Optimization and Algorithms

Risk-Constrained Kelly Gambling

Group 2

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***Abstract*—For this assignment we consider the classic Kelly gambling problem with a general distribution of outcomes and the Kelly problem with an additional risk constraint previously discussed in [1]. The constraint limits the probability of a drawdown of wealth to a given undesirable level. The method presented is parameterized by a single parameter that has a natural interpretation as a risk-aversion parameter. Additionally we formulated the problem for a strategy following the same Kelly problem, this time without knowledge on the returns. A optimal solution for a transactions based problem is also presented.**

***Index Terms*—Kelly Gambling problem, Optimization, CVX, Risk aversion, Optimal bets**

I. Introduction

In Kelly Gambling, a fraction of our wealth is placed on n bets. The n bets have a random non-negative return and a probability associated to it. We consider IID (Independent and Identically Distributed) returns and uniformly distributed probabilities. We consider two problems. Firstly the computation of the optimal Kelly bets that will maximize the growth rate of our wealth and then the insertion of an additional risk aversion parameter and analyze how the problem behaves. This parameter will bound the drawdown risk. Drawdown can be seen as the amount that our wealth dropped since it's initial value (one), before eventually increasing (happens with probability one). Our goal is to perform a trade-off so we can balance bets risks (probabilities) with bets payoffs and maximize our wealth. Adding a drawdown risk constraint to the Kelly gambling problem becomes a difficult optimization problem in general. In this paper we explain how to develop a bound on the drawdown risk that results in an approachable convex constraint on b. We will later evaluate it throw simulations.

The paper is organized in the following way. In Section II, we introduce the Kelly gambling problem and discuss the results of the simulations done. Section III describes the formulation of a possible solution of the Kelly gambling problem without the previous knowledge of the returns matrix. Finally, in Section IV, we approach a different problem related to transactions were we discuss a optimal solution.

II. Kelly problems formulation

A. Kelly gambling

Let $b \in \mathbb{R}^n$ be the fraction of our wealth $\boldsymbol{w}$ we gamble. Bets have a return $r \in \mathbb{R}^n_+$ such that after betting, our wealth becomes $r^T b$. Some considerations were made in order to proceed with the problem formulation:

1) $b \geq 0$, non-negative bets $\mathbf{1}^T b = 1$, where $\mathbf{1}^T$ is a vector with all components 1
2) Finite returns for all bets, i.e., $\mathbf{E} r_i < \infty$
3) The bet n has a certain return of one, i.e., $r_n = 1$. This means that b_n represents the amount of our wealth we do not gamble

The gamble is repeated at epochs $t = 1, 2, ...$ With an initial wealth $w_1 = 1$, the wealth at time time t is given by

$$w_t = (r_1^T b)...(r_{t-1}^T b)$$

or by taking it's logarithm

$$v_t = log(r_1^T b) + ... + log(r_{t-1}^T b)$$

w_t and v_t are both random walks with distributions determined by the choice of the bet $\mathbf{b}$, as well as the distribution of return vector$\mathbf{r}$.

Let us interpret $\mathbf{E} log(r^T b)$ as the growth rate of the wealth; it represents the deviation of v_t. In Kelly Gambling we choose $\mathbf{b}$ in order to maximize it. This leads to the optimization problem

$$\begin{array}{ll} \text{maximize} & \mathbf{E} log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \end{array} \tag{1}$$

When the return distribution is finite the Kelly gambling problem reduces to

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{K} \pi_i log(r_i^T) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \end{array} \tag{2}$$

which is readily solved using convex optimization. r_i stands for the return at epoch $t = i$ and π_i for the probability of return vector r_i.

B. Risk-Constrained Kelly gambling

Besides studying this problem, we also take a look at risk-constrained Kelly gambling. We define the minimum wealth as the infimum of the wealth trajectory over time,

$$W^{min} = \inf_{t=1,2,..} w_t \tag{3}$$

With $b = e_n$ (all components equal 0 except $b_n = 1$), we have $w_t = 1$ for all t, so $W_{min} = 1$. With b for which $\mathbf{E} log(r^T b) > 0, W_{min}$ takes values in [0,1]. The drawdown is given by $1 - W_{min}$. A drawdown of 0.4 means that our wealth has decreased to less 40% of it's initial value before increasing (which it eventually must do, since $v_t \to \infty$ for certain). The drawdown risk is defined as $\mathbf{Prob}(W_{min} < \alpha)$ where $\alpha \in [0, 1]$. As an example a drawdown risk of 0.2 for $\alpha = 0.7$ means the probability of experiencing more than 30% drawdown is only 20%. The smaller the drawdown risk the better. This risk depends on the bet vector b in a very complicated way. By adding a constraint, β, to the risk we obtain the problem

$$\begin{array}{ll} \text{maximize} & \mathbf{E}log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & \mathbf{Prob}(W_{min} < \alpha) < \beta \end{array} \tag{4}$$

with variable b, where $\alpha, \beta \in [0, 1]$ are given parameters. β limits the probability of a drop in wealth to value α.

