Lecture 6:

Elementary Graph Algorithms

Algorimiek (INFOAL)

Graph: an Abstract Model of the World

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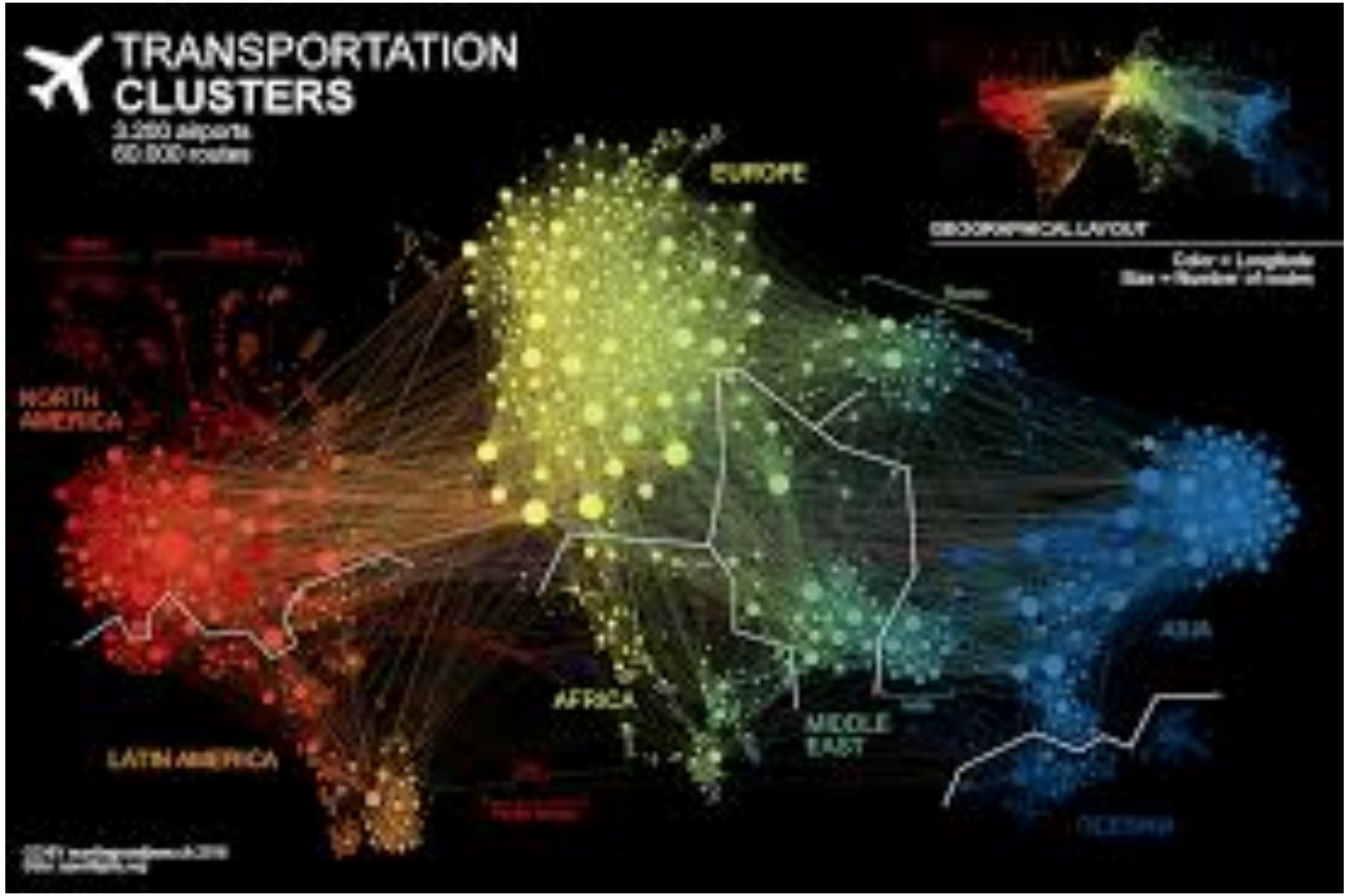

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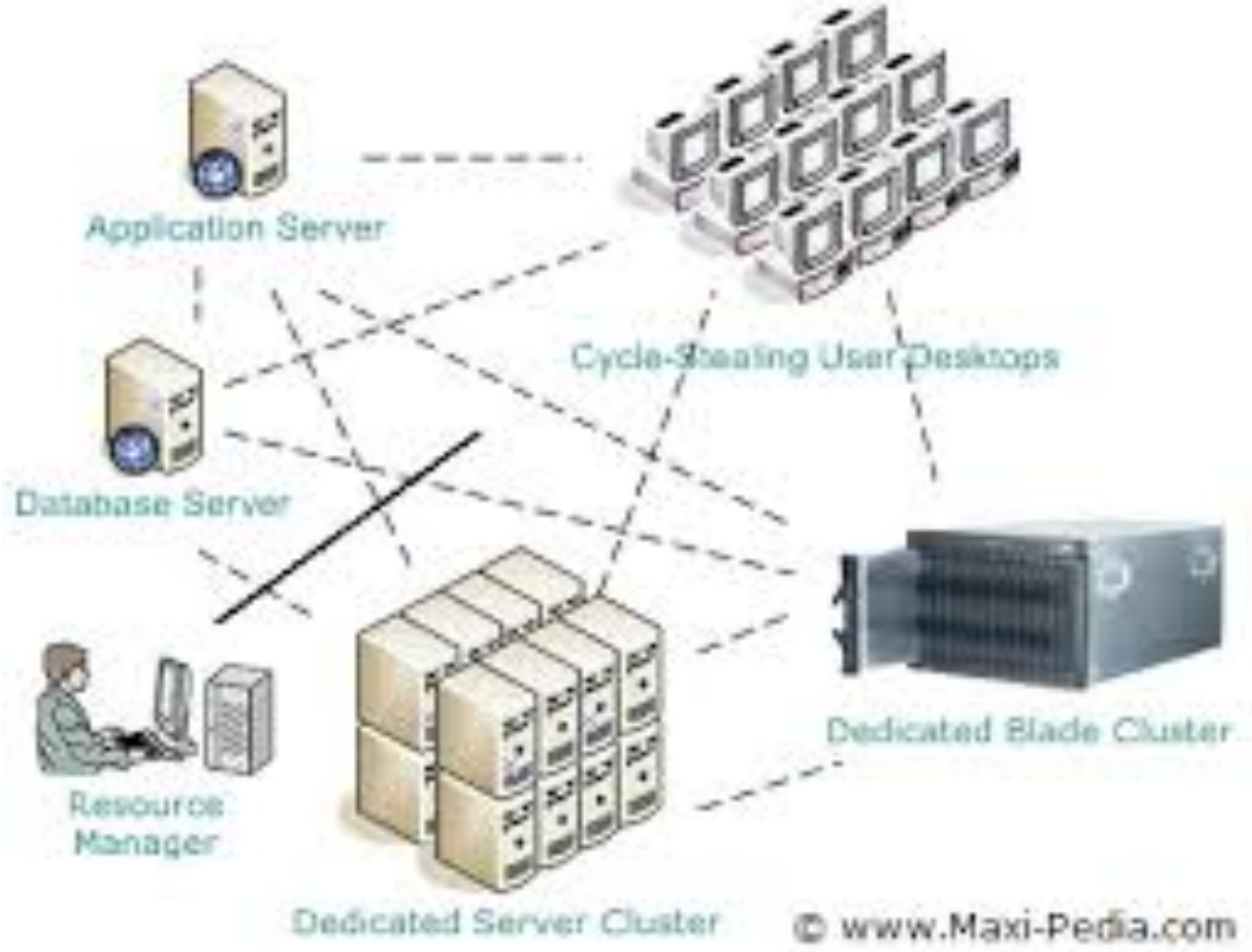

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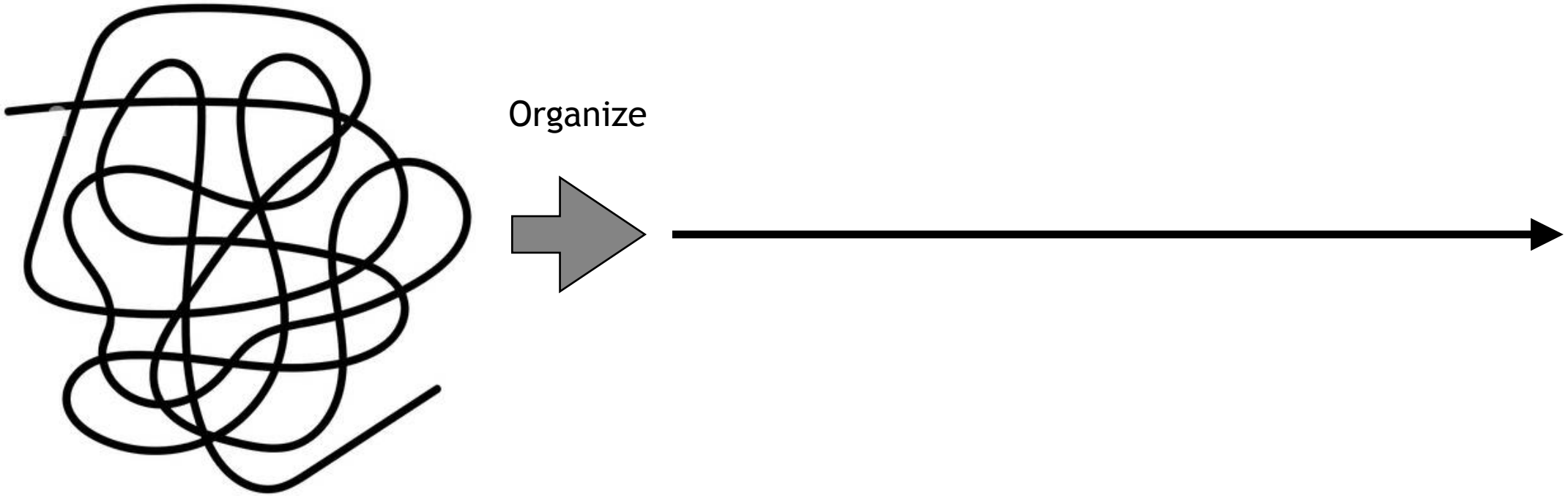

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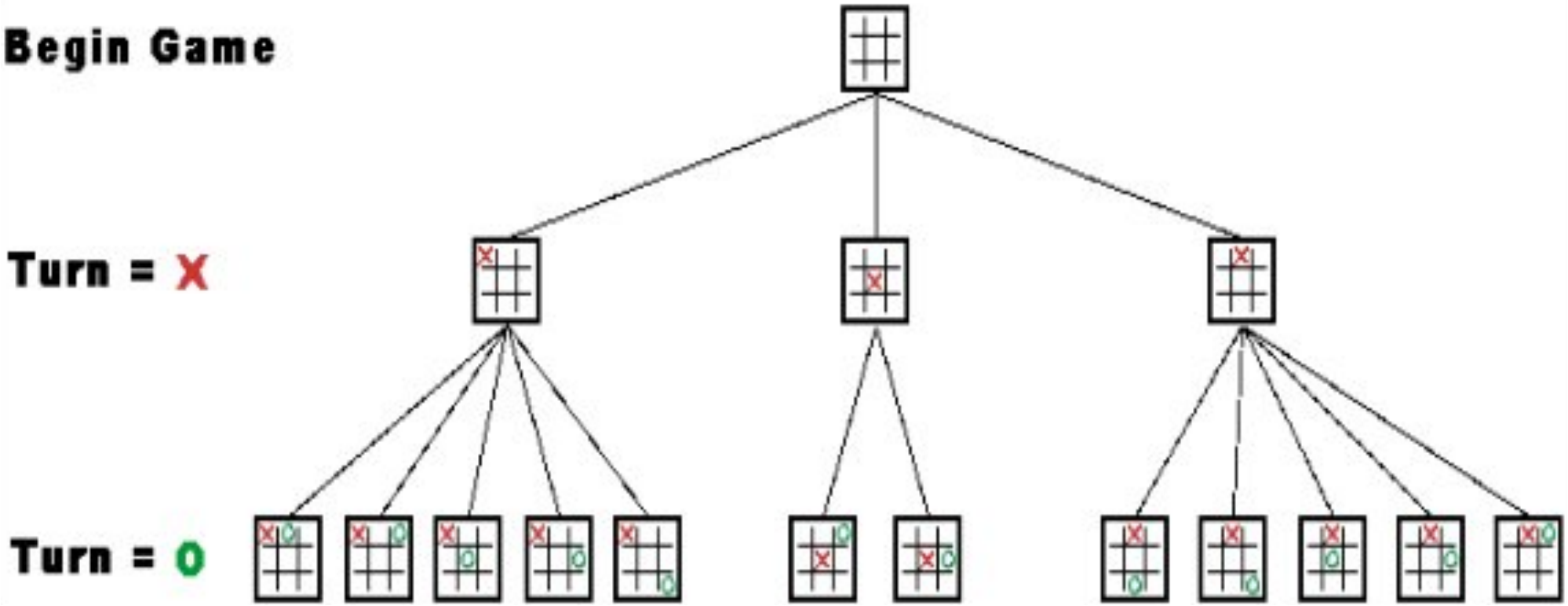

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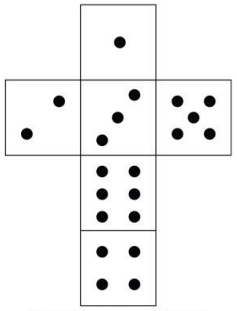

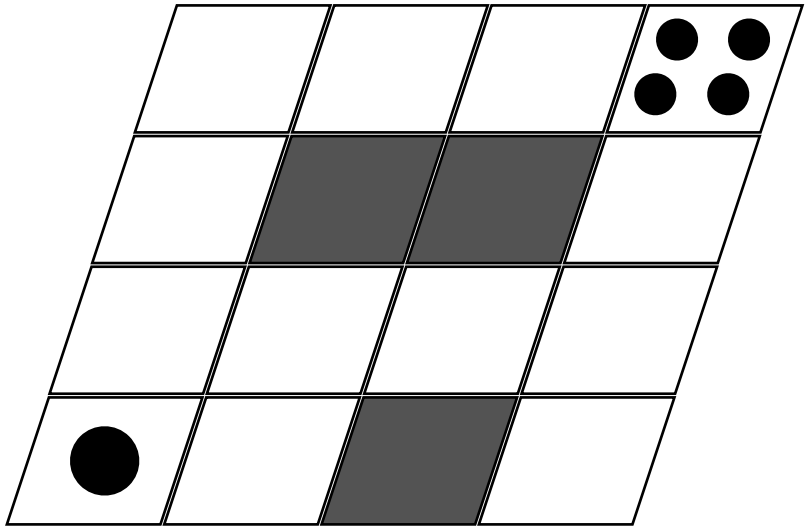

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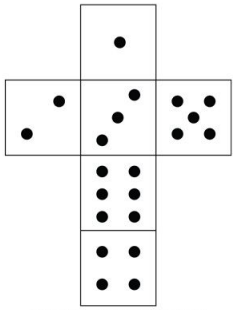

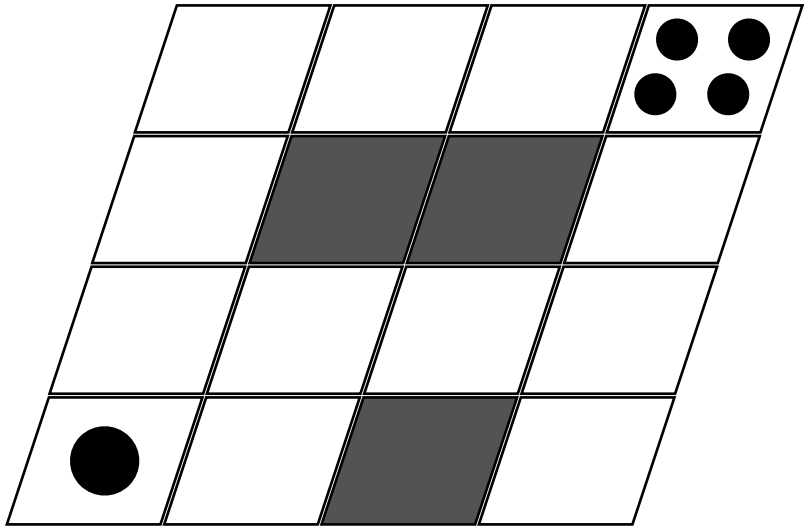

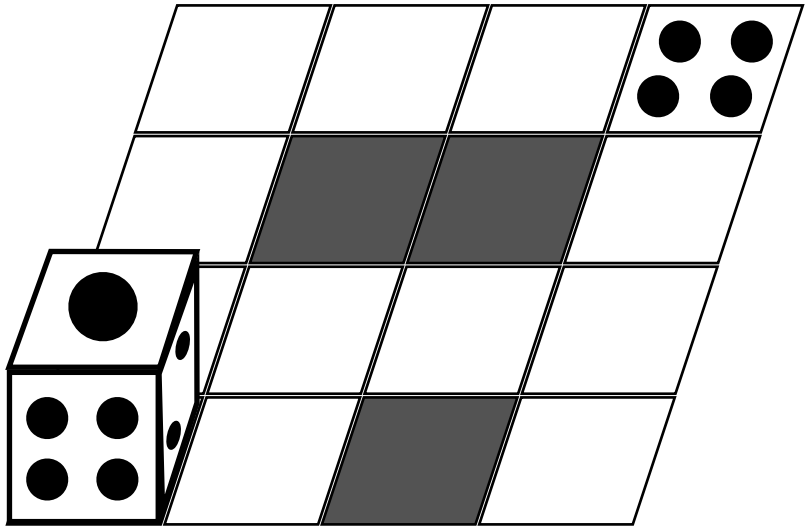

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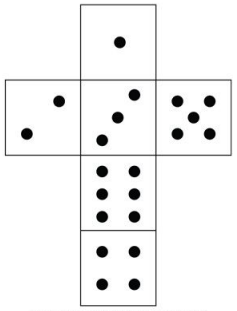

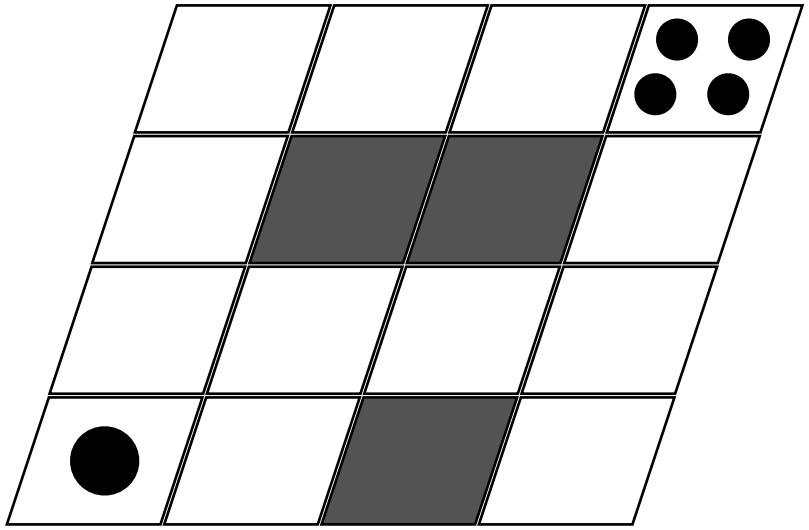

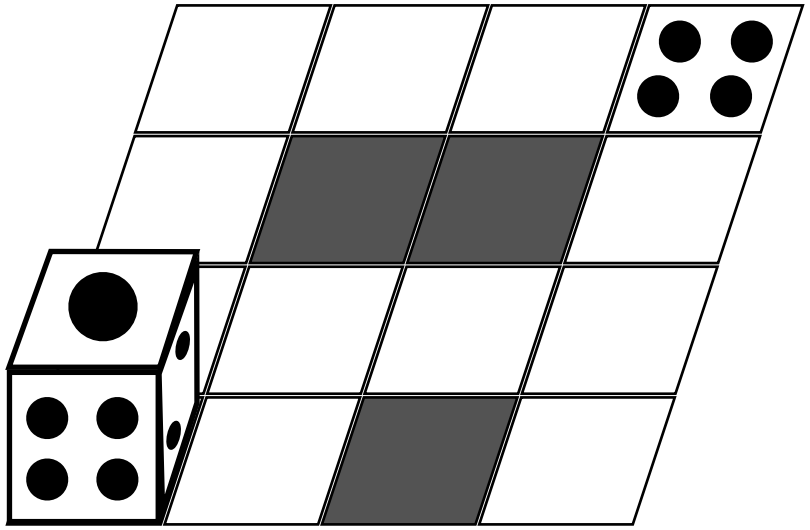

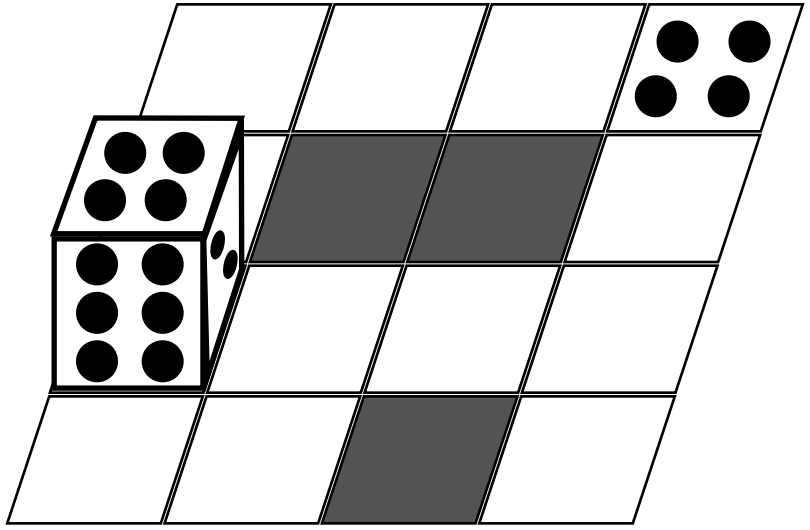

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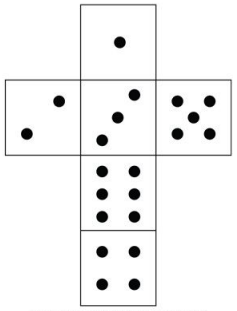

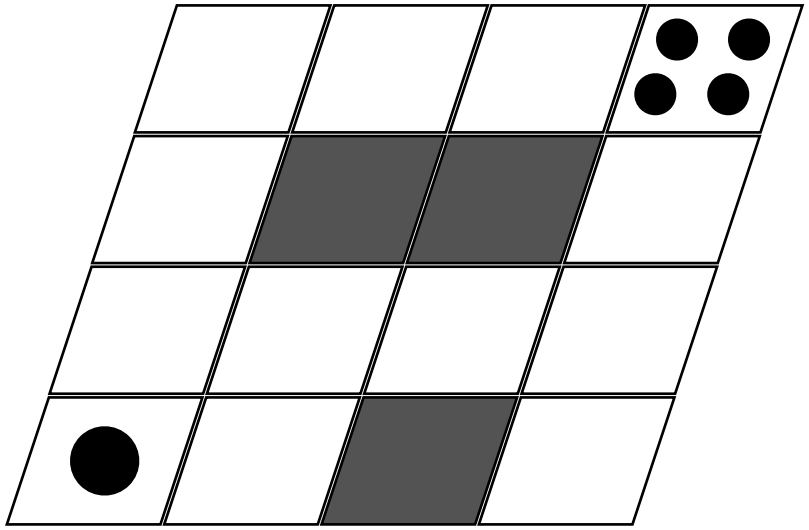

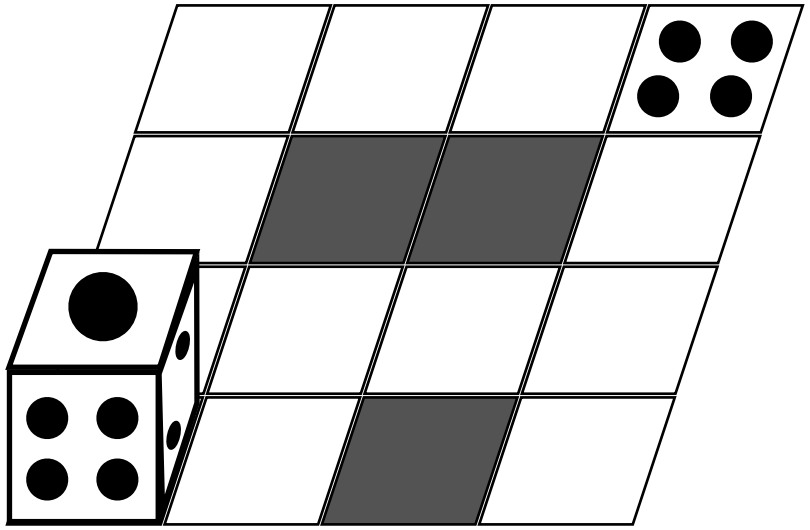

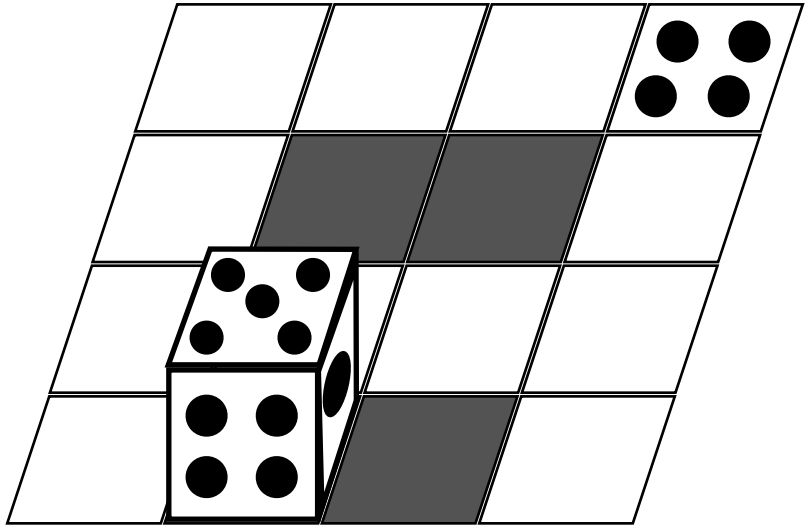

Outline

- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

How to Represent a Graph

- $G = (V, E)$

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  - V: set of vertices in the graph

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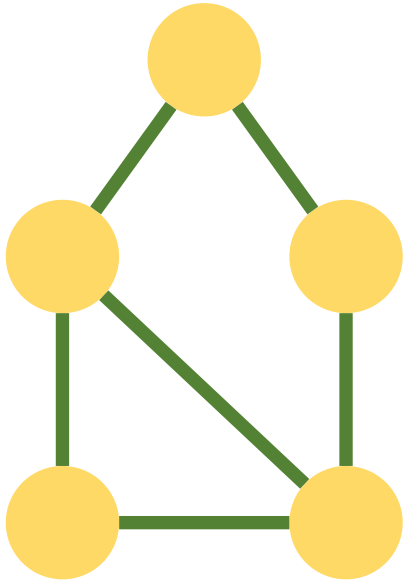
  - E: set of edges in the graph

How to Represent a Graph

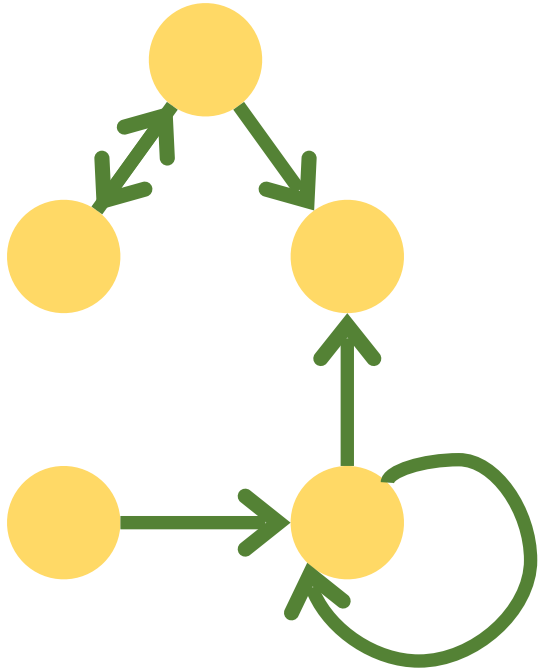
- $G = (V, E)$
  - V: set of vertices in the graph
    - $n = |V|$
  - E: set of edges in the graph
    - $m = |E|$

Undirected Graph and Directed Graph

- In directed graphs, the edges are ordered set. That is, $(u, v) \neq (v, u)$



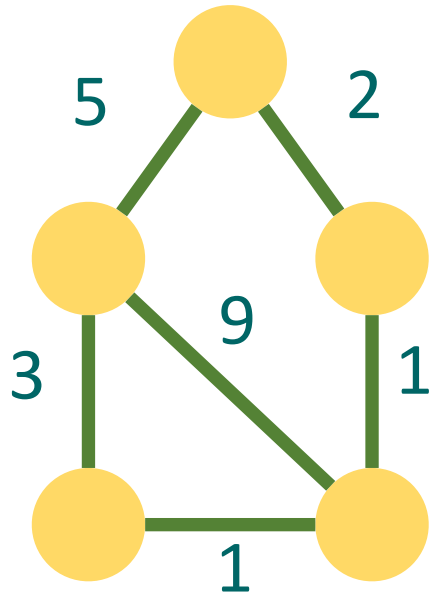
undirected graph

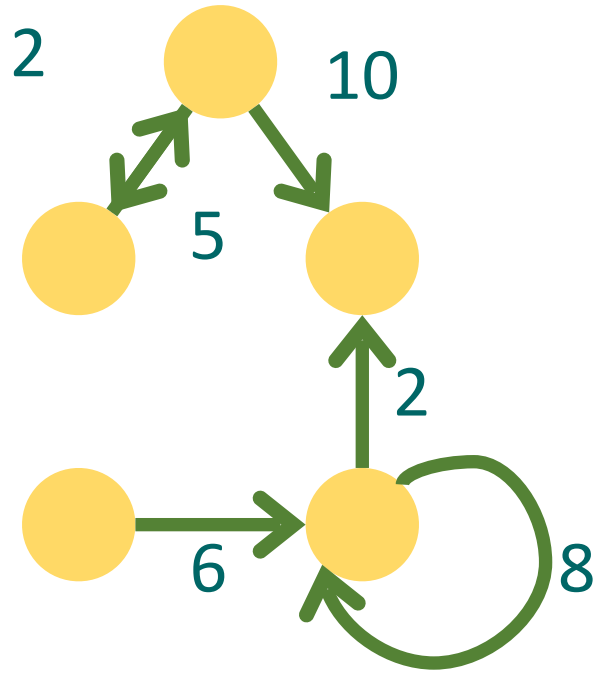


directed graph

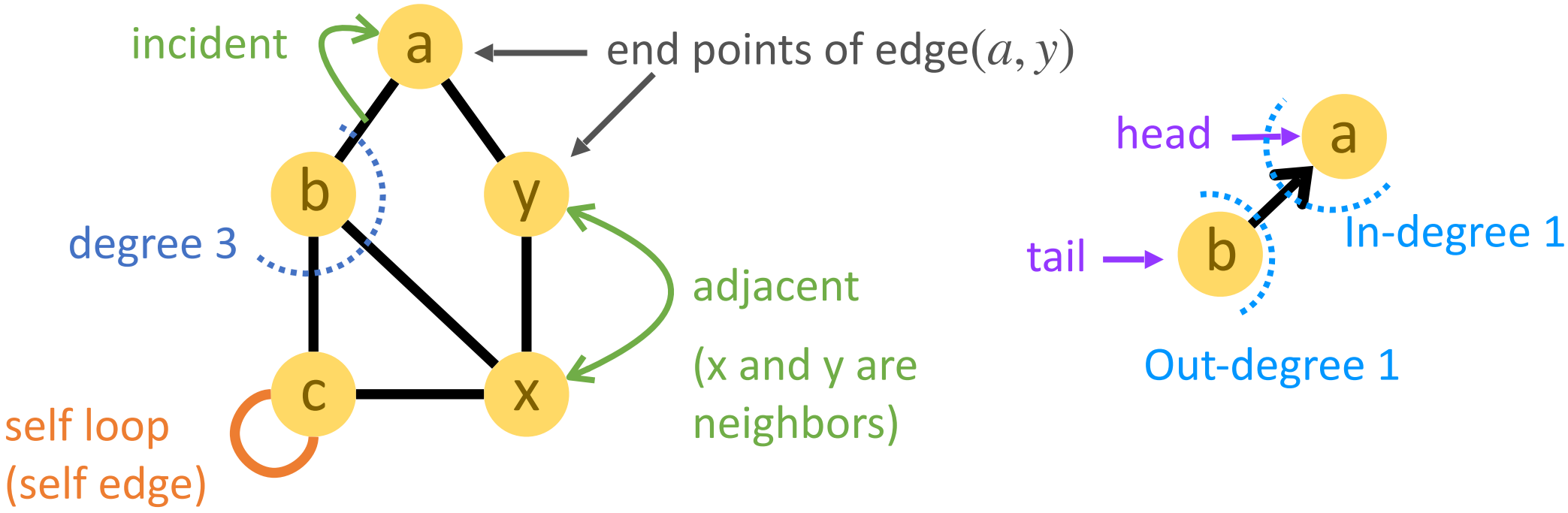
Weighted Graph

- The edges can be associated with **weights**.



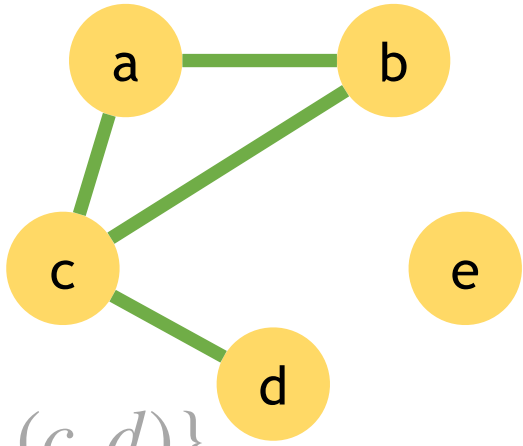


Basic Terminologies



How to Represent a Graph

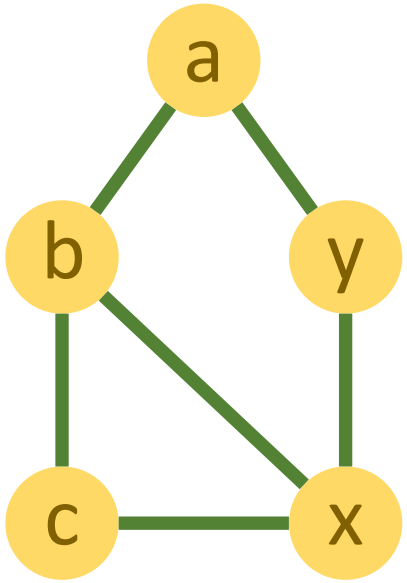
- $G = (V, E)$
  - V: set of vertices in the graph $V = \{a, b, c, d, e\}$
    - $n = |V| n = 5$
  - E: set of edges in the graph $E = \{(a, b), (a, c), (b, c), (c, d)\}$
    - $m = |E| m = 4$
- To represent the relationship between vertices and edges, we can use
  - **Adjacency list**
  - **Adjacency matrix**

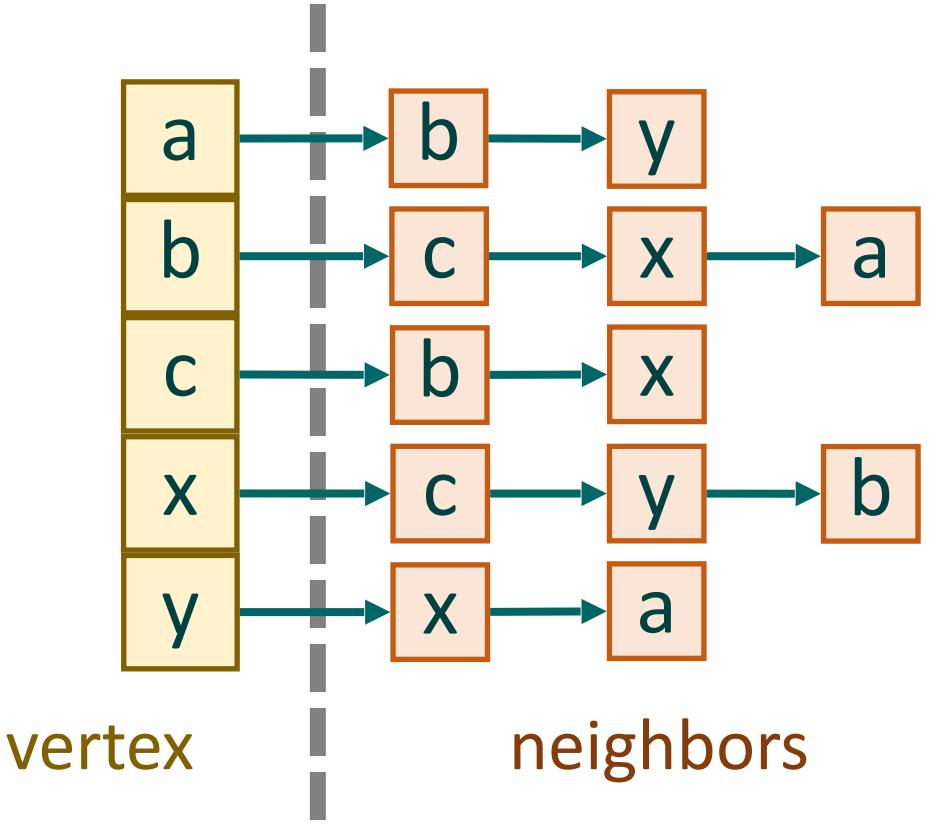


Adjacency List

Adjacency List

- For each vertex, store its neighbors in a linked list





$$V = \{a, b, c, x, y\}$$

$$E = \{(a, b), (a, y), (b, c), (b, x), (c, x), (x, y)\}$$

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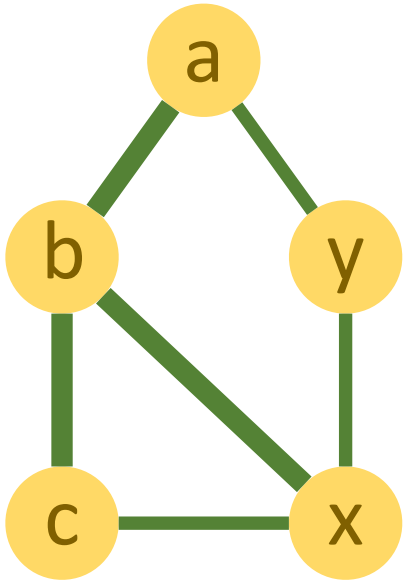

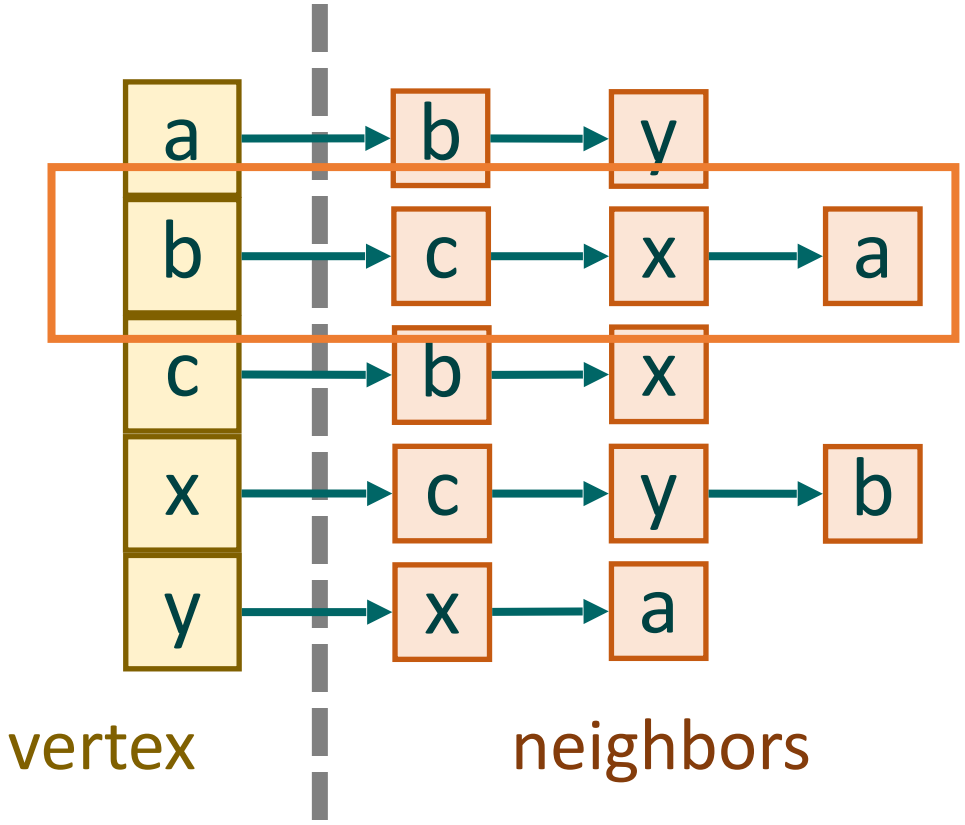

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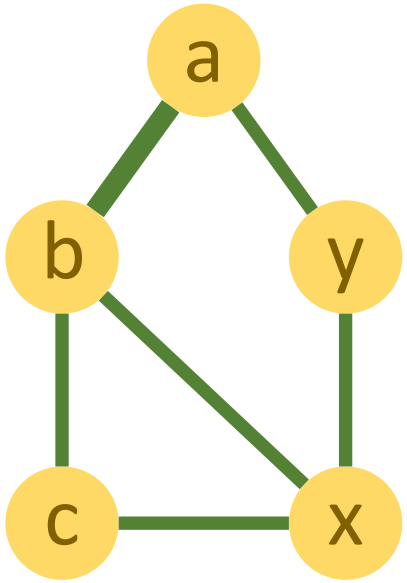

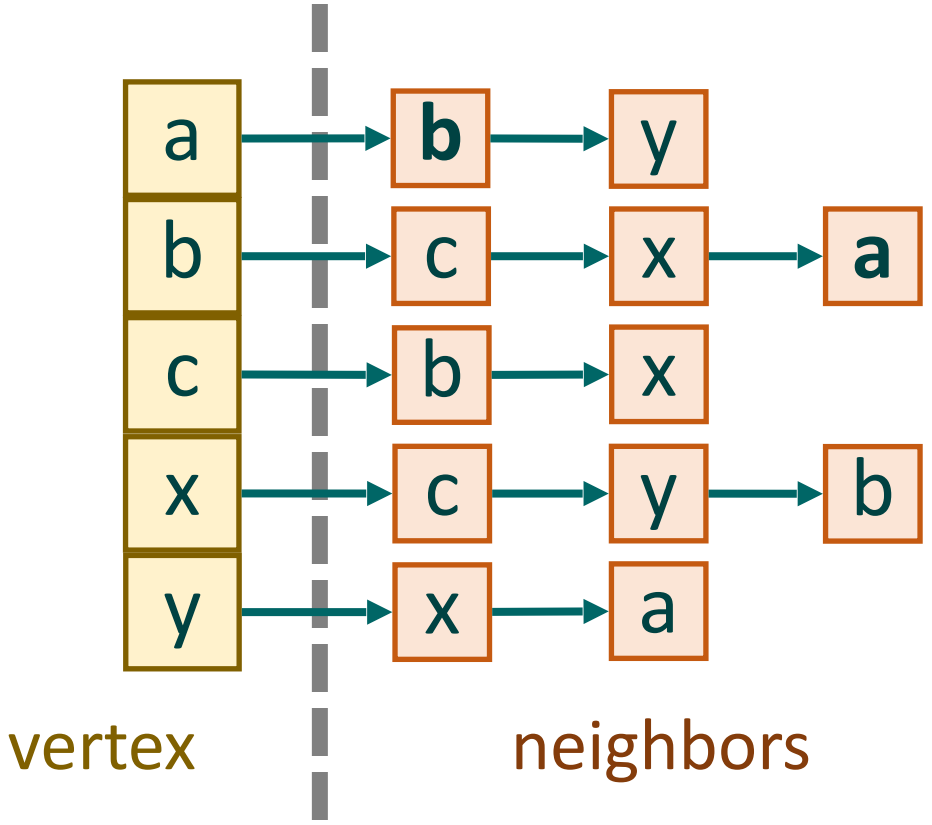

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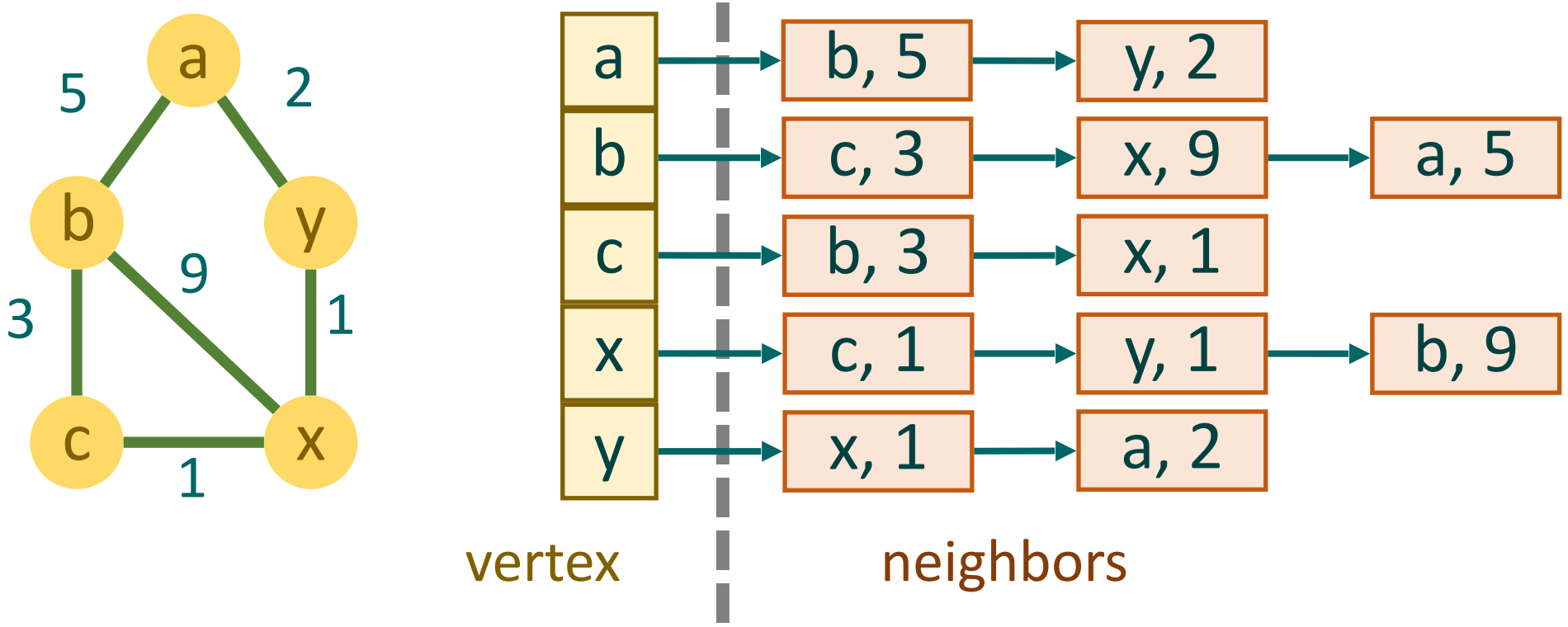

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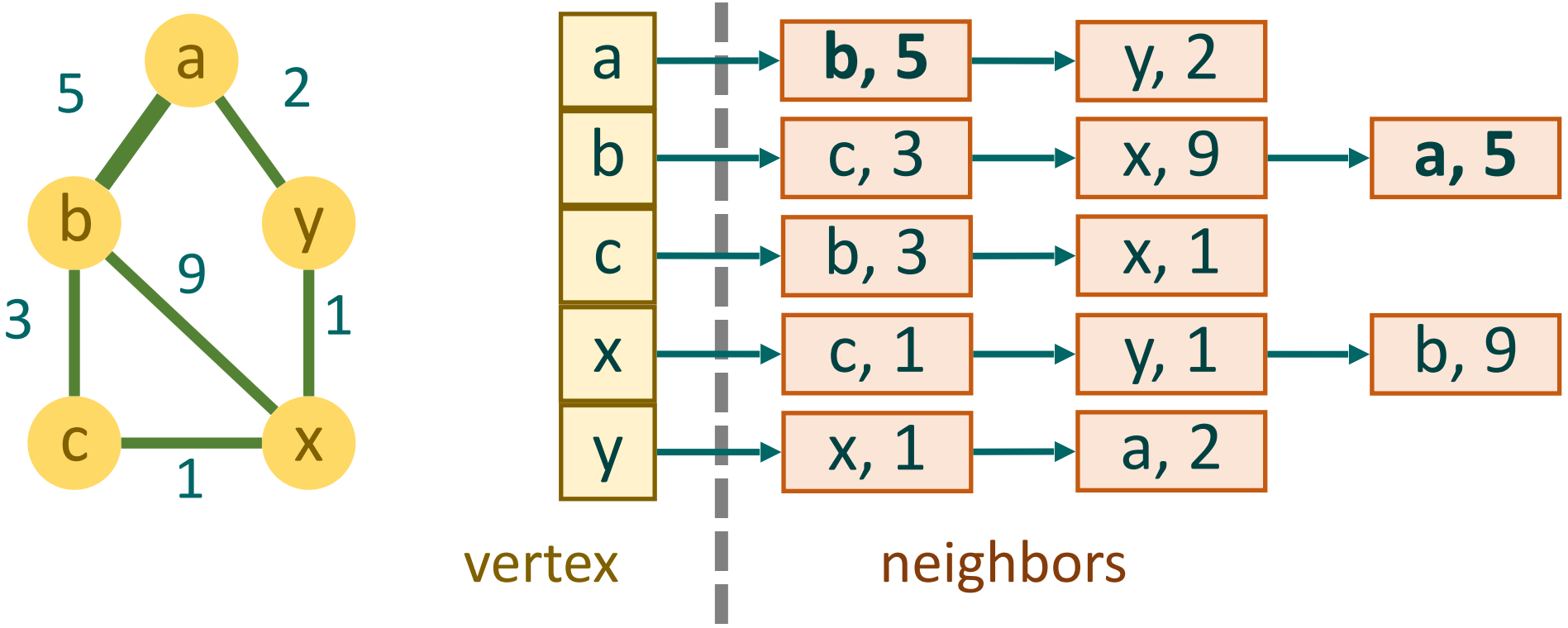

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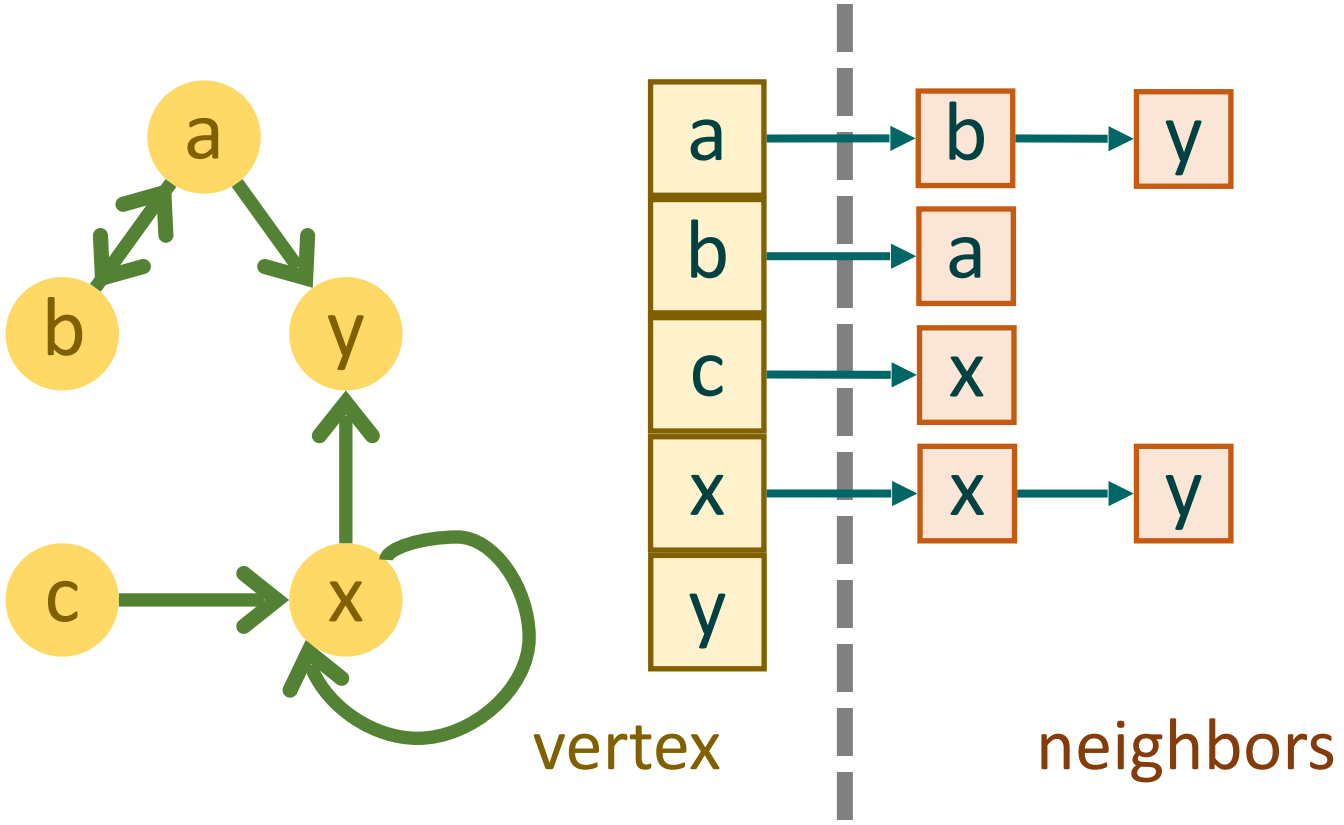

$$V = \{a, b, c, x, y\}$$

$$E = \{(a, b), (b, a), (a, y), (x, y), (x, x), (c, x)\}$$

Adjacency List

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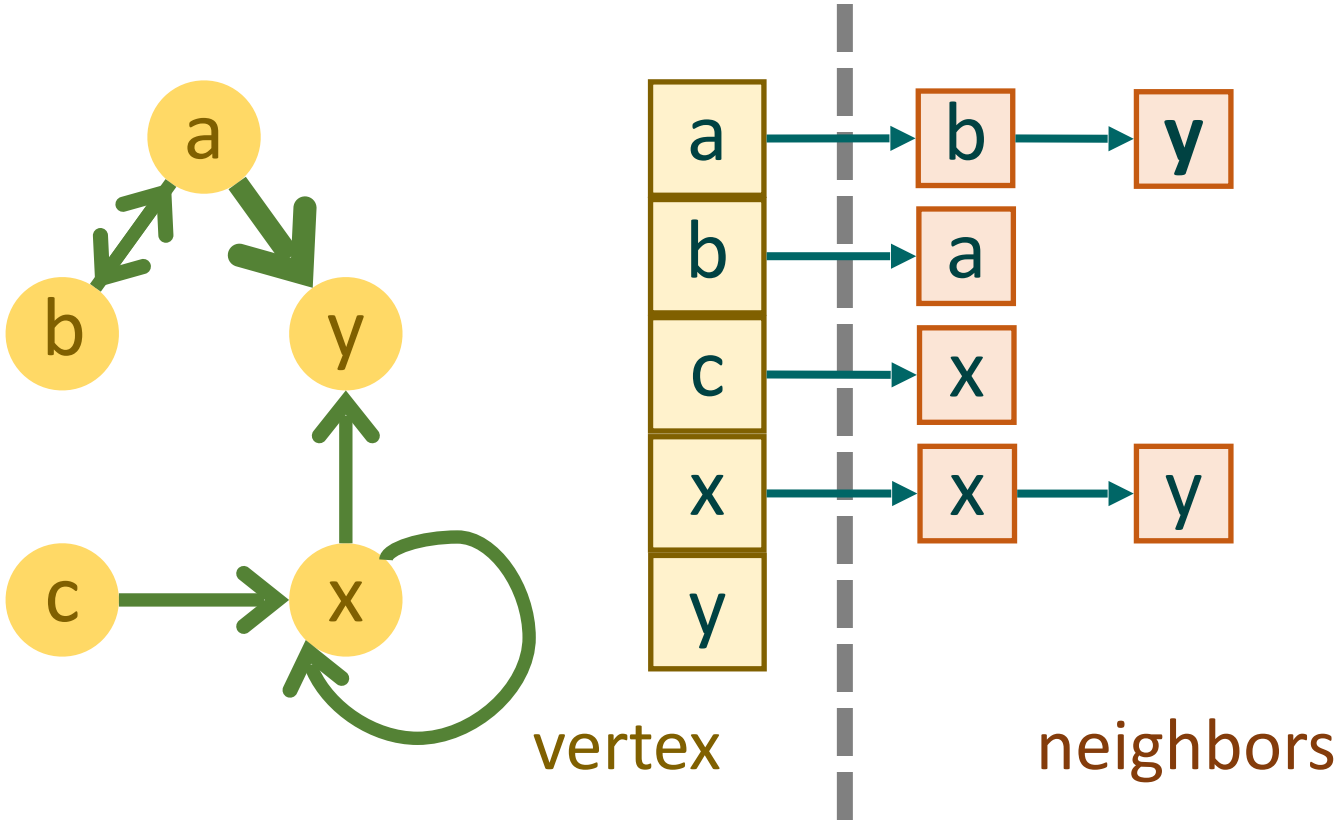
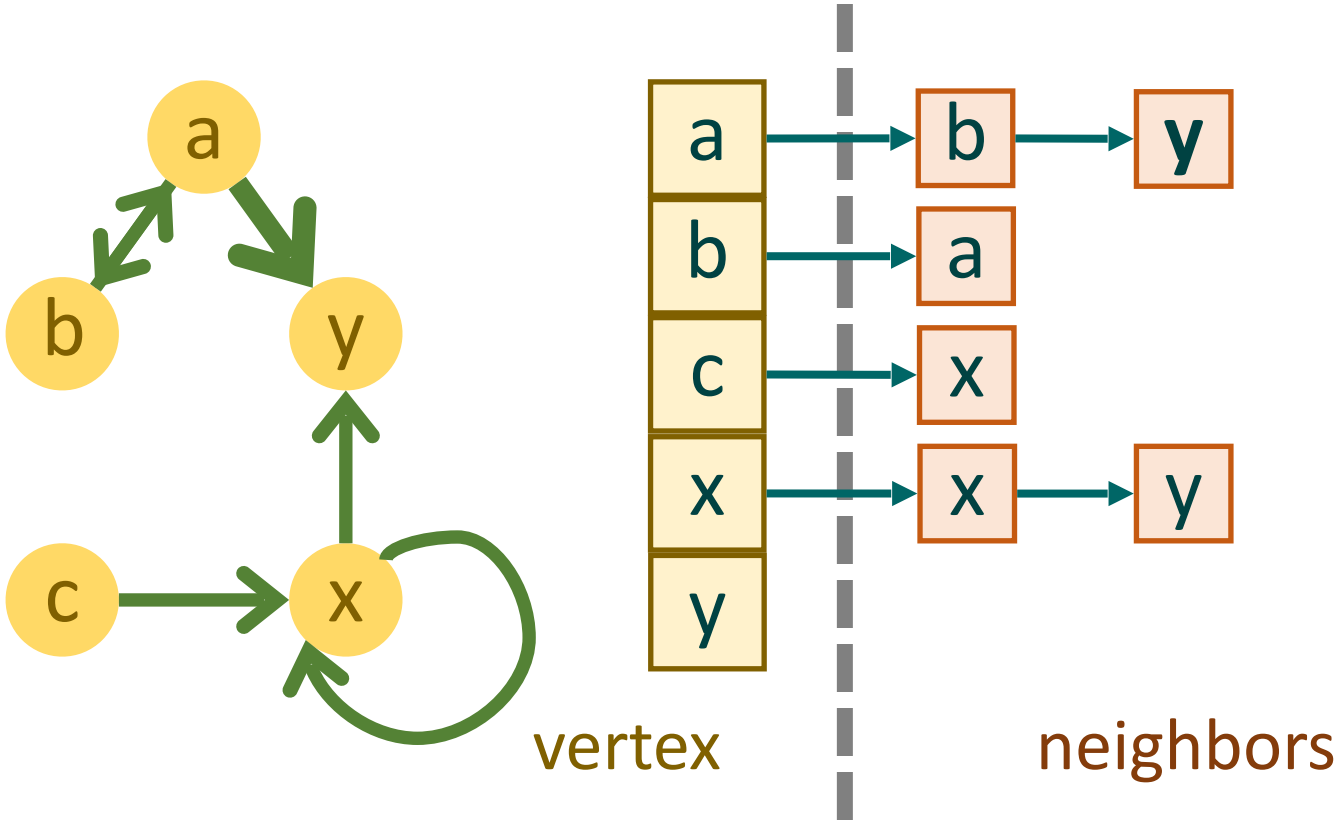


- Space complexity: $O(|E|)$

Adjacency List

- For each vertex, store its neighbors in a linked list



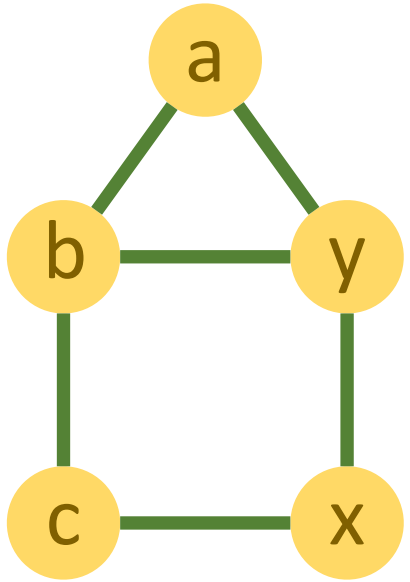
check if u and v are adjacent

- Space complexity: $O(|E|)$
- Querying complexity: $O(\text{max degree})$
  - For each vertex, store its neighbors in a linked list

Adjacency Matrix

Adjacency Matrix

- Use a $|V| \times |V|$ matrix A such that
  - $A(u, v) = 1$ iff $(u, v) \in E$
  - $A(u, v) = 0$ otherwise
- Space complexity: $O(|V|^2)$

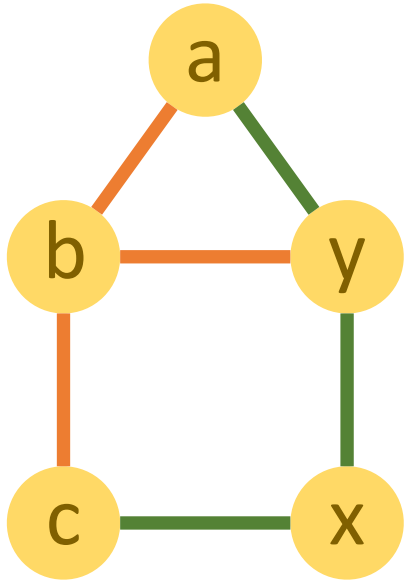


	a	b	c	x	y
a	0	1	0	0	1
b	1	0	1	0	1
c	0	1	0	1	0
x	0	0	1	0	1
y	1	1	0	1	0

Adjacency Matrix

- Use a $|V| \times |V|$ matrix A such that
  - $A(u, v) = 1$ iff $(u, v) \in E$
  - $A(u, v) = 0$ otherwise
- Space complexity: $O(|V|^2)$
- Querying complexity: $O(1)$
  - Check adjacency of vertices u and v

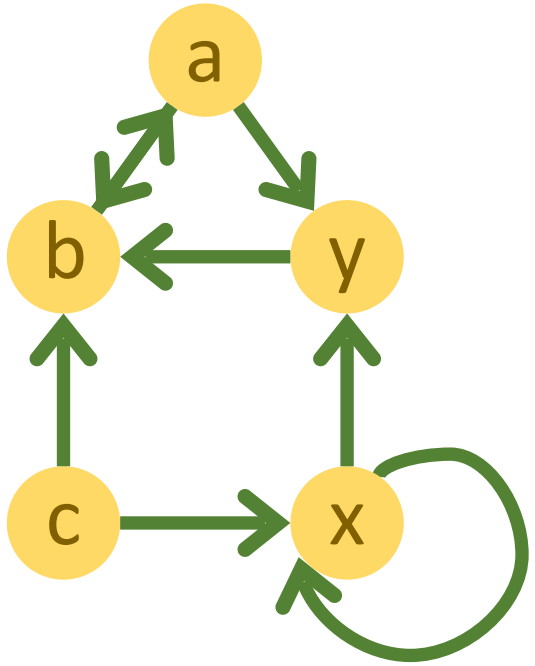
check if u and v are adjacent



	a	b	c	x	y
a	0	1	0	0	1
b	1	0	1	0	1
c	0	1	0	1	0
x	0	0	1	0	1
y	1	1	0	1	0

Adjacency Matrix

- Use a $|V| \times |V|$ matrix A such that
  - $A(u, v) = 1$ iff $(u, v) \in E$
  - $A(u, v) = 0$ otherwise
- Space complexity: $O(|V|^2)$
- Querying complexity: $O(1)$
  - Check adjacency of vertices u and v
- For directed graphs, A is not necessary symmetric



	a	b	c	x	y
a	0	1	0	0	1
b	1	0	0	0	0
c	0	1	0	1	0
x	0	0	0	1	1
y	0	1	0	0	0

Adjacency List and Adjacency Matrix

	Adjacency list	Adjacency matrix
Space	$O(\|V\|+\|E\|)$	$O(\|V\|^2)$
Add an edge		
Delete an edge		
List all neighbors of a vertex		
Check adjacency of vertices u and v		
List all edges		

Adjacency List and Adjacency Matrix

	Adjacency list	Adjacency matrix
Space	$O(\|V\|+\|E\|)$	$O(\|V\|^2)$
Add an edge	$O(1)$	$O(1)$
Delete an edge		
List all neighbors of a vertex		
Check adjacency of vertices u and v		
List all edges		

Adjacency List and Adjacency Matrix

	Adjacency list	Adjacency matrix
Space	$O(\|V\|+\|E\|)$	$O(\|V\|^2)$
Add an edge	$O(1)$	$O(1)$
Delete an edge	$O(\text{max degree}) = O(\|V\|)$	$O(1)$
List all neighbors of a vertex	$O(\#\text{ of neighbors}) = O(\|V\|)$	$O(\|V\|)$
Check adjacency of vertices u and v		
List all edges		

Adjacency List and Adjacency Matrix

	Adjacency list	Adjacency matrix
Space	$O(\|V\|+\|E\|)$	$O(\|V\|^2)$
Add an edge	$O(1)$	$O(1)$
Delete an edge	O(max degree) = $O(\|V\|)$	$O(1)$
List all neighbors of a vertex	O(# of neighbors) = $O(\|V\|)$	$O(\|V\|)$
Check adjacency of vertices u and v	O(max degree) = $O(\|V\|)$	$O(1)$
List all edges	$O(\|E\|)$	$O(\|V\|^2)$

Outline

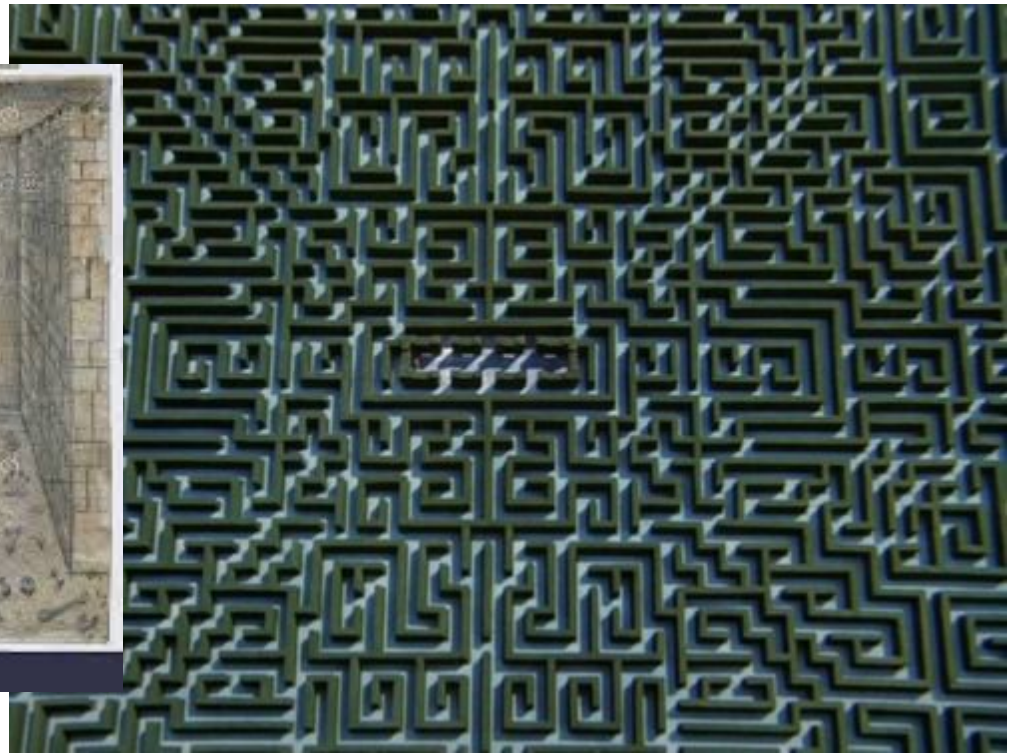
- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

Outline

- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
    - Parenthesis theorem
    - Edge classification
    - White-path theorem
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

Depth-First Search

- Recursively explore the graph starting from the current position
  - Walk as far as possible
  - Back track when there is no further vertex



Depth-First Search Initialization

```
For each u ∈ V[G]
    Mark u as not-discovered
    π[v] ← NIL
time ← 0
For each vertex u ∈ V[G]
    If u is not-discovered
        DFS-VISIT(u)
```

- Each vertex is marked as
  - White: **not-discovered** by the algorithm
  - Gray: **discovered** by the algorithm
  - Black: **finished**; all its neighbors are discovered

- For each of the vertices v, we take information:
  - $d[v]$: **discover time**, timestamp when v is first discovered
  - $f[v]$: **finish time**, timestamp when the search finishes examining v's adjacency list
    - $d[v] < f[v] \leq 2|V|$
  - $\pi[v]$: the predecessor of v when we traverse the graph

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
    d[u] ← time
    For each neighbor v of u
        If v is not-discovered
            π[v] ← u
            DFS-VISIT(v)
    Mark u as finished
    time ← time + 1
    f[u] ← time
```

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
    d[u] ← time
    For each neighbor v of u
        If v is not-discovered
            π[v] ← u
            DFS-VISIT(v)
    Mark u as finished
    time ← time + 1
    f[u] ← time
```

Time stamp:

At any time t, a vertex is discovered or a vertex is finished

Not-discovered
Discovered
Finished

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
    d[u] ← time
    For each neighbor v of u
        If v is not-discovered
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            DFS-VISIT(v)
    Mark u as finished
    time ← time + 1
    f[u] ← time
```

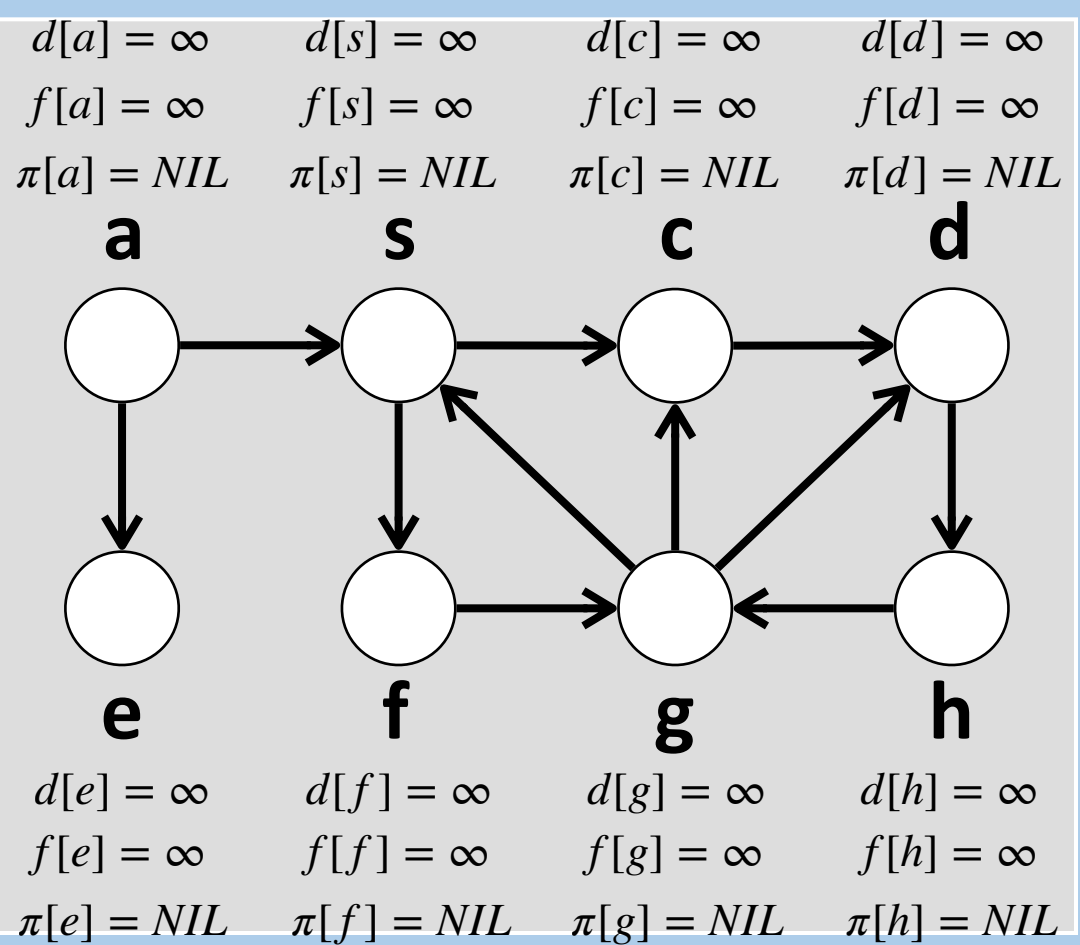

Not-discovered
Discovered
Finished

Depth-First Search

