UNIVERSITÀ DEGLI STUDI DI PAVIA

DIPARTIMENTO DI FISICA

Corso di Laurea Magistrale in Scienze Fisiche

On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

Tesi per la Laurea Magistrale di:
Rubens Longhi

Relatore:
Prof. Claudio Dappiaggi

Correlatore:
Dott. Nicoló Drago

A.A. 2018-2019

"*The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen.* "

Paul A.M. Dirac

UNIVERSITY OF PAVIA

Abstract

Department of Physics

Master Degree

On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

by Rubens Longhi

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too. . .

Contents

To my family

Introduction

Blabla

Chapter 1

Geometric preliminaries

In this chapter, we begin by recalling the basic definitions in order to fix the geometric setting in which we will work.
Globally hyperbolic spacetimes (M, g) are used in context of geometric analysis and mathematical relativity because in them there exists a smooth and spacelike Cauchy hypersurface Σ and that ensures the well-posedness of the Cauchy problem. Moreover, as shown by Bernal and Sánchez [BS05, Th. 1.1], in such spacetimes there exists a splitting for the full spacetime M as an orthogonal product $\mathbb{R} \times \Sigma$. These results corroborate the idea that in globally hyperbolic spacetimes one can preserve the notion of a global passing of time. In a globally hyperbolic spacetime, the entire future and past history of the universe can be predicted from conditions imposed at a fixed instant represented by the hypersurface Σ.

1.1 Globally Hyperbolic spacetimes with timelike boundary

The main goal of this section is to analyse the main properties of globally hyperbolic spacetimes and to generalise them to a natural class of spacetimes where boundary values problems can be formulated. This class is that of globally hyperbolic spacetimes with timelike boundary.While, in the case of $\partial M = \emptyset$ global hyperbolicity is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18].

Manifolds with boundary. From now on M will denote a smooth manifold with boundary with dimension $m > 1$. M is then locally diffeomorphic to open subsets of the closed half space of $\mathbb{R}^n$. We will assume that the boundary ∂M is smooth and, for simplicity, connected. A point $p \in M$ such that there exists an open neighbourhood U containing p, diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$.
In addition we endow M with a smooth Lorentzian metric g of signature $(-, +, ..., +)$ so that $\iota^* g$ identifies a Lorentzian metric on ∂M and we require (M, g) to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M, g) has a *timelike*

boundary.

For any $p \in M$, we denote by $J^+(p)$ the set all points that can be reached by future-directed causal curves emanating from p. For any subset $A \subset M$ we put $J^+(A) := \bigcup_{p\in A} J^+(p)$. If A is closed so is $J_+(A)$. We denote by $I^+(p)$ the set of all points in M that can be reached by future-directed timelike curves emanating from p. The set $I_+(p)$ is the interior of $J^+(p)$; in particular, it is an open subset of M. Interchanging the roles of future and past, we similarly define $J^-(p)$, $J^-(A)$, $I^-(p)$.

Definition 1.1.1.

- *A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.*
- *A spacetime with timelike boundary is* causal *if it possesses no closed, causal curve,*
- *A causal spacetime with timelike boundary M such that for all $p, q \in M$ $J^+(p) \cap J^-(q)$ is compact is called **globally hyperbolic**.*

These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 1.1.2. *Let (M, g) be a spacetime of dimension m. Then*

1. *(M, g) is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve,*
2. *if (M, g) is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the metric*

$$g = -\beta d\tau^2 + h_\tau, \tag{1.1}$$

where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1], whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a Cauchy surface for (M, g).

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M, g), we work directly with (1.1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M.

Definition 1.1.3. *A spacetime (M, g) is* static *if it possesses a timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to ∂M is tangent to the boundary, i.e. $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where ν is the unit vector, normal to the boundary at p.*

[1] Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

With reference to (1.1) this translates simply into the request that both β and h_τ are independent from τ.

Example 1.1.4. We first consider some examples of globally hyperbolic spacetimes without boundary ($\partial M = \emptyset$).

- The Minkowski spacetime $M = (\mathbb{R}^m, \eta)$ is stati and globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-1}$, endowed with the time-independent Euclidean metric.

- Let Σ be a Riemannian manifold with time independent metric h and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$, called **cosmological spacetime**, is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold, see [BGP15, Lem A.5.14]. This applies in particular if (Σ, h) is compact.

- The interior and exterior **Schwarzschild spacetimes**, that represent non-rotating black holes of mass $\mathrm{m} > 0$ are static and globally hyperbolic. Denoting S^2 the 2-dimensional sphere embedded in $\mathbb{R}^3$, we set

$$M_{\text{ext}} := \mathbb{R} \times (2\mathrm{m}, +\infty) \times S^2,$$

$$M_{\text{int}} := \mathbb{R} \times (0, 2\mathrm{m}) \times S^2.$$

The metric is given by

$$g = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2},$$

where $f(r) = 1 - \frac{2\mathrm{m}}{r}$, while $g_{S^2} = r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2$ is the polar coordinates metric on the sphere. For the exterior Schwarzschild spacetime we have $M_{\text{ext}} = \mathbb{R} \times \Sigma$ with $\Sigma = (2\mathrm{m}, +\infty) \times S^2$, $\beta = f$ and $h = \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2}$.

Example 1.1.5. Now we consider some examples of globally hyperbolic spacetimes with timelike boundary in which the boundary is not empty.

- The Half Minkowski spacetime $M = (\mathbb{R}^{m-1} \times [0, +\infty), \eta)$ is static and globally hyperbolic. Every spacelike half-hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-2} \times [0, +\infty)$, endowed with the time-independent Euclidean metric.

- Let Σ be a Riemannian manifold with boundary with time independent metric h and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$ is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold with boundary.

A particular role will be played by the support of the functions that we consider. In the following definition we introduce the different possibilities that we will consider - *cf.* [Bär15].

Definition 1.1.6. *Let (M, g) be a Lorentzian spacetime with timelike boundary and $E \to M$ a vector bundle on M. We denote with*

1. *$C^\infty_{\mathrm{c}}(M, E)$ the space of smooth sections of E with compact support in M while with $C^\infty_{\mathrm{cc}}(M, E) \subset C^\infty_{\mathrm{c}}(M)$ the collection of smooth and compactly supported sections f of E such that supp$(f) \cap \partial M = \emptyset$.*

2. *$C^\infty_{\mathrm{spc}}(M, E)$ (resp. $C^\infty_{\mathrm{sfc}}(M, E)$) the space of strictly past compact (resp. strictly future compact) sections of E, that is the collection of $f \in C^\infty(M, E)$ such that there exists a compact set $K \subseteq M$ for which J^+(supp(f)) $\subseteq J^+(K)$ (resp. J^-(supp(f)) $\subseteq J^-(K)$), where $J^\pm$ denotes the causal future and the causal past in M. Notice that $C^\infty_{\mathrm{sfc}}(M, E) \cap C^\infty_{\mathrm{spc}}(M, E) = C^\infty_{\mathrm{c}}(M, E)$.*

3. *$C^\infty_{\mathrm{pc}}(M, E)$ (resp. $C^\infty_{\mathrm{fc}}(M, E)$) denotes the space of future compact (resp. past compact) sections of E, that is, $f \in C^\infty(M, E)$ for which $\mathrm{supp}(f) \cap J^-(K)$ (resp. $\mathrm{supp}(f) \cap J^+(K)$) is compact for all compact $K \subset M$.*

