Introduction

Algebraic quantum field theory is a mathematically rigorous framework that allows to quantize classical fields. It does not give a quantum theory from first principles, but it is a technique that, whenever one is able to construct a classical relativistic field theory, permits to characterize the observables and the states of the associated quantum system. Moreover, this framework is suitable for a generalization to curved backgrounds within the Heisenberg picture formulation of quantum mechanics. In curved spacetimes, it is necessary to resort to Heisenberg formulation: The states can not evolve in time, since in general there is no uniquely identified notion of time. In addition, when dealing with quantum field theories, a formulation in terms of a fixed Hilbert space is inadequate, since the system possesses an infinite number of degrees of freedom. Hence, in the algebraic formulation, a physical system is described no more by self-adjoint operators on a Hilbert space, but by self-adjoint elements of an abstract $*$-algebra called the *algebra of observables*, which encodes the collection of physical quantities that can be measured on the system.

Our aim in this thesis is to build this algebra for a free electromagnetic system in a suitable class of curved spacetimes with boundary. This is based on the construction of the classical Maxwell field in terms of differential k-forms, proving the existence of distinguished advanced and retarded fundamental solutions for the Maxwell operator under suitable boundary conditions.

A fundamental solution or Green operator G for a differential operator P is an *inverse* of P with prescribed support properties. It allows to solve the equation $Pu = f$ for any compactly supported source f. More precisely, fundamental solutions for P acting on sections of a vector bundle E are defined as $\mathrm{G} : \Gamma_c(E) \to \Gamma(E)$ satisfying

$$P \circ \mathrm{G} = \mathrm{G} \circ P = \mathrm{id}|_{\Gamma_c(E)} \,.$$

We plan to construct explicitly, for each choice of boundary conditions, two operators $\mathrm{G}^{\pm}$ for Maxwell operator, called *advanced* and *retarded* fundamental solutions, such that $\mathrm{supp}\,(\mathrm{G}^{\pm}(f)) \subseteq J^{\pm}(\mathrm{supp}(f))$ for any f, where the symbols $J^{\pm}$ denote the causal future $(+)$ and the causal past $(-)$. These conditions ensures a *finite speed of propagation*.
Analyzing the support properties of the Green operators gives us information about the propagation of initial data. Moreover fundamental solutions are important for the construction of

(causal) propagators, that allow to implement quantum commutation relations at the level of the algebra of observables for a quantum field theory.
We will construct such operators choosing physically meaningful classes of boundary conditions, namely those that ensure that the flux of physically relevant quantities, such as those built from the stress-energy tensor, through the boundary is vanishing. This, combined with the requirement of unitary evolution, translates mathematically in the condition of self-adjointness of the operators involved.

From a geometric point of view, we will focus on globally hyperbolic spacetimes with timelike boundary (M, g), which have been introduced in a very recent paper by Aké, Flores and Sanchez – *cf.* [AFS18]. They are the natural class of spacetimes where boundary conditions can be assigned.
In the *ultrastatic* case, when there exists a global irrotational timelike Killing field, M splits smoothly as a product $\mathbb{R} \times \Sigma$, where the metric admits a decomposition $g = -\mathrm{d}t^2 + h$, where Σ is a Riemannian manifold with boundary such that the Cauchy surface $\{t\} \times \Sigma$ has Riemannian metric h for any time t.

We focus initially on homogeneous Maxwell equations for the Faraday 2-form F.
Maxwell's equations read

$$\begin{cases} \mathrm{d}F = 0 \\ \delta F = 0, \end{cases} \tag{1}$$

where d, δ are the exterior derivative and the codifferential, respectively, while $F \in \Omega^2(M)$ is the Faraday 2-form, which in a static case has the following decomposition in terms of electric and magnetic time-dependent differential forms $E \in C^\infty(\mathbb{R}, \Omega^1(\Sigma))$ and $B \in C^\infty(\mathbb{R}, \Omega^2(\Sigma))$:

$$F = B + \mathrm{d}t \wedge E. \tag{2}$$

In the second chapter of this thesis we will be able to construct fundamental solutions to Maxwell's equations for F with prescribed conditions at an interface between two media, which will be regarded as two different globally hyperbolic spacetimes with timelike boundary.

