Risk-Constrained Kelly Gambling

Optimization and Algorithms Project Presentation

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Outline

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Part I - Implementation in CVX

- 1. Kelly Problem
- 2. Risk-Constrained Kelly Problem
- 3. Our implementation
- 4. Results

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1. Kelly problem

- We consider the classic Kelly gambling problem with general distributions of outcomes.

- John Kelly proposed a systematic way to allocate a total wealth across a number of bets so as to maximize the long term growth rate when the gamble is repeated [1].

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1. Kelly problem

- With this strategy – Kelly optimal bets – there is a risk of the wealth dropping substantially from its original value before increasing.

- The wealth growth is given by

$$w_t = (r_1^T b) \ldots (r_{t-1} b)$$

with t equal to 1,2,… with I.I.D. returns.

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1. Kelly problem

- The goal is to chose *b* so that the wealth becomes larger.

- In Kelly gambling we want to find a b in order to maximize $\boldsymbol{E}\log(r^T b)$, the growth rate of wealth.

$$\begin{array}{ll} maximize & \boldsymbol{E}\log(r^T b) \\ subject\ to & \mathbf{1}^T b = 1, b \geq 0 \end{array}$$

- The solution of b optimal values is called *Kelly Optimal bets*.

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2. Risk-Constrained Kelly Problem

- In Risk-Contrained Kelly we add a drawdown risk contraint:

$$\begin{aligned} &maximize && \boldsymbol{E}\log(r^T b) \\ &subject\ to && \boldsymbol{1}^T b = 1, b \geq 0 \\ & && \boldsymbol{Prob}\left(W^{min} < \alpha\right) < \beta \end{aligned}$$

- The new contraint limits the probability of a drop in wealth to value α to be no more than β.

2. Risk-Constrained Kelly Problem

- A bound of the drawdown risk is developed to make the problem easily solvable in CVX.

$$\boldsymbol{Prob}(W^{min} < \alpha) < \alpha^{\lambda} = \beta$$

- It is valid as long α and β are between 0 and 1 and $\lambda = \frac{\log(\beta)}{\log(\alpha)}$.

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2. Risk-Constrained Kelly Problem

- The parameter λ is the risk aversion parameter.

- A higher value means the problem only considers risk-free bets while a low value limits the ratio of variance to the mean growth.

Part I - Implementation in CVX

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3. Our implementation

- The Kelly gambling problem is a convex optimization problem since the objective is concave and the contraints are convex.

- We inserted the function to maximize and the contraints directly on CVX.

3. Our implementation

- For the Kelly problem we used

$$\begin{aligned} &\mathit{maximize} && \sum_{i=1}^{K} \pi_i \log(r^T b) \\ &\mathit{subject\ to} && \mathbf{1}^T b = 1, b \geq 0 \end{aligned}$$

- And for the Risk-constrained Kelly problem we used

$$\begin{aligned} &\mathit{maximize} && \sum_{i=1}^{K} \pi_i \log(r^T b) \\ &\mathit{subject\ to} && \mathbf{1}^T b = 1, b \geq 0 \\ & && \log\left(\sum_{i=1}^{K} \exp(\log \pi_i - \lambda \log\left(r_i^T b\right))\right) \leq 0 \end{aligned}$$

Part I - Implementation in CVX

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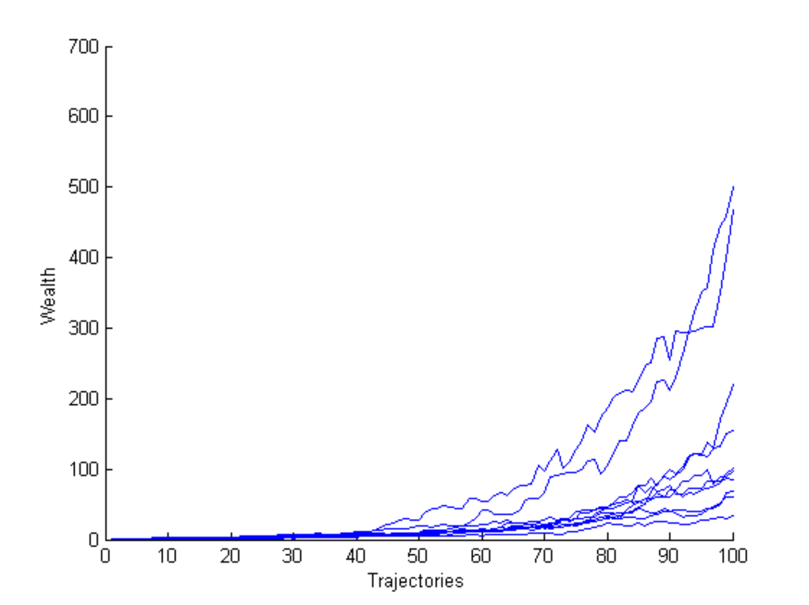
4. Results

- Wealth trajectories for the Kelly optimal bet.