4. *$C^\infty_{\mathrm{tc}}(M, E) := C^\infty_{\mathrm{fc}}(M, E) \cap C^\infty_{\mathrm{pc}}(M, E)$, the space of timelike compact sections.*

5. *$C^\infty_{\mathrm{sc}}(M, E) := C^\infty_{\mathrm{sfc}}(M, E) \cap C^\infty_{\mathrm{spc}}(M, E)$, the space of spacelike compact sections.*

1.2 Differential forms and operators on manifolds with boundary

To treat Maxwell equations properly and to be able to generalise them, we will use the language of differential forms. In this section (M, g) will denote a generic oriented pseudo-Riemannian manifold with boundary with signature $(-, +, \dots, +, +)$ or $(+, +, \dots, +, +)$. In the former case, when the manifold is Lorentzian, it is understood that the boundary is timelike in the sense of Definition 1.1.1. We present the following definitions in such a general framework since we will work both on spacetimes (M, g) with timelike boundary and on their Cauchy hypersurfaces (Σ, h), which are Riemannian manifolds with boundary in virtue of Theorem 1.1.2.

On top of a pseudo-Riemannian Hausdorff, connected, oriented and paracompact manifold (M, g) with boundary we consider the spaces of complex valued k-forms $\Omega^k(M)$ as smooth sections of $\Lambda^k T^*M$. Since (M, g) is oriented, we can identify a unique, metric-induced, Hodge operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \mathrm{d}\mu_g$, where $\wedge$ is the exterior product of forms and $\mathrm{d}\mu_g$ the metric induced volume form. We endow $\Omega^k(M)$ with the standard, metric induced, pairing

$$(\alpha, \beta) := \int_M \overline{\alpha} \wedge *\beta \,, \tag{1.2}$$

Remark 1.2.1. In case $E = \Lambda^k T^*M$, the spaces with support properties defined in Definition 1.1.6 will be denoted respectively by the following spaces of k-forms: $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{cc}}(M)$, $\Omega^k_{\mathrm{spc/sfc}}(M)$, $\Omega^k_{\mathrm{pc/fc}}(M)$, $\Omega^k_{\mathrm{tc/sc}}(M)$. If the regularity required for any of these spaces is different than smoothness, it will be denoted putting it in front of the space. For example, the space of square integrable k-forms will be denoted with $\mathrm{L}^2\Omega^k(M)$.

We indicate the exterior derivative with $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. A differential form α is called closed when $\mathrm{d}\alpha = 0$ and exact when $\alpha = \mathrm{d}\beta$ for some differential form β. Since M is endowed with a pseudo-Riemannian metric it holds that, when acting on smooth k-forms, $*^{-1} = (-1)^{k(m-k)}*$. Combining these data we define the *codifferential* operator $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta \doteq *^{-1} \circ \mathrm{d} \circ *$.

To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its projection along the direction normal to ∂M.

Remark 1.2.2. Since we will be considering not only form lying in $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, but also those in $\Omega^k(\partial M)$, we shall distinguish the operators acting on this space with a subscript $_\partial$, *e.g.* d_∂, $*_\partial$, δ_∂ or $(,)_\partial$.

Definition 1.2.3. *Let (M, g) be a pseudo-Riemannian manifold with boundary together with the embedding map $\iota_\partial : \partial M \hookrightarrow M$. We call* tangential *and* normal *maps*

$$\mathrm{t} \colon \Omega^k(M) \to \Omega^k(\partial M) \qquad \omega \mapsto \mathrm{t}\omega \doteq \iota_\partial^* \omega \tag{1.3a}$$

$$\mathrm{n} \colon \Omega^k(M) \to \Omega^{k-1}(\partial M) \qquad \omega \mapsto \mathrm{n}\omega \doteq *_\partial^{-1} \circ \mathrm{t} \circ *_M \,, \tag{1.3b}$$

In particular, for all $k \in \mathbb{N} \cup \{0\}$ we define

$$\Omega^k_{\mathrm{t}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\} \,, \qquad \Omega^k_{\mathrm{n}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\} \,. \tag{1.4}$$

Remark 1.2.4. The normal map $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ can be equivalently read as the restriction to ∂M of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(M)$ and the vector field $\nu \in \Gamma(TM)|_{\partial M}$ which corresponds pointwisely to the unit vector, normal to ∂M.

As last step, we observe that (1.3) together with (1.4) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$* \, \delta = (-1)^k \mathrm{d} * \,, \quad \delta * = (-1)^{k+1} * \mathrm{d} \,, \tag{1.5a}$$

$$*_\partial \, \mathrm{n} = \mathrm{t} * \,, \quad *_\partial \mathrm{t} = (-1)^k \mathrm{n} * \,, \quad \mathrm{d}_\partial \mathrm{t} = \mathrm{t}\mathrm{d} \,, \quad \delta_\partial \mathrm{n} = \mathrm{n}\delta \,. \tag{1.5b}$$

A notable consequence of (1.5b) is that, while on manifolds with empty boundary, the operators

d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. Indeed, a direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial \qquad \forall \alpha \in \Omega^k_{\mathrm{c}}(M)\,,\ \forall \beta \in \Omega^{k+1}_{\mathrm{c}}(M)\,, \tag{1.6}$$

where the pairing in the right-hand side is the one associated to forms living on ∂M.

1.3 Green operators

In this section we will follow mainly [Bär15]. Let $E_1, E_2 \to (M, g)$ be vector bundles over a globally hyperbolic spacetime (we will discuss separately the case of non-empty timelike boundary). Let $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ be a linear differential operator.

Definition 1.3.1. *An* advanced Green operator *of* P*, or* advanced fundamental solution *for* P*, is a linear map* $G^+ : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty(M, E_1)$ *such that*

(i) $G^+P = \mathrm{Id}_{C^\infty_{\mathrm{c}}(M,E_1)}$,

(ii) $PG^+ = \mathrm{Id}$,

(iii) $\mathrm{supp}(G^+f) \subset J^+(\mathrm{supp}\, f)$*, for all* $f \in C^\infty_{\mathrm{c}}(M, E_2)$.

Analogously, a linear map $G^- : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty(M, E_1)$ *is called a* retarded Green operator *of* P*, or* retarded fundamental solution *for* P *if* (i) *and* (ii) *hold, while holds*

(iii') $\mathrm{supp}(G^-f) \subset J^-(\mathrm{supp}\, f)$*, for all* $f \in C^\infty_{\mathrm{c}}(M, E_2)$.