Otherwise, Maxwell's equations can be written in terms of a generic vector potential $A \in \Omega^k(M)$, $0 < k < \dim M$ that is locally defined as a primitive of $F \in \Omega^{k+1}(M)$, in other words $F = \mathrm{d}A$ locally. In this case the equations of motion are $\delta \mathrm{d}A = 0$ and we have to take into account the *gauge invariance* of the theory and the interplay between the gauge freedom and the choice of boundary conditions on a globally hyperbolic spacetime with timelike boundary.
At first, we will prove the existence of fundamental solutions for the D'alembert–de Rham wave operator $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ in this framework for a certain class of boundary conditions in static spacetimes. Usually, in solving Maxwell's equations in a spacetime with no boundary, one works in

the so called *Lorenz gauge* $\delta A = 0$ so to recast the problem into an hyperbolic form $\Box A = 0$. We shall employ the same technique, which is available only for two restricted classes of boundary conditions for A that we named $\delta\mathrm{d}$-tangential (vanishing tangential component) and $\delta\mathrm{d}$-normal (vanishing normal derivative). Relying on these results we will be able to characterize the space of solutions of Maxwell's equations, but it will become clear that two distinct notions of gauge invariance have to be defined for each of the aforementioned boundary conditions.

In conclusion, as we mentioned earlier, we construct the algebra of observables for Maxwell's equations for the vector potential A under those boundary conditions. We will prove that the algebras so constructed are physically sound: they are *optimal*. This means that they contain enough and no more elements to distinguish between different configurations of the field.
Furthermore, we will show that, in analogy with the case without boundary, the algebra possesses a non-trivial centre: This topological obstruction is a feature which is common in Abelian gauge theories such as electromagnetism and it translates the impossibility to interpret such models as *locally covariant quantum field theories*. In fact, electromagnetism is not a local theory: It possesses non-local observables which measure the electric flux through surfaces that include monopoles. – *cf.* [SDH14].

The thesis is organized as follows.
The first chapter is devoted to establishing the geometric and analytic framework in which we will work. In particular, we define globally hyperbolic spacetimes with timelike boundary and we recall the notion of differential forms. Subsequently, we give an account of Sobolev spaces for Riemannian manifolds of bounded geometry. Then we recall the definition of fundamental solutions (or Green operators) and we give some example. In conclusion we state Maxwell's equations and we outline the problems that we tackle in the following chapters.

In the second chapter we analyze Maxwell's equations for the field strength F in a spacetime with a codimension 1 interface $Z \subset \Sigma$ between two media, regarded as manifolds with timelike boundary. We will separate the equations in a non-dynamical part, that will be treated using the so-called Hodge decomposition (*cf.* [Sch95]) and a dynamical part, whose fundamental solutions are constructed using a technique from functional analysis, namely that of *Lagrangian subspaces*. These allow to discriminate physically sound interface conditions. In the end, we give an account of the possible extensions of the results to the construction of the algebra of quantum observables for F.

The third and final chapter starts with the proof of the existence of advanced and retarded fundamental solutions for $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ operating on k-forms in ultrastatic spacetimes with timelike boundary using a functional analysis technique called *boundary triples*, under certain classes of boundary conditions. We apply our results to identify the space of solutions of $\delta\mathrm{d}A = 0$ under the aforementioned $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions. In addition we distinguish two different notions of gauge invariance and we show that within the two gauge equivalence classes is always possible to find a representative that abides Lorenz gauge. We give an account of the notion of algebra of observables within the framework of algebraic quantum field theory and we identify, for the two boundary conditions, the optimal algebras.

Part of the content of this thesis has appeared as an independent publication, available on the ArXiv: [DDL19].

