Universidade Federal de Santa Catarina
Centro Tecnológico de Joinville

CÃ¡lculo Vetorial

ExercÃcios Resolvidos

1 - Encontrar o trabalho realizado pelo campo $\vec{F}(x,y,z) = \dfrac{k}{x^2+y^2+z^2}(x\vec{i}+y\vec{j}+z\vec{k})$ ao longo da curva $\gamma(t) = (\cos t, \sin t, t)$

ResoluÃ§Ã£o: Procuraremos $f = f(x,y,z)$ tal que:

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{k_x}{x^2+y^2+z^2} \tag{1}$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{k_y}{x^2+y^2+z^2} \tag{2}$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{k_z}{x^2+y^2+z^2} \tag{3}$$

Integrando (1) em relaÃ§Ã£o a x obtemos:

$$f(x,y,z) = \int \frac{k_x}{x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘}\, dx + \phi(y,z) = \frac{k}{2}\ln(x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘}) + \phi(y,z),}$$

Portanto,

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{k_y}{x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘} + \dfrac{\partial \phi}{\partial y}(y,z) \overset{(2)}{=} \dfrac{k_y}{x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘} \Longrightarrow \dfrac{\partial \phi}{\partial y}(y,z) = 0 \Longrightarrow \phi(y,z) = \phi(z),}}$$

isto ï¿½ ϕ nï¿½o depende de y. Calculando,

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{k_z}{x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘} + \dfrac{\partial \phi}{\partial z}(z) \overset{(3)}{=} \dfrac{k_z}{x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘} \Longrightarrow \dfrac{\partial \phi}{\partial z}\phi(z) = 0 \Longrightarrow \phi(z) = C.}}$$

isto ï¿½ ϕ tambï¿½m nï¿½o depende de x, y, z.

Se tomarmos $\phi = 0$ temos $f(x,y,z) = \dfrac{k}{2}\ln(x\text{ï?‘} + y\text{ï?‘} + z\text{ï?‘}) + \phi(y,z)$, portanto,

$$W = \int_\gamma \vec{F}\cdot d\vec{r} = f(1,0,2\pi) - f(1,0,0) = \frac{k}{2}\ln(1+4\pi\text{ï?‘})$$

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2 - Mostre que a integral $\int_c (2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘})\,dx + (4xy - 9x^4y\text{ï?‘})dy$ ï¿½ independente do caminho e calcule essa integral, sendo C qualquer caminho de $(1,1)$ a $(3,2)$. **Soluï¿½ï¿½o:**

$$\int_C \overline{f}\,d\overline{r} = \int_C f_1dx + f_2dy$$

Em que:

$$\overline{f} = (f_1, f_2)$$

Nesse caso, $\overline{f}(x,y) = (2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘})\vec{\imath} + (4xy - 9x^4y\text{ï?‘})\vec{\jmath}$, $D(\overline{f}) = \mathbb{R}\text{ï?‘}$ ï¿½ simplismente conexo (nï¿½o tem buracos).

Verificando se o campo $\overline{f}$ ï¿½ conservativo:

$$\frac{\partial f_1}{\partial y} = 4y - 36x\text{ï?‘}y\text{ï?‘} \ ; \ \frac{\partial f_2}{\partial x} = 4y - 36x\text{ï?‘}y\text{ï?‘};$$
$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow \overline{f} \text{ ï¿½ conservativo.}$$

Entï¿½o, existe uma funï¿½ï¿½o $\phi(x,y)$, tal que $\Delta\phi(x,y) = \overline{f}(x,y)$.

Portanto, pelo Teorema 5.3, $\int_c (2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘})\,dx + (4xy - 9x^4y\text{ï?‘}\,dy$ ï¿½ independente do caminho em $\mathbb{R}\text{ï?‘}$.

Vamos encontrar a funï¿½ï¿½o $\phi(x,y)$ e usar o Teorema Fundamental para integrais de linha

$\Delta\phi(x,y) = \overline{f}(x,y) \Rightarrow \dfrac{\partial\phi}{\partial x} = 2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘}$ e $\dfrac{\partial\phi}{\partial y} = 4xy - 9x^4y\text{ï?‘}$.