Definition 1.3.2. *The operator* P *is called* Green hyperbolic *if* P *and* P^{t} *have advanced and retarded Green operator, where* $P^{\mathrm{t}} : C^\infty(M, E_2^*) \to C^\infty(M, E_1^*)$*, known as the* formal dual *of* P*, is the unique linear differential operator such that*

$$(\varphi, Pf)_M = (P^{\mathrm{t}}\varphi, f)_M, \quad \textit{i.e.} \quad \int_M \langle \varphi, Pf \rangle \,\mathrm{d}\mu_g = \int_M \left\langle P^{\mathrm{t}}\varphi, f \right\rangle \mathrm{d}\mu_g, \tag{1.7}$$

for all $f \in C^\infty(M, E_1)$ *and* $\varphi \in C^\infty(M, E_2^*)$ *such that* $\mathrm{supp}\, f \cap \mathrm{supp}\, \varphi$ *is compact.*

Remark 1.3.3. If (M, g) has empty boundary, the Green operators of a Green hyperbolic operator P are unique, see [Bär15, Cor. 3.12]. If the spacetime has a boundary, the differential operators must be given together with boundary conditions. These conditions are encoded in the domain of the operator, that is replaced by the subset $C^\infty_{\mathrm{b.c.}}(M, E_1) \subset C^\infty(M, E_1)$ of sections that satisfies the boundary conditions. Hence, in the case of non-empty boundary, the codomain $C^\infty(M, E_1)$ of G must be replaced, in Definitions 1.3.1 and 1.3.2, with the corresponding subspace $C^\infty_{\mathrm{b.c.}}(M, E_1)$.

Example 1.3.4. An important example of Green-hyperbolic operators are *wave operators*, or *normally hyperbolic operators*. Locally they are of the form

$$P = g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a^j(x)\frac{\partial}{\partial x^j} + b(x), \tag{1.8}$$

where g^{ij} denote the components of the inverse metric tensor, and a_j and b are smooth functions of x. Physically relevant examples of such operators are the *d'Alembert wave operator* acting on scalars ($E_1 = E_1 = \mathbb{R}$) $P = \Box$ and the Klein-Gordon operator $P = \Box + m^2$, $m > 0$. Moreover, in case $E_1 = E_2 = \Lambda^k T^* M$, we have the *d'Alembert-De Rham-Beltrami operator* $P = \Box_k = \mathrm{d}\delta + \delta\mathrm{d}$ acting on k-forms as well as the *Proca operator* $P = \Box_k + m^2$.
It is shown in [BGP15, Cor. 3.4.3] that (M, g) is a globally hyperbolic spacetime with empty boundary, wave operators as well as their formal duals (since they are wave operator themselves) have retarded and advanced Green operators. Hence, they are Green Hyperbolic.

Definition 1.3.5. *The operator $G := G^+ - G^- : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty(M, E_1)$ is called the* causal propagator *or* advanced minus retarded *Green operator.*

Remark 1.3.6. Recalling Definition 1.1.6 and the support properties of $G^\pm$ in Definition 1.3.1, we see that Green operators of P are in fact linear maps between the following spaces:

$$G^+ : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty_{\mathrm{spc}}(M, E_1), \tag{1.9}$$
$$G^- : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty_{\mathrm{sfc}}(M, E_1), \tag{1.10}$$
$$G : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty_{\mathrm{sc}}(M, E_1). \tag{1.11}$$

Moreover, as shown in [Bär15, Thm. 3.8, Cor. 3.10, 3.11], there are unique continuous linear extensions of $G^\pm$:

$$\overline{G}_+ : C^\infty_{\mathrm{pc}}(M, E_2) \to C^\infty_{\mathrm{pc}}(M, E_1) \quad \text{and} \quad \overline{G}_- : C^\infty_{\mathrm{fc}}(M, E_2) \to C^\infty_{\mathrm{fc}}(M, E_1), \tag{1.12}$$
$$\widetilde{G}_+ : C^\infty_{\mathrm{spc}}(M, E_2) \to C^\infty_{\mathrm{spc}}(M, E_1) \quad \text{and} \quad \widetilde{G}_- : C^\infty_{\mathrm{sfc}}(M, E_2) \to C^\infty_{\mathrm{sfc}}(M, E_1). \tag{1.13}$$

Proposition 1.3.7 (see Cor. 3.9, [Bär15])**.** *Let P be a Green hyperbolic operator. Then there are no nontrivial solutions $u \in C^\infty(M, E_1)$ of $Pu = 0$ with past-compact or future-compact support. In other words if u has past-compact or future-compact support, $Pu = 0$ implies $u = 0$. Moreover, for any $f \in C^\infty_{\mathrm{pc}}(M, E_2)$ or $f \in C^\infty_{\mathrm{fc}}(M, E_2)$ there exists a unique $u \in C^\infty(M, E_1)$ solving $Pu = f$ and such that $\mathrm{supp}(u) \subset J^+(\mathrm{supp}\, f)$ or $\mathrm{supp}(u) \subset J^-(\mathrm{supp}\, f)$, respectively.*

Remark 1.3.8. The solutions $u^\pm$ with different support properties of the equation $Pu = f$ provided in the former Proposition are given explicitly by $u^\pm = G^\pm(f)$. Hence u^+ is the unique solution to the following initial value problem:

$$\begin{cases} Pu = f \text{ in } M, \ f \in C^\infty_{\mathrm{pc}}(M, E_2), \\ u|_\Sigma = 0, \end{cases} \tag{1.14}$$

where $\Sigma \overset{\iota}{\hookrightarrow} M$ is any Cauchy surface that lies in the past of $\operatorname{supp} f$, i.e. $\iota(\Sigma) \subset J^-(\operatorname{supp} f)$. Analogously u^- is the unique solution with vanishing final data on any Cauchy surface in the future of $f \in C^\infty_{\mathrm{fc}}(M, E_2)$.

This discussion extends in the case of a spacetime with non-empty timelike boundary. Indeed, Proposition 1.3.7 extends, provided the existence of Green operators for a specified boundary condition. In this case, for example $u^+ = G^+_{\text{b.c.}}(f)$ is the solution to the initial data/boundary value problem

$$\begin{cases} Pu = f \text{ in } M,\ f \in C^\infty_{\mathrm{pc}}(M, E_2), \\ \text{boundary conditions on } \partial M, \\ u|_\Sigma = 0, \end{cases} \tag{1.15}$$

where, as before, Σ is any Cauchy surface such that $\Sigma \subset J^-(\operatorname{supp} f)$.

The following is an important theorem that will be generalized in case of non-empty timelike boundary. (see [BG12, Thm. 3.5])

Theorem 1.3.9. *Let G be the causal propagator of a Green-hyperbolic operator P on a spacetime with empty boundary. Then the following is an exact sequence:*

$$0 \longrightarrow C^\infty_{\mathrm{c}}(M, E_1) \xrightarrow{P} C^\infty_{\mathrm{c}}(M, E_2) \xrightarrow{G} C^\infty_{\mathrm{sc}}(M, E_1) \xrightarrow{P} C^\infty_{\mathrm{sc}}(M, E_2) \longrightarrow 0. \tag{1.16}$$

In the case of non-empty boundary, the existence of Green operators and all their properties must be proven for any suitable class of boundary conditions, and that will be the main focus of Chapters 2 and **??** in case P is Maxwell operator.