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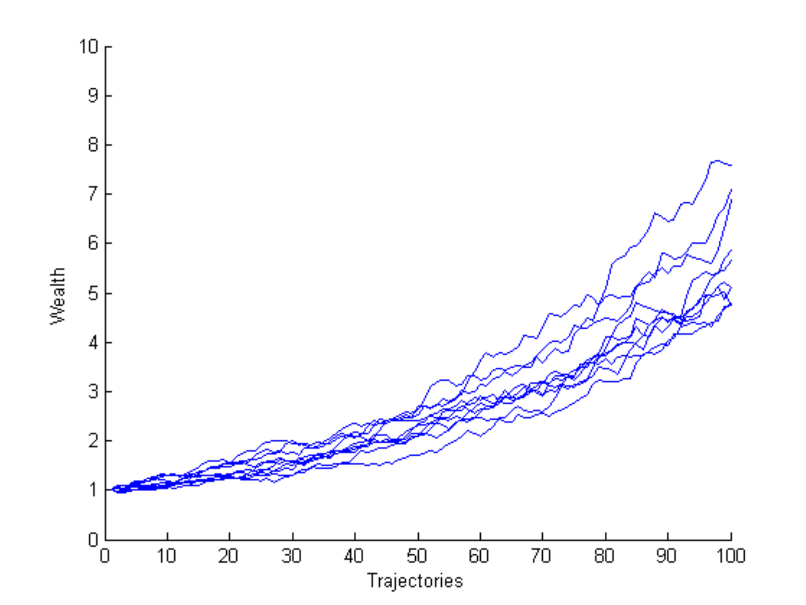
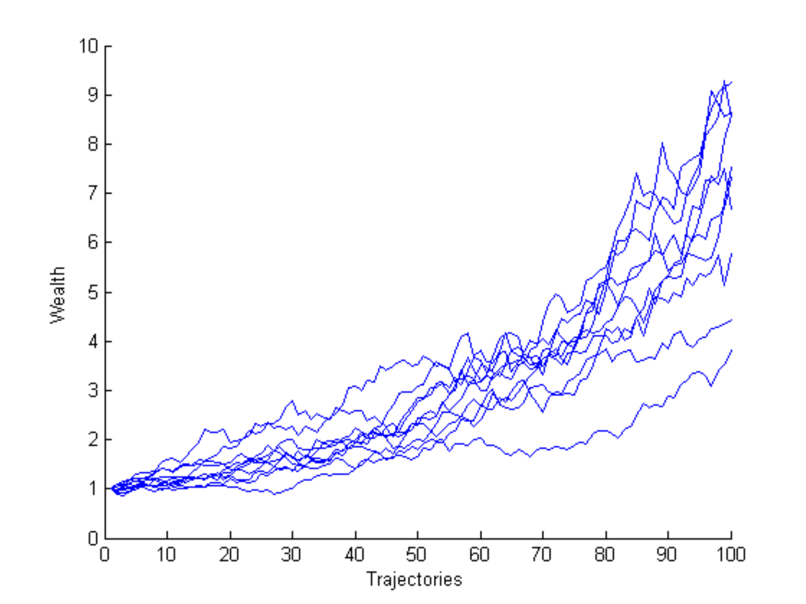
4. Results

- Wealth trajectories for the Risk-Constrained Kelly optimal bet.



4. Results

- Wealth trajectories for a random bet.



4. Results

- We can conclude that:
  - The Risk-Contrained Kelly with the selected parameters represent a **conservative approach** for betting;
  - Solving an optimization problem translates in a **better final result** given the contraints.