```
Procedure DFS-VISIT(u)
  Mark u as discovered
  time ← time + 1
  d[u] ← time
  For each neighbor v of u
     If v is not-discovered
        π[v] ← u
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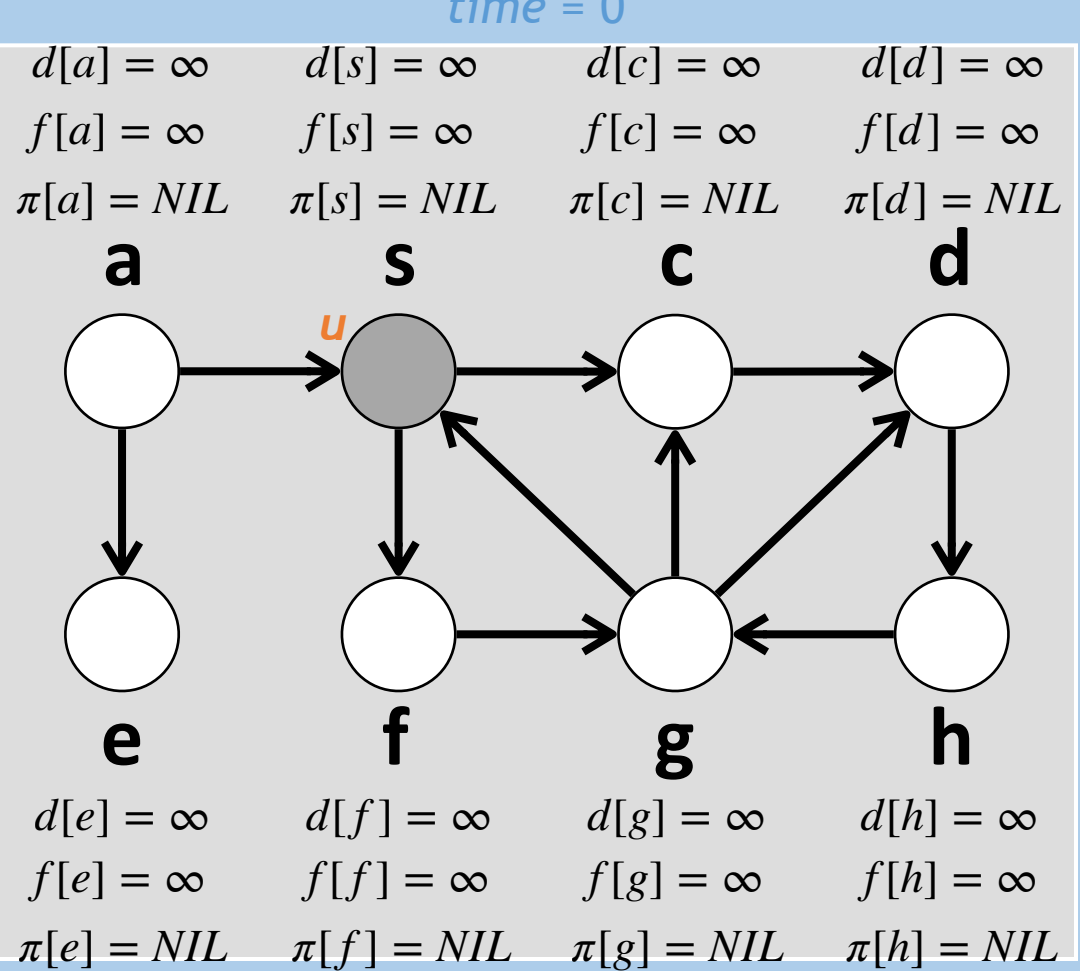

Not-discovered
Discovered
Finished

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
    d[u] ← time
    For each neighbor v of u
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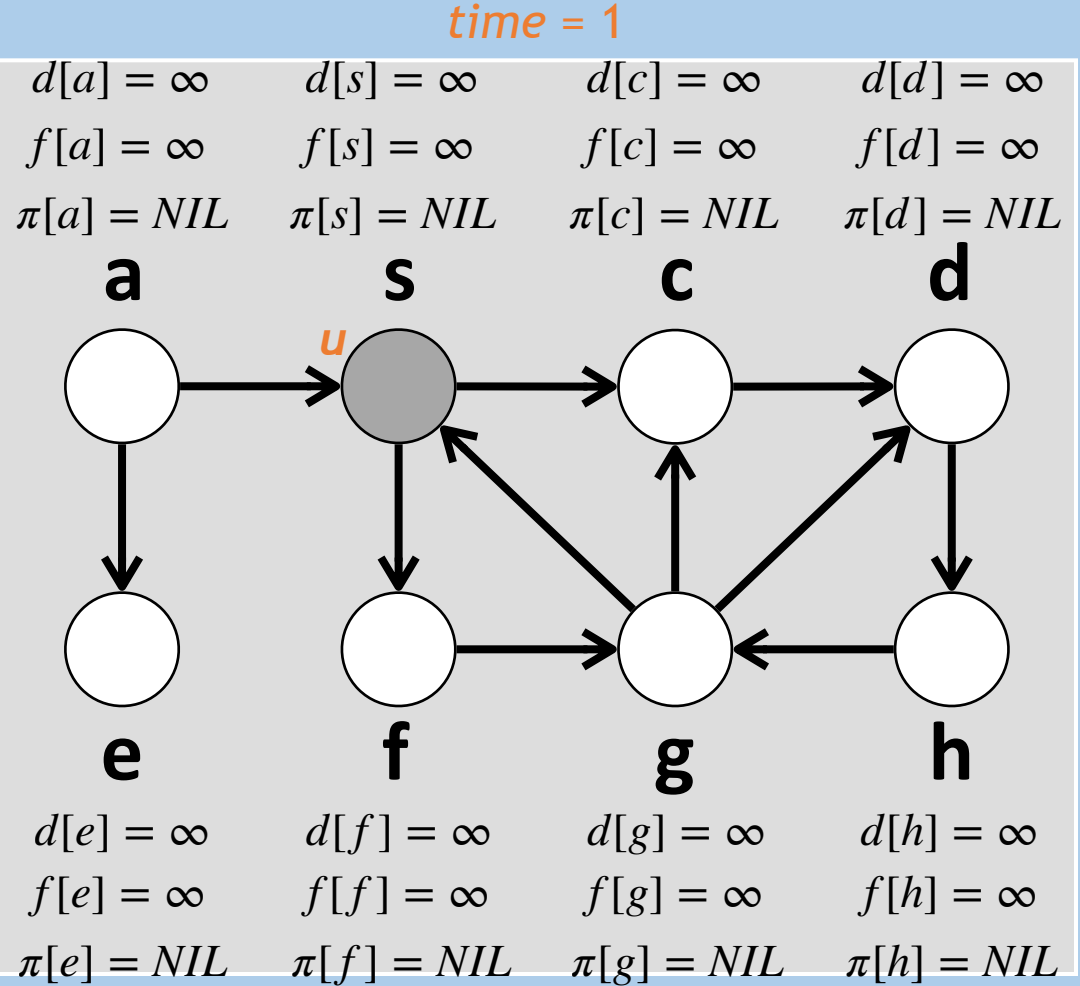

Not-discovered
Discovered
Finished

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
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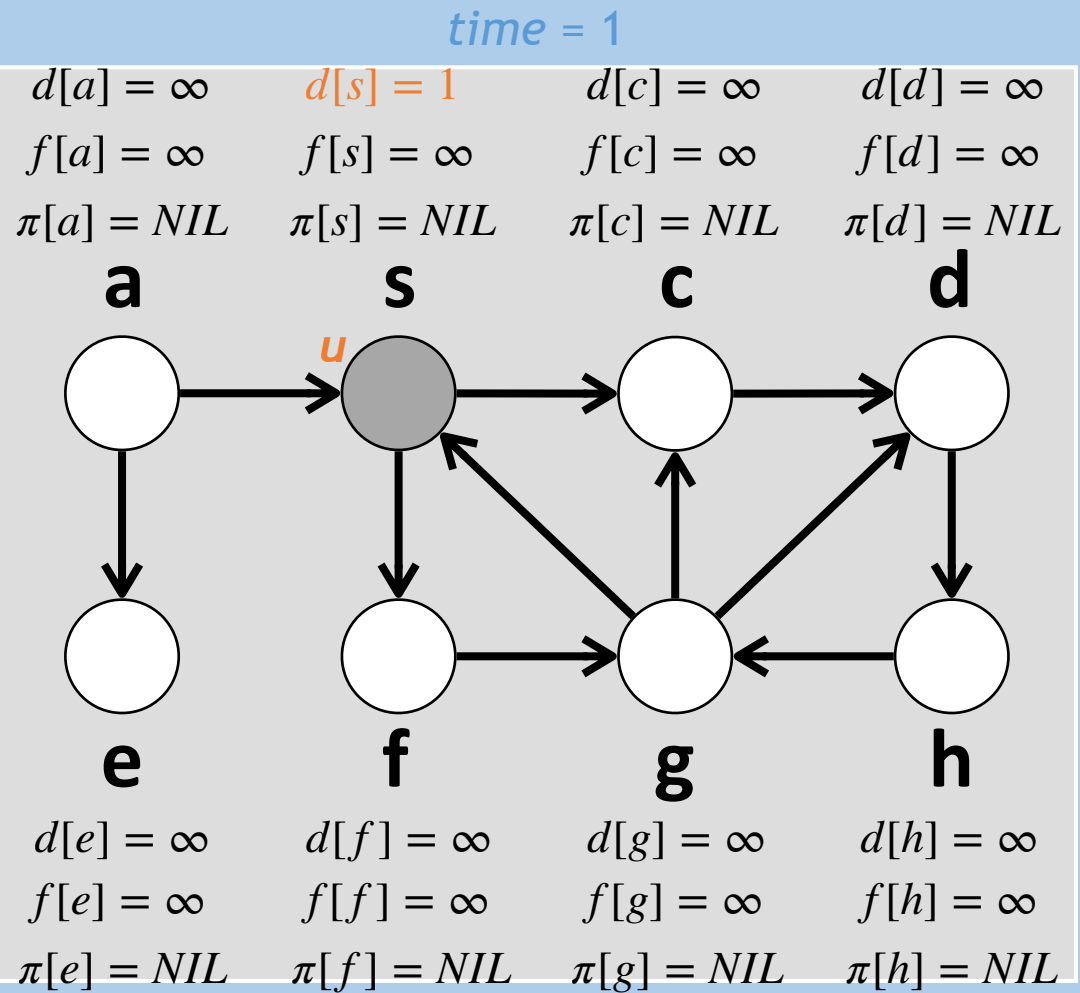

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
    Mark u as discovered
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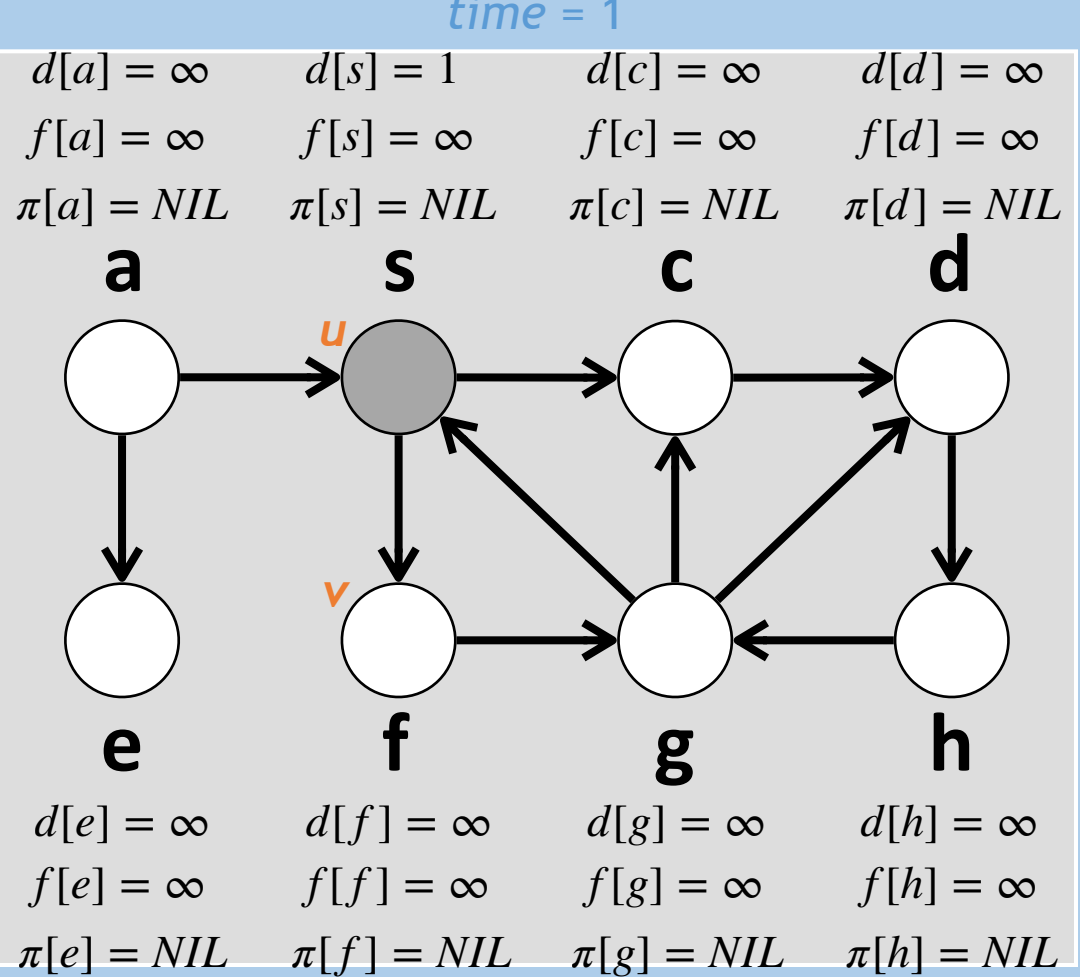

Not-discovered
Discovered
Finished

Depth-First Search

```
Procedure DFS-VISIT(u)
    Mark u as discovered
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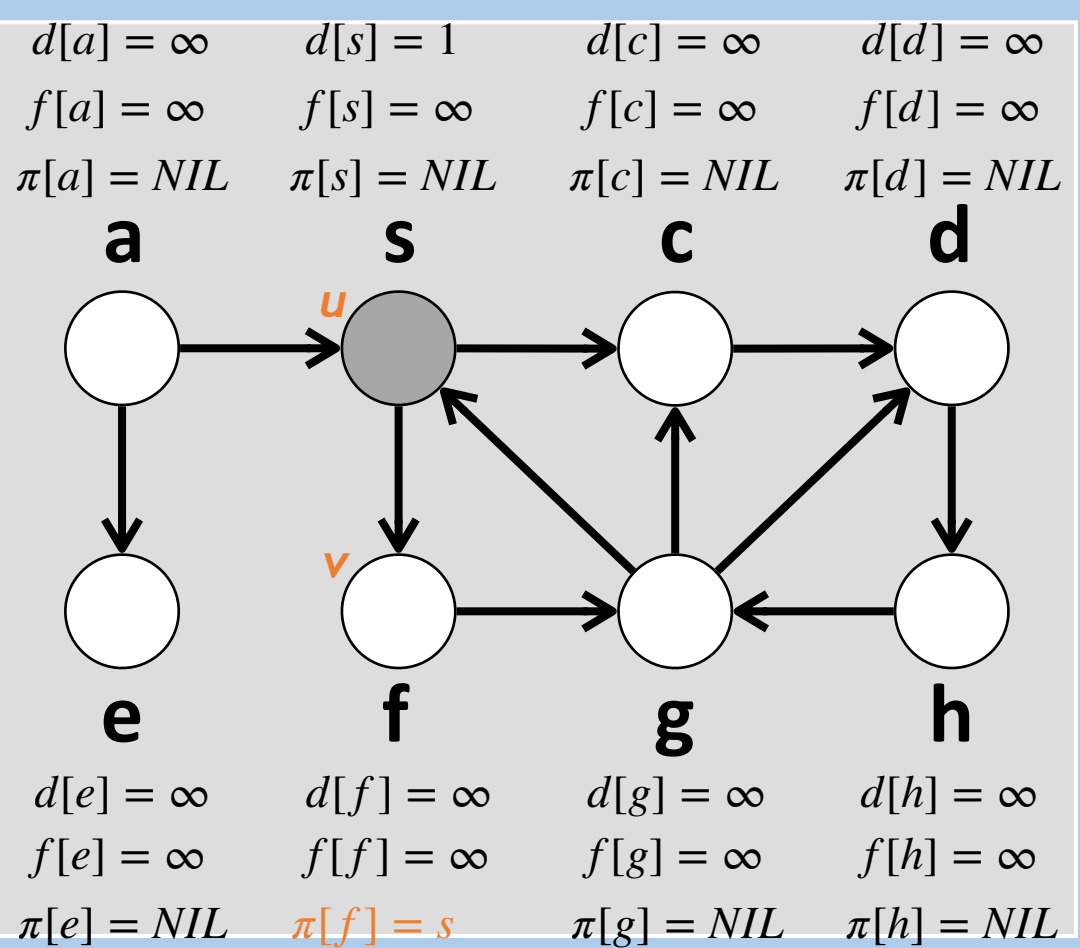

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
    Mark u as discovered
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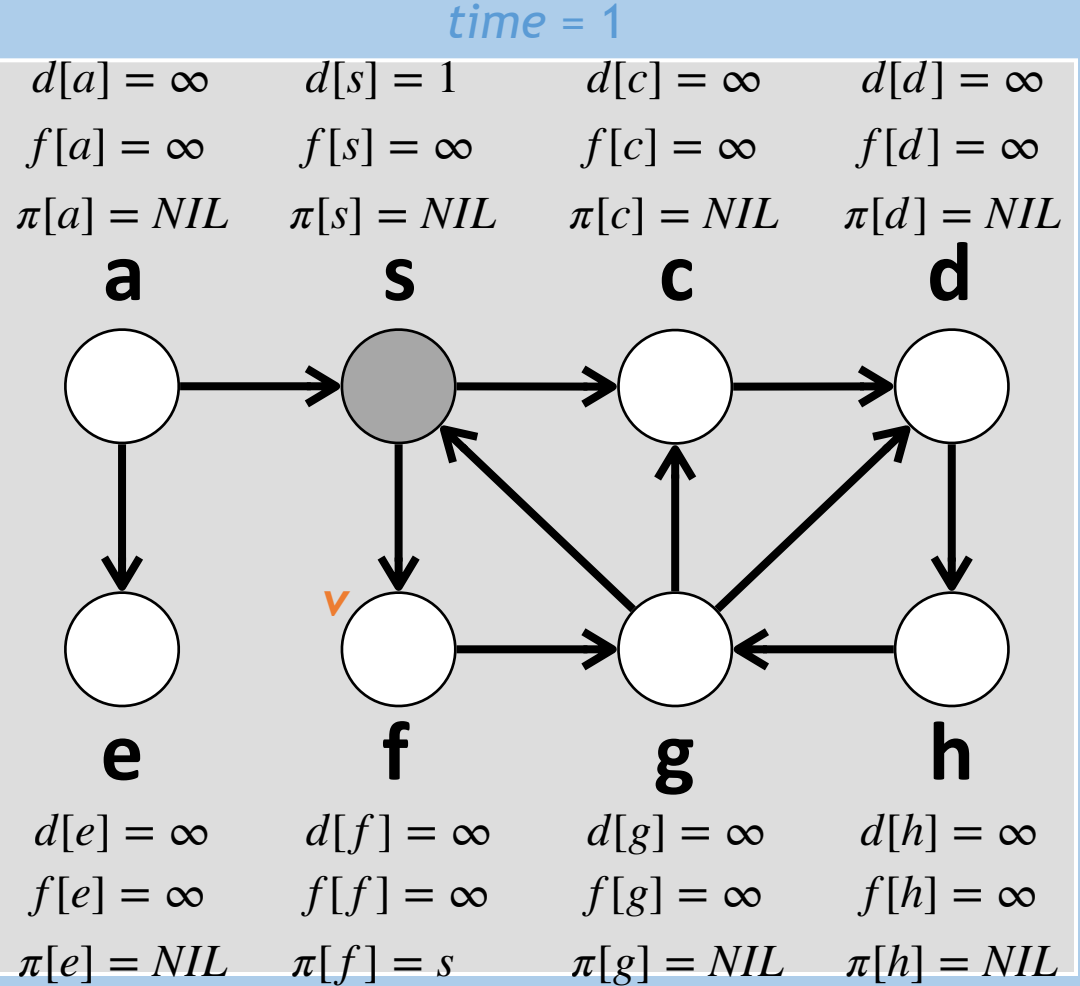

Not-discovered
Discovered
Finished

Depth-First Search

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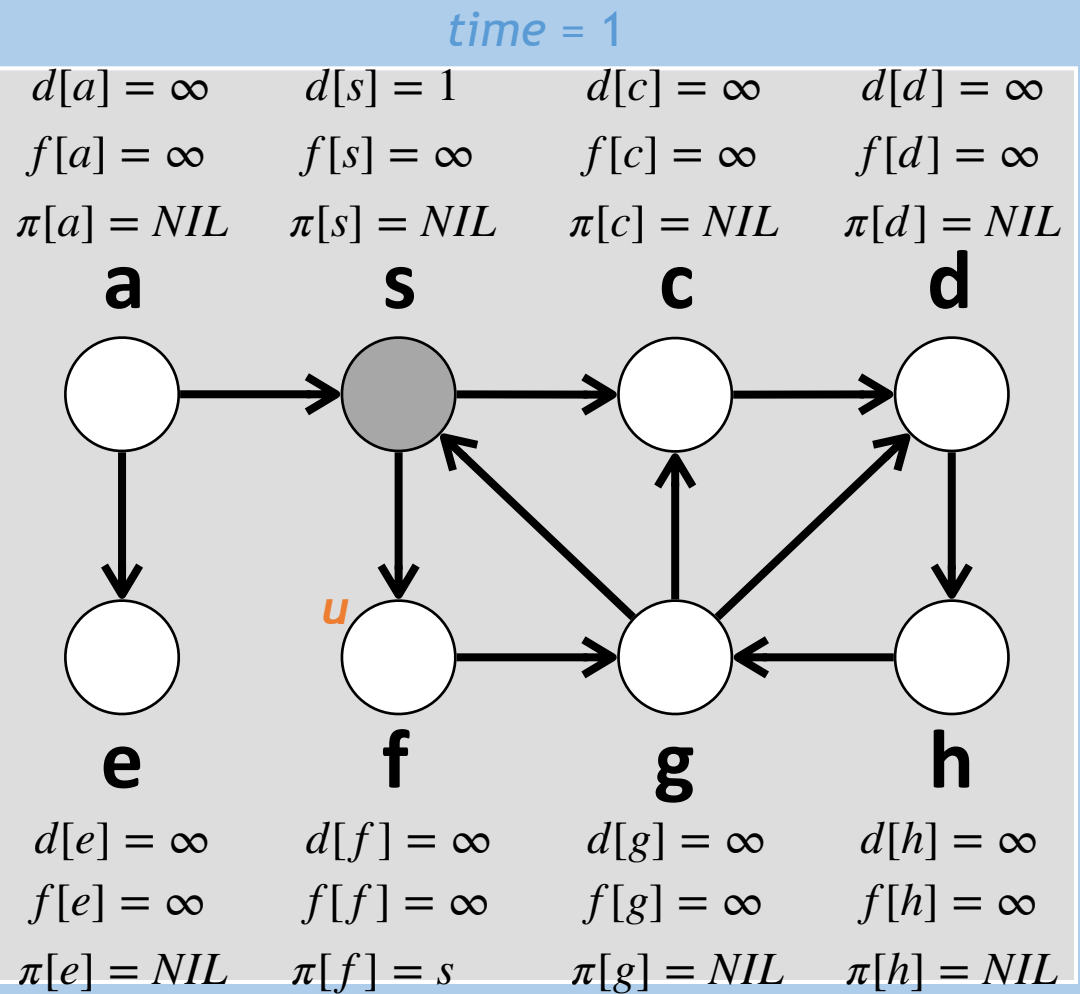

Not-discovered
Discovered
Finished

Depth-First Search

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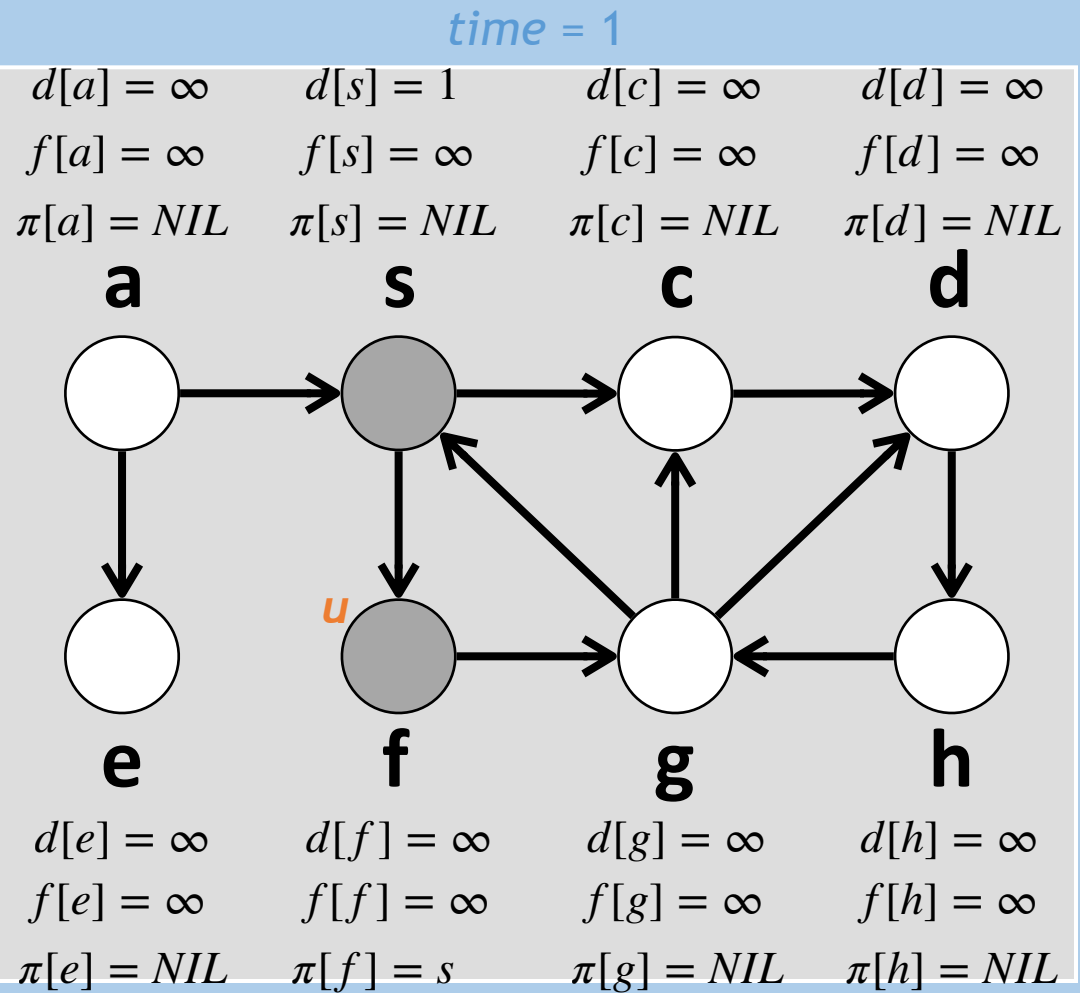

Not-discovered
Discovered
Finished

Depth-First Search

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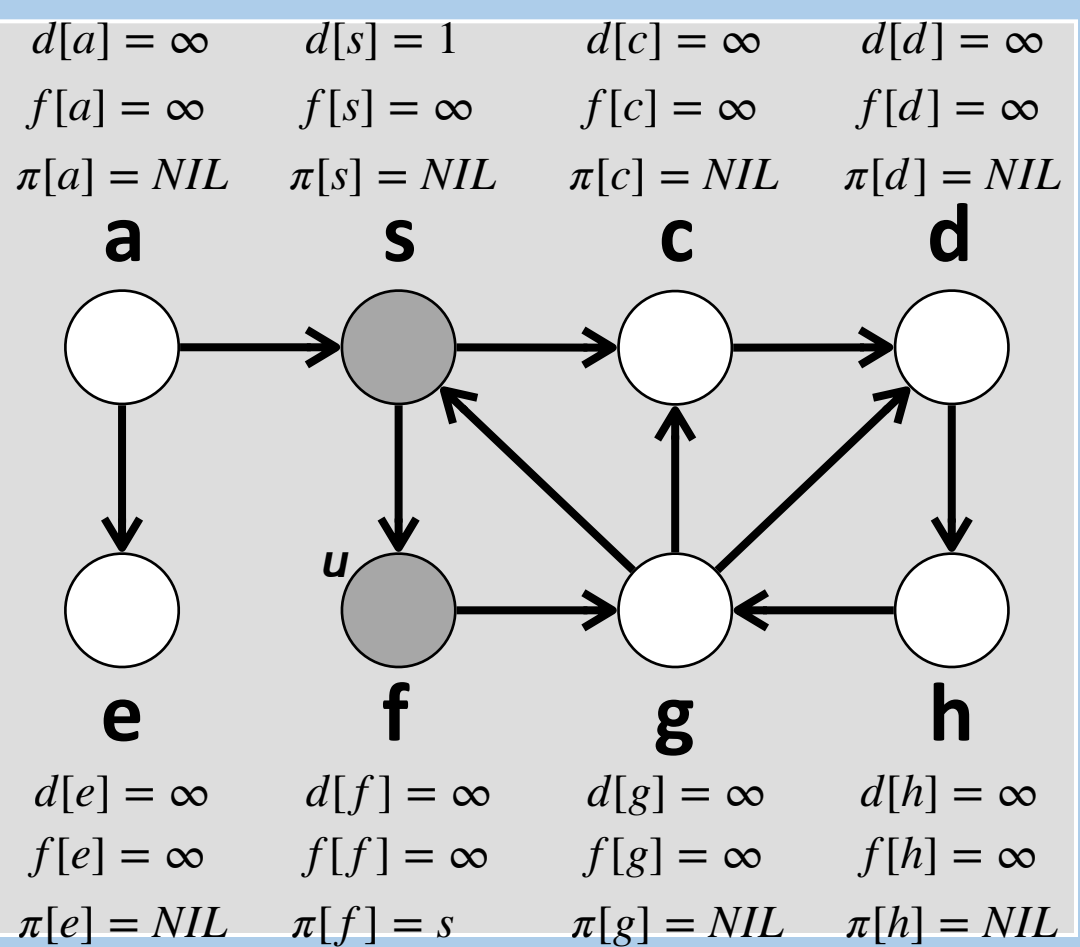

Not-discovered
Discovered
Finished

Depth-First Search

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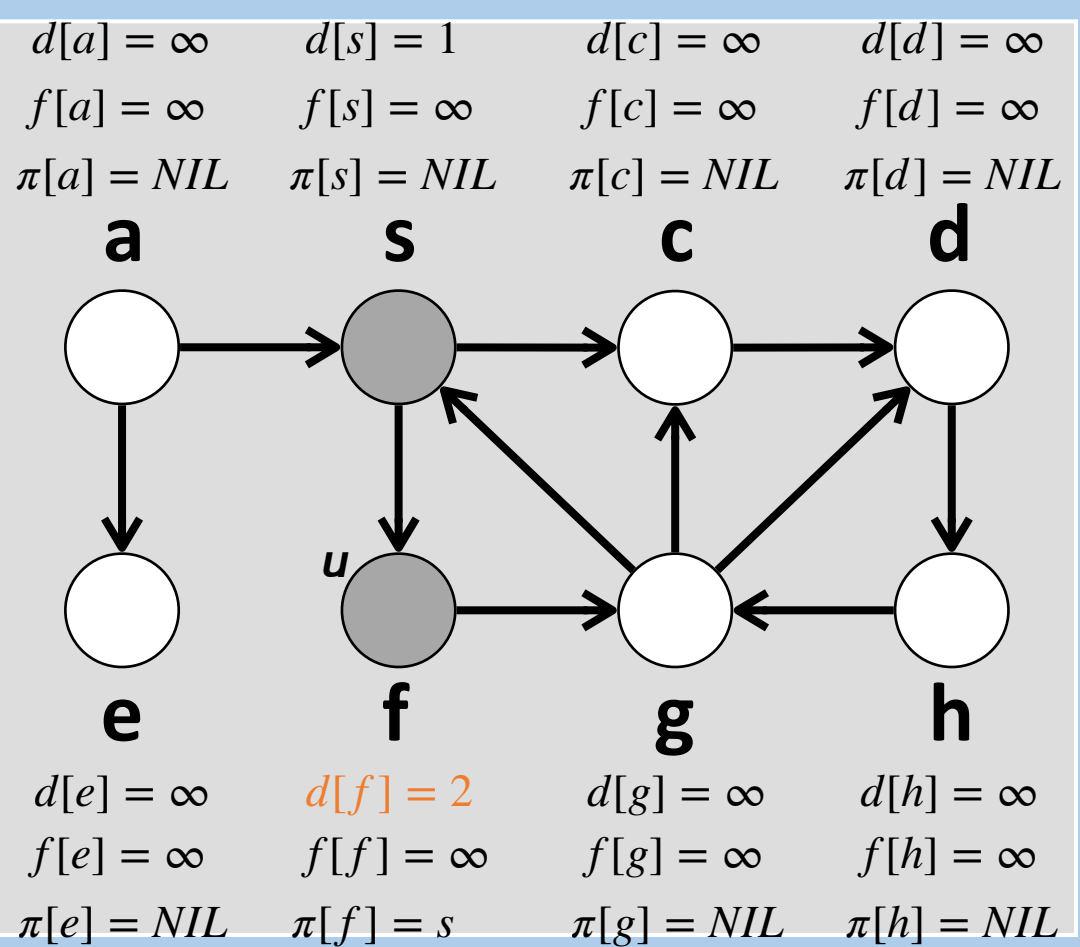

Not-discovered
Discovered
Finished

Depth-First Search

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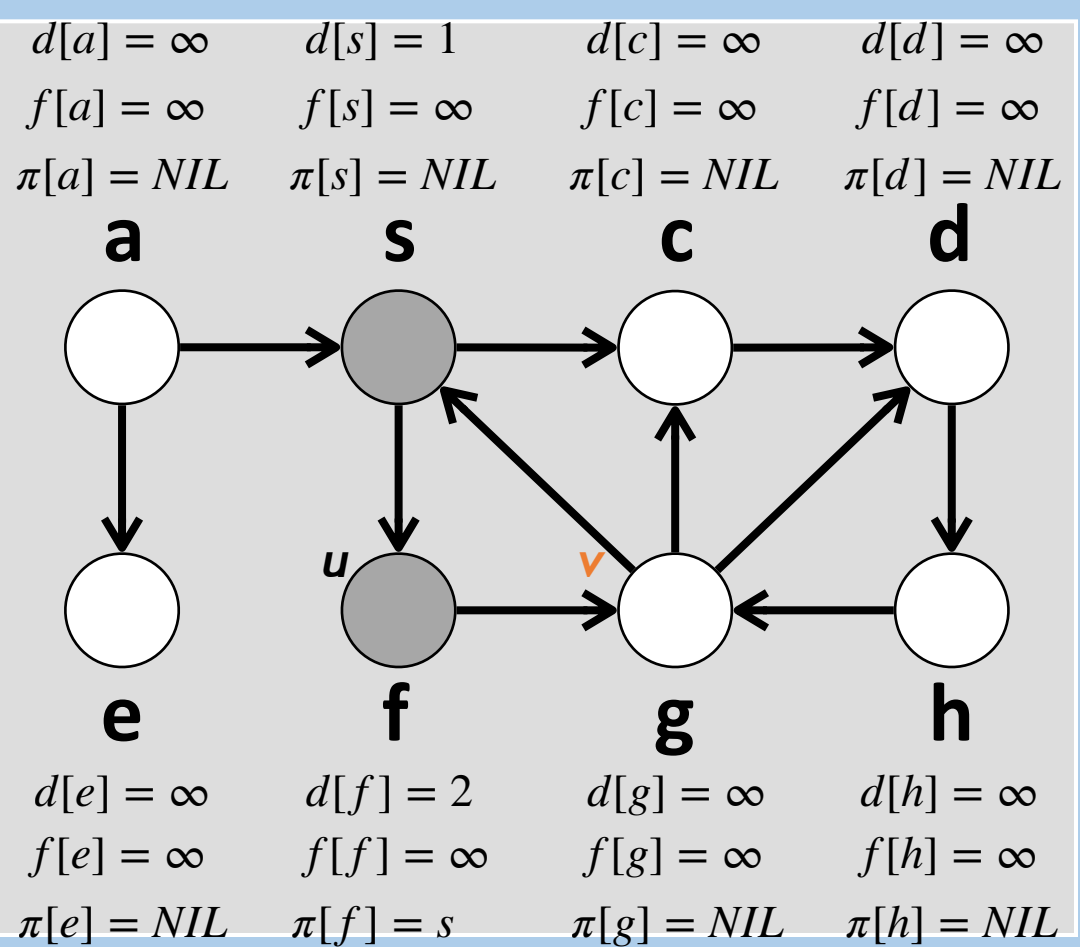

Not-discovered
Discovered
Finished

Depth-First Search

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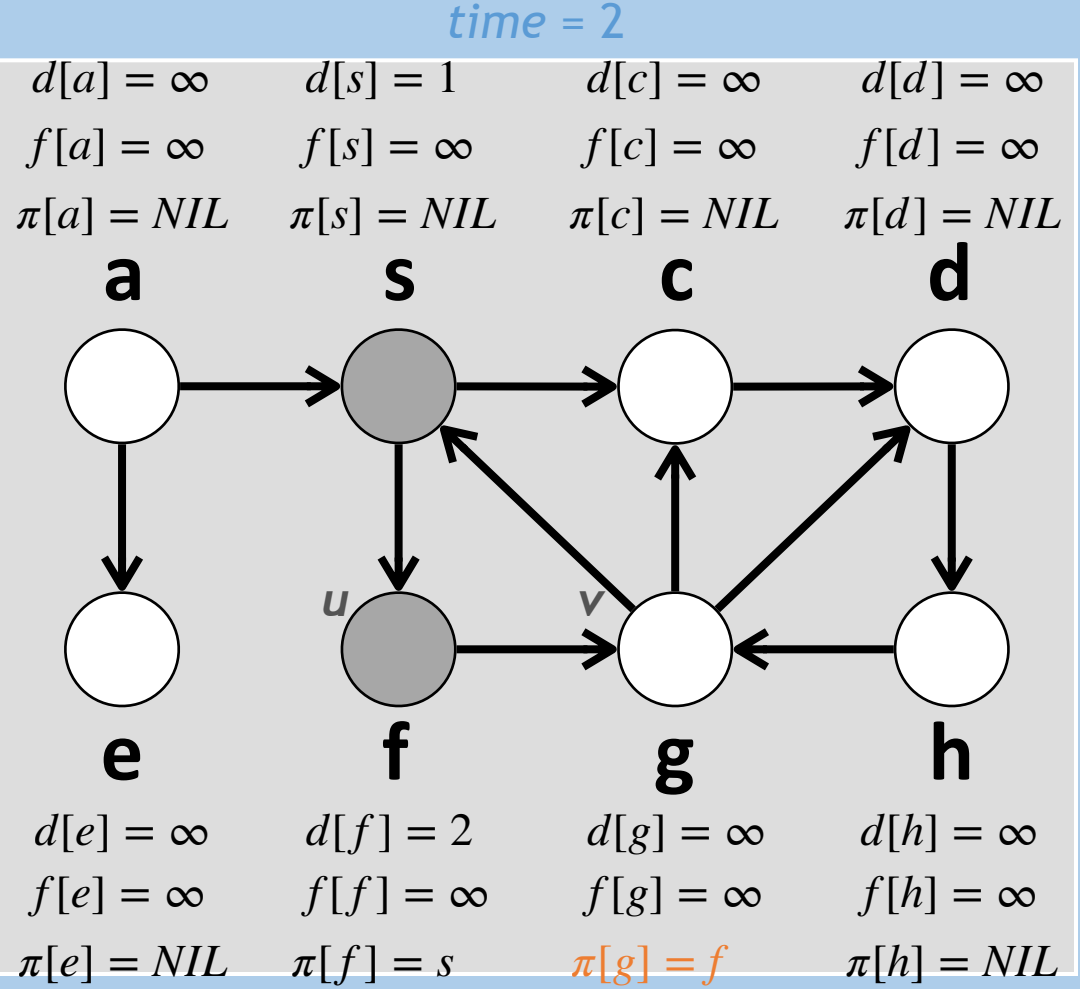

Not-discovered
Discovered
Finished

Depth-First Search

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    Mark u as discovered
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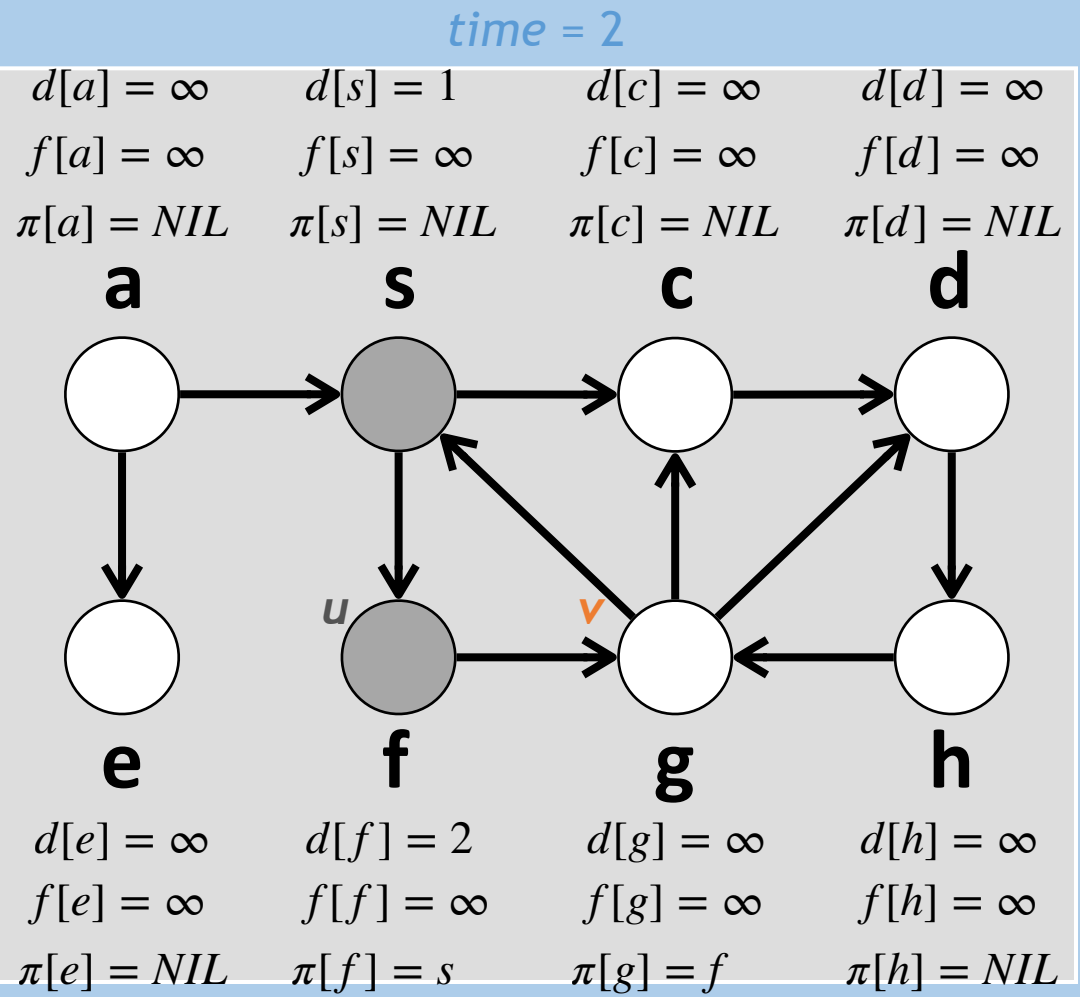

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
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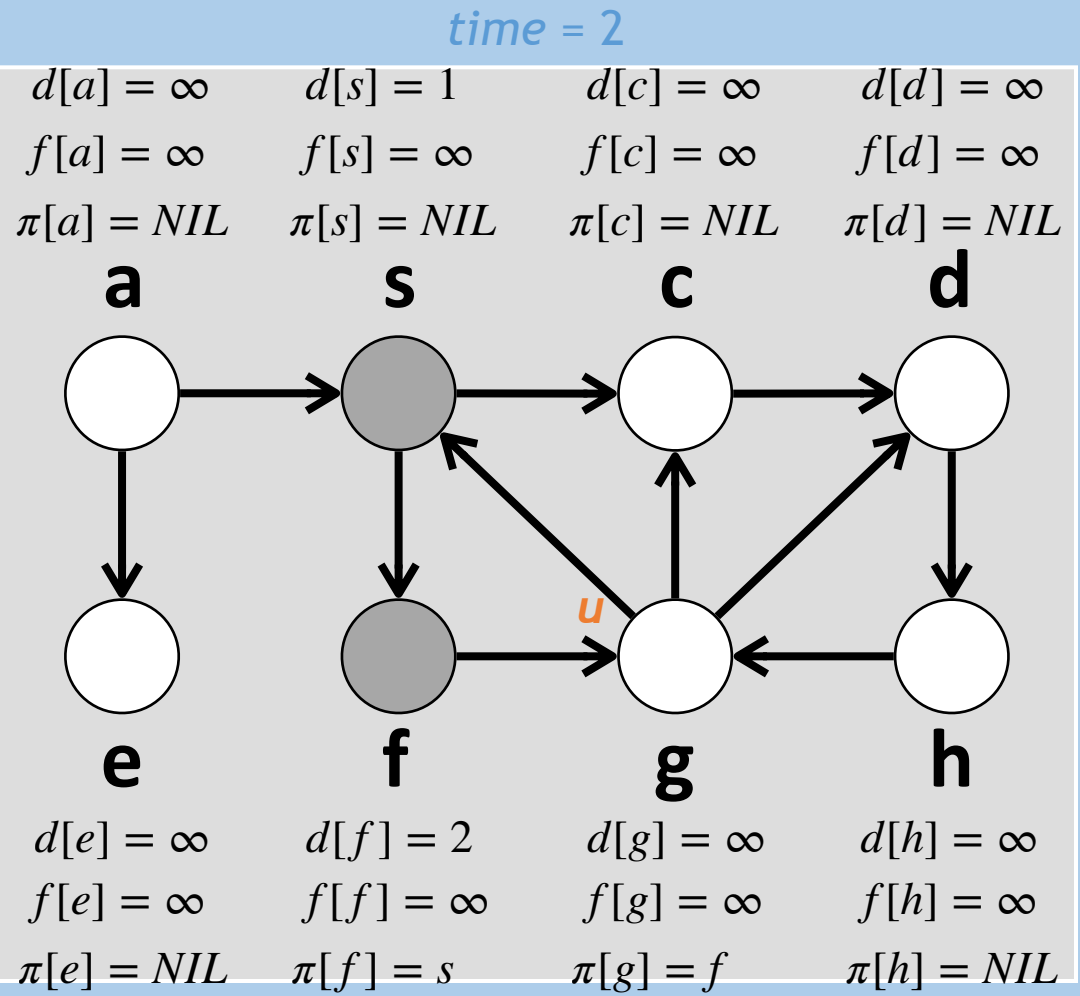

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
  Mark u as discovered
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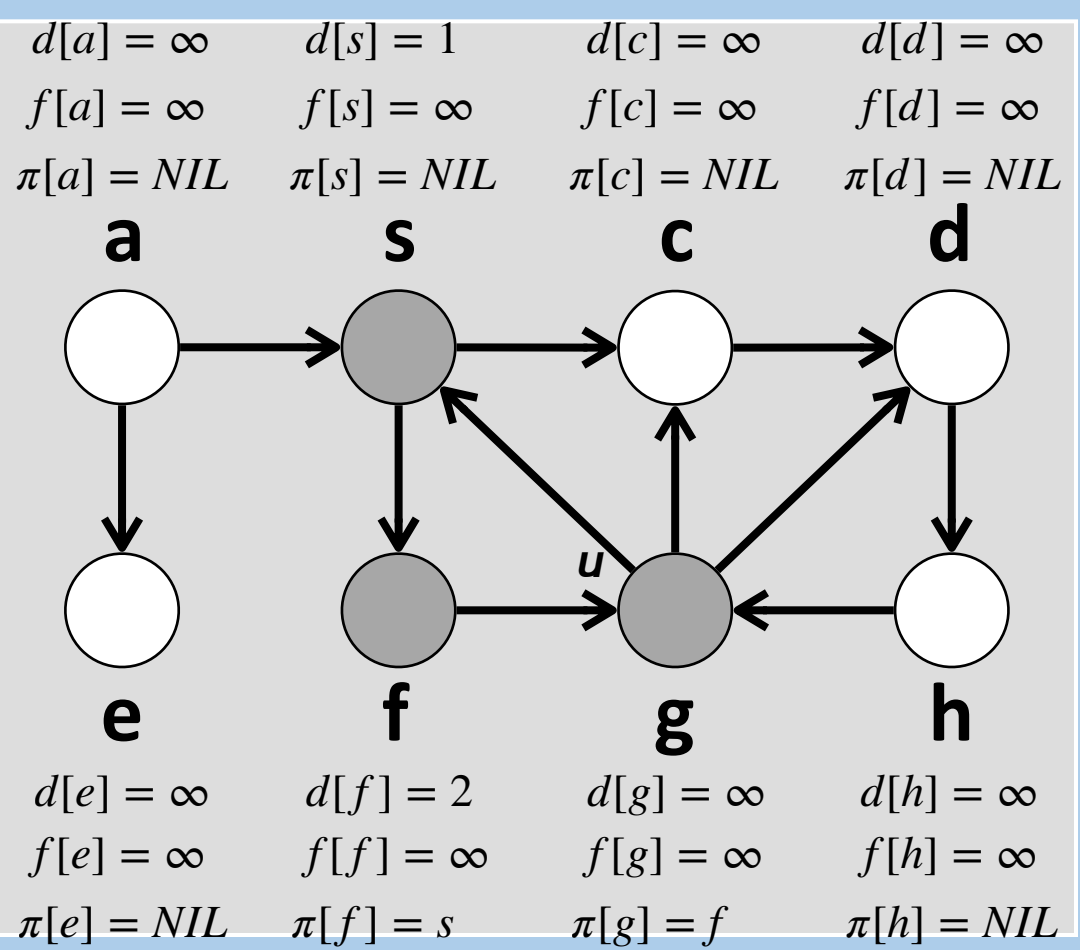

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
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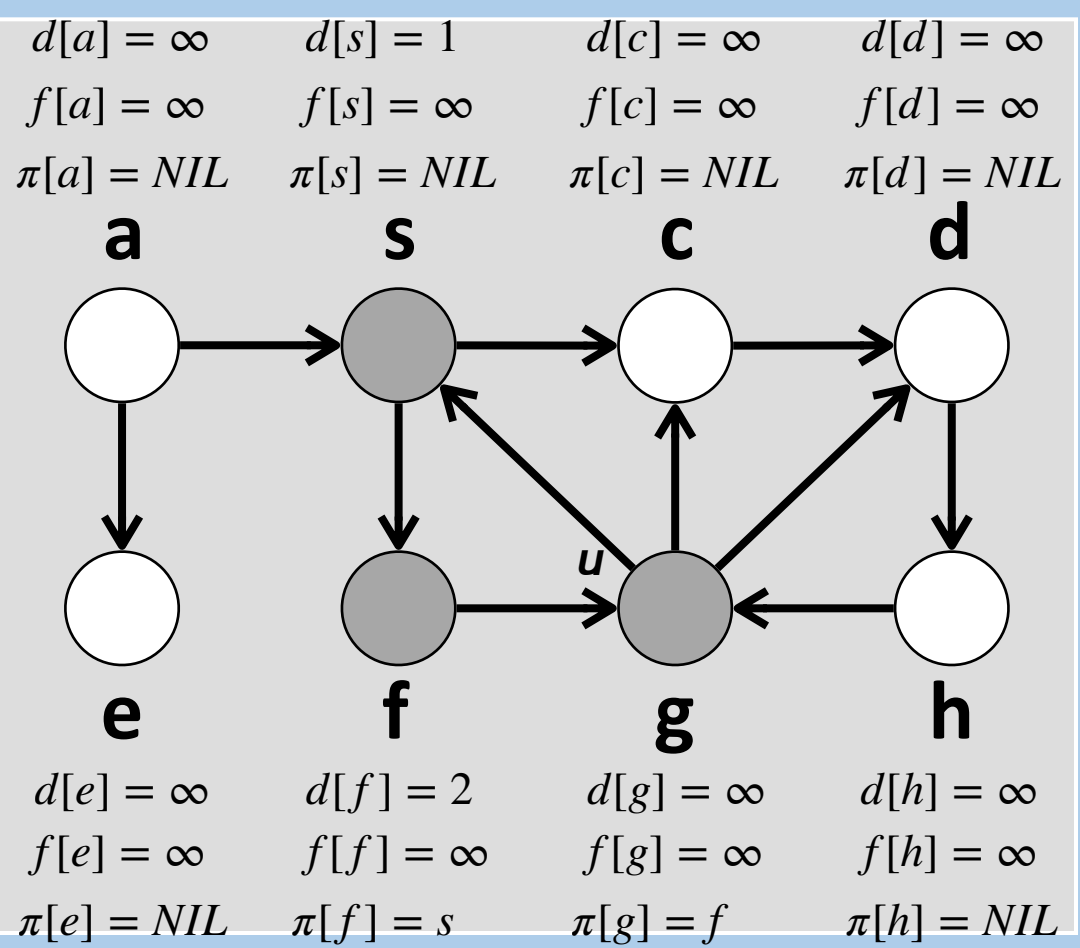

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
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    time ← time + 1
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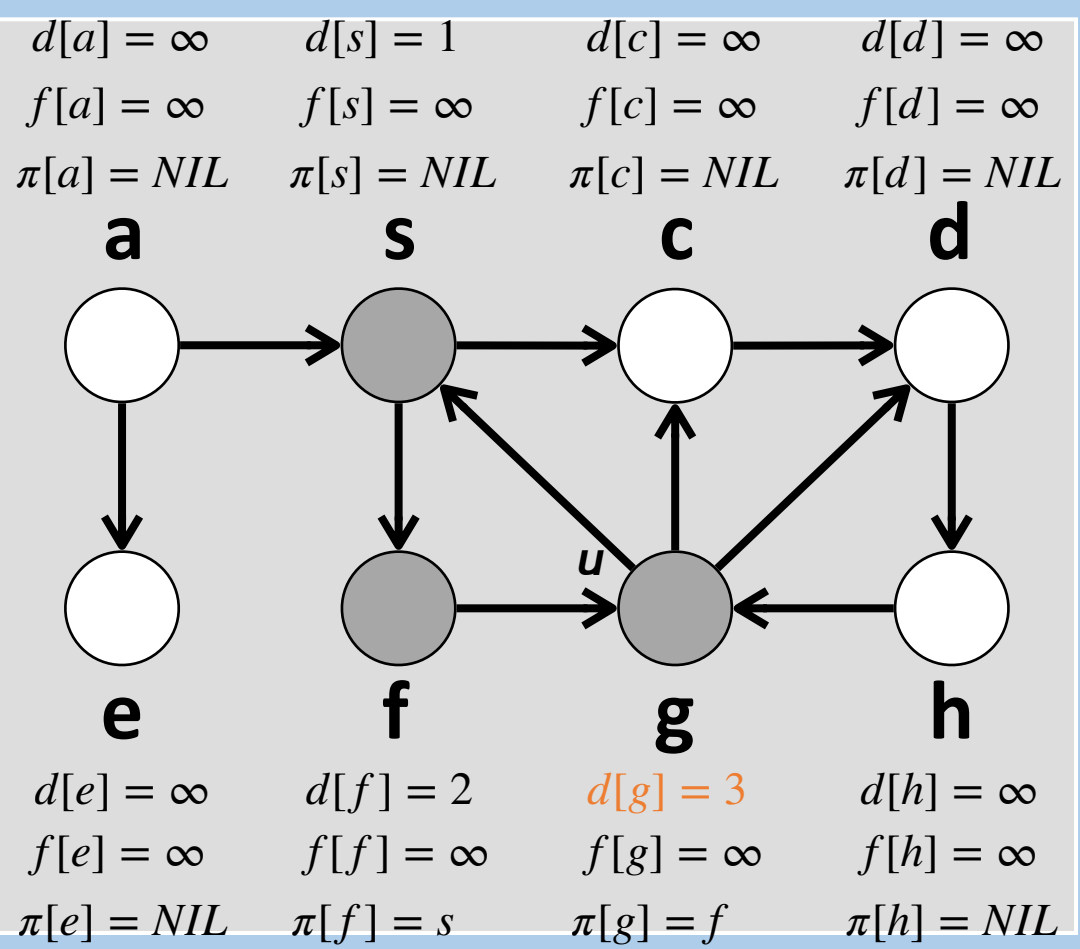

Not-discovered
Discovered
Finished

Depth-First Search

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Procedure DFS-VISIT(u)
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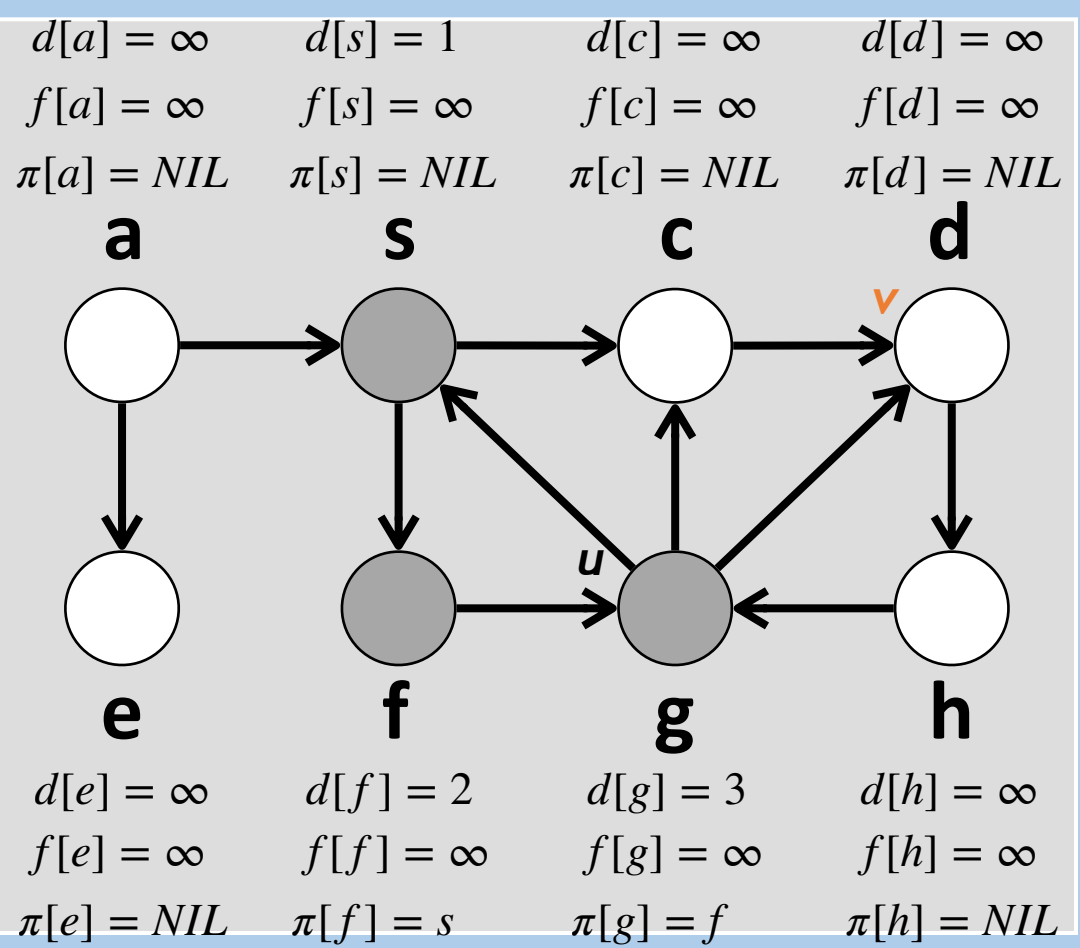

Not-discovered
Discovered
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Depth-First Search

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    Mark u as discovered
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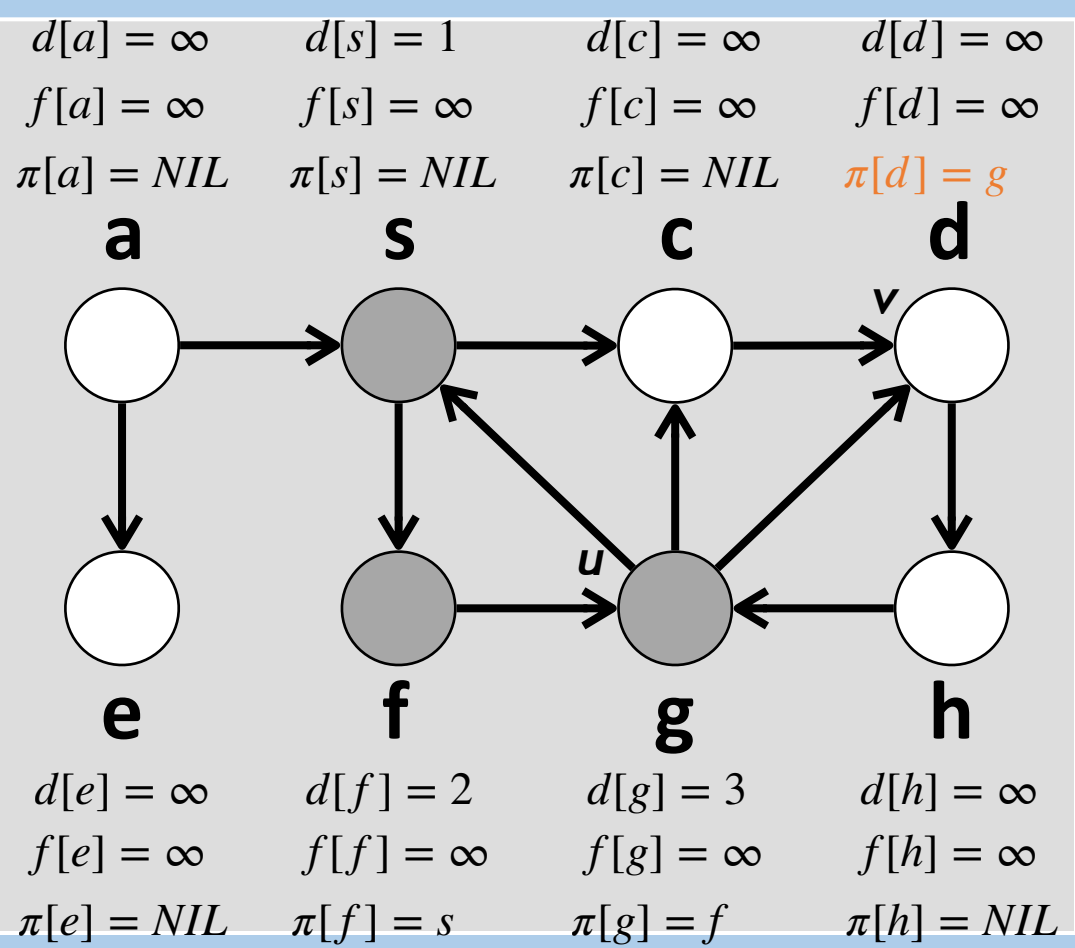

Not-discovered
Discovered
Finished

Depth-First Search

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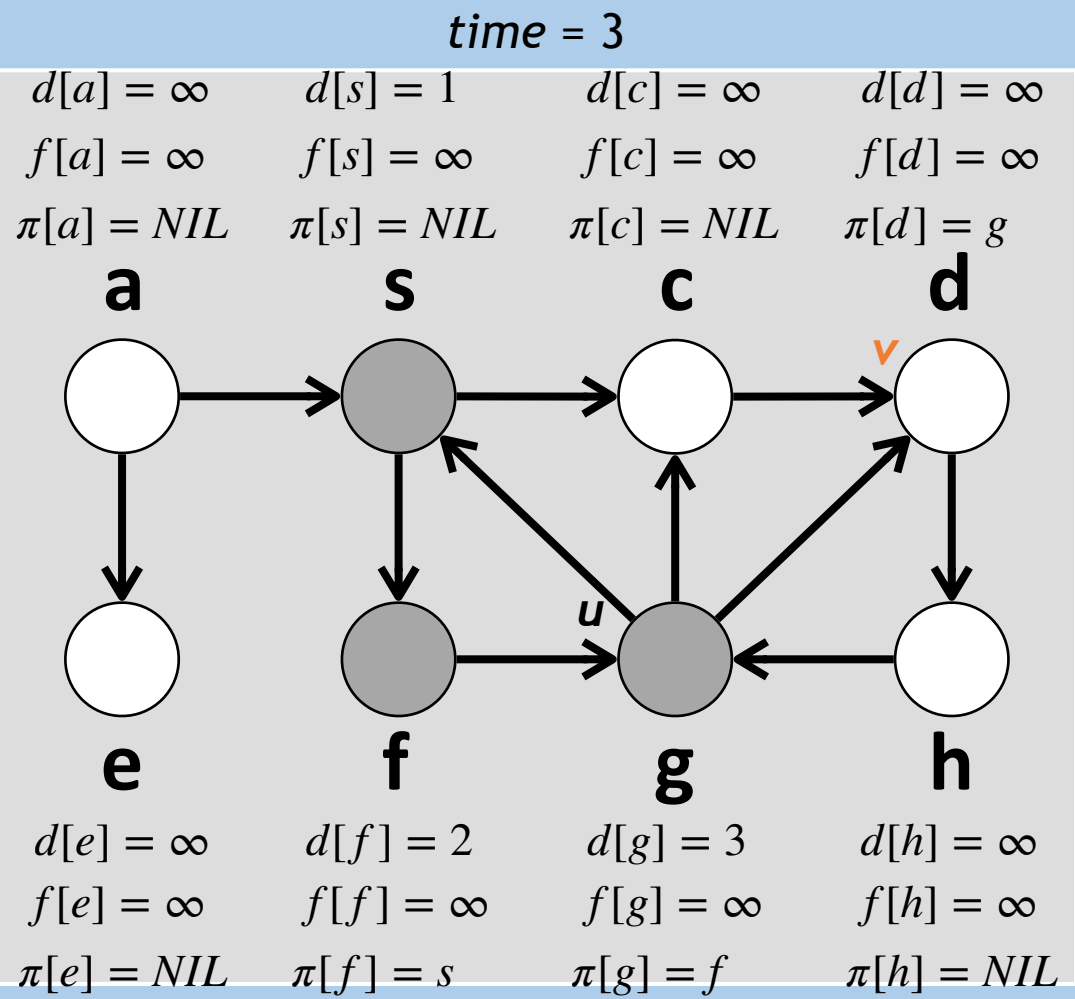

Not-discovered
Discovered
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Depth-First Search

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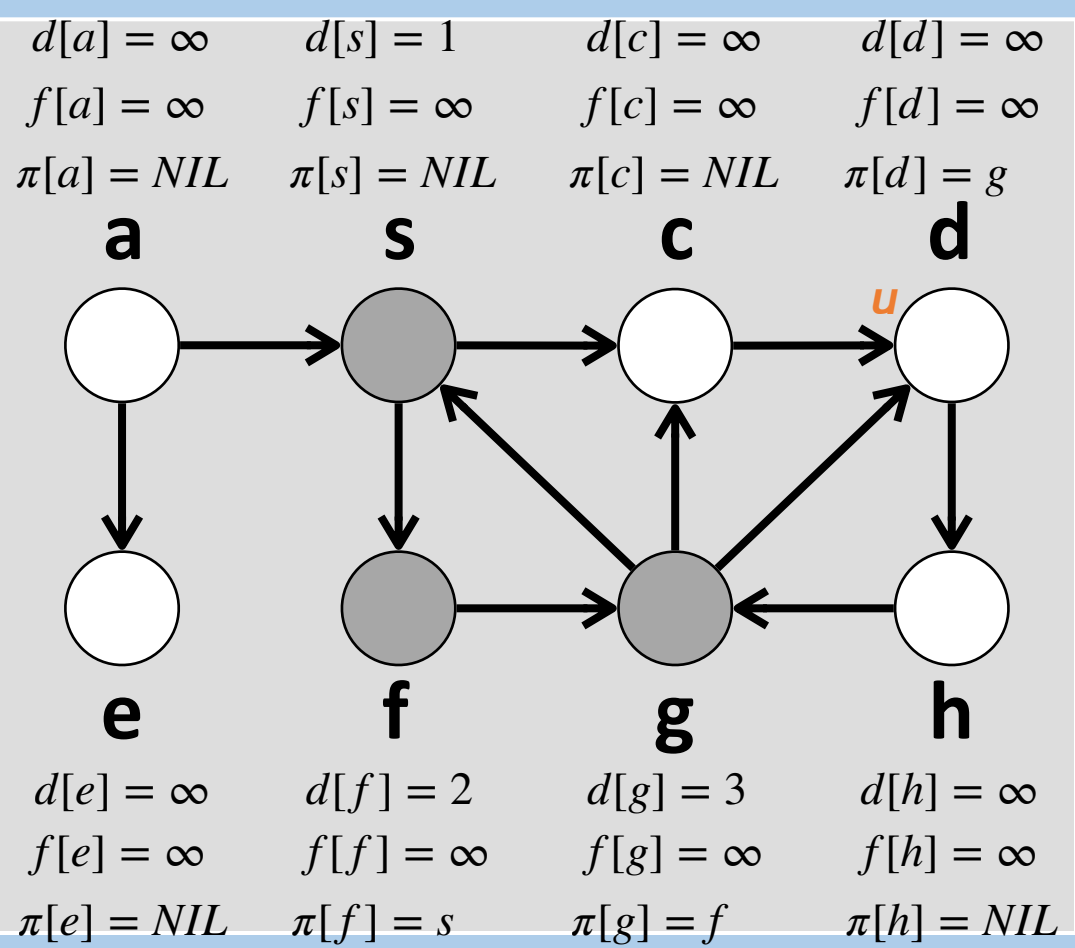

Not-discovered
Discovered
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Depth-First Search

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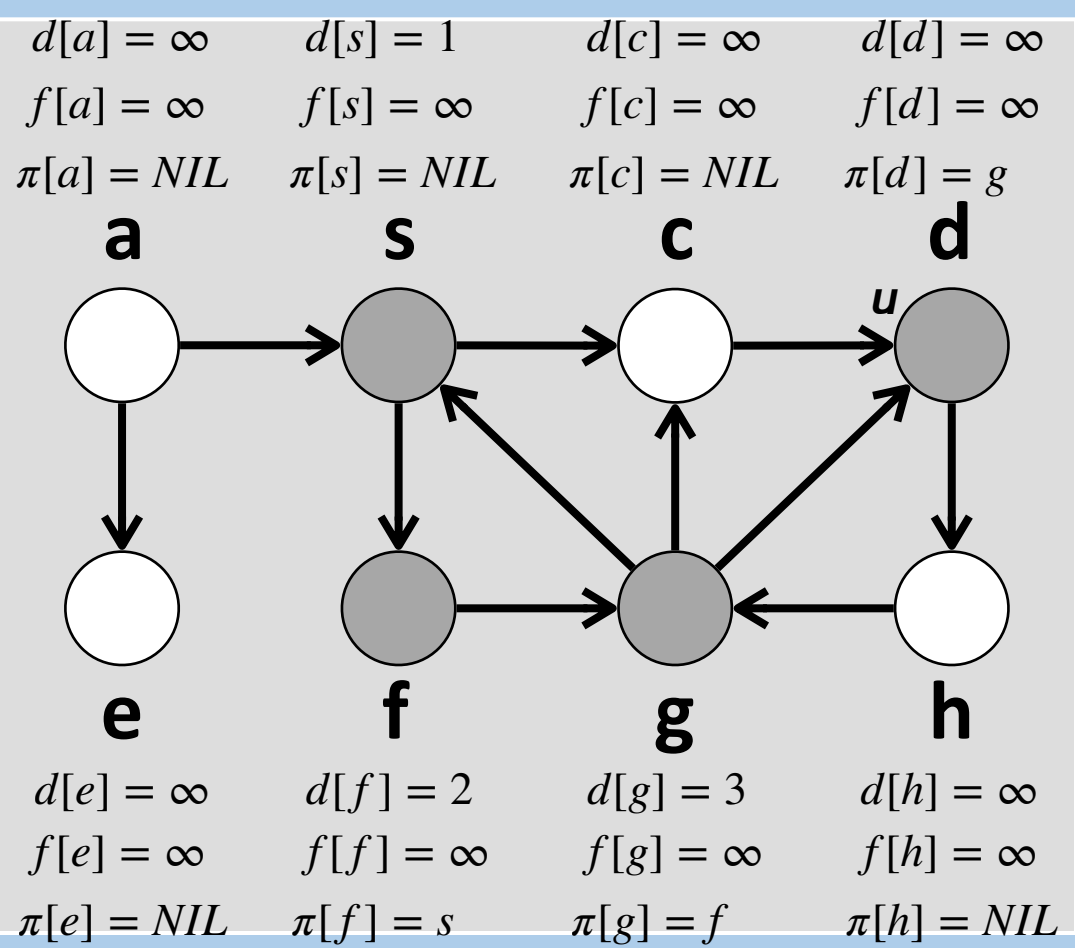

Not-discovered
Discovered
Finished

Depth-First Search

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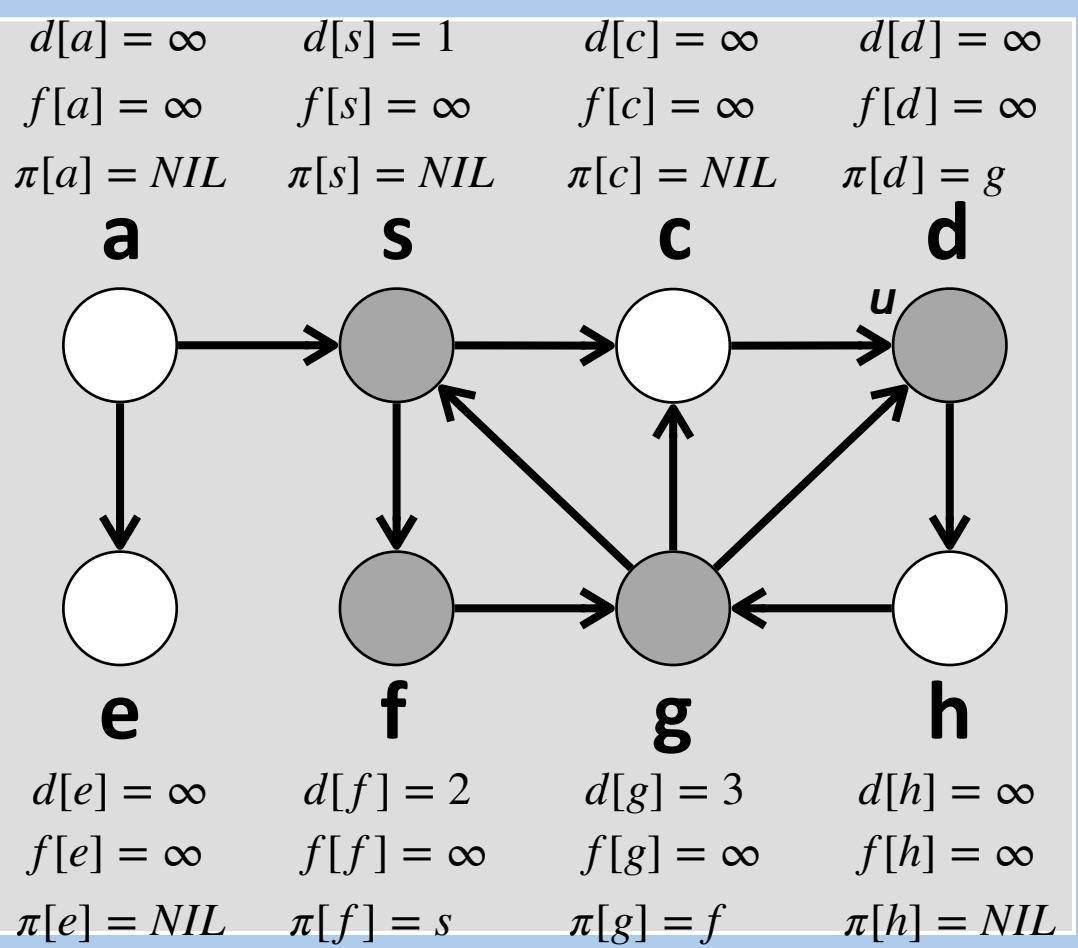

Not-discovered
Discovered
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Depth-First Search

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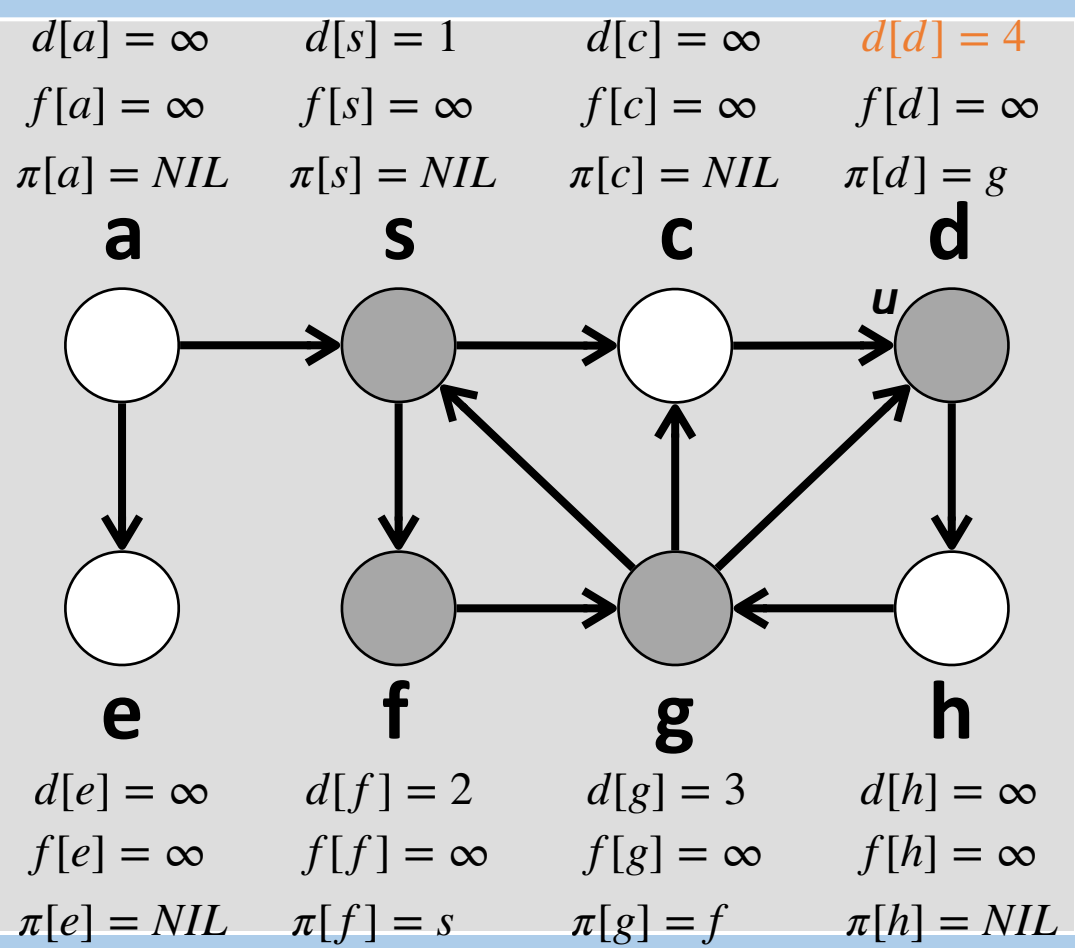

Not-discovered
Discovered
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Depth-First Search

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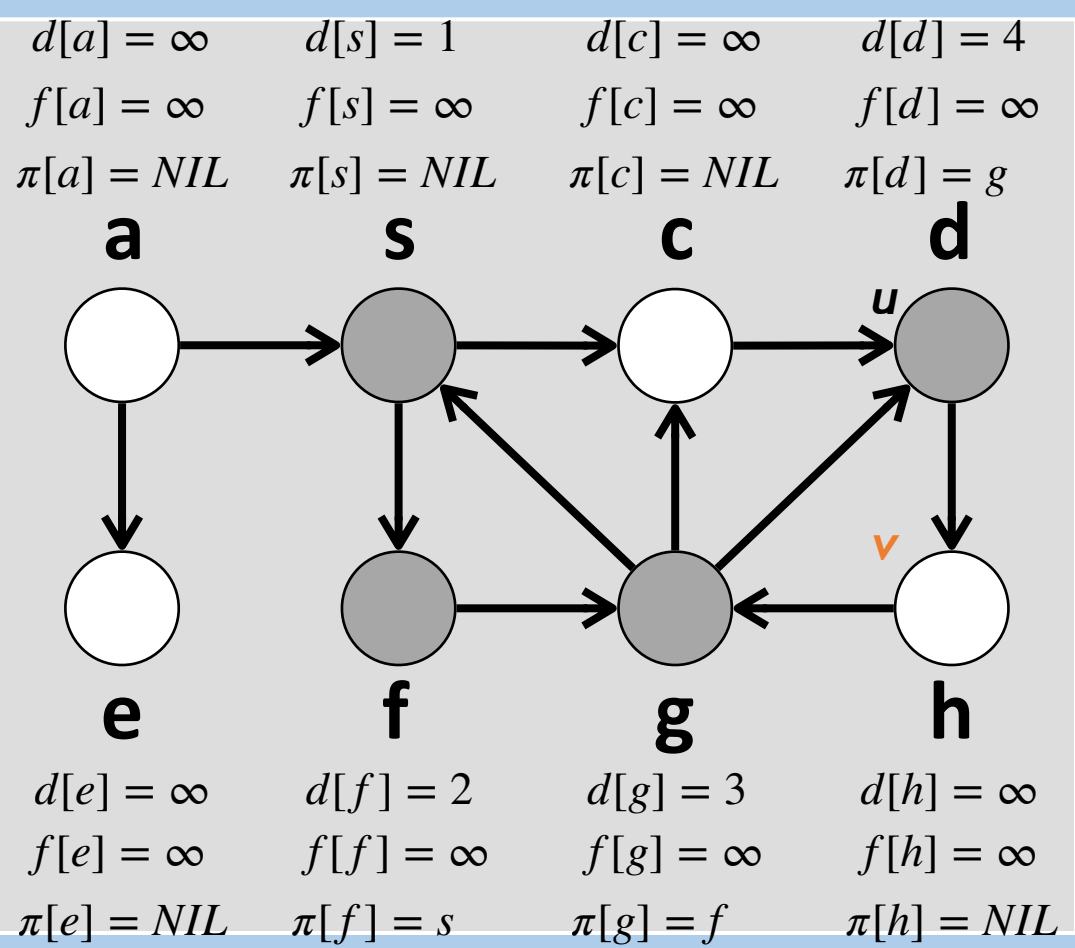

Not-discovered
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Depth-First Search

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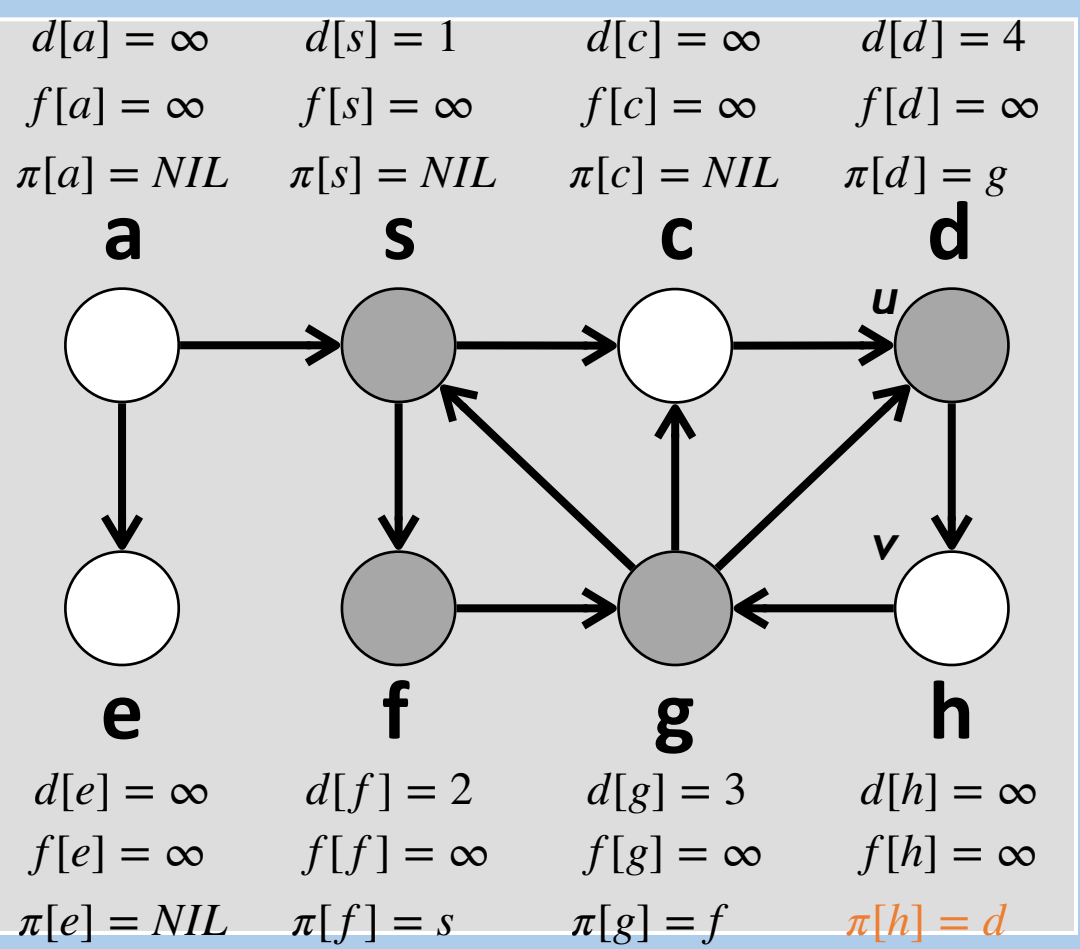

Not-discovered
Discovered
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Depth-First Search

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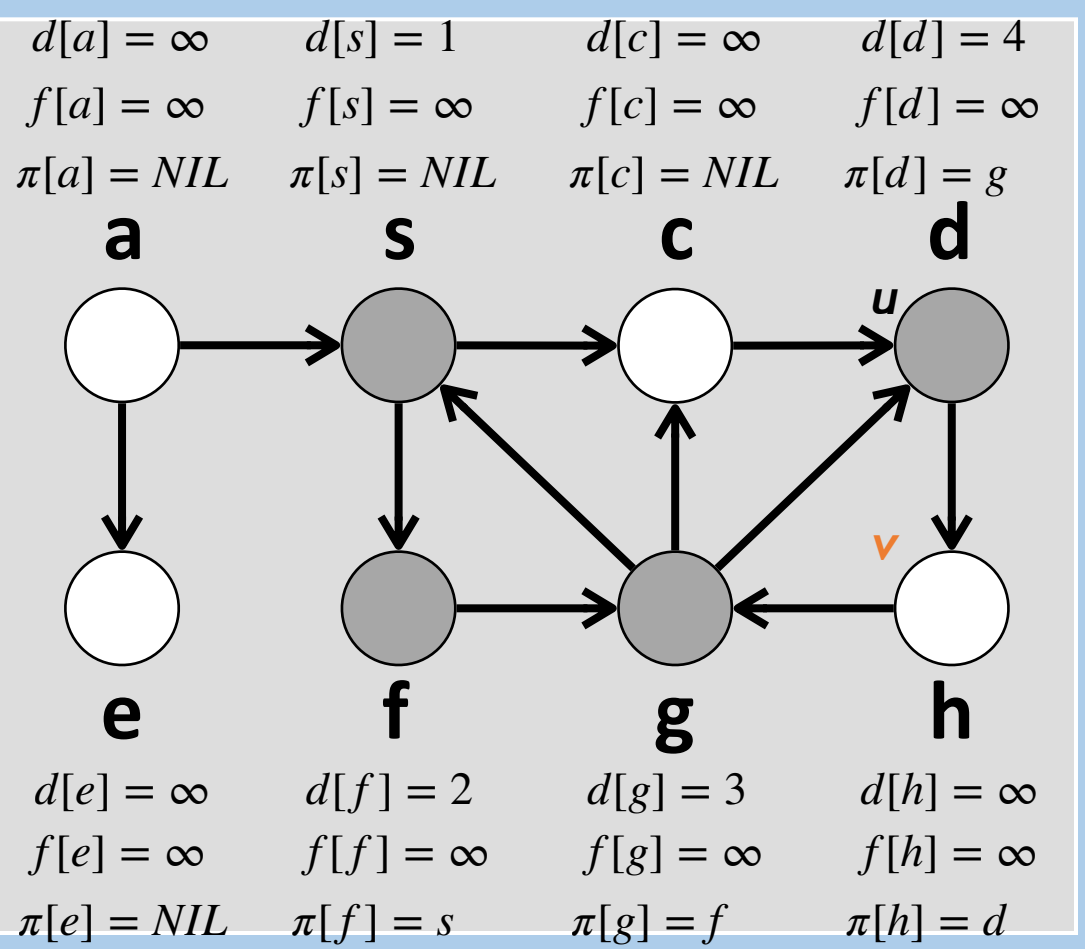

Not-discovered
Discovered
Finished

Depth-First Search