Example 1.3.10. (Wave operator on $\mathbb{R} \times \mathbb{R}_+$)
We consider the problem of the existence and the construction of advanced and retarded Green operators of $\Box = -\partial_t^2 + \partial_x^2$ on $M = \mathbb{R} \times \mathbb{R}_+ \ni (t, x)$. Clearly M is a globally hyperbolic spacetime with timelike boundary, endowed with the usual Minkowski metric $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2$. The boundary is the set $\{x = 0\}$. Given some initial condition, the differential equation $\Box u = f$, with $f \in C^\infty(M)$, is well posed (i.e. there exists a unique solution) provided one require u to satisfy some suitable boundary condition. We now construct explicitly the Green operators for $\Box$ on M with Dirichlet and Neumann boundary conditions using the Green operators for $\Box$ on $(\mathbb{R}^2, \eta)$, which existence is well known.
We define $\Box_D : C^\infty_D(M) \to C^\infty(M)$ and $\Box_N : C^\infty_N(M) \to C^\infty(M)$, with $C^\infty_D(M) := \{u \in C^\infty(M) \,|\, u|_{x=0} = 0\} = \Omega^0_{\mathrm{t}}(M)$ and $C^\infty_N(M) := \{u \in C^\infty(M) \,|\, \partial_x u|_{x=0} = 0\}$. The problem is to find the following advanced and retarded Green operators

$$G^\pm_D : C^\infty_{\mathrm{c}}(M) \to C^\infty_D(M), \quad G^\pm_N : C^\infty_{\mathrm{c}}(M) \to C^\infty_N(M). \tag{1.17}$$

As stated in [Bär15, Ex. 3.4], advanced and retarded Green operators for $\Box$ on $\mathbb{R}^2$ exist and have the following explicit expression

$$G^{\pm}(f)(t,x) = -\frac{1}{2}\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y. \tag{1.18}$$

This expression entails that the integral kernel of $G^{\pm}$ (also known as Green function or fundamental solution) is $-\frac{1}{2}$ times the characteristic function of $\{(t,x,s,y)\in\mathbb{R}^4 \mid (s,y)\in J^{\mp}(t,x)\}$. The ansatz, based on the method of images ([Jac99, p. 480]), is that the Dirichlet and Neumann Green operators will be respectively of the form

$$\begin{aligned}
G^{\pm}_D(f)(t,x) &= G^{\pm}(f)(t,x) - G^{\pm}(f)(t,-x) = \\
&= -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y - \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M,\\
G^{\pm}_D(f)(t,x) &= G^{\pm}(f)(t,x) + G^{\pm}(f)(t,-x) = \\
&= -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y + \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M.
\end{aligned}$$

It is a very easy calculation to verify $G^{\pm}_{D/N}(f)\in C^{\infty}_{D/N}(M)$ (i.e. $G^{\pm}_D(f)(t,x)|_{x=0}=0$ and $\partial_x G^{\pm}_D(f)(t,x)|_{x=0}=0$) and clearly the support properties still hold.
Focusing on the Dirichlet Green operators, they are constructed by imagining to extend the manifold M by reflection to be the entire $\mathbb{R}^2$ and, to enforce $G^{\pm}(f)$ to vanish on $x=0$, add a negative reflected source $-f(t,-x)$. This gives the desired result.

1.4 Maxwell equations for k-forms with empty boundary

non sono troppo sicuro che questa parte abbia il suo senso.

We now focus our attention on a m-dimensional spacetime (M,g) with empty boundary. Classically, electromagnetism is the theory of electric and magnetic fields E, B encoded in the Faraday 2-form F. The equations for $F\in\Omega^2(M)$ read

$$\begin{aligned}
\mathrm{d}F &= 0,\\
\delta F &= -J,
\end{aligned} \tag{1.19}$$

where J is the current 1-form, subject to the condition $\delta J = 0$, which encodes the behavior of electric charges and currents. Indeed, if M is static with $M=\mathbb{R}\times\Sigma$, the decomposition $F = B + \mathrm{d}t\wedge E$ holds, where $E\in\Omega^1(M)$ and $B\in C^{\infty}(\mathbb{R},\Omega^2(\Sigma))$, in agreement with the fact that the magnetic field B is usually referred to as a *pseudo-vector*.
The first equation imposes a geometric constraint: it ensures that the 2-form F is closed. Hence, in virtue of Poincaré lemma, whenever the second cohomology group $H^2(M)$ is trivial, there

exists a global 1-form A such that $F = \mathrm{d}A$. One can object that the choice of $A \in \Omega^1(M)$ is not unique. Indeed if we assume M to be globally hyperbolic with empty boundary, the configuration $A' := A + \mathrm{d}\chi$, $\chi \in \Omega^0(M)$ is equivalent to A since it gives rise to the same Faraday field F. This freedom in the choice of A is extensively used and it is called **gauge freedom** or gauge invariance. In this case A, A' are said to be gauge-equivalent.

Thanks to gauge invariance we can therefore write Maxwell equations for A, $\delta\mathrm{d}A = -J$, in a more convenient way. In fact, taken any fixed $A \in \Omega^1(M)$, and imposing the so-called **Lorenz gauge**, one can substitute the problem $\delta\mathrm{d}A = -J$ with the following hyperbolic system of equations

$$\begin{cases} \Box A = -J, \\ \delta A = 0. \end{cases} \tag{1.20}$$

where $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ is the wave operator. Moreover the second can be seen as a constraint on the first and it is called the *Lorenz gauge condition*. This system can be obtained by requiring a 1-form A', gauge-equivalent to A, to satisfy the Lorenz gauge condition $\delta A' = 0$. This is always possible in a globally hyperbolic spacetime with empty boundary since the equation $\Box\chi = \delta A$ has always at least a solution $\chi \in \Omega^0(M)$ for any fixed $A \in \Omega^1(M)$.

One could argue that the most general possible gauge transformation between A and A' is of the form $A' = A + \omega$ for a closed form $\omega \in \Omega^1(M)$. That is certainly true in the sense that the equations of motion (1.19) are unchanged by this transformation. Anyway we will refer to gauge-invariance exclusively in the sense previously defined since electromagnetism can be seen as an abelian **gauge theory** with structure group $U(1)$. In this framework, the classical vector potential A is a principal connection on a principal $U(1)$-bundle E over M (for more details see [Nak90, Ch. 10]). Then we identify (this choice is non-unique) the connection A with a 1-form $A \in \Omega^1(M)$. Locally, this principal connection can be expressed as an $U(1)$-valued operator $D = \mathrm{d} + A$ and the Faraday field can be recovered as the curvature of this connection: $F = D \circ D$. A gauge transformation for A in this context is of the form

$$A' = g^{-1}Ag - ig^{-1}\mathrm{d}g, \tag{1.21}$$

for any $g \in C^\infty(M, U(1))$. If now we express $g = e^{i\chi}$, for $\chi \in \Omega^0(M)$, we recover the transformation $A' := A + \mathrm{d}\chi$.

From a physical point of view, one wonders whether it is A or it is F the observable field of the dynamical system. Hence one can regard the electromagnetism as a theory for $F \in \Omega^2(M)$ or as a theory for a non-unique $A \in \Omega^1(M)$ and wonder if the initial and boundary value problem for Maxwell equations is well-posed in both cases. The former case for F will be covered in Chapter 2 and the latter for A in Chapter **??**.

In the homogeneous case ($J = 0$), one can generalise the Maxwell field to be $F \in \Omega^k(M)$, imposing $\mathrm{d}F = 0$ and $\delta F = 0$ and the equation for $A \in \Omega^{k-1}(M)$ becomes $\delta \mathrm{d} A = 0$. In this case gauge freedom is understood as a transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-2}(M)$.