$\dfrac{\partial\phi}{\partial x} = 2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘} \Rightarrow \phi(x,y) = \int(2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘})\,dx = 2y\text{ï?‘}x - 3x^4y\text{ï?‘} + c(y)$

$\dfrac{\partial\phi}{\partial y} = 4xy - 9x^4y\text{ï?‘} + c'(y) = 4xy - 9x^4y\text{ï?‘}$.

Entï¿½o, $c'(y) = 0$ e $c(y) = k$ (constante). Assim: $\phi(x,y) = (2y\text{ï?‘}x - 3x^4y^3) + k$ e

$\int_c (2y\text{ï?‘} - 12x\text{ï?‘}y\text{ï?‘})\,dx + (4xy - 9x^4y\text{ï?‘})\,dy = \phi(3,2) - \phi(1,1) = 1920 + 1 = 1921.$

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3 - Sejam γ_1, γ_2 e γ os seguintes caminhos ligando os pontos $A(1,1)$ e $B(2,4)$

$$\gamma_1 : x = 1 + t,\ y = 1 + 3t,\ 0 \leqslant t \leqslant 1 \text{ (segmento de reta)}$$
$$\gamma_2 : x = t,\ y = t\text{ï?‘},\ 0 \leqslant t \leqslant 2 \text{ (parï¿½bola)}$$

e γ consiste do segmento horizontal, do ponto $A(1,1)$ ao ponto $C(2,1)$, seguido do seguimento vertical do ponto $C(2,1)$ ao ponto $B(2,4)$.

1 . Ao longo de γ_1 , temos

$$\int_{\gamma_1} ydx + 2xdy = \int_0^1 [(1+3t) + 2(1+t)3]dt = \int_0^1 (7+9t)dt = \frac{23}{2}.$$

2 . Sobre γ_2, fazemos as substituiï¿½ï¿½es $x = t$, $y = t\text{ï?‘}$, $dx = dt$ e $dy = 2tdt$ e obtemos:

$$\int_{\gamma_2} ydx + 2xdy = \int_1^2 [t\text{ï?‘} + (2t)2t]dt = \int_1^2 5t\text{ï?‘}dt = \frac{35}{3}.$$

3 . A poligonal γ ï¿½ composta dos seguimentos γ_3 e γ_4 e observando que $y = 1$, sobre γ_3 e $x = 2$, sobre γ_4, obtemos:

$$\int_{\gamma} ydy + 2xdy = \int_{\gamma_3} ydx + 2xdy + \int_{\gamma_4} ydx + 2xdy = \int_1^2 dx + \int_1^4 4dy = 13.$$

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4 - Use o Teorema de Green para calcular a integral de linha ao longo da curva dada com orientaï¿½ï¿½o positiva.

(a) $\int_c \cos y dx + x\text{ï?‘} \sin y dy$, C ï¿½ o retï¿½ngulo com vï¿½rtices $(0,0)$, $(5,0)$, $(5,2)$ e $(0,2)$

(b) $\int_c (y + e^{\sqrt{x}})dx + (2x + \cos y\text{ï?‘})dy$, C ï¿½ o limite da regiï¿½o englobada pelas parï¿½bolas $y = x\text{ï?‘}$ e $x = y\text{ï?‘}$

Resoluï¿½ï¿½o:

(a)

A regiï¿½o D cercada por C ï¿½: $[0,5] \times [0,2]$, entï¿½o

$$\int_c \cos y\, dx + x\text{ï?‘} \sin y\, dy = \iint_D \left[\frac{\partial}{\partial x}(x\text{ï?‘} \sin y) - \frac{\partial}{\partial y}(\cos y)\right] dA = \int_0^5 \int_0^2 [2x \sin y - (-\sin y)]\, dy\, dx$$
$$= \int_0^5 (2x+1)\, dx \int_0^2 \sin y\, dy = [x\text{ï?‘} + x]_0^5\ [-\cos y]_0^2 = 30(1 - \cos 2)$$

(b)

$$\int_c (y + e^{\sqrt{x}})\, dx + (2x + \cos y\text{ï?‘})dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y\text{ï?‘}) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right] dA$$
$$= \int_0^1 \int_{y\text{ï?‘}^{\sqrt{y}}(2-1)\, dx\, dy = \int_0^1 (y^{1/2} - y\text{ï?‘})\, dy = \frac{1}{3}}$$

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5 - Use o teorema de Green para calcular $\int_c \mathbf{F} \cdot d\mathbf{r}$ (Verifique a orientaï¿½ï¿½o da curva antes de aplicar o teorema).

