}, \boldsymbol{Mod}, \Vdash \rangle$ be the starting institution. We now proceed by defining the envisaged global institution I^g and then showing, by means of a power-model comorphism, that it extends I in a conservative way.

Definition 3.1. *The global institution*

The *global institution* based on I is $I^g = \langle \mathbf{Sig}^g, \mathbf{Sen}^g, \mathbf{Mod}^g, \Vdash^g \rangle$, defined as follows:

- $\mathbf{Sig}^g = \mathbf{Sig}$;

- $\mathbf{Sen}^g$ is the functor defined by

    - for each signature Σ, $\mathbf{Sen}^g(\Sigma)$ is the least set such that
        * $\mathbf{Sen}(\Sigma) \subseteq \mathbf{Sen}^g(\Sigma)$,
        * $(\boxminus \delta) \in \mathbf{Sen}^g(\Sigma)$ if $\delta \in \mathbf{Sen}^g(\Sigma)$, and
        * $(\delta_1 \sqsupset \delta_2) \in \mathbf{Sen}^g(\Sigma)$ if $\delta_1, \delta_2 \in \mathbf{Sen}^g(\Sigma)$,
    - for each signature morphism $\sigma : \Sigma \to \Sigma'$, the function $\mathbf{Sen}^g(\sigma)$ is defined inductively by
        * $\mathbf{Sen}^g(\sigma)(\varphi) = \mathbf{Sen}(\sigma)(\varphi)$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
        * $\mathbf{Sen}^g(\sigma)(\boxminus \delta) = (\boxminus \mathbf{Sen}^g(\sigma)(\delta))$, and
        * $\mathbf{Sen}^g(\sigma)(\delta_1 \sqsupset \delta_2) = (\mathbf{Sen}^g(\sigma)(\delta_1) \sqsupset \mathbf{Sen}^g(\sigma)(\delta_2))$;

- $\mathbf{Mod}^g$ is the functor defined by

    - for each signature Σ, $\mathbf{Mod}^g(\Sigma) = \{M : \emptyset \neq M \subseteq \mathbf{Mod}(\Sigma)\}$,
    - for each signature morphism $\sigma : \Sigma \to \Sigma'$, the contravariant map $\mathbf{Mod}^g(\sigma)$ is such that $\mathbf{Mod}^g(\sigma)(M') = \mathbf{Mod}(\sigma)[M']$; and

- for each signature Σ, $\Vdash_\Sigma^g$ is defined inductively by

    - $M \Vdash_\Sigma^g \varphi$ iff $M \Vdash_\Sigma \varphi$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
    - $M \Vdash_\Sigma^g (\boxminus \delta)$ iff $M \nVdash_\Sigma^g \delta$, and
    - $M \Vdash_\Sigma^g (\delta_1 \sqsupset \delta_2)$ iff $M \nVdash_\Sigma^g \delta_1$ or $M \Vdash_\Sigma^g \delta_2$.

Proposition 3.2.

I^g is an institution.

Proof. The functoriality of $\mathbf{Sen}^g$ and $\mathbf{Mod}^g$ is straightforward. The satisfaction condition of I^g can be established by a simple induction on formulas. The only interesting case is the base case, that we analyze below, the other cases being immediate by induction hypotheses.

Let $\sigma : \Sigma \to \Sigma'$ be a signature morphism, $\varphi \in \mathbf{Sen}(\Sigma)$ and $M' \in \mathbf{Mod}^g(\Sigma')$. Then, by definition of $\mathbf{Sen}^g$ and $\Vdash^g$, $M' \Vdash_{\Sigma'}^g \mathbf{Sen}^g(\sigma)(\varphi)$ iff $M' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)$, that is, $m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)$ for every $m' \in M'$. Therefore, using the satisfaction condition of I, this is equivalent to having $\mathbf{Mod}(\sigma)(m') \Vdash_\Sigma \varphi$ for every $m' \in M'$, that is, $\mathbf{Mod}^g(\sigma)(M') \Vdash_\Sigma^g \varphi$. $\qquad\square$

In the resulting logic, the connectives $\boxminus$ and $\sqsupset$ correspond to global negation and global implication, respectively. Other connectives can be easily introduced, like global conjunction $(\delta_1 \sqcap \delta_2) \equiv \boxminus(\delta_1 \sqsupset (\boxminus \delta_2))$. If the base institution

has a negation $\neg$ and an implication $\Rightarrow$, which can be understood as local, these connectives do not collapse with the global ones. For implication, for instance, we have that $\{(\varphi_1 \Rightarrow \varphi_2)\} \vDash^g_\Sigma (\varphi_1 \sqsupset \varphi_2)$, but the converse does not hold in general, given two base formulas $\varphi_1, \varphi_2 \in \mathbf{Sen}(\Sigma)$. Namely, assume that I is the institution of classical propositional logic, $\pi_1, \pi_2 \in \Sigma$ are two propositional symbols and $v_1, v_2 \in \mathbf{Mod}(\Sigma)$ are two classical valuations such that $v_1(\pi_1) = 0$, $v_1(\pi_2) = 0$, $v_2(\pi_1) = 1$ and $v_2(\pi_2) = 0$. Then, $\{v_1, v_2\} \Vdash^g_\Sigma (\pi_1 \sqsupset \pi_2)$ but $\{v_1, v_2\} \nVdash^g_\Sigma (\pi_1 \Rightarrow \pi_2)$. The logic resulting from globalizing classical propositional logic was carefully studied in [19], where a sound and complete calculus could be obtained by capitalizing on a calculus for classical logic and adding an axiomatization of the new connectives. It is an open question if the same sort of enterprise can be done in the general case. However, it seems possible to generalize the technique used there, at least if the base logic enjoys an expressibility property analogous to the disjunctive normal form of classical logic.

More interesting, at the moment, is to establish the precise relationship between the institutions I and I^g.

Definition 3.3. *The C^g power-model comorphism*

The power-model comorphism $C^g : I \to I^g$ is such that $C^g = \langle \Phi^g, \alpha^g, \beta^g \rangle$ with:

- Φ^g is the identity functor on $\mathbf{Sig}$;

- for each signature Σ, α^g_Σ translates each formula $\varphi \in \mathbf{Sen}(\Sigma)$ to φ itself, now seen as a formula in $\mathbf{Sen}^g(\Sigma)$; and

- for each signature Σ, β^g_Σ translates each model $M \in \mathbf{Mod}^g(\Sigma)$ to M itself, now seen as a set of $\mathbf{Mod}(\Sigma)$ models.

It is straightforward to prove that C^g is indeed a power-model comorphism, which furthermore fulfills the surjectivity condition of Proposition 2.3.

Proposition 3.4.

C^g *is a power-model comorphism, and fulfills the surjectivity condition.*

Proof. The naturality of the transformations α^g and β^g is straightforward. Namely, given a signature morphism $\sigma : \Sigma \to \Sigma'$ and $\varphi \in \mathbf{Sen}(\Sigma)$, we have that $\mathbf{Sen}^g(\sigma)(\alpha^g_\Sigma(\varphi)) = \mathbf{Sen}^g(\sigma)(\varphi) = \mathbf{Sen}(\sigma)(\varphi) = \alpha^g_{\Sigma'}(\mathbf{Sen}(\sigma)(\varphi))$. Similarly, given $M' \in \mathbf{Mod}^g(\Sigma')$, we have $\mathbf{Pw}(\mathbf{Mod}(\sigma))(\beta^g_{\Sigma'}(M')) = \mathbf{Pw}(\mathbf{Mod}(\sigma))(M') = \mathbf{Mod}(\sigma)[M'] = \mathbf{Mod}^g(\sigma)(M') = \beta^g_\Sigma(\mathbf{Mod}^g(\sigma)(M'))$.