Conclusions

In this thesis we established, for a particular class of spacetimes with timelike boundary and for suitable boundary conditions, the existence of advanced and retarded fundamental solutions or Green operators for Maxwell's equations stated both in terms of the Faraday tensor $F \in \Omega^2(M)$ and in terms of the vector potential A. Subsequently we constructed the optimal quantum algebra of observables for the free electromagnetic field in terms of A for two selected boundary conditions.
In particular, for the equations in terms of $F \in \Omega^2(M)$, we analyzed Maxwell's equations in a framework in which the spacetime M could be split into $\mathbb{R} \times \Sigma$, with Σ being a closed Riemannian manifold with a codimension 1 interface $Z \subset \Sigma$. We separated the equations in a non-dynamical and in a dynamical part. The former has been treated using the so-called Hodge decomposition, while for the latter the fundamental solutions have been constructed using the technique of Lagrangian subspaces. These are at the hearth of a method to select boundary conditions that ensure the self-adjointness of the dynamical part of Maxwell's equations. Furthermore we gave an example of such conditions.
In the case of Maxwell's equations for a generic vector potential $A \in \Omega^k(M)$, $0 < k < \dim M$ we proceeded as follows. At first we proved of the existence of advanced and retarded fundamental solutions for $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ acting on k-forms in ultrastatic spacetimes with timelike boundary. We used the technique of boundary triples, imposing suitable classes of boundary conditions, dubbed $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal. Subsequently, we applied these results identifying the space of solutions of $\delta\mathrm{d}A = 0$ under $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions, distinguishing two different notions of gauge invariance and showing that within the two gauge equivalence classes it is always possible to find a representative that abides to Lorenz gauge. We identified, for the two boundary conditions, the optimal algebras of observables, showing that in general they do possess a non-trivial centre.

Possible extensions to this work arise for both the formulations of Maxwell's equations in terms of $F \in \Omega^2(M)$ and in terms of $A \in \Omega^k(M)$.

In particular, focusing on the results of Chapter 2, in the discussion about the non-dynamical part of Maxwell's equations for F (*cf.* Section 2.2), we assumed that the underlying Cauchy

hypersurface was closed, i.e. compact and with no boundary, in order to use Hodge decomposition. Hodge decomposition was used to give to the non-dynamical equations a mathematically rigorous framework to find a solution. It is natural to ask whether a generalization of Hodge decomposition can be used on a larger class of Cauchy hypersurfaces (*cf.* Subsection 2.2.3).
Moreover, we did not construct explicitly the algebra of observables for the Faraday tensor F, while we only gave an account on the possible strategy in Section 2.4.

On the other side, the results of Chapter 3 can be generalized in various directions. At first, we considered the wave operator $\Box$ and we indicated a class of boundary conditions, encoded by $\Omega^k_{f,f'}(M)$ which ensured the operator to be closed or, in other words, formally self-adjoint. It is important to stress that the boundary conditions $\Omega^k_{f,f'}(M)$ are not the largest class which makes the operator closed. As a matter of fact one can think of other possibilities, for example those similar to the Wentzell boundary conditions, which were studied in the scalar scenario in [DDF19; DFJA18; Zah18] and that could be interesting from a physical viewpoint. Therefore it arises the question whether there is a larger class of boundary conditions for $\Box$ such that the existence of fundamental solutions can be established.
Subsequently, in Subsection 3.1.2, we assumed the existence of distinguished fundamental solutions for the wave operator and studied their properties, which remain valid whenever the hypothesis of Assumption 3.1.4 hold. To show that the Assumption can be verified, we used a particular technique from functional analysis which is well-suited for static spacetimes. Hence, a possible follow-up is to investigate whether there are other methods that allow the construction of fundamental solutions for $\Box$ in non-ultrastatic spacetimes.
Regarding the construction of the spaces of solutions for Maxwell's equations $\delta \mathrm{d} A = 0$, we restricted the discussion to a particular pair of boundary conditions, $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal, for which it was possible to find a gauge-equivalent solution which satisfied the wave equation $\Box A = 0$. We wonder if there are other boundary conditions for the Maxwell operator $\delta\mathrm{d}$ such that the construction of fundamental solutions is possible without resorting to the Lorenz gauge and thus relying on the fundamental solutions for $\Box$ – *cf.* Section 3.2.
To conclude, we think that the work should be recast regarding electromagnetism as a Yang-Mills Abelian theory for the connections on a $U(1)$-principal bundle, rather than choosing $\mathbb{R}$ as a structure group. This should affect the choice of the gauge groups and hence the formulation of the spaces of solutions and of the algebra of observables – *cf.* the discussion in Subsection 3.2.1.