13.7 Example 11 – vehicle location estimation using radar

I introduced this example earlier (subsection 13.5.1) when I explained the multivariate uncertainty projection concept.

We want to track the vehicle using radar. The radar is located at the plane origin, and it measures the vehicle range (r) and the bearing angle (φ).

The radar measurement error distribution is Gaussian. We assume constant acceleration dynamics.

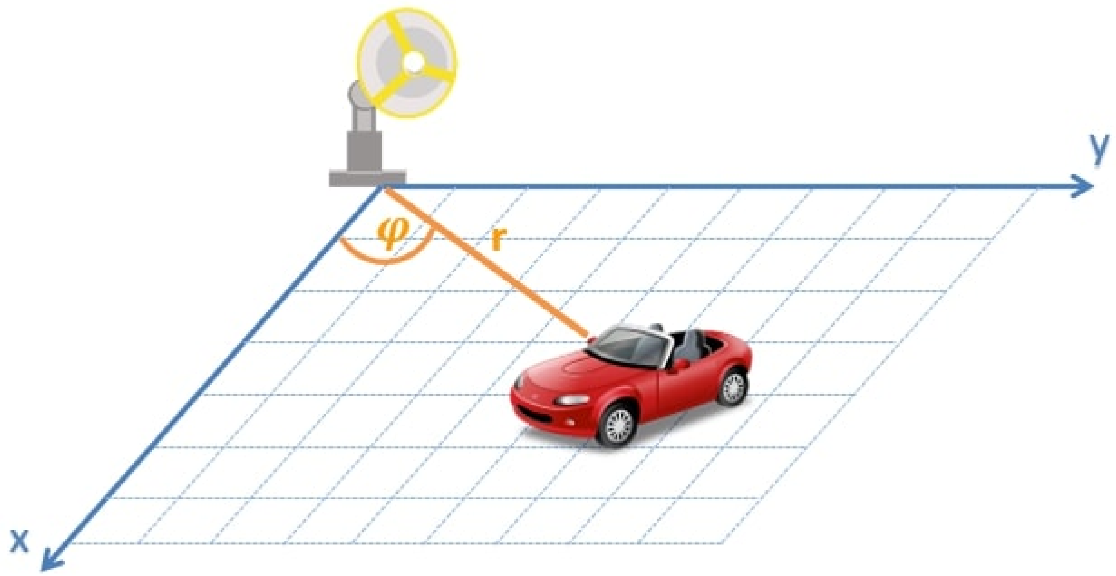

Figure 13.5: *Vehicle location estimation using radar.*

13.7.1 Kalman Filter equations

The state extrapolation equation

Similarly to example 9 (section 9.1), the state extrapolation equation is:

$$\hat{\boldsymbol{x}}_{n+1,n} = \boldsymbol{F}\hat{\boldsymbol{x}}_{n,n} \tag{13.57}$$

The system state $\boldsymbol{x}_n$ is defined by:

$$\boldsymbol{x}_n = \begin{bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \\ y_n \\ \dot{y}_n \\ \ddot{y}_n \end{bmatrix} \tag{13.58}$$

The extrapolated vehicle state for time $n+1$ can be described as follows:

$$\begin{bmatrix} \hat{x}_{n+1,n} \\ \hat{\dot{x}}_{n+1,n} \\ \hat{\ddot{x}}_{n+1,n} \\ \hat{y}_{n+1,n} \\ \hat{\dot{y}}_{n+1,n} \\ \hat{\ddot{y}}_{n+1,n} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & 0.5\Delta t^2 & 0 & 0 & 0 \\ 0 & 1 & \Delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta t & 0.5\Delta t^2 \\ 0 & 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{n,n} \\ \hat{\dot{x}}_{n,n} \\ \hat{\ddot{x}}_{n,n} \\ \hat{y}_{n,n} \\ \hat{\dot{y}}_{n,n} \\ \hat{\ddot{y}}_{n,n} \end{bmatrix} \tag{13.59}$$

The dynamic model of the system (the second type of non-linearity) in this example is linear! There is no need to calculate the Jacobian $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$.

