Matlab Exercise I: Simulating Brownian Motion

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Question 1

$$x[n] + \beta_1 x[n-1] + \beta_2 x[n-2] = \beta_3 \tilde{w}[n] \tag{1}$$

$$\beta_1 = -\frac{2m + \gamma\Delta t}{m + \gamma\Delta t} = -2.0 \tag{2}$$

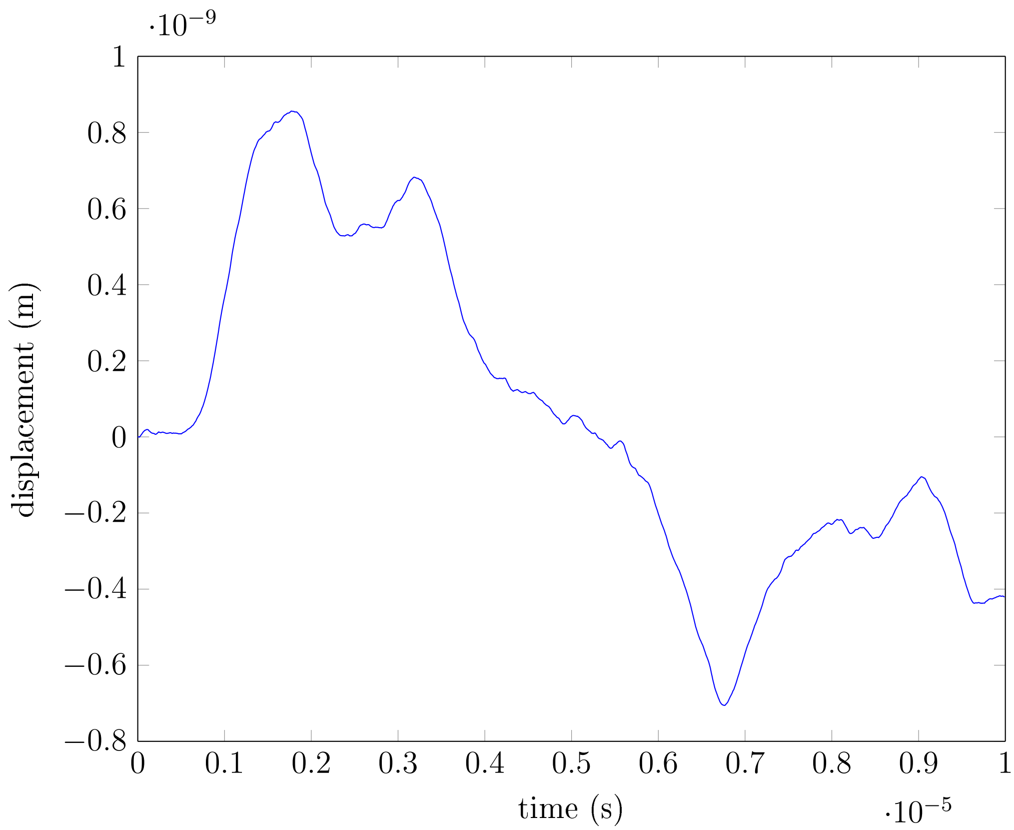
$$\beta_2 = \frac{m}{m + \gamma\Delta t} = 0.98 \tag{3}$$

$$\beta_3 = \frac{\sqrt{2k_B T\gamma\Delta t}}{\frac{m}{\Delta t} + \gamma} = 1.1 \cdot 10^{-12} \tag{4}$$