The coherence condition is trivial. $\qquad\square$

As a corollary, by Proposition 2.3, C^g shows that I^g is in fact a conservative extension of I.

4 Probabilization

Let us now characterize the exogenous enrichment of a given logic with probabilistic reasoning. We start by introducing the essential definitions and properties of probability spaces.

4.1 Probability spaces

A *probability space* over a non-empty set Ω of *outcomes* is a pair $P = \langle \mathcal{B}, \mu \rangle$ where:

- $\mathcal{B}$ is a Borel field over Ω, that is, $\mathcal{B} \subseteq 2^\Omega$ contains Ω and is closed for complements and countable unions; and

- $\mu : \mathcal{B} \to [0,1]$ is a measure with unitary mass, that is, $\mu(M) = 1$ and $\mu(\bigcup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \mu(B_i)$ if $\{B_i\}_{i=1}^\infty \subseteq \mathcal{B}$ is a family of pairwise disjoint sets.

In due course, we will need to map probability spaces along functions on their outcomes. Let $f : U \to U'$ be a function, $\Omega \subseteq U$ and $P = \langle \mathcal{B}, \mu \rangle$ a probability space over Ω. The *image of P along f* is the probability space $f(P) = \langle \mathcal{B}', \mu' \rangle$ over $\Omega' = f[\Omega]$ where:

- $\mathcal{B}' = \{B' \subseteq \Omega' : f^{-1}(B') \cap \Omega \in \mathcal{B}\}$; and

- μ' is such that $\mu'(B') = \mu(f^{-1}(B') \cap \Omega)$.

The image of a probability space is of course well defined.

4.2 Probabilistic institution

Let $I = \langle \mathbf{Sig}, \mathbf{Sen}, \mathbf{Mod}, \Vdash \rangle$ be the starting institution. As before, we shall first define the envisaged probabilistic institution I^p and then show, using power-model comorphisms, that I^p extends conservatively both I and I^g. Indeed, the whole idea is to work with sets of models of the original institution, as in the global case, but now endow them with a certain probability measure. Of course, also the linguistic resources of the logic will be augmented to allow probabilistic assertions and resoning. For that sake, we assume fixed a set X of variables. We shall also denote by R the set of all computable real numbers.

Definition 4.1. *The probabilistic institution*

The *probabilistic institution* based on I is $I^p = \langle \mathbf{Sig}^p, \mathbf{Sen}^p, \mathbf{Mod}^p, \Vdash^p \rangle$, defined as follows:

- $\mathbf{Sig}^p = \mathbf{Sig}$;

- $\mathbf{Sen}^p$ is the functor defined by

  - for each signature Σ, $\mathbf{Sen}^p(\Sigma)$ is the least set such that
    * $\mathbf{Sen}(\Sigma) \subseteq \mathbf{Sen}^p(\Sigma)$,
    * $(\boxminus \delta) \in \mathbf{Sen}^p(\Sigma)$ if $\delta \in \mathbf{Sen}^p(\Sigma)$,
    * $(\delta_1 \sqsupset \delta_2) \in \mathbf{Sen}^p(\Sigma)$ if $\delta_1, \delta_2 \in \mathbf{Sen}^p(\Sigma)$, and
    * $(t_1 \leq t_2) \in \mathbf{Sen}^p(\Sigma)$ if $t_1, t_2 \in \mathit{Trm}^p(\Sigma)$,

    where $\mathit{Trm}^p(\Sigma)$ is the least set (of *probabilistic terms*) such that
    * $X, R \subseteq \mathit{Trm}^p(\Sigma)$,

* $(\int \varphi) \in Trm^p(\Sigma)$ if $\varphi \in \mathbf{Sen}(\Sigma)$, and
* $(t_1 + t_2), (t_1.t_2) \in Trm^p(\Sigma)$ if $t_1, t_2 \in Trm^p(\Sigma)$,

 – for each signature morphism $\sigma : \Sigma \to \Sigma'$, the function $\mathbf{Sen}^p(\sigma)$ is defined inductively by

* $\mathbf{Sen}^p(\sigma)(\varphi) = \mathbf{Sen}(\sigma)(\varphi)$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
* $\mathbf{Sen}^p(\sigma)(\boxminus \delta) = (\boxminus \mathbf{Sen}^p(\sigma)(\delta))$,
* $\mathbf{Sen}^p(\sigma)(\delta_1 \sqsupset \delta_2) = (\mathbf{Sen}^p(\sigma)(\delta_1) \sqsupset \mathbf{Sen}^p(\sigma)(\delta_2))$, and
* $\mathbf{Sen}^p(\sigma)(t_1 \leq t_2) = (Trm^p(\sigma)(t_1) \leq Trm^p(\sigma)(t_2))$,

 where $Trm^p(\sigma) : Trm^p(\Sigma) \to Trm^p(\Sigma')$ is the function defined inductively by

* $Trm^p(\sigma)(x) = x$, for $x \in X$,
* $Trm^p(\sigma)(r) = r$, for $r \in R$,
* $Trm^p(\sigma)(\int \varphi) = (\int \mathbf{Sen}(\sigma)(\varphi))$,
* $Trm^p(\sigma)(t_1 + t_2) = (Trm^p(\sigma)(t_1) + Trm^p(\sigma)(t_2))$, and
* $Trm^p(\sigma)(t_1.t_2) = (Trm^p(\sigma)(t_1). Trm^p(\sigma)(t_2))$;

- **$\mathbf{Mod}^p$** is the functor defined by

 – for each signature Σ, $\mathbf{Mod}^p(\Sigma)$ is the class of all triples $\langle M, P, \rho \rangle$ where

* $\emptyset \neq M \subseteq \mathbf{Mod}(\Sigma)$,
* $P = \langle \mathcal{B}, \mu \rangle$ is a *Nilsson structure*[1], that is, a probability space over M such that $\{m \in M : m \Vdash_\Sigma \varphi\} \in \mathcal{B}$ for every $\varphi \in \mathbf{Sen}(\Sigma)$, and
* $\rho : X \to \mathbb{R}$ is an assignment,

 – for each $\sigma : \Sigma \to \Sigma'$, the contravariant map $\mathbf{Mod}^p(\sigma)$ is such that $\mathbf{Mod}^p(\sigma)(\langle M', P', \rho' \rangle) = \langle \mathbf{Mod}(\sigma)[M'], \mathbf{Mod}(\sigma)(P'), \rho' \rangle$; and

- for each signature Σ and $S = \langle M, \langle \mathcal{B}, \mu \rangle, \rho \rangle \in \mathbf{Mod}^p(\Sigma)$, $\Vdash_\Sigma^p$ is defined inductively by

 – $S \Vdash_\Sigma^p \varphi$ iff $M \Vdash_\Sigma \varphi$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
 – $S \Vdash_\Sigma^p (\boxminus \delta)$ iff $S \nVdash_\Sigma^p \delta$,
 – $S \Vdash_\Sigma^p (\delta_1 \sqsupset \delta_2)$ iff $S \nVdash_\Sigma^p \delta_1$ or $S \Vdash_\Sigma^p \delta_2$, and
 – $S \Vdash_\Sigma^p (t_1 \leq t_2)$ iff $[\![t_1]\!]^S \leq [\![t_2]\!]^S$,

 where the denotation of probabilistic terms $[\![\_]\!]^S : Trm^p(\Sigma) \to \mathbb{R}$ is defined inductively by

 – $[\![x]\!]^S = \rho(x)$, for $x \in X$,
 – $[\![r]\!]^S = r$, for $r \in R$,

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[1]Nilsson structures provide the semantic basis for several probabilistic logics used for reasoning with uncertainty [23, 10, 24].