The covariance extrapolation equation

The Covariance Extrapolation Equation is similar to example 9 (section 9.1):

$$\boldsymbol{P}_{n+1,n} = \boldsymbol{F}\boldsymbol{P}_{n,n}\boldsymbol{F}^T + \boldsymbol{Q} \tag{13.60}$$

The estimate covariance is:

$$\boldsymbol{P} = \begin{bmatrix} p_x & p_{x\dot{x}} & p_{x\ddot{x}} & 0 & 0 & 0 \\ p_{\dot{x}x} & p_{\dot{x}} & p_{\dot{x}\ddot{x}} & 0 & 0 & 0 \\ p_{\ddot{x}x} & p_{\ddot{x}\dot{x}} & p_{\ddot{x}} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_y & p_{y\dot{y}} & p_{y\ddot{y}} \\ 0 & 0 & 0 & p_{\dot{y}y} & p_{\dot{y}} & p_{\dot{y}\ddot{y}} \\ 0 & 0 & 0 & p_{\ddot{y}y} & p_{\ddot{y}\dot{y}} & p_{\ddot{y}} \end{bmatrix} \tag{13.61}$$

The elements on the main diagonal of the matrix are the variances of the estimation:

- p_x is the variance of the X coordinate position estimation
- $p_{\dot{x}}$ is the variance of the X coordinate velocity estimation
- $p_{\ddot{x}}$ is the variance of the X coordinate acceleration estimation
- p_y is the variance of the Y coordinate position estimation
- $p_{\dot{y}}$ is the variance of the Y coordinate velocity estimation
- $p_{\ddot{y}}$ is the variance of the Y coordinate acceleration estimation
- The off-diagonal entries are covariances

The process noise matrix

The process noise matrix is also similar to example 9 (section 9.1):

$$\boldsymbol{Q} = \begin{bmatrix} \sigma_x^2 & \sigma_{x\dot{x}}^2 & \sigma_{x\ddot{x}}^2 & 0 & 0 & 0 \\ \sigma_{\dot{x}x}^2 & \sigma_{\dot{x}}^2 & \sigma_{\dot{x}\ddot{x}}^2 & 0 & 0 & 0 \\ \sigma_{\ddot{x}x}^2 & \sigma_{\ddot{x}\dot{x}}^2 & \sigma_{\ddot{x}}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_y^2 & \sigma_{y\dot{y}}^2 & \sigma_{y\ddot{y}}^2 \\ 0 & 0 & 0 & \sigma_{\dot{y}y}^2 & \sigma_{\dot{y}}^2 & \sigma_{\dot{y}\ddot{y}}^2 \\ 0 & 0 & 0 & \sigma_{\ddot{y}y}^2 & \sigma_{\ddot{y}\dot{y}}^2 & \sigma_{\ddot{y}}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} & 0 & 0 & 0 \\ \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t & 0 & 0 & 0 \\ \frac{\Delta t^2}{2} & \Delta t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} \\ 0 & 0 & 0 & \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t \\ 0 & 0 & 0 & \frac{\Delta t^2}{2} & \Delta t & 1 \end{bmatrix} \sigma_a^2 \tag{13.62}$$

Where:

- Δt is the time between successive measurements
- σ_a^2 is a random variance in acceleration

The measurement equation

The measurement equation is different from example 9.

The measurement vector $\boldsymbol{z}_n$ is:

$$\boldsymbol{z}_n = \begin{bmatrix} r_n \\ \varphi_n \end{bmatrix} \tag{13.63}$$

The system state vector $\boldsymbol{x}_n$ is defined by:

$$\boldsymbol{x}_n = \begin{bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \\ y_n \\ \dot{y}_n \\ \ddot{y}_n \end{bmatrix} \tag{13.64}$$

Let us find the relation between the measurement vector and the state vector. The vehicle range (r) can be expressed by x and y using the Pythagorean theorem:

$$r = \sqrt{x^2 + y^2} \tag{13.65}$$

The vehicle bearing angle (φ) can be expressed by x and y using a trigonometrical function:

$$\varphi = tan^{-1}\frac{y}{x} \tag{13.66}$$

Since the state-to-measurement relation (the first type of non-linearity) is non-linear, the measurement equation is a type of $\boldsymbol{z}_n = \boldsymbol{h}(\boldsymbol{x}_n)$:

$$\begin{bmatrix} r \\ \varphi \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2} \\ tan^{-1}\frac{y}{x} \end{bmatrix} \tag{13.67}$$

Jacobian derivation:

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x} & \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial \dot{x}} & \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial \ddot{x}} & \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y} & \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial \dot{y}} & \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial \ddot{y}} \\ \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial x} & \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial \dot{x}} & \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial \ddot{x}} & \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial y} & \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial \dot{y}} & \frac{\partial\left(tan^{-1}\frac{y}{x}\right)}{\partial \ddot{y}} \end{bmatrix} \tag{13.68}$$

I derived the partial derivatives earlier when I explained the multivariate uncertainty projection concept (subsection 13.5.1).

