This exam has 7 problems on 9 pages. There are no calculators, phones, or other electronic devices allowed during this exam. Be sure to show all your work.

Name: ______________________ **Score:**

id-number:

Problem 1. Solve the following limits when possible. (*Hints: try to add and subtract, multiply and divide by the same number; make use of the complement i.e. complement of (a+b) is (a-b)* and write numbers as quotients if possible)

(a) $\lim_{h\to 0} \frac{5}{x-1}$

$$\lim_{x\to 0} \frac{5}{x-1} = \frac{5}{-1} = 5$$

Operating

(b) $\lim_{x\to\infty} \frac{162000x^2+7800x+10}{1+x+x^2+x^3}$

$$\lim_{x\to\infty} \frac{162000x^2+7800x+10}{1+x+x^2+x^3} = \lim_{x\to\infty} \frac{\frac{162000x^2}{x^3}+\frac{7800x}{x^3}+\frac{10}{x^3}}{\frac{1}{x^3}+\frac{x}{x^3}+\frac{x^2}{x^3}+\frac{x^3}{x^3}}$$

Dividing the numerator and denominator by x^3

$$= \lim_{x\to\infty} \frac{\frac{162000}{x}+\frac{7800}{x^2}+\frac{10}{x^3}}{\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1}$$

Common factors cancel

$$= \frac{0}{1} = 0$$

The limit term by term is 0, hence the result

(c) $\lim_{x\to 2} \frac{x^2-7x+10}{x^2-5x+6}$

$$\lim_{x\to 2} \frac{x^2-7x+10}{x^2-5x+6} = \lim_{x\to 2} \frac{(x-2)(x-5)}{(x-2)(x-3)}$$

Factorising

$$= \lim_{x\to 2} \frac{(x-5)}{(x-3)}$$

Cancelling common factors

$$= \lim_{x\to 2} \frac{2-5}{2-3} = \frac{-3}{-1} = 3$$

Operating

(d) $\lim_{x\to 0} \frac{x}{|x|}$

Left-hand limit: $\lim_{x\to 0^-} \frac{x}{|x|} = -1$

Right-hand limit: $\lim_{x\to 0^+} \frac{x}{|x|} = 1$

$$\lim_{x\to 0^-} \frac{x}{|x|} \neq \lim_{x\to 0^+} \frac{x}{|x|} \Rightarrow \lim_{x\to 0} \frac{x}{|x|}$$

Does not exist

(e) $\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right) &= \lim_{x\to\infty}\frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)} && \text{Multiplying and Dividing by the conjugate}\\
&= \lim_{x\to\infty}\frac{x+1-x}{\left(\sqrt{x+1}+\sqrt{x}\right)} && \text{Operating the numerator}\\
&= \lim_{x\to\infty}\frac{1}{\left(\sqrt{x+1}+\sqrt{x}\right)} && \text{Evaluating the limit}\\
&= 0
\end{aligned}
$$

(f) $\lim_{x\to\infty}\frac{2^x-2^{-x}}{2^x+2^{-x}}$

$$
\begin{aligned}
\lim_{x\to\infty}\frac{2^x-2^{-x}}{2^x+2^{-x}} &= \lim_{x\to\infty}\frac{2^x-\frac{1}{2^x}}{2^x+\frac{1}{2^x}} && \text{Rewriting the expression}\\
&= \lim_{x\to\infty}\frac{2^x-0}{2^x+0} && \text{Evaluating the limit}\\
&= 1
\end{aligned}
$$

Problem 2. Find the $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x\partial y}$ of the following functions:

(a) $z(x,y)=x^{\ln 4}e^{x^2+y^2}$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \ln 4x^{\ln 4-1}e^{x^2+y^2}+2x^{\ln 4+1}e^{x^2+y^2} && \text{Using the product and chain rules}\\
\frac{\partial z}{\partial y} &= 2yx^{\ln 4}e^{x^2+y^2} && \text{Using the power rule}\\
\frac{\partial^2 z}{\partial x\partial y} &= 2y\left(\ln 4x^{\ln 4-1}+2x^{\ln 4+1}\right)e^{x^2+y^2}
\end{aligned}
$$