```
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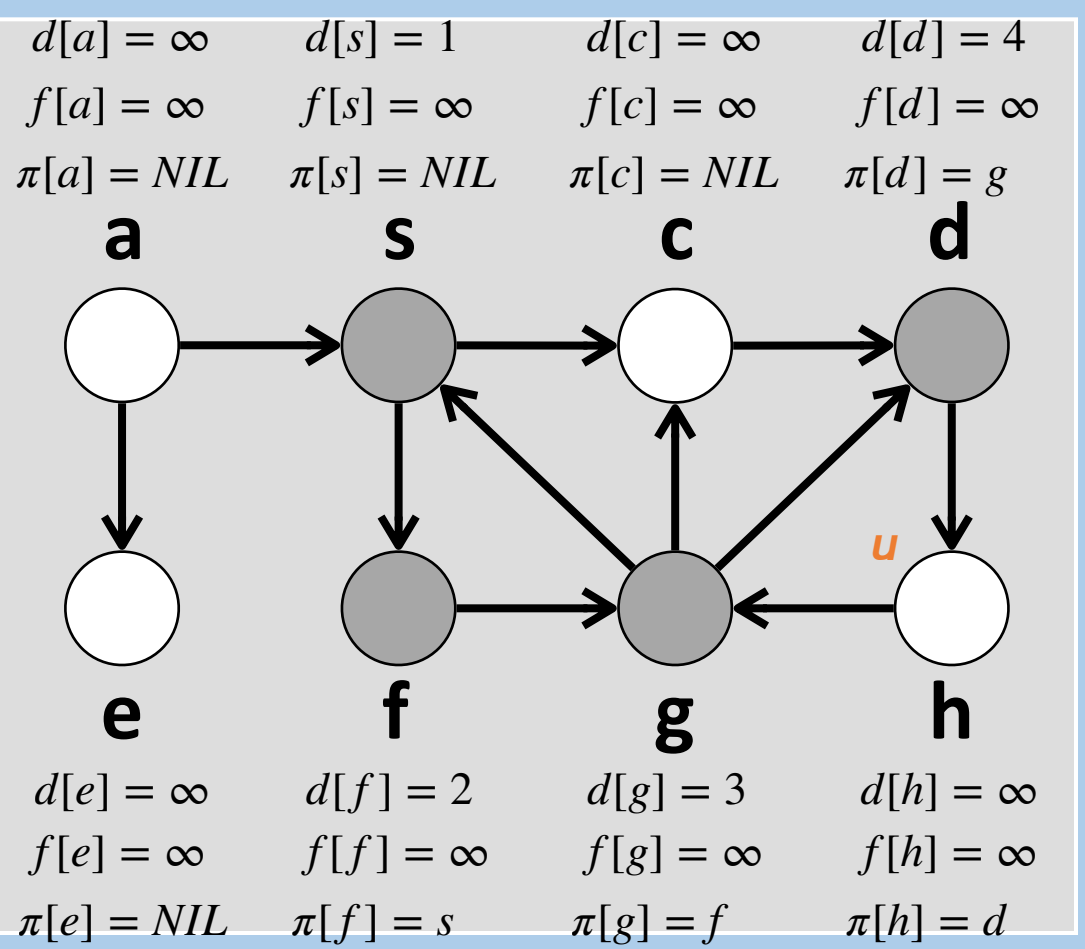

Not-discovered
Discovered
Finished

Depth-First Search

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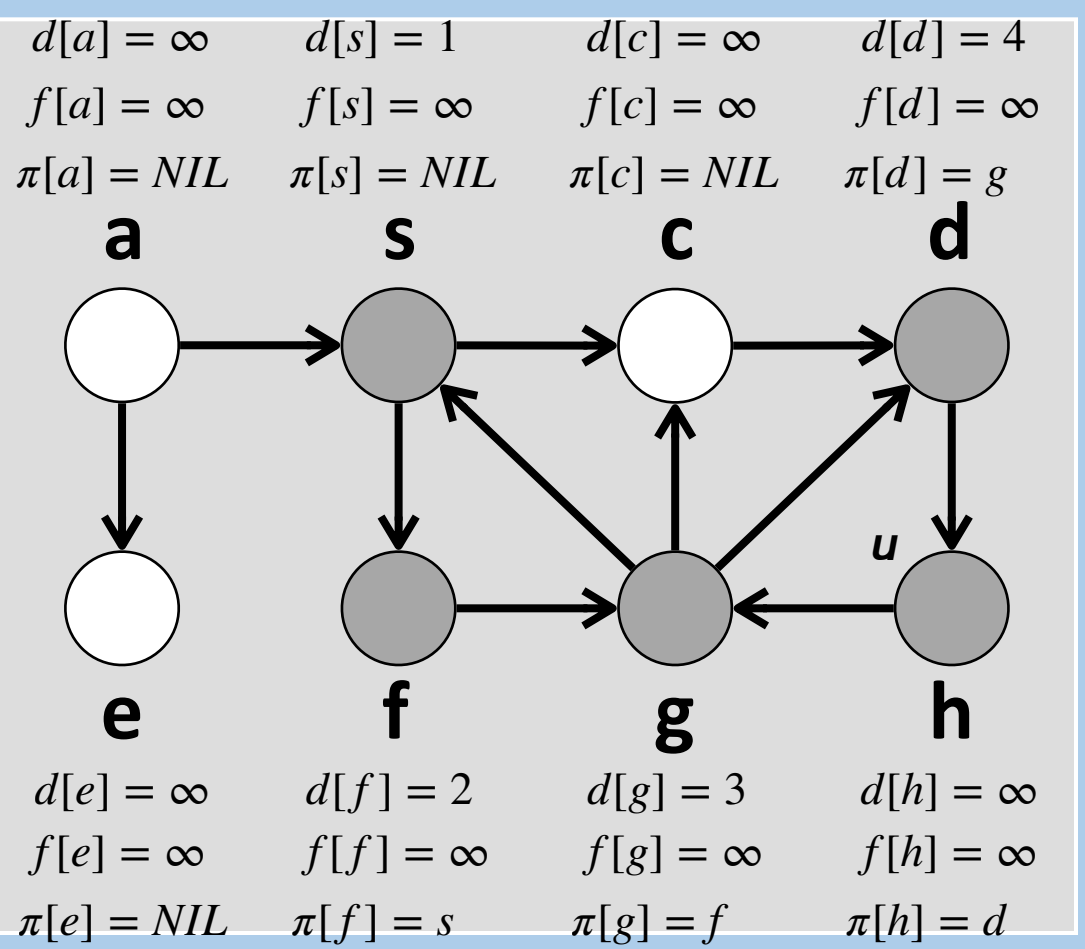

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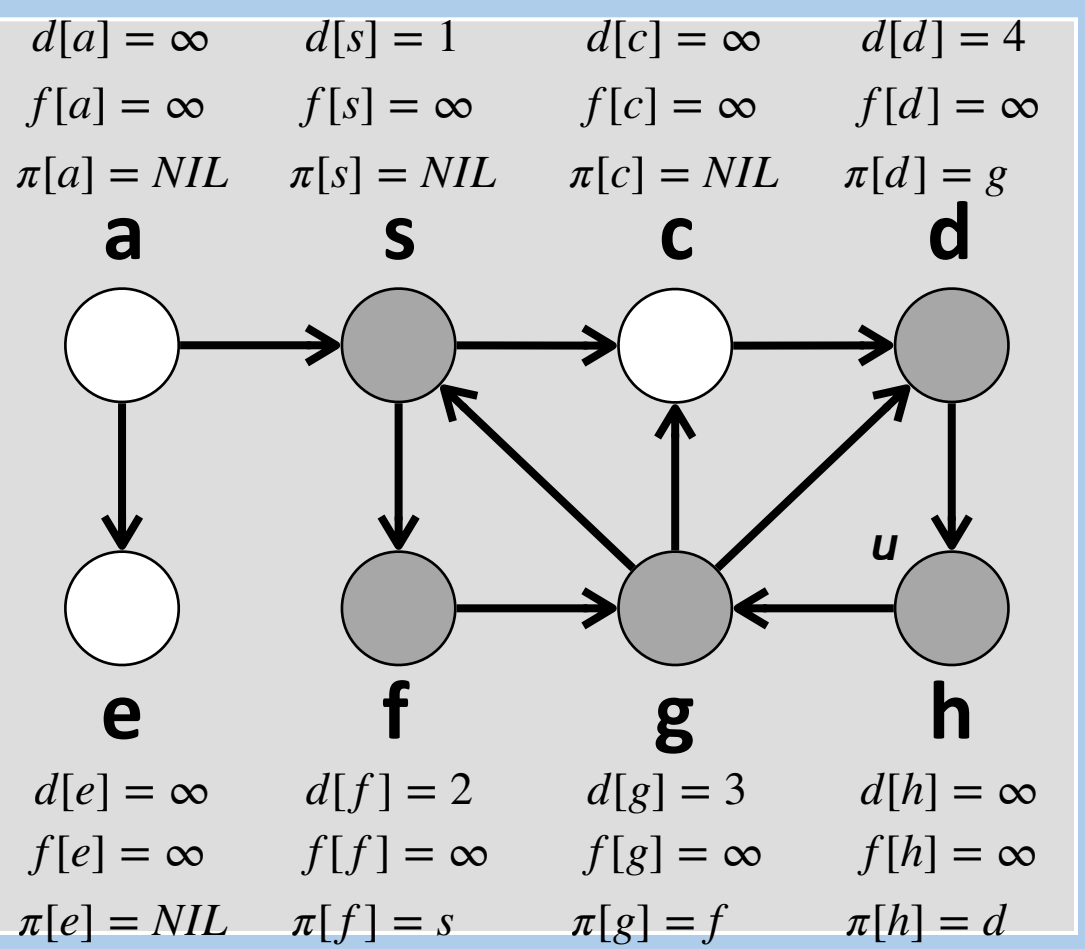

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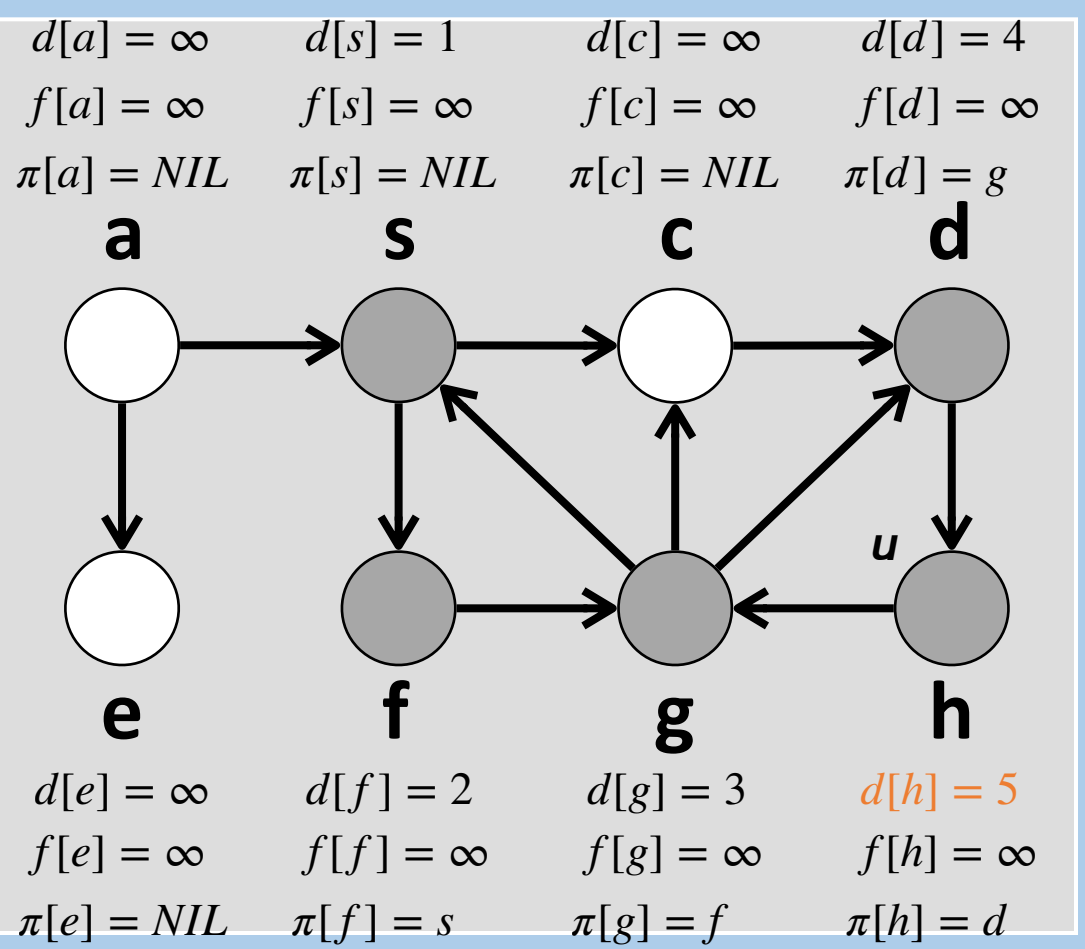

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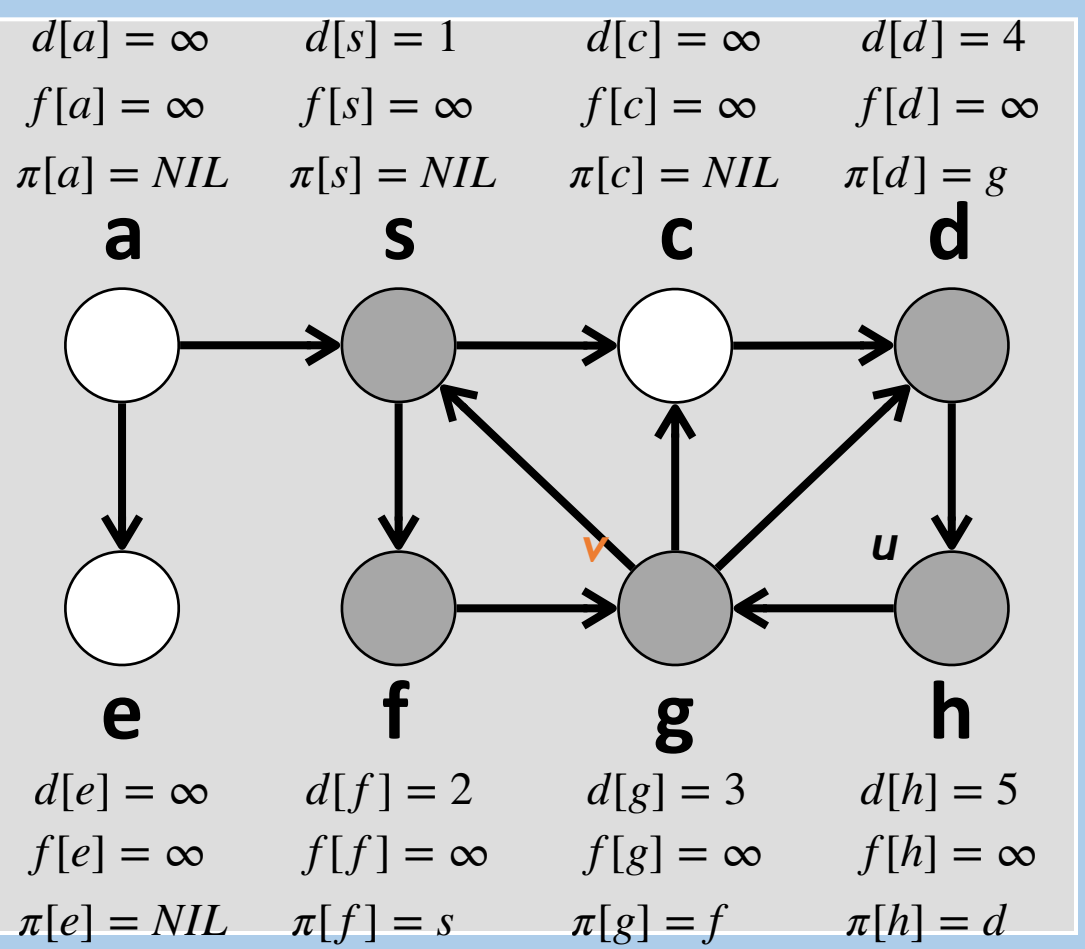

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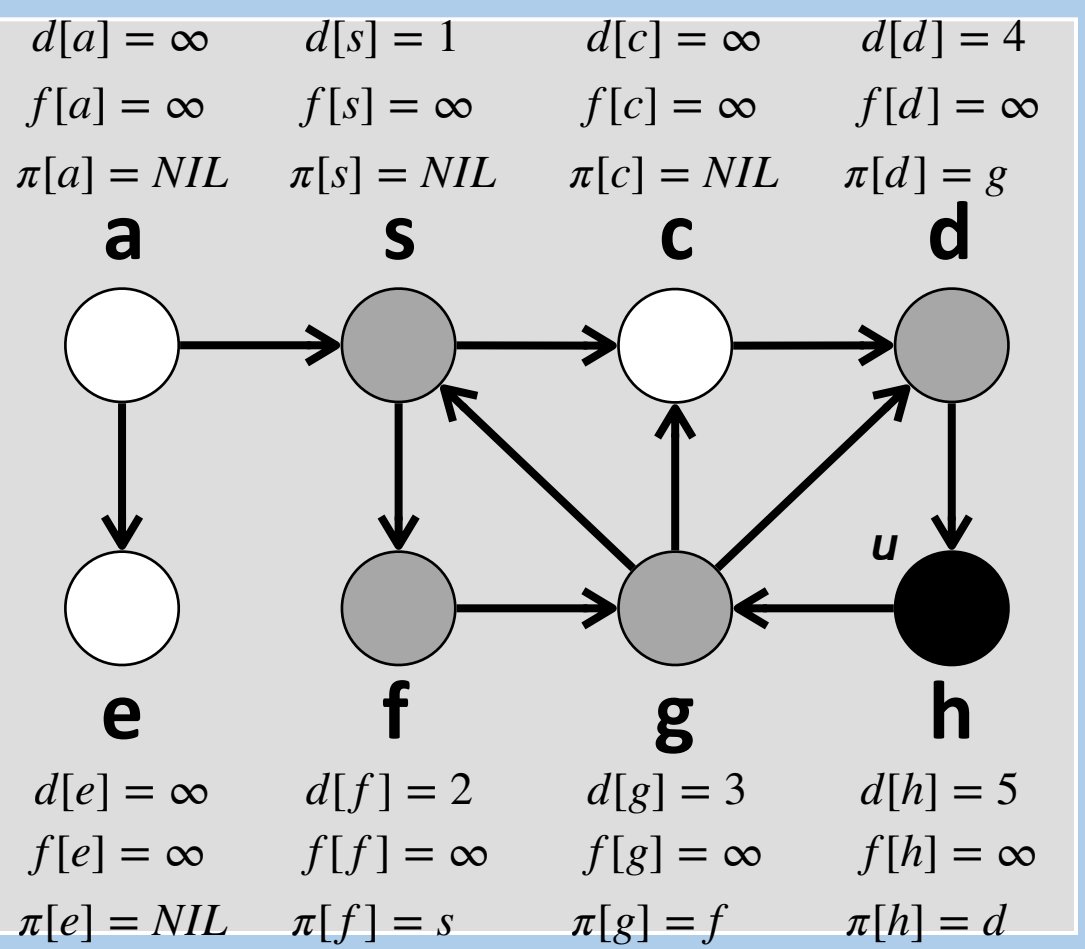

Not-discovered
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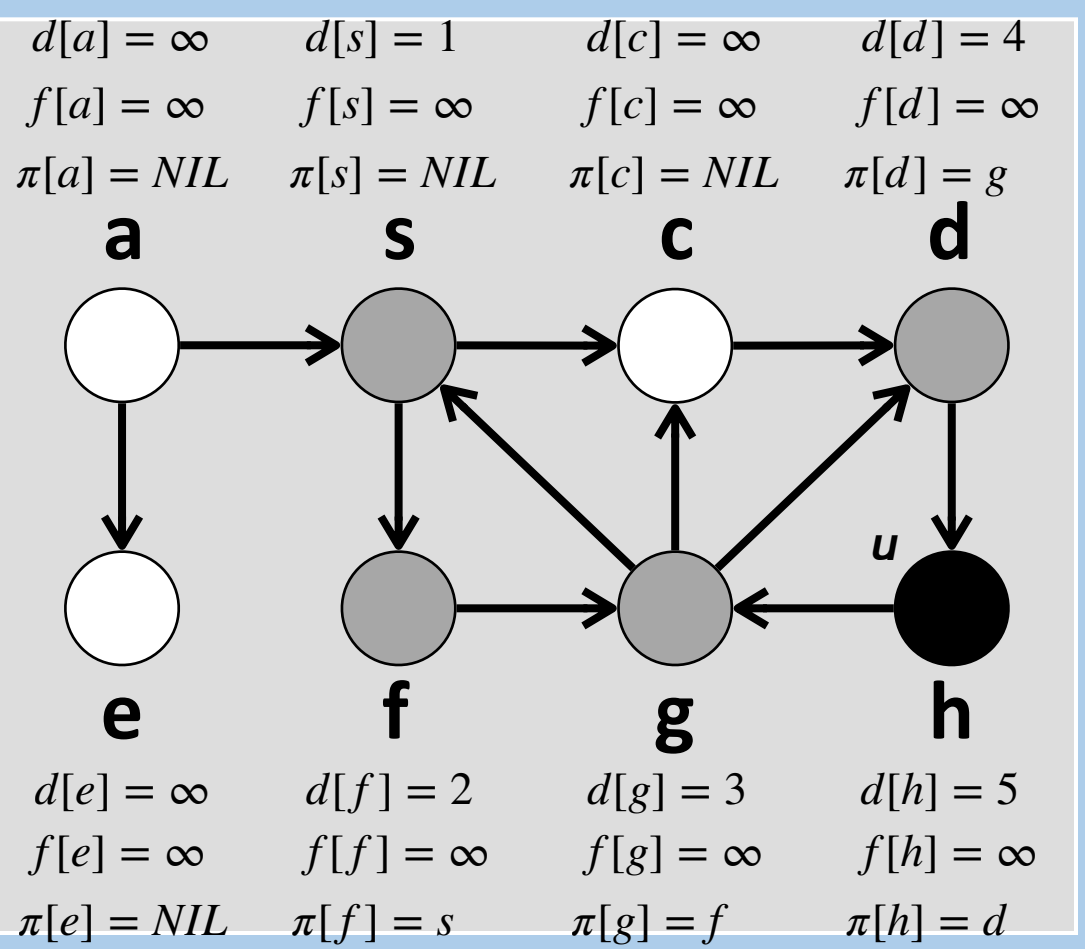

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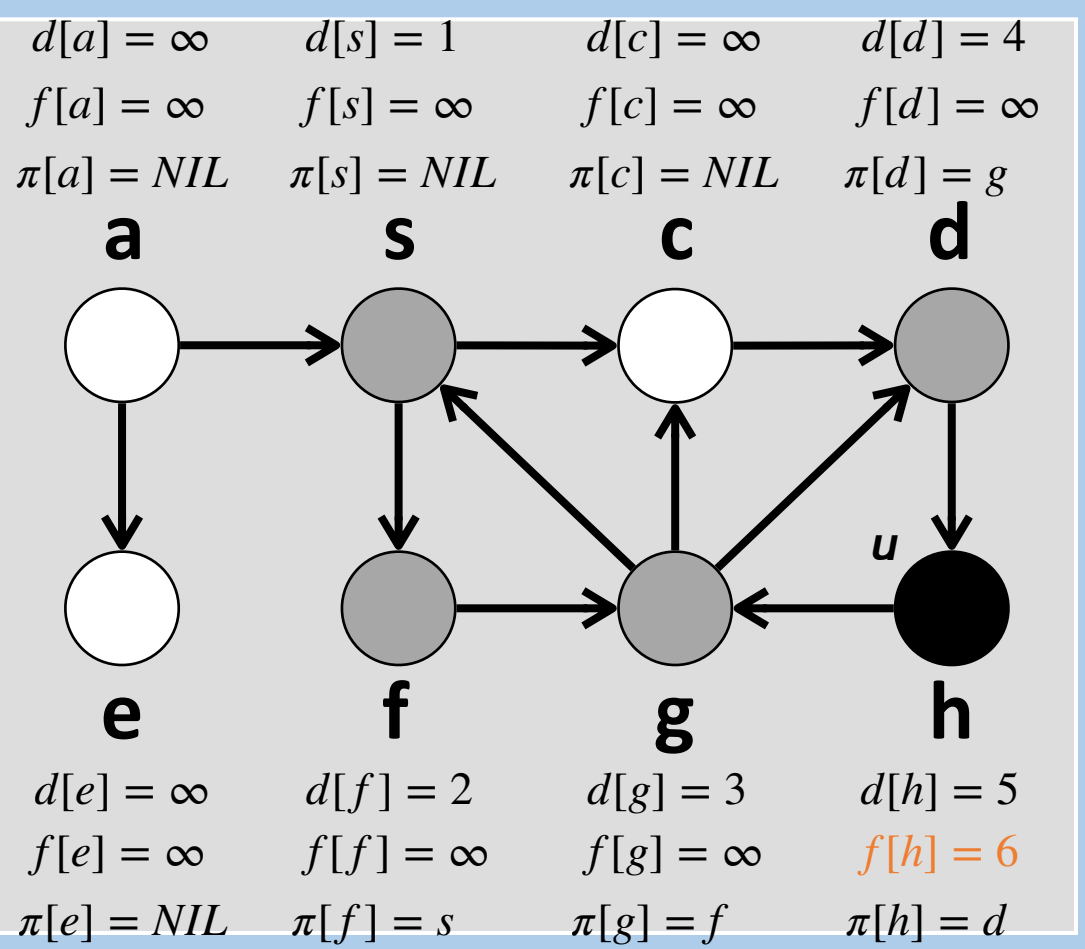

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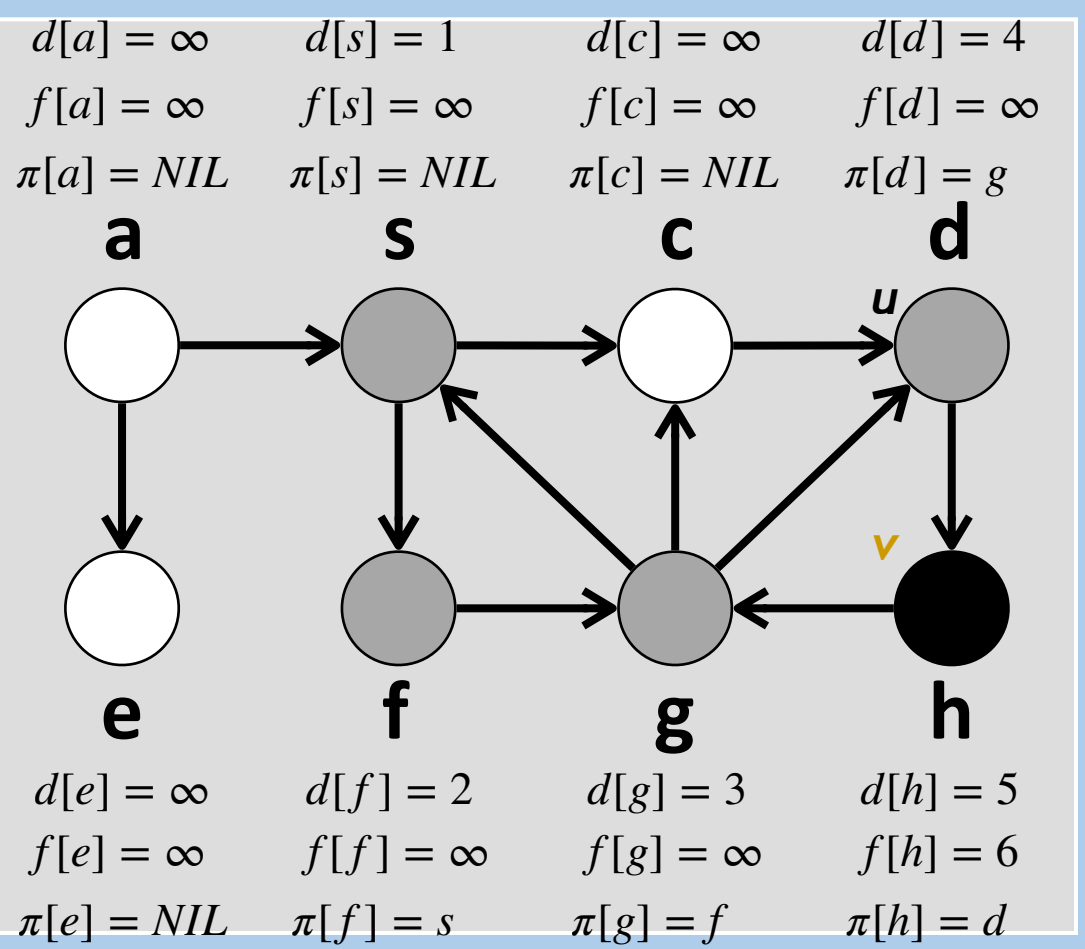

Not-discovered
Discovered
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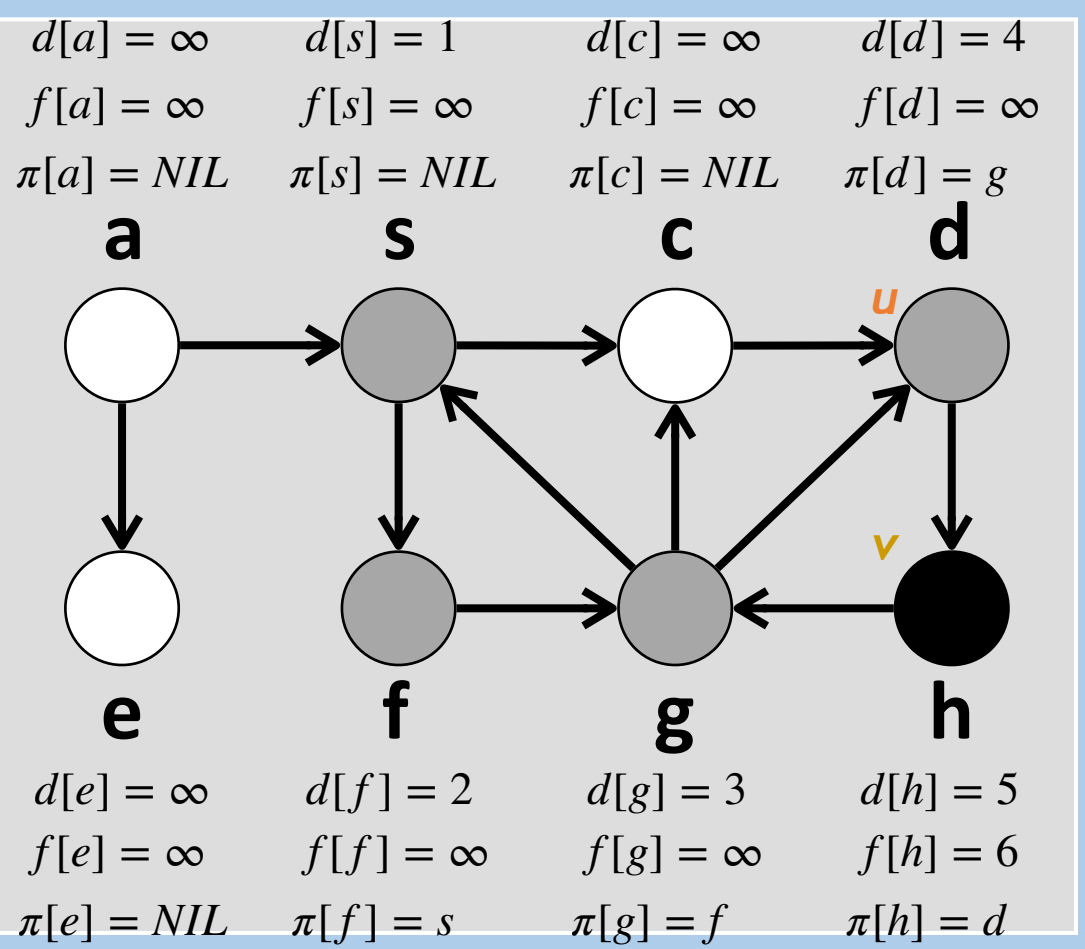

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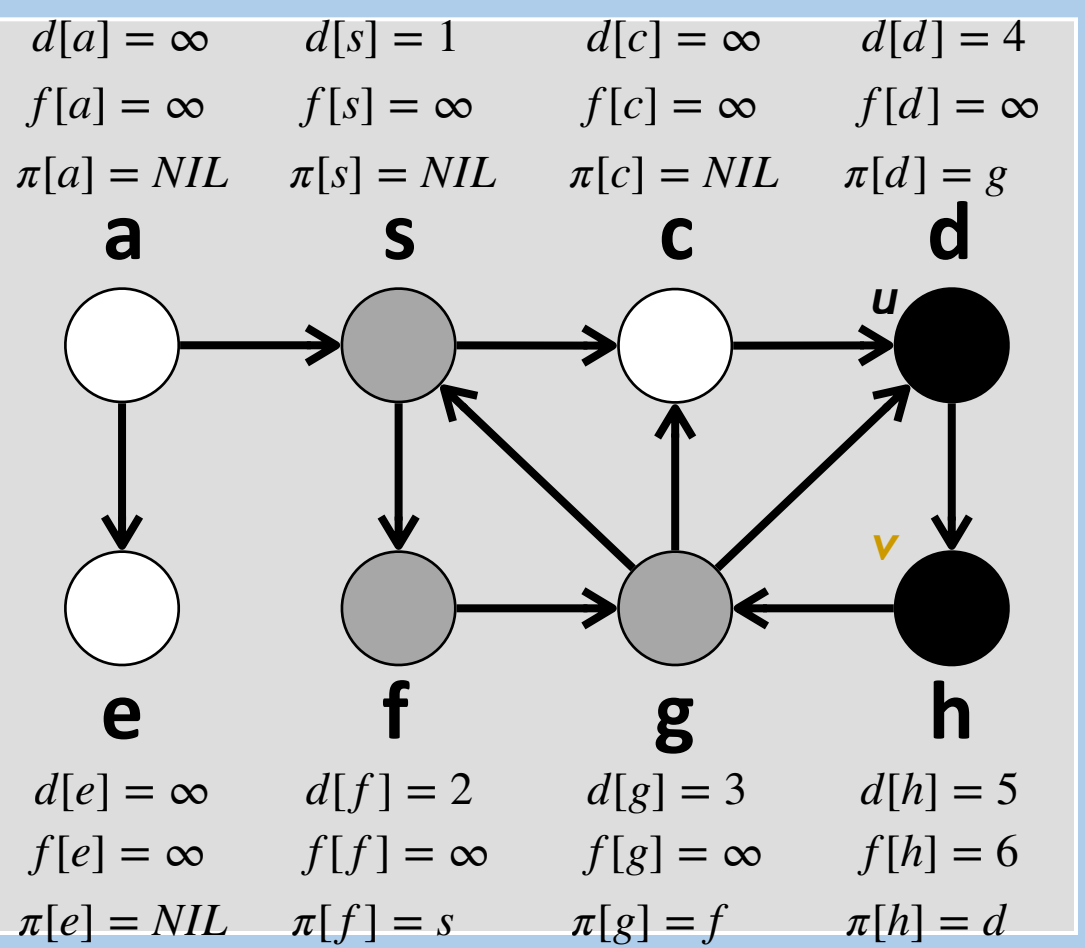

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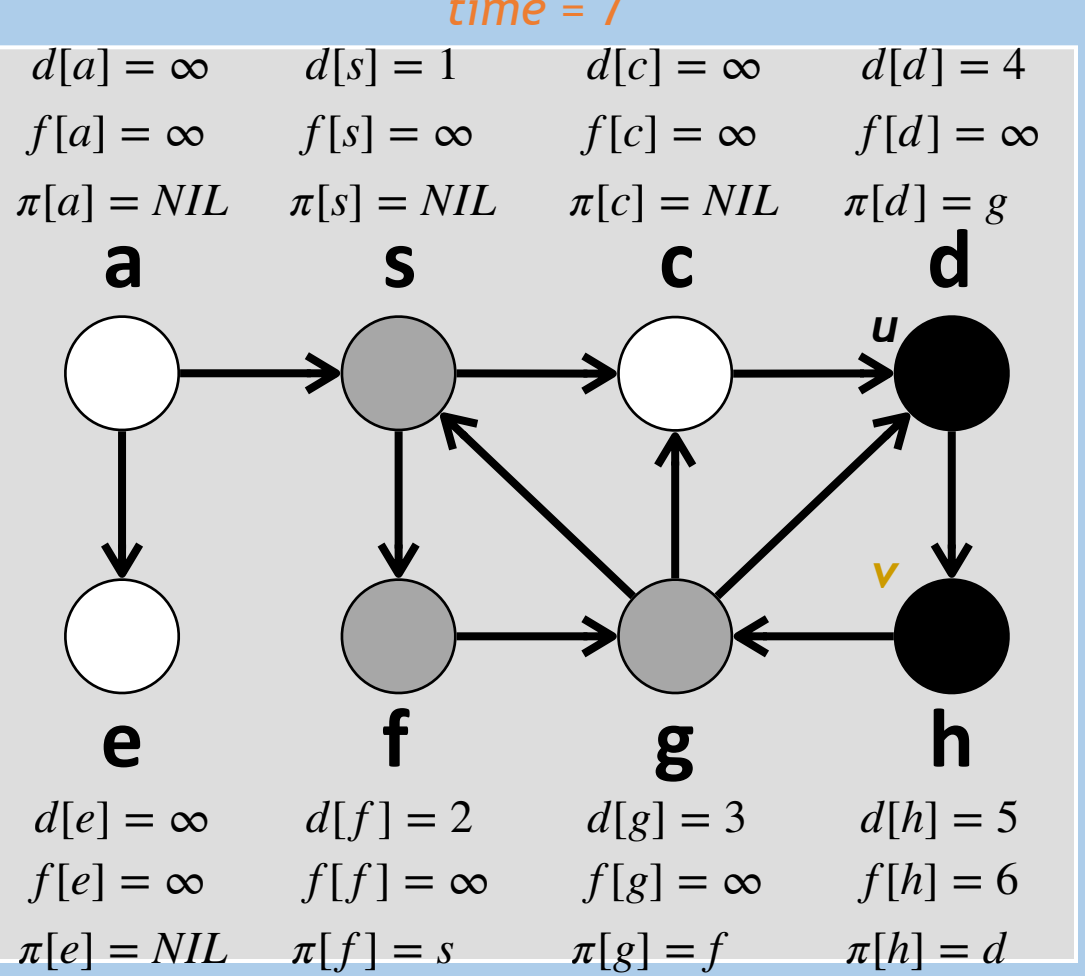

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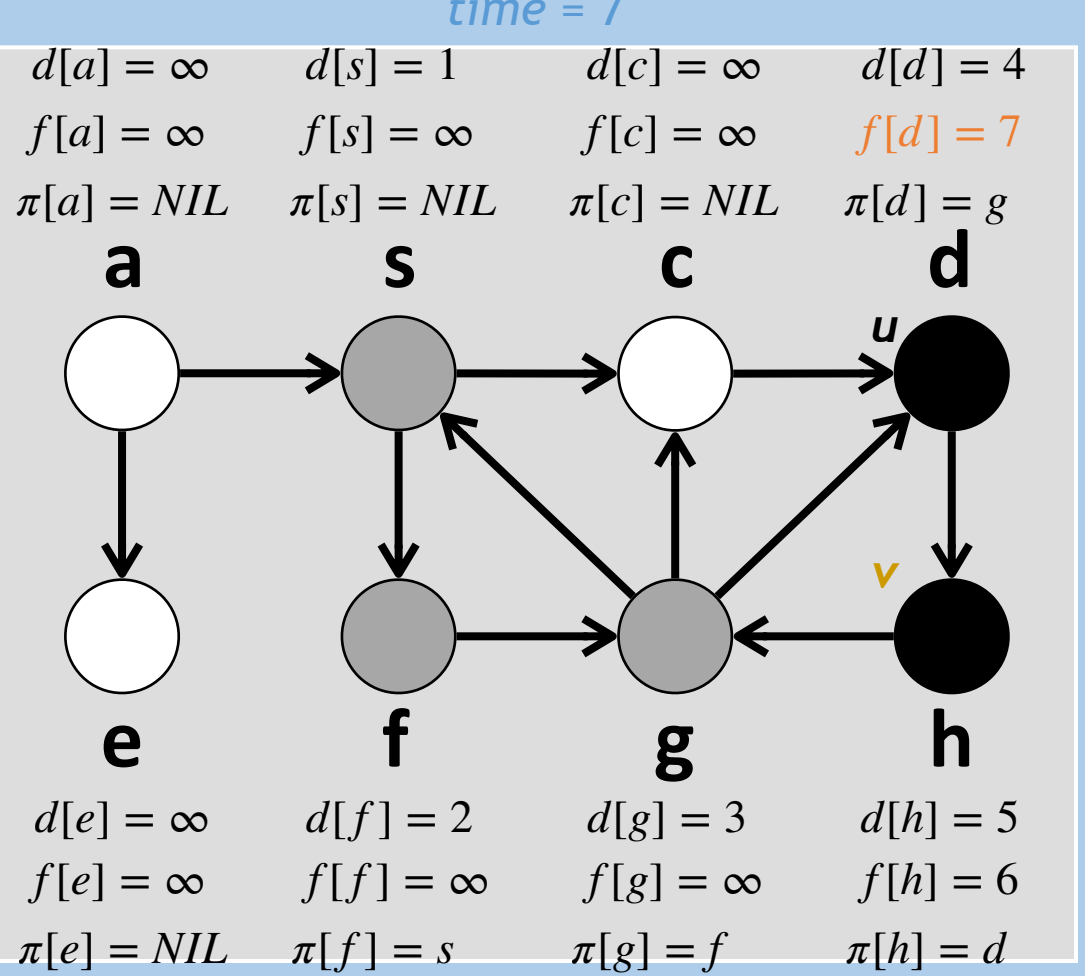

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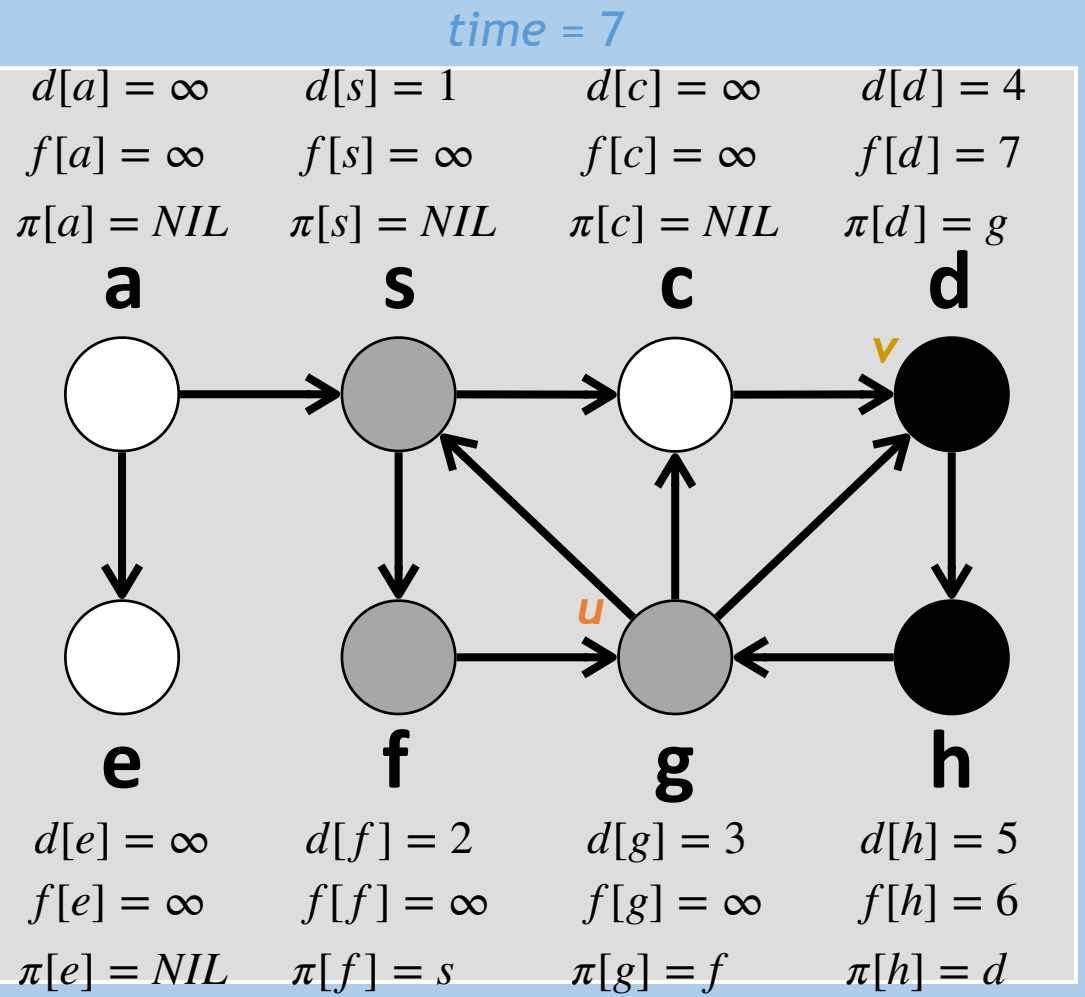

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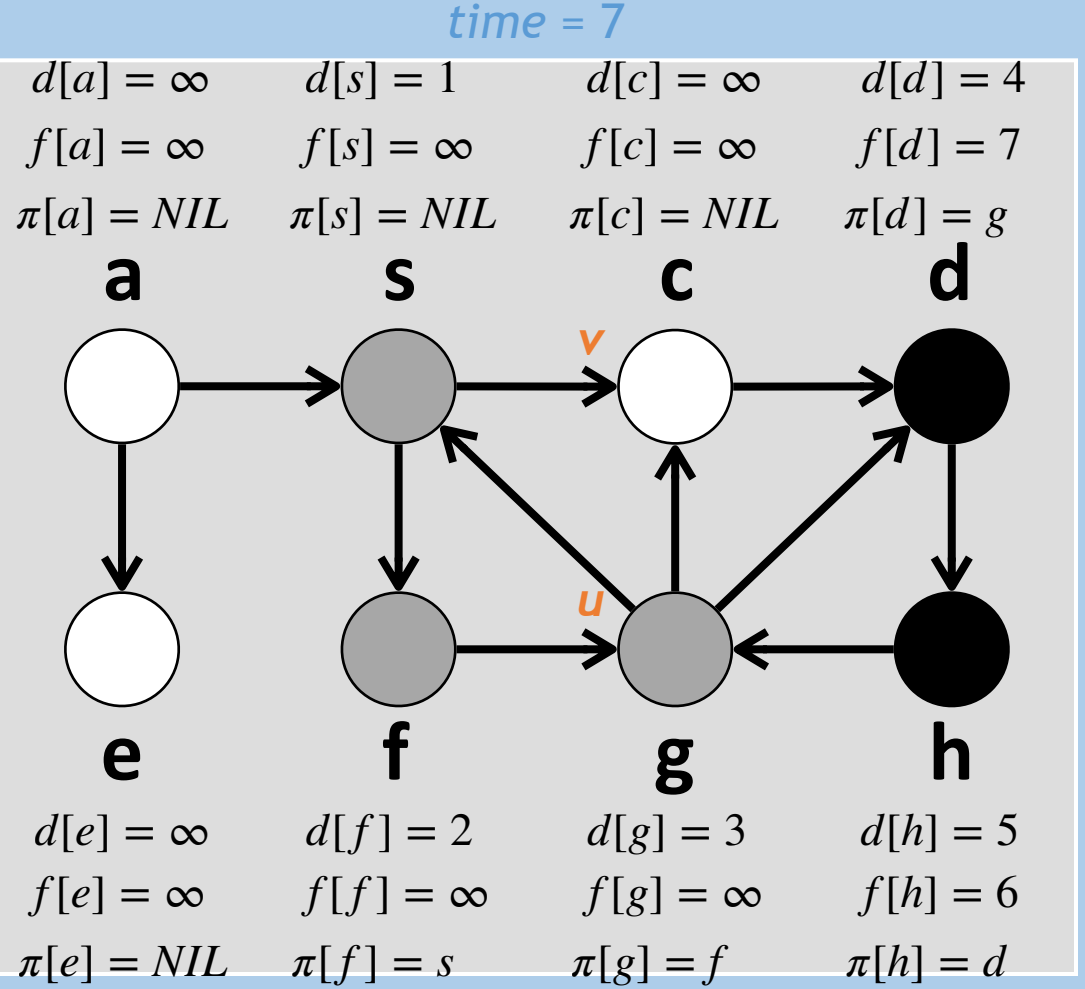

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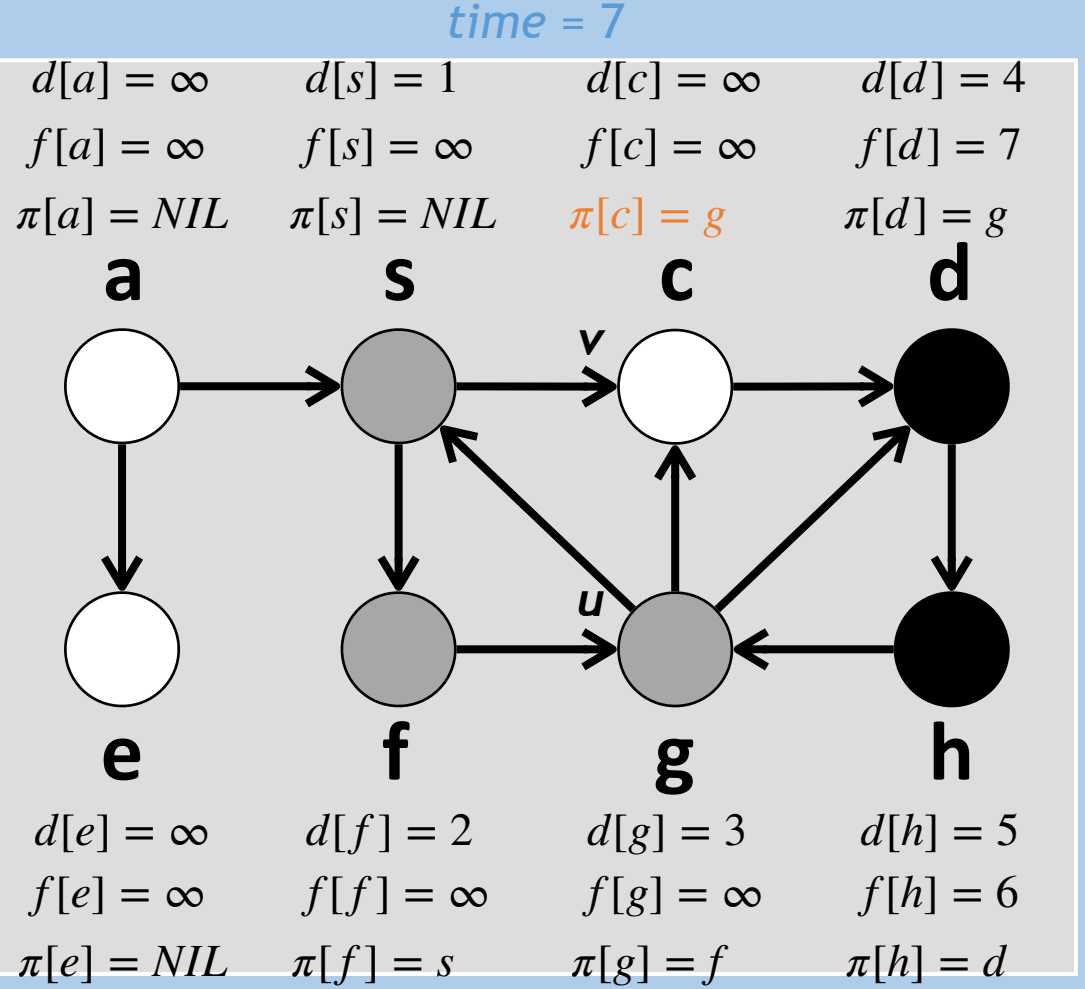

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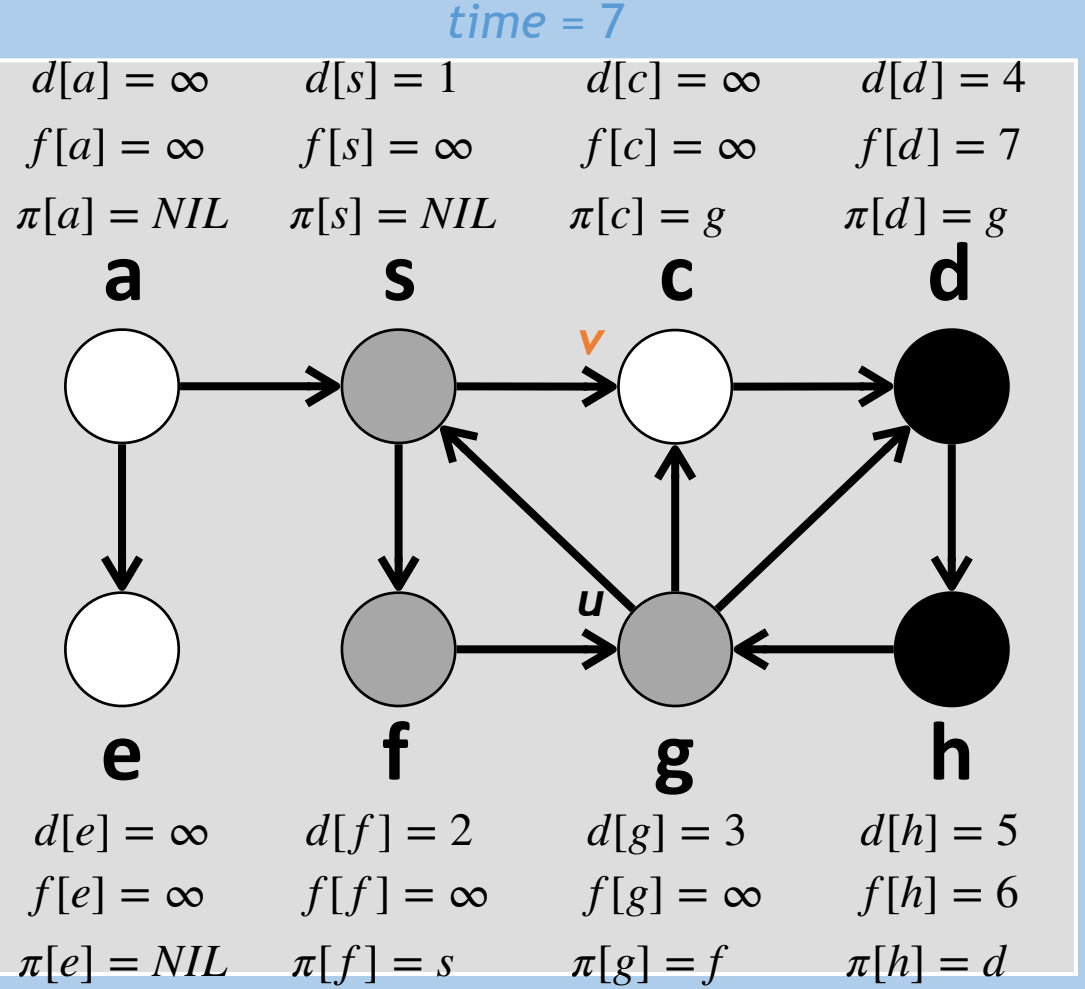