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & 0 & 0 & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 \\ \frac{-y}{x^2+y^2} & 0 & 0 & \frac{x}{x^2+y^2} & 0 & 0 \end{bmatrix} \tag{13.69}$$

The measurement uncertainty

The measurement covariance matrix is:

$$\boldsymbol{R}_n = \begin{bmatrix} \sigma_{r_m}^2 & 0 \\ 0 & \sigma_{\varphi_m}^2 \end{bmatrix} \tag{13.70}$$

The Kalman Gain

The Kalman Gain in for EKF is given by:

$$\boldsymbol{K}_n = \boldsymbol{P}_{n,n-1} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}^T \left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \boldsymbol{P}_{n,n-1} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}^T + \boldsymbol{R}_n \right)^{-1} \tag{13.71}$$

Where:

$\boldsymbol{K}_n$	is the Kalman Gain
$\boldsymbol{P}_{n,n-1}$	is a prior estimate covariance matrix of the current state (predicted at the previous state)
$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}$	is the linearized Observation Function
$\boldsymbol{R}_n$	is the measurement noise covariance matrix

The state update equation

The State Update Equation is given by:

$$\hat{\boldsymbol{x}}_{n,n} = \hat{\boldsymbol{x}}_{n,n-1} + \boldsymbol{K}_n \left(\boldsymbol{z}_n - \boldsymbol{h}(\hat{\boldsymbol{x}}_{n,n-1}) \right) \tag{13.72}$$

Where:

$\hat{\boldsymbol{x}}_{n,n}$	is the estimated system state vector at time step n
$\hat{\boldsymbol{x}}_{n,n-1}$	is the predicted system state vector at time step $n-1$
$\boldsymbol{K}_n$	is the Kalman Gain
$\boldsymbol{h}(\hat{\boldsymbol{x}}_{n,n-1})$	is the Observation Function

The covariance update equation

The Covariance Update Equation in a matrix form is given by:

$$\boldsymbol{P}_{n,n} = \left(\boldsymbol{I} - \boldsymbol{K}_n \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right) \boldsymbol{P}_{n,n-1} \left(\boldsymbol{I} - \boldsymbol{K}_n \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{K}_n \boldsymbol{R}_n \boldsymbol{K}_n^T \tag{13.73}$$

Where:

$\boldsymbol{P}_{n,n}$	is the estimate covariance matrix of the current state
$\boldsymbol{P}_{n,n-1}$	is the prior estimate covariance matrix of the current state (predicted at the previous state)
$\boldsymbol{K}_n$	is the Kalman Gain
$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}$	is the linearized Observation Function
$\boldsymbol{R}_n$	is the measurement noise covariance matrix

13.7.2 The numerical example

The vehicle trajectory is similar to example 9 (section 9.1). The vehicle moves straight in the Y direction with a constant velocity. After traveling 400 meters, the vehicle turns left with a turning radius of 300 meters. During the turning maneuver, the vehicle experiences acceleration due to the circular motion (angular acceleration).

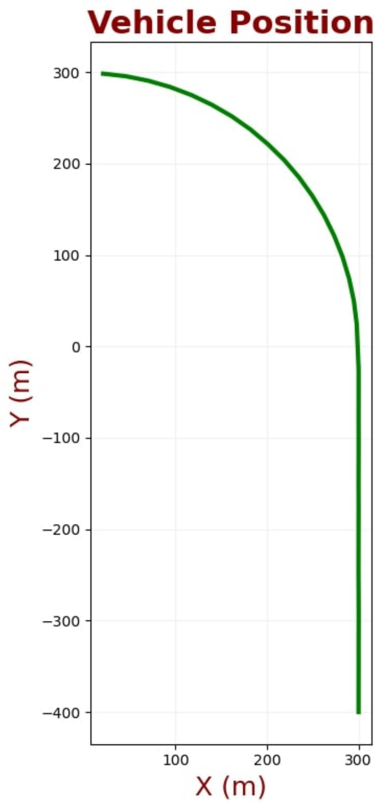

Figure 13.6: *Vehicle trajectory.*

- The measurements period: $\Delta t = 1s$
- The random acceleration standard deviation: $\sigma_a = 0.2\frac{m}{s^2}$
- The range measurement error standard deviation: $\sigma_{r_m} = 5m$
- The bearing angle measurement error standard deviation: $\sigma_{\varphi_m} = 0.0087rad$
- The state transition matrix $\boldsymbol{F}$ is:

$$\boldsymbol{F} = \begin{bmatrix} 1 & \Delta t & 0.5\Delta t^2 & 0 & 0 & 0 \\ 0 & 1 & \Delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta t & 0.5\Delta t^2 \\ 0 & 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- The process noise matrix $\boldsymbol{Q}$ is:

$$\boldsymbol{Q} = \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} & 0 & 0 & 0 \\ \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t & 0 & 0 & 0 \\ \frac{\Delta t^2}{2} & \Delta t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} \\ 0 & 0 & 0 & \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t \\ 0 & 0 & 0 & \frac{\Delta t^2}{2} & \Delta t & 1 \end{bmatrix} \sigma_a^2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 1 \end{bmatrix} 0.2^2$$