Chapter 2

Maxwell equations with interface conditions

As outlined in Section 1.4, the very nature of Maxwell equations allows us to use both F and A as variables with which to describe electromagnetic phenomena. Whenever the second cohomology group $H^2(M)$ is trivial, the two theories are equivalent, since $F = \mathrm{d}A$.

In this chapter, we regard $F \in \Omega^2(M)$ as the true physical dynamical variable which describes electromagnetism. The aim of this chapter is to present a technique which allows to characterize, in a class of manifolds with the presence of an interface between two media, the existence of fundamental solutions for Maxwell equations, written in terms of the Faraday form $F \in \Omega^2(M)$. The presence of an interface on the one hand generalizes the idea of the presence of a timelike boundary, allowing to recover the geometric setting outlined in Chapter 1 in case on one side of the interface lies a perfect insulator. On the other hand, in order to make use of geometric techniques such as Hodge decomposition, we will have to make several geometric assumptions which ensure global hyperbolicity, but unfortunately are way less general.

2.1 Geometrical set-up

The physical and practical situation we want to approach is that of a manifold split into two parts, filled with two media, each of them with different electromagnetic properties. The two media will be separated by an hypersurface, on which our aim will be that of putting *jump conditions*.

We consider a static Lorentzian manifold (M, g) with **empty boundary**, such that M can be decomposed as $\mathbb{R} \times \Sigma$, where the Cauchy hypersurface (Σ, h) is assumed to be a complete, connected, odd-dimensional, **closed** Riemannian manifold. The assumptions we made so far on imply that (M, g) is a globally hyperbolic spacetime without boundary.
Maxwell equations, recalling formulas (**??**) and (**??**), for $F \in \Omega^2(M)$ are simply

$$\mathrm{d}F = 0\,, \qquad \delta F = 0\,, \tag{2.1}$$

The geometrical assumptions on M permit us to split F into electric and magnetic components

$$F = *_\Sigma B + \mathrm{d}t \wedge E\,, \tag{2.2}$$

where $E, B \in \Omega^1(\Sigma)$ while $*_\Sigma$ is the Hodge dual on Σ.
Maxwell equations are then reduced to the usual

$$\partial_t E - \mathrm{curl} B = 0\,, \qquad \partial_t B + \mathrm{curl} E = 0\,, \tag{2.3a}$$

$$\mathrm{div}(E) = \mathrm{div}(B) = 0\,, \tag{2.3b}$$

where $\mathrm{div} = \delta_\Sigma$ is the co-differential of Σ, while curl is defined in equation (2.40) – in particular $\mathrm{curl} = *_\Sigma \mathrm{d}_\Sigma$ if $\dim \Sigma = 3 \mod 4$.
To model the presence of an interface that divides M in two distinct regions, we also let Z be a codimension 1 smooth embedded hypersurface of Σ.

We denote with d, δ the differential and co-differential over M, while $\mathrm{d}_\Sigma, \delta_\Sigma$ denote the differential and co-differential over Σ.

In this setting we would like to consider Maxwell equations with Z-interface boundary conditions. This means that we will consider Maxwell equations on $M \setminus (\mathbb{R} \times Z)$, allowing for jump discontinuities to occur on $\mathbb{R} \times Z$. Hence, we split $\Sigma = \Sigma_+ \cup \Sigma_-$, such that

$$\Sigma_Z := \Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-\,, \tag{2.4}$$

and we refer to Σ_- (*resp.* Σ_+) as the left (*resp.* right) component of Σ. Moreover, $\Sigma_\pm$ are compact manifolds with boundary $\partial\Sigma_\pm = \pm Z$, and the orientation on Z induced by Σ_+ is the opposite of the one induced by Σ_-. Hence, the manifolds $(\mathbb{R} \times \Sigma_\pm, g = -\mathrm{d}t^2 + h)$ are **globally hyperbolic spacetimes with timelike boundary**, which is $\mathbb{R} \times Z$.

Whenever $Z \neq \emptyset$ the system (2.3) has to be modified, in particular the non-dynamical equations (2.3b) involving the divergence operator div have to be suitably interpreted – *cf.* Subsection 2.2.4. In particular one expects that the condition $\mathrm{div}(E) = \mathrm{div}(B) = 0$ should be interpreted weakly, leading to a constraint on the normal jump of E across Z. Moreover, the dynamical equations (2.3a) have to be combined with boundary conditions at the interface Z – *cf.* [Jac99, Sec. I.5].

In what follows we will state the precise meaning of the problem (2.3) with interface Z with the help of Hodge theory and Lagrangian subspaces [EM99, EM03, EM05].

2.2 Constraint equations: Hodge theory with interface

In this section we present a Hodge decomposition for the closed Riemannian manifold (Σ, h) with interface Z. This generalizes the known results on classical Hodge decomposition on manifolds with possible non-empty boundary [Ama17, AM04, Gaf55, Gro91, Kod49, Li09, Sch95, Sco95, ZS00].

Hodge theory comes as a generalization of Helmholtz decomposition. He first formulated a result on the splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the *Hodge decomposition*. The idea behind Helmholtz decomposition is that any vector field in $\mathbb{R}^3$ can be decomposed as a sum of an irrotational field, i.e. $\mathrm{curl} = \mathrm{d}_\Sigma = 0$, and a solenoidal field, i.e. $\mathrm{div} = \delta_\Sigma = 0$. In other words, for $\mathbf{F} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$, one can write

$$\mathbf{F} = -\nabla\Phi + \mathrm{curl}\,\mathbf{A}. \tag{2.5}$$

In what follows $\mathrm{L}^2\Omega^k(\Sigma)$ will denote the space of k-forms (see Section 1.2) which are square integrable with respect to the pairing induced by the metric h

$$(\alpha, \beta)_\Sigma := \int_\Sigma \overline{\alpha} \wedge *_\Sigma \beta\,, \tag{2.6}$$

where $*_\Sigma$ is the Hodge dual.

Definition 2.2.1. *We shall define the following*

- $C^\infty_\mathrm{c}\Omega^k(\Sigma)$ *the space of smooth and compactly supported k-forms (if Σ is compact, it coincides with the space of smooth k-forms $C^\infty\Omega^k(\Sigma)$, but we will still use $C^\infty_\mathrm{c}\Omega^k(\Sigma)$ in view of possible generalizations),*
- $\mathrm{H}^\ell\Omega^k(\Sigma)$ *the space of k-forms with weak L^2-derivatives up to order $\ell \in \mathbb{N} \cup \{0\}$ with respect to one (hence all) connection over Σ ($\mathrm{H}^\ell\Omega^k(\Sigma)$ is independent of the choice of the connection whenever Σ is compact or of bounded geometry),*
- $\mathrm{H}^{-\ell}\Omega^k(\Sigma) := \mathrm{H}^\ell_0\Omega^k(\Sigma)^*$, *where $*$ indicates the dual with respect to the scalar product $(\,,\,)_\Sigma$.*

2.2.1 Hodge decomposition on compact manifold with non-empty boundary

The Hodge theorem for a closed manifold Σ states that there is an L^2-orthogonal decomposition

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}(\Sigma) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}(\Sigma) \oplus \ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}\,, \tag{2.7}$$

where $\Delta = \mathrm{d}_\Sigma\delta_\Sigma + \delta_\Sigma\mathrm{d}_\Sigma$ is the Laplacian and $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ denotes the space of *harmonic forms*. If Σ has no an empty boundary, the space of harmonic forms $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ coincides with that of **harmonic fields**, defined (following [Kod49] and [Sch95]) as

$$\{\omega \in \mathrm{H}^1\Omega^k(\Sigma) |\ \mathrm{d}_\Sigma\omega = 0\,,\ \delta_\Sigma\omega = 0\} = \ker(\delta_\Sigma)_{\mathrm{H}^1\Omega^k(\Sigma)} \cap \ker(\mathrm{d}_\Sigma)_{\mathrm{H}^1\Omega^k(\Sigma)}\,. \tag{2.8}$$

The last result can be stated as follows and it is very easy to prove.