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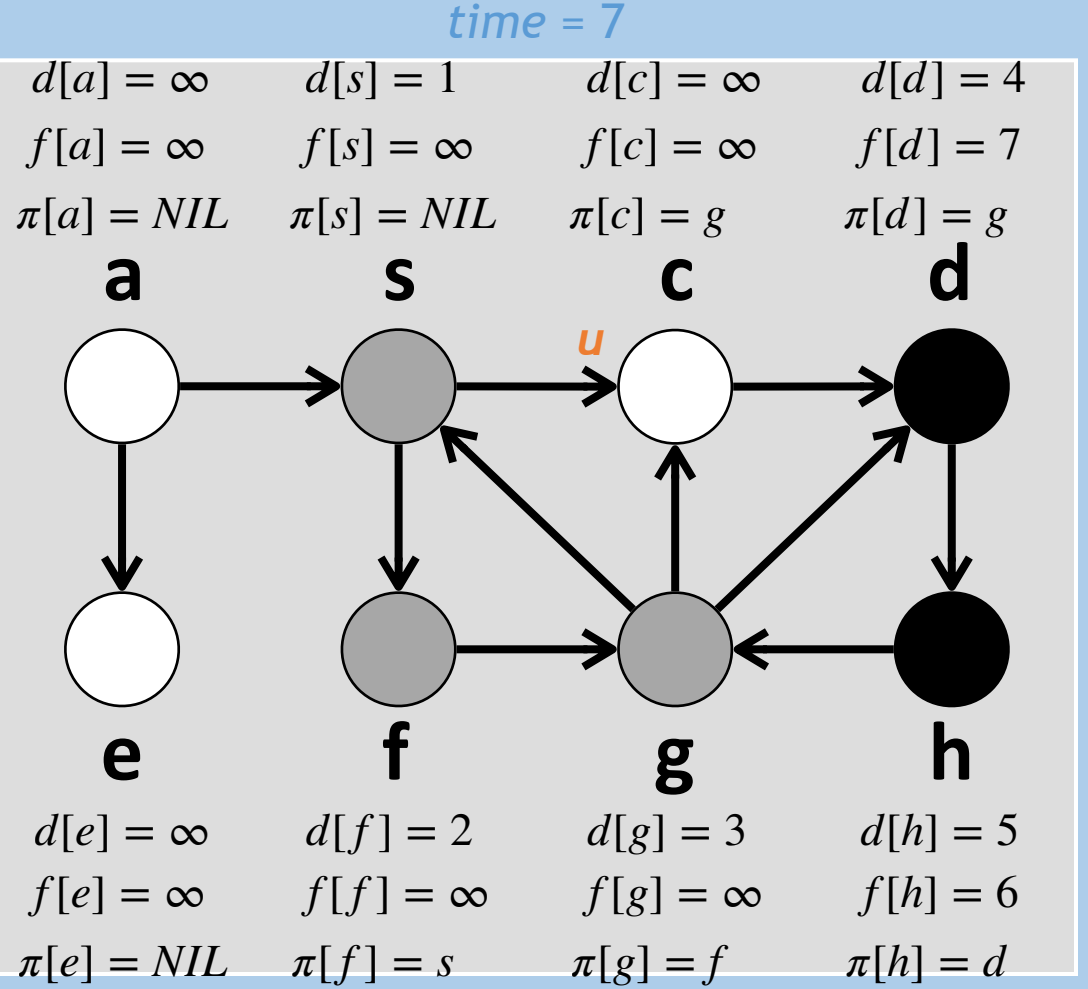

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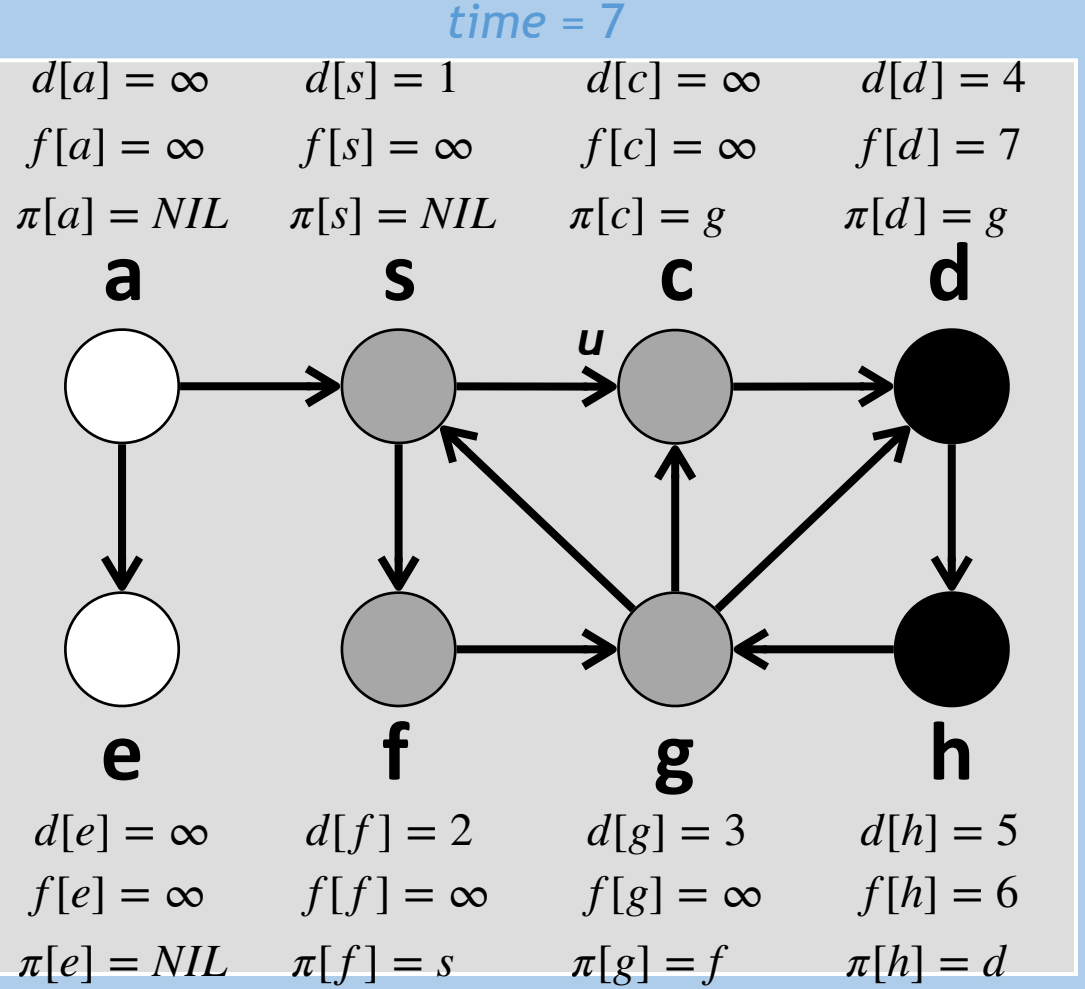

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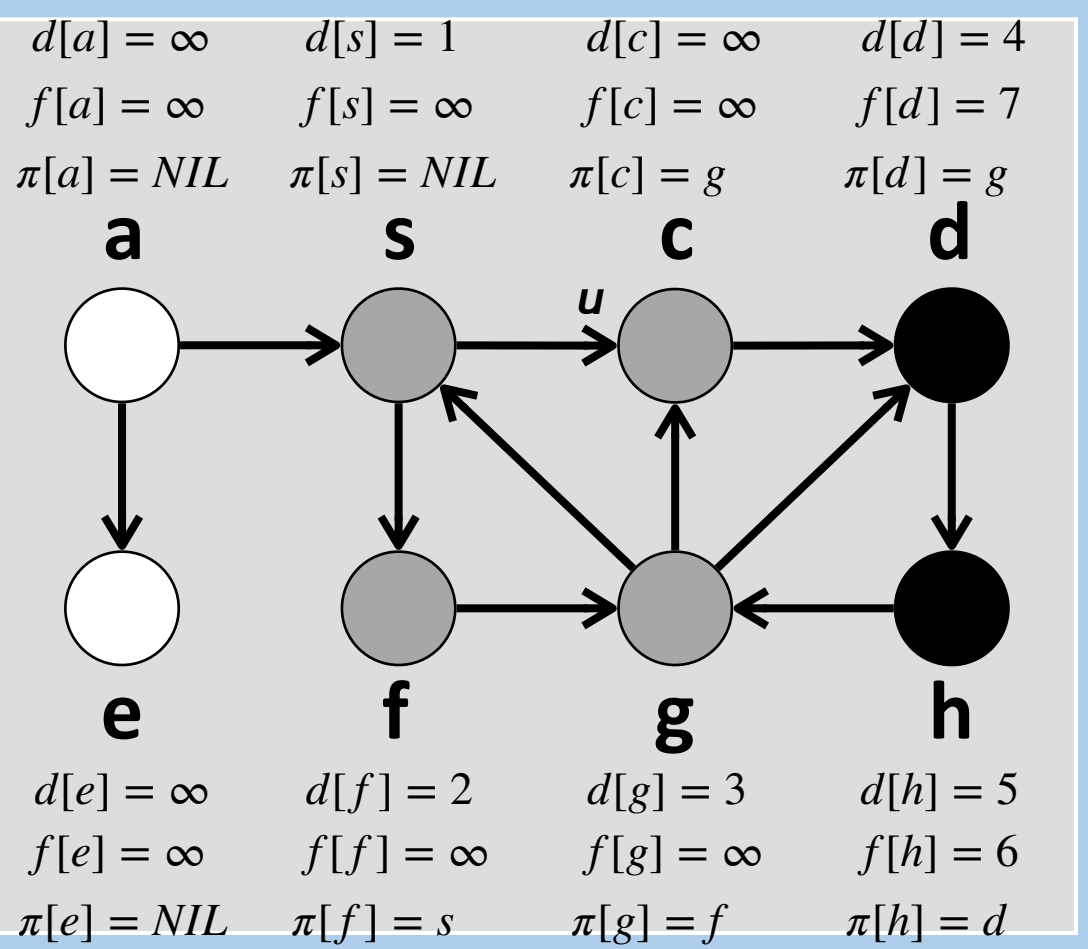

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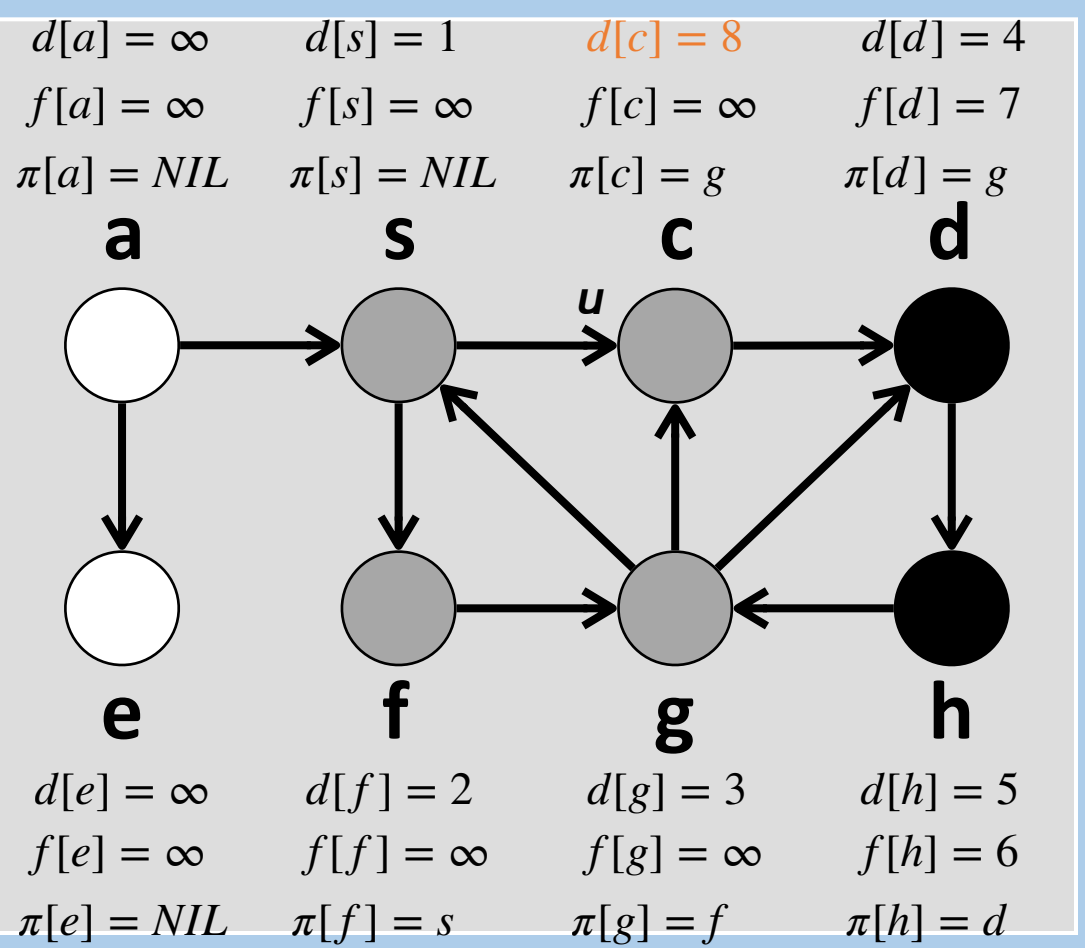

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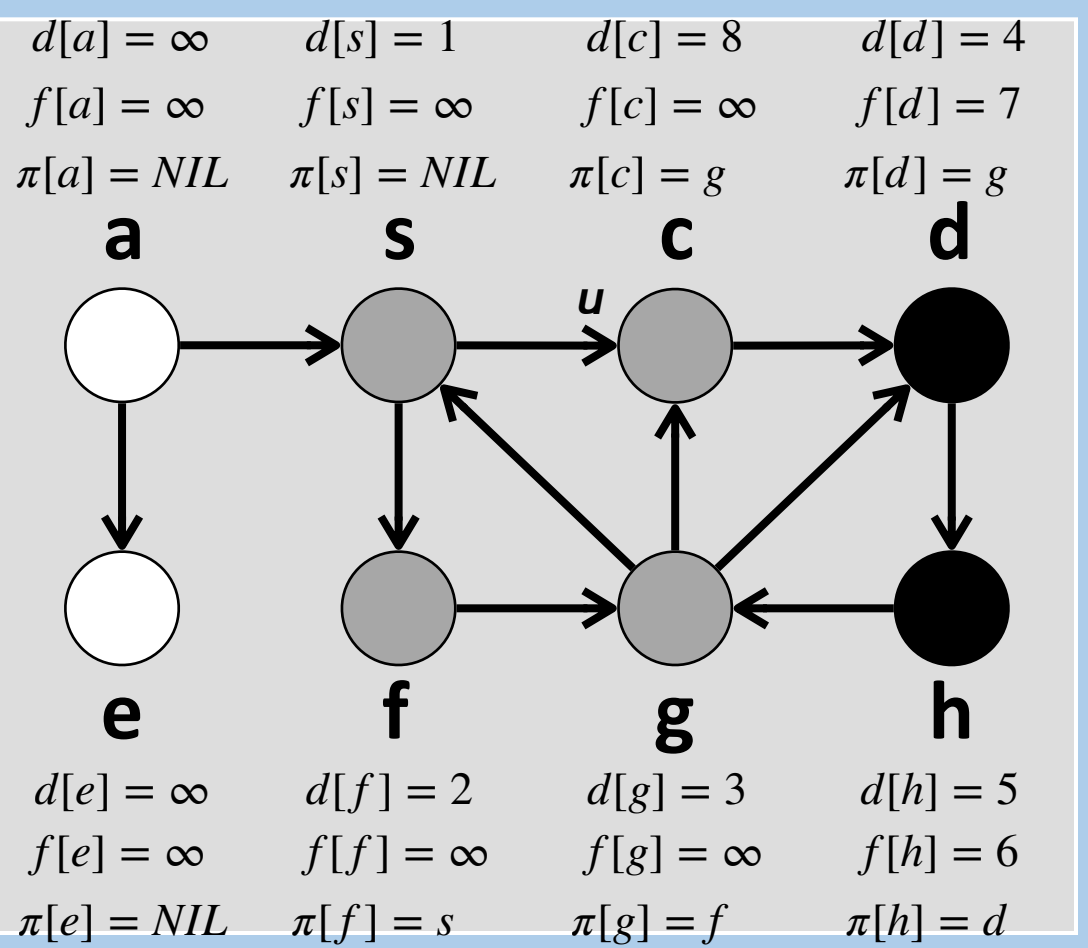

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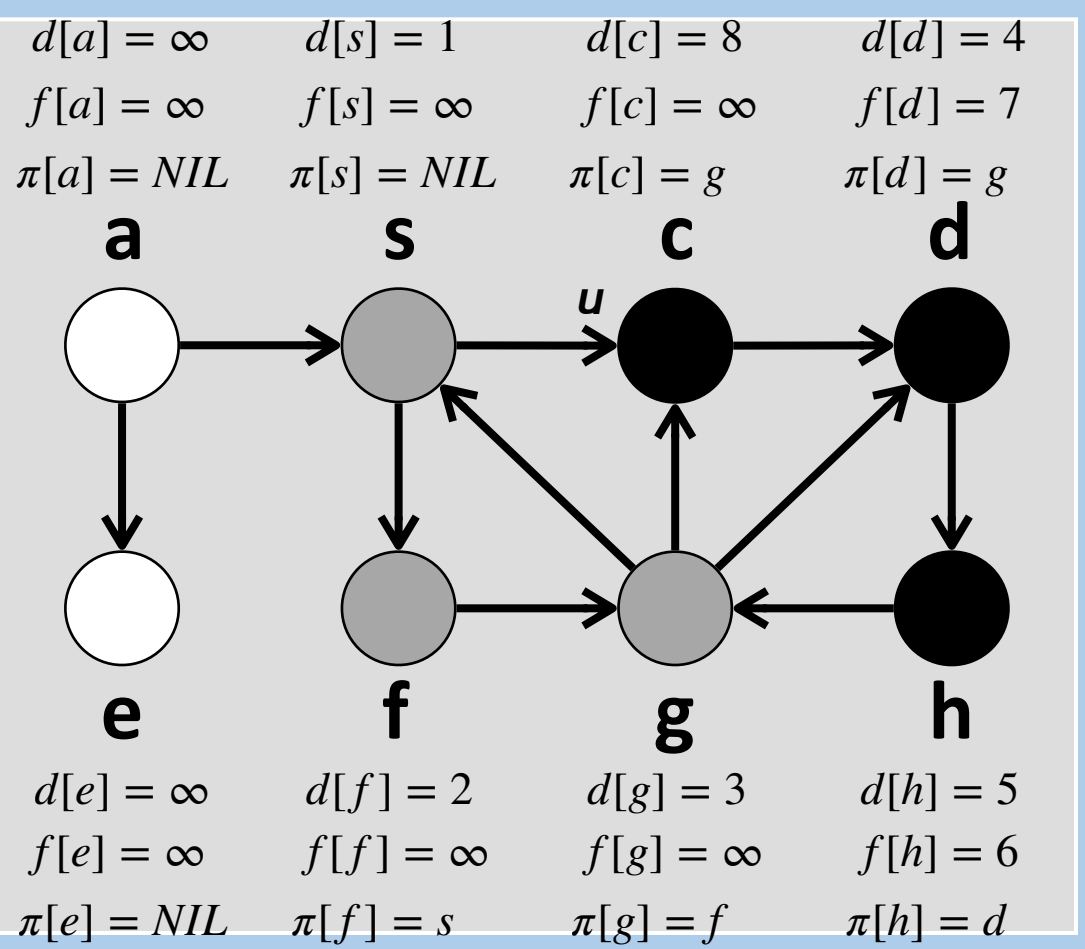

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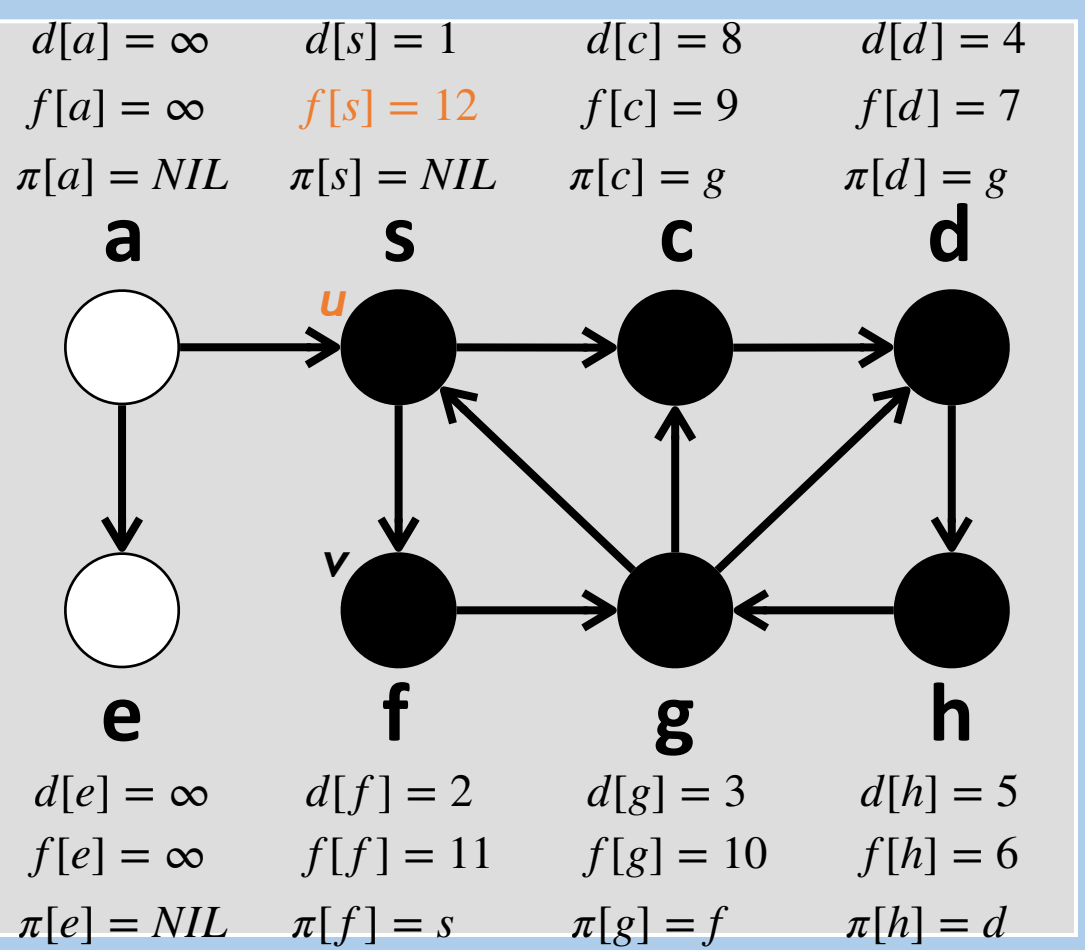

Not-discovered
Discovered
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Depth-First Search

```
For each u ∈ V[G]
    Mark u as not-discovered
    π[v] ← NIL
time ← 0
For each vertex u ∈ V[G]
    If u is not-discovered
        DFS-VISIT(u)
```

```
Procedure DFS-VISIT(u)
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    time ← time + 1
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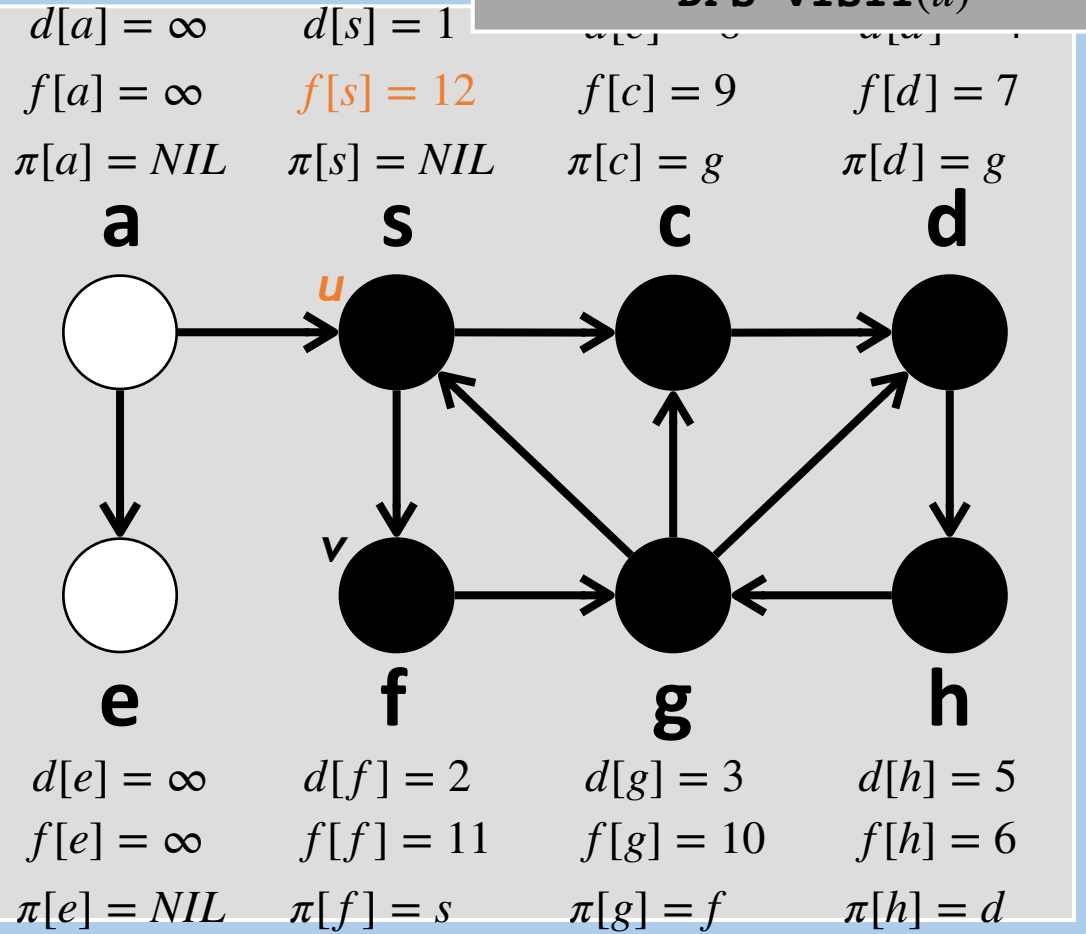

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    Mark u as not-discovered
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time ← 0
For each vertex u ∈ V[G]
    If u is not-discovered
        DFS-VISIT(u)
```

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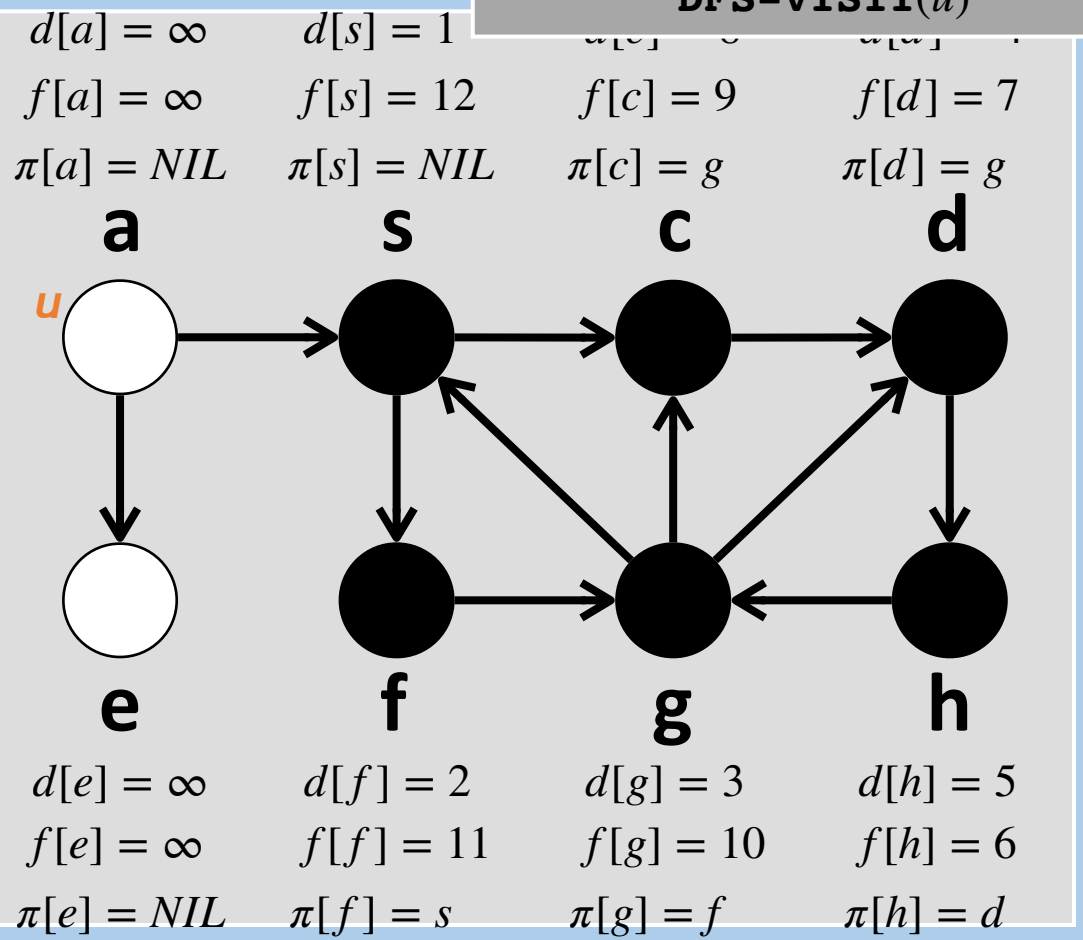

DFS Performance

- Counting:

DFS Performance

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  - Each vertex is marked to white, gray, and black once: $O(|V|)$ time

DFS Performance

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  - Each edge is considered at most twice: $O(|E|)$ time

DFS Performance

- Counting:
  - Each vertex is marked to white, gray, and black once: $O(|V|)$ time
  - Each edge is considered at most twice: $O(|E|)$ time
- Total time complexity: $O(|V| + |E|)$

Linear time on graphs:
$O(|V| + |E|)$

What's Happening

```
Procedure DFS-VISIT(u)
    Mark u as discovered
    time ← time + 1
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    For each neighbor v of u
        If v is not-discovered
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            DFS-VISIT(v)
    Mark u as finished
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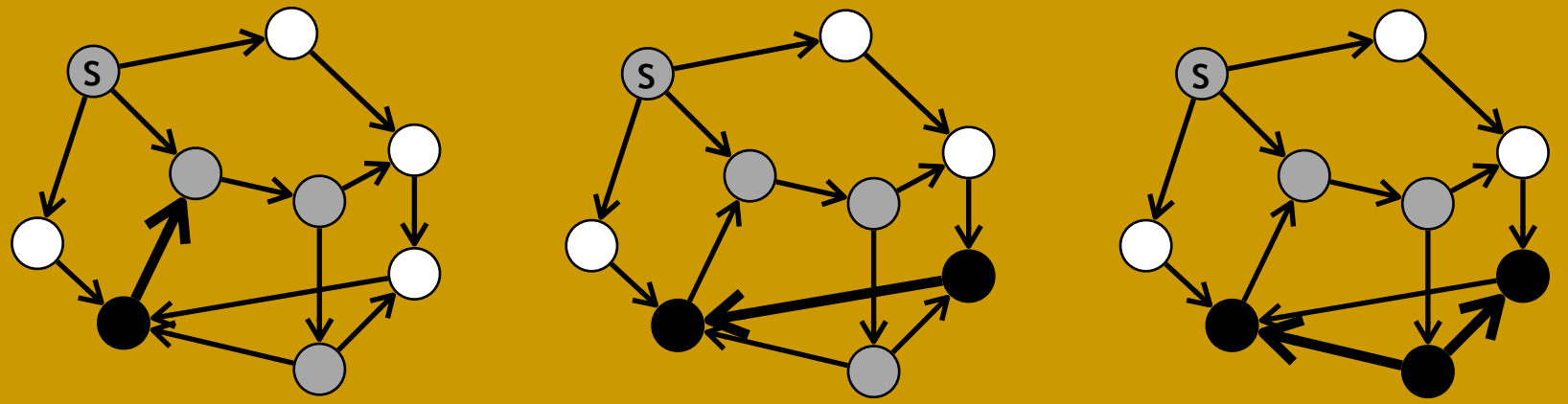
    f[u] ← time
```

- Each vertex is marked as
  - White: **not-discovered** by the algorithm
  - Gray: **discovered** by the algorithm
  - Black: **finished**; all its neighbors are discovered



A vertex is marked in black if every neighbor of it is discovered (gray or black).

The stack contains all gray vertices. Once a vertex turns black, it is popped from the stack.

(Stack: from system function call)

Outline

- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
    - Parenthesis theorem
    - Edge classification
    - White-path theorem
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

Depth-First Search Properties

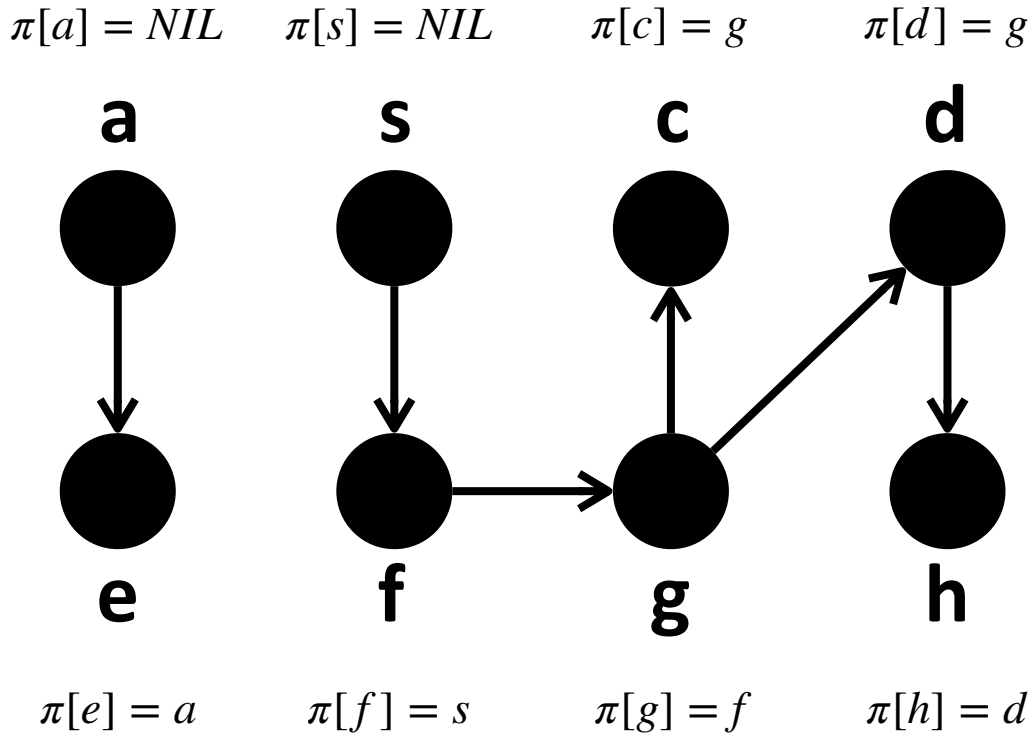
- Parenthesis theorem
- Edge classification
- White-path theorem

Depth-First Search Properties

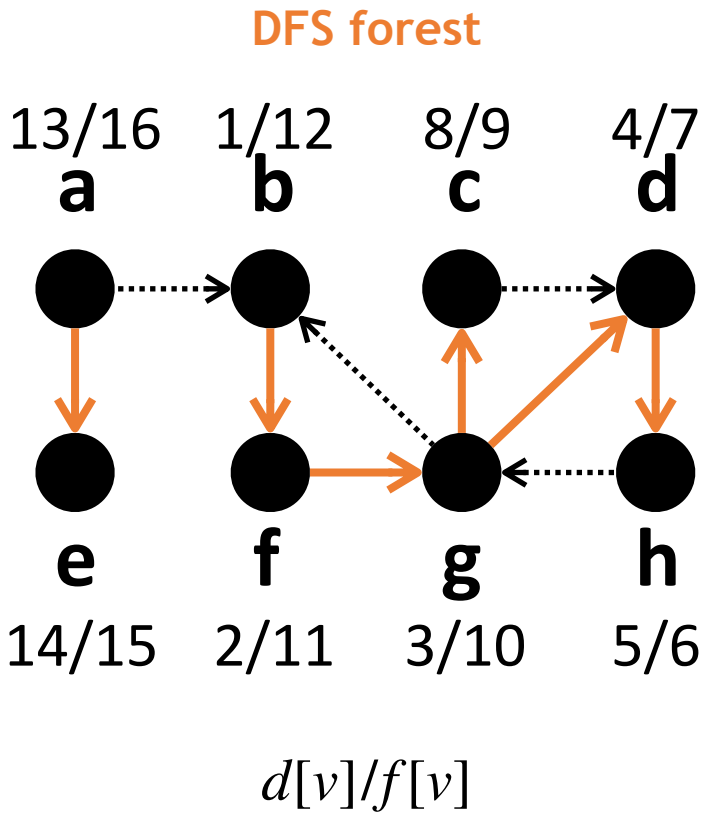
- Parenthesis theorem
- Edge classification
- White-path theorem

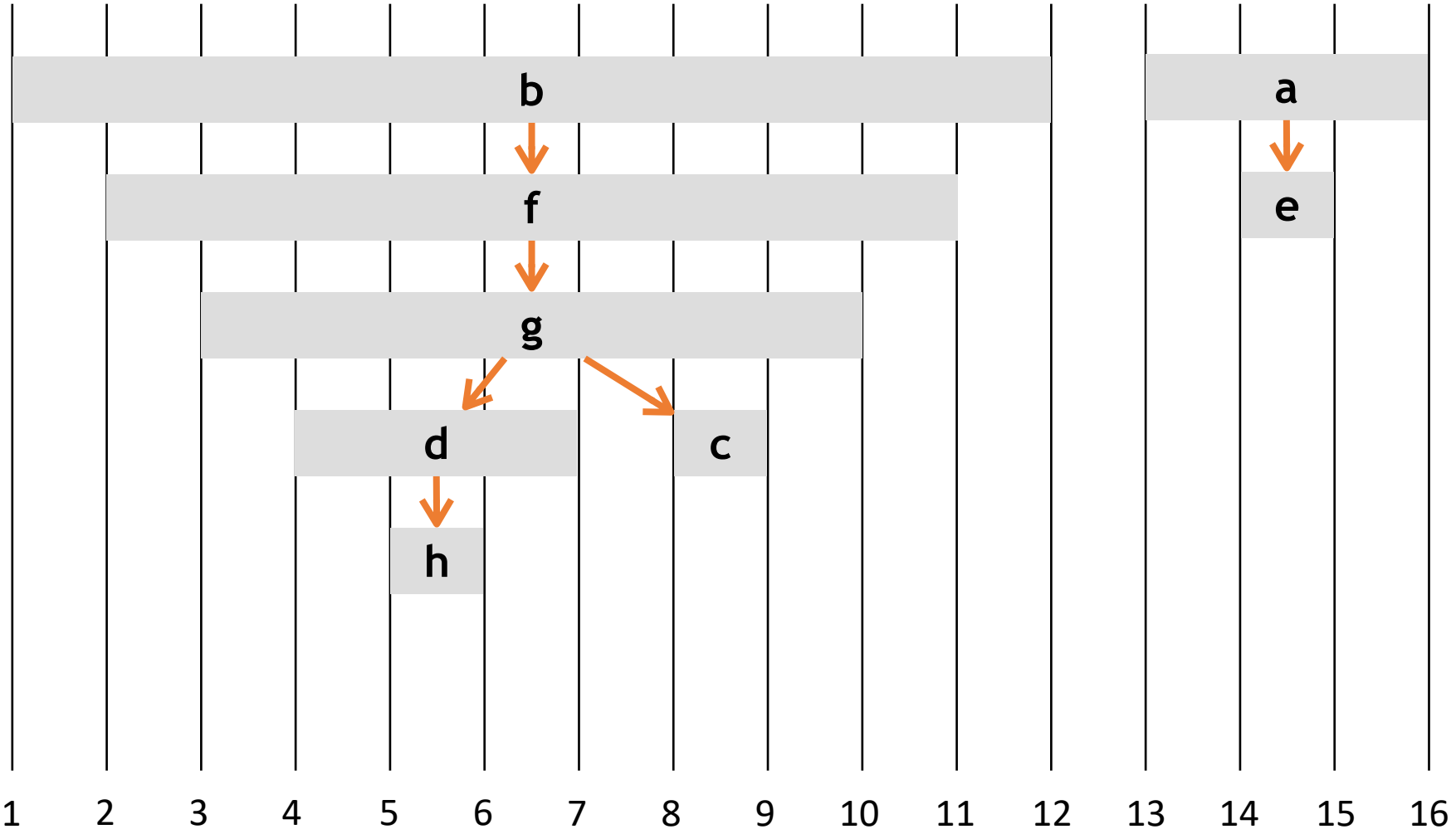
Depth-First Forest

- The predecessor subgraph of a DFS $G_\pi = (V, E_\pi)$, where $E_\pi = \{(\pi[v], v) \mid v \in V \text{ and } \pi[v] \neq \phi\}$, is a forest



Parenthesis Structure of DFS





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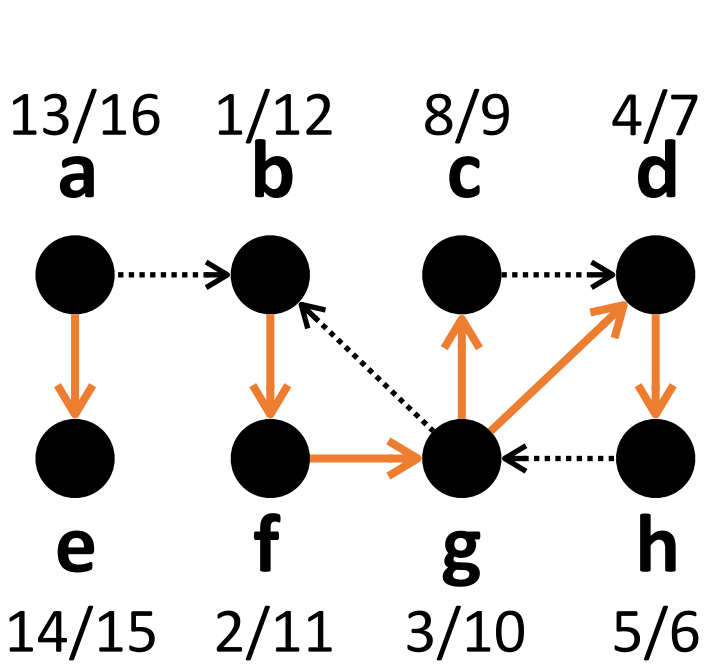

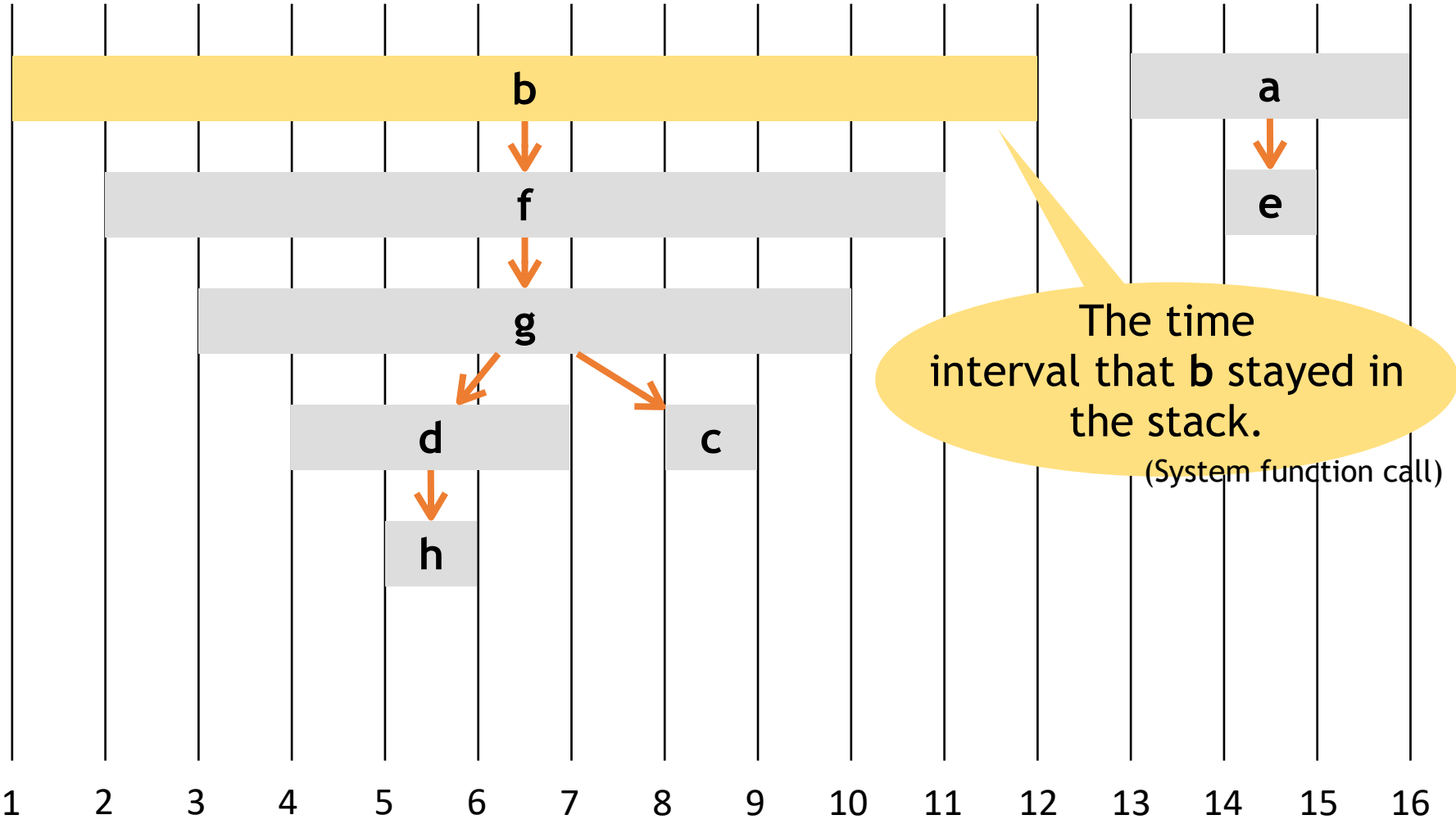

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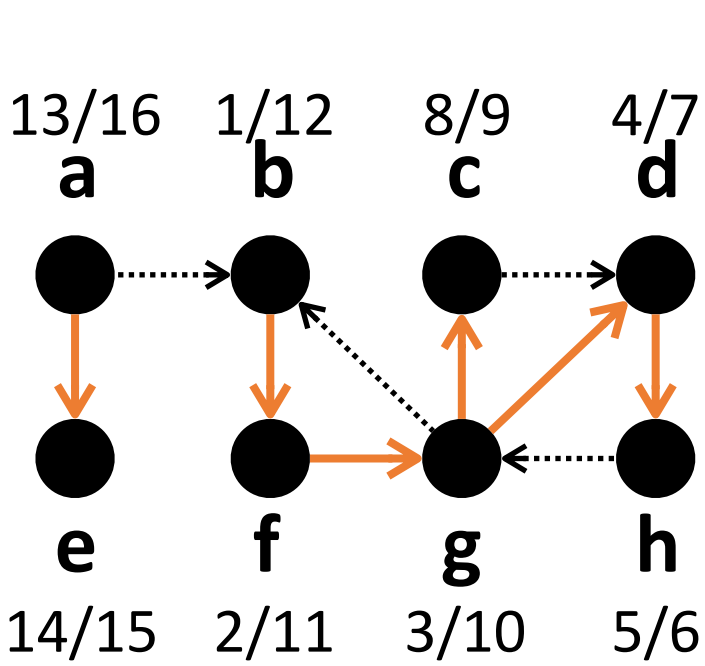
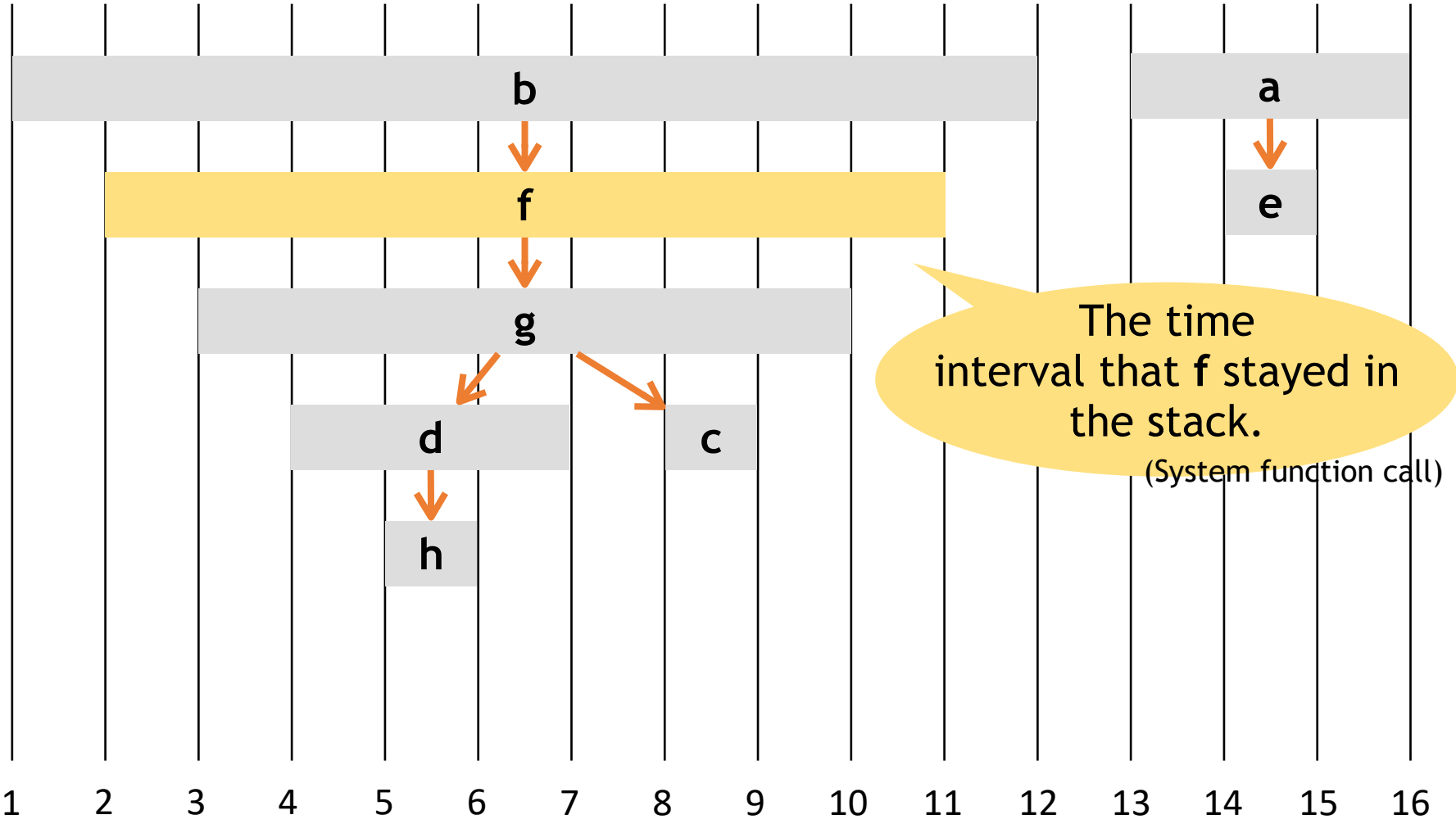




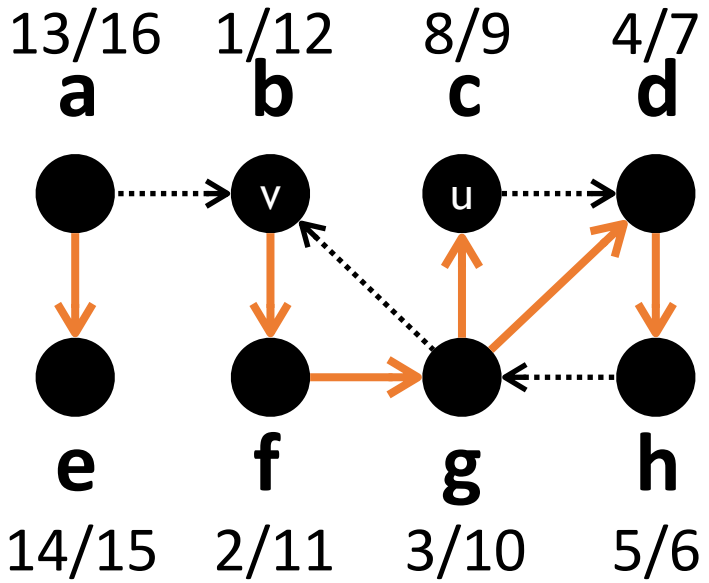
The time interval that **f** stayed in the stack.

(System function call)

Parenthesis Theorem

In any DFS of a graph $G = (V, E)$, for any two vertices $u, v \in V$, there are three cases of the positions of u and v in the depth-first forest:

- u is a descendant of v,



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In any DFS of a graph $G = (V, E)$, for any two vertices $u, v \in V$, there are three cases of the positions of u and v in the depth-first forest:

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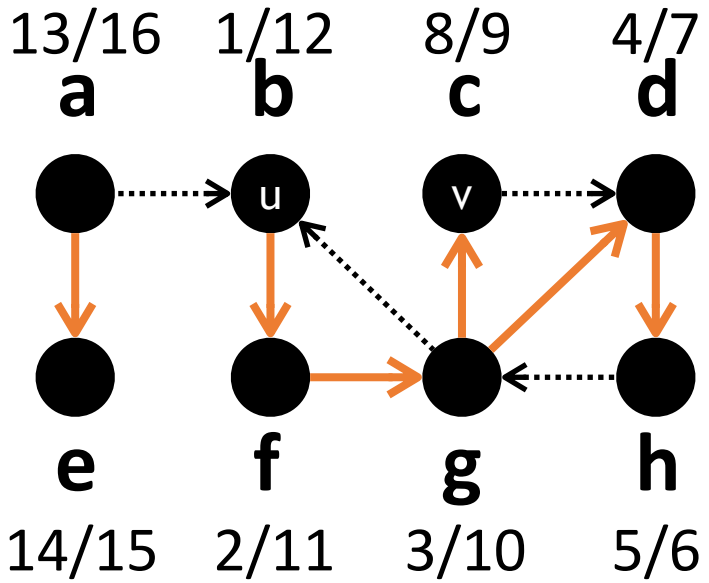

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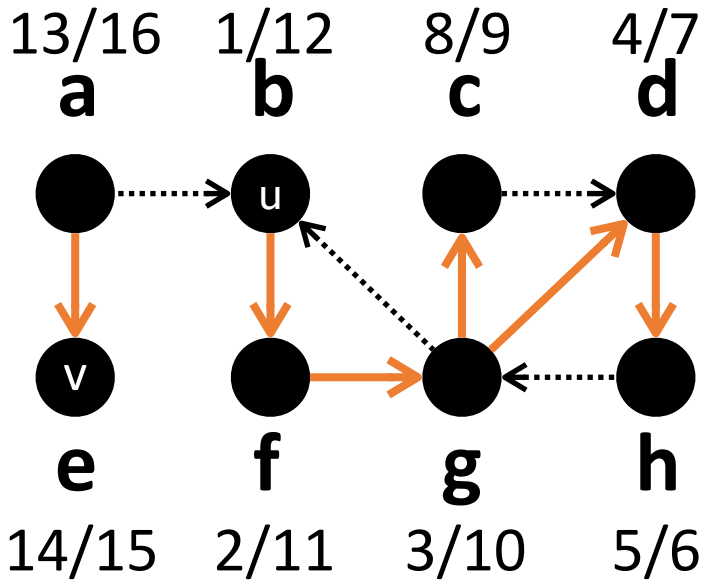

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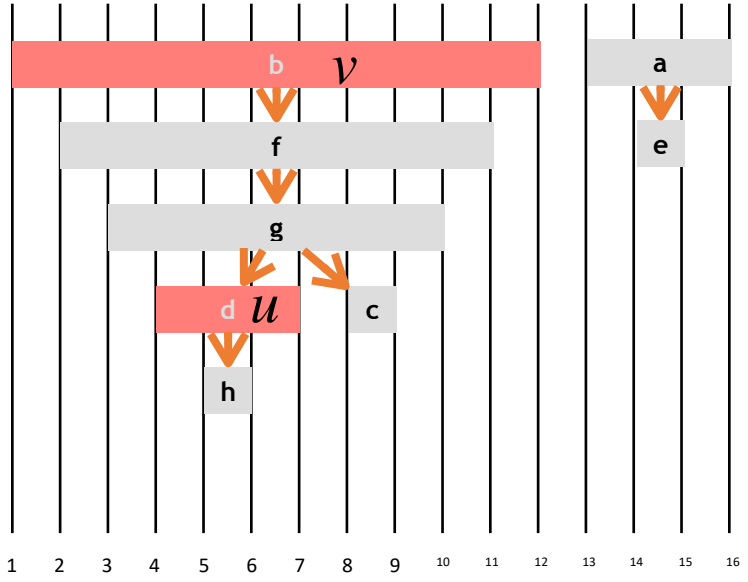

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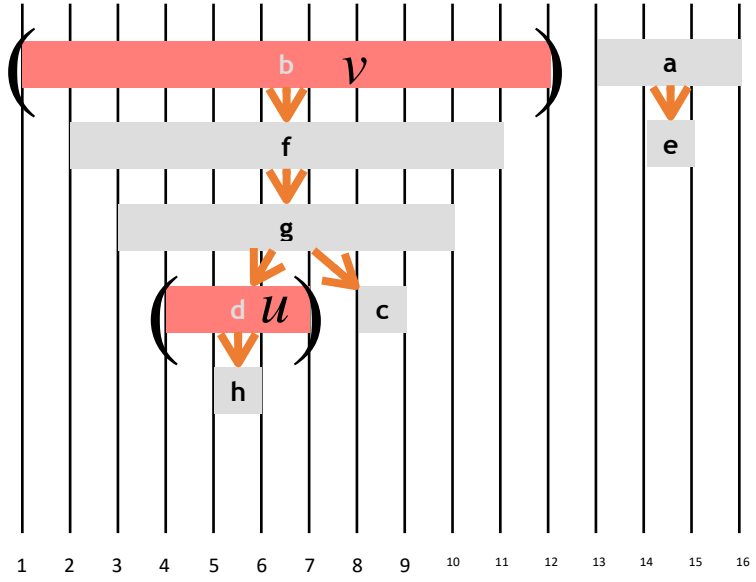

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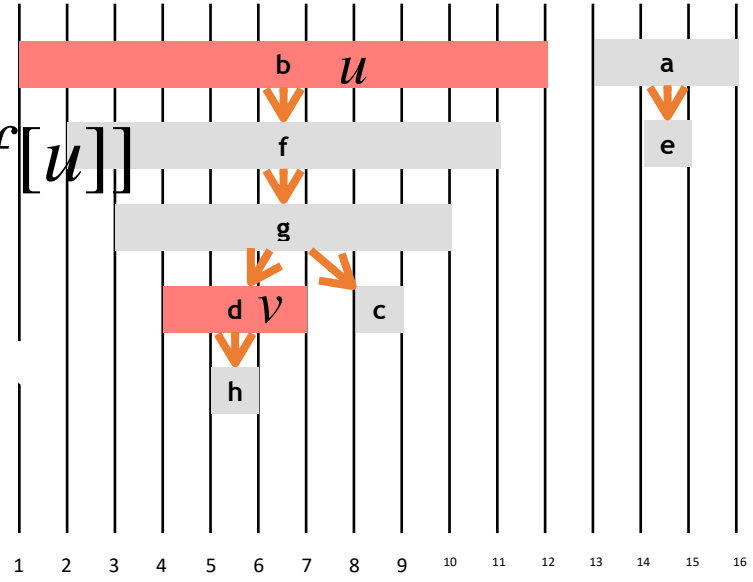

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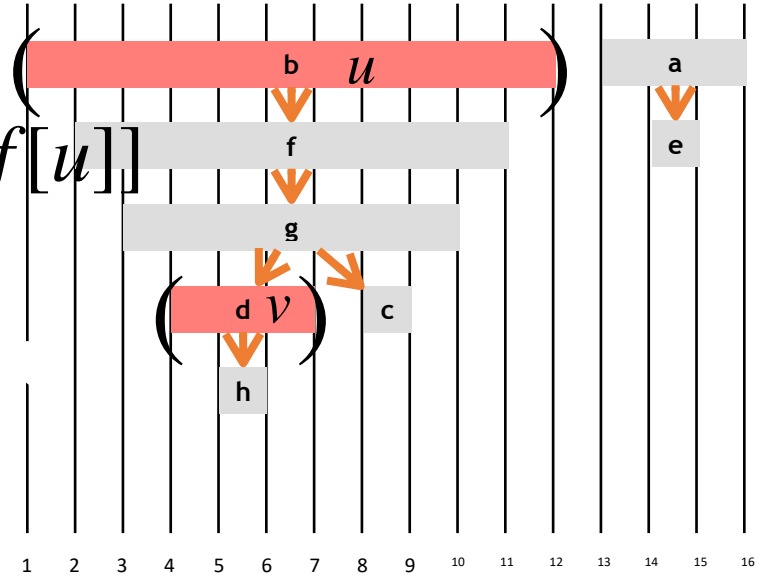

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- either u nor v is as descendant of the other
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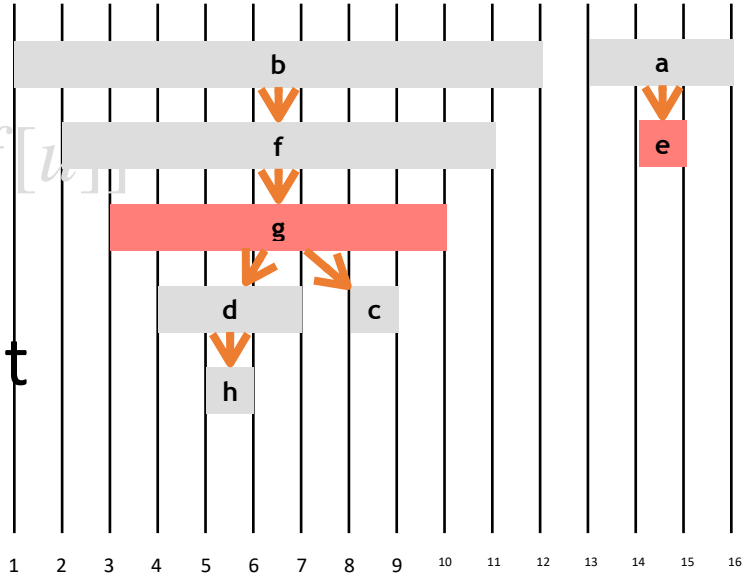

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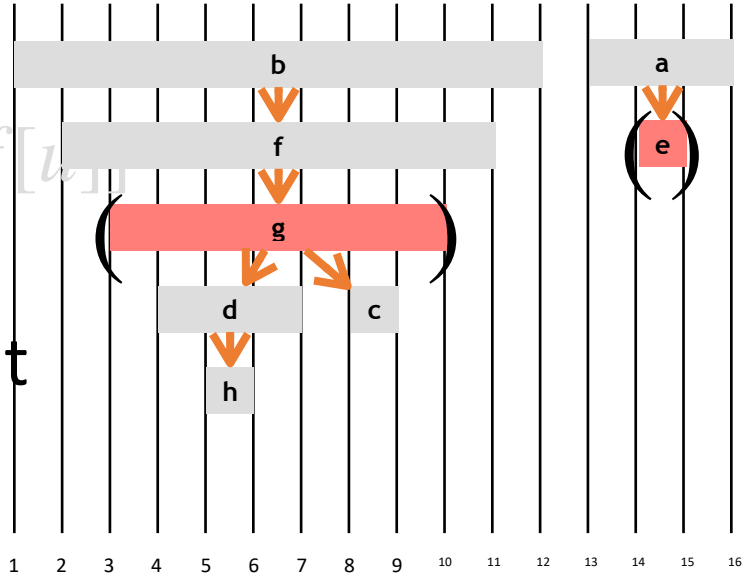


Parenthesis Theorem

In any DFS of a graph $G = (V, E)$, for any two vertices $u, v \in V$,
v is a proper descendant of u in the depth-first forest if and only if

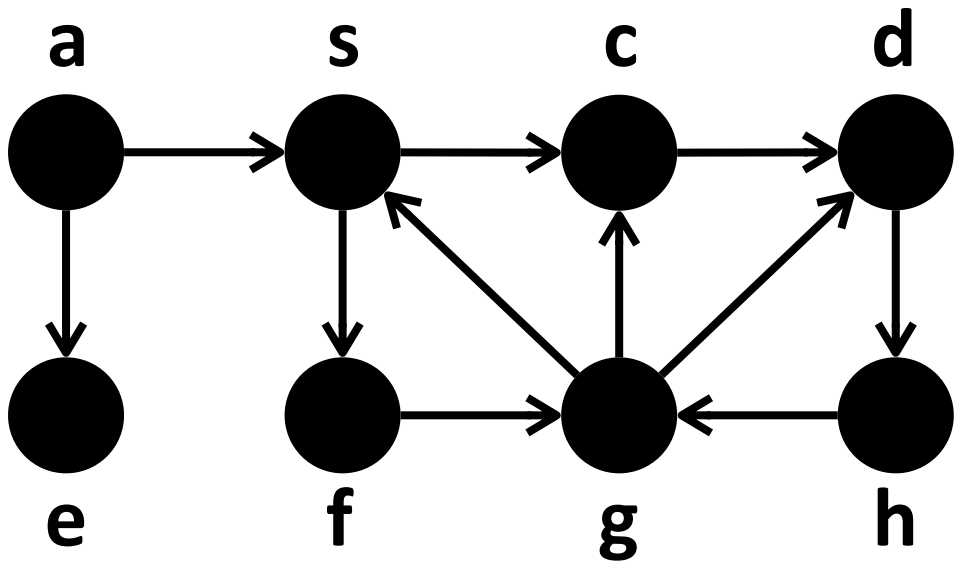
$$d[u] < d[v] < f[v] < f[u]$$

Depth-First Search Properties

- Parenthesis theorem
- Edge classification
- White-path theorem

Edges Classification

- According to the Depth-first forest $G_\pi = (V, E_\pi)$, each edge $(u, v) \in E$ is one of the four types:



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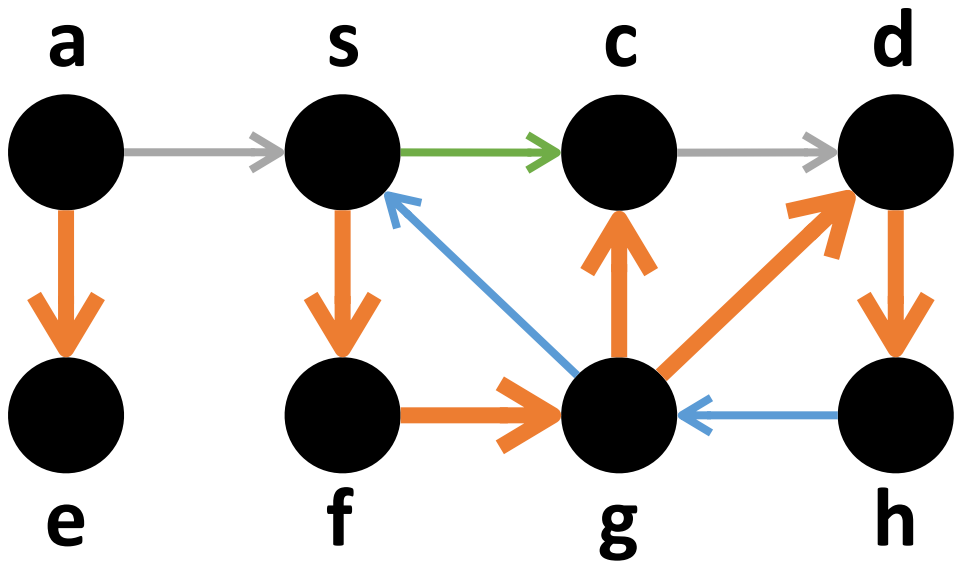

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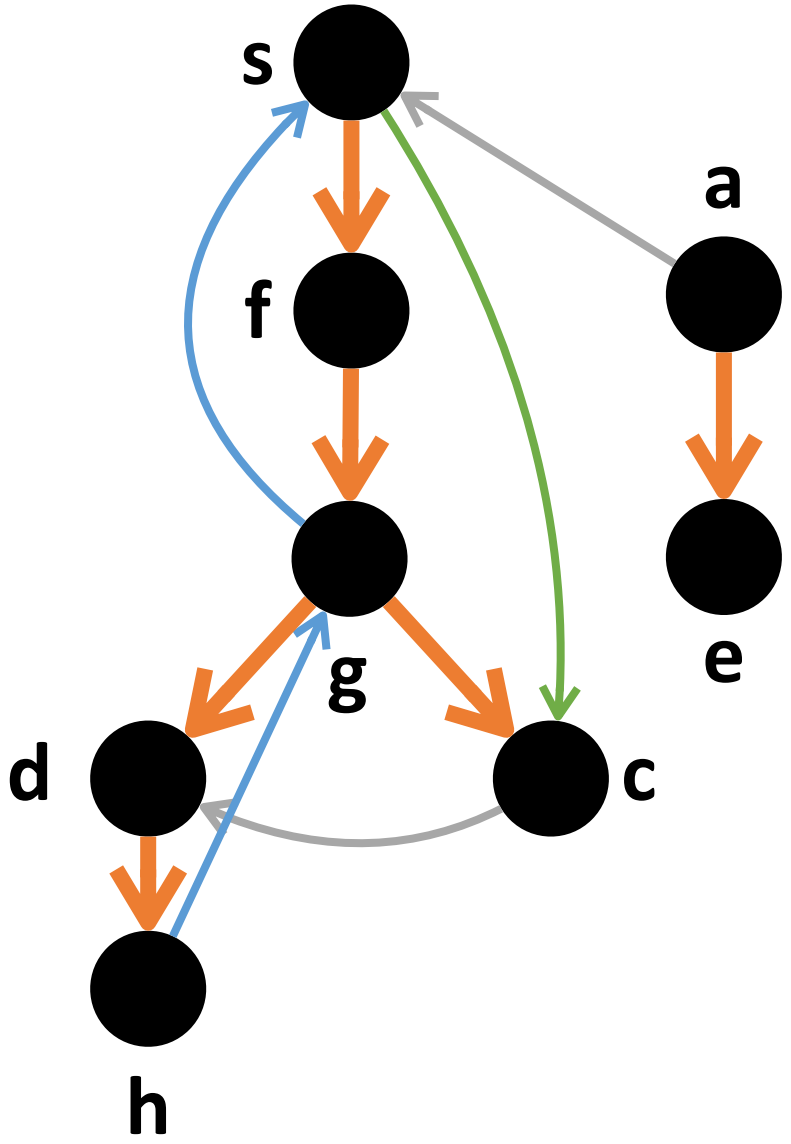