From [1] we find a condition that bounds the drawdown risk. For any $\alpha \in$ [0,1] and $\beta \in$ [0,1] that satisfies $\lambda = log\beta/log\alpha$ we have

$$\mathbf{E}(r^T b)^{-\lambda} \leq 1 \Longrightarrow Prob(W_{min} < \alpha) < \alpha^\lambda = \beta \tag{5}$$

With this we get the next optimization problem

$$\begin{array}{ll} \text{maximize} & \mathbf{E}log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & \mathbf{E}(r^T b)^{-\lambda} \leq 1 \end{array} \tag{6}$$

For the finite outcomes case we can restate the just mentioned RCK problem in a convenient and tractable form. We first take the log of the last constraint and get

$$log \sum_{i=1}^{K} \pi_i (r_i^T b)^{-\lambda} \leq 0 \tag{7}$$

that can be written as

$$log \left(\sum_{i=1}^{K} exp(log\pi_i) - \lambda log(r_i^T b) \right) \leq 0 \tag{8}$$

We can finally define the risk constrained optimization problem as

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{K} \pi_i log(r_i^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & log\left(\sum_{i=1}^{K} exp(log\pi_i) - \lambda log(r_i^T b)\right) \leq 0 \end{array} \tag{9}$$

The presented problem is convex, since the objective is concave and the contraints are convex, so it was used a convex optimization software "CVX: Software for Disciplined Convex Programming" [2] to handle it.

C. Simulation results

After testing the problems with CVX software we obtained the figures 1, 2 and 3. For all the simulations, 1000 iterations were done. To improve the readability of the figures, only 10 out of 1000 trajectories are displayed. Since the main objective of plotting these graphics is to compare the wealth growth rates of each method, every trajectory is computed against the same returns vector at each time instant.

In Figure 1 we test the Kelly problem in CVX while in Figure 2 the Risk-Contraint Kelly is tested. Figure 3 presents a random betting problem, containing the same conditions (the sum of b values must be equal to 1 and $b > 0$), in order to test the value of having an optimization problem instead of having a random decision in each instant.

For the risk-constrained Kelly we used the given values $\alpha = 0.9$ and $\beta = 0.05$ and obtained the RCK bets for $\lambda = 28.4332$. For this values we observed that the wealth increases at a much higher rate (more than 10 times) for the first case when comparing to the risk-constrained case. Consider an average iteration line used for guidance in the three cases. We can see that in 2 and 3, the ten iterations are more dispersed from it than in 1. This comes from introducing the risk aversion parameter α. We could also check that, for smaller values of λ (α closer to 0 and β closer to 1) the bets vector b from RCK gets closer to the optimal one (dispersion increases), and vice-versa. Finally we can conclude that this value of λ offers a conservative approach for betting.

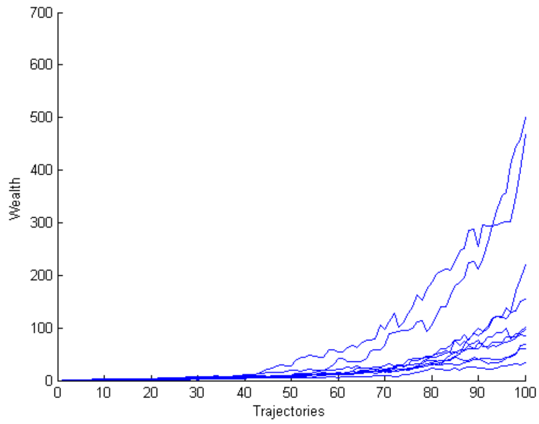

Fig. 1. Result of 10 trajectories out of 1000 simulated for the Kelly gambling problem.

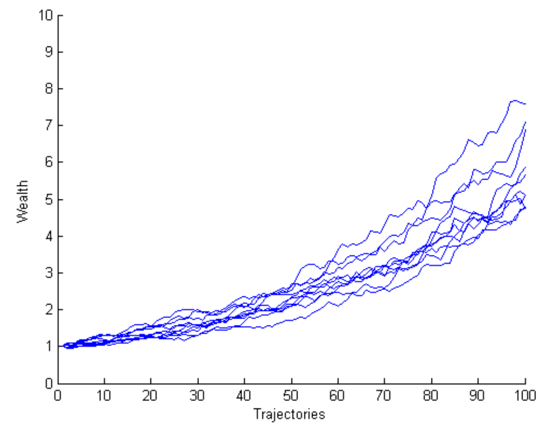

Fig. 2. Result of 10 trajectories out of 1000 simulated for the Risk-Constrained Kelly gambling problem.