Part II – The Optimal Strategy

- 1. Problem Formulation
- 2. Our implementation
- 3. Results

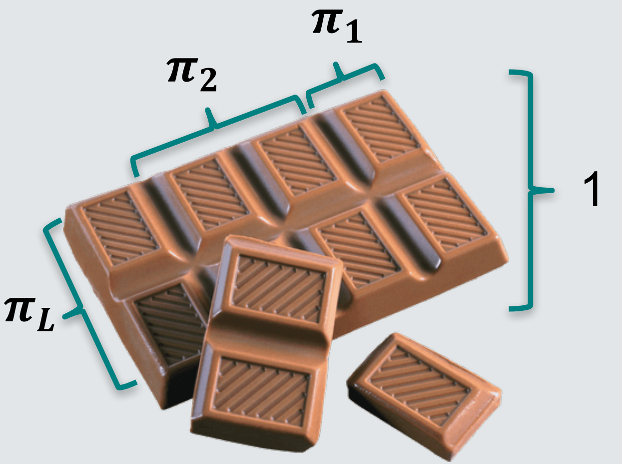
1. Problem Formulation

- Only one bet;

- We cannot evaluate the performance of the optimal strategy;
- Still using Kelly gambling;

1. Problem Formulation

- We have L hypothesis for vector $\boldsymbol{\pi_i}$



- *Consider no drawdown* λ = 0

1. Problem Formulation

- The goal is to chose b so that the wealth becomes larger.

- In Kelly gambling we wanted to find a b in order to maximize $\boldsymbol{E}\log(r^T b)$, the growth rate of wealth.

$$\begin{aligned} &maximize\ \boldsymbol{\pi_i}\log(r^T b) \\ &subject\ to \quad \mathbf{1}^T b = 1, b \geq 0 \end{aligned}$$

- The solution of b optimal values cannot depend on $\boldsymbol{\pi_i}$.

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1. Problem Formulation

- We came up with 2 solutions

Since we bet just one…

$$\dots r \in R_+$$

- The First one:

$$\overline{\boldsymbol{\pi}_i} = \sum_{i=1}^{L} \left(\frac{1}{L} * \boldsymbol{\pi}_i \right) \Rightarrow \begin{array}{ll} \underset{b}{maximize} & \overline{\boldsymbol{\pi}_i} * \log(r^T b) \\ subject\ to & \mathbf{1}^T b = 1, b \geq 0 \\ & \mathbf{1}^T \boldsymbol{\pi}_i = 1, \boldsymbol{\pi}_i \geq 0 \end{array}$$

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1. Problem Formulation

- It was an aproximation of the L, $\boldsymbol{\pi_i}$ vectors;
- So in average we could still get the best probability outcome for the best value of the return matrix but with a low probability to happen;

1. Problem Formulation

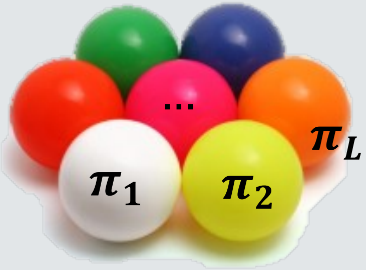
- The Second one:

$$\sum_{i=1}^{L}\left(\sum_{i=1}^{K}\boldsymbol{\pi_i} * log(r_i{}^T\boldsymbol{b})\right)$$

$$\underset{b}{maximize} \sum_{i=1}^{L}\left(\sum_{i=1}^{K}\boldsymbol{\pi_i} * log(r_i{}^T\boldsymbol{b})\right)$$

$$subject\ to \quad \mathbf{1}^T b = 1, b \geq 0$$

$$\mathbf{1}^T\boldsymbol{\pi_i} = 1, \boldsymbol{\pi_i} \geq 0$$



1. Problem Formulation

- Wrapping up our final formulation of the problem is:

$$\underset{\boldsymbol{b}}{\boldsymbol{maximize}} \sum_{i=1}^{L} \left(\sum_{i=1}^{K} \boldsymbol{\pi_i} * log(r_i{}^T \boldsymbol{b}) \right)$$

$$\boldsymbol{subject\ to} \quad \mathbf{1}^T b = 1, b \geq 0$$
$$\mathbf{1}^T \boldsymbol{\pi_i} = 1, \boldsymbol{\pi_i} \geq 0$$

- Which would give us a sub-optimal solution, instead of na optimal, since it depends on the $\boldsymbol{\pi_i}$ vector.