- The measurement variance $\boldsymbol{R}$ is:

$$\boldsymbol{R}_n = \begin{bmatrix} \sigma_{r_m}^2 & 0 \\ 0 & \sigma_{\varphi_m}^2 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 0.0087^2 \end{bmatrix}$$

The following table contains the set of 35 noisy measurements:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
r(m)	502.55	477.34	457.21	442.94	427.27	406.05	400.73	377.32
φ(rad)	-0.9316	-0.8977	-0.8512	-0.8114	-0.7853	-0.7392	-0.7052	-0.6478
9	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
360.27	345.93	333.34	328.07	315.48	301.41	302.87	304.25	294.46
-0.59	-0.5183	-0.4698	-0.3952	-0.3026	-0.2445	-0.1626	-0.0937	0.0085
18	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**
294.29	299.38	299.37	300.68	304.1	301.96	300.3	301.9	296.7
0.0856	0.1675	0.2467	0.329	0.4149	0.504	0.5934	0.667	0.7537
27	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**
297.07	295.29	296.31	300.62	292.3	298.11	298.07	298.92	298.04
0.8354	0.9195	1.0039	1.0923	1.1546	1.2564	1.3274	1.409	1.5011

Table 13.2: *Example 11 measurements.*

13.7.2.1 Iteration Zero

Initialization

We don't know the vehicle location, so we approximate the initial position at about 100m from the true vehicle position ($\hat{x}_{0,0} = 400m, \hat{y}_{0,0} = -300m$)

$$\boldsymbol{\hat{x}}_{0,0} = \begin{bmatrix} 400 \\ 0 \\ 0 \\ -300 \\ 0 \\ 0 \end{bmatrix}$$

Since our initial state vector is a guess, we set a very high estimate uncertainty. The high estimate uncertainty results in a high Kalman Gain by giving a high weight to the measurement.

$$\boldsymbol{P}_{0,0} = \begin{bmatrix} 500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 500 \end{bmatrix}$$

Prediction

$$\hat{\boldsymbol{x}}_{1,0} = \boldsymbol{F}\hat{\boldsymbol{x}}_{0,0} = \begin{bmatrix} 400 \\ 0 \\ 0 \\ -300 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{P}_{1,0} = \boldsymbol{F}\boldsymbol{P}_{0,0}\boldsymbol{F}^T + \boldsymbol{Q} = \begin{bmatrix} 1125 & 750 & 250 & 0 & 0 & 0 \\ 750 & 1000 & 500 & 0 & 0 & 0 \\ 250 & 500 & 500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1125 & 750 & 250 \\ 0 & 0 & 0 & 750 & 1000 & 500 \\ 0 & 0 & 0 & 250 & 500 & 500 \end{bmatrix}$$

13.7.2.2 First Iteration

Step 1 - Measure

The measurement values:

$$\boldsymbol{z}_1 = \begin{bmatrix} 502.55 \\ -0.9316 \end{bmatrix}$$

Step 2 - Update

Observation matrix $(\boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0}))$ calculation.

$$\boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0}) = \begin{bmatrix} \sqrt{x_{1,0}^2 + y_{1,0}^2} \\ tan^{-1}\dfrac{y_{1,0}}{x_{1,0}} \end{bmatrix} = \begin{bmatrix} \sqrt{400^2 + (-300)^2} \\ tan^{-1}\dfrac{-300}{400} \end{bmatrix} = \begin{bmatrix} 500 \\ 0.644 \end{bmatrix}$$

Observation matrix Jacobian $\left(\dfrac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}}\right)$ calculation:

$$\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{x_{1,0}}{\sqrt{x_{1,0}^2 + y_{1,0}^2}} & 0 & 0 & \dfrac{y_{1,0}}{\sqrt{x_{1,0}^2 + y_{1,0}^2}} & 0 & 0 \\ \dfrac{-y_{1,0}}{x_{1,0}^2 + y_{1,0}^2} & 0 & 0 & \dfrac{x_{1,0}}{x_{1,0}^2 + y_{1,0}^2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0 & 0 & -0.6 & 0 & 0 \\ 0.0012 & 0 & 0 & 0.0016 & 0 & 0 \end{bmatrix}$$