- $[\![\int\varphi]\!]^S = \mu(\{m \in M : m \Vdash_\Sigma \varphi\})$,
- $[\![t_1 + t_2]\!]^S = [\![t_1]\!]^S + [\![t_2]\!]^S$, and
- $[\![t_1.t_2]\!]^S = [\![t_1]\!]^S.[\![t_2]\!]^S$,

with the usual order, addition and multiplication on the real numbers.

Proposition 4.2.

I^p is an institution.

Proof. The functoriality of $\mathbf{Sen}^p$ is straightforward. Concerning $\mathbf{Mod}^p$, and given $\sigma : \Sigma \to \Sigma'$, just note that indeed $\langle M, P, \rho \rangle = \mathbf{Mod}^p(\sigma)(\langle M', P', \rho' \rangle) \in \mathbf{Mod}^p(\Sigma)$. In particular, given $\varphi \in \mathbf{Sen}(\Sigma)$, $\{m \in M : m \Vdash_\Sigma \varphi\}$ is measurable, because $M = \mathbf{Mod}(\sigma)[M']$ and it is also measurable

$\mathbf{Mod}(\sigma)^{-1}(\{m \in M : m \Vdash_\Sigma \varphi\}) \cap M' =$
$\{m' \in M' : \mathbf{Mod}(\sigma)(m') \Vdash_\Sigma \varphi\} = \{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)\}.$

The satisfaction condition of I^p can be established by a simple induction on formulas. The only interesting case is now that of inequalities. Let $t \in Trm^p(\Sigma)$. For ease of notation let also $S' = \langle M', P', \rho' \rangle$ and $P' = \langle \mathcal{B}', \mu' \rangle$, $S = \mathbf{Mod}^p(\sigma)(S') = \langle M, P, \rho \rangle$ and $P = \mathbf{Mod}(\sigma)(P') = \langle \mathcal{B}, \mu \rangle$. We need to show that $[\![t]\!]^S = [\![Trm^p(\sigma)(t)]\!]^{S'}$. This fact can be shown by a simple induction on terms. The interesting case concerns the terms $(\int\varphi)$. Given $\varphi \in \mathbf{Sen}(\Sigma)$, using the definitions of term denotation and of image of a probability space, the satisfaction condition of the base institution I, and the definition of term translation, along with a little set-theoretical manipulation, we have that

$[\![\int\varphi]\!]^S = \mu(\{m \in M : m \Vdash_\Sigma \varphi\}) =$
$\mu'(\mathbf{Mod}(\sigma)^{-1}(\{m \in M : m \Vdash_\Sigma \varphi\}) \cap M') =$
$\mu'(\{m' \in M' : \mathbf{Mod}(\sigma)(m') \Vdash_\Sigma \varphi\}) =$
$\mu'(\{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)\}) = [\![\int\mathbf{Sen}(\sigma)(\varphi)]\!]^{S'} =$
$[\![Trm^p(\sigma)(\int\varphi)]\!]^{S'}.$ $\qquad\square$

The term $(\int\varphi)$ denotes the probability of φ, interpreted as the probability of the models of the base institution that satisfy φ. The logic resulting from the probabilization of classical propositional logic was carefully studied in [19], where a sound and weak complete calculus could be obtained. The calculus extends the one for the globalization of classical propositional logic by exploring the interplay between the classical connectives and probability, and uses an oracle rule for reasoning with real numbers. Although the logic enjoys the deduction theorem with respect to global implication, strong completeness is out of reach simply because the logic is not compact. Take, for instance, $\Delta = \{(r \le x) : r < \frac{1}{2}\}$. Clearly, $\Delta \vDash_\Sigma^p (\frac{1}{2} \le x)$ but no finite subset of Δ does. Another interesting relevant remark is the fact that the operators $\square$ and $\lozenge$ defined by $(\square\varphi) \equiv (1 \le \int\varphi)$ and $(\lozenge\varphi) \equiv \boxminus(\int\varphi \le 0)$ behave as normal modalities.

In the general case depicted here, however, our aim is to establish the precise relationship between the institutions I, I^g and I^p.

Definition 4.3. *The C^{gp} comorphism*

The comorphism $C^{gp} : I^g \to I^p$ is such that $C^{gp} = \langle \Phi^{gp}, \alpha^{gp}, \beta^{gp} \rangle$ with:

- Φ^{gp} the identity functor on **Sig**;

- for each signature Σ, α_Σ^{gp} translates each formula $\delta \in \mathbf{Sen}^g(\Sigma)$ to δ itself, now seen as a formula in $\mathbf{Sen}^p(\Sigma)$; and

- for each signature Σ, β_Σ^{gp} translates each $\langle M, P, \rho \rangle \in \mathbf{Mod}^p(\Sigma)$ to $M \in \mathbf{Mod}^g(\Sigma)$.

Proposition 4.4.

C^{gp} is a comorphism, and fulfills the surjectivity condition.

Proof. The naturality of the transformations α^{gp} and β^{gp} is straightforward. The coherence condition is trivial. As for surjectivity, given a non-empty $M \subseteq \mathbf{Mod}(\Sigma)$ and $m \in M$, take for instance the triple $S = \langle M, \langle \mathcal{B}, \mu \rangle, \rho \rangle$ where $\mathcal{B} = 2^M$, $\mu(B) =$ and ρ is any assignment. It is easy to see that $\langle \mathcal{B}, \mu \rangle$ is a probability space over M, and $\beta_\Sigma^{gp}(S) = M$. $\qquad\square$

As a corollary, by Proposition 2.3 and the observations therein, C^{gp} shows that I^p is a conservative extension of I^g. By transitivity, I^p is also a conservative extension of I. Indeed, by composition, we also obtain a power-model comorphism $C^p = C^{gp} \circ C^g : I \to I^p$ that fulfills the surjectivity condition.

5 Quantization

Finally, we turn our attention to the exogenous enrichment of a given logic with quantum reasoning. In order to materialize the key idea of adopting superpositions of models of the given logic as the models of the envisaged quantum logic, let us start by recalling the essential concepts of quantum systems.

5.1 Quantum systems

Let us recall the relevant postulates of quantum physics (following closely [22]) and set up some important mathematical structures.

Postulate 5.1. *Associated to any isolated quantum system is a Hilbert space[2]. The state of the system is completely described by a unit vector $|w\rangle$ in the Hilbert space.*

For example, a quantum bit or *qubit* is associated to a Hilbert space of dimension two: a state of a qubit is a vector $\alpha_0|0\rangle + \alpha_1|1\rangle$ where $\alpha_0, \alpha_1 \in \mathbb{C}$ and $|\alpha_0|^2 + |\alpha_1|^2 = 1$. That is, the quantum state is a *superposition* of the two

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[2]Recall that a Hilbert space is a complex vector space with inner product which is complete for the induced norm. Its elements are usually denoted using the *ket* Dirac notation $|w\rangle$.

classical states $|0\rangle$ and $|1\rangle$ of a classical bit. Therefore, from a logical point of view, representing the qubit by a propositional constant, a *quantum valuation* is a superposition of the two classical valuations.

Postulate 5.2. *The Hilbert space associated to a quantum system composed of finitely many independent component systems is the tensor product[3] of the component Hilbert spaces.*

For instance, a system composed of two independent qubits is associated to a Hilbert space of dimension four: a state of such a system is a vector $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ where $\alpha_{00}, \alpha_{10}, \alpha_{01}, \alpha_{11} \in \mathbb{C}$ and $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$. Again, representing the two qubits by two propositional constants, a *quantum valuation* is a superposition of the four classical valuations. So, the Hilbert space of the system composed of two independent qubits is indeed the tensor product of the two Hilbert spaces, each corresponding to a single qubit.