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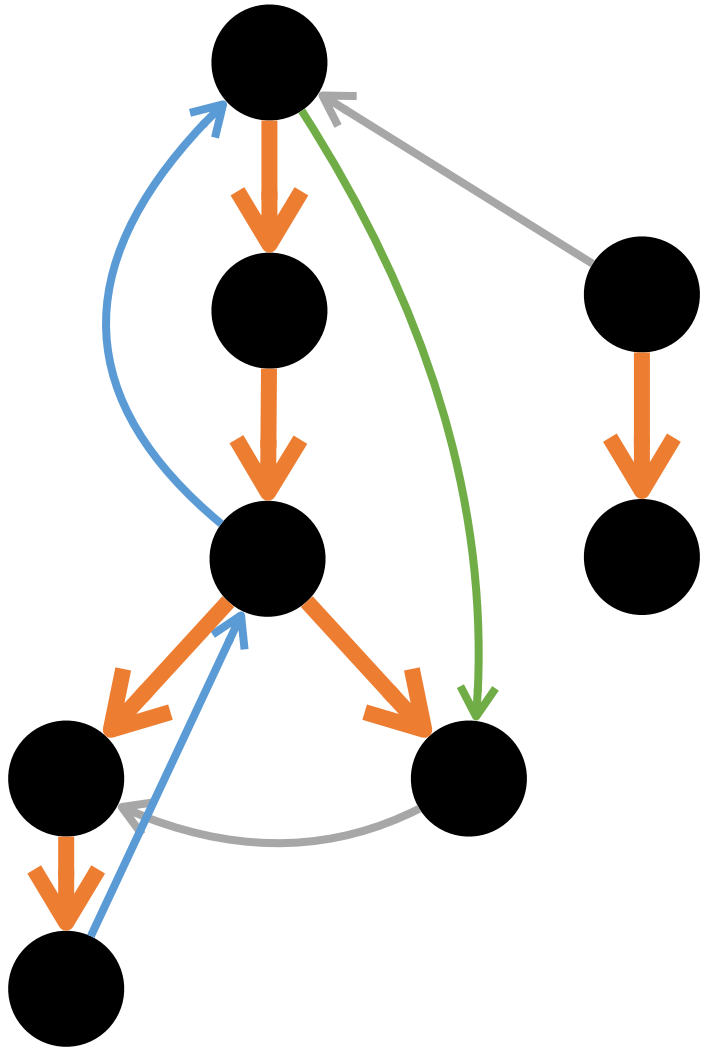

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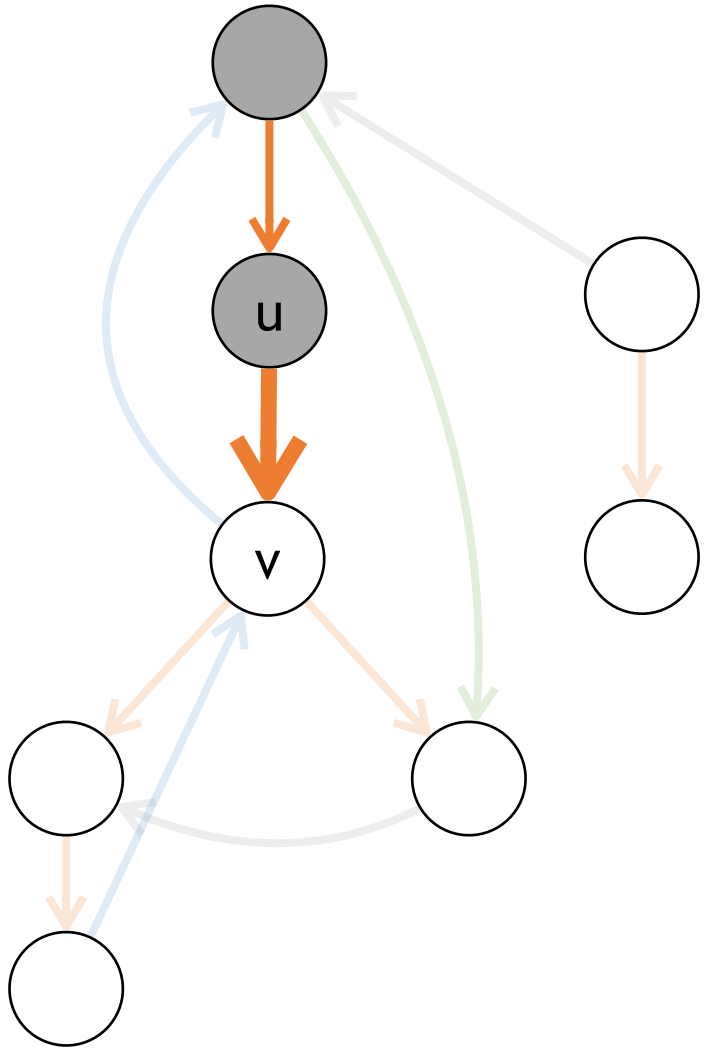

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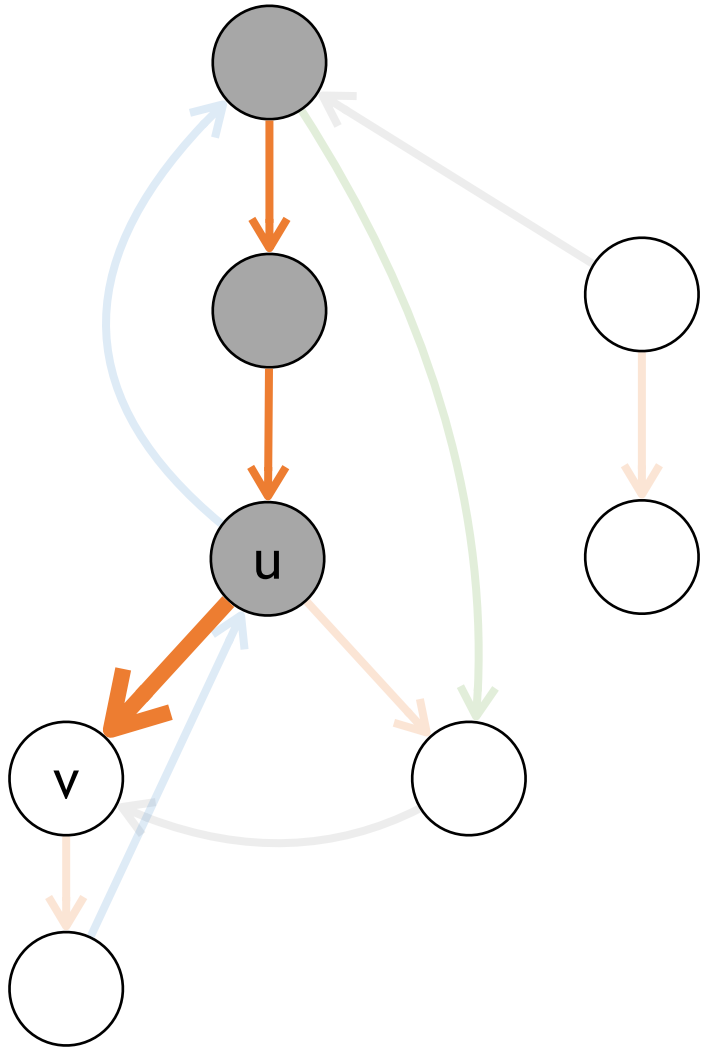

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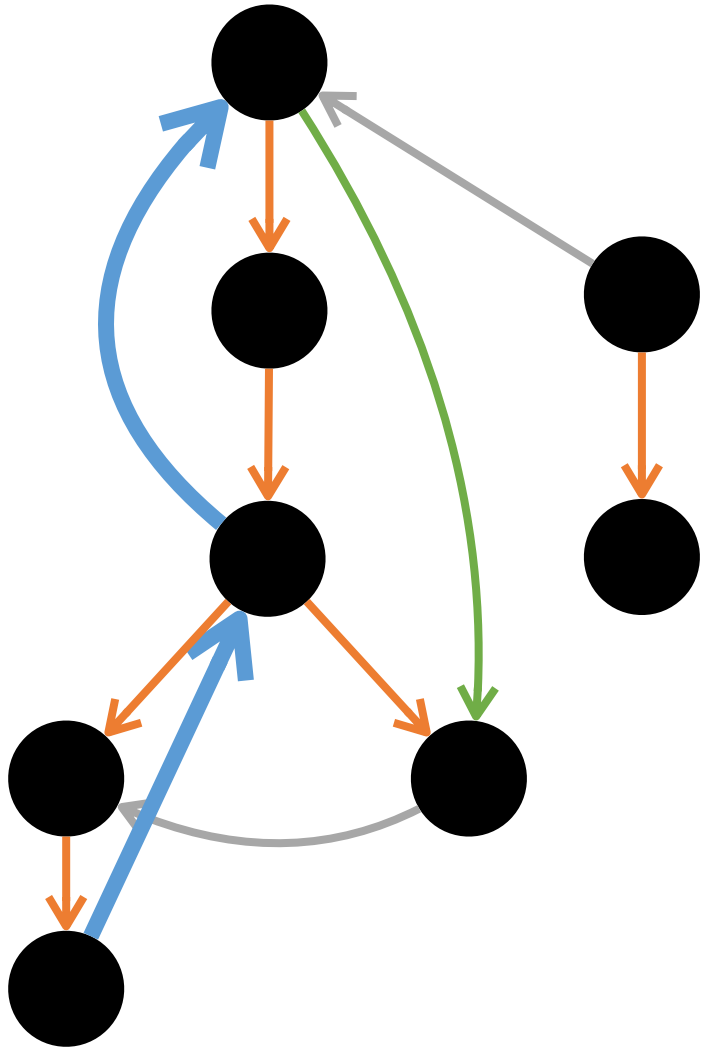

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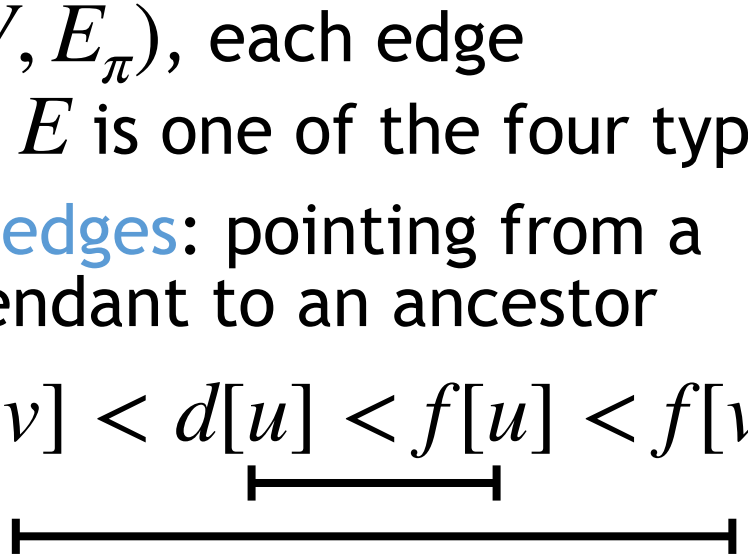
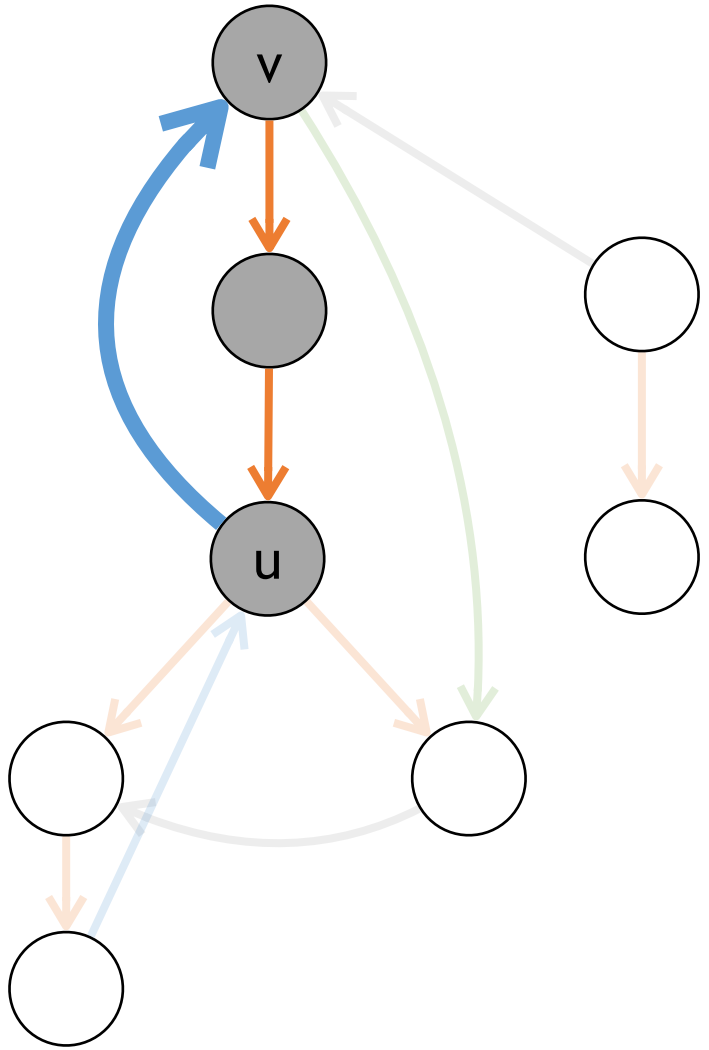


    - 



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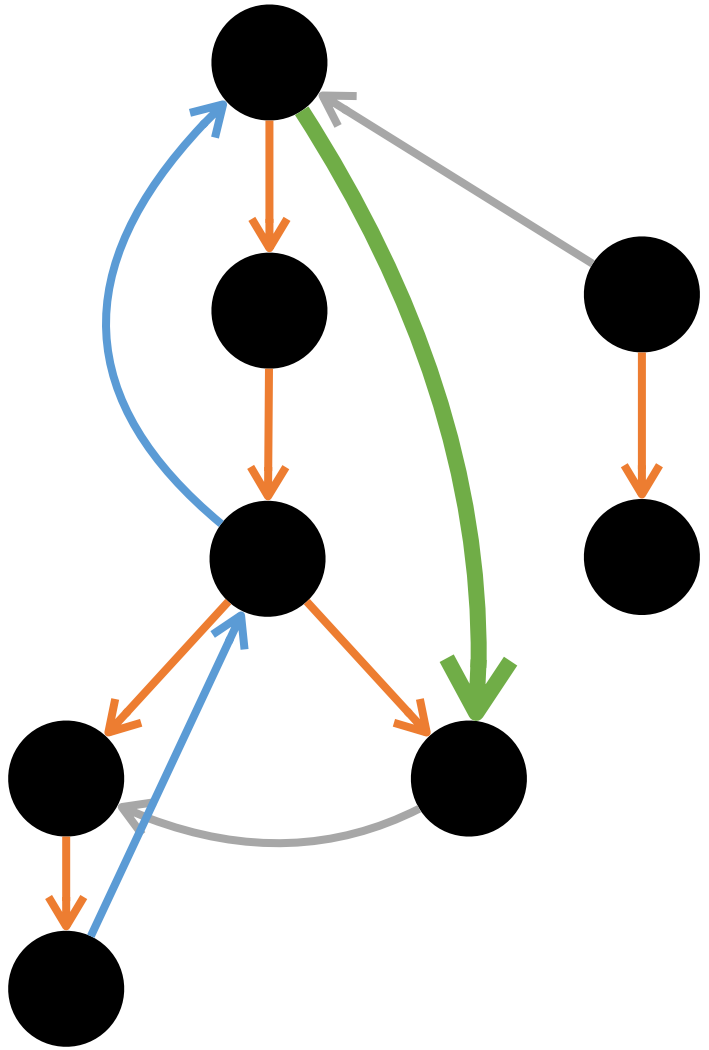

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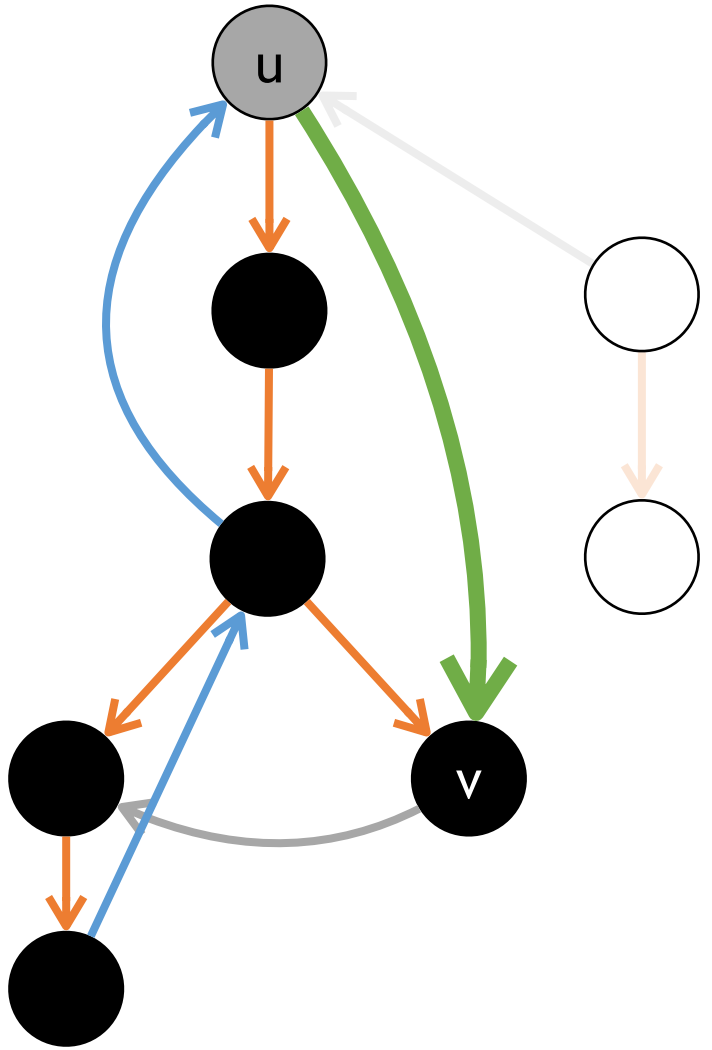

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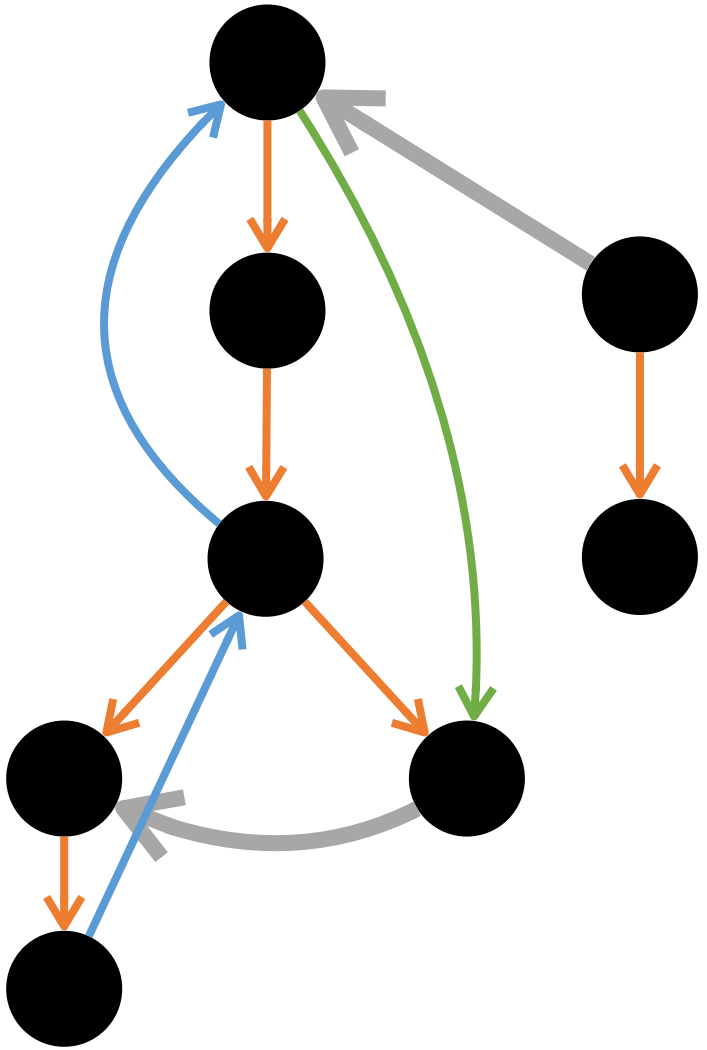
    - $d[v] < f[v] < d[u] < f[u]$



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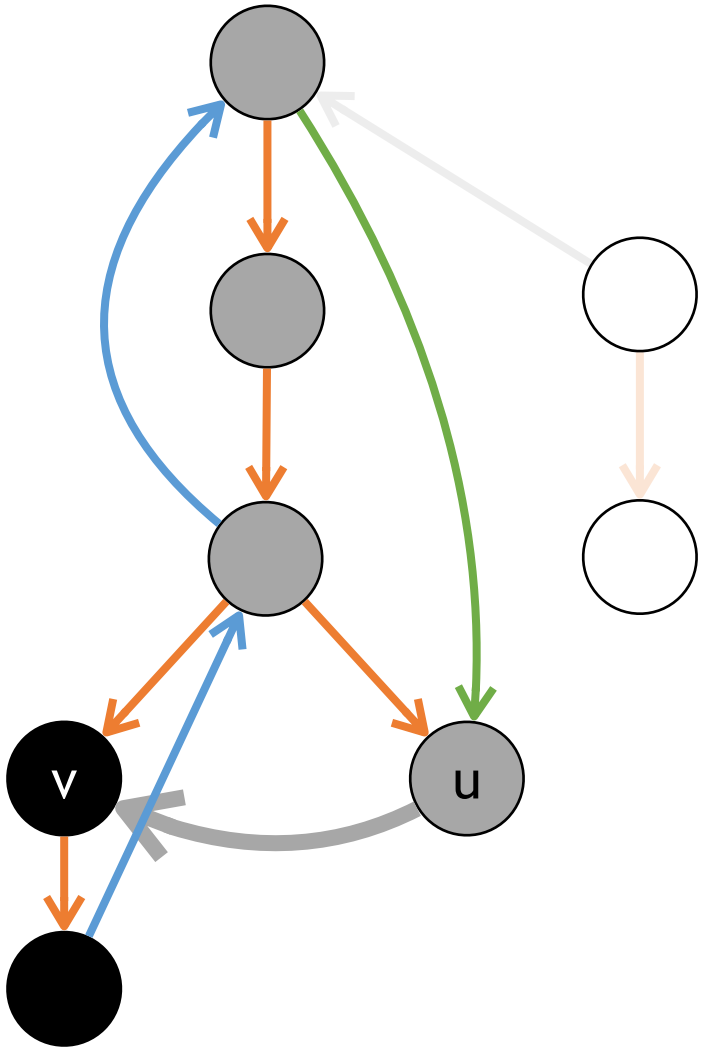

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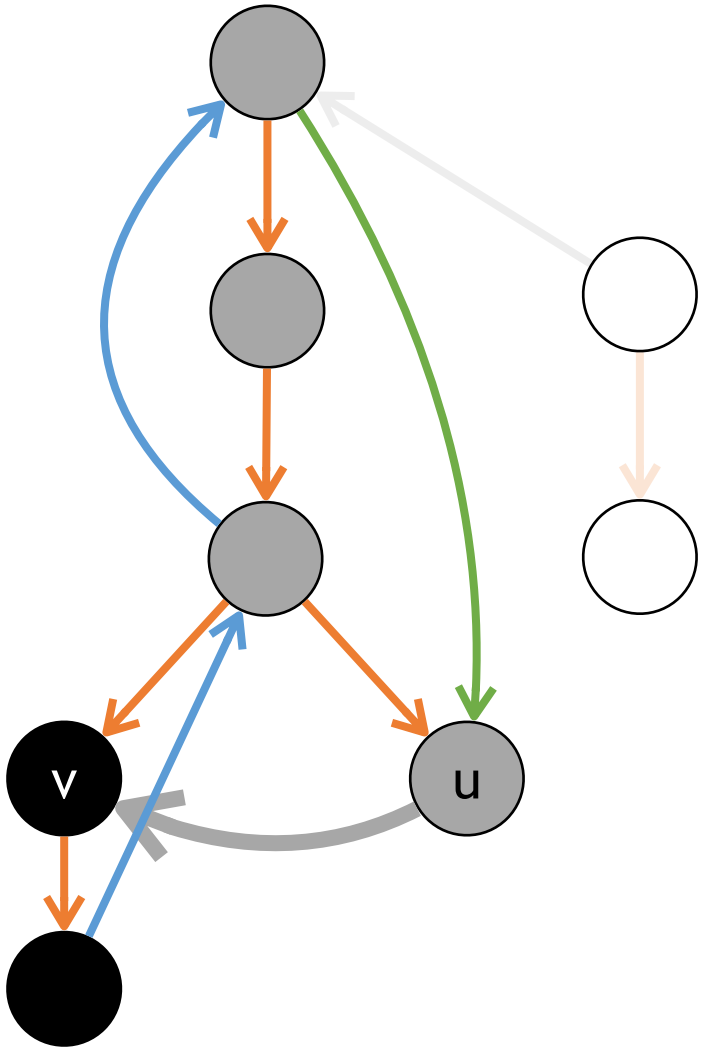
    - $d[v] < f[v] < d[u] < f[u]$

    - If $d[u] < d[v]$, (u, v) would be a forward edge



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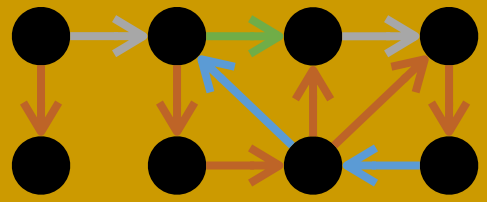
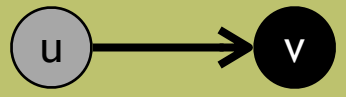
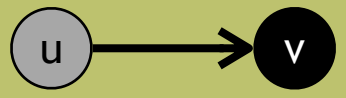


Type of (u, v)	Relation between d[u], f[u], d[v], f[v]	Intervals [d[u], f[u]] and [d[v], f[v]]	Color of v at the time (u, v) is explored
Tree edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	u → v
Back edge	d[v] < d[u] < f[u] < f[v]	[d[u], f[u]] [d[v], f[v]]	u → v
Forward edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	u → v
Cross edge	d[v] < f[v] < d[u] < f[u]	[d[u], f[u]] [d[v], f[v]]	u → v

During the DFS, we know which class the visited edge belongs to

Nesting of descendants' Intervals

Type of (u, v)	Relation between d[u], f[u], d[v], f[v]	Intervals [d[u], f[u]] and [d[v], f[v]]	Color of v at the time (u, v) is explored
Tree edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	
Back edge	d[v] < d[u] < f[u] < f[v]	[d[u], f[u]] [d[v], f[v]]	
Forward edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	
Cross edge	d[v] < f[v] < d[u] < f[u]	[d[u], f[u]] [d[v], f[v]]	

Vertex v is a descendant of vertex u in the depth-first forest **if and only if** $d[u] < d[v] < f[v] < f[u]$

Depth-First Search Properties

- Parenthesis theorem
- Edge classification
- White-path theorem

White-Path Theorem

In a depth-first forest of a graph G, vertex v is a descendant of vertex u **if and only if** at the time that u is discovered by DFS, vertex v can be reached from u along a path consisting entirely of white vertices

White-Path Theorem

In a depth-first forest of a graph G, vertex v is a descendant of vertex u if and **only if** at the time that u is discovered by DFS, vertex v can be reached from u along a path consisting entirely of white vertices

If vertex v is a descendant of vertex u,
v can be reached from u along a white path

White-Path Theorem

In a depth-first forest of a graph G, vertex v is a descendant of vertex u if and **only if** at the time that u is discovered by DFS, vertex v can be reached from u along a path consisting entirely of white vertices

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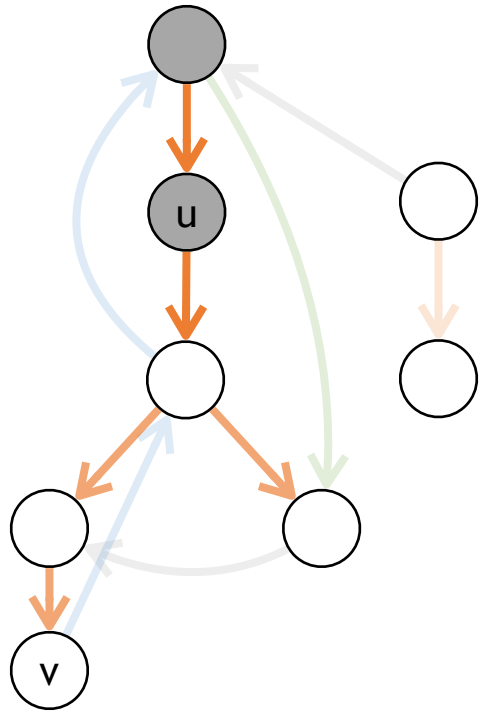

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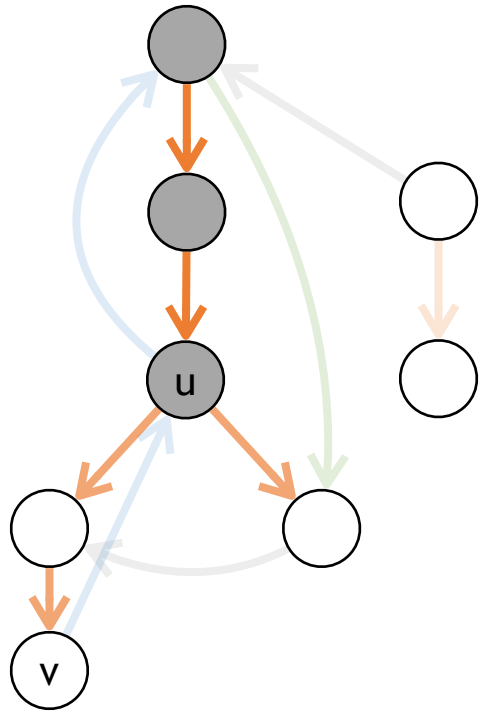

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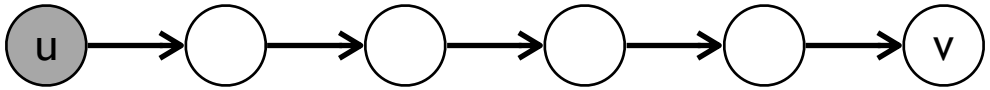

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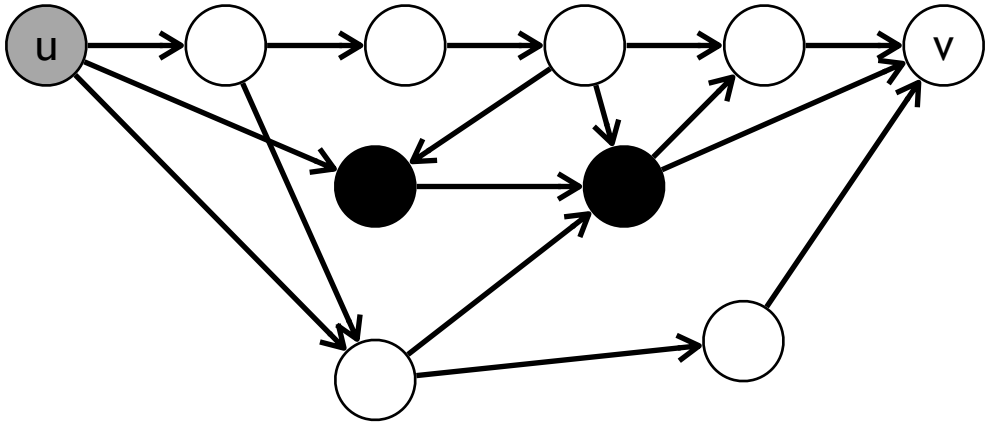

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Given a directed graph G, there is a directed cycle **if and only if** a DFS of G yields at least one back edge.

Application: Cycle Detection

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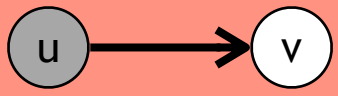
Type of (u, v)	Relation between d[u], f[u], d[v], f[v]	Intervals [d[u], f[u]] and [d[v], f[v]]	Color of v at the time (u, v) is explored
Tree edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	
Back edge	d[v] < d[u] < f[u] < f[v]	[d[u], f[u]] [d[v], f[v]]	
Forward edge	d[u] < d[v] < f[v] < f[u]	[d[u], f[u]] [d[v], f[v]]	
Cross edge	d[v] < f[v] < d[u] < f[u]	[d[u], f[u]] [d[v], f[v]]	

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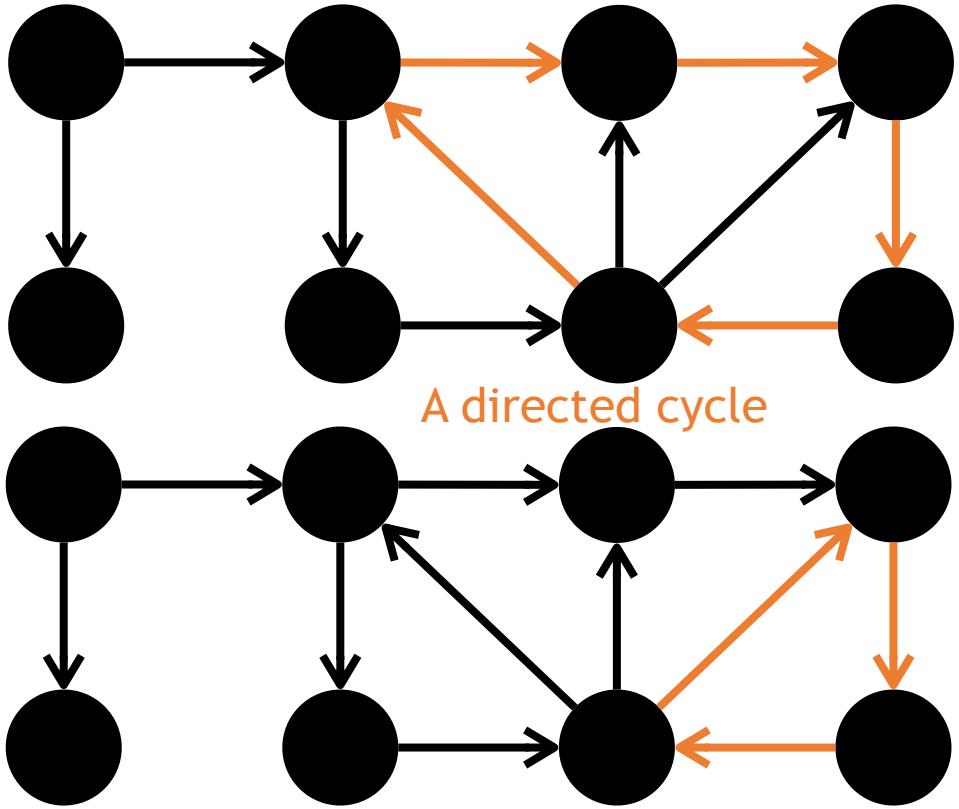

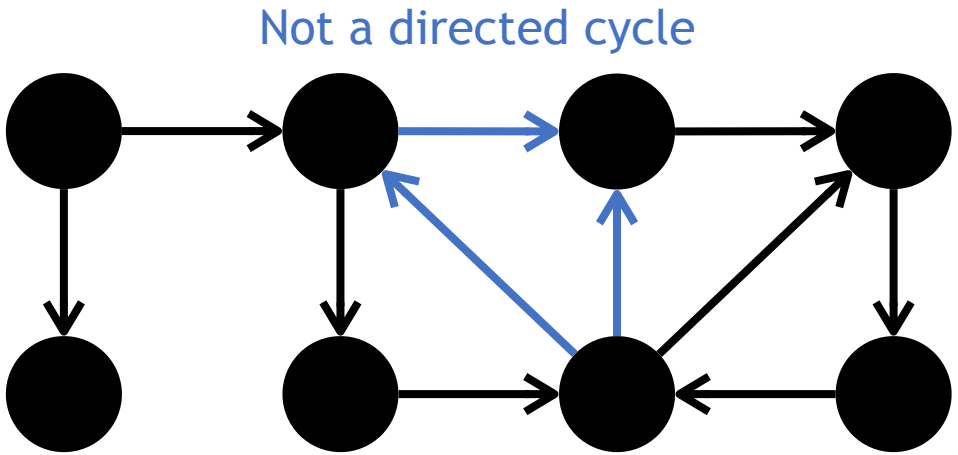

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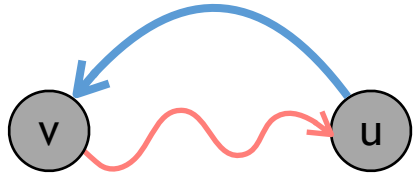

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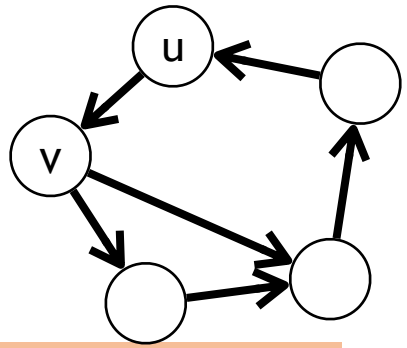

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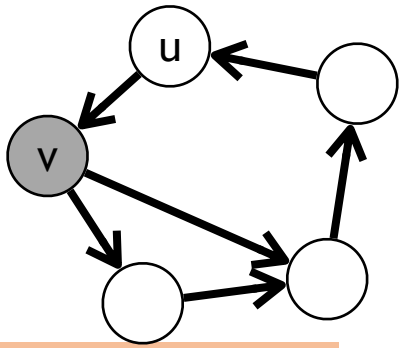

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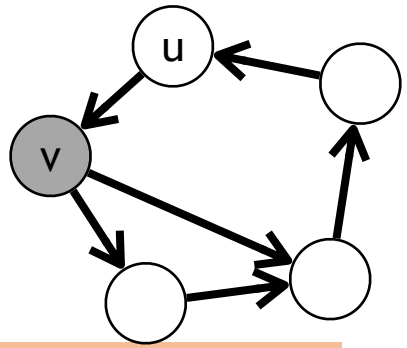

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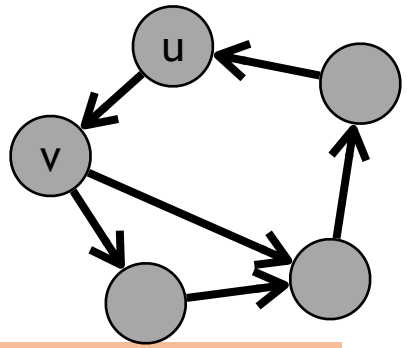

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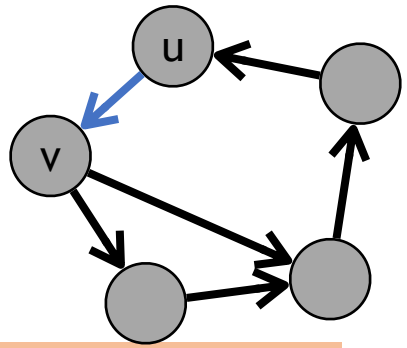

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What's Happening

- Use the edge-classification, we can detect if there is a directed cycle in the graph by checking if there is a back edge.
- The correctness proof uses the White-Path theorem.

Outline

- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

Directed Acyclic Graph

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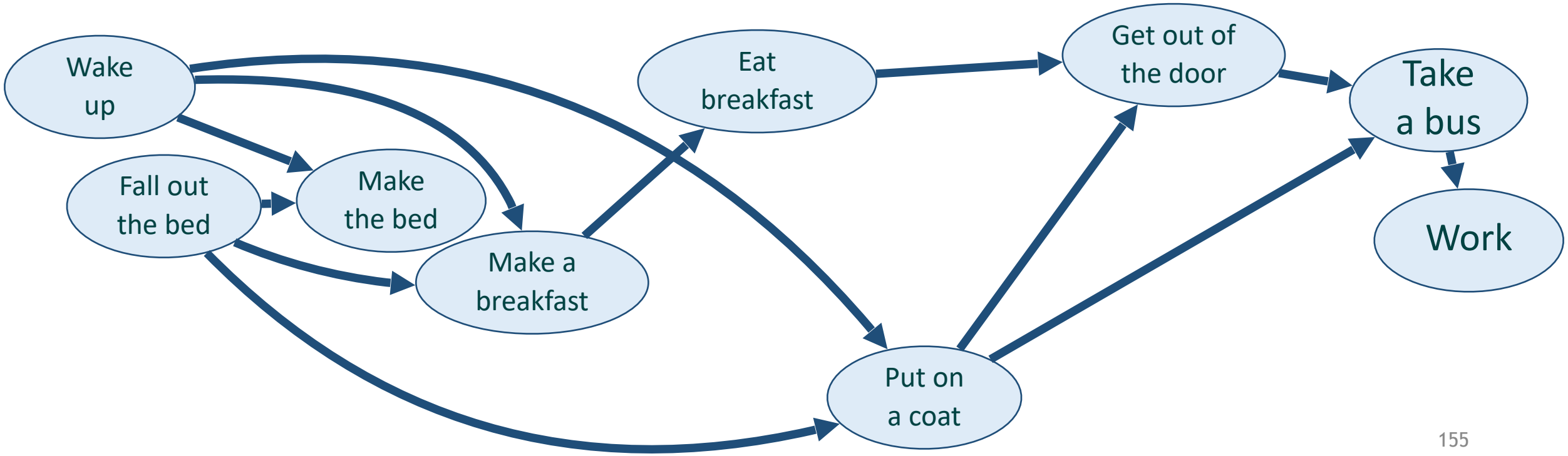

Directed Acyclic Graph

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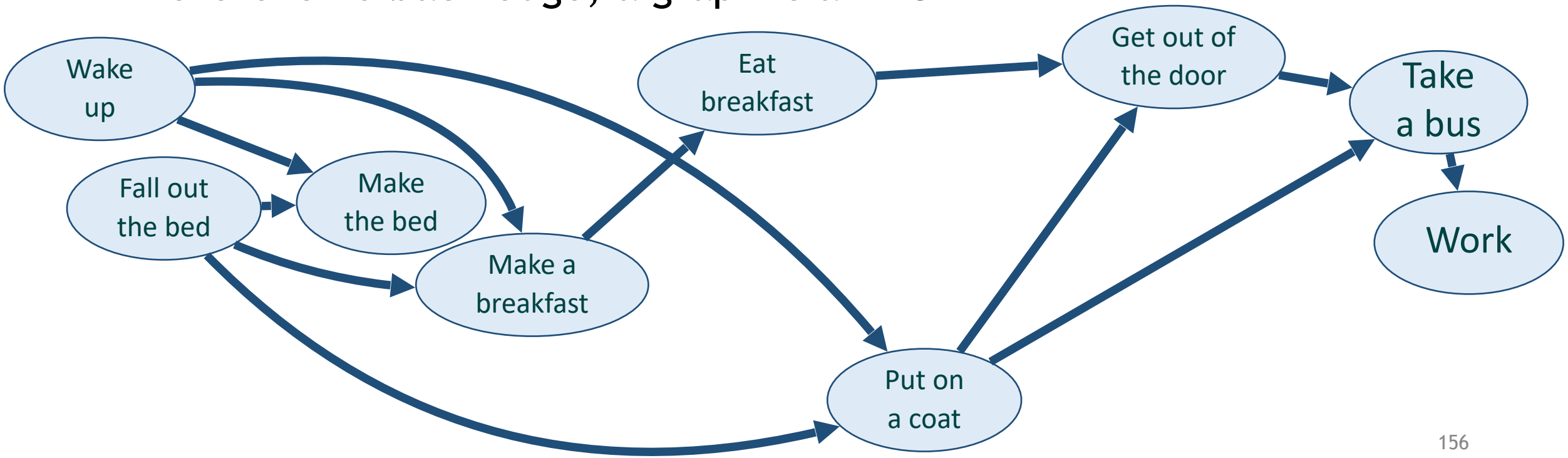
- If there is no back edge, a graph is a DAG



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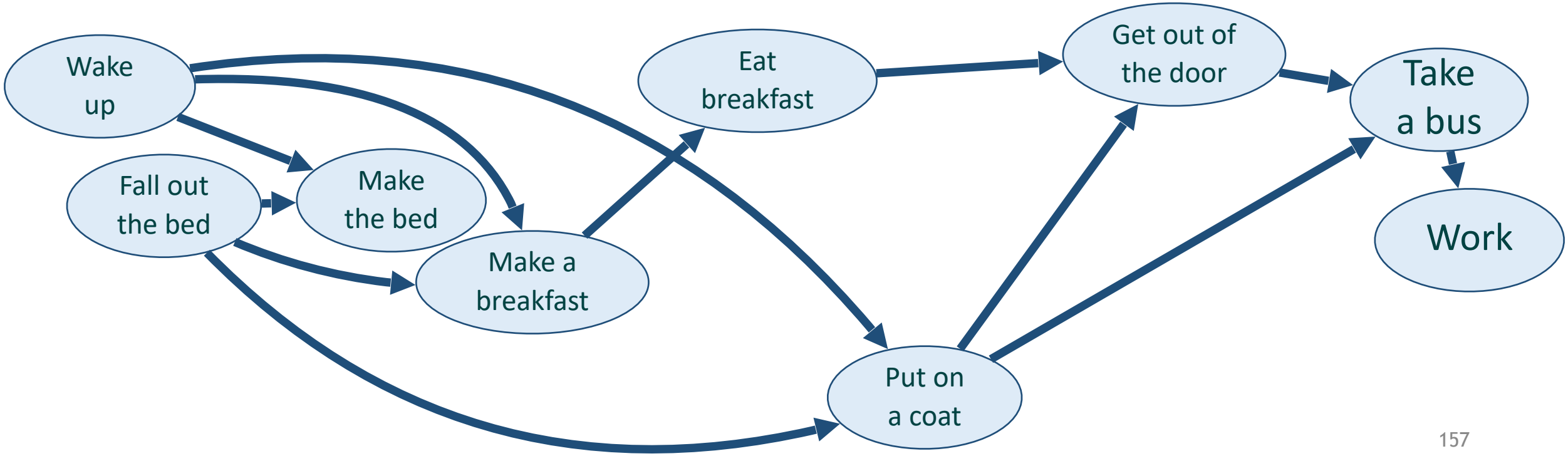

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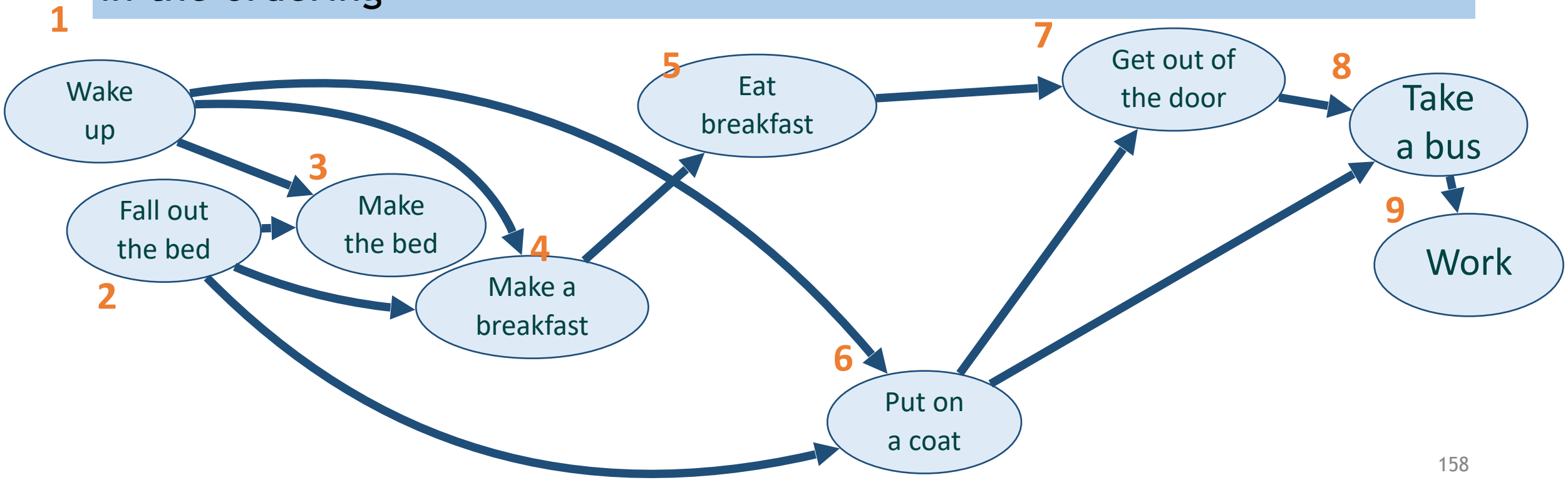

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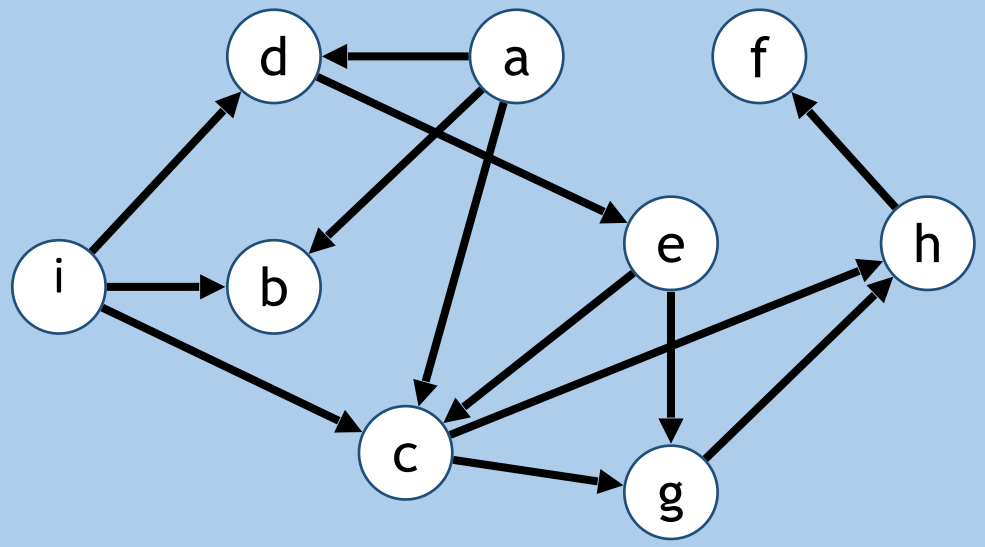

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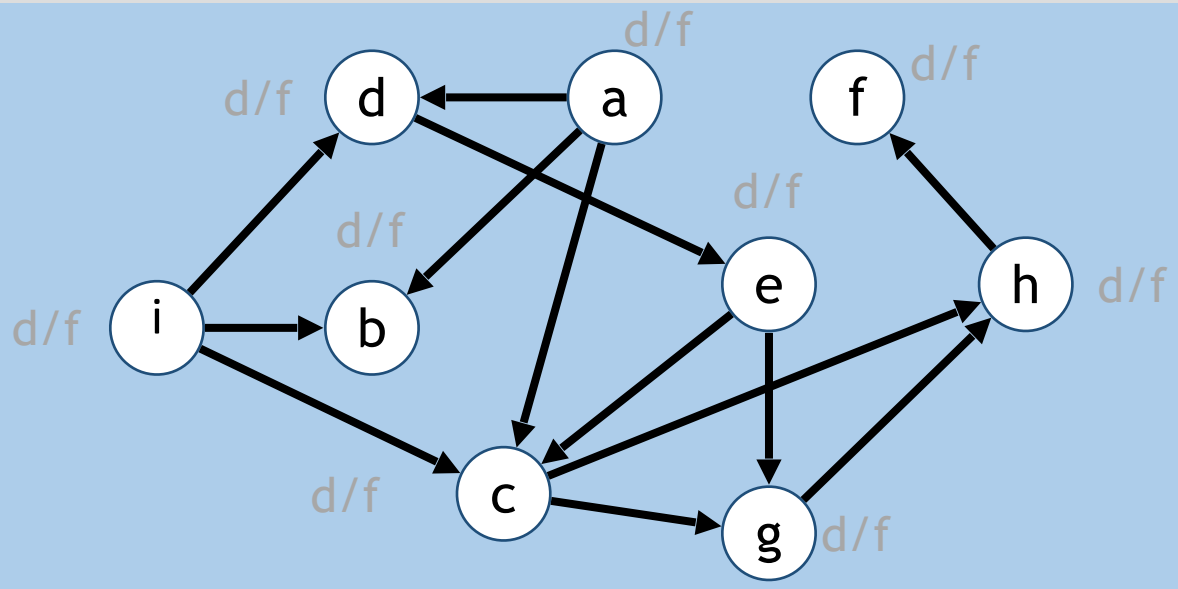

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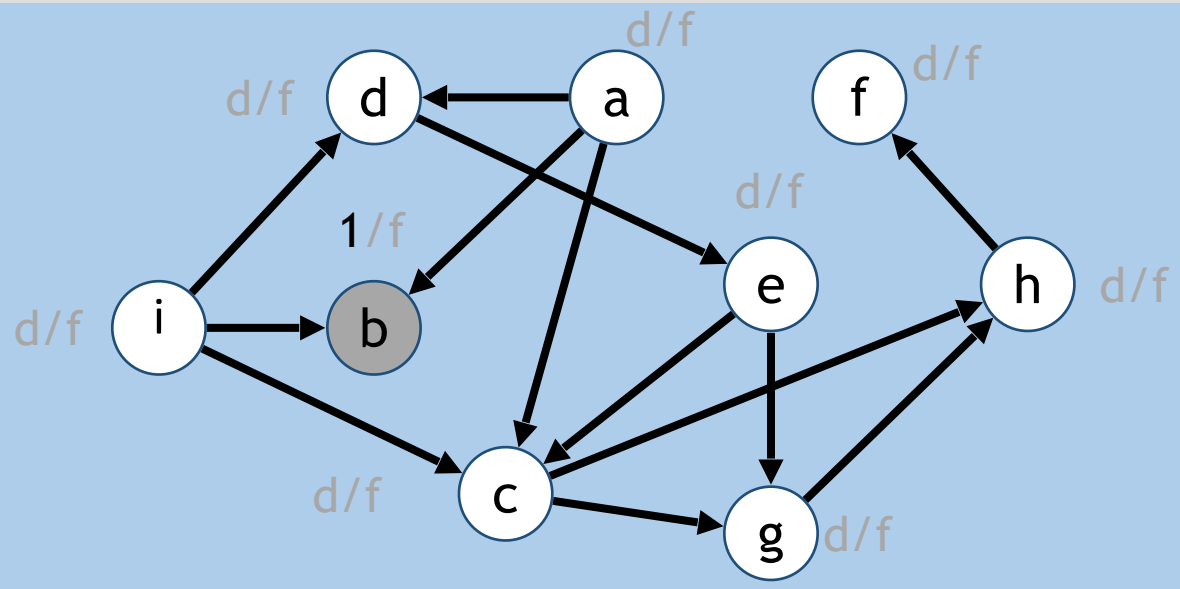

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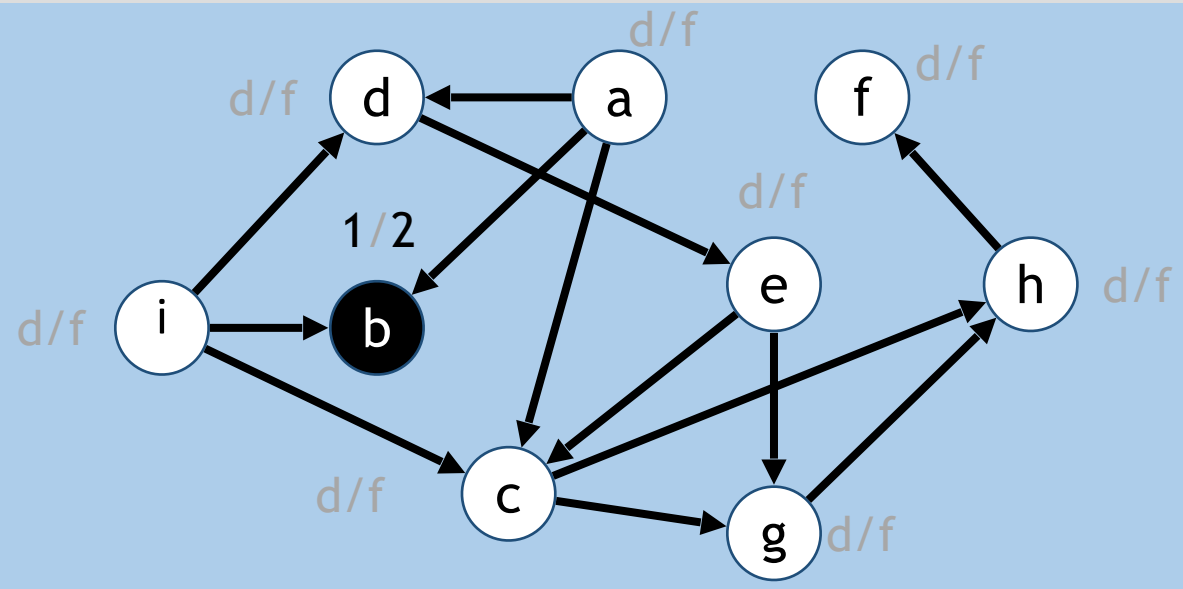

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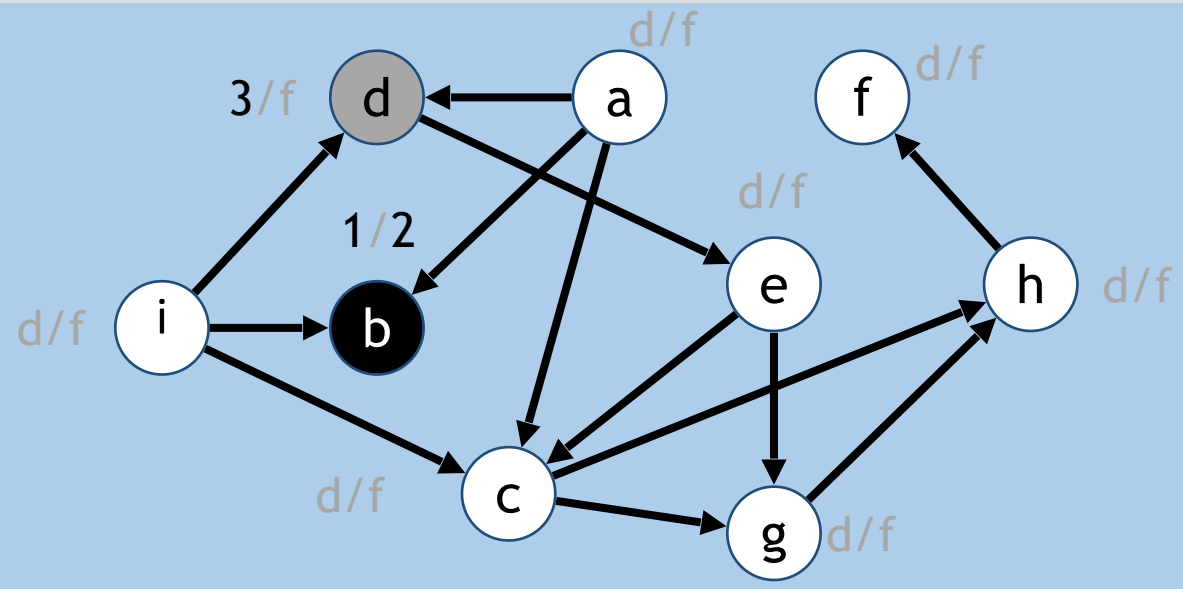

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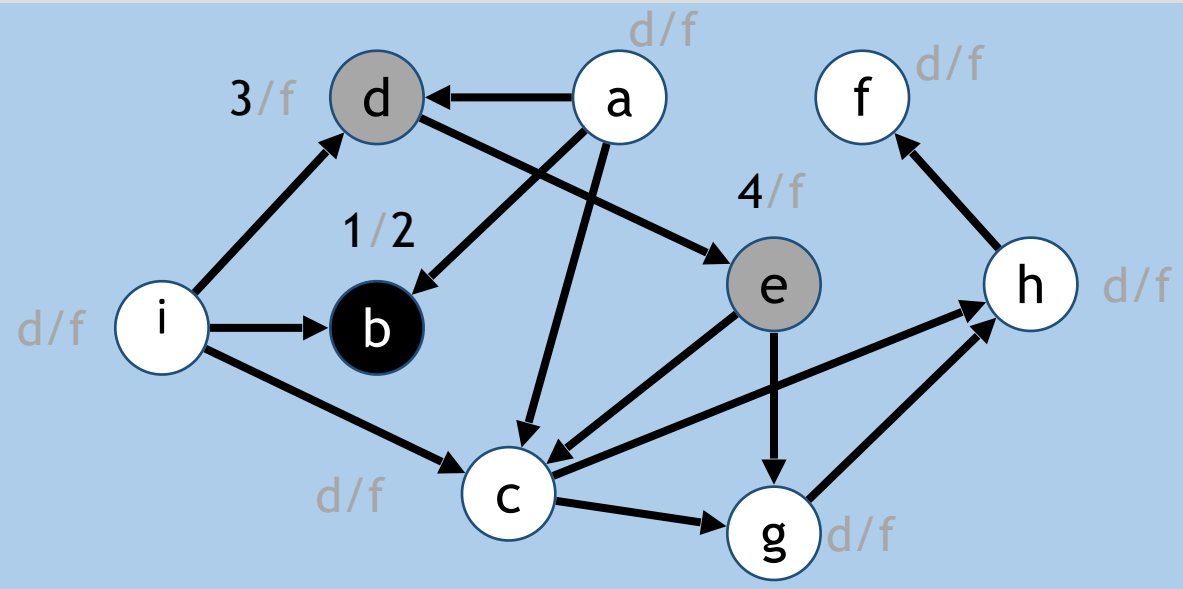

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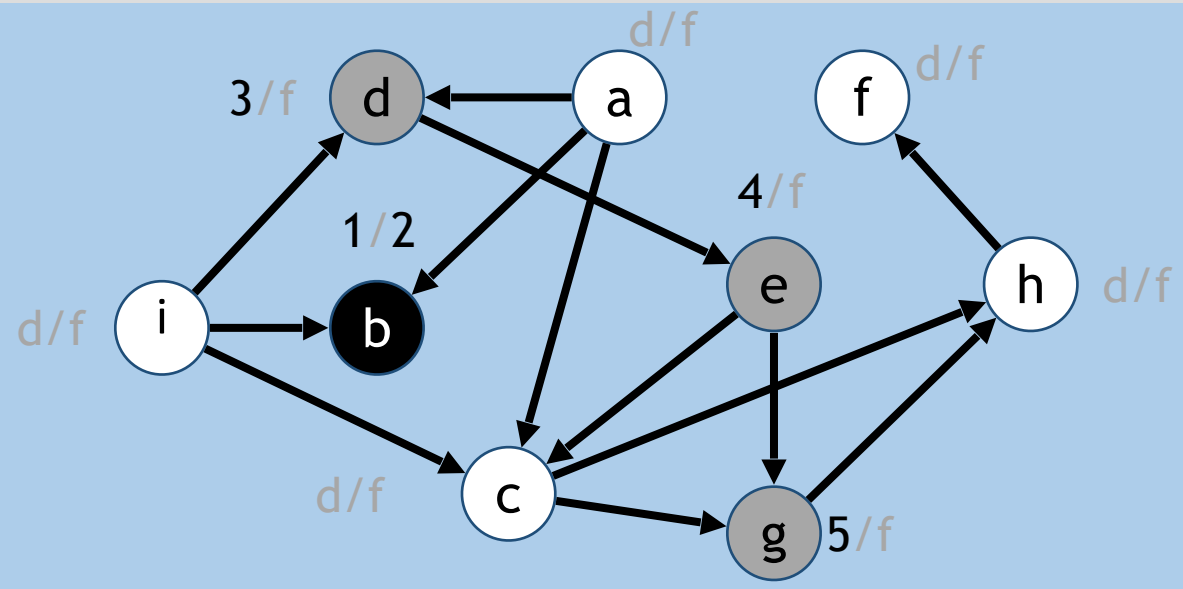

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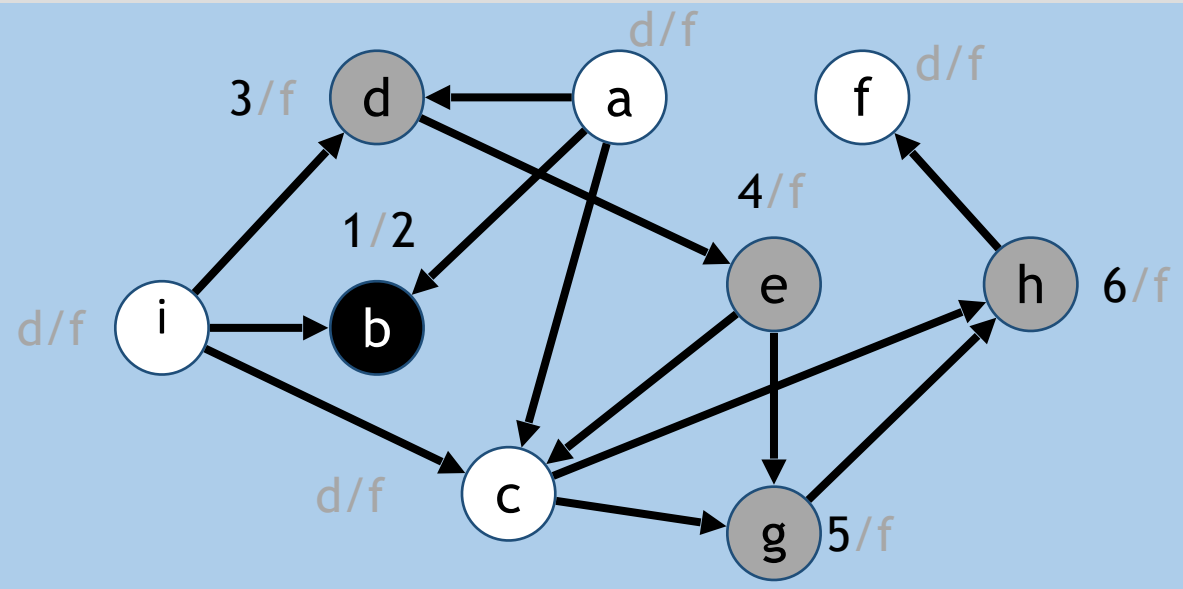

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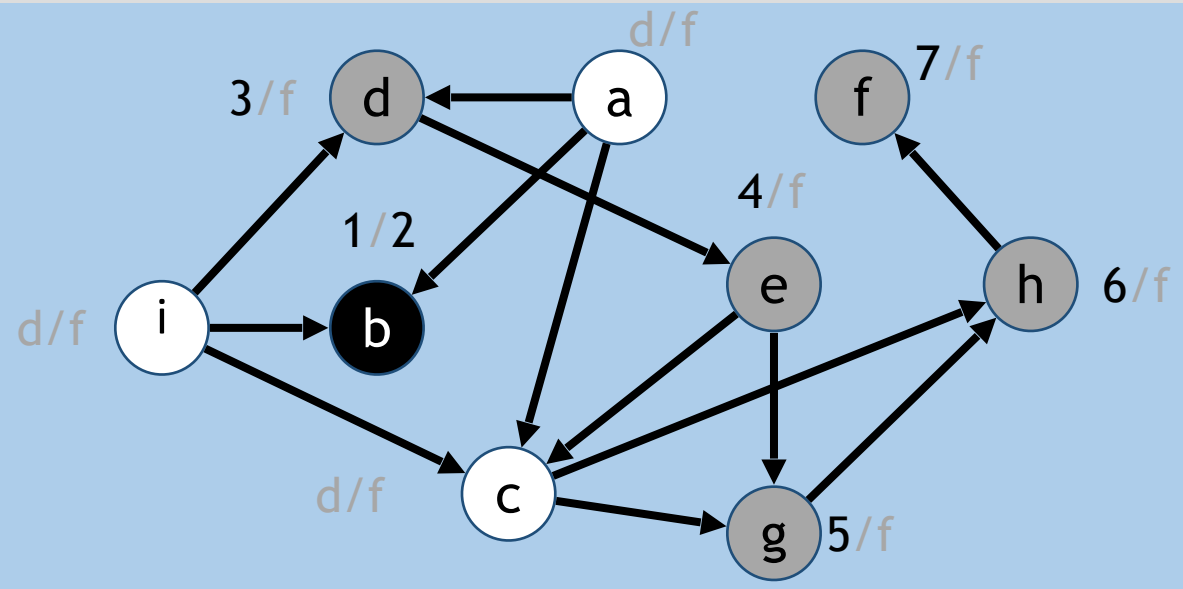

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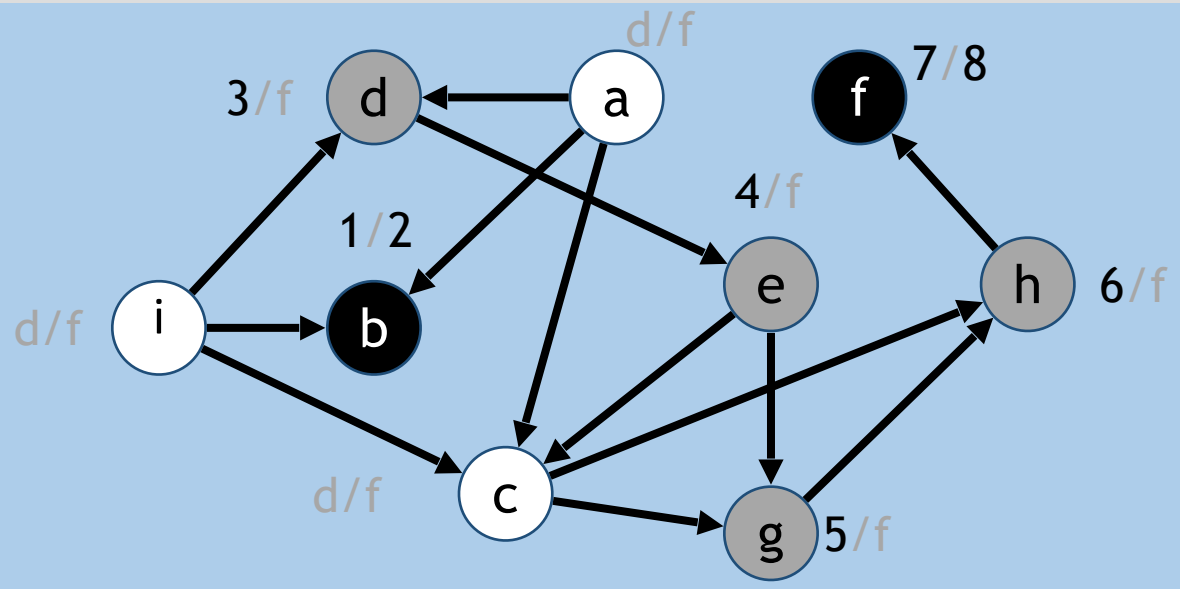

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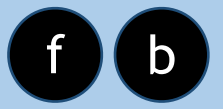

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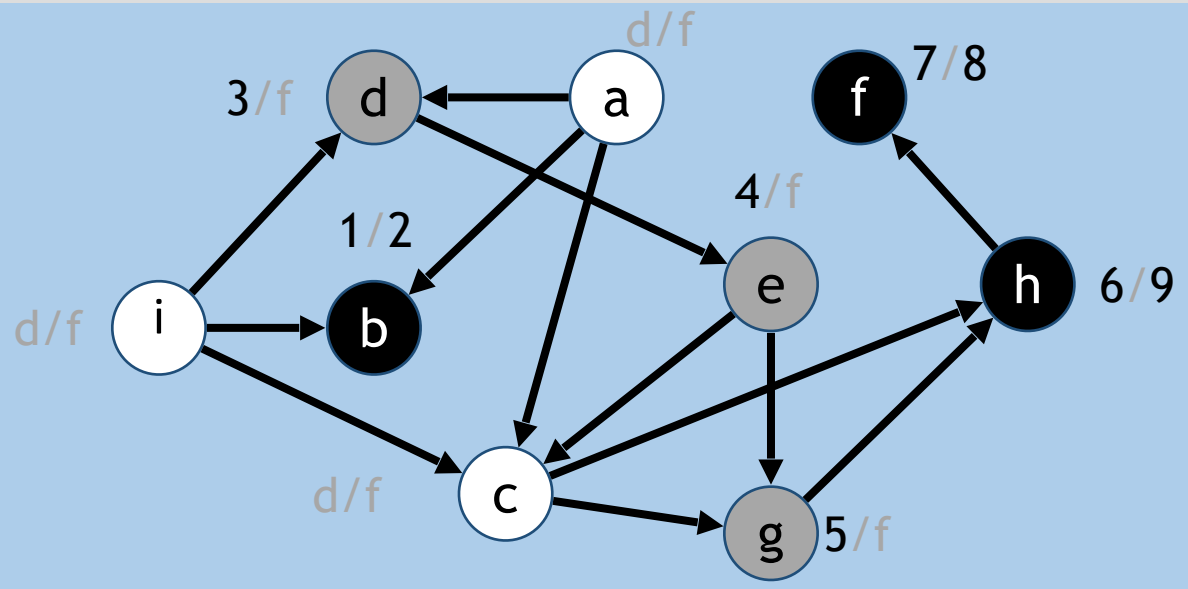