$$m = (4/3) * \pi * R^3 * \rho \tag{5}$$

Question 2

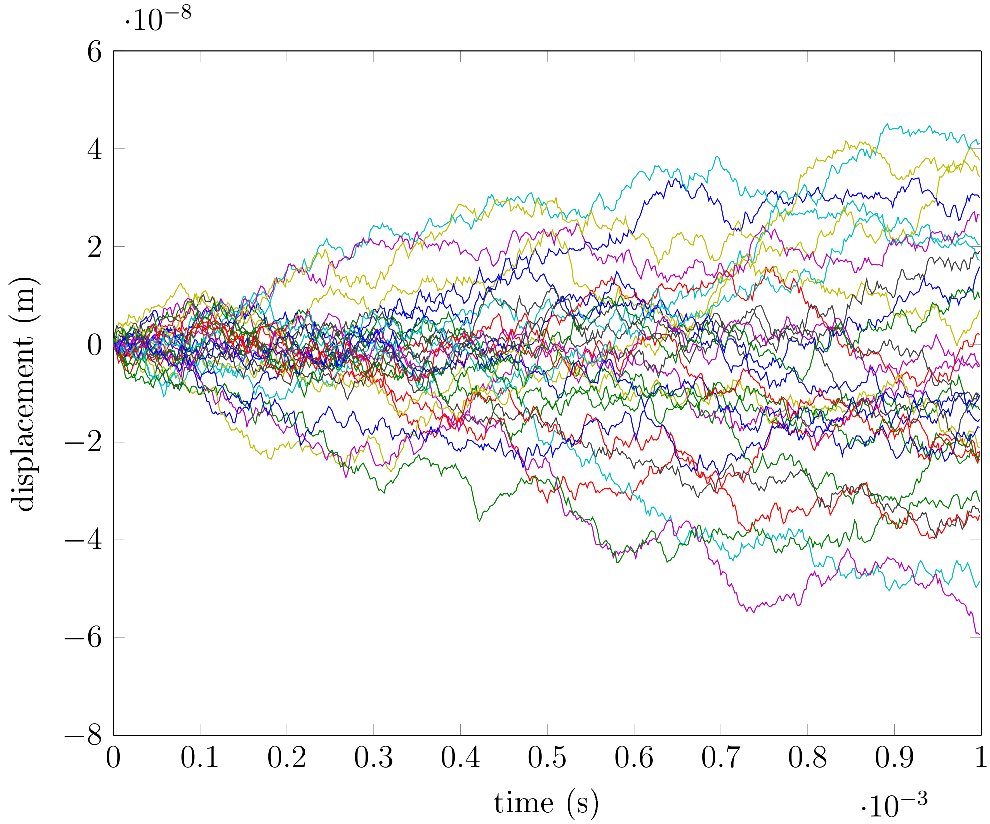
```
%% Constants

N = 1e5;
dt = 1e-8; % s
R = 1e-6; % m
kB = 1.38e-23; % J/K
T = 300; % K
eta = 1e-3; % Pa s
rho = 2.6e3; % kg/m^3
gamma = 6*pi*R*eta; % Pa m s

% Compute particle mass in kg (nb. particles are spheres)
m = (4/3)*pi*R^3*rho;

% Expressions for the coefficients in terms of the constants given above
beta1 = -(2*m + gamma*dt)/(m + gamma*dt);
beta2 = m/(m + gamma*dt);
beta3 = sqrt(2*kB*T*gamma/dt)/(m/dt^2 + gamma/dt);

% Initialize signal vector (x) and generate white noise samples vector (w)
N2 = 1e3;
x = zeros(N2,1);
w = randn(N2,1);

% Simulate the difference equation
for k = 3:N2
    x(k) = - beta1*x(k-1) - beta2*x(k-2) + beta3*w(k);
end

% Plot the result as a function of time
time = (0:dt:dt*(N2-1));
plot(time,x);
xlabel('time (s)');
ylabel('displacement (m)');
```



$\cdot 10^{-9}$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
displacement (m)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)
$\cdot 10^{-5}$

Question 3

```
% Initialize number of simulation (L) and the signal and noise matrices (x
    and w)
L = 30;
x = zeros(N,L);
w = randn(N,L);

% Simulate the difference equation
for l = 1:L
    for k = 3:N
        x(k,l) = - beta1*x(k-1,l) - beta2*x(k-2,l) + beta3*w(k,l);
    end
end

% Plot the results
time = (0:dt:dt*(N-1));
plot(time,x);
xlabel('time (s)');
ylabel('displacement (m)');
```