(b) $z(x,y)=\frac{x^{\frac{1}{4}}y^{\frac{1}{3}}}{e^x}$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\frac{1}{4}y^{\frac{1}{3}}x^{-\frac{3}{4}}e^x-x^{\frac{1}{4}}y^{\frac{1}{3}e^{x}}}{\left(e^x\right)^2} && \text{Using product and chain rules}\\
&= \frac{\frac{1}{4}y^{\frac{1}{3}}\left(x^{-\frac{3}{4}}-4x^{\frac{1}{4}}\right)}{e^x} && \text{Factorising and cancelling}\\
\frac{\partial z}{\partial y} &= \frac{1}{3}\frac{x^{\frac{1}{4}}}{e^x}y^{-\frac{2}{3}} && \text{By the power rule}\\
\frac{\partial^2 z}{\partial x\partial y} &= \frac{1}{12}y^{-\frac{2}{3}}\frac{\left(x^{-\frac{3}{4}}-4x^{\frac{1}{4}}\right)}{e^x}
\end{aligned}
$$

(c) $z(x,y) = \ln x^3y^2$

$$\frac{\partial z}{\partial x} = \frac{3x^2y^2}{x^3y^2} = \frac{3}{x} \qquad \text{Using the product rule and factorising}$$
$$\frac{\partial z}{\partial y} = \frac{2yx^3}{x^3y^2} = \frac{2}{y}$$
$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

Problem 3. Solve the following integrals by direct methods, substitution and parts respectively:

(a) $\int_0^1 x^{-4} dx$

$$\begin{aligned}\int_0^1 x^{-4}dx &= \left.\frac{x^{-3}}{-3}\right]_0^1 && \text{Using power rule}\\ &= -\frac{1}{3} + \lim_{x\to 0}\frac{1}{5x^3} && \text{Evaluating the limit}\\ &= \nexists\end{aligned}$$

(b) $\int_0^{\frac{3}{2}} \frac{xdx}{\sqrt{9-4x^2}}$

$$\begin{aligned}\int_0^{\frac{3}{2}} \frac{xdx}{\sqrt{9-4x^2}} &= \left\{\begin{matrix} u & = 9-4x^2 \\ du & = -8xdx \\ u(\frac{3}{2}) & = 0 \\ u(0) & = 9\end{matrix}\right\} && \text{Doing the change of variables}\\ &= -\frac{1}{8}\int_9^0 \frac{du}{\sqrt{u}}\\ &= \frac{1}{8}\int_0^9 \frac{du}{\sqrt{u}} && \text{Changing limits of integration}\\ &= \frac{1}{4}\left[u^{\frac{1}{2}}\right]_0^9 && \text{Using the power rule}\\ &= \frac{1}{4}\left[\sqrt{9}-0\right]\\ &= \frac{3}{4}\end{aligned}$$

(c) $\int x \ln x dx$

$$\begin{aligned}
\int x \ln x dx &= \left\{ \begin{array}{c} x = f'(x) \\ \ln x = g(x) \end{array} \right\} && \text{Defining the primitives and derivatives} \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 \cdot \frac{1}{x} dx && \text{Integrating by parts} \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} && \text{By direct methods} \\
&= \frac{x^2}{2} \left[\ln x - \frac{1}{2} \right] + C && \text{Rearranging and adding the constant}
\end{aligned}$$

Problem 4. Find the derivative of w with respect to z of the following function $\ln\left(x^4 y^3 z^6 t^2 w\right)$ by making use of the implicit function theorem.

By the implict function theorem if we have a function $F(x_1, ..., x_j, ..., x_n) = c$ that can be written out as $F\left(x_1, ..., f(\mathbf{x_{-j}}), ...x_n\right) = c$, then its derivative $\frac{dx_j}{dx_i}$ can be worked out as

$$\frac{dx_j}{dx_i} = -\frac{\frac{dF(\mathbf{x})}{dx_j}}{\frac{dF(\mathbf{x})}{dx_i}}.$$

Then turning the focus to the function in question $f(x, y, z, t, w) = \ln\left(x^4 y^3 z^6 t^2 w\right)$ we can calculate the derivative of w w.r.t. z as follows:

$$\begin{aligned}
\frac{dw}{dz} &= -\frac{\frac{df}{dz}}{\frac{df}{dw}} && \text{By the IFT} \\
&= -\frac{\frac{6x^4y^3z^5t^2w}{x^4y^3z^6t^2w}}{\frac{x^4y^3z^6t^2}{x^4y^3z^6t^2w}} && \text{Cancelling common factors} \\
&= -\frac{\frac{6}{z}}{\frac{1}{w}} = -\frac{6w}{z} && \text{Rearranging}
\end{aligned}$$

Problem 5. Sketch the following function e^{-x^2} by showing analytically over what intervals the functions is increasing and decreasing, over what intervals the function is convex or concave and where are the maxima and minima of the function if at all.

