This exam has 6 problems on 7 pages. There are no calculators, phones, or other electronic devices allowed during this exam. Be sure to show all your work.

Name: ______________________________ **Score:**

id-number:

Problem 1. Solve the following limits if posible:

(a) $\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h} &= \lim_{h\to 0}\frac{(\sqrt{h+1}-1)(\sqrt{h+1}+1)}{h(\sqrt{h+1}+1)} && \text{Multiplying by the conjugate}\\
&= \lim_{h\to 0}\frac{h+1-1}{h(\sqrt{h+1}+1)} && \text{Common factors cancel}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{h+1}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2} && \text{Operating}
\end{aligned}$$

(b) $\lim_{x\to\infty}\frac{34000x^2+6000x+25000}{1+x+x^2+x^3}$

$$\begin{aligned}
\lim_{x\to\infty}\frac{34000x^2+6000x+25000}{1+x+x^2+x^3} &= \lim_{x\to\infty}\frac{\frac{34000x^2}{x^3}+\frac{6000x}{x^3}+\frac{25000}{x^3}}{\frac{1}{x^3}+\frac{x}{x^3}+\frac{x^2}{x^3}+\frac{x^3}{x^3}} && \text{Dividing the numerator and denominator by } x^3\\
&= \lim_{x\to\infty}\frac{\frac{34000}{x}+\frac{6000}{x^2}+\frac{25000}{x^3}}{\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1} && \text{Common factors cancel}\\
&= \frac{0}{1} = 0 && \text{The limit term by term is } 0\text{, hence the result}
\end{aligned}$$

(c) $\lim_{x\to 5}\frac{2x^2-7x-15}{3x^2-11x-20}$

$$\begin{aligned}
\lim_{x\to 5}\frac{2x^2-7x-15}{3x^2-11x-20} &= \lim_{x\to 5}\frac{(2x+3)(x-5)}{(3x+4)(x-5)} && \text{Factorising}\\
&= \lim_{x\to 5}\frac{(2x+3)}{(3x+4)} && \text{Common factors cancel}\\
&= \frac{13}{19} && \text{Then evaluate}
\end{aligned}$$

(d) $\lim_{x\to 0} \frac{x}{|x|}$

$$\text{Left-hand limit: } \lim_{x\to 0^-} \frac{x}{|x|} = -1$$
$$\text{Right-hand limit: } \lim_{x\to 0^+} \frac{x}{|x|} = 1$$
$$\lim_{x\to 0^-} \frac{x}{|x|} \neq \lim_{x\to 0^+} \frac{x}{|x|} \Rightarrow \lim_{x\to 0} \frac{x}{|x|} \quad \text{Does not exist}$$

Problem 2. Derive w.r.t. both variables:

(a) $x^{\ln 4} e^{x^2+y^2}$

$$\frac{\partial x^{\ln 4} e^{x^2+y^2}}{\partial x} = \ln 4 x^{\ln 4 - 1} e^{x^2+y^2} + 2x^{\ln 4+1} e^{x^2+y^2} \quad \text{Using the product and chain rules}$$
$$\frac{\partial x^{\ln 4} e^{x^2+y^2}}{\partial y} = 2yx^{\ln 4} e^{x^2+y^2} \quad \text{Using the power rule}$$

(b) $\frac{x^{\frac{1}{4}} y^{\frac{1}{3}}}{e^x}$

$$\frac{\partial \frac{x^{\frac{1}{4}} y^{\frac{1}{3}}}{e^x}}{\partial x} = \frac{\frac{1}{4} y^{\frac{1}{3}} x^{-\frac{3}{4}} e^x - x^{\frac{1}{4}} y^{\frac{1}{3}} e^x}{(e^x)^2} \quad \text{Using product and chain rules}$$
$$= \frac{\frac{1}{4}\left(x^{-\frac{3}{4}} - x^{\frac{1}{4}}\right) y^{\frac{1}{3}}}{e^x} \quad \text{Factorising and cancelling}$$
$$\frac{\partial \frac{x^{\frac{1}{4}} y^{\frac{1}{3}}}{e^x}}{\partial y} = \frac{1}{3} \frac{x^{\frac{1}{4}}}{e^x} y^{-\frac{2}{3}} \quad \text{By the power rule}$$