Question 4

```
% Initialize the number of simulation (vector of L values)
% and the number of samples in each simulation (vector of H values)
Ls = [30,300,3000];
Hs = [10^3,10^4,10^5];

% Take L the maximum number of simulations we need to compare
% (we can take subsets for the lower values of L)
L = max(Ls);

% Note that N (defined above) is equal to the maximum h

% Initialize signal and noise matrices (x and w)
x = zeros(N,L);
w = randn(N,L);

% Simulate the difference equation L times
for l = 1:L
    for k = 3:N
        x(k,l) = - beta1*x(k-1,l) - beta2*x(k-2,l) + beta3*w(k,l);
    end
end

% Take different subsets of the data for each combination of the
% L and h parameters and plot as a histogram
for i = 1:length(Ls)
    L = Ls(i);

    for j = 1:length(Hs)
        h = Hs(j);

        % Select appropriate subplot
        subplot(length(Ls), length(Hs), (i-1)*length(Hs)+j);

        % Plot the histogram
        hist(x(h,1:L),sqrt(L));
        xlim([min(min(x)) max(max(x))]);
        xlabel('displacement (m)');
        title(sprintf('L=%d, h=%d', L, h));

    end
end
```

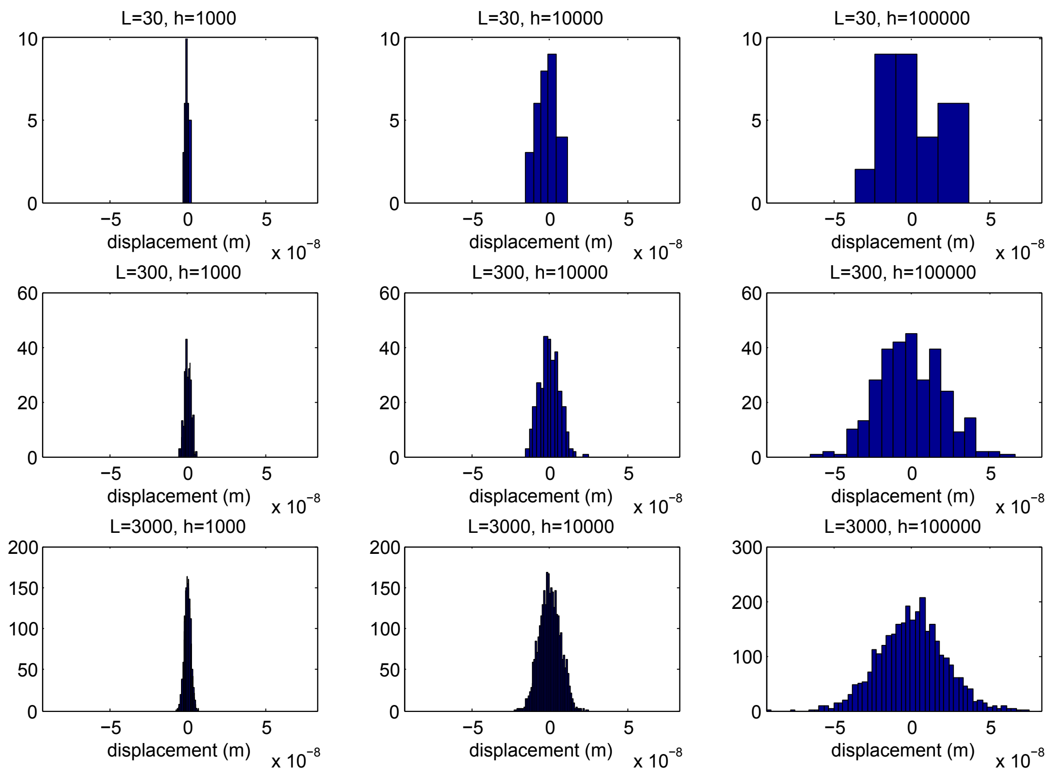


Figure 1: The histograms of the distribution of particles at varying distances (h) and different amount of particles (L)