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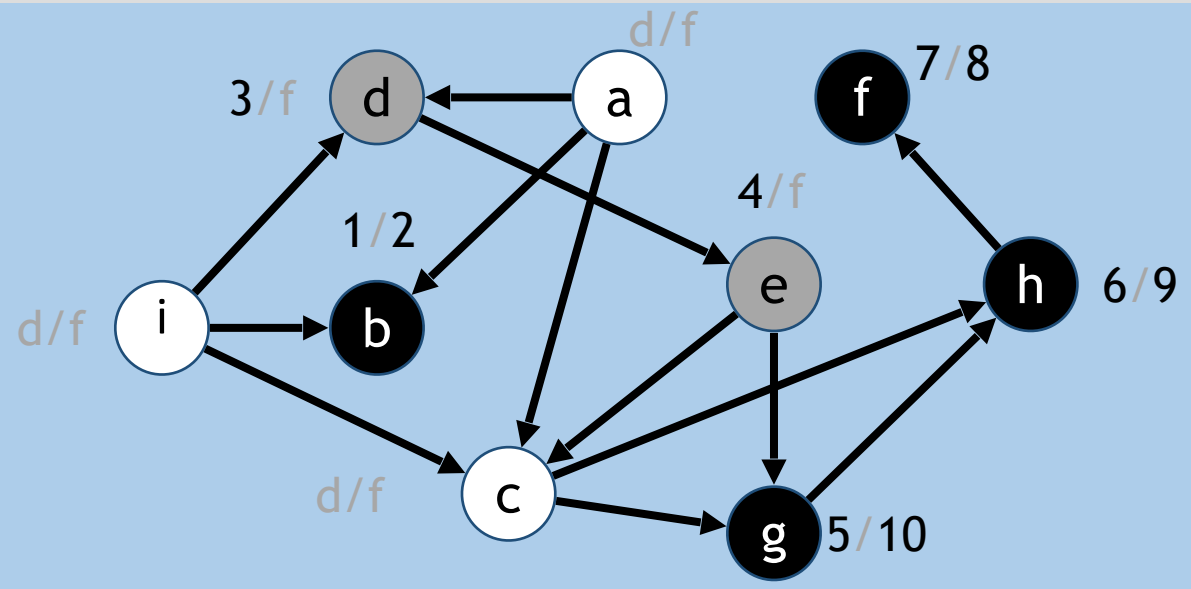

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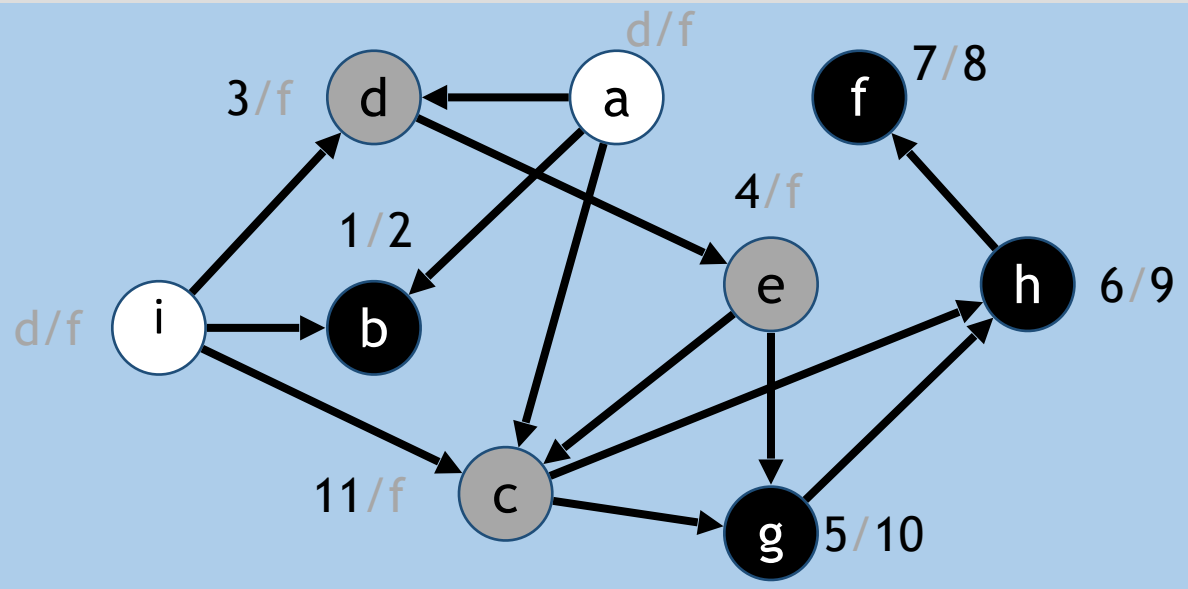

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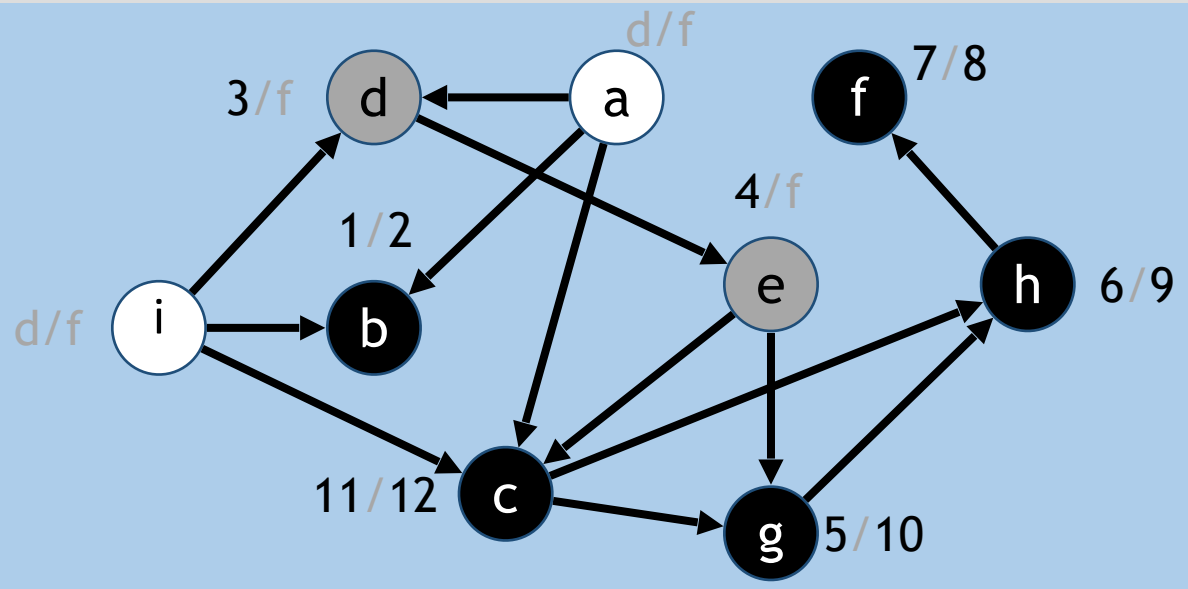

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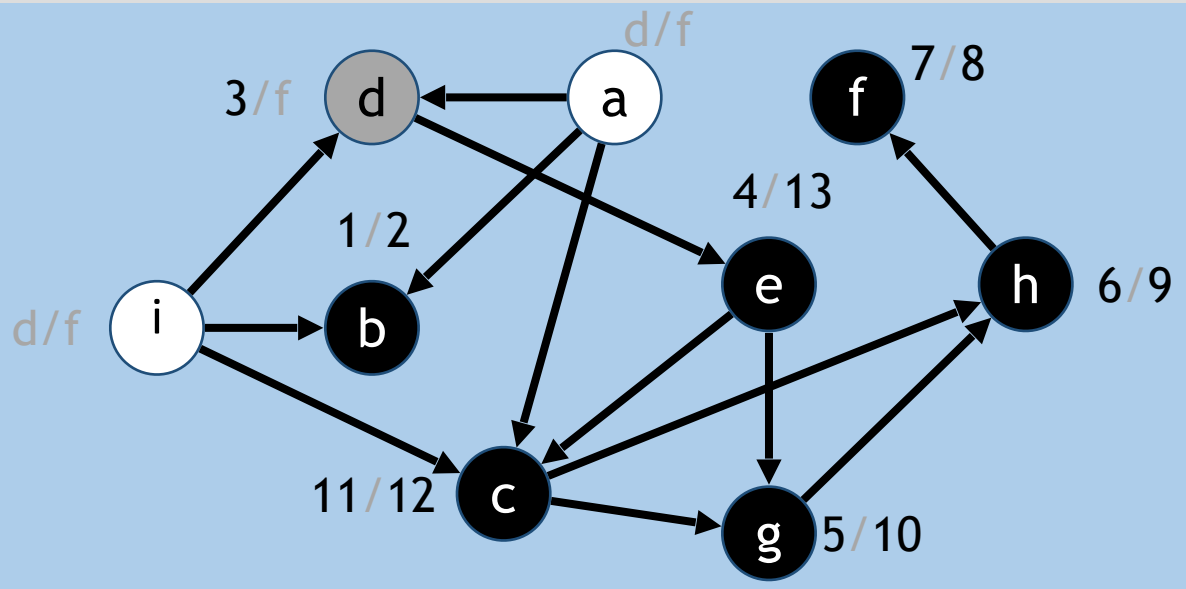

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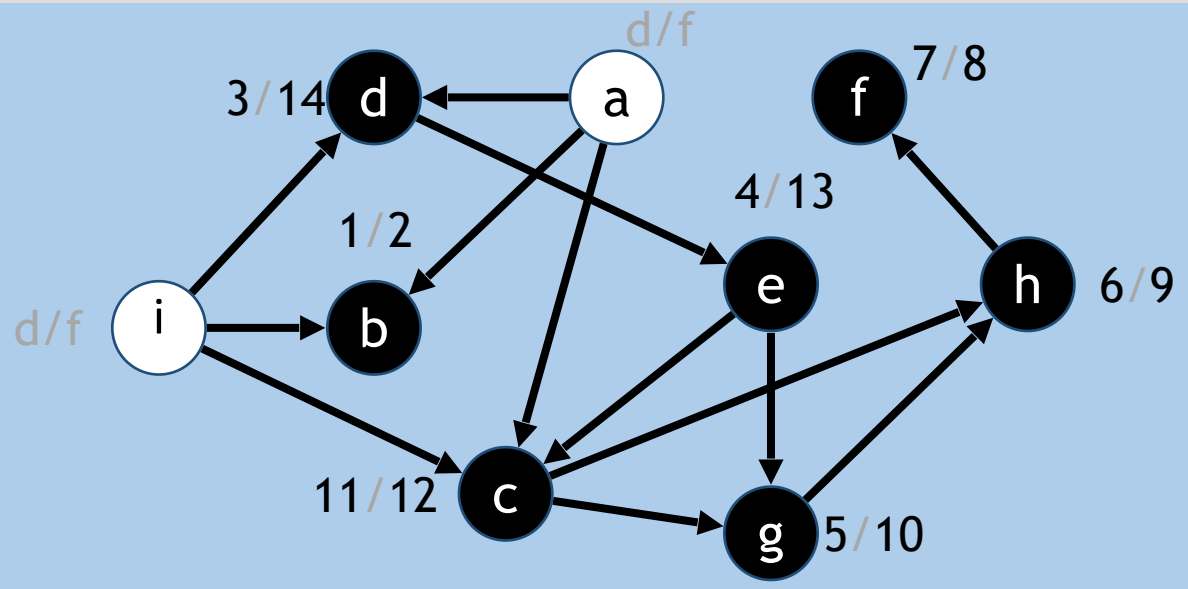

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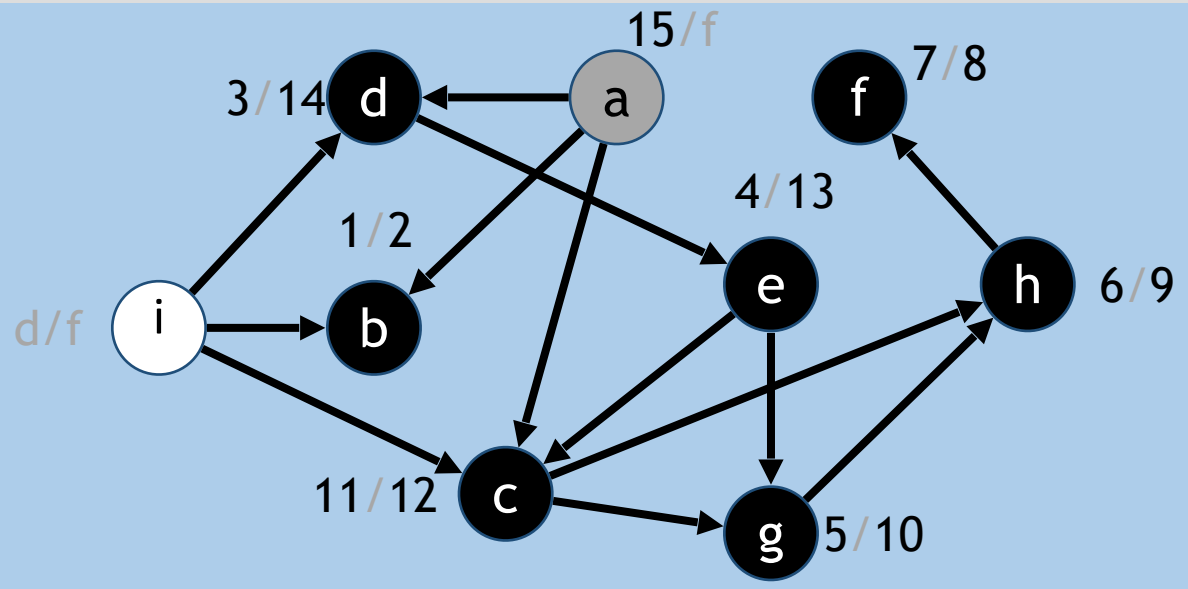

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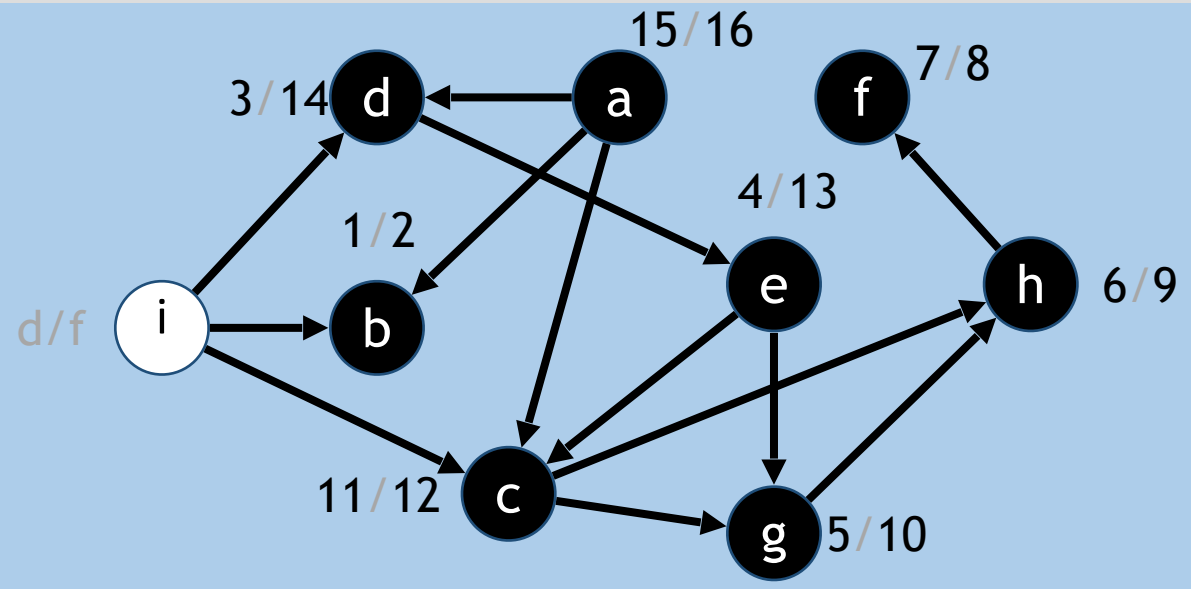

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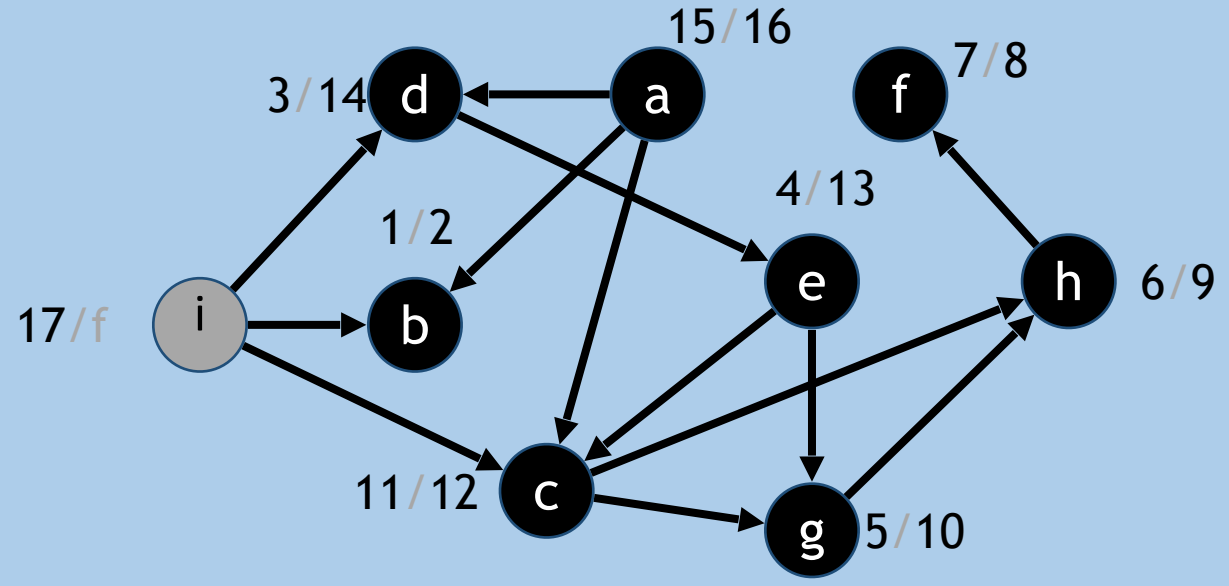

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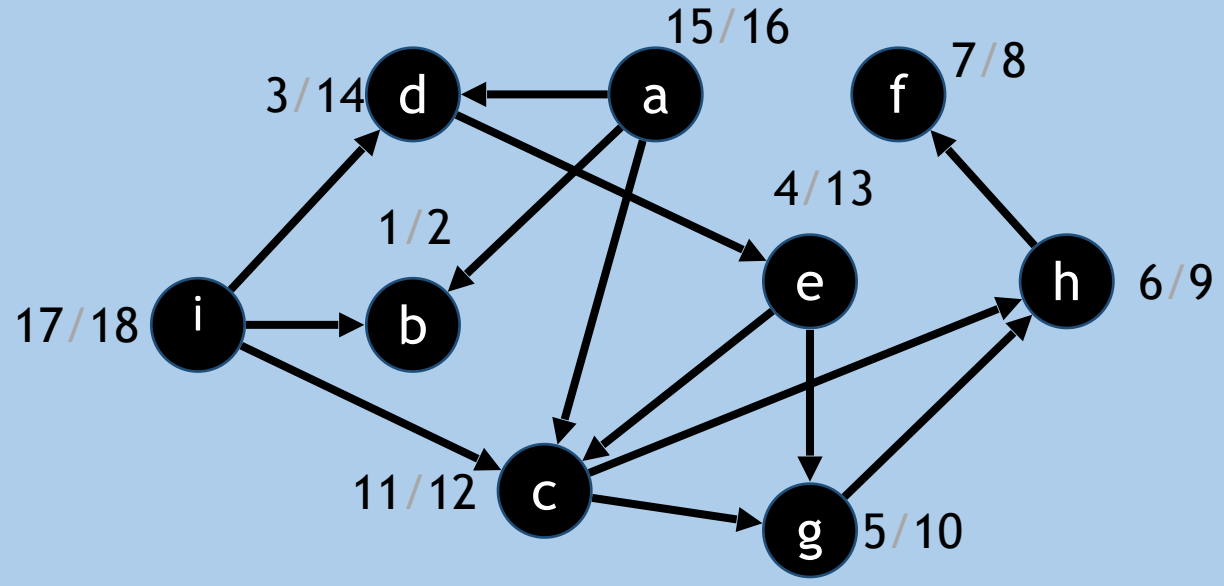

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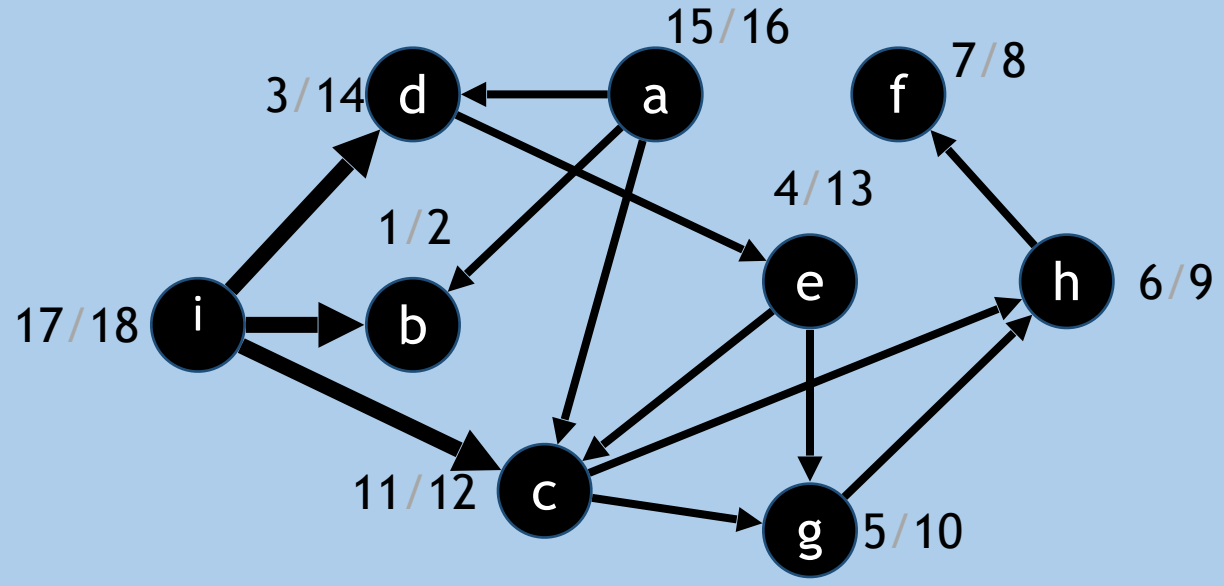

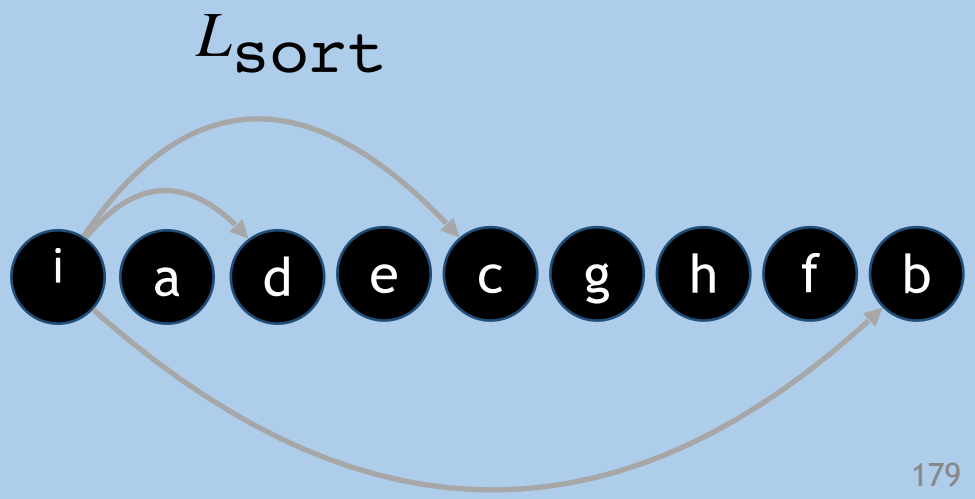

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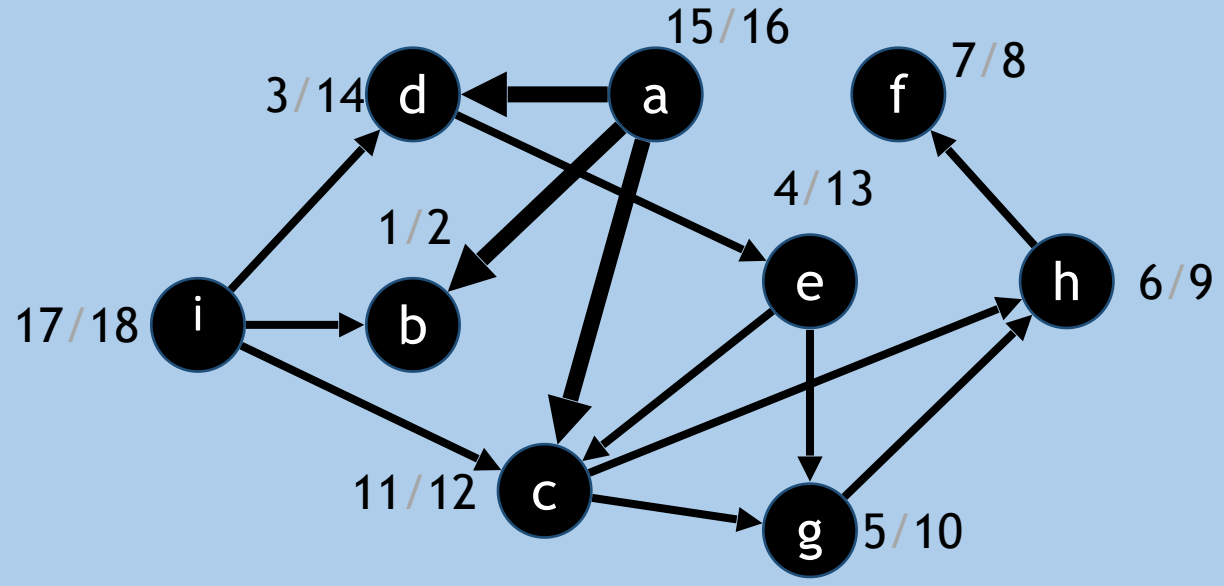

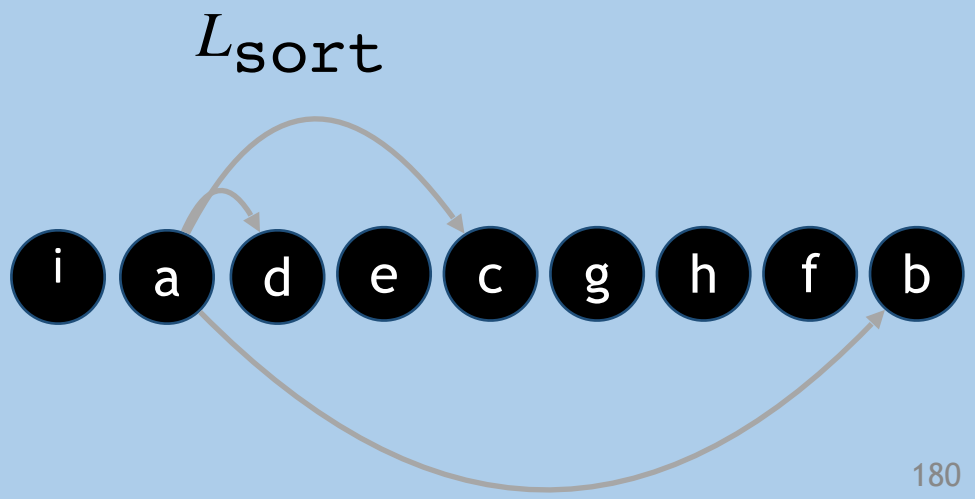

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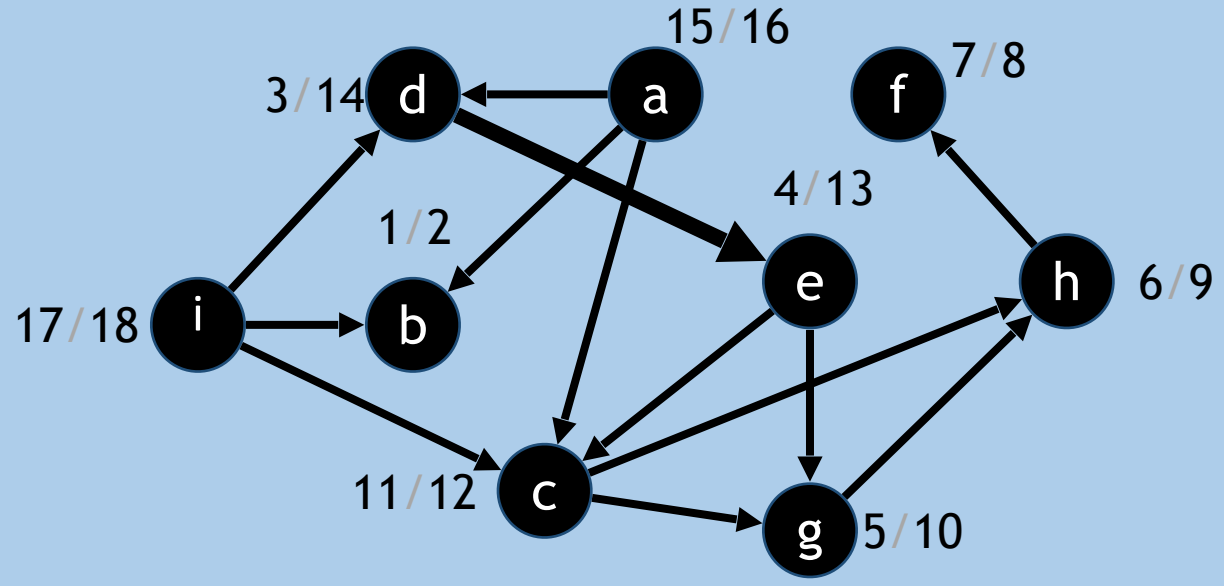

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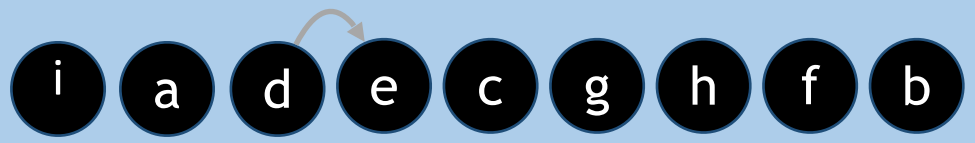

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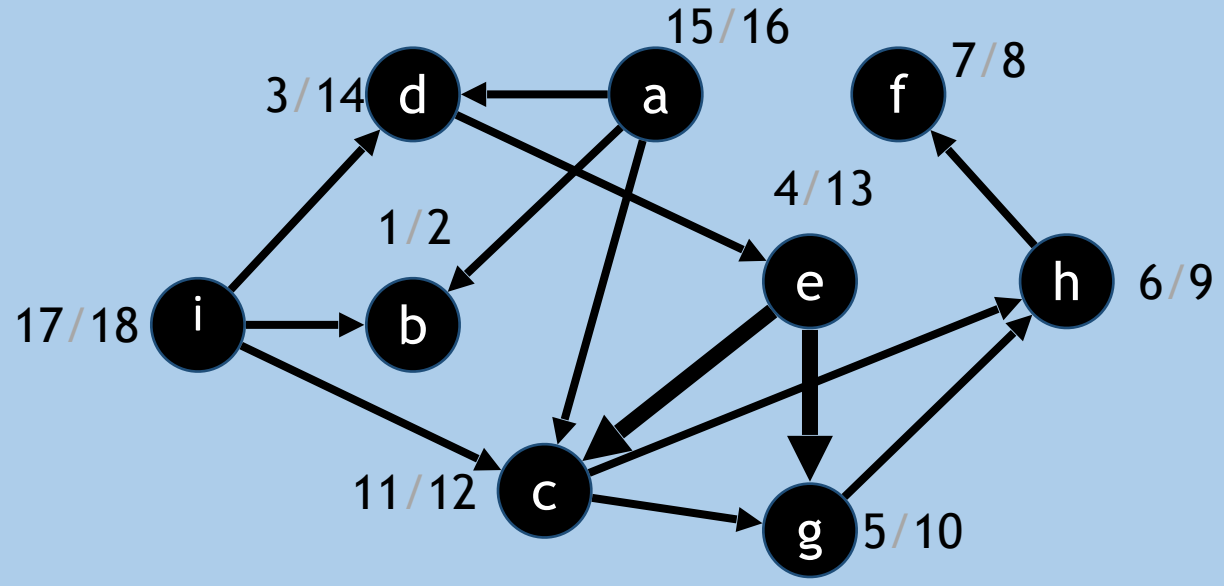

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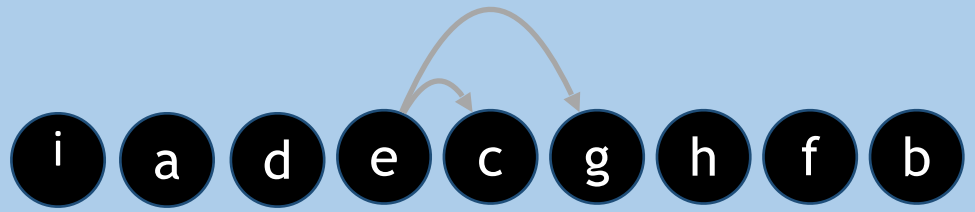

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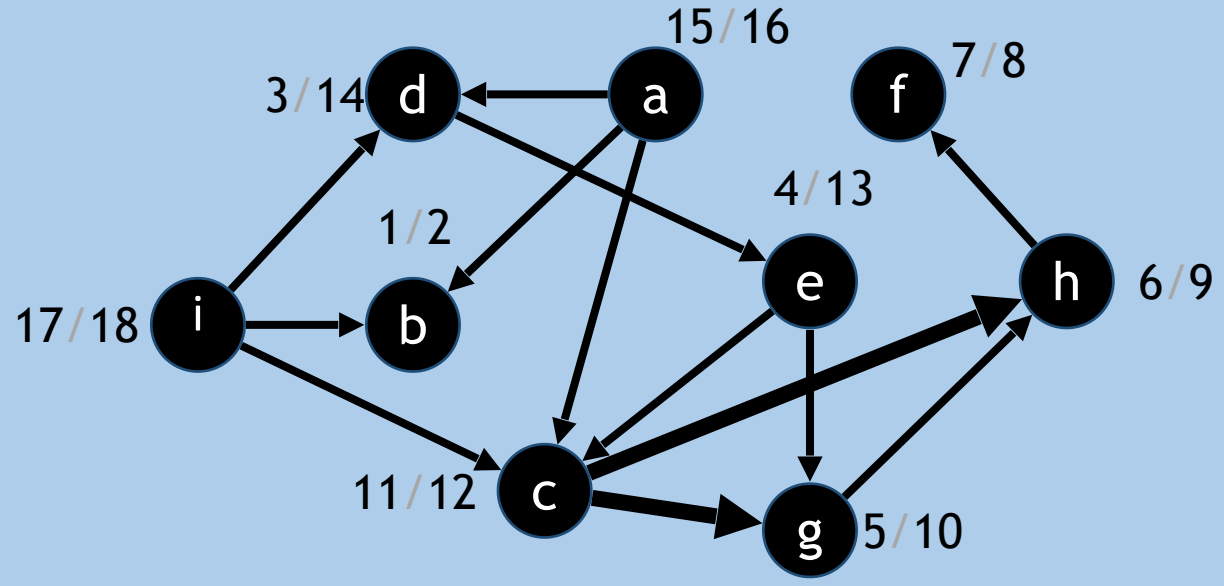

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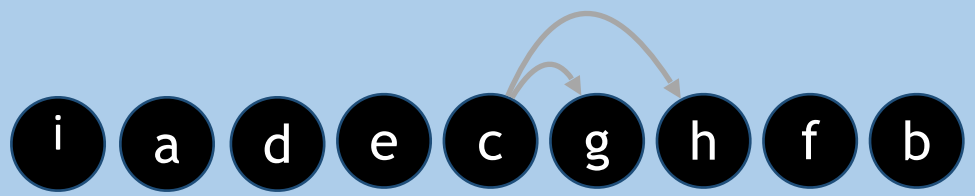

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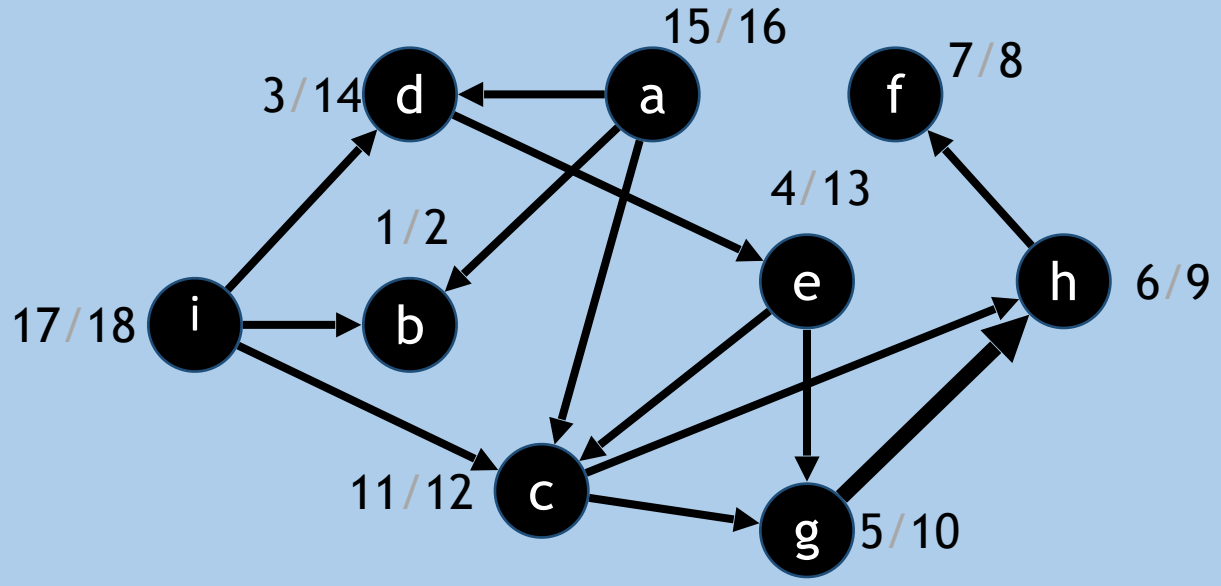

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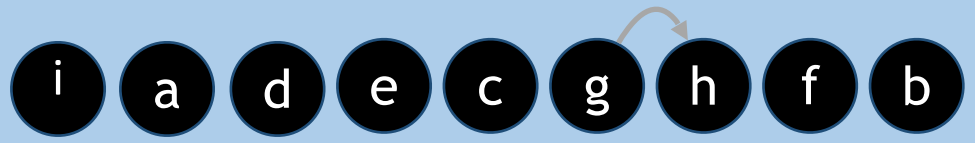

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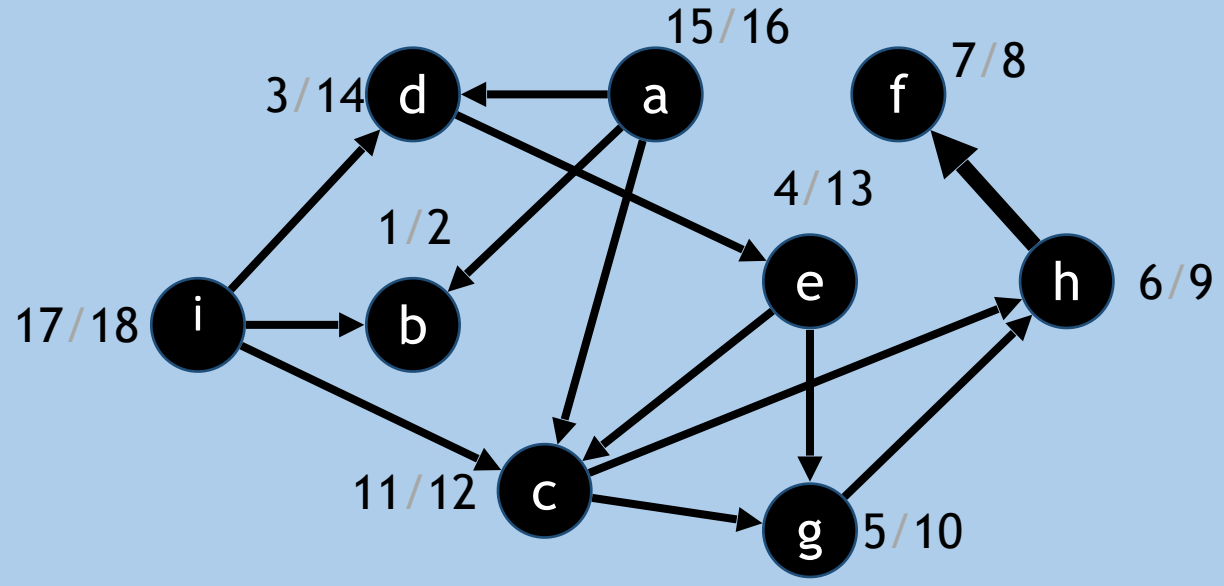

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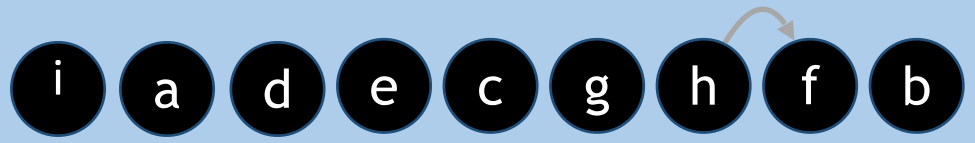

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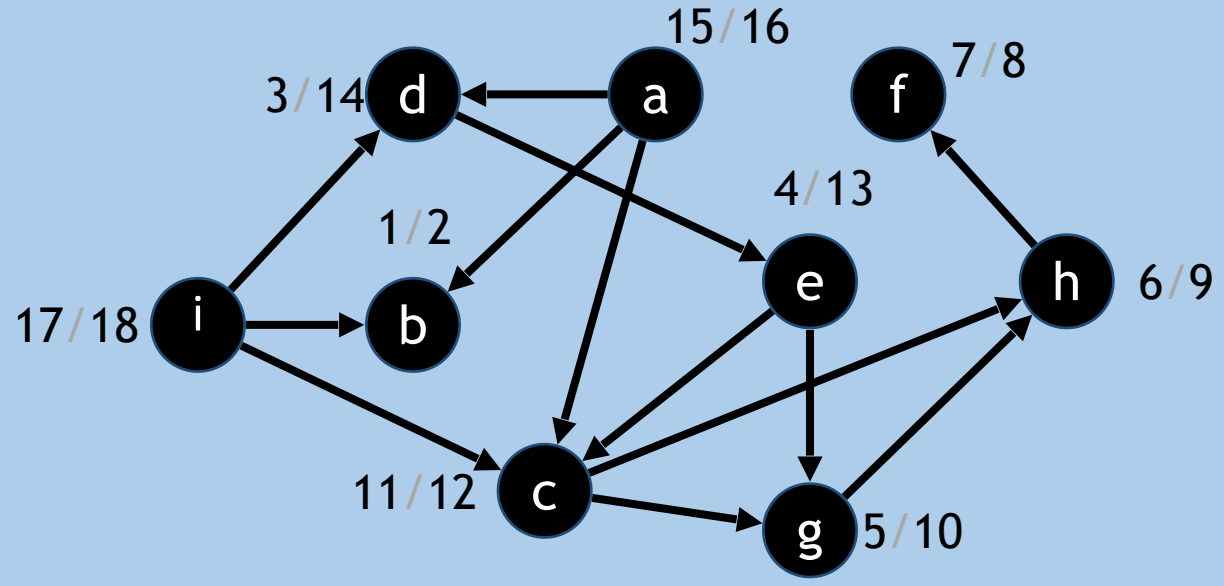

L_{sort}

Topological Sort Performance

```
Procedure Topological-Sort(G)
   Linked list L_sort ← NIL
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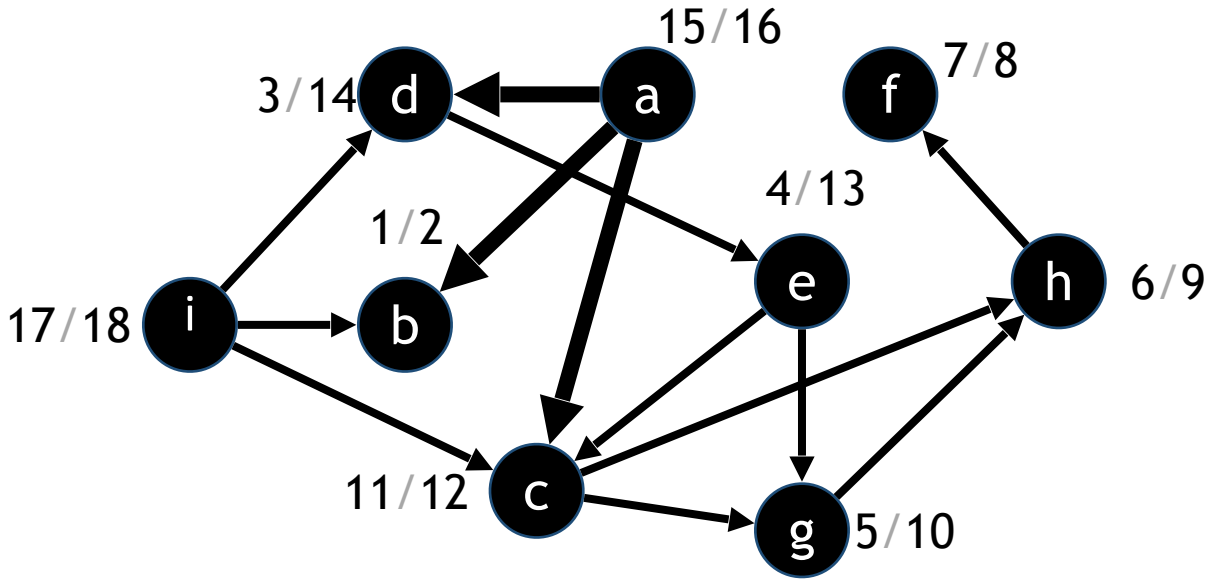
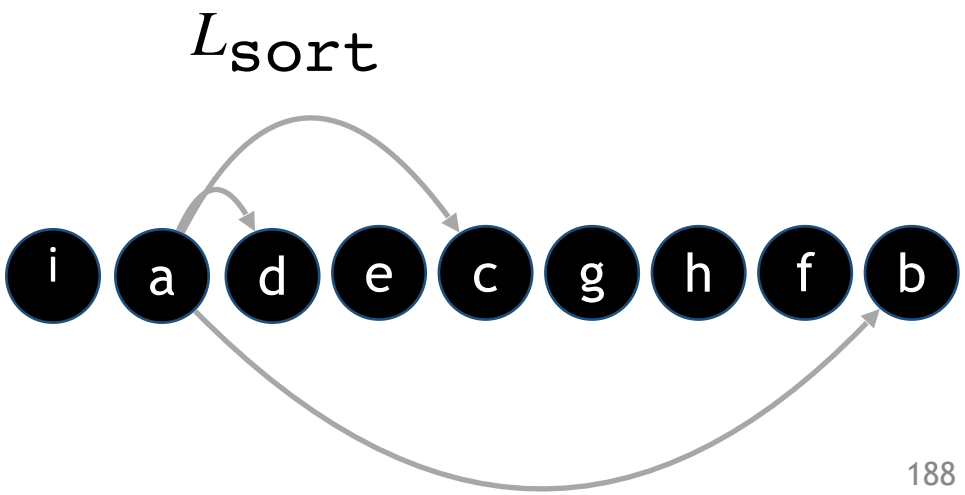
```

- DFS: $O(|V| + |E|)$ time
  - We put the v to L_{sort} once it is marked as finished (no need to sort the vertices by their finish time): total $O(|V|)$ time

Topological Sort Correctness

Theorem: After the procedure **Topological-Sort**, for any two distinct vertices $u, v \in V$, if there is an edge $(u, v) \in E$, in the list L_{sort}, u appears before v.





Topological Sort Correctness

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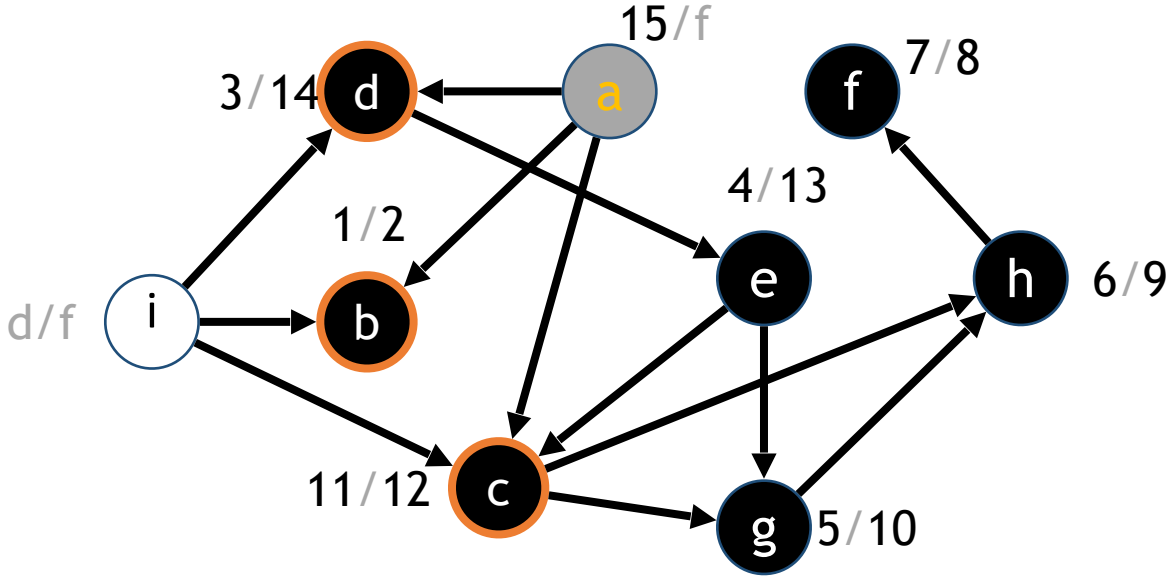
<Proof Idea>: when a vertex is finished, all its children have been marked as finished $\Rightarrow$ all the children appears after it in L_{sort}



L_{sort}

Proof: exercise

What's Happening

- Given a directed acyclic graph, we can find a linear ordering of the vertices by performing a DFS and sort the vertices backwards according to their finish time.

- You don't really need a sorting. The time complexity is $O(|V| + |E|)$

Outline

- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph traversal
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
- Two applications on DFS:
  - Topological sort
  - Strongly connected components

Strongly Connected Component

Definition: A **strongly connected component** of a directed graph $G = (V, E)$ is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u and v in C, there is a path from u to v and a path from v to u. That is, u and v are reachable from each other.

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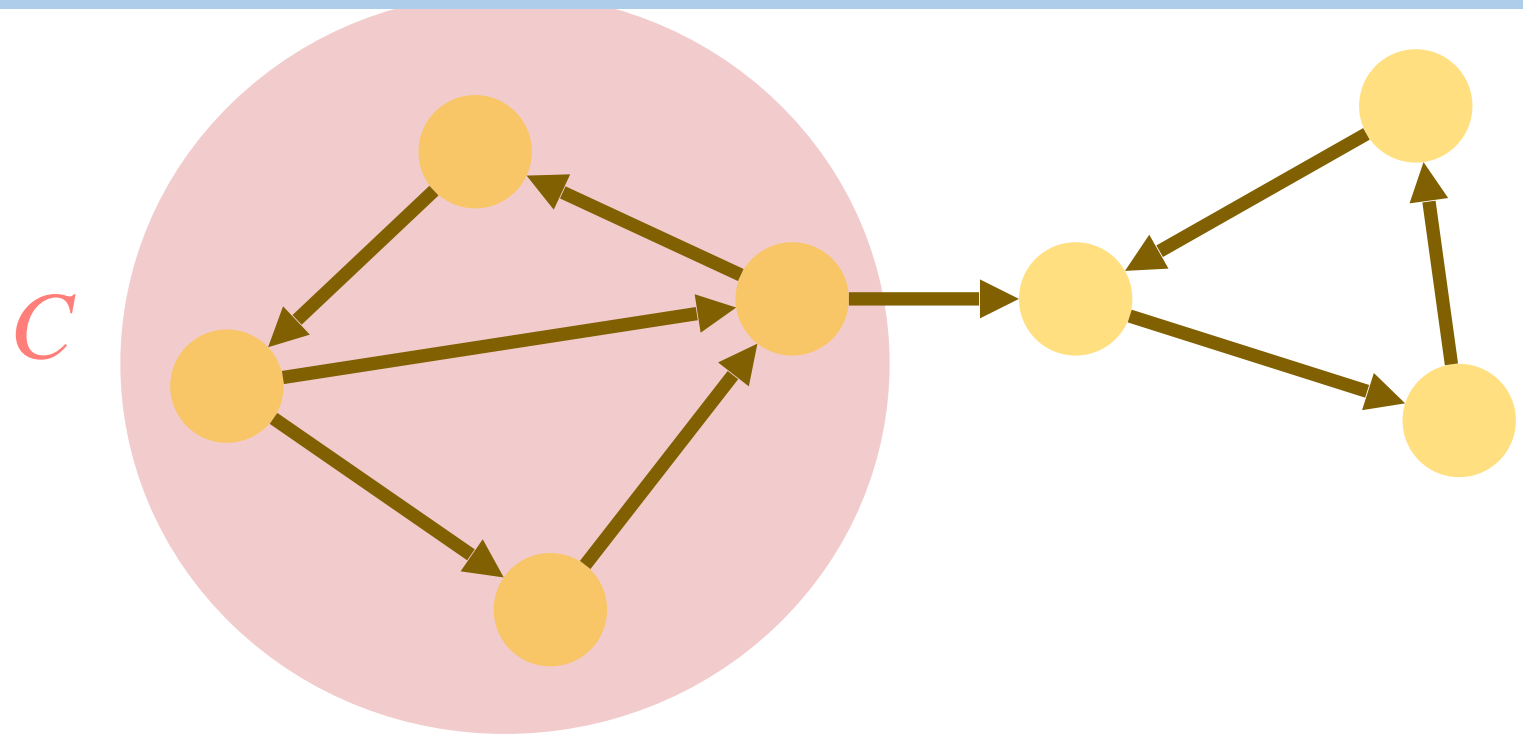

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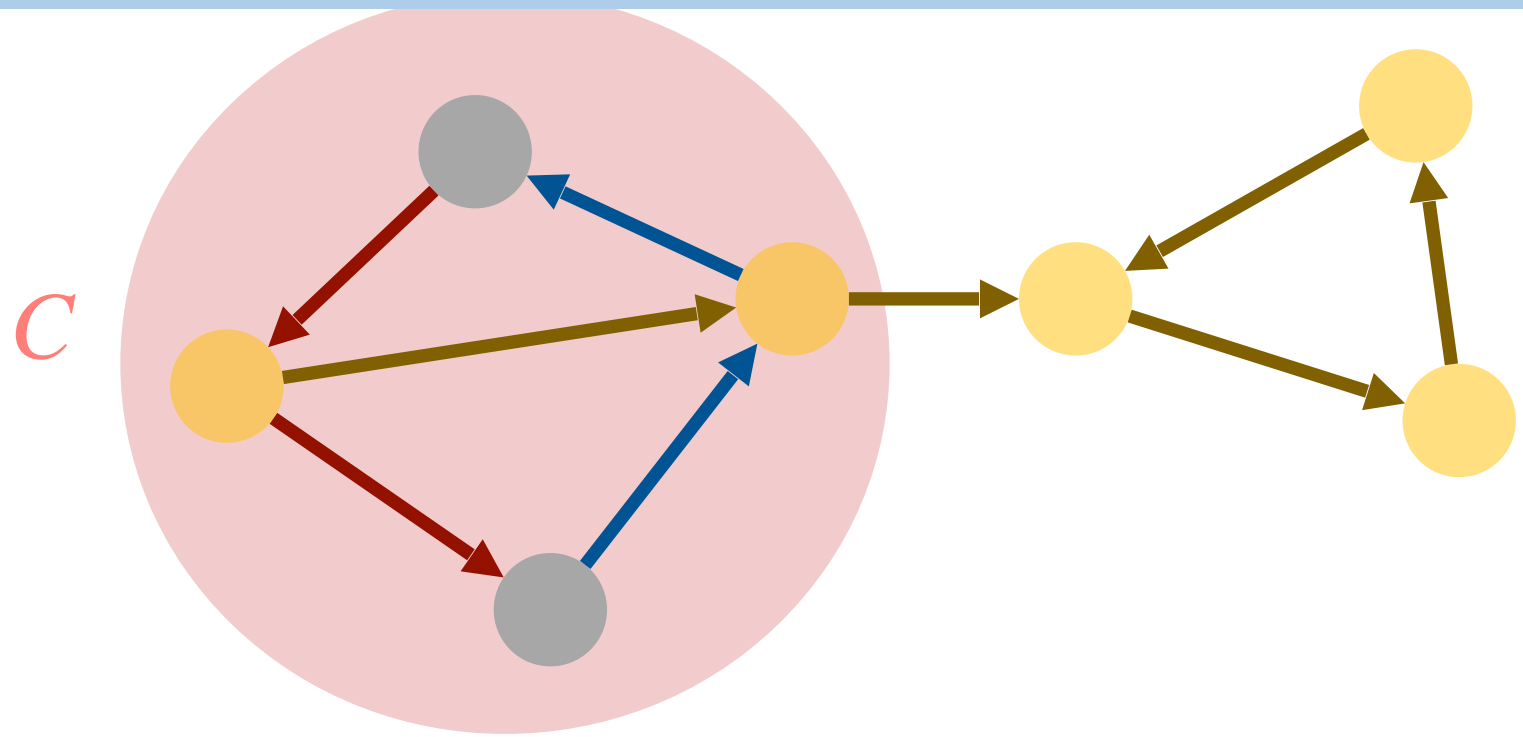

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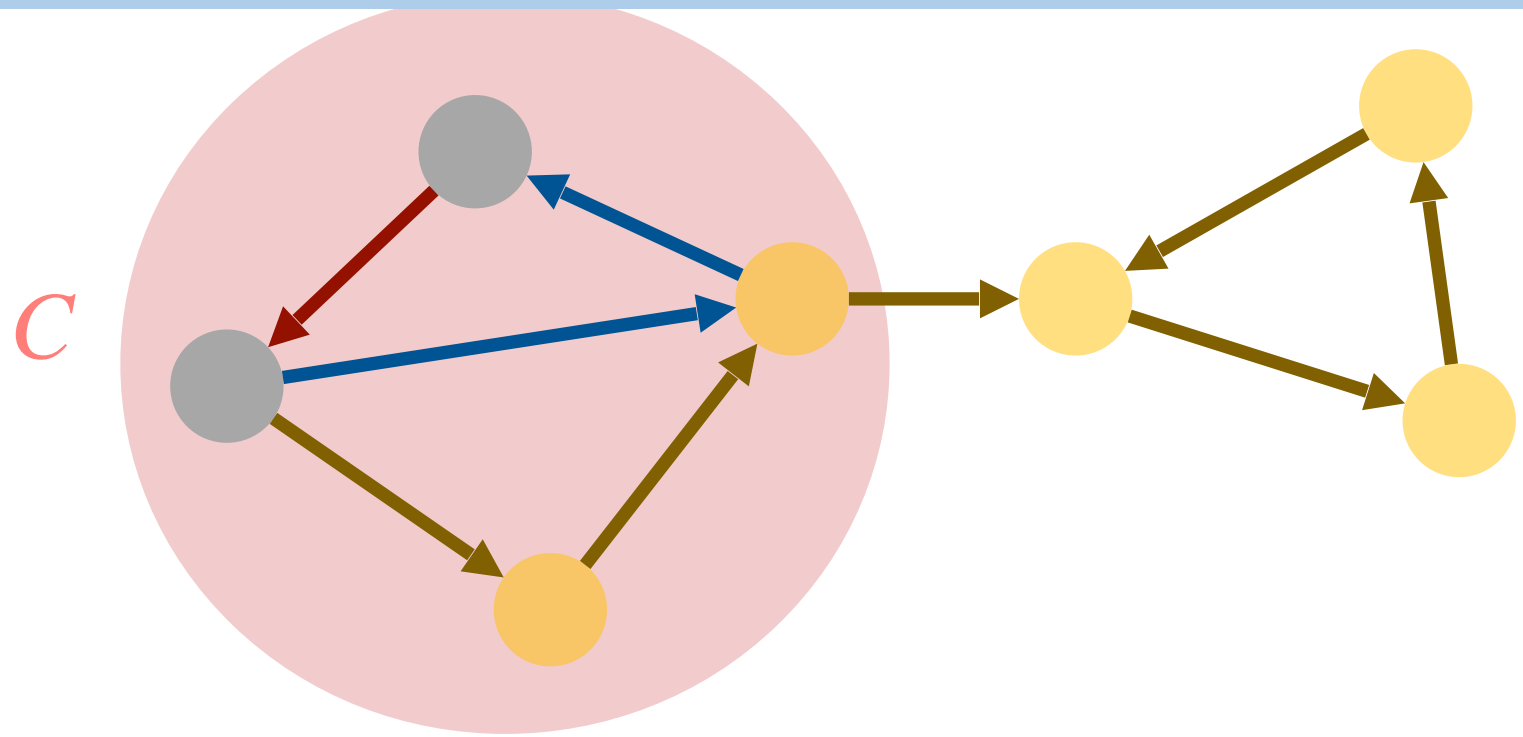

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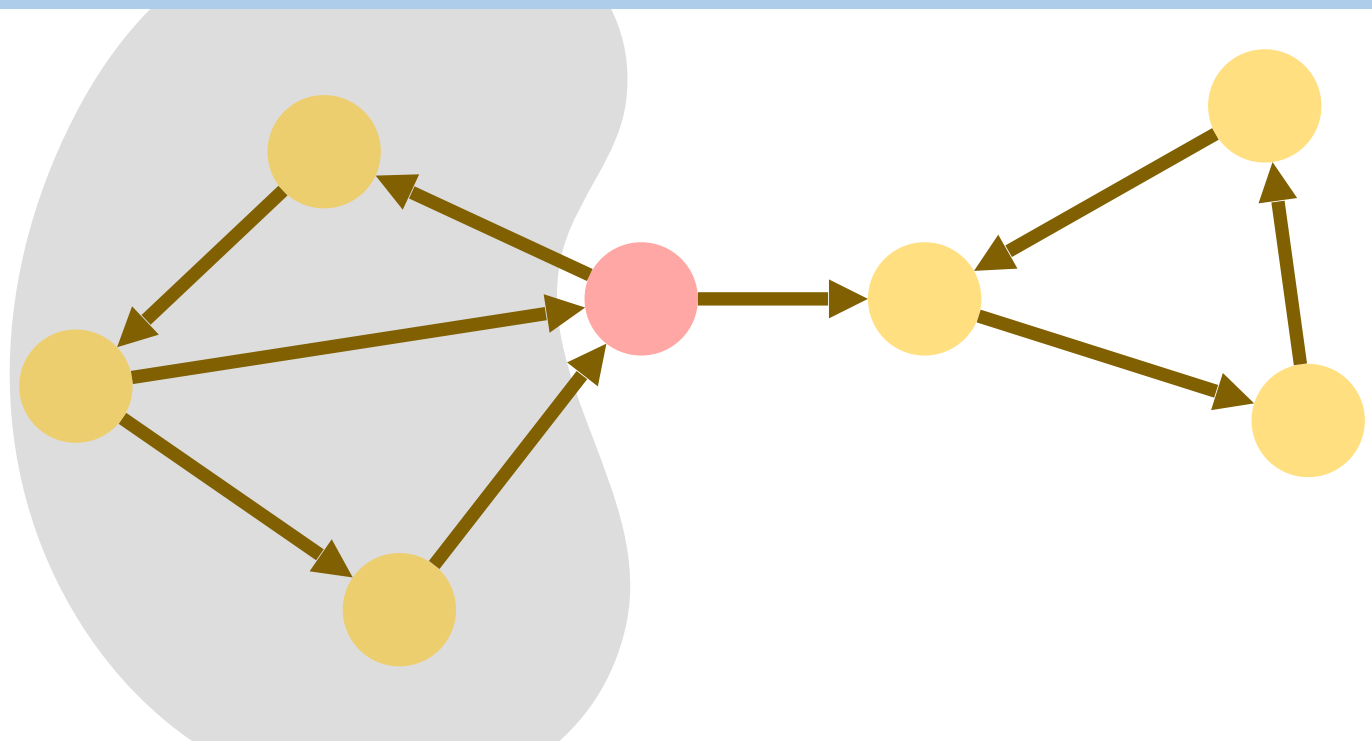
Not maximal



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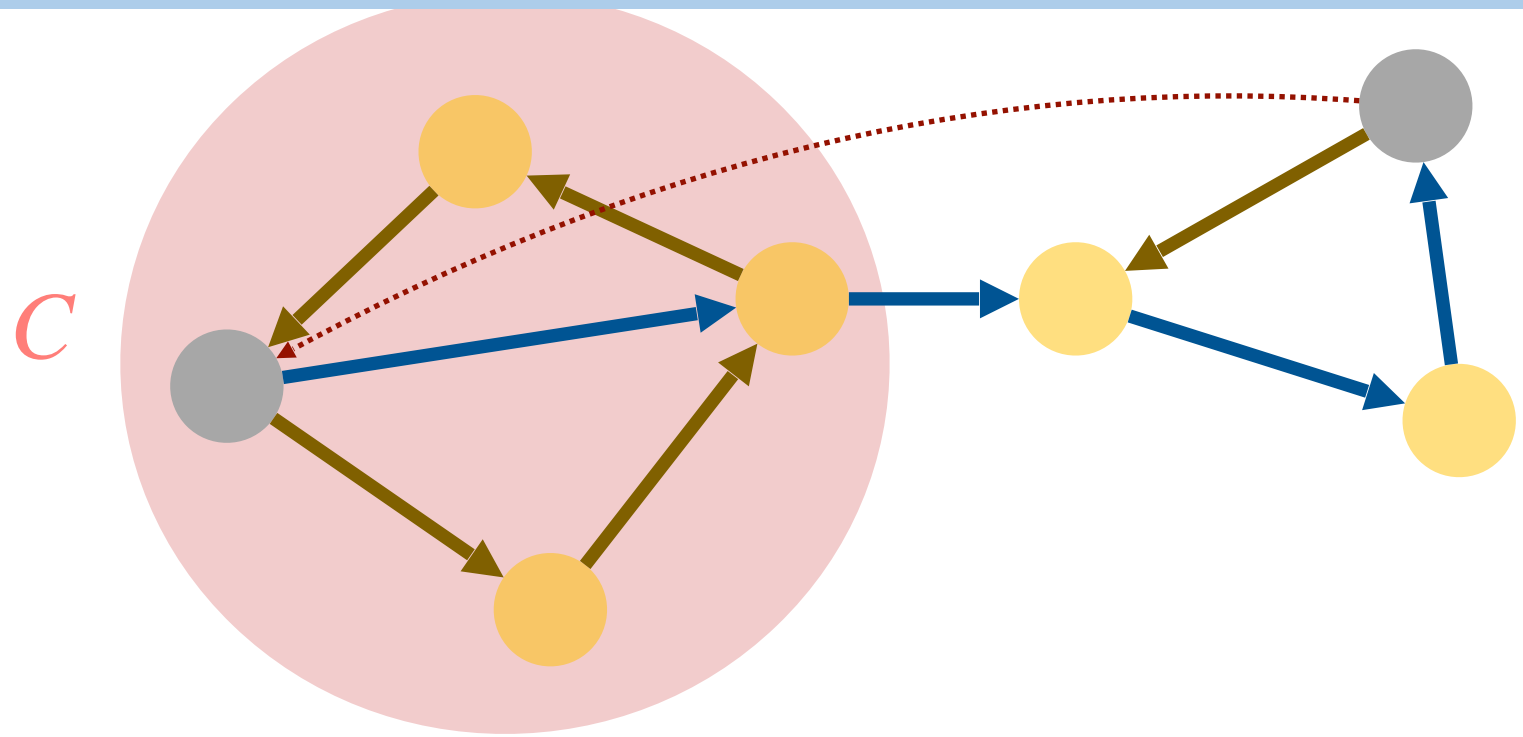

Find Strongly Connected Component

Find Strongly Connected Component

A **transpose** of a directed graph $G = (V, E)$ is a directed graph $G^T = (V, E^T)$, where $E^T = \{(u, v) \mid (v, u) \in E\}$

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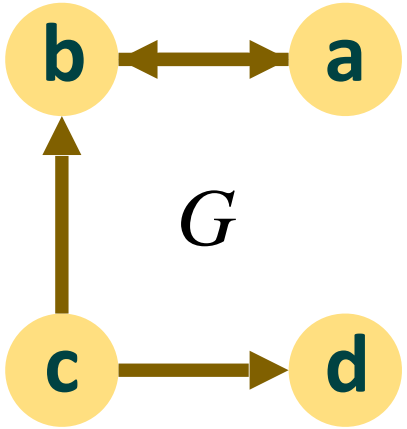

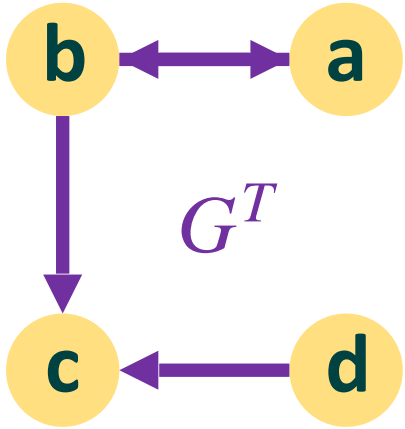

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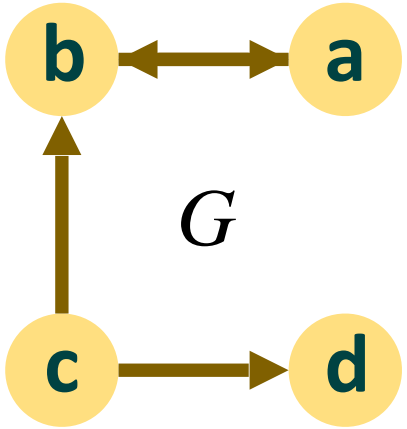

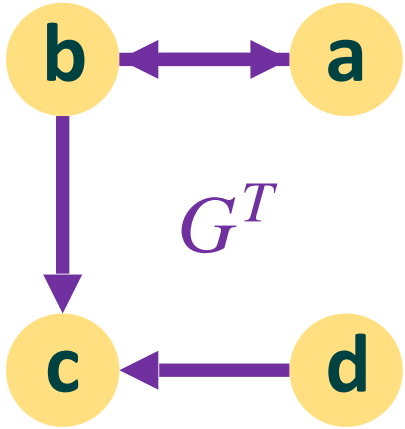

- G^T can be constructed in $O(|V| + |E|)$ time

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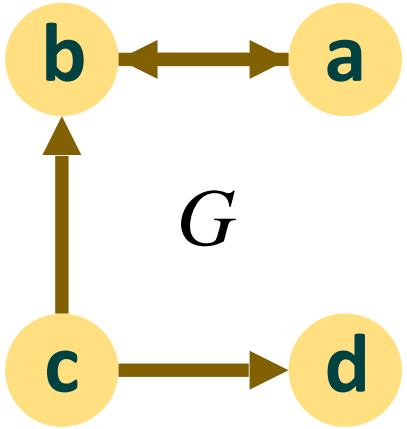

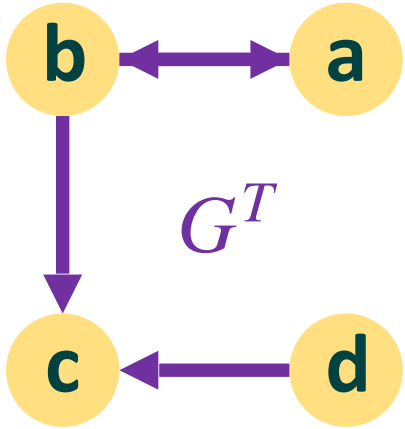

- If there is a path from u to v in G, there is a path from v to u in G^T
- For two vertices u and v in a strongly connected component, there is a path from u to v, and a path from v to u

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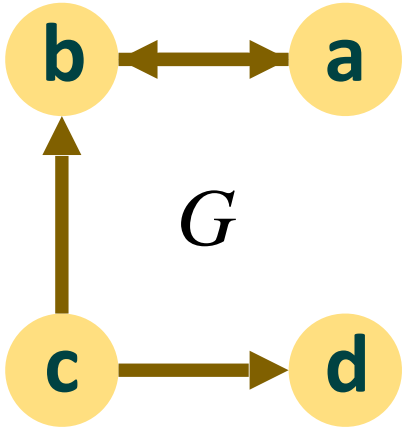

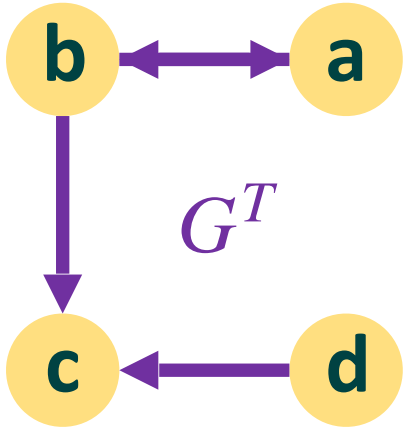

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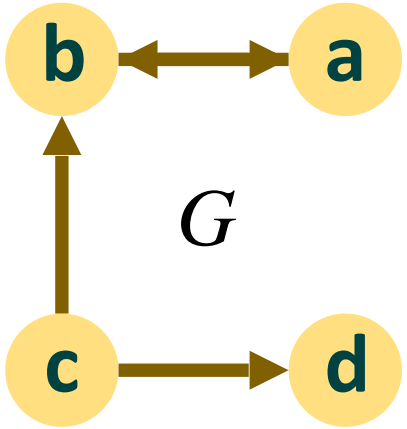
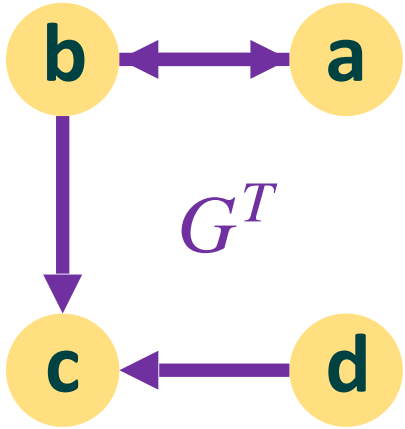
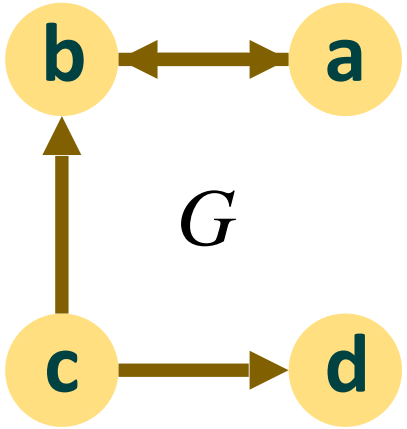


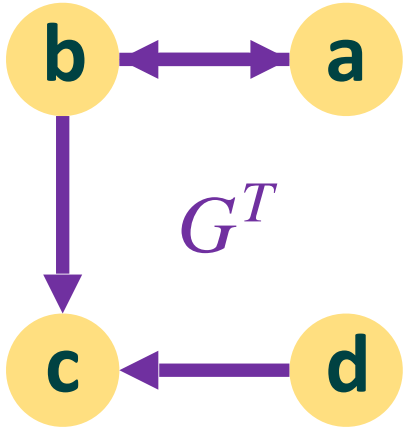


- Maybe we can run a DFS on G, and another DFS on G^T to get the reachable sets of vertices in both graphs and find their intersection?