Since we want to work with an arbitrary set of qubits, we will need the following general construction. Given a nonempty set E, the *free Hilbert space* over E is $\mathcal{H}(E)$, the *inner product* space over $\mathbb{C}$ defined as follows:

- each element is a map $|w\rangle : E \to \mathbb{C}$ such that

    - $\{e \in E : |w\rangle(e) \neq 0\}$ is countable, and
    - $\sum_{e \in E} ||w\rangle(e)|^2 < \infty$;

- addition, scalar multiplication and inner product are defined by

    - $|w_1\rangle + |w_2\rangle = \lambda e. |w_1\rangle(e) + |w_2\rangle(e)$,
    - $\alpha|w\rangle = \lambda e. \alpha|w\rangle(e)$, and
    - $\langle w_1|w_2\rangle = \sum_{e \in E} |w_1\rangle(e)\overline{|w_2\rangle(e)}$.

As usual, the inner product induces the *norm* $|||w\rangle|| = \sqrt{\langle w|w\rangle}$, which on its turn induces the *distance* $d(|w_1\rangle, |w_2\rangle) = |||w_1\rangle - |w_2\rangle||$. Since $\mathcal{H}(E)$ is complete for this distance, $\mathcal{H}(E)$ is a Hilbert space. Clearly, $\{|e\rangle : e \in E\}$ is an *orthonormal basis* of $\mathcal{H}(E)$, where $|e\rangle(e) = 1$ and $|e\rangle(e') = 0$ for every $e' \neq e$. A *unit vector* of $\mathcal{H}(E)$ is just a vector $|w\rangle \in \mathcal{H}(E)$ such that $|||w\rangle|| = 1$.

Let Q be the set of qubits in hand. If there are no dependencies between the qubits then the system is described by the Hilbert space $\mathcal{H}(2^Q)$, where 2^Q is the set of all classical valuations. However, in many cases, we will be given a finite partition $\mathcal{S} = \{Q_1, \ldots, Q_n\}$ of Q, giving rise to n independent subsystems. In the sequel, we will use $\bigcup \mathcal{S}$ to denote the set $\{\bigcup_{Q_i \in \mathcal{R}} Q_i : \mathcal{R} \subseteq \mathcal{S}\}$. Moreover, it may also be that the qubits Q_i of each isolated subsystem are also

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[3]Recall that the tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is the Hilbert space composed by the pairs $|w_1\rangle \otimes |w_2\rangle$ such that the following equalities hold for all $\alpha \in \mathbb{C}, |w_1\rangle, |w_1'\rangle \in \mathcal{H}_1$ and $|w_2\rangle, |w_2'\rangle \in \mathcal{H}_2$: $\alpha(|w_1\rangle \otimes |w_2\rangle) = \alpha|w_1\rangle \otimes |w_2\rangle = |w_1\rangle \otimes \alpha|w_2\rangle$; $(|w_1\rangle + |w_1'\rangle) \otimes |w_2\rangle = (|w_1\rangle \otimes |w_2\rangle) + (|w_1'\rangle \otimes |w_2\rangle)$; $|w_1\rangle \otimes (|w_2\rangle + |w_2'\rangle) = (|w_1\rangle \otimes |w_2\rangle) + (|w_1\rangle \otimes |w_2'\rangle)$. The operation is clearly associative.

constrained and some of the classical valuations in 2^{Q_i} are impossible. Any set $V \subseteq 2^Q$ of *admissible* classical valuations induces a set of admissible classical valuations for each subsystem, that is, $V_i = \{v_i : v \in V\}$ with $v_i = v|_{Q_i}$. Analogously, we will use v_R to denote the restriction $v|_R$ of a valuation v to $R \in \bigcup S$, and $V_R = \{v_R : v \in V\}$. Then, the space describing the corresponding quantum system will be the tensor product $\bigotimes_{i=1}^n \mathcal{H}(V_i)$. Still, note that although $(2^Q)_i = 2^{Q_i}$ and $2^Q = \prod_{i=1}^n 2^{Q_i}$, in general $V \subsetneq \prod_{i=1}^n V_i$. Moreover, although $\mathcal{H}(2^Q) = \bigotimes_{i=1}^n \mathcal{H}(2^{Q_i}) = \bigotimes_{i=1}^n \mathcal{H}(\prod_{i=1}^n 2^{Q_i})$, in general we have that $\mathcal{H}(V) \subsetneq \bigotimes_{i=1}^n \mathcal{H}(V_i) \subsetneq \mathcal{H}(\prod_{i=1}^n V_i)$.

Hence, we should only consider quantum states of $\bigotimes_{i=1}^n \mathcal{H}(V_i)$ that are compatible with V. Given the subspace relations stated above, we shall call a *structured quantum state* over V and $\mathcal{S}$ to a family $|w\rangle = \{|w_i\rangle\}_{i=1}^n$ such that:

- each $|w_i\rangle$ is a unit vector of $\mathcal{H}(V_i)$; and

- $\langle v | \bigotimes_{i=1}^n |w_i\rangle\rangle = \prod_{i=1}^n \langle v_i | w_i\rangle = 0$ if $v \notin V$.

Note that it is easy to identify $\bigotimes_{i=1}^n |w_i\rangle$ with a unique unit vector in $\mathcal{H}(V)$ since all the amplitudes on valuations not in V are null. Hence, by abuse of notation, we shall also use $|w\rangle$ to denote $\bigotimes_{i=1}^n |w_i\rangle$.

Now, we turn our attention to the postulates concerning measurements of physical quantities.

Postulate 5.3. *Every measurable physical quantity of an isolated quantum system is described by an observable[4] acting on its Hilbert space.*

Postulate 5.4. *The possible outcomes of the measurement of a physical quantity are the eigenvalues of the corresponding observable. When the physical quantity is measured using observable A on a system in a state $|w\rangle$, the resulting outcomes are ruled by the probability space $\mathrm{Prob}_{|w\rangle}^A = \langle \Omega, \mathcal{B}|_\Omega, \mu_{|w\rangle}^A \rangle$ where in the case of a countable spectrum*

$$\mu_{|w\rangle}^A = \lambda B. \sum_{\lambda \in \Omega} \chi_B(\lambda) |P_\lambda|w\rangle|^2.$$

For the applications we have in mind in quantum computation and information, only *logical projective measurements* are relevant. In general, the stochastic result of making a logical projective measurement of the system at a structured quantum state $|w\rangle$ determined as above is fully described by the probability space $\langle 2^V, \mu_{|w\rangle}\rangle$ over V where $\mu_{|w\rangle}(B) = \sum_{v \in B} |\langle v|w\rangle|^2$ for every $B \subseteq 2^V$.

In the sequel, we will need to be able to map quantum systems and states across qubit maps. Let $f : U \to U'$, $Q \subseteq U$ and $Q' = f[Q]$. Then, the function

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[4]Recall that an observable is a Hermitian operator such that the direct sum of its eigensubspaces coincides with the underlying Hilbert space. Since the operator is Hermitian, its spectrum Ω (the set of its eigenvalues) is a subset of $\mathbb{R}$. For each $\lambda \in \Omega$, we denote the corresponding eigensubspace by E_λ and the projector onto E_λ by P_λ.