Question 5

```
Kappa = zeros(3,3);
Sigma = zeros(3,3);
for i = 1:length(Ls)
    L = Ls(i);

    for j = 1:length(Hs)
        h = Hs(j);

        % Repeat hist command to get the data
        [counts, centers] = hist(x(h,(1:L)),sqrt(L));

        % Calculate initial estimates of sigma and kappa
        sigma0 = std(x(h,(1:L)));
        kappa0 = counts(round(length(counts)/2));

        % Define the objective function which we'll try to minimize
        % p is a vector with fit parameters:
        %     p(1) standard deviation (sigma)
        %     p(2) scale factor (kappa)
        fobj = @(p) sum( (counts - p(2)*exp(-centers.^2/2/p(1)^2)).^2 );

        % Fit the Gaussian through the histogram data
        p_opt = fminsearch(fobj, [sigma0, kappa0]);
        sigma = p_opt(1);
        kappa = p_opt(2);
        Sigma(i,j) = sigma;
        Kappa(i,j) = kappa;
        % Select appropriate subplot
        subplot(length(Ls), length(Hs), (i-1)*length(Hs)+j);

        % Plot the Gaussian on top of the histogram
        hold all;
        xg = linspace(min(min(x)), max(max(x)), 1000);
        yg = kappa.*exp(-(xg.^2)./(2.*sigma.^2));
        plot(xg, yg, 'r');
        xlabel(sprintf('sigma = %.2e (m)', sigma));
        hold off;

    end
end
```

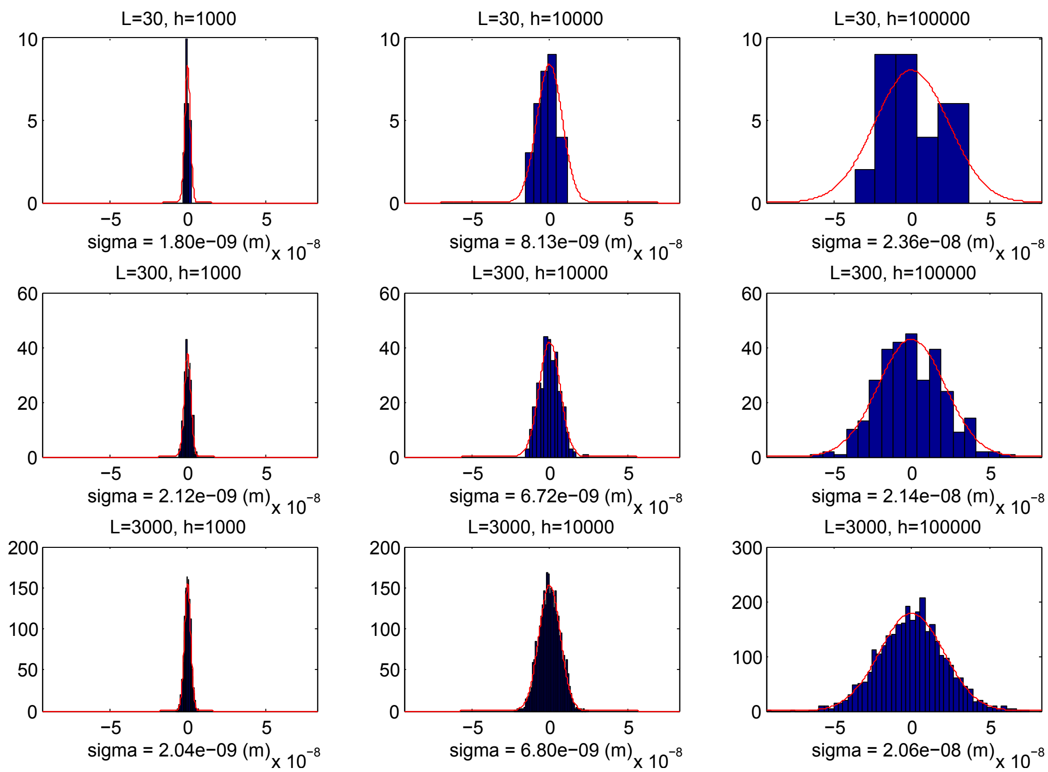


Figure 2: The histograms of the distribution of particles at varying distances (h) and different amount of particles (L), with the Gaussian fits as estimated by $\hat{\kappa}$ and $\hat{\sigma}$

Question 6

Table 1: h = 10^3

Parameter \ L	30	300	3000
$\hat{\kappa}$	8.3	3.8×10^1	1.6×10^2
$\hat{\sigma}$	1.8×10^{-9}	2.1×10^{-9}	2.0×10^{-9}