Find Strongly Connected Component

A **transpose** of a directed graph $G = (V, E)$ is a directed graph $G^T = (V, E^T)$, where $E^T = \{(u, v) \mid (v, u) \in E\}$





- Maybe we can run a DFS on G, and another DFS on G^T to get the reachable sets of vertices in both graphs and find their intersection?
  - Finding the intersections can be tricky…

Strongly Connected Component

```
Procedure STRONGLY-CONNECTED-COMPONENTS(G)
   Call DFS(G) and compute finishing times f[u] for each u
   Compute G^T
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   The depth-first trees in DFS(G^T) are SCCs
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```
Procedure DFS(G)
     For each u ∈ V[G]
          Mark u as not-discovered
          π[v] ← NIL
     time ← 0
     For each vertex u ∈ V[G]
          If u is not-discovered
               DFS-VISIT(u)
```

Strongly Connected Component

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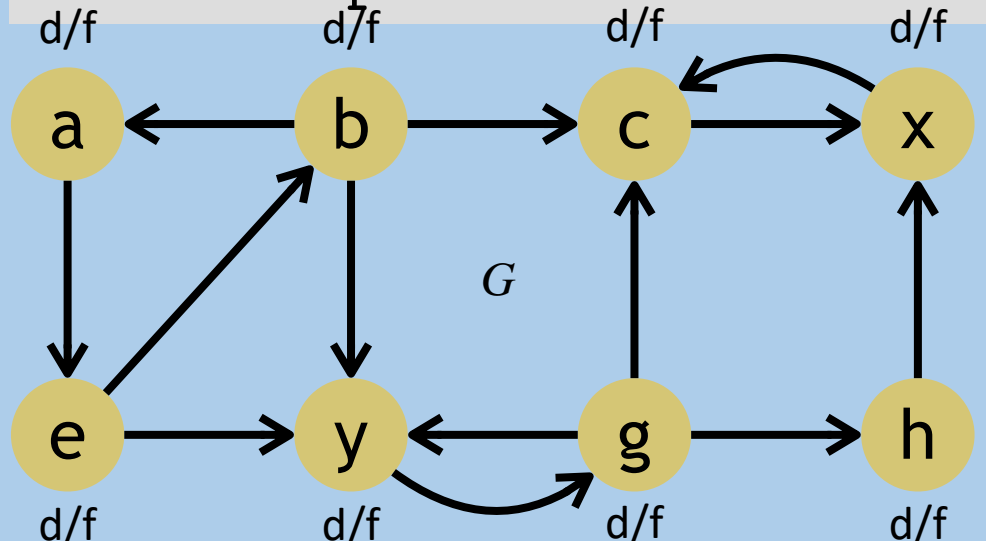

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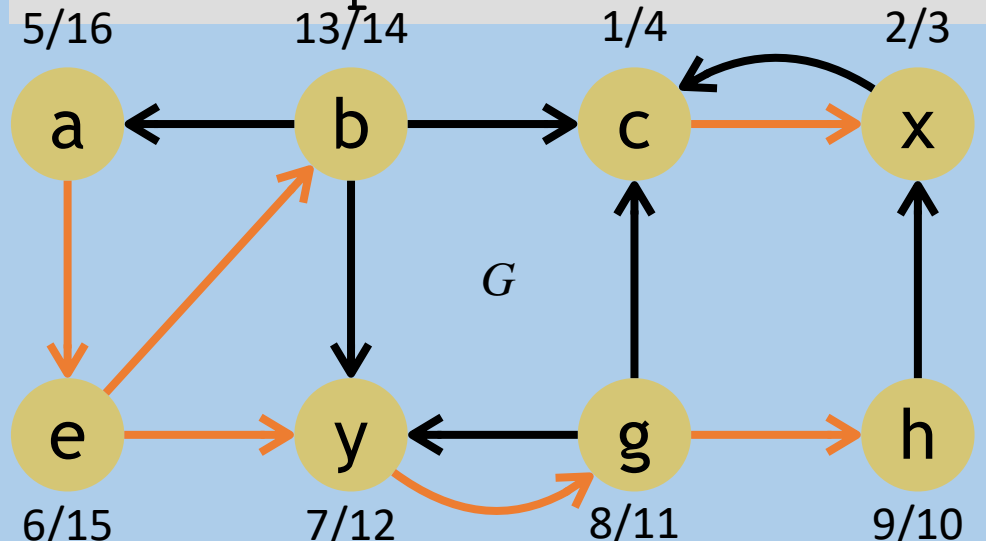

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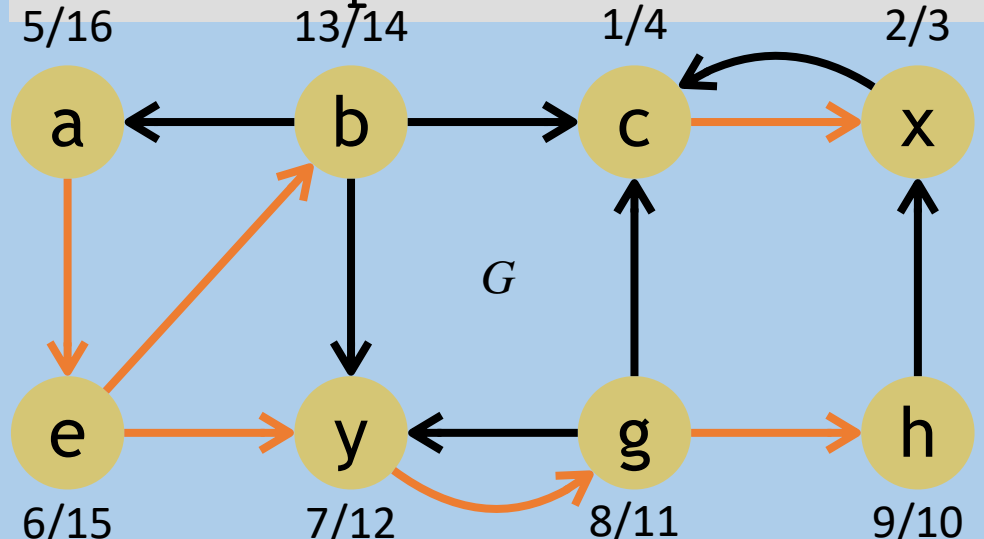

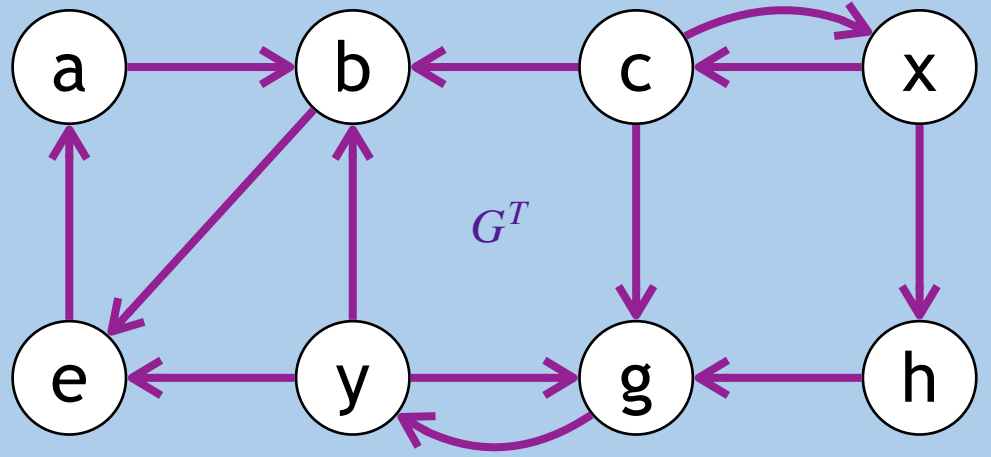

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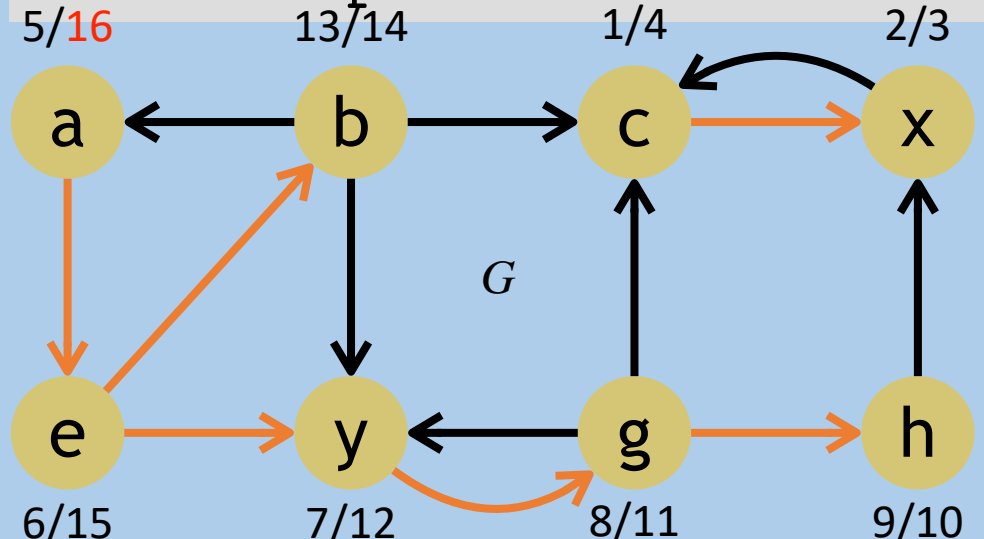

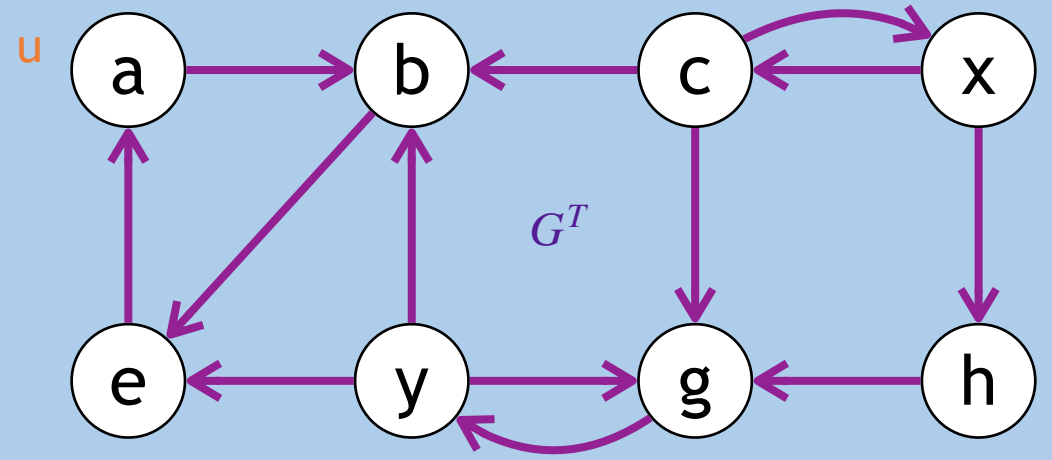

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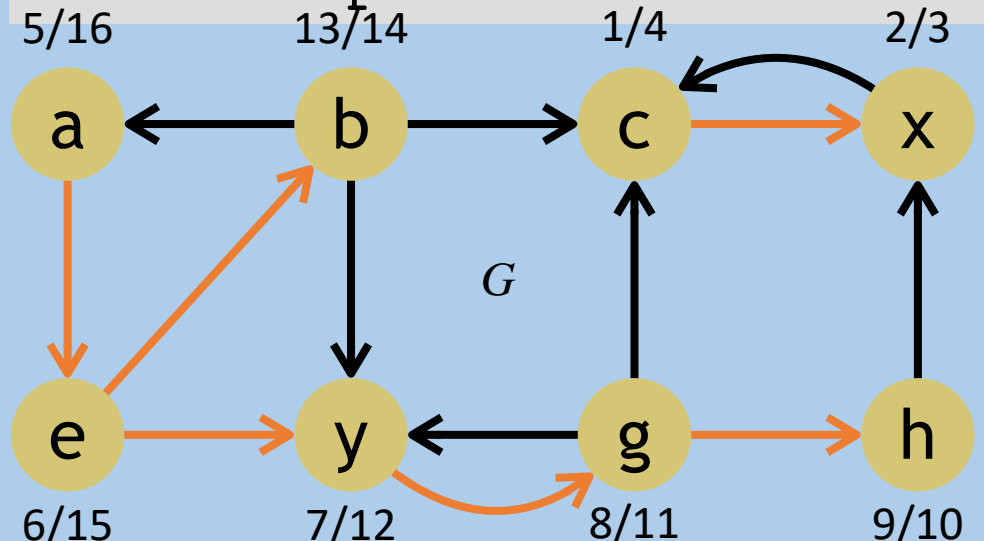

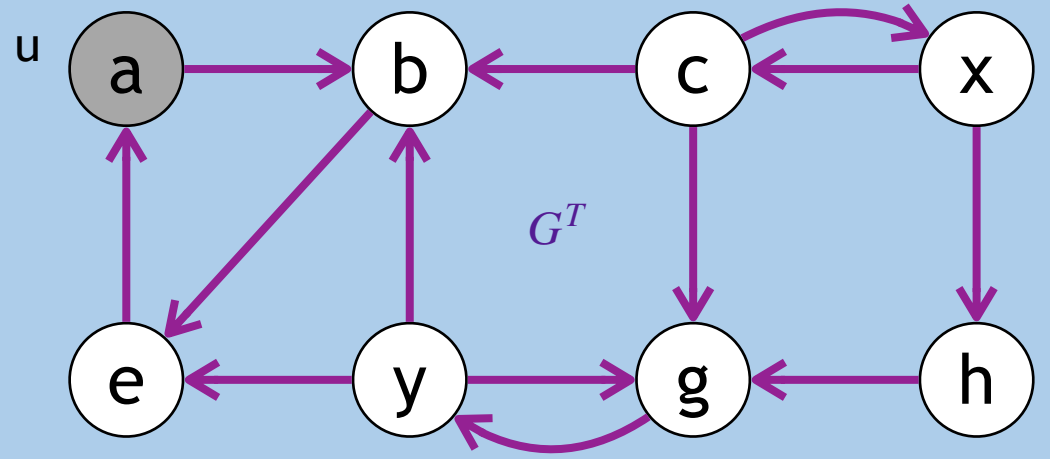

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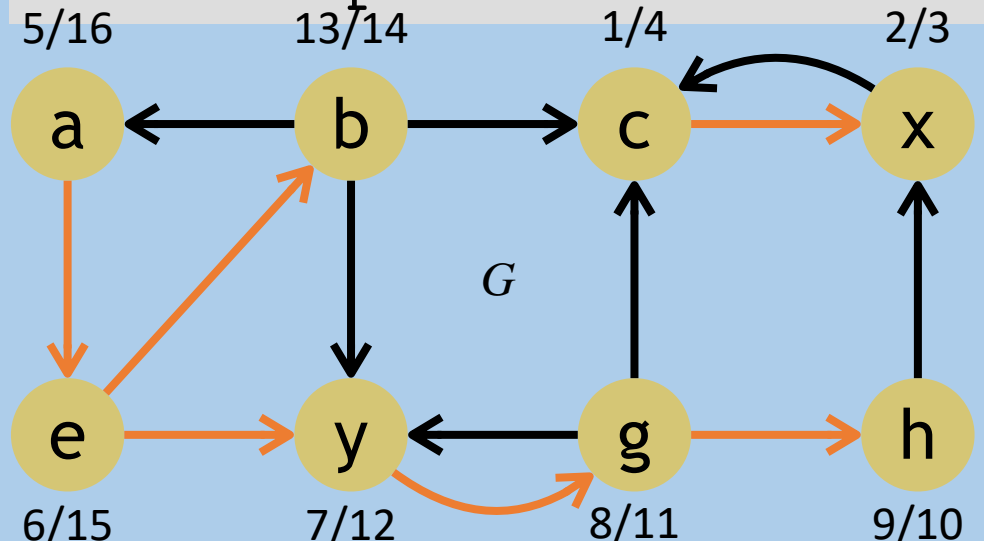

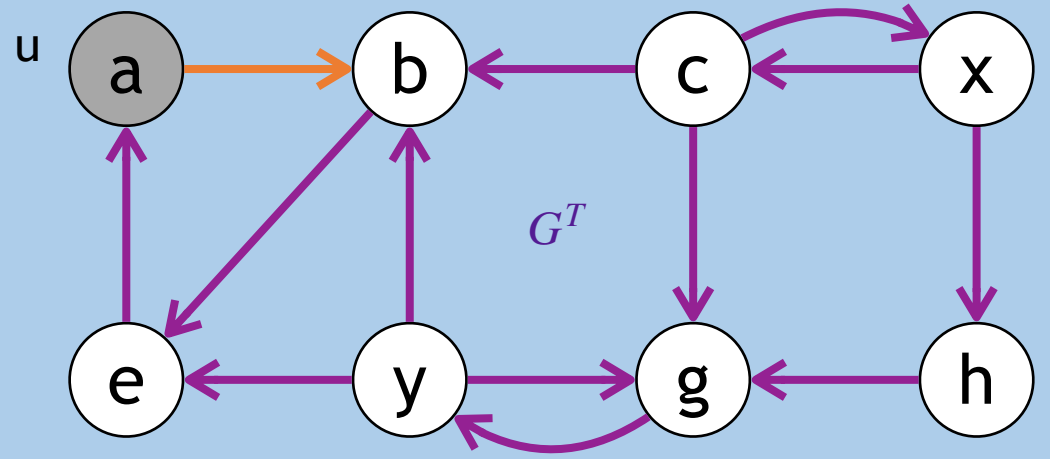

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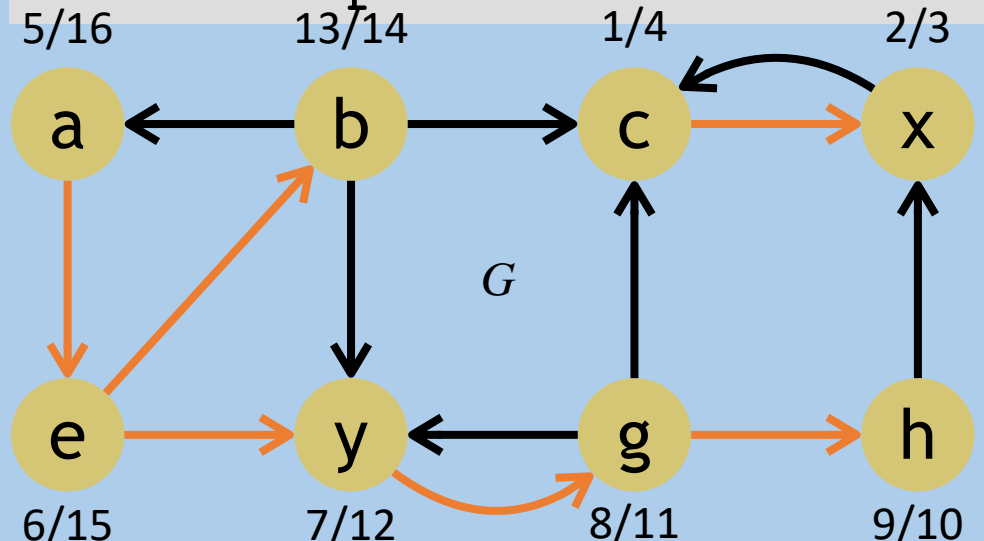

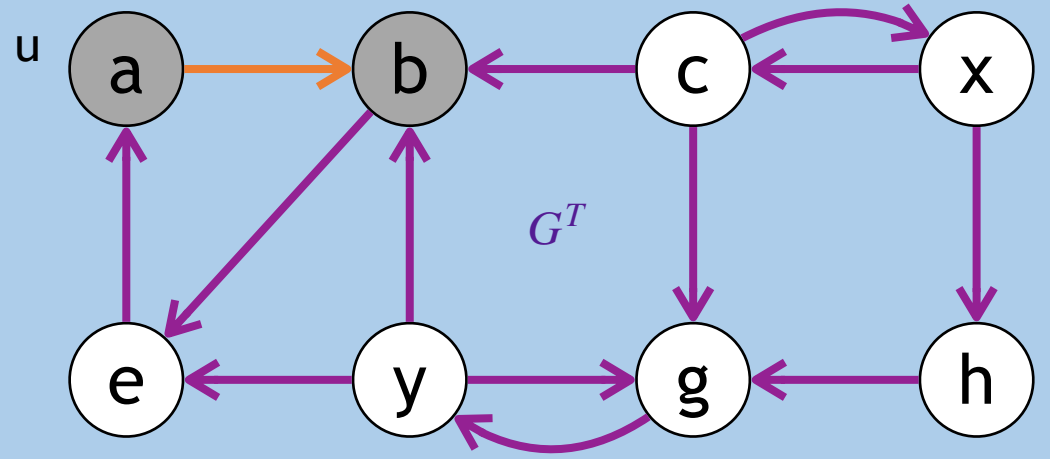

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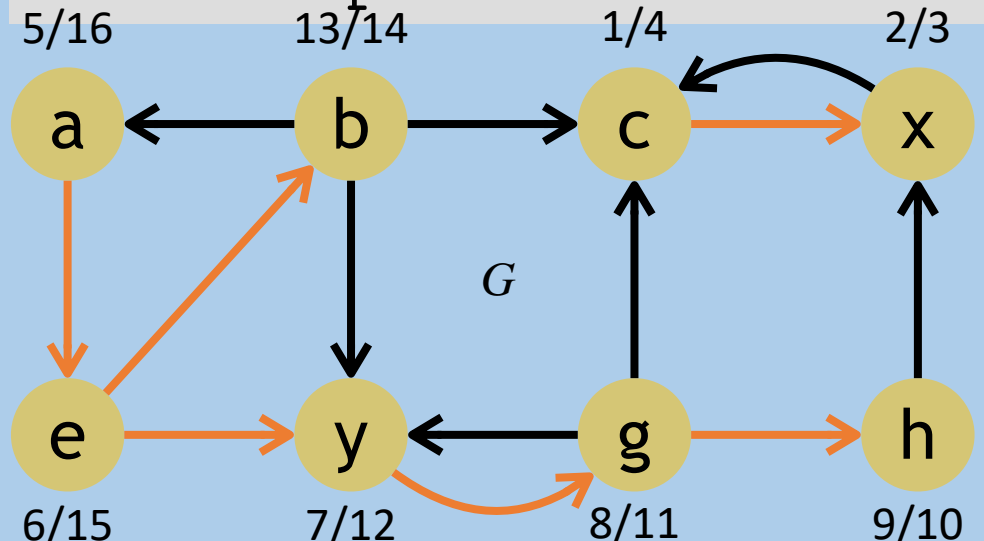

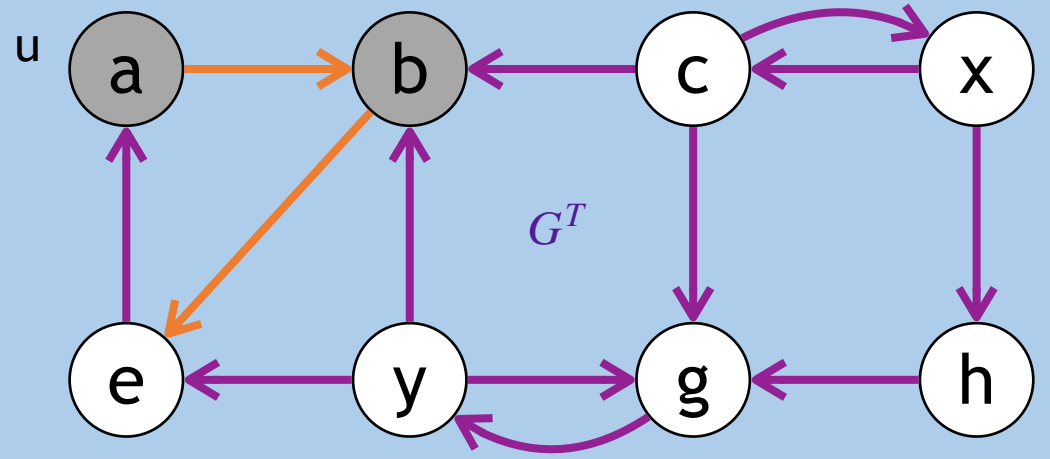

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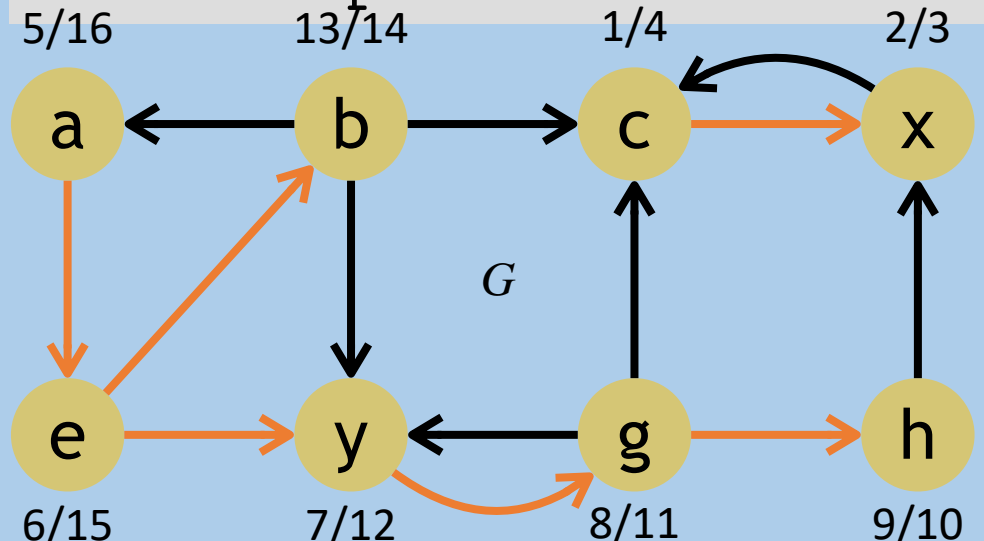

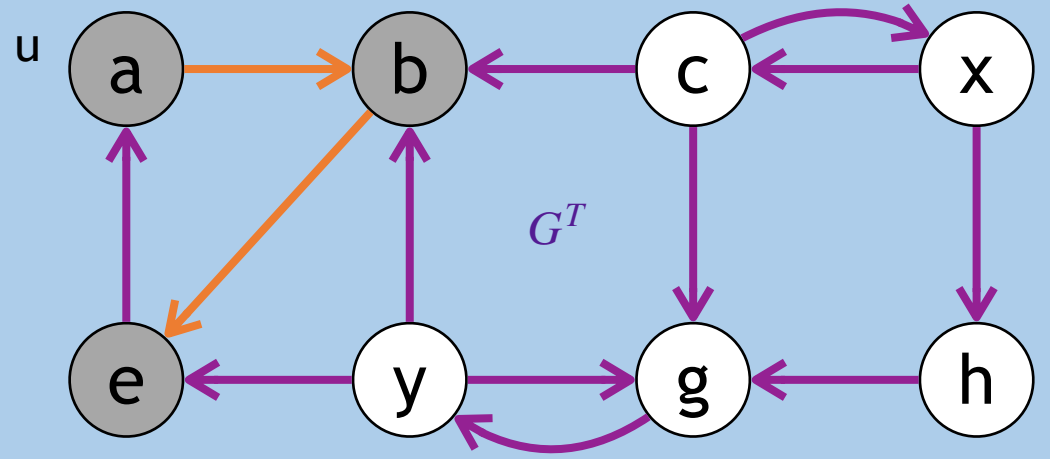

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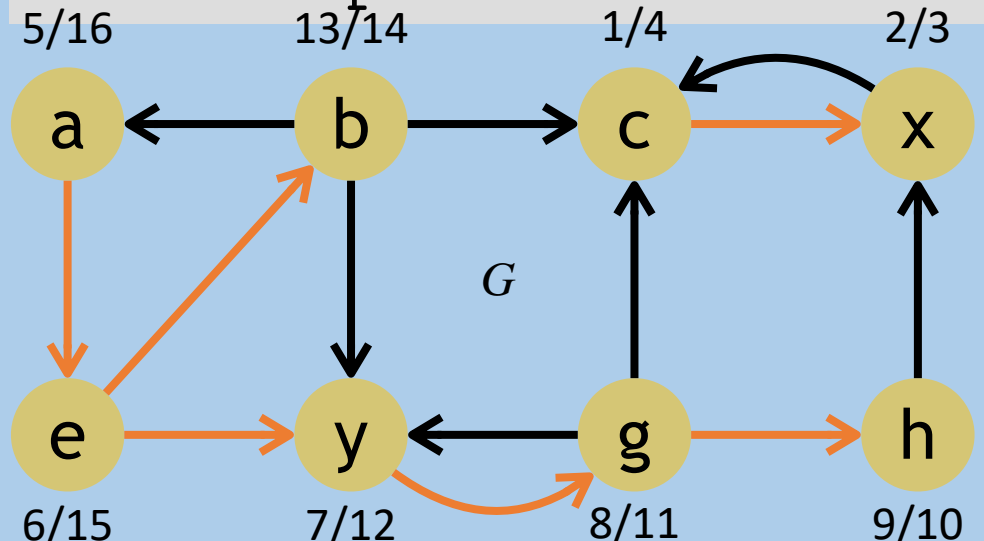

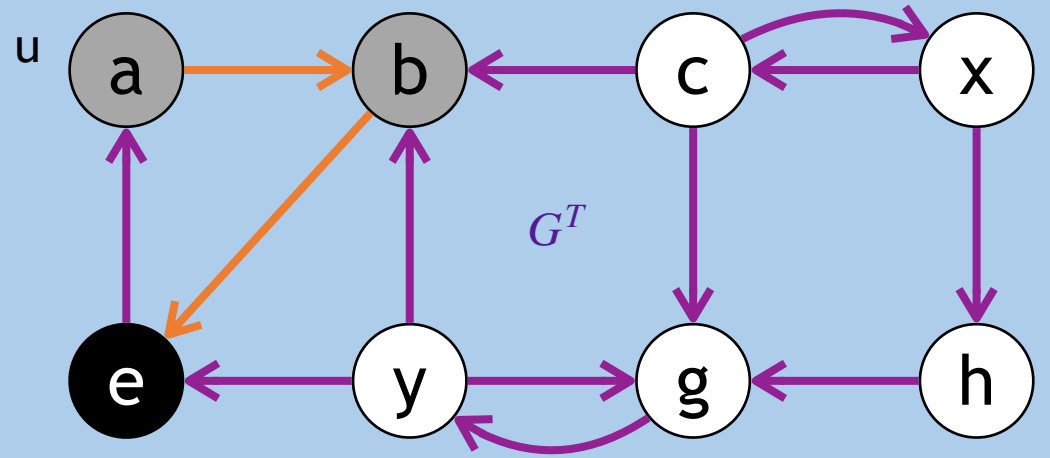

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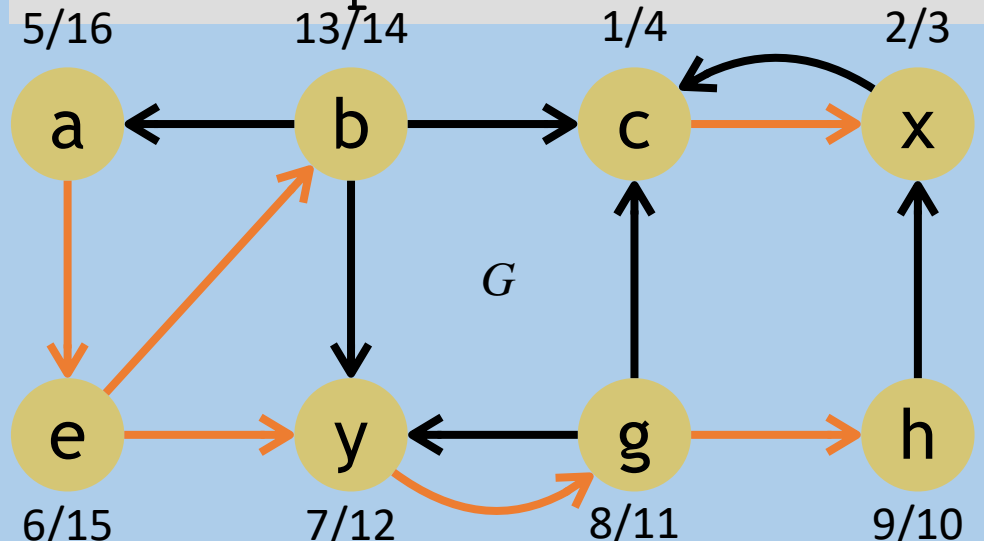

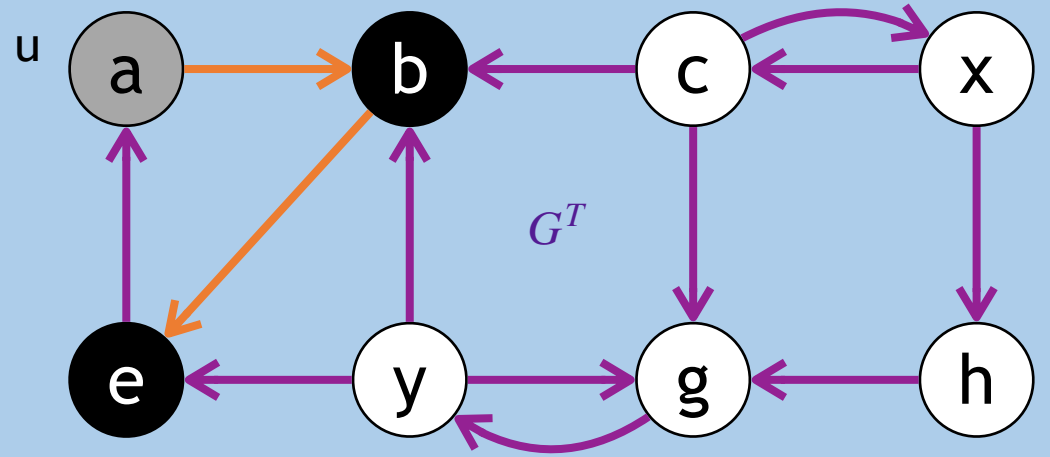

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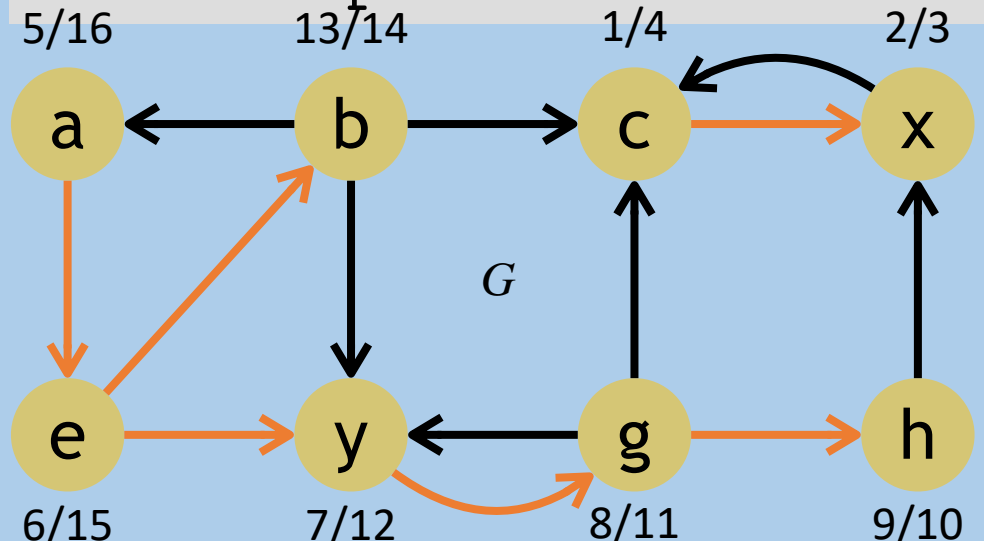

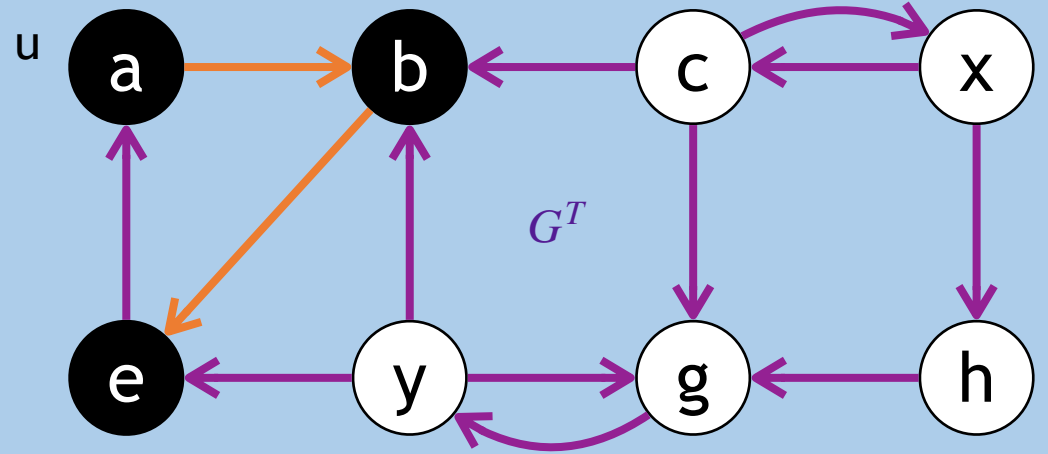

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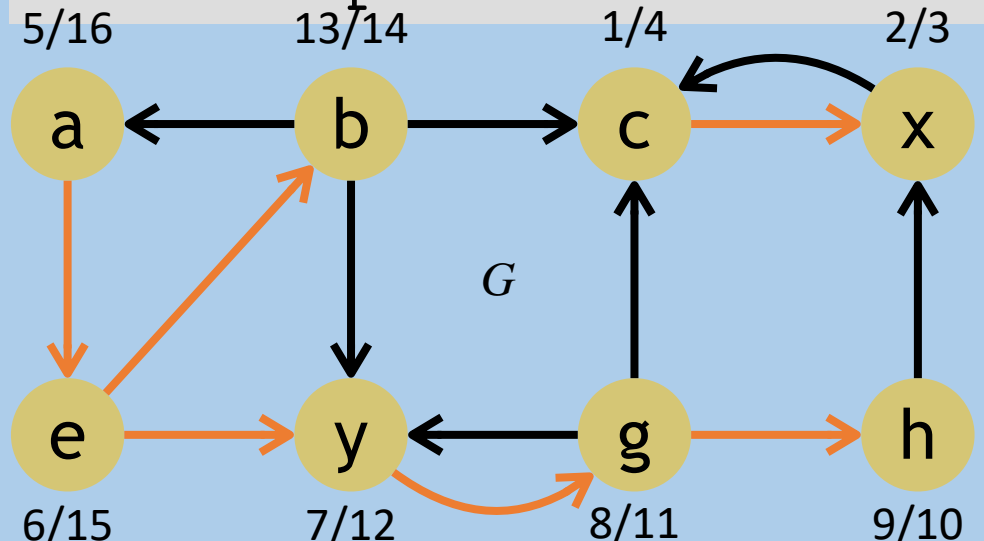

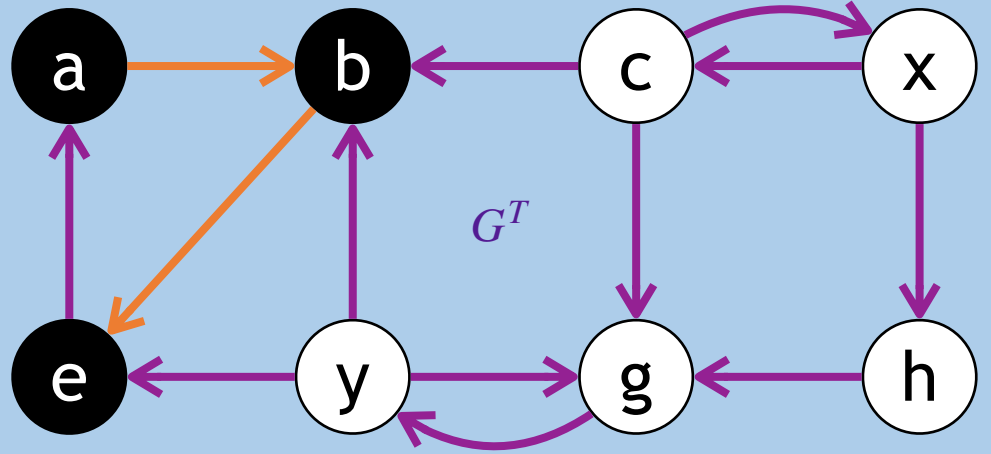

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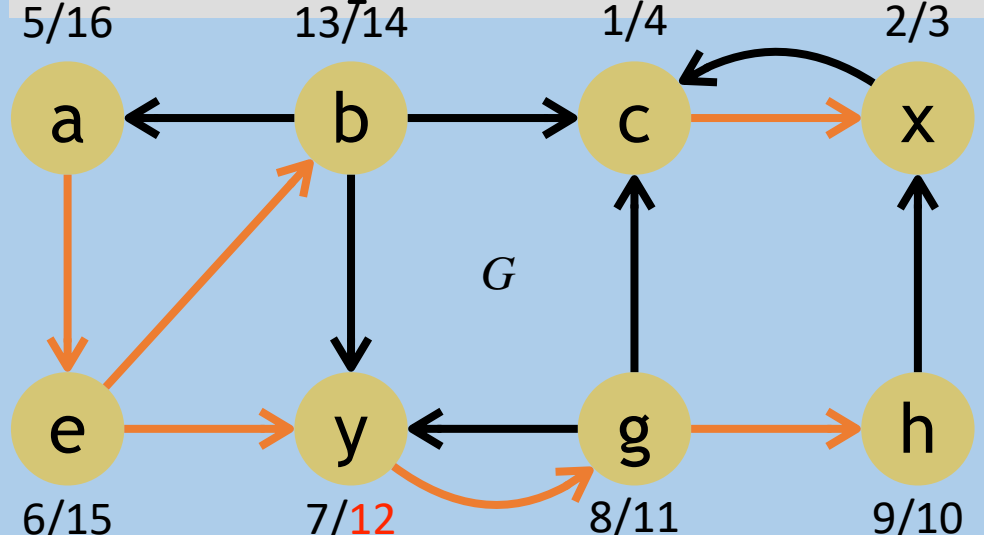

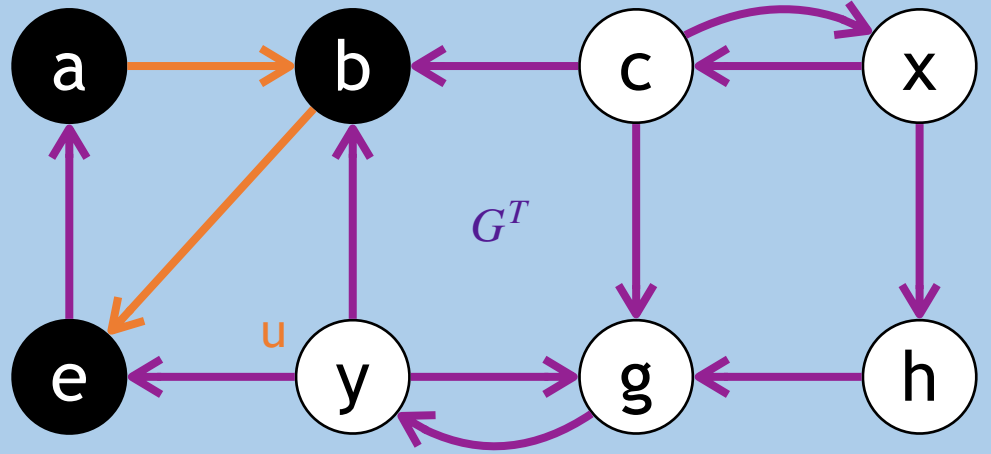

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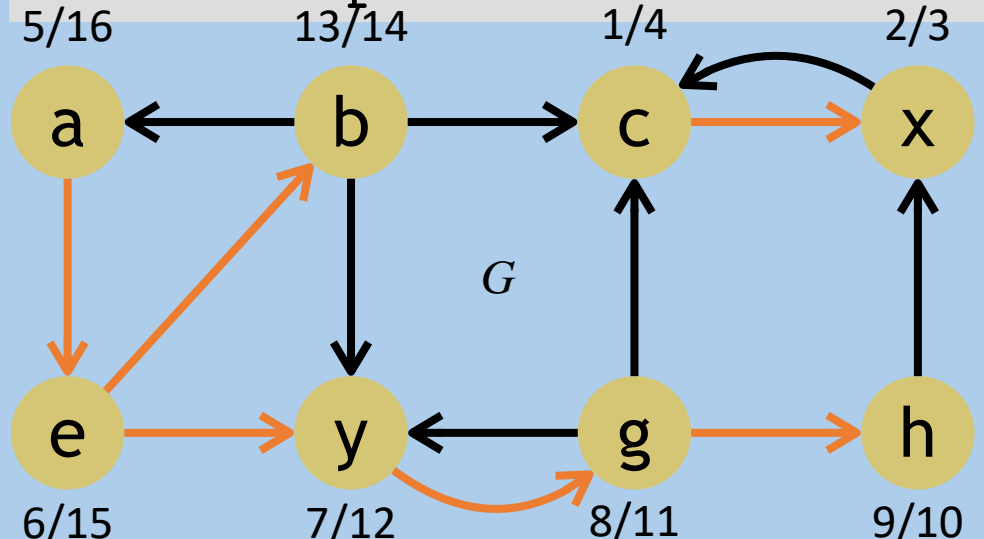

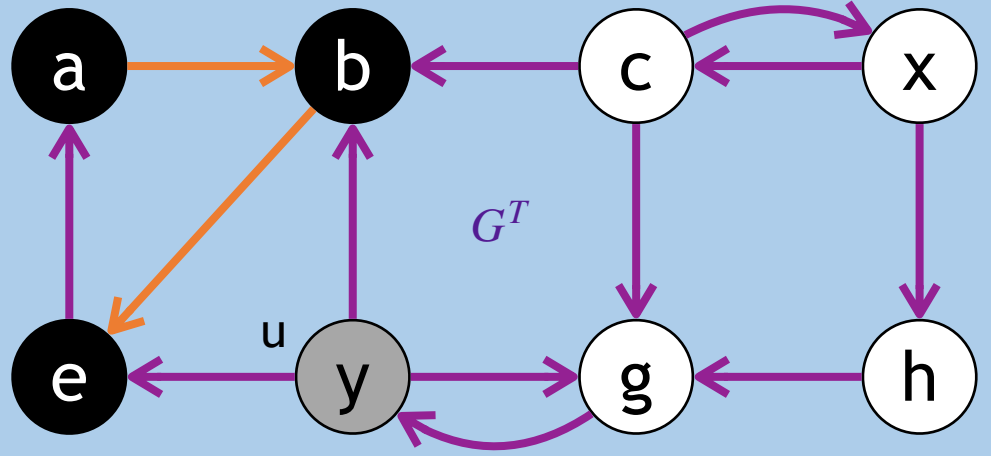

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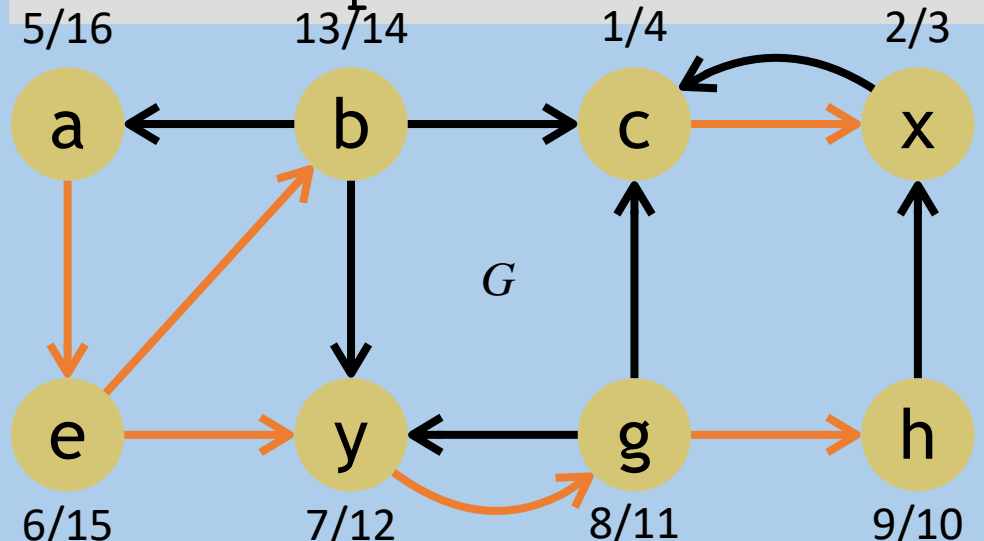

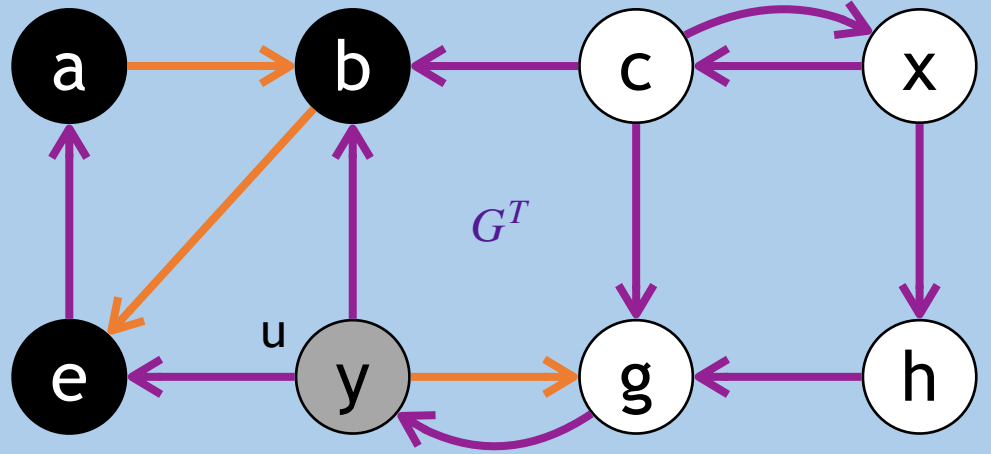

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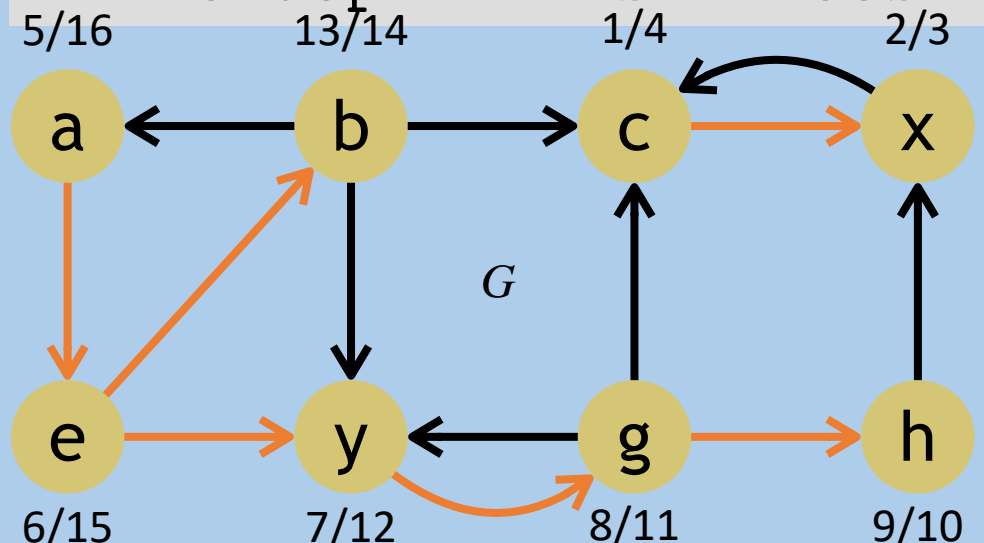

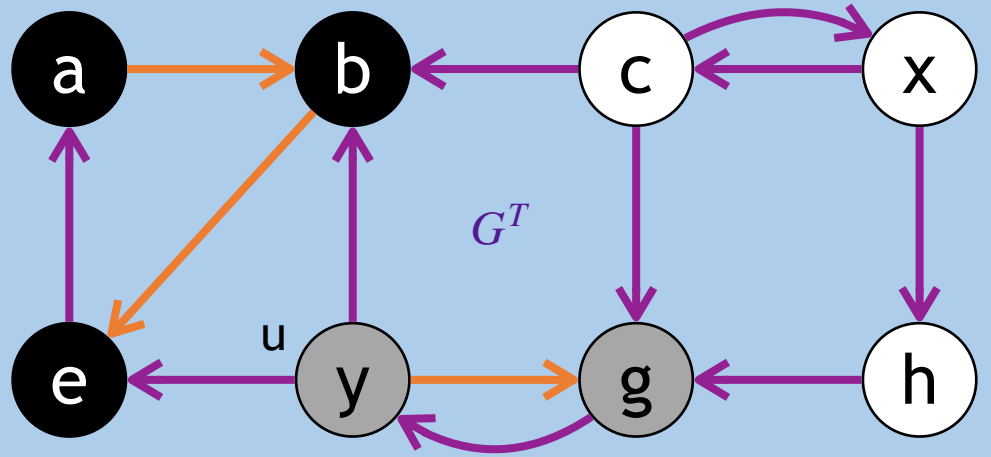

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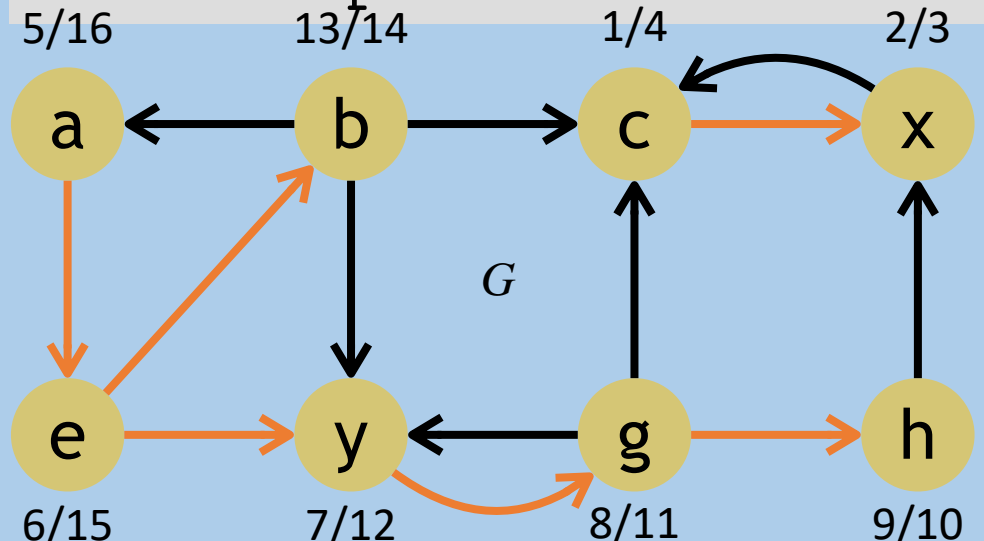

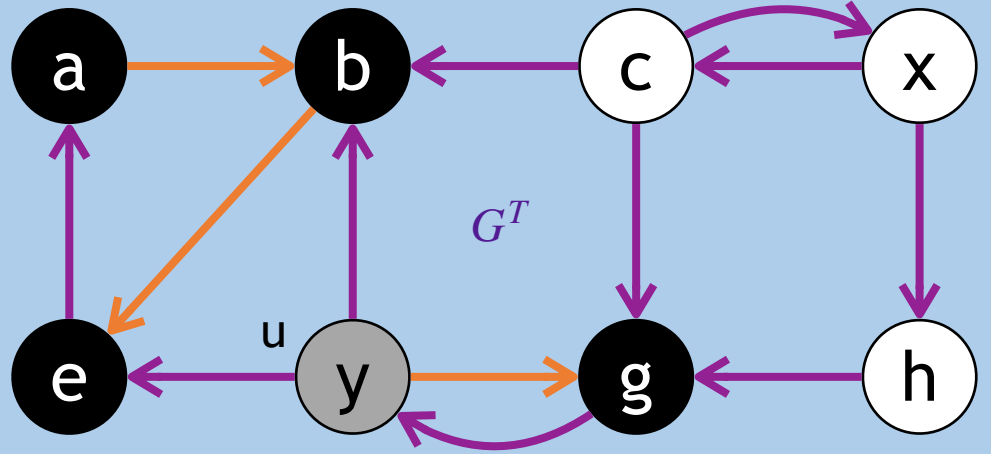

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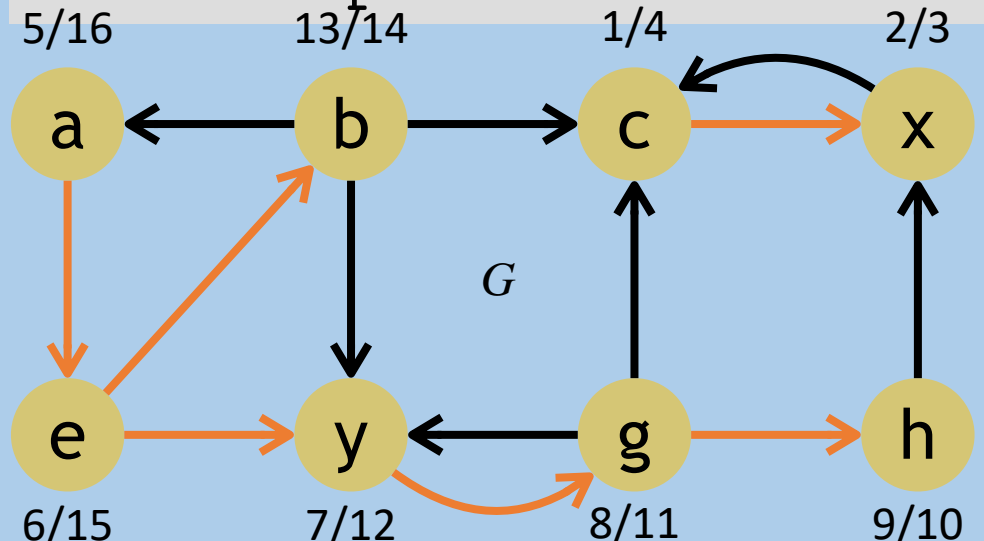

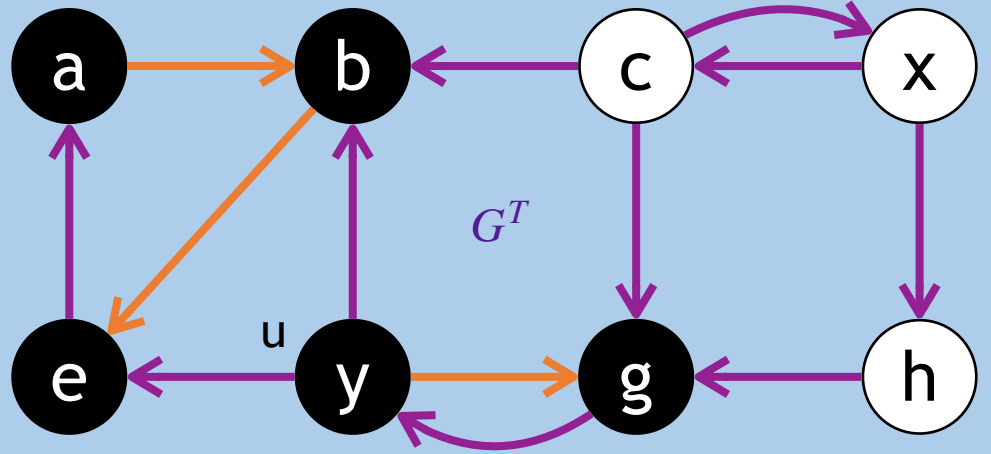

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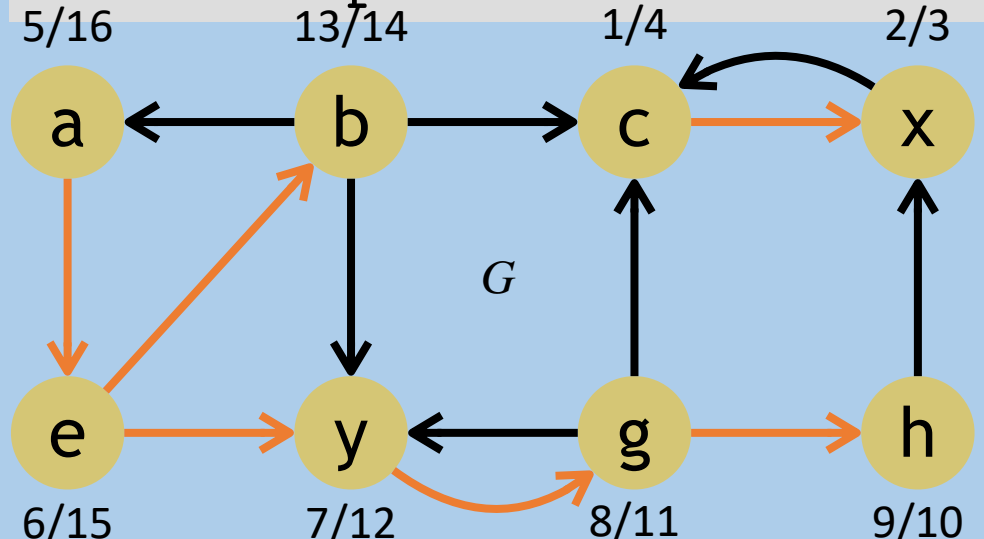

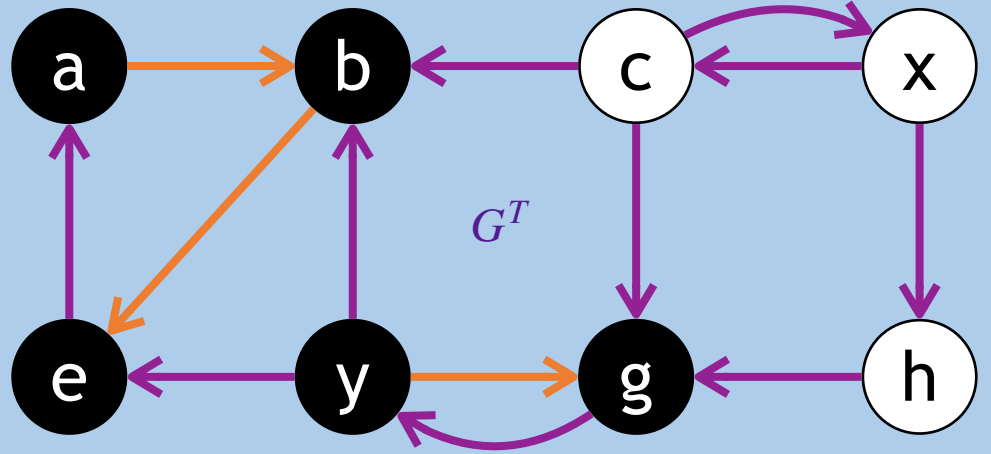

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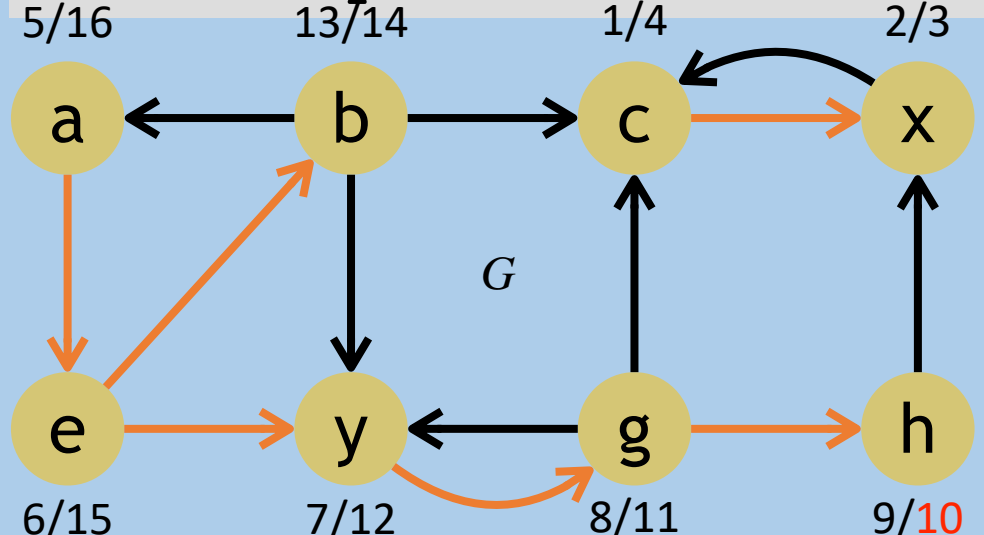

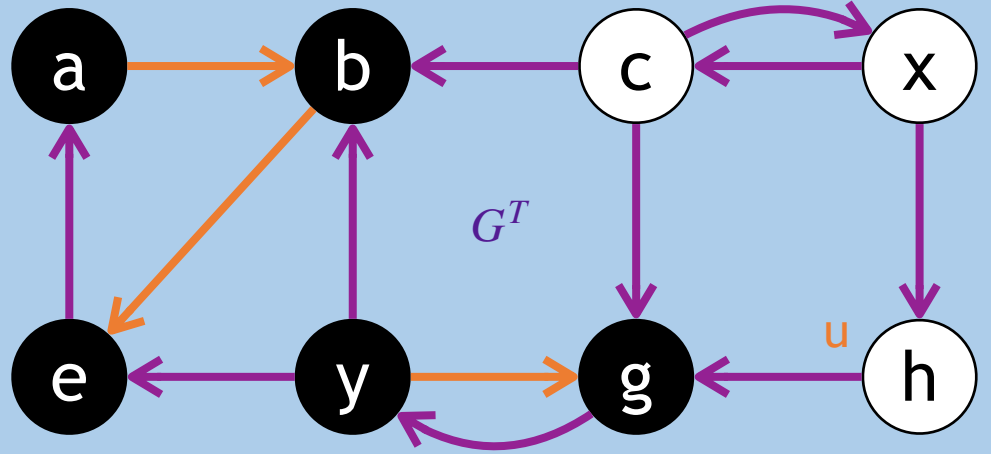

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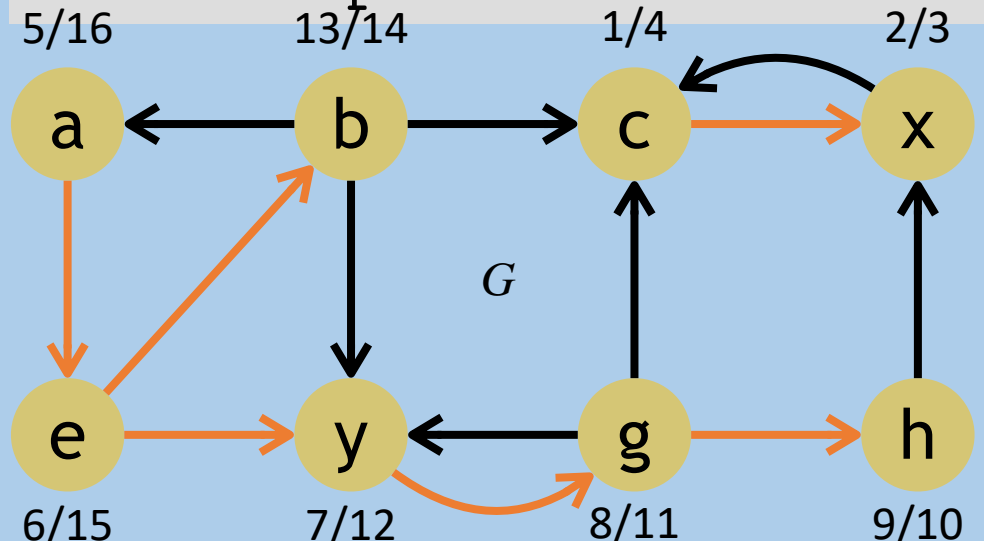