Part II – The Optimal Strategy

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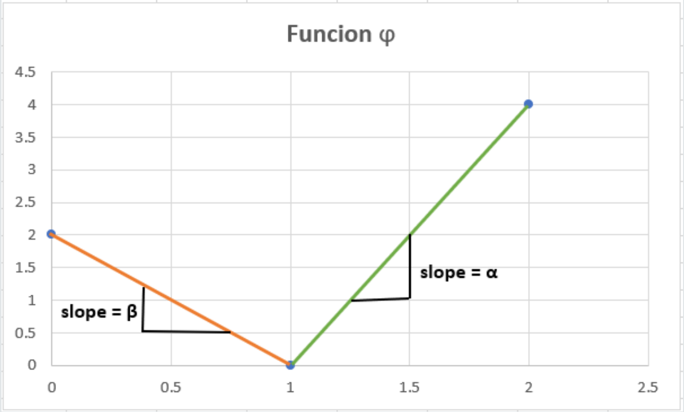
2. Our Implementation

$$maximize\ Elog(r^T b) - \sum_{K=1}^{N-1} \emptyset_K(b_K)$$

$$subject\ to \sum b = 1\ , b \geq 0$$

How to compute Ø ???



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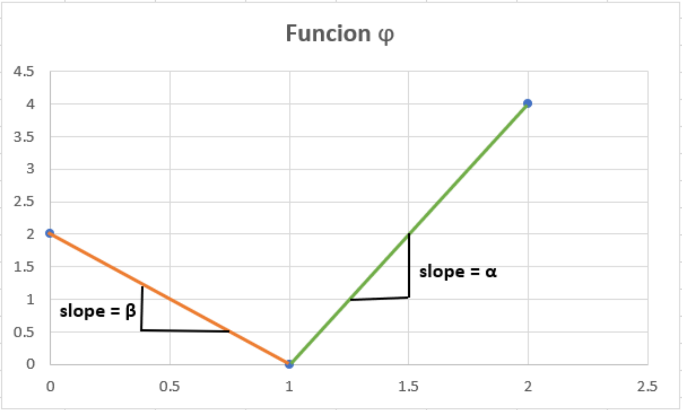
2. Our Implementation

$$\left.\begin{aligned} \emptyset_{k\,left} &= \beta * x + b_{left} \\ \emptyset_{k\,right} &= \beta * x + b_{right} \end{aligned}\right\} \quad b_{left}\,?\, b_{right}?$$

$$\emptyset_k = 0 \quad \Rightarrow \quad b_{left} = -\beta * b_{ref}\;;\; b_{right} = -\alpha * b_{ref}$$

2. Our Implementation

$$\phi_k = \frac{\phi_{k\,left} + abs(\phi_{k\,left})}{2} + \frac{\phi_{k\,right} + abs(\phi_{k\,right})}{2}$$



Part II – The Optimal Strategy

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3. Results

β	α	b_{ref}	b	ϕ	f
-0,0091	0,00222	3,15E-02	5,58E-10	1,44E-05	0,02599
-0,0126	0,00313	3,80E-02	3,90E-02	6,32E-11	0,02599
-0,0026	0,00164	1,03E-11	3,14E-10	6,22E-05	0,02267
-0,0062	0,00901	6,71E-02	1,02E-01	1,82E-04	0,02530
-0,0043	0,00816	4,35E-02	1,14E-09	2,34E-05	0,02596
-0,0057	0,00780	5,36E-11	3,07E-10	2,96E-04	0,02267
-0,0075	0,00175	2,40E-02	1,14E-01	2,45E-05	0,02510
-0,0068	0,00136	2,49E-02	8,61E-10	1,79E-05	0,02517
-0,0165	0,00439	3,60E-02	8,07E-10	8,88E-06	0,02597
-0,0084	0,00036	7,34E-12	3,09E-10	1,35E-05	0,02267
-0,0124	0,00364	3,14E-02	7,27E-10	2,42E-05	0,02572
-0,0193	0,00194	7,27E-02	3,37E-01	6,69E-04	0,02491
-0,0159	0,00843	5,83E-12	2,66E-10	3,20E-04	0,02267
-0,0025	0,00382	1,47E-11	2,77E-10	1,45E-04	0,02267
-0,0005	0,00227	5,23E-12	2,33E-10	8,61E-05	0,02267
-0,0133	0,00901	3,92E-11	8,52E-10	3,42E-04	0,02267
-0,0153	0,00580	5,52E-02	2,50E-02	2,64E-04	0,02576
-0,0159	0,00884	3,56E-02	8,36E-02	2,07E-05	0,02596
-0,0084	0,00767	6,72E-02	3,00E-01	2,46E-04	0,02529
0,0000	0,00000	4,73E-01	4,51E-10	0	-0,01161

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Thank you for your attention

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