(a) $\mathbf{F}(x,y) = \langle y\cos x - xy\,\text{sen}\,x, xy + x\cos x\rangle$, C ï¿½ triangulo de $(0,0)$ a $(0,4)$ a $(2,0)$ a $(0,0)$

(b) $\mathbf{F}(x,y) = \langle e^{-x} + y^2, e^{-y} + x^2\rangle$, C consiste no da curva $y = \cos x$ de $(-\frac{\pi}{2}, 0)$ a $(\frac{\pi}{2}, 0)$

Soluï¿½ï¿½o:

(a)

$\mathbf{F}(x,y) = \langle y\cos x - xy\,\text{sen}\,x, xy + x\cos x\rangle$ e a regiï¿½o D cercada por C ï¿½ dada por: $\{(x,y)\,|\,0 \leq x \leq 2\ , 0 \leq y \leq 4 - 2x\}$. C ï¿½ percorrido no sentido horï¿½rio, entï¿½o $-C$ fornece a orientaï¿½ï¿½o positiva.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} (y\cos x - xy\,\text{sen}\,x)\,dx + (xy + x\cos x)\,dy \\
&= -\iint_D \left[\frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\,\text{sen}\,x)\right] dA \\
&= -\iint_D (y - x\,\text{sen}\,x + \cos x - \cos x + x\,\text{sen}\,x)\,dA = -\int_0^2 \int_0^{4-2x} y\,dy\,dx \\
&= -\int_0^2 \left[\frac{1}{2}y\text{ï?‘}\right]_{y=0}^{y=4-2x} dx = \int_0^2 \frac{1}{2}(4-2x)\text{ï?‘}\,dx = -\int_0^2 (8 - 8x + 2x\text{ï?‘})dx = -\left[8x - 4x\text{ï?‘} + \frac{2}{3}x\right. \\
&= -\left(16 - 16 + \frac{16}{3} - 0\right) = -\frac{16}{3}
\end{aligned}$$

(b)

$\mathbf{F}(x,y) = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$ e a regiï¿½o D cercada por C ï¿½ dada por: $(x,y)\,|\,-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\ , 0 \leq y \leq \cos x$. C ï¿½ percorrido no sentido horï¿½rio, entï¿½o $-C$ fornece a orientaï¿½ï¿½o positiva.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} (e^{-x} + y^2)\,dx + (e^{-y} + x^2)\,dy \\
&= -\iint_D \left[\frac{\partial}{\partial x}(e^{-y} + x^2) - \frac{\partial}{\partial y}(e^{-x} + y^2)\right] dA \\
&= -\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} (2x - 2y)\,dy\,dx = -\int_{-\pi/2}^{\pi/2} [2xy - y\text{ï?‘}]_{y=0}^{y=\cos x}\ dx \\
&= -\int_{-\pi/2}^{\pi/2} (2x\cos x - \cos^2 x)\,dx = -\int_{-\pi/2}^{\pi/2} \left(2x\cos x - \frac{1}{2}(1 + \cos 2x)\right) dx \\
&= -\left[2x\,\text{sen}\,x + 2\cos x - \frac{1}{2}\left(x + \frac{1}{2}\,\text{sen}\,2x\right)\right]_{-\pi/2}^{\pi/2} \quad [\textit{integrar por partes no primeiro termo}] \\
&= -\left(\pi - \frac{1}{4}\pi - \pi - \frac{1}{4}\pi\right) = \frac{1}{2}\pi
\end{aligned}$$