Table 2: h = 10^4

Parameter \ L	30	300	3000
$\hat{\kappa}$	8.4	4.2×10^1	1.5×10^2
$\hat{\sigma}$	8.1×10^{-9}	6.7×10^{-9}	6.8×10^{-9}

Table 3: h = 10^5

Parameter \ L	30	300	3000
$\hat{\kappa}$	8.1	4.3×10^1	1.8×10^2
$\hat{\sigma}$	2.4×10^{-8}	2.1×10^{-8}	2.1×10^{-8}

Question 7

From Table 1, Table 2 and Table 3 it can be concluded that at a set time $h\Delta t$, the estimator for the height of the Gaussian ($\hat{\kappa}$) increases with the amount of realizations L, but the estimator for the standard deviation ($\hat{\kappa}$) does not vary much. The estimator for the height increases because the the histograms are not normalized, and therefore the Gaussian is not normalized. The standard deviation seems to converge to the "real" value σ with more drawings.

These statements are visualized in Figure 2. In the top row, it shows the Gaussian fits for $L = 30$ where the Gaussian crudely follow the histogram. For instance, at h = 100000 the estimator $\hat{\kappa}$ for the best Gaussian fit is 8.3, while the actual height of the middle bin is 10. The last row shows $L = 3000$, where it shows that the Gaussian follows the histograms much more accurately, with most bins neatly within the Gaussian curve. This shows that $\hat{\sigma}$ can be more accurately predicted by increasing the amount of realizations from the process (L).

A necessary condition for a process to be wide sense stationary (WSS) is for its correlation function to satisfy the following equation:

$$r_x(k,l) = r_x(k-l)\forall k,l. \tag{6}$$

Taking $k = l$, the auto-correlation function becomes:

$$r_x(k,k) = r_x(0) = E\{(x[k] - E\{x[k]\})^2\} = \mathrm{Var}\{x\}. \tag{7}$$

Since k can be any value, the variance of a WSS process must be constant over time. This process is not WSS, because the variance increases with the time. The longer the process goes on, the more widespread the results are.

We will now look at the Z transform of Equation 1 to see whether from this it can also be concluded the the system is not WSS. Assuming white noise can be Z-transformed, the Z-transform of Equation 1 is:

$$X(z) + \beta_1 z^{-1} X(z) + \beta_2 z^{-2} X(z) = \beta_3 W(z). \tag{8}$$

The system function $H(z)$ can then be written as:

$$H(z) = \frac{X(z)}{W(z)} = \frac{\beta_3}{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}. \tag{9}$$

A necessary condition for a system to be a stable, and thus to be WSS, is that the poles of the system function $H(z)$ are within the unit circle. The poles can be found by finding z for which the divider of Equation 9 is 0:

$$1 + \beta_1 z^{-1} + \beta_2 z^{-2} = 0. \tag{10}$$

The value of z with the biggest magnitude which satisfies Equation 10 will be denoted by z_1 and can be found using the ABC-formula:

$$z_1 = \frac{-\beta_1 + \sqrt{\beta_2^2 - 4\beta_2}}{2}. \tag{11}$$

substituting Equation 2 and Equation 3 in Equation 11, the following expression for z_1 is found:

$$\begin{aligned}
z_1 =& \frac{1}{2}\left(\frac{2m+\gamma\Delta t}{m+\gamma\Delta t} + \sqrt{\left(\frac{2m+\gamma\Delta t}{m+\gamma\Delta t}\right)^2 - \frac{4m}{m+\gamma\Delta t}}\right) = \\
&\frac{1}{2}\left(\frac{2m+\gamma\Delta t}{m+\gamma\Delta t} + \sqrt{\frac{4m + 4m\gamma\Delta t + \gamma^2\Delta t^2}{(m+\gamma\Delta t)^2} - \frac{4m + 4m\gamma\Delta t}{(m+\gamma\Delta t)^2}}\right) = \\
&\frac{1}{2}\left(\frac{2m+\gamma\Delta t}{m+\gamma\Delta t} + \frac{\gamma\Delta t}{m+\gamma\Delta t}\right) = 1.
\end{aligned}$$

Since $z_1 = 1$, the unit circle is not part of the region of convergence of $H(z)$, so $H(z)$ is not stable, and therefore the system is not WSS.