$f^\bullet : 2^{Q'} \to 2^Q$ defined by $f^\bullet(v')(q) = v'(f(q))$ is injective: if $f^\bullet(v_1') = f^\bullet(v_2')$ then, for each $q \in Q$, $v_1'(f(q)) = v_2'(f(q))$, which implies that $v_1' = v_2'$ since $Q' = f[Q]$. Hence, $f^\bullet$ establishes a bijection between any given set of classical valuations $V' \subseteq 2^{Q'}$ and $V = f^\bullet[V'] \subseteq 2^Q$. Therefore, $f^\bullet$ also establishes an isomorphism between the Hilbert spaces $\mathcal{H}(V')$ and $\mathcal{H}(V)$ obtained my mapping $|w'\rangle \in \mathcal{H}(V')$ to $|w\rangle = f^\bullet(|w'\rangle)$ such that $|w\rangle(f^\bullet(v')) = |w'\rangle(v')$. Moreover, note that every finite partition $\mathcal{S}' = \{Q_1', \ldots, Q_n'\}$ induces a partition $\mathcal{S} = f^{-1}[\mathcal{S}'] = \{Q_1, \ldots, Q_n\}$ of Q with each $Q_i = f^{-1}(Q_i') \cap Q$. Hence, since surjectivity gurantees that each $Q_i' = f[Q_i]$, the Hilbert space isomorphism established in the preceding paragraph by $f^\bullet$ also applies to the subsystems, that is, $\mathcal{H}(V_i')$ and $\mathcal{H}(V_i)$ are isomorphic.

5.2   Quantum institution

We now characterize the exogenous enrichment of a given institution $I = \langle \mathbf{Sig}, \mathbf{Sen}, \mathbf{Mod}, \Vdash \rangle$ with quantum reasoning. As in the previous cases, we shall first define the envisaged quantum institution I^q and then characterize its relationship to I, as well as to the institutions previously built. To this end, qubits will be selected formulas of the original logic, that induce upon observation a probability distribution on models of the original institution. The notation I^q is a little abusive here, since the enrichment will be parameterized by a functor that chooses the qubits of interest. Hence, we consider fixed a functor $\mathbf{Qb} : \mathbf{Sig} \to \mathbf{Set}$ such that, for every signature Σ, $\mathbf{Qb}(\Sigma) \subseteq \mathbf{Sen}(\Sigma)$ and, for every signature morphism $\sigma : \Sigma \to \Sigma'$, $\mathbf{Qb}(\sigma) = \mathbf{Sen}(\sigma)|_{\mathbf{Qb}(\Sigma)}$ and $\mathbf{Sen}(\sigma)[\mathbf{Qb}(\Sigma)] = \mathbf{Qb}(\Sigma')$. Note that $\mathbf{Sen}(\sigma)$ is required to be surjective on qubits, and that this requirement is essential in the subsequent development of the I^q institution.

Clearly, models of the given institution induce classical valuations on the qubits. We denote by $V_\Sigma : \mathbf{Mod}(\Sigma) \to 2^{\mathbf{Qb}(\Sigma)}$ defined, for each qubit $\varphi \in \mathbf{Qb}(\Sigma)$, by

$$V_\Sigma(m)(\varphi) = \begin{cases} 1 & \text{if } m \Vdash_\Sigma \varphi \\ 0 & \text{otherwise} \end{cases} .$$

To fulfill the original idea of working with quantum *superpositions* of models of the original institution, we will have to restrict our attention to sets of models $M \subseteq \mathbf{Mod}(\Sigma)$ on which V_Σ is injective, that is, if $m_1, m_2 \in M$ and $m_1 \neq m_2$ then $V_\Sigma(m_1) \neq V_\Sigma(m_2)$. In this way, we have a bijection between M and $V_\Sigma[M]$.

Given $A \subseteq F \subseteq \mathbf{Qb}(\Sigma)$, we shall denote by $v_A^F \in 2^F$ the classical valuation of the qubits in F defined by

$$v_A^F(\varphi) = \begin{cases} 1 & \text{if } \varphi \in A \\ 0 & \text{if } \varphi \in F \setminus A \end{cases} .$$

The syntax of the logic will also be augmented, not only with probabilistic reasoning, but also in order to allow us to manipulate complex amplitudes and to talk about qubit independence. Hence, besides for the set X of (real valued) variables, we also assume fixed a set Z of (complex valued) variables. As before, R denotes the set of computable real numbers.

Definition 5.5. *The quantum institution*

The *quantum institution* based on I (and **Qb**) is $I^q = \langle \mathbf{Sig}^q, \mathbf{Sen}^q, \mathbf{Mod}^q, \Vdash^q \rangle$, defined as follows:

- $\mathbf{Sig}^q = \mathbf{Sig}$;

- $\mathbf{Sen}^q$ is the functor defined by

  - for each signature Σ, $\mathbf{Sen}^q(\Sigma)$ is the least set such that
    * $\mathbf{Sen}(\Sigma) \subseteq \mathbf{Sen}^q(\Sigma)$,
    * $(\boxminus \delta) \in \mathbf{Sen}^q(\Sigma)$ if $\delta \in \mathbf{Sen}^q(\Sigma)$,
    * $(\delta_1 \sqsupset \delta_2) \in \mathbf{Sen}^q(\Sigma)$ if $\delta_1, \delta_2 \in \mathbf{Sen}^q(\Sigma)$,
    * $[F] \in \mathbf{Sen}^q(\Sigma)$ if $F \subseteq \mathbf{Qb}(\Sigma)$, and
    * $(t_1 \le t_2) \in \mathbf{Sen}^q(\Sigma)$ if $t_1, t_2 \in \mathit{Trm}_R^q(\Sigma)$,

    where the sets $\mathit{Trm}_R^q(\Sigma)$ and $\mathit{Trm}_C^q(\Sigma)$ (of *real valued* and *complex valued* terms, respectively) are defined by mutual induction as follows:
    * $X, R \subseteq \mathit{Trm}_R^q(\Sigma)$,
    * $(\int \varphi) \in \mathit{Trm}_R^q(\Sigma)$ if $\varphi \in \mathbf{Sen}(\Sigma)$,
    * $(t_1 + t_2), (t_1.t_2) \in \mathit{Trm}_R^q(\Sigma)$ if $t_1, t_2 \in \mathit{Trm}_R^q(\Sigma)$, and
    * $\mathrm{Re}(u), \mathrm{Im}(u), \arg(u), |u| \in \mathit{Trm}_R^q(\Sigma)$ if $u \in \mathit{Trm}_C^q(\Sigma)$,

    and
    * $Z \subseteq \mathit{Trm}_C^q(\Sigma)$,
    * $|\top\rangle_{FA} \in \mathit{Trm}_C^q(\Sigma)$ if $A \subseteq F \subseteq \mathbf{Qb}(\Sigma)$,
    * $(t_1 + it_2), (t_1.e^{it_2}) \in \mathit{Trm}_C^q(\Sigma)$ if $t_1, t_2 \in \mathit{Trm}_R^q(\Sigma)$,
    * $\overline{u} \in \mathit{Trm}_C^q(\Sigma)$ if $u \in \mathit{Trm}_C^q(\Sigma)$,
    * $(u_1 + u_2), (u_1.u_2) \in \mathit{Trm}_C^q(\Sigma)$ if $u_1, u_2 \in \mathit{Trm}_C^q(\Sigma)$, and
    * $(\varphi \rhd u_1; u_2) \in \mathit{Trm}_C^q(\Sigma)$ if $\varphi \in \mathbf{Sen}(\Sigma)$ and $u_1, u_2 \in \mathit{Trm}_C^q(\Sigma)$,