(c) $\ln x^3 y^2$

$$\frac{\partial \ln x^3 y^2}{\partial x} = \frac{3x^2 y^2}{x^3 y^2} = \frac{3}{x} \quad \text{Using the product rule and factorising}$$
$$\frac{\partial \ln x^3 y^2}{\partial y} = \frac{2yx^3}{x^3 y^2} = \frac{2}{y}$$

Problem 3. Solve the following integrals by direct methods, substitution and parts respectively:

(a) $\int_0^1 x^{\frac{1}{2}} dx$

$$\int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} \Big]_0^1 = \frac{2}{3}$$

(b) $\int xe^{-x^2}dx$

$$\int xe^{-x^2}dx = \begin{Bmatrix} u & = -x^2 \\ du & = -2xdx \Leftrightarrow -\frac{du}{2} = xdx \end{Bmatrix} \qquad \text{Doing the change of variables}$$
$$= -\frac{1}{2}\int e^u du \qquad \text{By direct methods}$$
$$= -\frac{1}{2}e^u \qquad \text{Reverting the change}$$
$$= -\frac{1}{2}e^{-x^2} + C \qquad \text{And adding the constant of integration}$$

(c) $\int x \ln x dx$

$$\int x\ln x dx = \begin{Bmatrix} x = f'(x) \\ \ln x = g(x) \end{Bmatrix} \qquad \text{Defining the primitives and derivatives}$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}dx \qquad \text{Integrating by parts}$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} \qquad \text{By direct methods}$$
$$= \frac{x^2}{2}\left[\ln x - \frac{1}{2}\right] + C \qquad \text{Rearranging and adding the constant}$$

Problem 4. Find the derivative of w with respect to z of the following function $\ln\left(x^4y^3z^6t^2w\right)$ by making use of the implicit function theorem.

By the implict function theorem if we have a function $f(x,y)$ then its derivative can be worked out as:

$$\frac{dy}{dx} = -\frac{\frac{df(x,y)}{dx}}{\frac{df(x,y)}{dy}}$$

Then turning the focus to the function in question $f(x,y,z,t,w) = \ln\left(x^4y^3z^6t^2w\right)$ we can calculate the derivative of w w.r.t. z as follows:

$$\frac{dw}{dz} = -\frac{\frac{df}{dz}}{\frac{df}{dw}} \qquad \text{By the IFT}$$
$$= -\frac{\frac{6x^4y^3z^5t^2w}{x^4y^3z^6t^2w}}{\frac{x^4y^3z^6t^2}{x^4y^3z^6t^2w}} \qquad \text{Cancelling common factors}$$
$$= -\frac{\frac{6}{z}}{\frac{1}{w}} = -\frac{6w}{z} \qquad \text{Rearranging}$$

Problem 5. Sketch the following function e^{-x^2} by showing analytically over what intervals the functions is increasing and decreasing, over what intervals the function is convex or concave and where are the maxima and minima of the function if at all.

First, to see where the function is increasing or decreasing we have to find where the first derivative is positive or negative:

$$\begin{aligned} f'(x) = \frac{\partial e^{-x^2}}{\partial x} > 0 \Leftrightarrow -2xe^{-x^2} > 0 \Leftrightarrow -2x > 0 \\ \Leftrightarrow x < 0 \Rightarrow f(x) \text{ increasing in } (-\infty, 0) \\ \text{By the same argument: } \text{if } x > 0 \Rightarrow f(x) \text{ decreasing in } (0, \infty) \\ \text{if } x = 0 \Rightarrow \text{Critical point: min, max or inflexion point} \end{aligned}$$

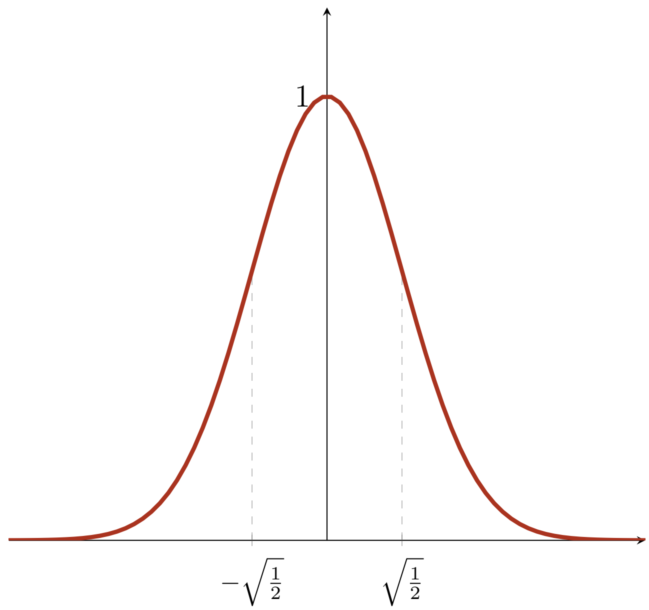
Second, we have to look where the function is concave up or concave down, i.e. where the second derivative is positive and where is negative.