First, to see where the function is increasing or decreasing we have to find where the first derivative is positive or negative:

$$\begin{aligned} f'(x) = \frac{\partial e^{-x^2}}{\partial x} > 0 \Leftrightarrow -2xe^{-x^2} > 0 \Leftrightarrow -2x > 0 & \\ \Leftrightarrow x < 0 \Rightarrow f(x) \text{ increasing in } (-\infty, 0) & \\ \text{By the same argument: } \text{if } x > 0 \Rightarrow f(x) \text{ decreasing in } (0, \infty) & \\ \text{if } x = 0 \Rightarrow \text{Critical point: min, max or inflexion point} & \end{aligned}$$

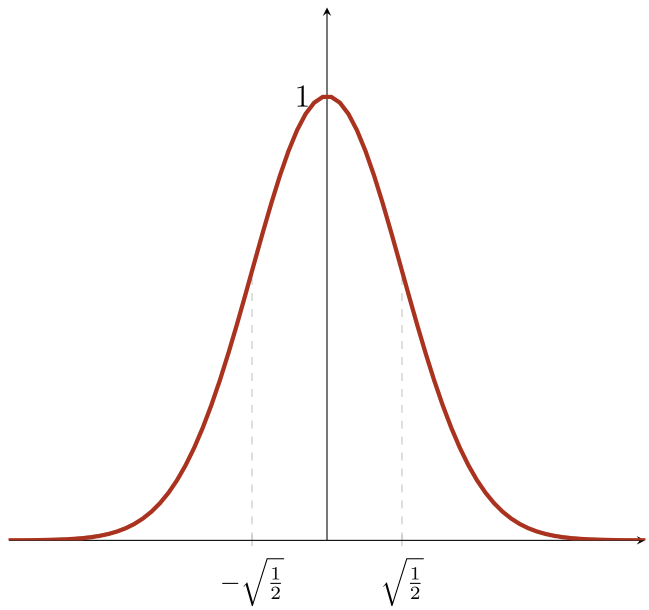
Second, we have to look where the function is concave up or concave down, i.e. where the second derivative is positive and where is negative.

$$\begin{aligned} f''(x) = \frac{\partial^2 e^{-x^2}}{\partial x^2} > 0 \Leftrightarrow 4x^2e^{-x^2} - 2d^{-x^2} > 0 & \\ \Leftrightarrow 2(2x^2 - 1)e^{-x^2} > 0 \Leftrightarrow 2x^2 - 1 > 0 & \\ \Leftrightarrow x^2 > \frac{1}{2} \Leftrightarrow x < -\sqrt{\frac{1}{2}} \cap x > \sqrt{\frac{1}{2}} & \quad \text{Interval over which the function is concave up} \\ \text{By the same argument if } x^2 < \frac{1}{2} \Leftrightarrow -\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}} & \quad \text{Interval over which the function is concave down} \\ \text{if } x^2 = \frac{1}{2} \Leftrightarrow x = -\sqrt{\frac{1}{2}} \text{ or } x = \sqrt{\frac{1}{2}} & \quad \text{Points of inflection} \end{aligned}$$

Now, with all of this information we can say that because the critical point $x = 0$ belongs to the interval where the function is concave down, i.e. $\left(-\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}}\right)$ it is a maximum. In order to draw a sketch of the function we need to know where exactly the critical and inflection points lay in the xy-plane, then:

$$\begin{aligned} \text{Critical point: } & x = 0 \Rightarrow f(0) = e^{-0^2} = 1 \Rightarrow \text{c.p. at } (x, y) = (0, 1) \\ \text{Inflexion points: } & x = -\sqrt{\frac{1}{2}} \approx -0.7 \Rightarrow f\left(-\sqrt{\frac{1}{2}}\right) \approx 0.6 \Rightarrow \text{i.p. at } (x, y) \approx (-0.7, 0.6) \\ & x = -\sqrt{\frac{1}{2}} \approx 0.7 \Rightarrow f\left(-\sqrt{\frac{1}{2}}\right) \approx 0.6 \Rightarrow \text{i.p. at } (x, y) \approx (0.7, 0.6) \end{aligned}$$