  - for each signature morphism $\sigma : \Sigma \to \Sigma'$, the function $\mathbf{Sen}^q(\sigma)$ is defined inductively by
    * $\mathbf{Sen}^q(\sigma)(\varphi) = \mathbf{Sen}(\sigma)(\varphi)$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
    * $\mathbf{Sen}^q(\sigma)(\boxminus \delta) = (\boxminus \mathbf{Sen}^q(\sigma)(\delta))$,
    * $\mathbf{Sen}^q(\sigma)(\delta_1 \sqsupset \delta_2) = (\mathbf{Sen}^q(\sigma)(\delta_1) \sqsupset \mathbf{Sen}^q(\sigma)(\delta_2))$,
    * $\mathbf{Sen}^q(\sigma)([F]) = [\mathbf{Sen}(\sigma)[F]]$, and
    * $\mathbf{Sen}^q(\sigma)(t_1 \le t_2) = (\mathit{Trm}_R^q(\sigma)(t_1) \le \mathit{Trm}_R^q(\sigma)(t_2))$,

    where $\mathit{Trm}_R^q(\sigma) : \mathit{Trm}_R^q(\Sigma) \to \mathit{Trm}_R^q(\Sigma')$ and $\mathit{Trm}_C^q(\sigma) : \mathit{Trm}_C^q(\Sigma) \to \mathit{Trm}_C^q(\Sigma')$ are the functions defined by mutual induction as follows:
    * $\mathit{Trm}_R^q(\sigma)(x) = x$, for $x \in X$,
    * $\mathit{Trm}_R^q(\sigma)(r) = r$, for $r \in R$,
    * $\mathit{Trm}_R^q(\sigma)(\int \varphi) = (\int \mathbf{Sen}(\sigma)(\varphi))$,
    * $\mathit{Trm}_R^q(\sigma)(t_1 + t_2) = (\mathit{Trm}_R^q(\sigma)(t_1) + \mathit{Trm}_R^q(\sigma)(t_2))$,

* $Trm_R^q(\sigma)(t_1.t_2) = (Trm_R^q(\sigma)(t_1).Trm_R^q(\sigma)(t_2))$,
* $Trm_R^q(\sigma)(\mathrm{Re}(u)) = \mathrm{Re}(Trm_C^q(\sigma)(u))$,
* $Trm_R^q(\sigma)(\mathrm{Im}(u)) = \mathrm{Im}(Trm_C^q(\sigma)(u))$,
* $Trm_R^q(\sigma)(\arg(u)) = \arg(Trm_C^q(\sigma)(u))$, and
* $Trm_R^q(\sigma)(|u|) = |Trm_C^q(\sigma)(u)|$,

and

* $Trm_C^q(\sigma)(z) = z$, for $z \in Z$,
* $Trm_C^q(\sigma)(|\top\rangle_{FA}) = |\top\rangle_{\mathbf{Sen}(\sigma)[F],\mathbf{Sen}(\sigma)(A)}$,
* $Trm_C^q(\sigma)(t_1 + it_2) = (Trm_R^q(\sigma)(t_1) + i\,Trm_R^q(\sigma)(t_2))$,
* $Trm_C^q(\sigma)(t_1.e^{it_2}) = (Trm_R^q(\sigma)(t_1).e^{i\,Trm_R^q(\sigma)(t_2)})$,
* $Trm_C^q(\sigma)(\overline{u}) = \overline{Trm_C^q(\sigma)(u)}$,
* $Trm_C^q(\sigma)(u_1 + u_2) = (Trm_C^q(\sigma)(u_1) + Trm_C^q(\sigma)(u_2))$,
* $Trm_C^q(\sigma)(u_1.u_2) = (Trm_C^q(\sigma)(u_1).Trm_C^q(\sigma)(u_2))$, and
* $Trm_C^q(\sigma)(\varphi \triangleright u_1; u_2) = (\mathbf{Sen}(\sigma)(\varphi) \triangleright Trm_C^q(\sigma)(u_1); Trm_C^q(\sigma)(u_2))$;

- **Mod^q** is the functor defined by

  - for each signature Σ, $\mathbf{Mod}^q(\Sigma)$ is the class of all tuples $\langle M, \mathcal{S}, |w\rangle, \nu, \rho\rangle$ where

    * $\emptyset \neq M \subseteq \mathbf{Mod}(\Sigma)$ such that V_Σ is injective on M,
    * $\mathcal{S}$ is a finite partition of $\mathbf{Qb}(\Sigma)$,
    * $|w\rangle$ is a structured quantum state over $V_\Sigma[M]$ and $\mathcal{S}$,
    * $\nu = \{\nu_{FA}\}_{A \subseteq F \subseteq \mathbf{Qb}(\Sigma)}$ is a family of complex numbers such that, whenever $F \in \bigcup \mathcal{S}$, $\nu_{FA} = \langle v_A^F | \bigotimes_{Q_i \subseteq F} w_i\rangle$ if $v_A^F \in V_F$, and $\nu_{FA} = 0$ if $v_A^F \notin V_F$, and
    * ρ is an assignment such that $\rho(x) \in \mathbb{R}$ for every $x \in X$, and $\rho(z) \in \mathbb{C}$ for every $z \in Z$,

  - for each signature morphism $\sigma : \Sigma \to \Sigma'$, the contravariant map $\mathbf{Mod}^q(\sigma)$ is such that $\mathbf{Mod}^q(\sigma)(\langle M', \mathcal{S}', |w'\rangle, \nu', \rho'\rangle) = \langle \mathbf{Mod}(\sigma)[M'], \mathbf{Sen}(\sigma)^{-1}(\mathcal{S}'), \mathbf{Sen}(\sigma)^\bullet[|w'\rangle], \nu, \rho'\rangle$ where $\nu_{FA} = \nu'_{\mathbf{Sen}(\sigma)[F]\mathbf{Sen}(\sigma)[A]}$; and

- for each signature Σ and $W = \langle M, \mathcal{S}, |w\rangle, \nu, \rho\rangle \in \mathbf{Mod}^q(\Sigma)$, $\Vdash_\Sigma^q$ is defined inductively by

  - $W \Vdash_\Sigma^q \varphi$ iff $M \Vdash_\Sigma \varphi$, for $\varphi \in \mathbf{Sen}(\Sigma)$,
  - $W \Vdash_\Sigma^q (\boxminus \delta)$ iff $W \not\Vdash_\Sigma^q \delta$,
  - $W \Vdash_\Sigma^q (\delta_1 \sqsupset \delta_2)$ iff $W \not\Vdash_\Sigma^q \delta_1$ or $W \Vdash_\Sigma^q \delta_2$,
  - $W \Vdash_\Sigma^q [F]$ iff $F \in \bigcup \mathcal{S}$, and
  - $W \Vdash_\Sigma^q (t_1 \leq t_2)$ iff $[\![t_1]\!]_R^W \leq [\![t_2]\!]_R^W$,

  where the denotations of real valued terms $[\![\_]\!]_R^W : Trm_R^q(\Sigma) \to \mathbb{R}$ and of complex valued terms $[\![\_]\!]_C^W : Trm_C^q(\Sigma) \to \mathbb{C}$ are defined by mutual induction as follows:

- $[\![x]\!]_R^W = \rho(x)$, for $x \in X$,
- $[\![r]\!]_R^W = r$, for $r \in R$,
- $[\![\int \varphi]\!]_R^W = \mu_{|w\rangle}(V_\Sigma[\{m \in M : m \Vdash_\Sigma \varphi\}])$,
- $[\![t_1 + t_2]\!]_R^W = [\![t_1]\!]_R^W + [\![t_2]\!]_R^W$,
- $[\![t_1.t_2]\!]_R^W = [\![t_1]\!]_R^W.[\![t_2]\!]_R^W$,
- $[\![\mathrm{Re}(u)]\!]_R^W = \mathrm{Re}([\![u]\!]_C^W)$,
- $[\![\mathrm{Im}(u)]\!]_R^W = \mathrm{Im}([\![u]\!]_C^W)$,
- $[\![\arg(u)]\!]_R^W = \arg([\![u]\!]_C^W)$, and
- $[\![|u|]\!]_R^W = |[\![u]\!]_C^W|$,

and

- $[\![z]\!]_C^W = \rho(z)$, for $z \in Z$,
- $[\![|\top\rangle_{FA}]\!]_C^W = \nu_{FA}$,
- $[\![t_1 + it_2]\!]_C^W = [\![t_1]\!]_R^W + i[\![t_2]\!]_R^W$,
- $[\![t_1.e^{it_2}]\!]_C^W = [\![t_1]\!]_R^W.e^{i[\![t_2]\!]_R^W}$,
- $[\![\overline{u}]\!]_C^W = \overline{[\![u]\!]_C^W}$,
- $[\![u_1 + u_2]\!]_C^W = [\![u_1]\!]_C^W + [\![u_2]\!]_C^W$,
- $[\![u_1.u_2]\!]_C^W = [\![u_1]\!]_C^W.[\![u_2]\!]_C^W$, and
- $[\![\varphi \triangleright u_1; u_2]\!]_C^W = \begin{cases} [\![u_1]\!]_C^W & \text{if } M \Vdash_\Sigma \varphi \\ [\![u_2]\!]_C^W & \text{otherwise} \end{cases}$,

with all the usual relations and operations on real and complex numbers.

Proposition 5.6.

I^q is an institution.

Proof. The functoriality of $\mathbf{Sen}^q$ is straightforward. Concerning $\mathbf{Mod}^q$, and given a signature morphism $\sigma : \Sigma \to \Sigma'$, note that indeed $W = \langle M, |w\rangle, \nu, \rho\rangle = \mathbf{Mod}^q(\sigma)(W') \in \mathbf{Mod}^q(\Sigma)$ if $W' = \langle M', |w'\rangle, \nu', \rho'\rangle \in \mathbf{Mod}^q(\Sigma')$. In particular, since $M = \mathbf{Mod}(\sigma)[M']$, $\mathbf{Sen}(\sigma)^\bullet[V_{\Sigma'}[M']] = V_\Sigma[M]$ just because $V_\Sigma(\mathbf{Mod}(\sigma)(m'))(\varphi) = V_{\Sigma'}(m')(\mathbf{Sen}(\sigma)(\varphi))$ for every $m' \in \mathbf{Mod}(\Sigma')$ and $\varphi \in \mathbf{Qb}(\Sigma)$, due to the satisfaction condition of the original institution. Moreover, if $A \subseteq F \subseteq \mathbf{Qb}(\Sigma)$, and we let $F' = \mathbf{Sen}(\sigma)[F]$ and $A' = \mathbf{Sen}(\sigma)[A]$, the definition of $\nu_{FA} = \nu'_{F'A'}$ is suitable. First note that $v_{A'}^{F'} \in V_{\Sigma'}[M']$ iff $v_A^F \in V_\Sigma[M]$ just because $\mathbf{Sen}(\sigma)^\bullet(v_{A'}^{F'}) = v_A^F$. Moreover, $F \in \bigcup\mathcal{S}$ iff $F' \in \bigcup\mathcal{S}'$.

The satisfaction condition of I^q can be established by a simple induction on formulas and on terms. The only interesting cases concern independence formulas $[F]$, plus probability $(\int \varphi)$ and amplitude $|\top\rangle_{FA}$ terms.

In the first case, we need to show that $W \Vdash_\Sigma^q [F]$ iff $W' \Vdash_{\Sigma'}^q [F']$. The result follows immediately because $F \in \cup\mathcal{S}$ iff $\mathbf{Sen}(\sigma)[F] \in \cup\mathcal{S}'$.

In the second case, we need to show that $[\![\int \varphi]\!]_R^W = [\![\int \mathbf{Sen}(\sigma)(\varphi)]\!]_R^{W'}$. Indeed, using the bijection between M and $V_\Sigma[M]$, the fact that $M = \mathbf{Mod}(\sigma)[M']$,

the satisfaction condition of the institution I, and as a result the fact that $\mathbf{Sen}(\sigma)^{\bullet}(V_{\Sigma'}(m')) = V_{\Sigma}(\mathbf{Mod}(\sigma)(m'))$, we have that

$$\llbracket \int \varphi \rrbracket_R^W = \mu_{|w\rangle}(V_{\Sigma}[\{m \in M : m \Vdash_{\Sigma} \varphi\}]) =$$
$$\sum_{m \in M : m \Vdash_{\Sigma} \varphi} |\langle V_{\Sigma}(m)|w\rangle|^2 = \sum_{m \in M : m \Vdash_{\Sigma} \varphi} ||w\rangle(V_{\Sigma}(m))|^2 =$$
$$\sum_{m \in M : m \Vdash_{\Sigma} \varphi} |\mathbf{Sen}(\sigma)^{\bullet}(|w'\rangle)(V_{\Sigma}(m))|^2 =$$
$$\sum_{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)} |\mathbf{Sen}(\sigma)^{\bullet}(|w'\rangle)(\mathbf{Sen}(\sigma)^{\bullet}(V_{\Sigma'}(m')))|^2 =$$
$$\sum_{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)} ||w'\rangle(V_{\Sigma'}(m'))|^2 =$$
$$\sum_{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)} |\langle V_{\Sigma'}(m')|m'\rangle|^2 =$$
$$\mu_{|w'\rangle}(V_{\Sigma'}[\{m' \in M' : m' \Vdash_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi)\}]) = \llbracket \int \mathbf{Sen}(\sigma)(\varphi) \rrbracket_R^{W'}.$$

In the third case, we need to show that $\llbracket |\top\rangle_{FA} \rrbracket_R^W = \llbracket |\top\rangle_{F'A'} \rrbracket_R^W$. Since we already know that $F \in \bigcup \mathcal{S}$ iff $F' \in \bigcup \mathcal{S}'$, and that $v_{A'}^{F'} \in V_{\Sigma'}[M']$ iff $v_A^F \in V_{\Sigma}[M]$, it suffices to verify that, when it makes sense,

$$\langle v_A^F| \bigotimes_{Q_i \subseteq F} w_i \rangle = \prod_{Q_i \subseteq F} \langle (v_A^F)_{Q_i}|w_i\rangle = \prod_{Q_i \subseteq F} ||w_i\rangle((v_A^F)_{Q_i})|^2 =$$
$$\prod_{Q_i \subseteq F} |\mathbf{Sen}(\sigma)^{\bullet}(|w_i'\rangle)(\mathbf{Sen}(\sigma)^{\bullet}((v_{A'}^{F'})_{Q_i'}))|^2 = \prod_{Q_i' \subseteq F'} ||w_i'\rangle((v_{A'}^{F'})_{Q_i'})|^2 =$$
$$\prod_{Q_i' \subseteq F'} \langle (v_{A'}^{F'})_{Q_i'}|w_i'\rangle = \langle v_{A'}^{F'}| \bigotimes_{Q_i' \subseteq F'} w_i' \rangle. \qquad \square$$