$$\begin{aligned} f''(x) = \frac{\partial^2 e^{-x^2}}{\partial x^2} > 0 \Leftrightarrow 4x^2e^{-x^2} - 2d^{-x^2} > 0 \\ \Leftrightarrow 2(2x^2 - 1)e^{-x^2} > 0 \Leftrightarrow 2x^2 - 1 > 0 \\ \Leftrightarrow x^2 > \frac{1}{2} \Leftrightarrow x < -\sqrt{\frac{1}{2}} \cap x > \sqrt{\frac{1}{2}} \quad \text{Interval over which the function is concave up} \\ \text{By the same argument if } x^2 < \frac{1}{2} \Leftrightarrow -\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}} \quad \text{Interval over which the function is concave down} \\ \text{if } x^2 = \frac{1}{2} \Leftrightarrow x = -\sqrt{\frac{1}{2}} \text{ or } x = \sqrt{\frac{1}{2}} \quad \text{Points of inflection} \end{aligned}$$

Now, with all of this information we can say that because the critical point $x = 0$ belongs to the interval where the function is concave down, i.e. $\left(-\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}}\right)$ it is a maximum. In order to draw a sketch of the function we need to know where exactly the critical and inflection points lay in the xy-plane, then:

$$\begin{aligned} \text{Critical point: } x = 0 \Rightarrow f(0) = e^{-0^2} = 1 \Rightarrow \text{c.p. at } (x, y) = (0, 1) \\ \text{Inflexion points: } x = -\sqrt{\frac{1}{2}} \approx -0.7 \Rightarrow f\left(-\sqrt{\frac{1}{2}}\right) \approx 0.6 \Rightarrow \text{i.p. at } (x, y) \approx (-0.7, 0.6) \\ x = -\sqrt{\frac{1}{2}} \approx 0.7 \Rightarrow f\left(-\sqrt{\frac{1}{2}}\right) \approx 0.6 \Rightarrow \text{i.p. at } (x, y) \approx (0.7, 0.6) \end{aligned}$$

Now we have all the ingredients to sketch the curve:



Problem 6. Solve the following problem using the Khun-Tuker conditions:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = -x^2 - xy - y^2 \\ s.t. \quad & x - 2y \leq -1 \\ & 2x + y \leq 2 \end{aligned}$$

As a first step in this kind of problems the Lagrangian has to be set:

$$\mathcal{L} = -x^2 - xy - y^2 - \lambda_1(x - 2y + 1) - \lambda_2(2x + y - 2)$$

Secondly, state the FOC together with the binding restrictions for the general case:

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - y - \lambda_1 - 2\lambda_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -x - 2y + 2\lambda_1 - \lambda_2 = 0$$
$$x - 2y = -1 \qquad (\lambda_1)$$
$$2x + y = 2 \qquad (\lambda_2)$$

After setting the general form, proceed by cases removing the irrelevant information in each case:

case 1: $\lambda_1 = \lambda_2 = 0$
As a result we are left only with the derivatives of the function:

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - y = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -x - 2y = 0$$

Solving the system of equations the result is $(x*, y*) = (0, 0)$. As a final step, for this solution to be a candidate it has to meet all the KKT-conditions (Krausch Kuhn Tucker), thus inserting the values in the restrictions:

$$\left.\begin{array}{ll} x^* - 2y^* & \leq -1 \\ 2x^* + y^* & \leq 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} 0 - 2 \cdot 0 = 0 & \leq -1 \lightning \\ 2 \cdot 0 + 0 = 0 & \leq 2 \end{array}\right.$$

Because a contradiction has been found, we can stop there and the point can be discarded.

case 2: $\lambda_1 > 0$ and $\lambda_2 = 0$
In this case we have to include the first restriction corresponding to λ_1:

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - y - \lambda_1 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -x - 2y + 2\lambda_1 = 0 \tag{2}$$

$$x - 2y = -1 \qquad (\lambda_1) \tag{3}$$

Solve for λ_1 in (1) to substitute it in (2) and get:

$$5x + 4y = 0 \tag{4}$$

Now solve the system of equations $(3), (4)$

$$\left.\begin{array}{ll} 5x + 4y & = 0 \\ x - 2y & = -1 \end{array}\right\} \Rightarrow (x^*, y^*) = \left(-\frac{2}{7}, \frac{5}{14}\right)$$

Let's see if it fulfills the second constrain and the $\lambda_1 > 0$ condition.

$$\left.\begin{array}{lll} 2x^* + y^* & \leq 2 & (\lambda_2) \\ -2x^* - y^* - \lambda_1 & = 0 & (1) \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} -2\frac{2}{7} + \frac{5}{14} = -\frac{3}{14} & \leq 2 \\ \lambda_1 = 2\frac{2}{7} - \frac{5}{14} = \frac{3}{14} & \geq 0 \end{array}\right.$$

Hence, the point $(x^*, y^*) = \left(-\frac{2}{7}, \frac{5}{14}\right)$ meets the KKT-contions and it will be a candidate for an optimum.

case 3: $\lambda_1 = 0$ and $\lambda_2 > 0$
In this case we have to include the second restriction corresponding to λ_2:

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - y - 2\lambda_2 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -x - 2y - \lambda_2 = 0 \tag{2}$$

$$2x + y = 2 \qquad (\lambda_2) \tag{3}$$

Solve for λ_2 in (2) to substitute it in (1) and get:

$$y^* = 0 \tag{4}$$

$$\lambda_2 = -x \tag{5}$$

Now plug the value of y^* in (3) and get the value of $x^* = 1$
Let's see if it fulfills the first constrain and the $\lambda_2 > 0$ condition.

$$x^* - 2y^* = -1 \leq -1 \qquad (\lambda_2)$$
$$\lambda_2 = -x^* = -1 > 0 \; \lightning \qquad (5)$$

Since we have arrived to a contradiction we can discard **case 3** and consequently point $(x^*, y^*) = (1, 0)$

case 4: $\lambda_1 > 0$ and $\lambda_2 > 0$
This case in a sense is much easier since we have to meet the constrains with equality, we can just solve the system of equations generated by the constraints and check if it meets the KKT-condtions, i.e. both λ's > 0:

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - y - \lambda_1 - 2\lambda_2 = 0 \qquad (1)$$
$$\frac{\partial \mathcal{L}}{\partial y} = -x - 2y + 2\lambda_1 - \lambda_2 = 0 \qquad (2)$$
$$x - 2y = -1 \qquad (\lambda_1) \qquad (3)$$
$$2x + y = 2 \qquad (\lambda_2) \qquad (4)$$

Solve the system of equations $(3), (4)$:

$$\left.\begin{aligned} x - 2y &= -1 \\ 2x + y &= 2 \end{aligned}\right\} \Rightarrow (x^*, y^*) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

Then use the FOC to solve for the λ's:

$$\left.\begin{aligned} -2x^* - y^* - \lambda_1^* - 2\lambda_2^* &= 0 \\ -x^* - 2y^* + 2\lambda_1^* - \lambda_2^* &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} -2\frac{3}{5} - \frac{4}{5} - \lambda_1^* - 2\lambda_2^* &= 0 \\ -\frac{3}{5} - 2\frac{4}{5} + 2\lambda_1^* - \lambda_2^* &= 0 \end{cases}$$
$$\begin{aligned} -\lambda_1^* - 2\lambda_2^* &= 2 \\ 2\lambda_1^* - \lambda_2^* &= \tfrac{11}{5} \end{aligned} \Rightarrow \begin{cases} \lambda_1^* &= \frac{12}{50} \\ \lambda_2^* &= -\frac{31}{25} \quad > 0 \; \lightning \end{cases}$$

Again we have arrived to a contradiction, being able to discard the point $(x^*, y^*) = \left(\frac{3}{5}, \frac{4}{5}\right)$

Second order conditions. All in all, we are left with just one possible solution $(x^*, y^*) = \left(-\frac{2}{7}, \frac{5}{14}\right)$. In order to check if it is a maximum, minimum or an saddle point we can use the SOC by means of the Hessian:

$$H(x^*, y^*) = \begin{pmatrix} f_{xx}(x^*, y^*) & f_{xy}(x^*, y^*) \\ f_{yx}(x^*, y^*) & f_{yy}(x^*, y^*) \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

As we see the hessian is positive definite regardless of where the function is (the function is globally concave) and thus the point is a maximum.