The Kalman Gain calculation:

$$\boldsymbol{K}_1 = \boldsymbol{P}_{1,0} \left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} \right)^T \left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} \boldsymbol{P}_{1,0} \left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} \right)^T + \boldsymbol{R} \right)^{-1} = \begin{bmatrix} 0.783 & 295 \\ 0.522 & 196.7 \\ 0.174 & 65.6 \\ -0.587 & 393.3 \\ -0.391 & 262.2 \\ -0.13 & 87.4 \end{bmatrix}$$

Estimate the current state:

$$\hat{\boldsymbol{x}}_{1,1} = \hat{\boldsymbol{x}}_{1,0} + \boldsymbol{K}_1 \left(\boldsymbol{z}_1 - \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0}) \right) = \begin{bmatrix} 317 \\ -55.3 \\ -18.4 \\ -414.8 \\ -76.5 \\ -25.5 \end{bmatrix}$$

Update the current estimate covariance:

$$\boldsymbol{P}_{1,1} = \left(\boldsymbol{I} - \boldsymbol{K}_1 \frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} \right) \boldsymbol{P}_{1,0} \left(\boldsymbol{I} - \boldsymbol{K}_1 \frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{1,0})}{\partial \boldsymbol{x}} \right)^T + \boldsymbol{K}_1 \boldsymbol{R} \boldsymbol{K}_1^T$$

$$= \begin{bmatrix} 22.39 & 14.93 & 4.98 & -2.75 & -1.84 & -0.61 \\ 14.93 & 509.97 & 336.67 & -1.84 & -1.22 & -0.41 \\ 4.98 & 336.67 & 445.58 & -0.61 & -0.41 & -0.14 \\ -2.75 & -1.84 & -0.61 & 22.39 & 14.93 & 4.98 \\ -1.84 & -1.22 & -0.41 & 14.93 & 509.97 & 336.67 \\ -0.61 & -0.41 & -0.14 & 4.98 & 336.67 & 445.58 \end{bmatrix}$$

Step 3 - Predict

$$\hat{\boldsymbol{x}}_{2,1} = \boldsymbol{F}\hat{\boldsymbol{x}}_{1,1} = \begin{bmatrix} 252.42 \\ -73.8 \\ -18.45 \\ -504.13 \\ -102.06 \\ -25.52 \end{bmatrix}$$

$$\boldsymbol{P}_{2,1} = \boldsymbol{F}\boldsymbol{P}_{1,1}\boldsymbol{F}^T + \boldsymbol{Q} = \begin{bmatrix} 1015.28 & 1257.7 & 564.46 & -8.7 & -4.35 & -1.09 \\ 1257.7 & 1628.94 & 782.3 & -4.35 & -2.18 & -0.54 \\ 564.46 & 782.3 & 445.62 & -1.09 & -0.54 & -0.14 \\ -8.7 & -4.35 & -1.09 & 1010.2 & 1255.16 & 563.83 \\ -4.35 & -2.18 & -0.54 & 1255.16 & 1627.67 & 781.98 \\ -1.09 & -0.54 & -0.14 & 563.83 & 781.98 & 445.54 \end{bmatrix}$$

13.7.2.3 Second Iteration

Step 1 - Measure

The measurement values:

$$\boldsymbol{z}_2 = \begin{bmatrix} 477.34 \\ -0.8977 \end{bmatrix}$$

Step 2 - Update

Observation matrix $(\boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1}))$ calculation.

$$\boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1}) = \begin{bmatrix} \sqrt{x_{2,1}^2 + y_{2,1}^2} \\ tan^{-1}\dfrac{y_{2,1}}{x_{2,1}} \end{bmatrix} = \begin{bmatrix} \sqrt{252.42^2 + (-504.13)^2} \\ tan^{-1}\dfrac{-504.13}{252.42} \end{bmatrix} = \begin{bmatrix} 563.8 \\ -1.1 \end{bmatrix}$$

Observation matrix Jacobian $\left(\dfrac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\right)$ calculation:

$$\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{x_{2,1}}{\sqrt{x_{2,1}^2 + y_{2,1}^2}} & 0 & 0 & \dfrac{y_{2,1}}{\sqrt{x_{2,1}^2 + y_{2,1}^2}} & 0 & 0 \\ \dfrac{-y_{2,1}}{x_{2,1}^2 + y_{2,1}^2} & 0 & 0 & \dfrac{x_{2,1}}{x_{2,1}^2 + y_{2,1}^2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.45 & 0 & 0 & -0.89 & 0 & 0 \\ 0.0016 & 0 & 0 & 0.0008 & 0 & 0 \end{bmatrix}$$