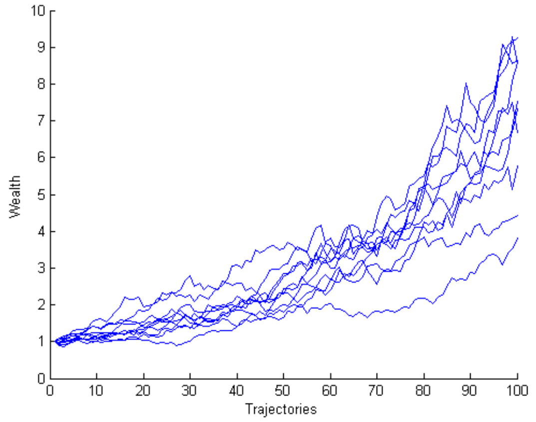

Fig. 3. Result of 10 trajectories out of 1000 simulated for a random gambling problem.

III. Kelly gambling without previous return knowledge.

In this part we were proposed to formulate the Kelly Gambling problem, but this time without knowing the probability of each and everyone of the π_i vector. It's given that we have L possible and finite hypothesis for vector π_i and we would just be betting once.
Because of this we have insufficient values to compute the maximum value of **b**. So we had to formulate the original problem (without the drawdown, $\lambda = 0$) and find out the optimal and robust strategy for each of the vector pi. A problem appeared when we could not evaluate the optimal strategy performance because we did not have the probability of each of the probability vectors, although it's assumed the problem giving the optimal strategy does not depend on the probability vector.
So in Kelly gambling we wanted to find a b in order to maximize the growth rate of wealth. We found 2 solutions for the problem. The second, we consider to be better than the the first, nonetheless we though it was a good way to explain our reasoning.

$$\begin{array}{ll} \text{maximize} & \mathbf{E} log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & \mathbf{1}^T \pi = 1, \quad \pi \geq 0 \end{array} \tag{10}$$

So we started to make the average of all values for the vector π_i, which gave us all the values for the vectors π_i summed with each other and then we divided them by L (the probability of π_i being picked).
We then come up to our final conclusion of:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{L} (\sum_{i=1}^{K} \pi_i log(r^T b)) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & \mathbf{1}^T \pi = 1, \quad \pi \geq 0 \end{array} \tag{11}$$

This represents the average of all the π_i with probability of L for each one (but different from one another), and computed for all K from 1 to $K = 100$. With this expression we could calculate the maximum value for b, based on the probability of the outcome.

This solution was not optimal, since we still use the π_i vector. So we only get a sub-optimal solution for this problem.

IV. Optimal strategy for transactions

Let ϕ_k be the risk function for a certain bet b_k. This function is composed of transaction costs α and β in case it is decided to change strategy after taking an initial one. We need to decide what strategy b_k is better knowing the risk associated to a change of strategy. We define this risk constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & \mathbf{E} log(r^T b) - \sum \phi_k(b_k) \\ \text{subject to} & \sum b = 1, \quad b \geq 0 \end{array} \tag{12}$$

We know that α and β are generic and we need to define the risk function associated to this transaction costs in a single expression since CVX can't use functions defined by branches. This function ϕ_k it's defined in Figure 4.

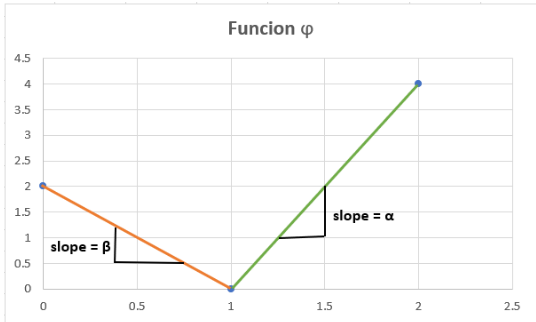

Fig. 4. ϕ Function

To get a single expression for ϕ_k we decide do analyze each part of the function independently. If we chose a change of strategy b_k to the left of the strategy of reference b_{ref} we need to see a line with slope β. On the other hand, if we chose a change of strategy b_k to the right of the strategy of reference b_{ref} we need to see a line with slope α.
The first step was to calculate the line equations parameters. We noticed that the two lines converged to the strategy of reference b_{ref} when $\phi_k = 0$ therefore we can obtain the the y-coordinate of the y-interception:

$$\phi_{kleft} = \beta * b_k + b_{left}; \phi_{kright} = \alpha * b_k + b_{right} \tag{13}$$