Proposition 2.2.2. *Let $\alpha \in \mathrm{H}^1\Omega^k(\Sigma)$, where Σ is a closed manifold. Then $\Delta\alpha = 0$ if and only if $\mathrm{d}_\Sigma\alpha = 0$ and $\delta_\Sigma\alpha = 0$.*

Proof. Clearly if $\mathrm{d}_\Sigma \alpha = 0$ and $\delta_\Sigma \alpha = 0$, $\Delta\alpha = 0$. On the other hand if $\Delta\alpha = 0$,

$$0 = (\Delta\alpha, \alpha)_\Sigma = ((\mathrm{d}_\Sigma\delta_\Sigma + \delta_\Sigma \mathrm{d}_\Sigma)\alpha, \alpha)_\Sigma = (\mathrm{d}_\Sigma\delta_\Sigma\alpha, \alpha)_\Sigma + (\delta_\Sigma \mathrm{d}_\Sigma\alpha, \alpha)_\Sigma = \tag{2.9}$$

$$= (\delta_\Sigma\alpha, \delta_\Sigma\alpha)_\Sigma + (\mathrm{d}_\Sigma\alpha, \mathrm{d}_\Sigma\alpha)_\Sigma = \|\delta_\Sigma\alpha\|^2 + \|\mathrm{d}_\Sigma\alpha\|^2. \tag{2.10}$$

So both $\mathrm{d}_\Sigma\alpha = 0$ and $\delta_\Sigma\alpha = 0$. □

For a compact manifold Σ with non-empty boundary $\partial\Sigma$ the decomposition (2.7) requires a slight adjustment and harmonic forms do not coincide with harmonic fields anymore. Because of boundary terms, $\ker\Delta$ no longer coincides with the closed and co-closed forms. It is clear that every harmonic field is a harmonic form, but the converse is false. To show this, consider the following example.

Example 2.2.3. Let U a bounded subset of $\mathbb{R}^2$, endowed with the standard euclidean metric. On U, the 1-form $\omega = x\,dy$ is clearly harmonic, since its second derivatives vanish, but it is not in $\ker \mathrm{d}$ as

$$\mathrm{d}(x\,\mathrm{d}y) = \partial_x\, x\,\mathrm{d}x \wedge \mathrm{d}y + \partial_y\, x\,\mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y.$$

ω is though in $\ker\delta$ as $*\mathrm{d} * (x\,\mathrm{d}y) = *\mathrm{d}(x\,\mathrm{d}x) = 0$.

Definition 2.2.4. *From now on, we define $\mathcal{H}^k(\Sigma)$ as the L^2-closure of the space of harmonic fields*

$$\mathcal{H}^k(\Sigma) = \overline{\{\omega \in \mathrm{H}^1\Omega^k(\Sigma) |\; \mathrm{d}_\Sigma\omega = 0\,,\;\; \delta_\Sigma\omega = 0\}}^{\mathrm{L}^2}\,. \tag{2.11}$$

In fact, the space of harmonic fields is infinite dimensional and so is much too big to represent the cohomology, and to recover the Hodge isomorphism one has to impose boundary conditions. Indeed the spaces $\mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}(\Sigma)$, $\delta_\Sigma H^1\Omega^{k+1}(\Sigma)$, $\mathcal{H}^k(\Sigma)$ are not orthogonal unless suitable boundary conditions are imposed. Therefore, one has to give a precise meaning to the boundary value of a differential form. As differential forms are not scalar quantities, there is either the possibility to trace tangentially or normally to the boundary.

Remark 2.2.5. As usual, trace maps have image that lies in a fractional Sobolev space $\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma)$, defined commonly as in [LM72]. According to [Geo79, p. 171] and [Wec04, Sec. 2], the tangential and normal maps defined in Definition 1.2.3 can be extended to continuous surjective maps

$$\mathrm{t}\oplus\mathrm{n}\colon \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \oplus \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \qquad \forall \ell \geq \frac{1}{2}\,. \tag{2.12}$$

We are now the Hodge decomposition for compact manifolds with boundary [Sch95, Thm. 2.4.2].

Theorem 2.2.6. *Let (Σ, h) be a compact, connected, Riemannian manifold with non-empty boundary*

1. *For all* $\omega \in C^\infty_c\Omega^{k-1}(\Sigma)$ *and* $\eta \in C^\infty_c\Omega^k(\Sigma)$ *it holds*

$$(\mathrm{d}_\Sigma\omega, \eta)_\Sigma - (\omega, \delta_\Sigma\eta)_\Sigma = (\mathrm{t}\omega, \mathrm{n}\eta)_{\partial\Sigma}\,, \tag{2.13}$$

where $(\ ,\ )_\Sigma$ *has been defined in equation* (2.6) *while* $(\ ,\ )_{\partial\Sigma}$ *is defined similarly. Equation* (2.13) *still holds true for* $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ *and* $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma)$ *– cf. remark* 2.2.5*.*

2. *The Hilbert space* $\mathrm{L}^2\Omega^k(\Sigma)$ *of square integrable* k*-forms splits into the* L^2*-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.14}$$

where $\mathcal{H}^k(\Sigma)$ *is the closure with respect to the* L^2 *norm of the space of harmonic fields, as defined per equation* (2.11) *and*

$$\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma)|\ \mathrm{t}\alpha = 0\}\,, \tag{2.15}$$
$$\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma)|\ \mathrm{n}\beta = 0\}\,, \tag{2.16}$$

following the definitions of Equation (1.4)*.*

Sketch of proof. We first observe that the decomposition is direct. The spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma)$, $\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)$ and $\mathcal{H}^k(\Sigma)$ are mutually orthogonal to each other with respect to the inner product on $\mathrm{L}^2\Omega^k(\Sigma)$, which is an immediate consequence of Equation (2.13). Hence the Hodge decomposition - if it is established - is a L^2-orthogonal splitting. It remains to show that the decomposition (2.14) is complete. In particular it suffices to show that

1. each $\omega \in \mathrm{L}^2\Omega^k(\Sigma)$ splits uniquely as $\omega = \mathrm{d}_\Sigma\alpha + \delta_\Sigma\beta + \kappa$, with $\alpha \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)$ and $\kappa \in \left(\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)\right)^\perp$;
2. the spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma)$ and $\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)$ are closed in the L^2 topology;
3. the L^2-orthogonal complement $\left(\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)\right)^\perp$ coincides with $\mathcal{H}^k(\Sigma)$.