The Kalman Gain calculation:

$$\boldsymbol{K}_2 = \boldsymbol{P}_{2,1}\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\right)^T\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\boldsymbol{P}_{2,1}\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{R}\right)^{-1} = \begin{bmatrix} 0.44 & 492.34 \\ 0.54 & 611.49 \\ 0.24 & 274.65 \\ -0.87 & 246.4 \\ -1.08 & 309.3 \\ -0.49 & 139.36 \end{bmatrix}$$

Estimate the current state:

$$\hat{\boldsymbol{x}}_{2,2} = \hat{\boldsymbol{x}}_{2,1} + \boldsymbol{K}_2\left(\boldsymbol{z}_2 - \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})\right) = \begin{bmatrix} 317.47 \\ 7.6 \\ 18.19 \\ -377.14 \\ 56.13 \\ 45.6 \end{bmatrix}$$

Update the current estimate covariance:

$$\boldsymbol{P}_{2,2} = \left(\boldsymbol{I} - \boldsymbol{K}_2\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\right)\boldsymbol{P}_{2,1}\left(\boldsymbol{I} - \boldsymbol{K}_2\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{2,1})}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{K}_2\boldsymbol{R}\boldsymbol{K}_2^T$$

$$= \begin{bmatrix} 23.8 & 29.48 & 13.23 & -0.31 & -0.23 & -0.08 \\ 29.48 & 107.4 & 99.42 & -0.23 & -4.87 & -2.8 \\ 13.23 & 99.42 & 139.14 & -0.08 & -2.8 & -1.62 \\ -0.31 & -0.23 & -0.08 & 24.24 & 30.12 & 13.53 \\ -0.23 & -4.87 & -2.8 & 30.12 & 105.54 & 98.22 \\ -0.08 & -2.8 & -1.62 & 13.53 & 98.22 & 138.39 \end{bmatrix}$$

Step 3 - Predict

$$\hat{\boldsymbol{x}}_{3,2} = \boldsymbol{F}\hat{\boldsymbol{x}}_{2,2} = \begin{bmatrix} 334.17 \\ 25.8 \\ 18.19 \\ -298.21 \\ 101.73 \\ 45.6 \end{bmatrix}$$

$$\boldsymbol{P}_{3,2} = \boldsymbol{F}\boldsymbol{P}_{2,2}\boldsymbol{F}^T + \boldsymbol{Q} = \begin{bmatrix} 337.6 & 368.83 & 182.2 & -8.92 & -10.19 & -3.69 \\ 368.83 & 445.42 & 238.6 & -10.19 & -12.09 & -4.42 \\ 182.2 & 238.6 & 139.18 & -3.69 & -4.42 & -1.62 \\ -8.92 & -10.19 & -3.69 & 336.39 & 365.74 & 180.97 \\ -10.19 & -12.09 & -4.42 & 365.74 & 440.41 & 236.65 \\ -3.69 & -4.42 & -1.62 & 180.97 & 236.65 & 138.43 \end{bmatrix}$$

At this point, I think it would be reasonable to jump to the last Kalman Filter iteration.

13.7.2.4 Thirty-Fifth Iteration

Step 1 - Measure

The measurement values:

$$\boldsymbol{z}_{35} = \begin{bmatrix} 298.04 \\ 1.5011 \end{bmatrix}$$

Step 2 - Update

Observation matrix $(\boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34}))$ calculation.

$$\boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34}) = \begin{bmatrix} \sqrt{x_{35,34}^2 + y_{35,34}^2} \\ tan^{-1}\dfrac{y_{35,34}}{x_{35,34}} \end{bmatrix} = \begin{bmatrix} \sqrt{307.8^2 + (-277.04)^2} \\ tan^{-1}\dfrac{-277.04}{307.8} \end{bmatrix} = \begin{bmatrix} 300.07 \\ 1.5 \end{bmatrix}$$

Observation matrix Jacobian $\left(\dfrac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right)$ calculation:

$$\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right) = \begin{bmatrix} \dfrac{x_{35,34}}{\sqrt{x_{35,34}^2 + y_{35,34}^2}} & 0 & 0 & \dfrac{y_{35,34}}{\sqrt{x_{35,34}^2 + y_{35,34}^2}} & 0 & 0 \\ \dfrac{-y_{35,34}}{x_{35,34}^2 + y_{35,34}^2} & 0 & 0 & \dfrac{x_{35,34}}{x_{35,34}^2 + y_{35,34}^2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.07 & 0 & 0 & 0.998 & 0 & 0 \\ -0.003 & 0 & 0 & 0.0002 & 0 & 0 \end{bmatrix}$$