$$\phi_k = 0 \Longrightarrow b_{left} = -\beta * b_k; b_{right} = -\alpha * b_k \tag{14}$$

$$b_k = -(b_{left}/\beta) = b_k = -(b_{right}/\alpha) = b_{ref} \tag{15}$$

$$b_{left} = -\beta * b_{ref}; b_{right} = -\alpha * b_{ref} \tag{16}$$

Now we have established the lines parameters to both sides of the function:

$$\phi_{kleft} = \beta * b_k - \beta * b_{ref} \tag{17}$$

$$\phi_{kright} = \alpha * b_k - \alpha * b_{ref} \tag{18}$$

A problem came across since we can't use this functions in CVX, the interesting and logical solution that was developed is:

$$\phi_k = 1/2 * (\phi_{kleft} + abs(\phi_{kleft}) + \phi_{kright} + abs(\phi_{kright})) \tag{19}$$

Note that if we choose a change of strategy b_k to the left of the strategy of reference b_{ref}, ϕ_{kleft} is positive and $abs(\phi_{kleft})$ is positive too. On other hand, ϕ_{kright} is negative but $abs(\phi_{kright})$ is positive so they cancel each other and the 1/2 factor normalizes ϕ_{kleft}. For the other side of the function, if we chose a change of strategy b_k to the right of the strategy of reference b_{ref}, ϕ_{kright} is positive and $abs(\phi_{kright})$ is positive too, in other hand, ϕ_{kleft} is negative but $abs(\phi_{kleft})$ is positive so they cancel each other and the 1/2 factor normalizes ϕ_{krigth}. This function satisfies both cases of the risk function ϕ_k so we are ready to use CVX.

A. Numerical results

With the data set provided by professor, we have 20 values for α, β and b_{ref} so we can compute on MatLab the function f that maximizes the average growth of wealth having in consideration the risk function ϕ_k. Each α and β ar associated to a b_{ref} so we can calculate each risk function for each case of the 20. The results from the risk function for each b_{ref} are in table I.

TABLE I
RESULTS OF OUR IMPLEMENTATION.

β	α	b_{ref}	b	ϕ	f
-0,0091	0,00222	3,15E-02	5,58E-10	1,44E-05	0,02599
-0,0126	0,00313	3,80E-02	3,90E-02	6,32E-11	0,02599
-0,0026	0,00164	1,03E-11	3,14E-10	6,22E-05	0,02267
-0,0062	0,00901	6,71E-02	1,02E-01	1,82E-04	0,02530
-0,0043	0,00816	4,35E-02	1,14E-09	2,34E-05	0,02596
-0,0057	0,00780	5,36E-11	3,07E-10	2,96E-04	0,02267
-0,0075	0,00175	2,40E-02	1,14E-01	2,45E-05	0,02510
-0,0068	0,00136	2,49E-02	8,61E-10	1,79E-05	0,02517
-0,0165	0,00439	3,60E-02	8,07E-10	8,88E-06	0,02597
-0,0084	0,00036	7,34E-12	3,09E-10	1,35E-05	0,02267
-0,0124	0,00364	3,14E-02	7,27E-10	2,42E-05	0,02572
-0,0193	0,00194	7,27E-02	3,37E-01	6,69E-04	0,02491
-0,0159	0,00843	5,83E-12	2,66E-10	3,20E-04	0,02267
-0,0025	0,00382	1,47E-11	2,77E-10	1,45E-04	0,02267
-0,0005	0,00227	5,23E-12	2,33E-10	8,61E-05	0,02267
-0,0133	0,00901	3,92E-11	8,52E-10	3,42E-04	0,02267
-0,0153	0,00580	5,52E-02	2,50E-02	2,64E-04	0,02576
-0,0159	0,00884	3,56E-02	8,36E-02	2,07E-05	0,02596
-0,0084	0,00767	6,72E-02	3,00E-01	2,46E-04	0,02529
0,0000	0,00000	4,73E-01	4,51E-10	0	-0,01161

After the analysis of the results obtained by the implementation of the optimal strategy we can conclude that the best strategy $b_k = 0.0380$ which means that the best solution for this problem is selecting the second inherited strategy $b_{ref} = 2$. But this could be easily predicted before, as one could watch the vector α_{20} and β_{20} and checked that their value was 0, meaning we were on a present of 2 functions (defined by branches) with no slope , so we would have no cost in changing our strategy. Our ϕ_{20} was zero for every strategy we choose. Concluding the best strategy is in fact $b_k = 0.0380$, because is the optimal strategy that gives us the maximum value for wealth and the lowest risk cost of ϕ_k.

REFERENCES

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