Remark 2.2.7. The previous decomposition generalizes to Sobolev spaces, in particular for all $\ell \in \mathbb{N} \cup \{0\}$ we have

$$\mathrm{H}^\ell\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^{\ell+1}\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma\mathrm{H}^{\ell+1}\Omega^{k+1}_{\mathrm{n}} \oplus \mathrm{H}^\ell\mathcal{H}^k(\Sigma)\,, \tag{2.17}$$

where $\mathrm{H}^\ell\mathcal{H}^k(\Sigma) = \mathcal{H}^k(\Sigma) \cap \mathrm{H}^\ell\Omega^k(\Sigma)$, since $\mathrm{H}^\ell\Omega^k(\Sigma) \hookrightarrow \mathrm{L}^2\Omega^k(\Sigma)$.

2.2.2 Hodge decomposition for compact manifold with interface

The aim of this section is to generalize Theorem 2.2.6 for the case of a closed Riemannian manifold Σ with interface Z.

Definition 2.2.8. *We define any space of k-forms on Σ_Z (for any kind of regularity) as follows:*

$$\Omega^k(\Sigma_Z) := \Omega^k(\Sigma_+) \oplus \Omega^k(\Sigma_-)\,, \tag{2.18}$$

where the pair $\omega+\eta \in \Omega^k(\Sigma_+)\oplus\Omega^k(\Sigma_-)$ identifies an element $\alpha \in \Omega^k(\Sigma_Z)$ such that $\alpha|_{\Sigma_+} = \omega$ and $\alpha|_{\Sigma_-} = \eta$.

Remark 2.2.9. The space $C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$, following the previous definition, is defined as

$$C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z) = C^\infty_{\mathrm{c}}\Omega^k(\Sigma_+) \oplus C^\infty_{\mathrm{c}}\Omega^k(\Sigma_-)\,. \tag{2.19}$$

This implies $\omega \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ if and only if ω is a smooth k-form in Σ_Z and $\mathrm{supp}_\Sigma\,\omega := \overline{\{x \in \Sigma_Z \,|\, \omega(x) \neq 0\}}^{\Sigma}$ is compact. Hence, forms in $C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ can touch the interface, but are allowed to have discontinuities.

The splitting of Σ in $\Sigma_\pm$ as in Equation (2.4) holds, therefore theorem 2.2.6 applies to both $\mathrm{L}^2\Omega^k(\Sigma_\pm)$. Since Z has zero measure the space of square integrable k-forms splits as

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{L}^2\Omega^k(\Sigma_Z) = \mathrm{L}^2\Omega^k(\Sigma_+) \oplus \mathrm{L}^2\Omega^k(\Sigma_-)\,. \tag{2.20}$$

We expect a Z-relative Hodge decomposition as in (2.14) to hold true in this situation, where the boundary conditions of the spaces $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\mathrm{H}^1\Omega^{k-1}_{\mathrm{n}}(\Sigma)$ should be replaced by appropriate jump conditions across Z. For that, notice that the splitting (2.20) does not generalize to the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$, in particular

$$\mathrm{H}^\ell\Omega^k(\Sigma) \subset \mathrm{H}^\ell\Omega^k(\Sigma_Z) = \mathrm{H}^\ell\Omega^k(\Sigma_+) \oplus \mathrm{H}^\ell\Omega^k(\Sigma_-)\,, \tag{2.21}$$

is a proper inclusion. Indeed, consider any regular form ω in Σ_Z which has $[\mathrm{t}\omega] \neq 0$. In this case ω can not have square integrable (weak) derivatives.

Definition 2.2.10. *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface $Z \hookrightarrow \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. For $\omega \in C^\infty\Omega^k(\Sigma_Z)$ we define the tangential jump $[\mathrm{t}\omega] \in C^\infty\Omega^k(Z)$ and normal jump $[\mathrm{n}\omega] \in C^\infty\Omega^{k-1}(Z)$ across Z by*

$$[\mathrm{t}\omega] := \mathrm{t}_+\omega - \mathrm{t}_-\omega\,, \qquad [\mathrm{n}\omega] := \mathrm{n}_+\omega - \mathrm{n}_-\omega\,, \tag{2.22}$$

where $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ denote the tangential and normal map on $\Sigma_\pm$ as per Definition 1.2.3.

Remark 2.2.11. As usual the tangential and normal traces $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ as well as the tangential and normal jump extend by continuity on $\mathrm{H}^1\Omega^k(\Sigma_Z)$ and are surjective if the codomain is $\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(Z)$ - *cf.* Remark 2.2.5. It is an immediate consequence of Definition 2.2.10 that

$$\mathrm{H}^1\Omega^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma_Z) |\; [\mathrm{t}\omega] = 0\,,\; [\mathrm{n}\omega] = 0\}\,. \tag{2.23}$$

An analogous equality does not hold for $C^\infty\Omega^k(\Sigma)$ because traces of higher order derivatives have to match at Z.

Theorem 2.2.12. *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface Z. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-$.*

1. *For all $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ it holds*

$$(\mathrm{d}_\Sigma\omega, \eta)_Z - (\omega, \delta_\Sigma\eta)_Z = ([\mathrm{t}\omega], \mathrm{n}_+\eta)_Z - (\mathrm{t}_-\omega, [\mathrm{n}\eta])_Z\,, \tag{2.24}$$

   where $(\ ,\ )_Z$ is the scalar product between forms on Z – cf. equation (2.6) – while $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ are the tangential and normal maps on $\Sigma_\pm$ as per definition 1.2.3. Equation (2.13) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for all $\ell \geq 1$.

2. *The Hilbert space $\mathrm{L}^2\Omega^k(\Sigma)$ of square integrable k-forms splits into the L^2-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.25}$$

   where $\mathcal{H}^k(\Sigma)$ is the closure with respect to the L^2 norm of the space of harmonic fields, as defined per equation (2.11) and

$$\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma_Z)|\ [\mathrm{t}\alpha] = 0\}\,, \tag{2.26}$$

$$\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma_Z)|\ [\mathrm{n}\beta] = 0\}\,. \tag{2.27}$$

Proof. Equation (2.24) is an immediate consequence of (2.13). In particular for $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ we decompose $\omega = \omega_+ + \omega_-$ and $\eta = \eta_+ + \eta_-$ where $\omega_\pm \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_\pm)$ and $\eta_\pm \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_\pm)$. (Notice that with this notation we have $\mathrm{t}_\pm\omega = \mathrm{t}_\pm\omega_\pm$.) Applying equation (2.13) we have

$$\begin{aligned}(\mathrm{d}_\Sigma\omega, \eta) - (\omega, \delta_\Sigma\eta) &= \sum_\pm ((\mathrm{d}_\Sigma\omega_\pm, \eta_\pm) - (\omega_\pm, \delta_\Sigma\eta_\pm)) = \int_Z \mathrm{t}_+\overline{\omega} \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *_\Sigma\mathrm{n}_-\eta \\ &= \int_Z [\mathrm{t}\overline{\omega}] \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *[\mathrm{n}\beta]\,.\end{aligned}$$

A density argument leads to the same identity for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for $\ell \geq 1$.