The Kalman Gain calculation:

$$\boldsymbol{K}_{35} = \boldsymbol{P}_{35,34}\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right)^T \left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\boldsymbol{P}_{35,34}\left(\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{R}\right)^{-1} = \begin{bmatrix} 0.04 & -174.67 \\ 0.01 & -73.57 \\ 0 & -15.06 \\ 0.5 & -4.83 \\ 0.16 & -1.32 \\ 0.03 & -0.3 \end{bmatrix}$$

Estimate the current state:

$$\hat{\boldsymbol{x}}_{35,35} = \hat{\boldsymbol{x}}_{35,34} + \boldsymbol{K}_{35}\left(\boldsymbol{z}_{35} - \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})\right) = \begin{bmatrix} 20.87 \\ -25.93 \\ -0.84 \\ 298.38 \\ 2.55 \\ -1.8 \end{bmatrix}$$

Update the current estimate covariance:

$$\boldsymbol{P}_{35,35} = \left(\boldsymbol{I} - \boldsymbol{K}_{35}\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right)\boldsymbol{P}_{35,34}\left(\boldsymbol{I} - \boldsymbol{K}_{35}\frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{35,34})}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{K}_{35}\boldsymbol{R}\boldsymbol{K}_{35}^T$$

$$= \begin{bmatrix} 4.07 & 1.7 & 0.34 & 0.95 & 0.31 & 0.04 \\ 1.7 & 1.3 & 0.38 & 0.16 & 0.2 & 0.04 \\ 0.34 & 0.38 & 0.16 & -0.01 & 0.03 & 0.01 \\ 0.95 & 0.16 & -0.01 & 12.02 & 4.05 & 0.7 \\ 0.31 & 0.2 & 0.03 & 4.05 & 2.29 & 0.56 \\ 0.04 & 0.04 & 0.01 & 0.7 & 0.56 & 0.19 \end{bmatrix}$$

Step 3 - Predict

$$\hat{\boldsymbol{x}}_{36,35} = \boldsymbol{F}\hat{\boldsymbol{x}}_{35,35} = \begin{bmatrix} -5.49 \\ -26.77 \\ -0.84 \\ 300.02 \\ 0.74 \\ -1.8 \end{bmatrix}$$

$$\boldsymbol{P}_{36,35} = \boldsymbol{F}\boldsymbol{P}_{35,35}\boldsymbol{F}^T + \boldsymbol{Q} = \begin{bmatrix} 9.53 & 4 & 0.82 & 1.68 & 0.61 & 0.09 \\ 4 & 2.25 & 0.57 & 0.41 & 0.27 & 0.05 \\ 0.82 & 0.57 & 0.2 & 0.02 & 0.04 & 0.01 \\ 1.68 & 0.41 & 0.02 & 23.74 & 8 & 1.38 \\ 0.61 & 0.27 & 0.04 & 8 & 3.63 & 0.79 \\ 0.09 & 0.05 & 0.01 & 1.38 & 0.79 & 0.23 \end{bmatrix}$$

13.7.3 Example summary

The following chart demonstrates the EKF location and velocity estimation performance.

The chart on the left compares the true, measured, and estimated values of the vehicle position. Two charts on the right compare the true, measured, and estimated values of x - axis velocity and y - axis velocity.