Most of the syntactic constructions introduced in I^q are self explanatory. The quantum specific constructs, besides all the operations on complex numbers, are the $[F]$ formulas and the $|\top\rangle_{FA}$ terms. Intuitively, $[F]$ holds if the qubits in F form an independent subsystem of the whole, whereas $|\top\rangle_{FA}$ evaluates, whenever it is meaningful, to the complex amplitude of the vector $|v_A^F\rangle$ in the current state of the systems. The logic resulting from the quantization of classical propositional logic was introduced and studied in [16, 18]. A sound and weak complete calculus for the logic was obtained in [17] using an iterated Henkin construction inspired by the technique in [10]. The qubits of interest in this case were the propositional symbols. Using the logic it is possible, for instance, to model and reason about quantum states corresponding to the famous case of Schrödinger's cat. The relevant attributes of the cat are: being inside or outside the box, alive or dead, and moving or still. These three attributes are represented by the qubits **cat-in-box**, **cat-alive**, **cat-moving**, respectively. The following formulas constrain the state of the cat at different levels of detail:

1. $[\mathbf{cat\text{-}in\text{-}box}, \mathbf{cat\text{-}alive}, \mathbf{cat\text{-}moving}]$;

2. $(\mathbf{cat\text{-}moving} \Rightarrow \mathbf{cat\text{-}alive})$;

3. $((\Diamond\, \mathbf{cat\text{-}alive}) \sqcap (\Diamond\, (\neg\, \mathbf{cat\text{-}alive})))$;

4. $(\boxminus[\mathbf{cat\text{-}alive}])$;

5. $(\int \mathbf{cat\text{-}alive} = \frac{1}{3})$.

Observe that the assertions above are jointly consistent. Intuitively, they state that the qubits **cat-in-box**, **cat-alive**, **cat-moving** are not entangled

with other qubits; that if the cat is moving then it is alive; that it is both possible that the cat is alive and that the cat is dead; that the qubit **cat-alive** is entangled with the others; and that the cat is in a state where the probability of observing it alive (after collapsing the wave function) is $\frac{1}{3}$.

Our aim is now to establish the precise relationship of I^q with the institutions I, I^g, I^p.

Definition 5.7. *The C^{pq} comorphism*

The comorphism $C^{pq} : I^p \to I^q$ is such that $C^{pq} = \langle \Phi^{pq}, \alpha^{pq}, \beta^{pq} \rangle$ with:

- Φ^{pq} the identity functor on **Sig**;

- for each signature Σ, α^{pq}_Σ translates each formula $\delta \in \mathbf{Sen}^p(\Sigma)$ to δ itself, now seen as a formula in $\mathbf{Sen}^q(\Sigma)$; and

- for each signature Σ, β^{pq}_Σ translates each $\langle M, \mathcal{S}, |w\rangle, \nu, \rho \rangle \in \mathbf{Mod}^q(\Sigma)$ to $\langle M, \langle 2^M, \mu \rangle, \rho|_X \rangle$ where $\mu(B) = \mu_{|w\rangle}(V_\Sigma[B])$.

Proposition 5.8.

C^{pq} *is a comorphism.*

Proof. The naturality of the transformation α^{pq} is straightforward. Concerning β^{pq} just note that given $W = \langle M, \mathcal{S}, |w\rangle, \nu, \rho \rangle \in \mathbf{Mod}^q(\Sigma)$, $\beta^{pq}(W)$ is well defined. The probability space $\langle 2^M, \mu \rangle$ over M is just an isomorphic copy of $\langle 2^{V_\Sigma[M]}, \mu_{|w\rangle} \rangle$ over $V_\Sigma[M]$. It is clearly a Nilsson structure, and its naturality follows easily. The coherence condition is trivial. $\qquad\square$

Note however that, in general, C^{pq} does not satisfy the surjectivity condition, and thus I^q is not a conservative extension of I^p. This happens for two essential reasons: first, the sets M of models that appear in quantum models must be in one-to-one correspondence with their induced classical valuations on the qubits; second, even for such an M, due to the independence partitions, not all Nilsson structures over M can be obtained from a quantum structure.

Of course, by composition, we also obtain a comorphism $C^{gq} = C^{pq} \circ C^{gp} : I^g \to I^q$, and a power-model comorphism $C^q = C^{pq} \circ C^p : I \to I^q$. It is very easy to check that C^q meets the necessary surjectivity condition, and therefore I^q is still a conservative extension of I. Given Σ, and a model m of I, we just need to consider any quantum structure of the form $\langle \{m\}, \{\mathbf{Qb}(\Sigma)\}, 1|V_\sigma(m)\rangle, \nu, \rho \rangle$ with $\nu_{FA} = 1$ if $F = \mathbf{Qb}(\Sigma)$ and $v_A^F = V_\sigma(m)$, or $F = \emptyset$, and $\nu_{FA} = 0$ otherwise.

On the other hand, it is easy to see that also C^{gq} will be surjective, and hence I^q a conservative extension of I^g, whenever the first of the above mentioned restrictions is trivial. That is, requiring that M is in one-to-one correspondence with its induced set of valuations should not exclude any possible set of models. For this condition to hold, it suffices to require that the qubit functor $\mathbf{Qb}$ is chosen in such a way that, for each signature Σ and $m_1, m_2 \in \mathbf{Mod}(\Sigma)$, if m_1 and m_2 coincide on the satisfaction of all qubits then $m_1 = m_2$. If the qubits are *representative* typically one ends up with logically equivalent models, but in many institutions it is possible to avoid having logically equivalent models.

The case of classical propositional logic is paradigmatic, once we take as qubits all the propositional symbols. But similar choices are possible in many other logics. In [4] it is shown how to do this choice in any suitable finitely-valued logic. For instance, in Lukasiewicz's three-valued logic it suffices to consider as qubits all propositional symbols and negations of propositional symbols. This possibility also helps in shedding light on the usefulness of considering restricted sets of admissible valuations.

6 Conclusion
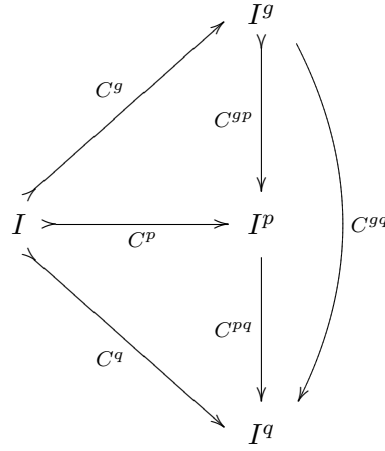
Figure 1: The diagram of institutions and (power-model) comorphisms.

Figure 1 provides a diagramatic view of the institutions and (power-model) comorphisms we have built, where $\rightarrowtail$ is used to distinguish the arrows that guarantee a conservative extension from their source to target.

Our main goal in bringing into the realm of instituions the exogenous approch to globalization, probabilization and quantization of logics was to assess how general these constructions were. The first two constructions are fully general, in the sense that nothing is assumed about the given institution and also that nothing else is needed. But quantization requires some additional information (the choice of qubit formulae). Furthermore, the resulting quantum logic seems to be improvable in some aspects (namely, injectivity of V_Σ on models, and surjectivity of the qubit translations) that were pointed out by the institutional framework. The solution seems to suggest a slight generalization of the exogenous approach towards working with multisets of models (as in Kripke structures), a promising line of further development of the approach. Futhermore, many interesting institution-theoretic questions remain open about these logics and the construction mechanisms discussed herein, like analyzing the properties of the constructions as functors on the category of institutions (or better, on some category of institutions), studying the underlying categories of models, and study their impact on the properties of the resulting categories of

specifications. From a logic-theoretic point of view, the next step is to attempt at extending the completeness results in [19, 17] for a generic base institution, with as little additional requirements as possible.

Acknowledgments

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