Now we have all the ingredients to sketch the curve:



Problem 6. Solve the following problem using the Khun-Tuker conditions:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = 3xy - x^3 \\ s.t. \quad & 2x - y = -5 \\ & 5x + 2y \geq 37 \\ & x, y \geq 0 \end{aligned}$$

First use the constraint $2x - y = -5$ to simplify the problem:

$$\left.\begin{array}{lc} \max_x & f(x) = 3x(2x+5) - x^3 \\ s.t. & 5x + 2(2x+5) \geq 37 \\ & x, y \geq 0 \end{array}\right\} \Rightarrow \left.\begin{array}{lc} \max_x & f(x) = -x^3 + 6x^2 + 15x \\ s.t. & x \geq 3 \end{array}\right\}$$

There are two cases:

Case 1: $\lambda = 0$ The FOC is

$$\begin{aligned} -3x^2 + 12x + 15 = 0 &\Leftrightarrow -x^2 + 4x + 5 = 0 \\ &\Leftrightarrow -x^2 - x + 5x + 5 = 0 \\ &\Leftrightarrow -x(x-1) + 5(x-1) = 0 \\ &\Leftrightarrow (5-x)(x-1) = 0 \end{aligned}$$

Then either $x = 1$ or $x = 5$, since $x \geq 3$ we can discard the former and take the latter as a possible solution.

Case 2: $\lambda > 0$

Then $x = 3$ as a possible solution.
Comparing the two cases:

1. $f(5) = -5^3 + 6 \cdot 5^2 + 15 \cdot 5 = (-5 + 6 + 3) \cdot 25 = 100$
2. $f(3) = -3^3 + 6 \cdot 3^2 + 15 \cdot 3 = (-3 + 6 + 5) \cdot 9 = 72$

Substituting the solution at $x = 5$ in the original problem $y = 15$. Hence the final solution would be

$$f(x^*, y^*) = 100$$
$$\text{at } (x^*, y^*) = (5, 15)$$

Problem 7. Solve the following system of equations using the inverse matrix.

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad &= 4 \\ x_1 - x_2 + x_3 \quad &= 5 \\ 2x_1 + 3x_2 - x_3 \quad &= 1 \end{aligned}$$

First you have to write the system in matrix form

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$$

Secondly, we have to transform the matrix of coefficients $\mathbf{A}$ into the $\mathbf{I}$ matrix and the $\mathbf{I}$ matrix into the inverse $\mathbf{A}^{-1}$, or in other words from $(\mathbf{A}|\mathbf{I})$ to $(\mathbf{I}|\mathbf{A}^{-1})$. Proceeding

1. Set the $n \times 2n$ matrix

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 & 1 \end{array}\right)$$

2. Subtract the 1st row from the 2nd row and twice the 1st from the 3rd

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right)$$

3. Change the sign of the 3rd and swap the 2nd and the 3rd rows

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & -3 & 0 & -1 & 1 & 0 \end{array}\right)$$

4. Subtract 3 times the 2nd row from the 3rd one

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 9 & 5 & 1 & -3 \end{array}\right)$$

5. Divide the 3rd row into 9

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{5}{9} & \frac{1}{9} & -\frac{3}{9} \end{array}\right)$$

6. Subtract 3 times the 3rd row from the 2nd and once form the 1st

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 0 & \frac{4}{9} & -\frac{1}{9} & \frac{3}{9} \\ 0 & 1 & 0 & \frac{3}{9} & -\frac{3}{9} & 0 \\ 0 & 0 & 1 & \frac{5}{9} & \frac{1}{9} & -\frac{3}{9} \end{array}\right)$$

7. Subtract 2 times the 2^{nd} row from the 1^{st}

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{9} & \frac{5}{9} & \frac{3}{9} \\ 0 & 1 & 0 & \frac{3}{9} & -\frac{3}{9} & 0 \\ 0 & 0 & 1 & \frac{5}{9} & \frac{1}{9} & -\frac{3}{9} \end{array}\right)$$

Then inverse matrix is $\frac{1}{9}$

$$\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & 3 \end{pmatrix}$$

Solving the system of equation is nothing else than

$$\mathbf{A}\mathbf{x} = \mathbf{b} \iff \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \iff \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

and thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 20 \\ -3 \\ 22 \end{pmatrix}$$