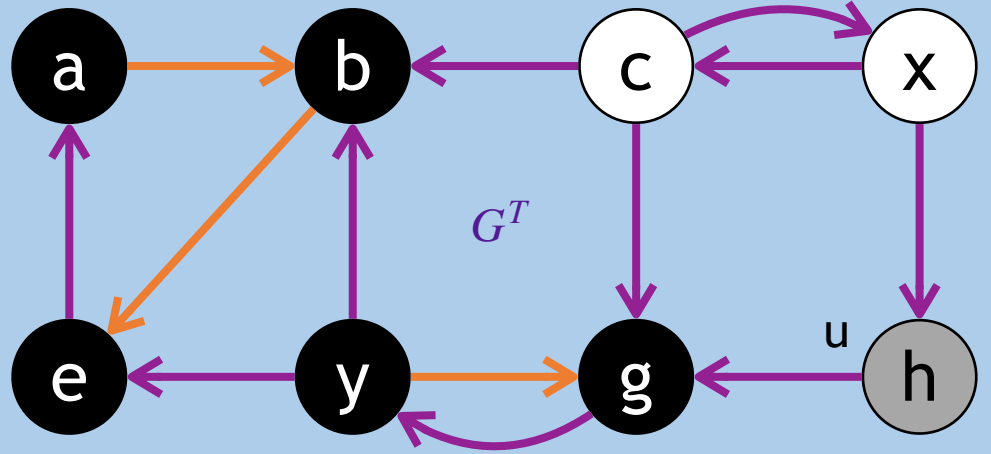

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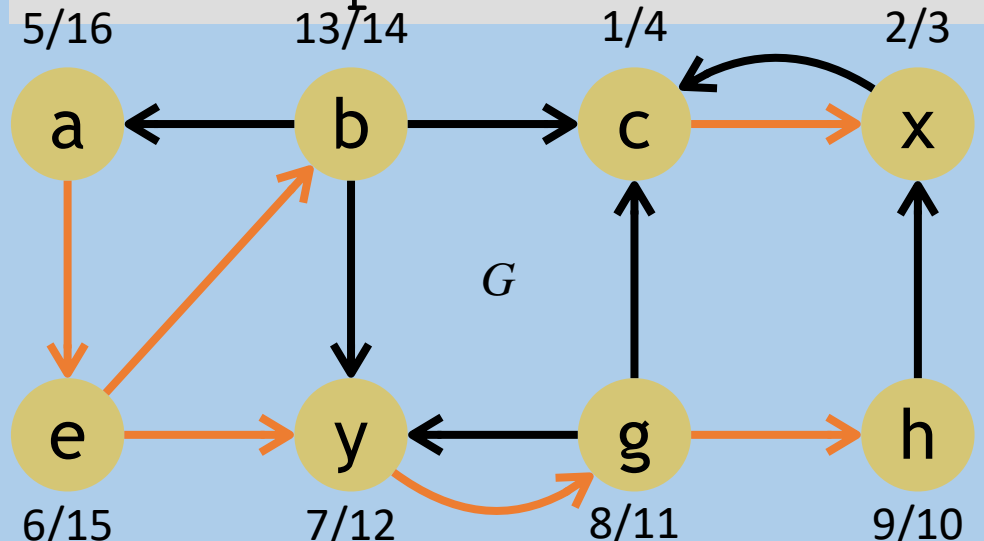

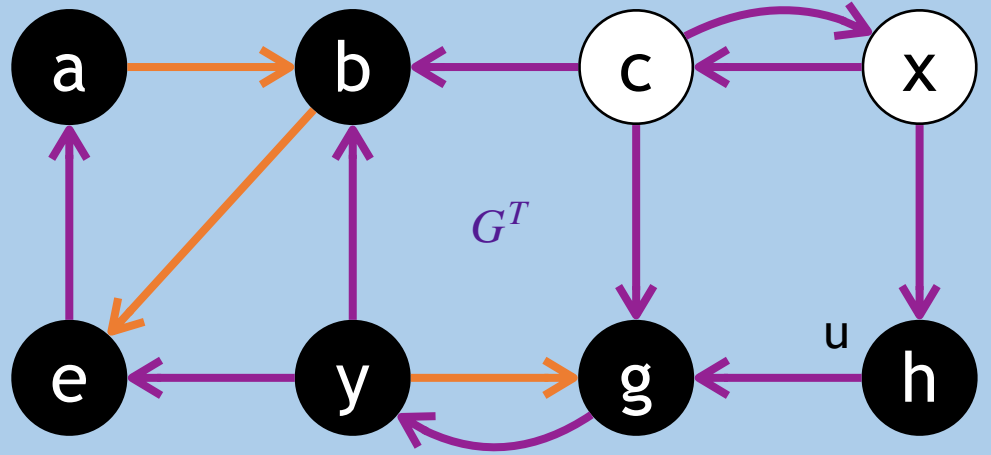

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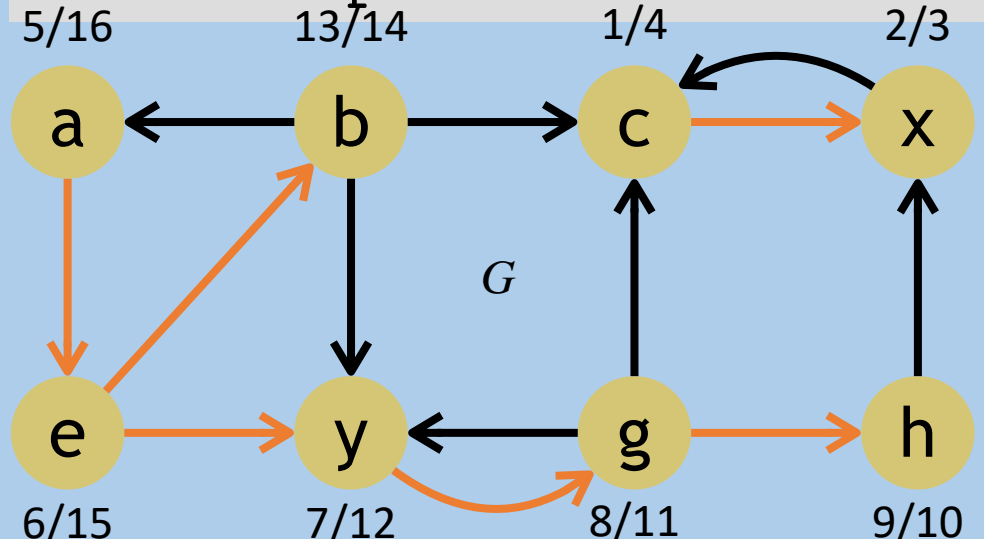

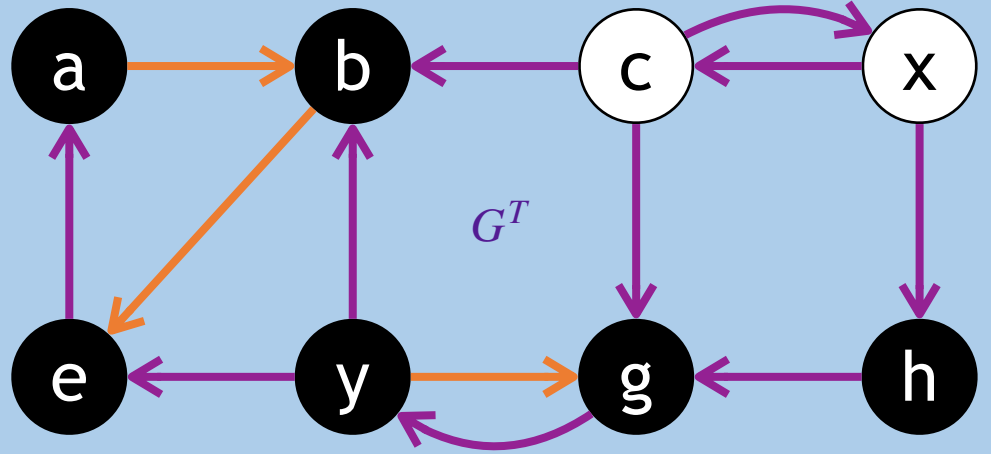

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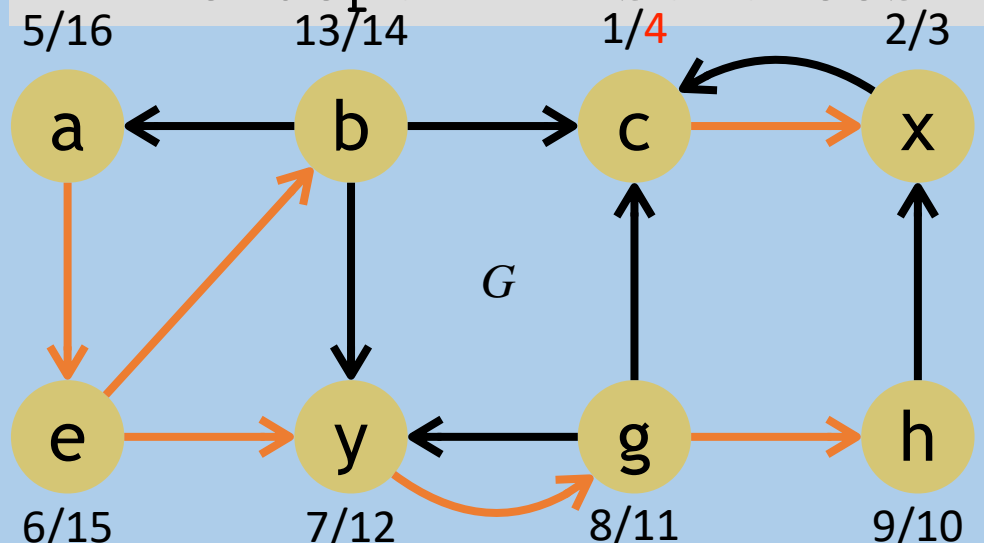

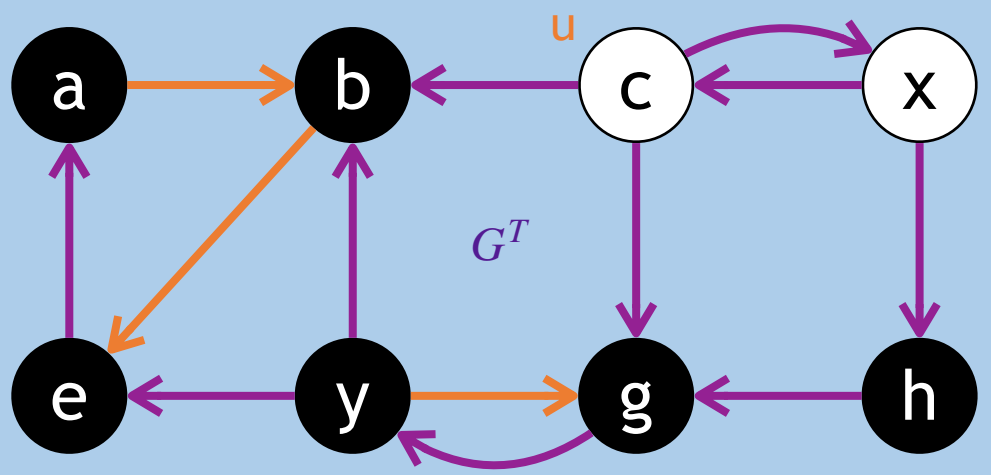

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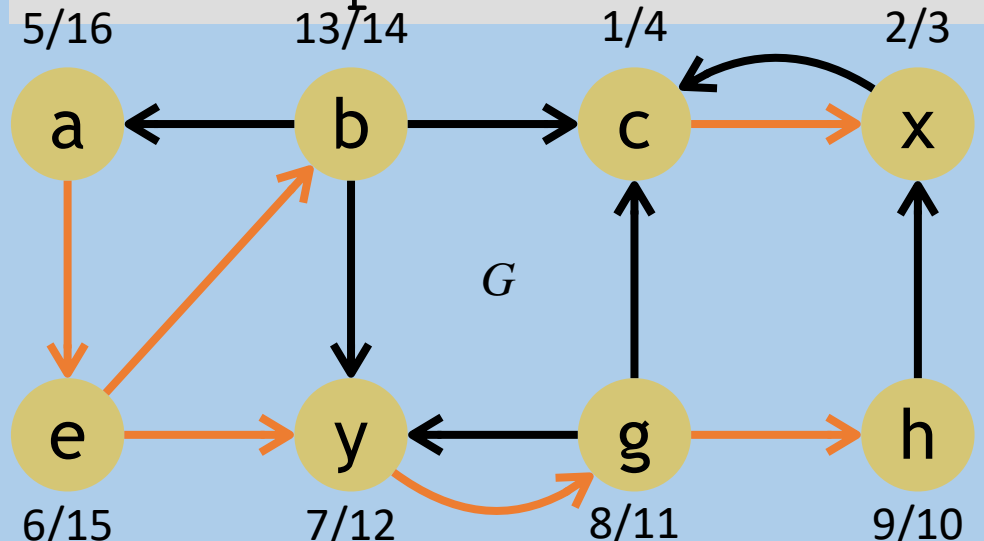

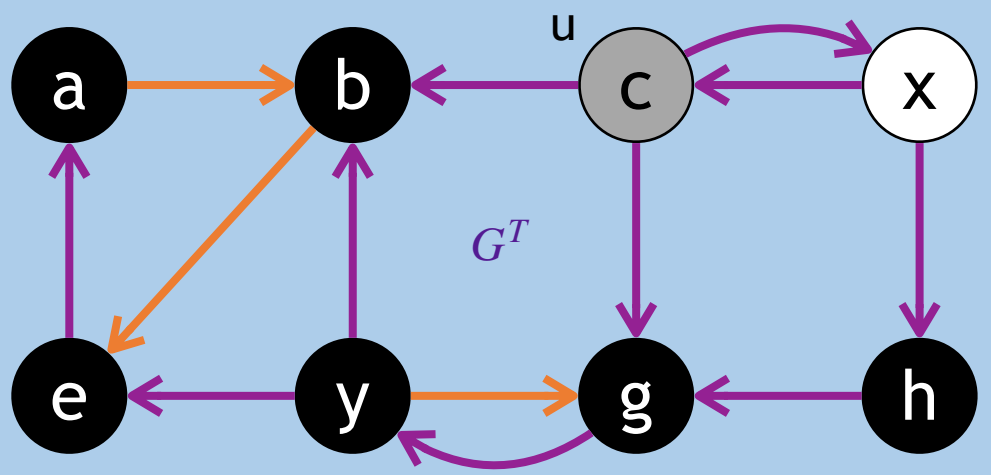

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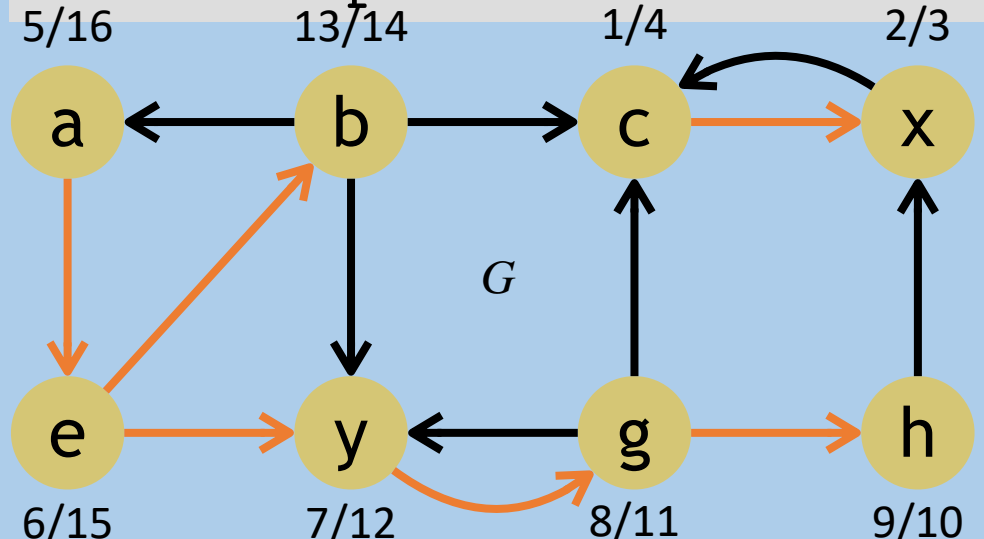

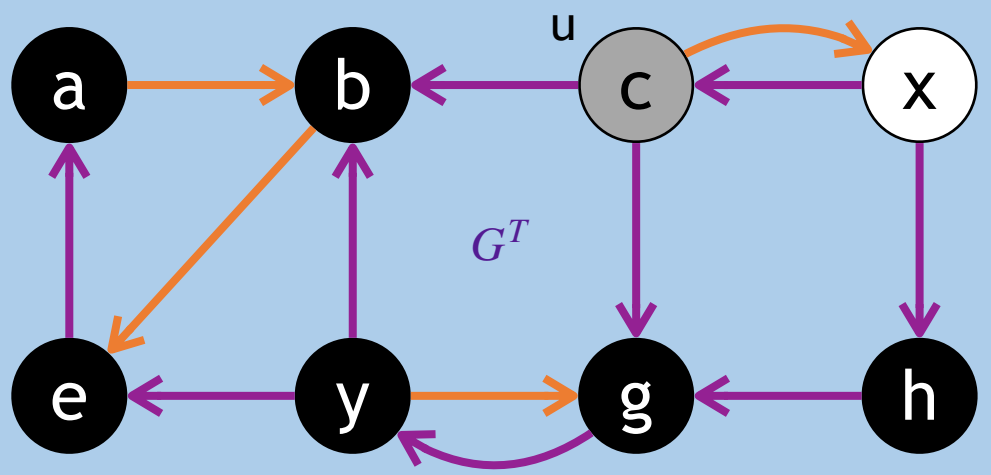

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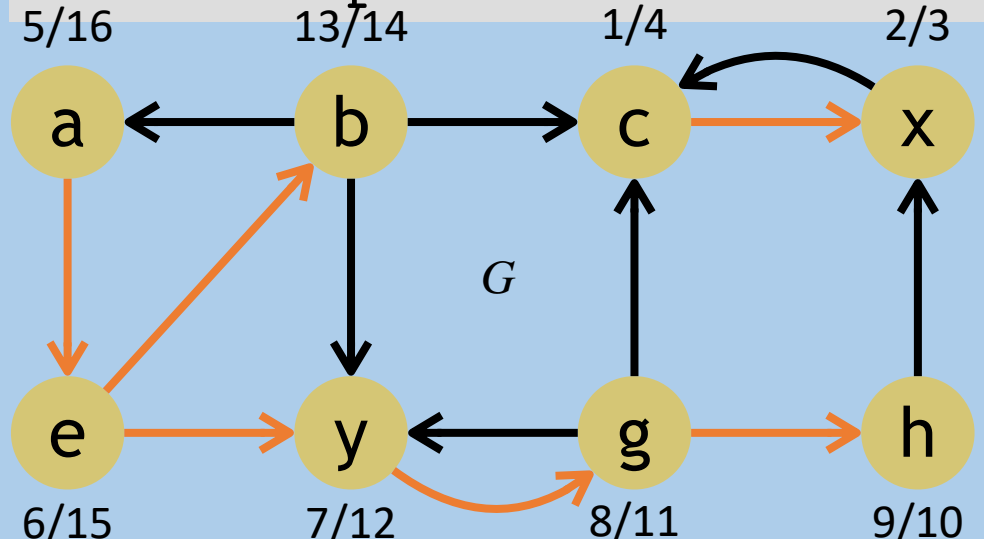

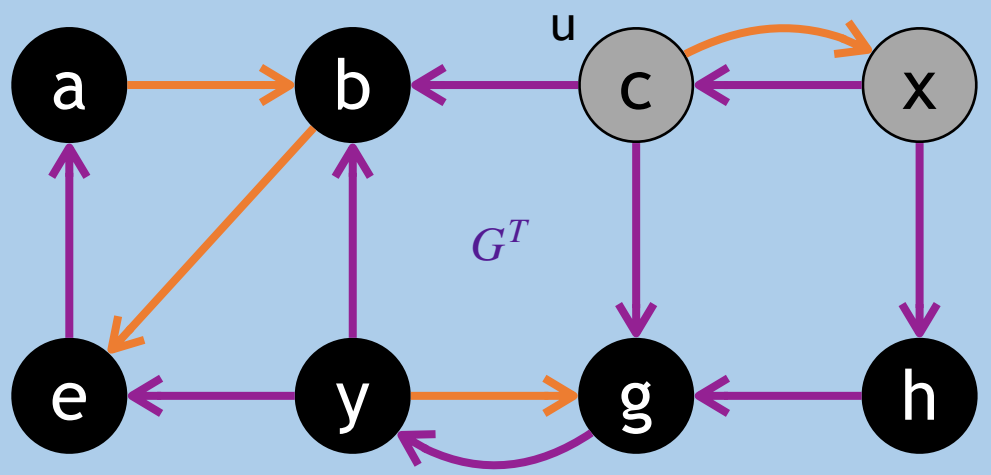

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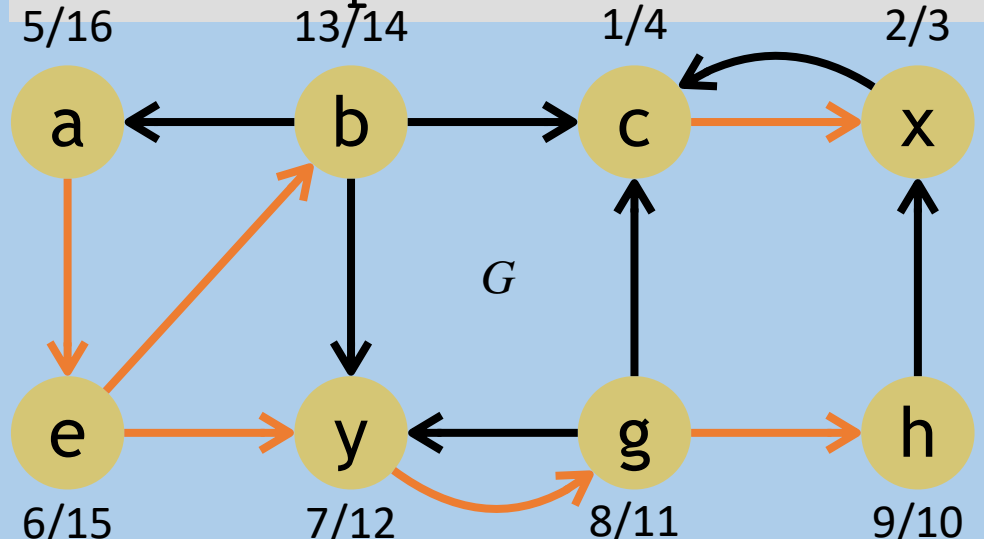

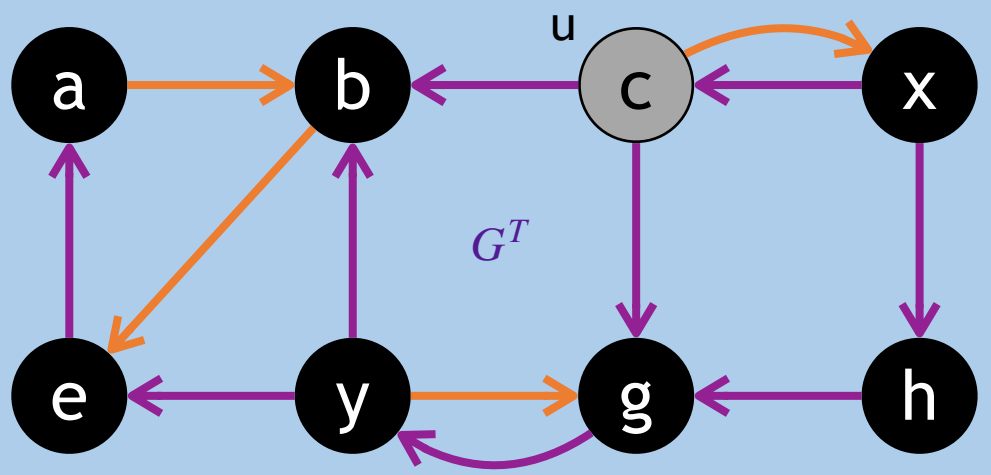

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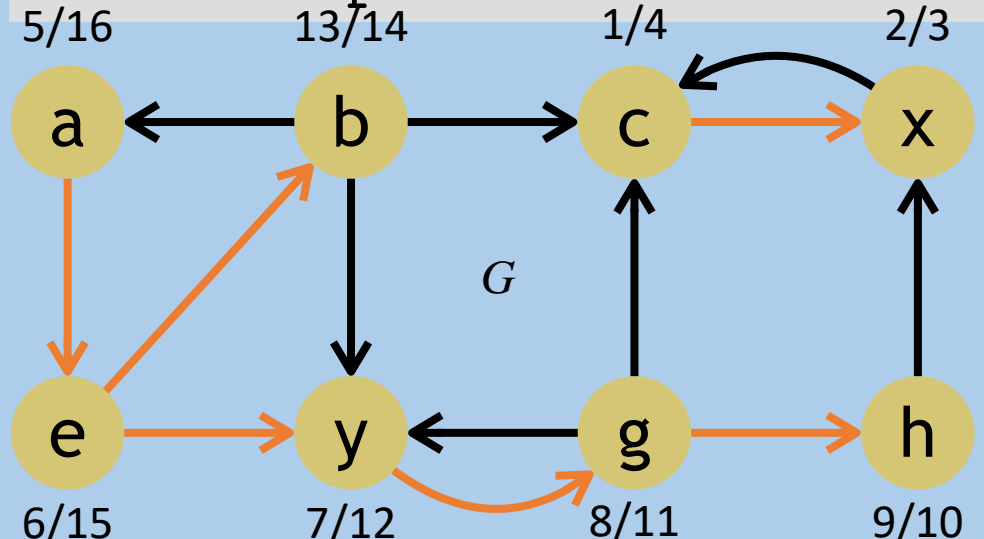

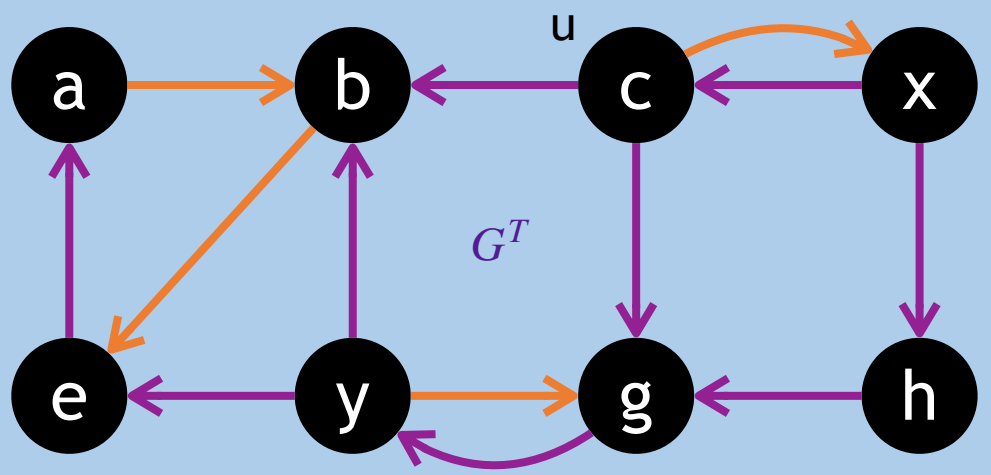

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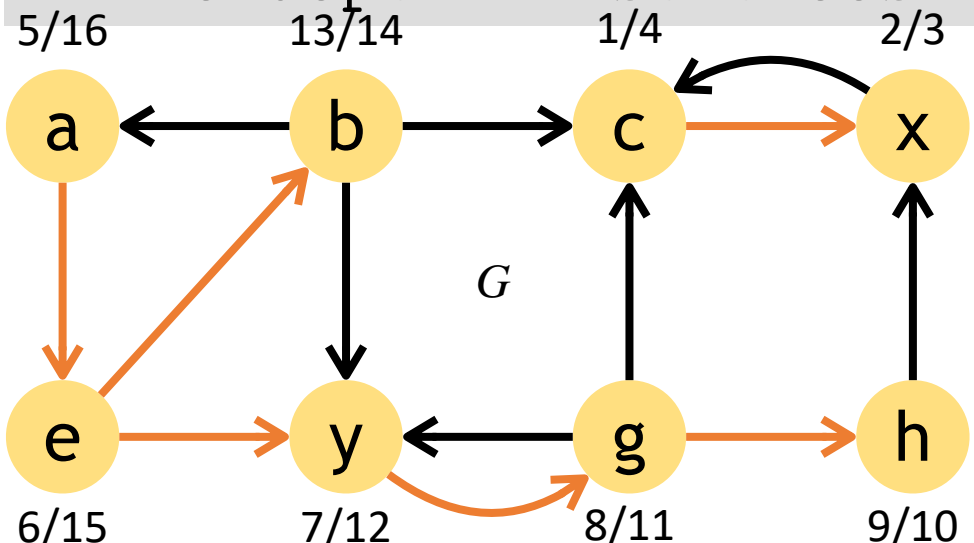

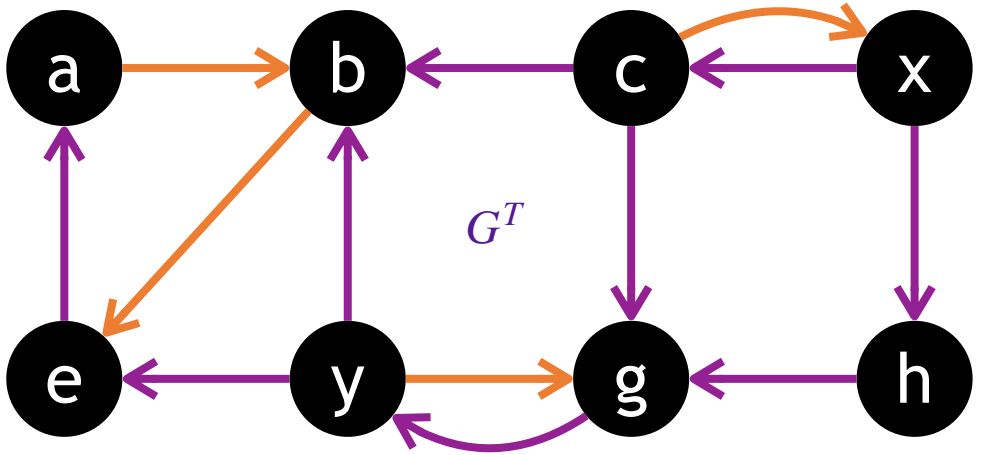

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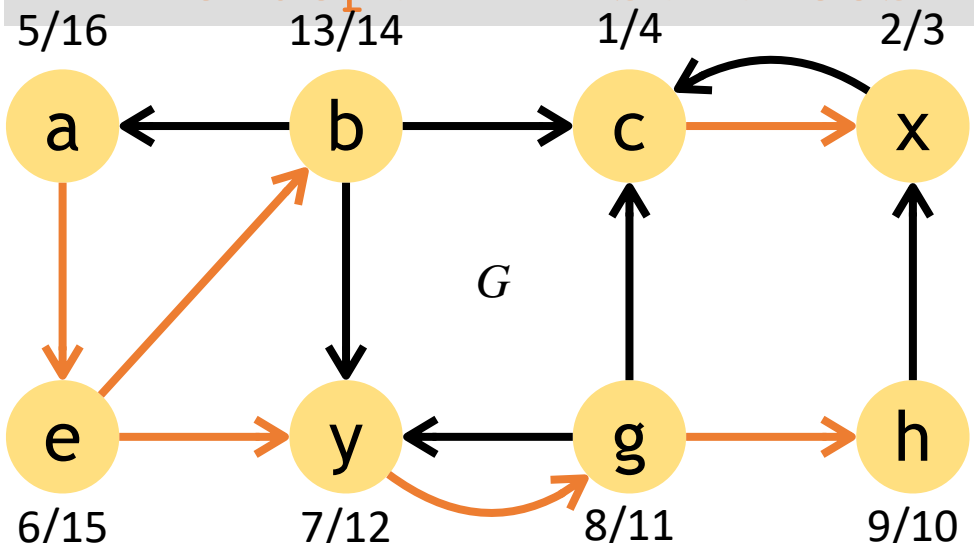

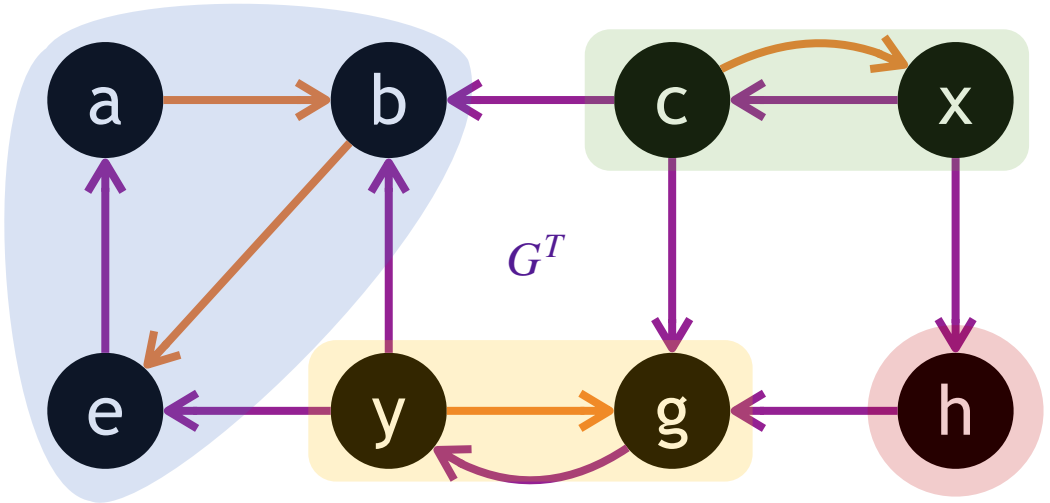

What's Happening

- Given a directed graph, we can find the strongly connect components by performing two DFSes: the first one is on the original graph, and the second one is on the transpose graph.

- In the second DFS on the transpose graph, it is important that when you are selecting the vertex to start a new DFS tree, you choose the one with the largest finish time in the first DFS.

Component graph

Definition: Given a graph $G = (V, E)$ and suppose G has strongly connected components $C_1, C_2, \cdots, C_k$. Its **component graph** $G^{\text{SCC}} = (V^{\text{SCC}}, E^{\text{SCC}})$ is a directed graph such that:

- $V^{\text{SCC}} = \{v_1, v_2, \cdots, v_k\}$, where v_i corresponds to C_i for each i
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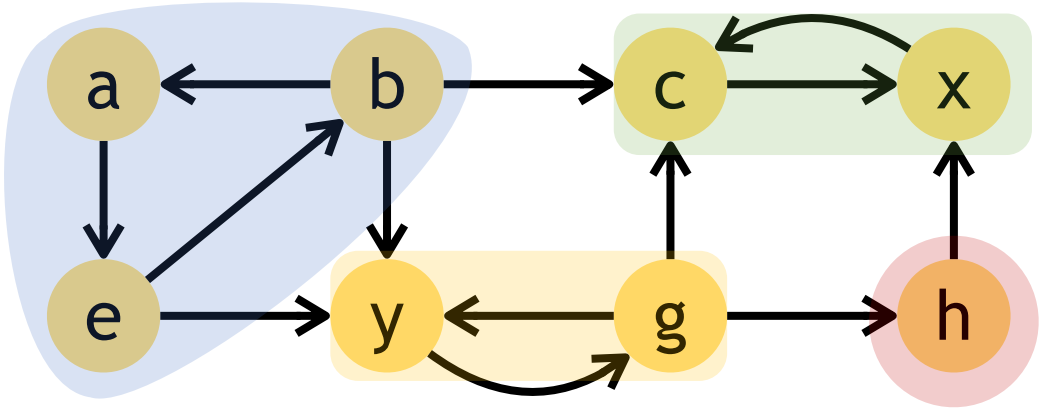
G



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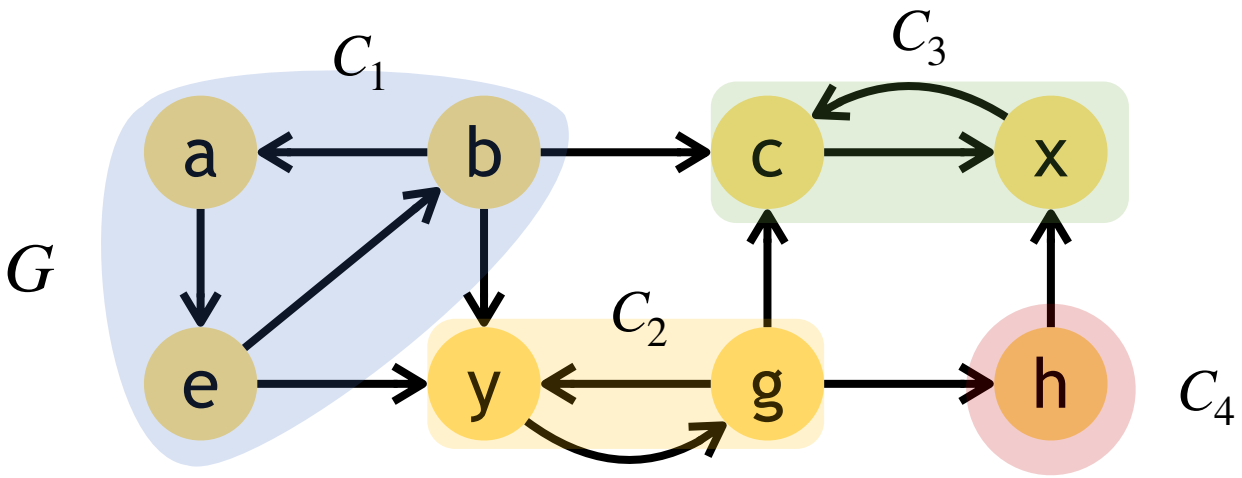

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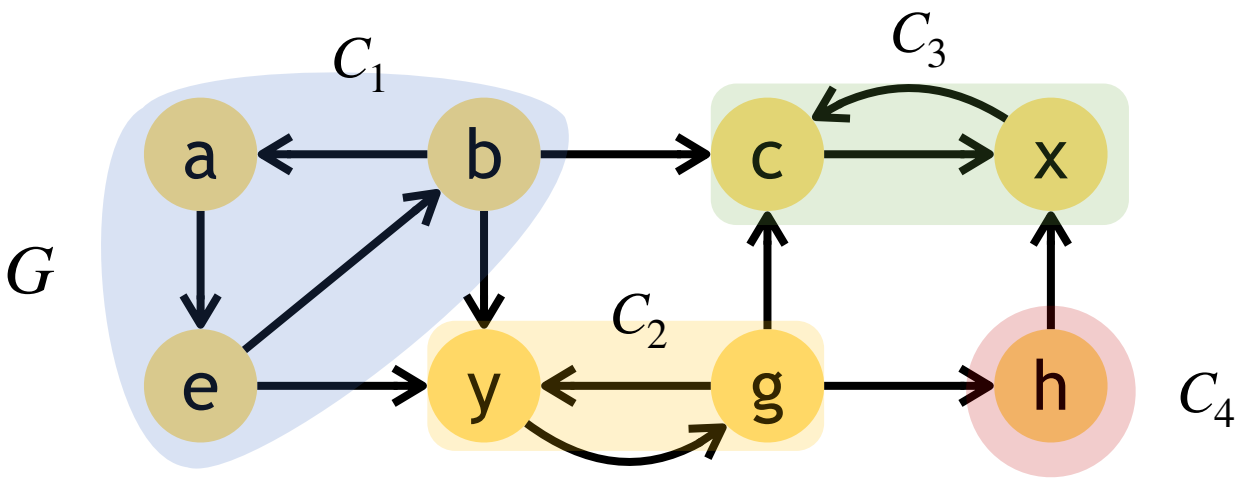

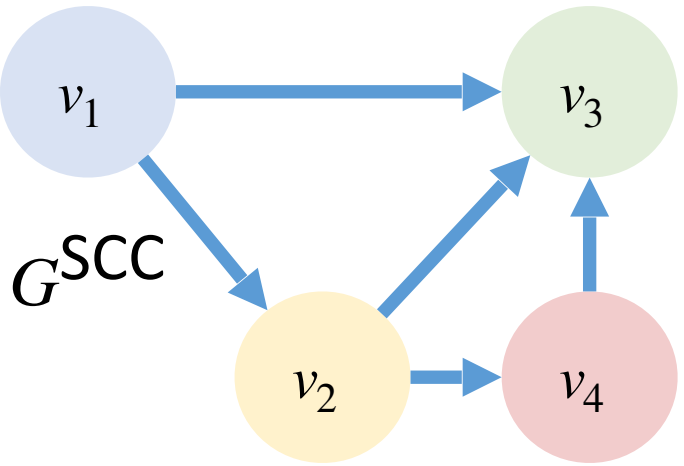

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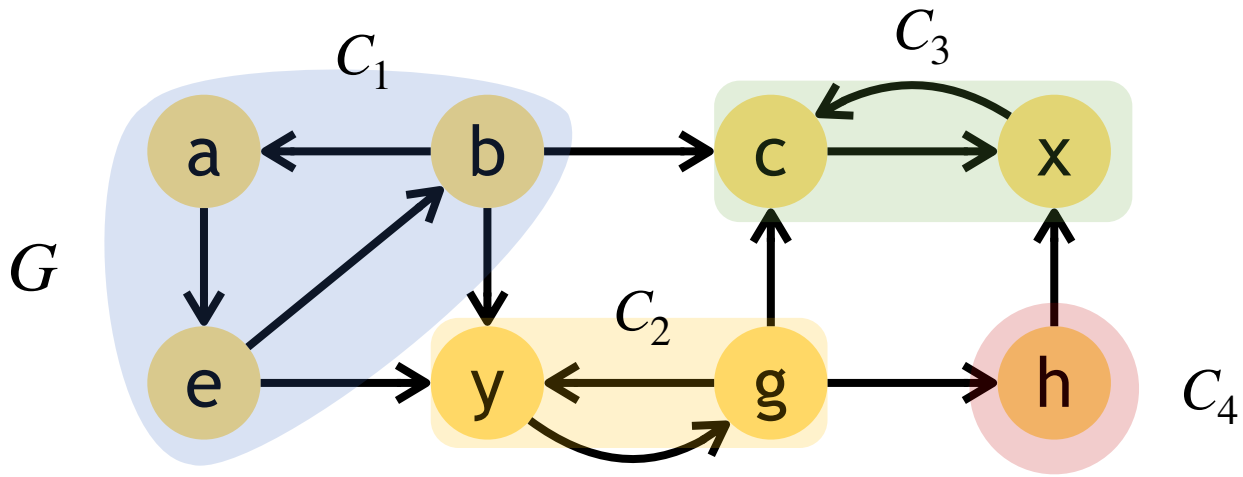

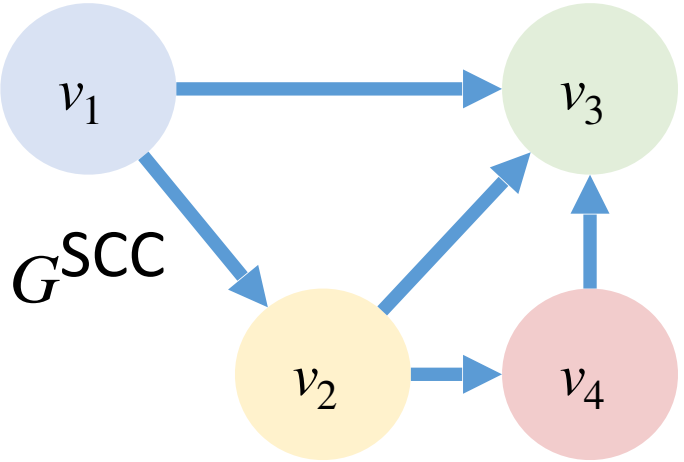

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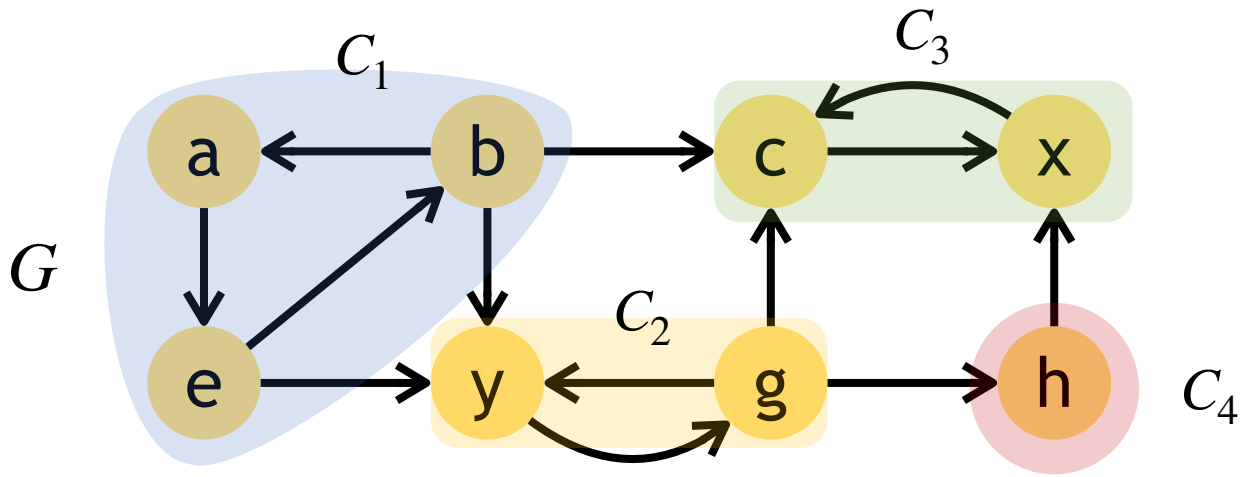
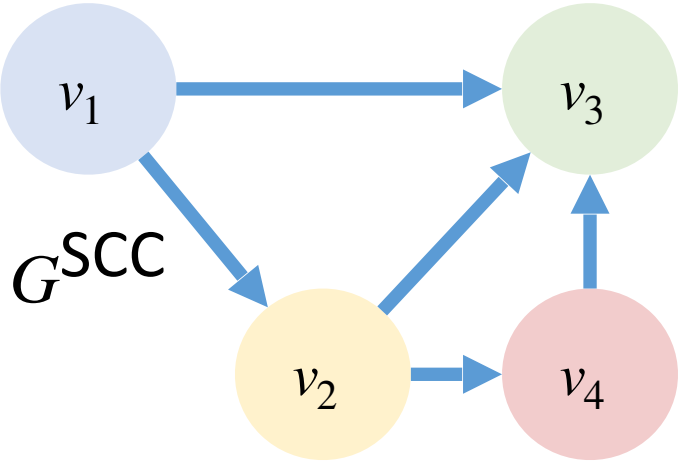
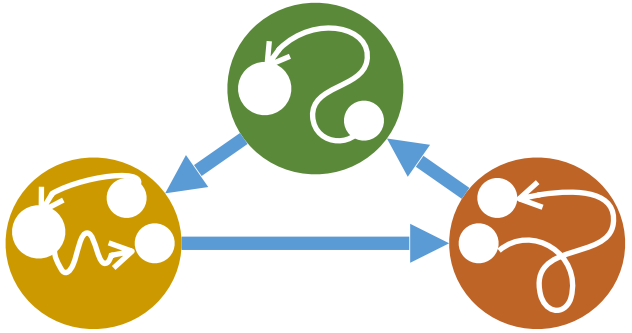


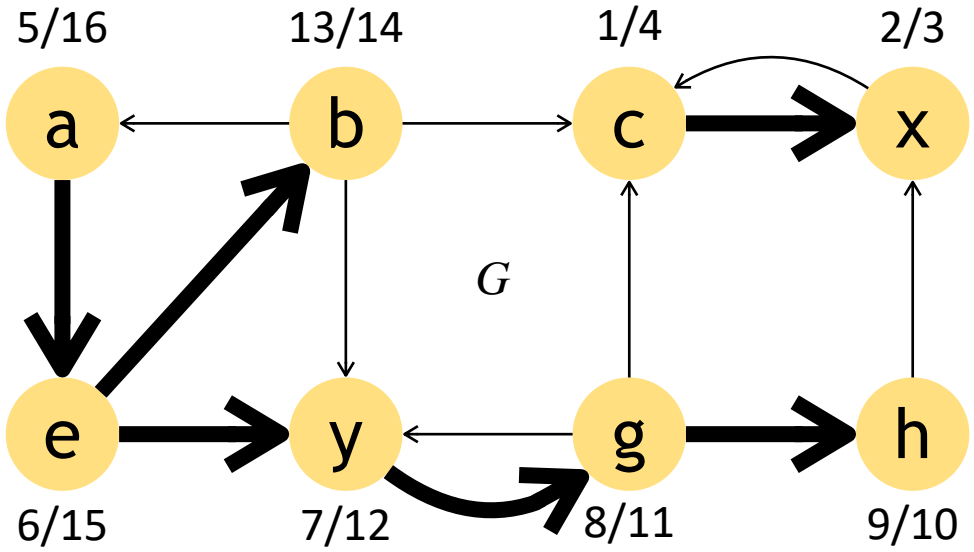


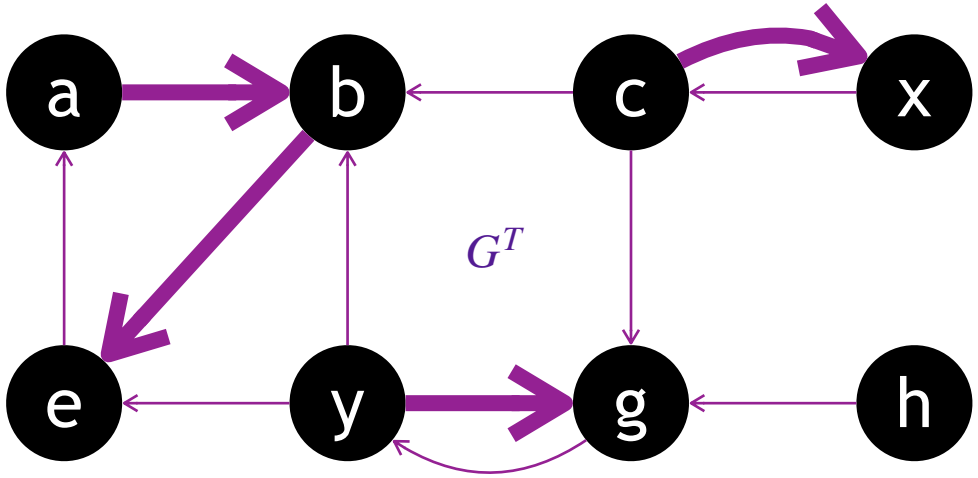
If there is a cycle in G^{SCC}, for any pair of involved components, there are mutual paths. It contradicts to the fact that the SCCs are maximal.



SCC Algorithm Correctness

Theorem: **STRONGLY-CONNECTED-COMPONENTS**(G) correctly computes the strongly connected components of a directed graph G.





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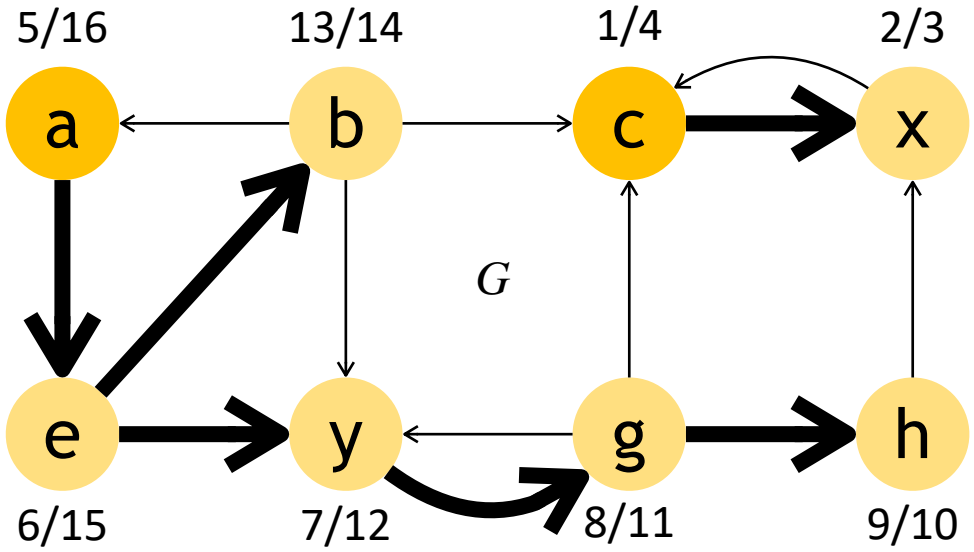

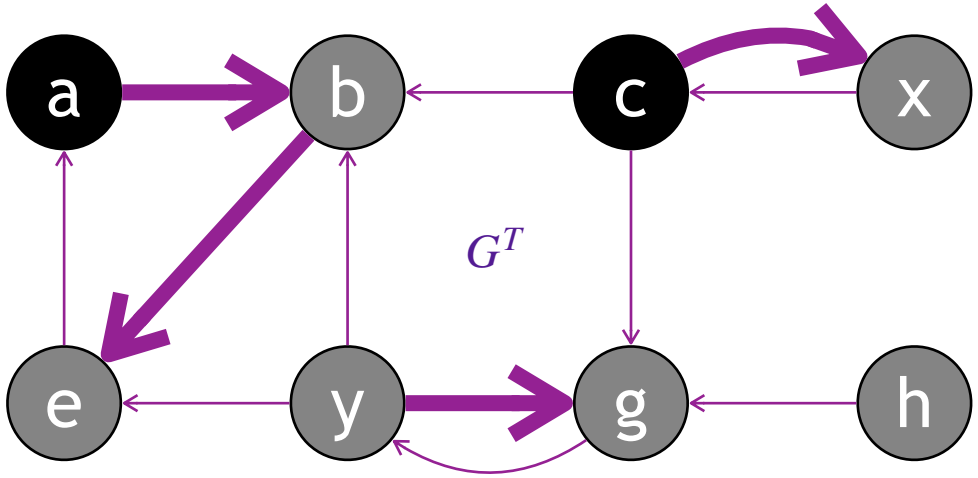

The vertex with the largest finishing time must be one of the roots of DFS trees in G

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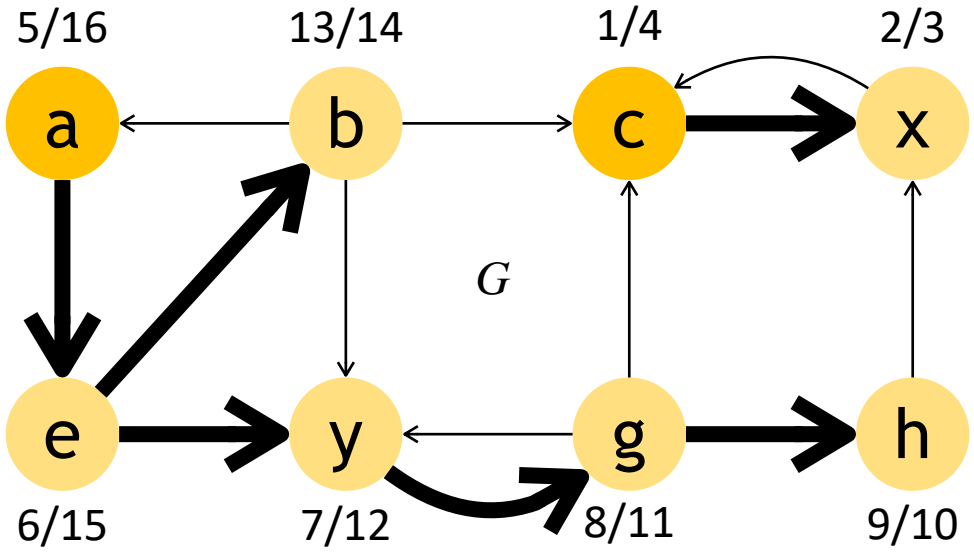

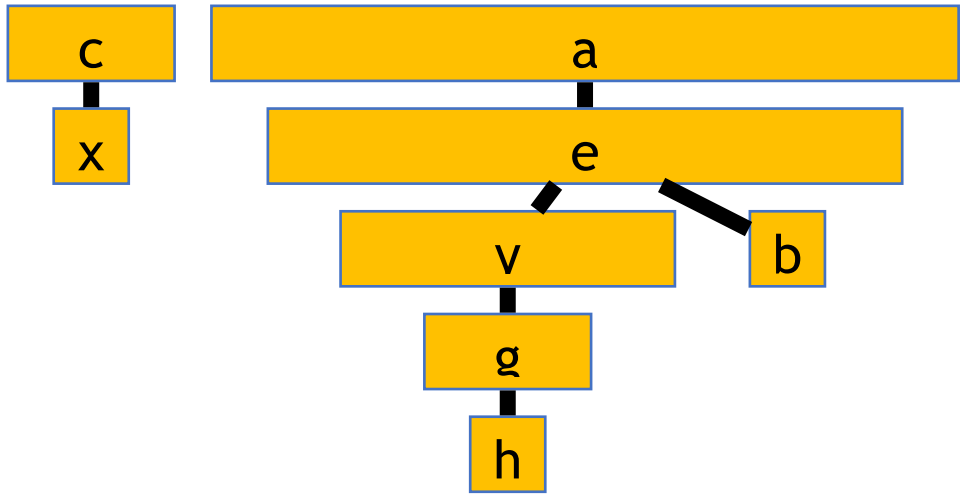

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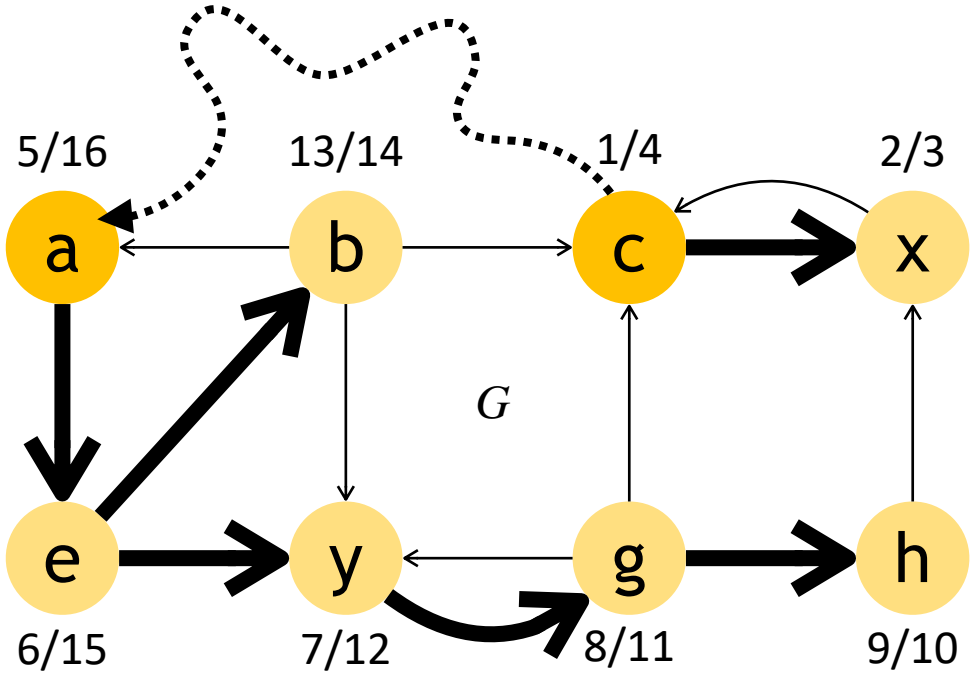
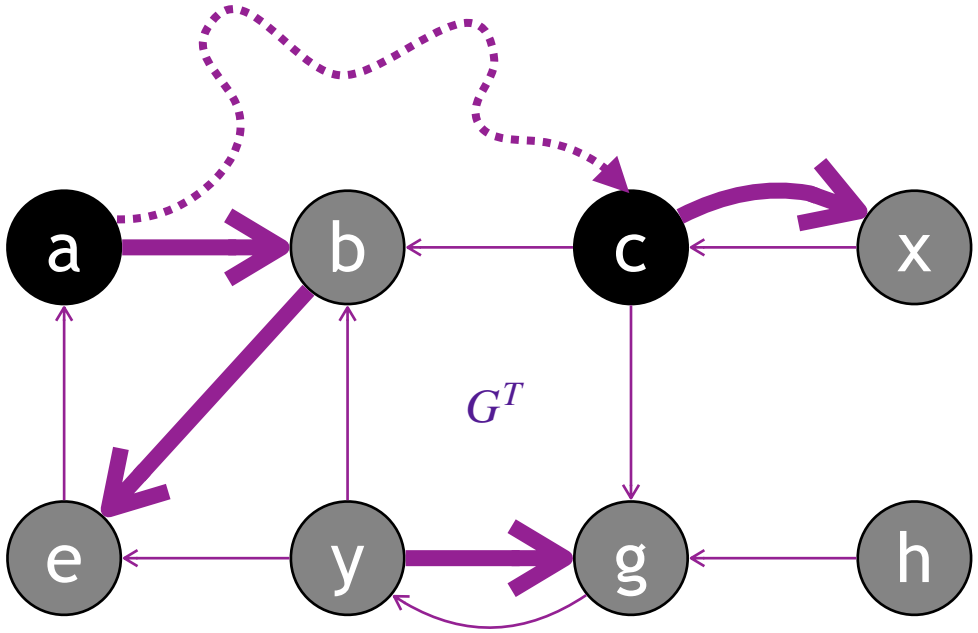


There is no path from c (root with a smaller finish time) to a (root with a larger finish time) in G



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$\Rightarrow$ DVS-VISIT(a) only reaches (some of) the vertices in the DFS tree rooted on a in G

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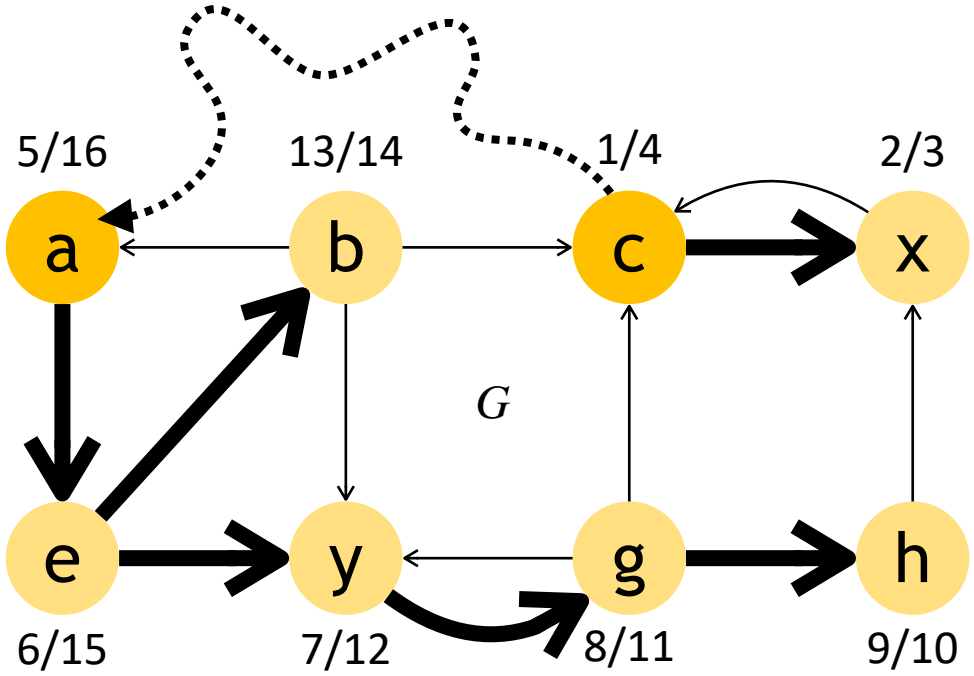

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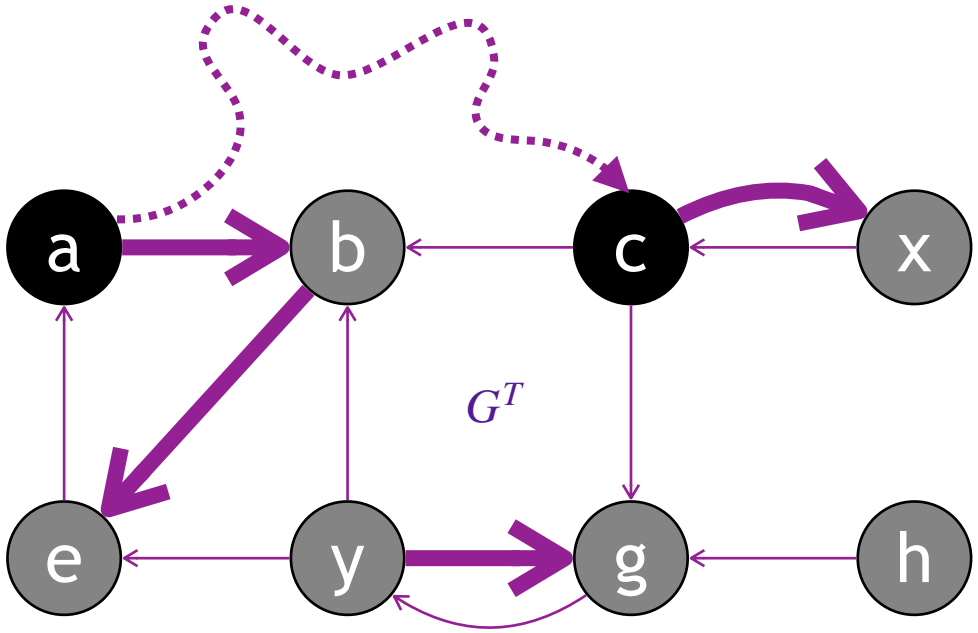

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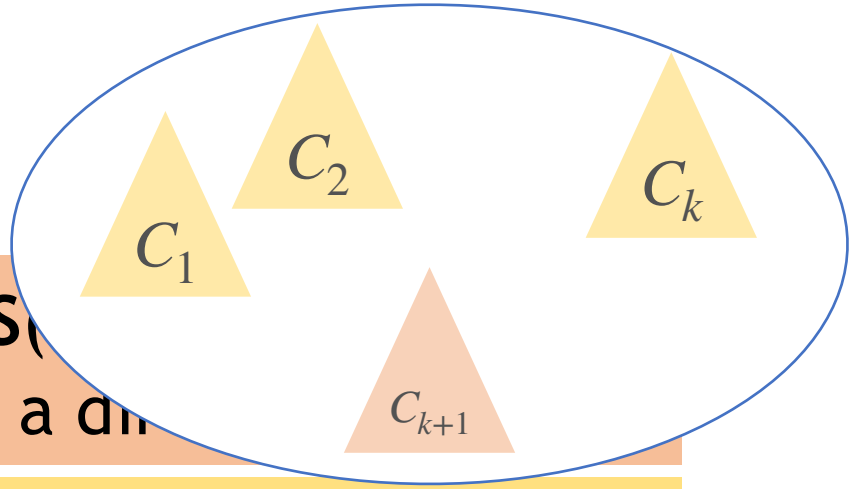

Theorem: **STRONGLY-CONNECTED-COMPONENTS(**
computes the strongly connected components of a di

Claim: The first k trees produced by **DFS**(G^T) are SCCs

Prove by induction on k

Given the inductive hypothesis, show that the $(k+1)$-st tree, T_{k+1} is also a strongly connected component, C_{k+1}

- Every vertex $v \in C_{k+1}$ will be in T_{k+1}
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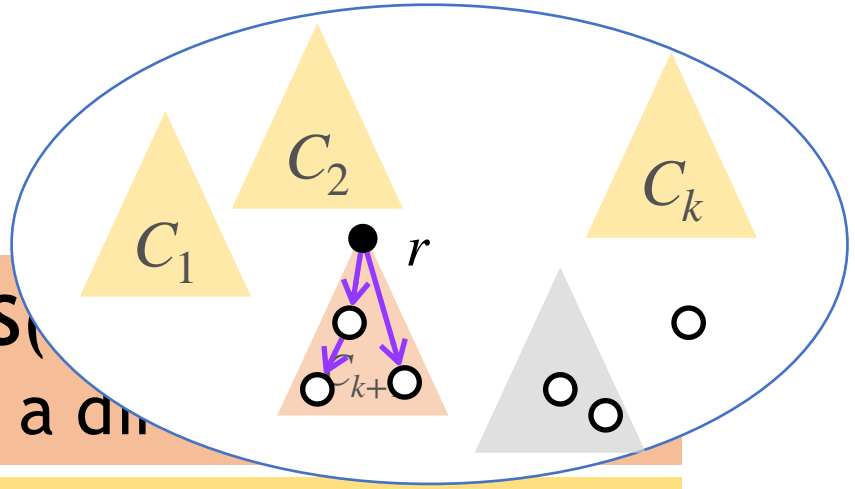

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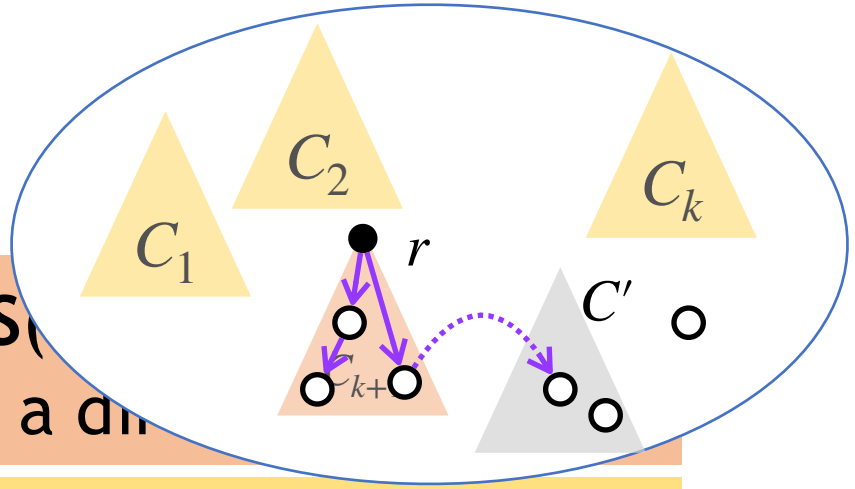
By White-Path theory, let r be the first vertex in C_{k+1} visited by **DFS**(G^T), all vertices in C_{k+1} are descendants of r

SCC Correctness



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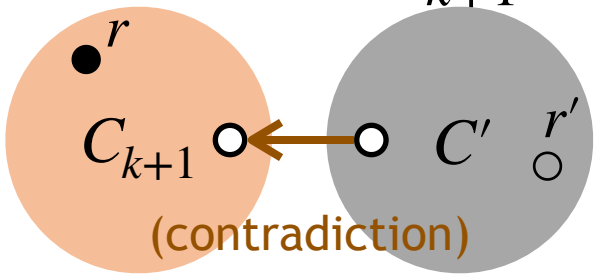
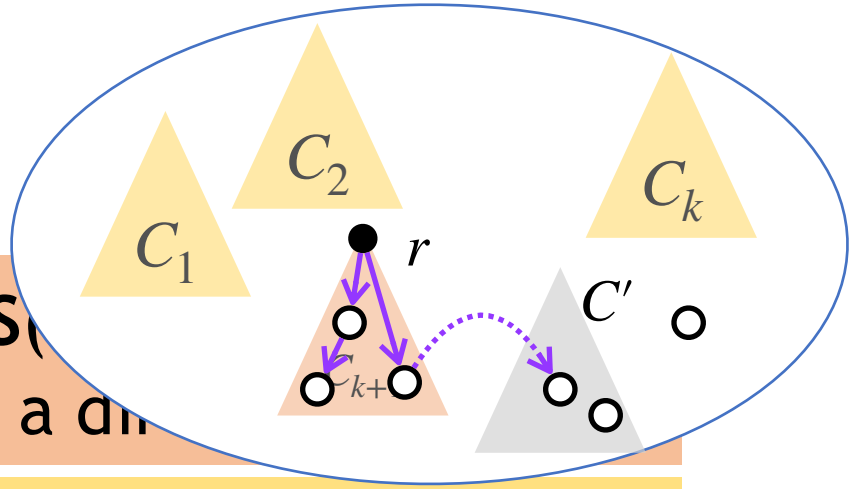


In G^T, there is no edge from C_{k+1} to SCC $C' \notin \{C_1, C_2, \cdots, C_{k+1}\}$

Otherwise, there is an edge from C' to C_{k+1} in G

- If C' is visited by DFS(G) first, there is $r' \in C'$ with $d[r'] < d[r] < f[r] < f[r']$

SCC Correctness



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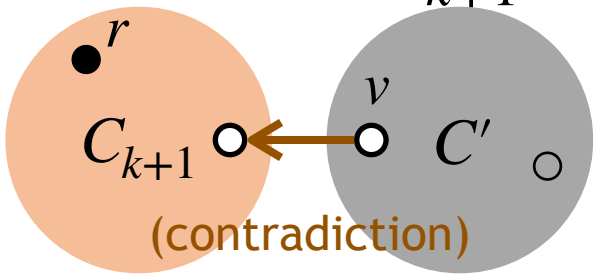

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- If C_{k+1} is visited by DFS(G) first, by the parenthesis property, $d[r] < f[r] < d[v] < f[v]$ for all $v \in C'$. (Contradiction)

Wrap Up

- Graph representation (exercise 4)
- DFS and its useful properties:
  - Parenthesis theorem
  - Classification of edges
  - White-path theorem (exercise 1)
- Applications of DFS:
  - Topological sort of DAGs (exercise 2)
  - Finding strongly connected components (exercise 3)