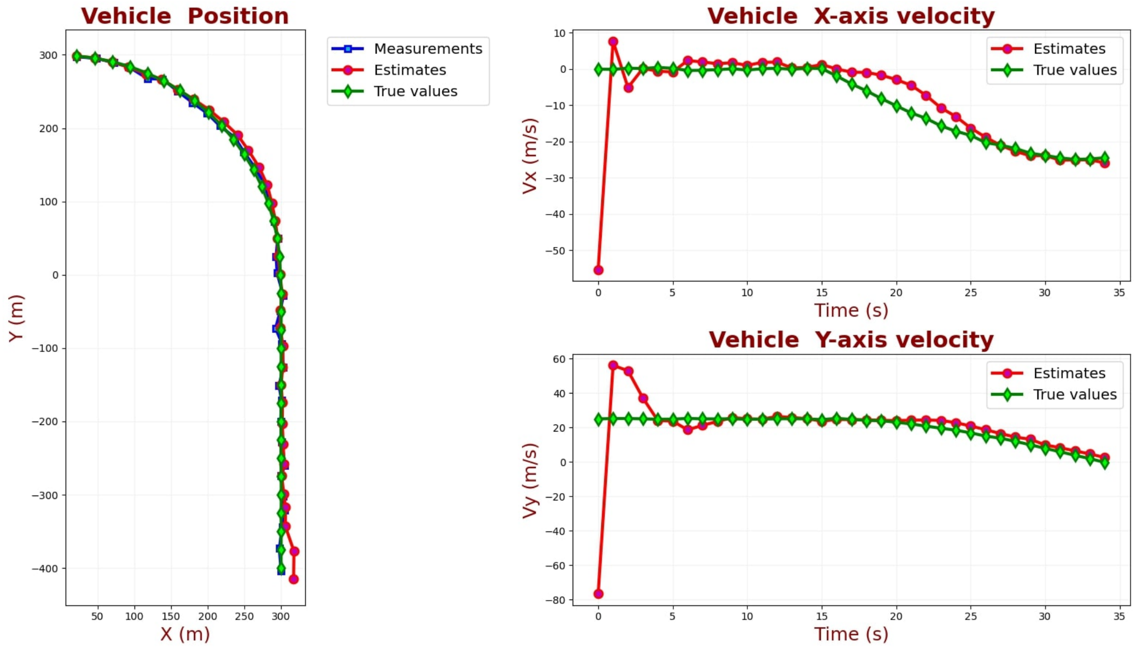

Figure 13.7: *Example 11: true value, measured values and estimates.*

We can see a satisfying performance of the EKF. Although the filter is roughly initiated at about 100 meters from the true position with zero initial velocity, it provides a good position estimation after taking two measurements and a good velocity estimation after taking four measurements.

Let us take a closer look at the vehicle position estimation performance. The following chart describes the true, measured, and estimated values of the vehicle position compared to the 95% confidence ellipses. We can see that the ellipses' size constantly decreases. That means that the EKF converges with time.

The following charts provide a zoom into the linear part of the vehicle motion and the turning maneuver part.

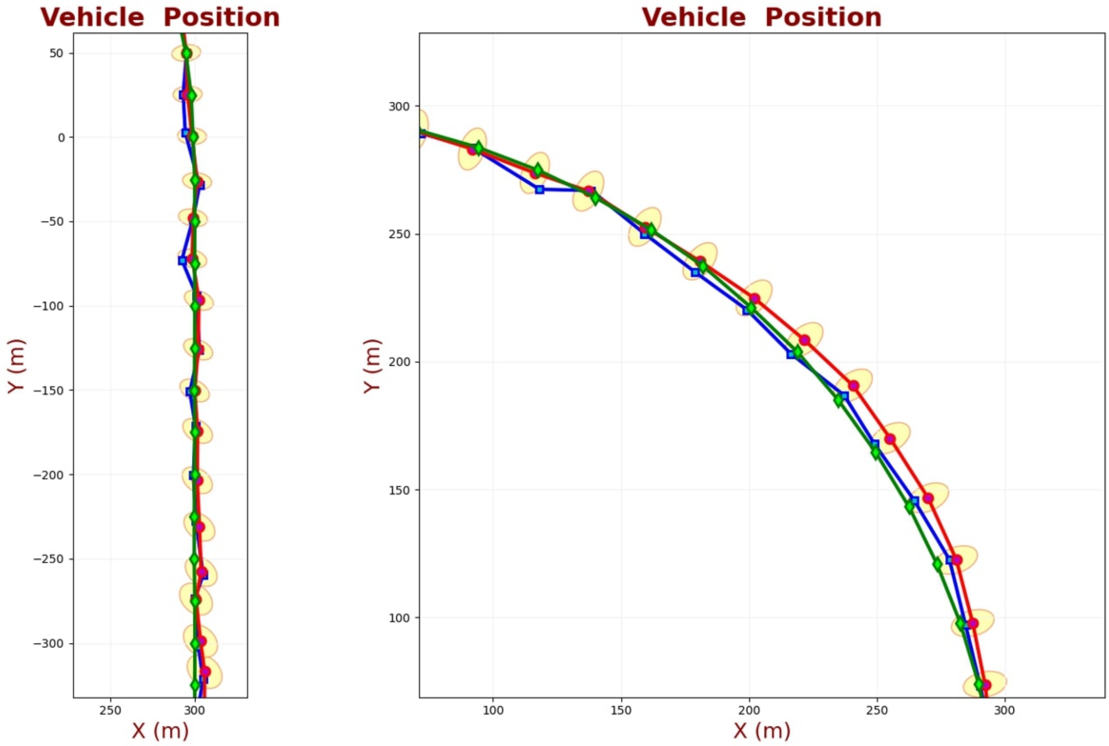

Figure 13.8: *Example 11: true value, measured values and estimates - zoom.*

We can see that at the linear part of the vehicle motion, the EKF copes with the noisy measurements and follows the true vehicle position. On the other hand, during the vehicle turning maneuver, the EKF estimates are quite away from the true vehicle position, although they are within the 90% confidence ellipse bounds.