We now prove the splitting (2.25). The spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)$, $\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$, $\mathcal{H}^k(\Sigma)$ are orthogonal because of equation (2.24). Let now ω be in the orthogonal complement of $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. We wish to show that $\omega \in \mathcal{H}^k(\Sigma)$. We split $\omega = \omega_+ + \omega_-$ with $\omega_\pm \in$

$\mathrm{L}^2\Omega^k(\Sigma_\pm)$ and apply theorem 2.2.6 to each component so that

$$\omega = \sum_\pm (\mathrm{d}_\Sigma\alpha_\pm + \delta_\Sigma\beta_\pm + \kappa_\pm)\,,$$

where $\alpha_\pm \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_\pm)$, $\beta_\pm \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_\pm)$ and $\kappa_\pm \in \mathcal{H}^k(\Sigma_\pm)$. Let now be $\hat{\alpha} \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_+)$: this defines an element in $\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ by considering its extension by zero on Σ_-. Since $\omega \perp \mathrm{d}_\Sigma\mathrm{H}^1\Omega_{[\mathrm{t}]}(\Sigma_Z)$ we have $0 = (\mathrm{d}_\Sigma\hat{\alpha}, \omega) = (\mathrm{d}_\Sigma\hat{\alpha}, \mathrm{d}_\Sigma\alpha_+)$, thus $\mathrm{d}_\Sigma\alpha_+ = 0$ by the arbitrariness of $\hat{\alpha}$. With a similar argument we have $\alpha_- = 0$ as well as $\beta_\pm = 0$.
Therefore $\omega \in \mathcal{H}^k(\Sigma_Z)$. In order to prove that $\omega \in \mathcal{H}^k(\Sigma)$ we need to show that $[\mathrm{t}\omega] = 0$ as well as $[\mathrm{n}\omega] = 0$ – *cf.* remark 2.2.11. This is a consequence of $\omega \perp \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)\oplus\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. Indeed, let $\alpha \in \mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$: applying equation (2.24) we find

$$0 = (\mathrm{d}_\Sigma\alpha, \omega) = -\int_Z \mathrm{t}_-\overline{\alpha} \wedge *[\mathrm{n}\omega]\,. \tag{2.28}$$

The arbitrariness of $\mathrm{t}_-\alpha$ (t_- is surjective) implies $[\mathrm{n}\omega] = 0$. Similarly $[\mathrm{t}\omega] = 0$ follows by $\omega \perp \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. □

Remark 2.2.13. The harmonic part of decomposition (2.25) contains harmonic k-forms which are continuous across the interface Z – *cf.* remark 2.2.11. One can also consider a decomposition which allows for a discontinuous harmonic component: in particular it can be shown that

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma_Z)\,,$$

where now $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)$ is the subspace of $\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ made of $(k-1)$-forms α such that $\mathrm{t}_\pm\omega = 0$ and similarly $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)$ if and only if $\beta \in \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$ and $\mathrm{n}_\pm\beta = 0$.

2.2.3 Further perspectives on Hodge decomposition

The results of theorem 2.2.6 generalize in several directions. In 1949, Kodaira (see [Kod49]) proved a *weak* L^2 orthogonal decomposition, for any (non-compact) Riemannian manifold (M, g) with no boundary, of the form

$$\mathrm{L}^2\Omega^k(M) = \overline{\mathrm{d}C^\infty_{\mathrm{c}}\Omega^{k-1}(M)} \oplus \overline{\delta C^\infty_{\mathrm{c}}\Omega^{k+1}(M)} \oplus \mathcal{H}^k(M)\,. \tag{2.29}$$

Gromov, in [Gro91], proved that under the assumption that the Laplacian has a spectral gap in $\mathrm{L}^2\Omega^k(M)$, i.e. there is no spectrum of Δ in an open interval $(0, \eta)$, with $\eta > 0$, the following strong L^2-orthogonal decomposition holds for any (non-compact) Riemannian manifold (M, g) with no boundary:

$$\mathrm{L}^2\Omega^k(M) = \mathrm{dH}^1\Omega^{k-1}(M) \oplus \delta\mathrm{H}^1\Omega^{k+1}(M) \oplus \mathcal{H}^k(M)\,. \tag{2.30}$$

For the case ∂M, the paper by Amar, [Ama17], recovers a strong L^p decomposition for complete non-compact manifolds, while both [Li09] and [ZS00] reach the strong L^p decomposition within the framework of weighted Sobolev spaces. In addition, [Sco95] provides a strong L^p-decomposition on compact manifolds. Finally, Schwartz [Sch95] himself provides an extension to the Hodge decomposition on non-compact manifolds with non-empty boundary in case M is the complement of an open bounded domain in $\mathbb{R}^n$ using weighted Sobolev spaces.

The papers by [AM04, Gaf55] are devoted to develop Hodge decomposition from the point of view of Hilbert space theory, thus arriving at it without the use of differential equation theory as in [Sch95]. For the case of a non-compact Riemannian manifold Σ one may follow the results of [AM04] in order to achieve the following weak-Hodge decomposition – *cf.* equation (2.14). We consider the operators $\mathrm{d}_{\Sigma,\mathrm{t}}, \delta_{\Sigma,\mathrm{n}}$ defined by

$$\mathrm{dom}(\mathrm{d}_{\Sigma,\mathrm{t}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ \mathrm{t}\omega = 0\} \qquad \mathrm{d}_{\Sigma,\mathrm{t}}\omega := \mathrm{d}_\Sigma\omega\,, \tag{2.31}$$
$$\mathrm{dom}(\delta_{\Sigma,\mathrm{n}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ \mathrm{n}\omega = 0\} \qquad \delta_{\Sigma,\mathrm{n}}\omega := \delta_\Sigma\omega\,. \tag{2.32}$$

Notice that $\mathrm{d}_{\Sigma,\mathrm{t}}$ as well as $\delta_{\Sigma,\mathrm{n}}$ are nihilpotent because of relations (1.5). These operators are closed and from equation (2.13) it follows that their adjoints are the following:

$$\mathrm{dom}(\mathrm{d}_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\}\,, \qquad \delta^*_{\Sigma,\mathrm{n}} = \mathrm{d}_\Sigma\,,$$
$$\mathrm{dom}(\delta_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\}\,, \qquad \mathrm{d}^*_{\Sigma,\mathrm{t}} = \delta_\Sigma\,.$$

It then follows immediately that $(\overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})})^\perp = \overline{\ker(\mathrm{d}) \cap \ker\delta} = \mathcal{H}^k(\Sigma)$ so that

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})} \oplus \mathcal{H}^k(\Sigma)\,. \tag{2.33}$$

Following the same steps of proof of theorem 2.2.12 it follows that a similar weak-Hodge decomposition holds for the case of non-compact Riemannian manifolds Σ with interface Z, actually

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,[\mathrm{t}]})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,[\mathrm{n}]})} \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.34}$$

where $\mathrm{d}_{\Sigma,[\mathrm{t}]}, \delta_{\Sigma,[\mathrm{n}]}$ are defined by

$$\mathrm{dom}(\mathrm{d}_{\Sigma,[\mathrm{t}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ [\mathrm{t}\omega] = 0\} \qquad \mathrm{d}_{\Sigma,[\mathrm{t}]}\omega := \mathrm{d}_\Sigma\omega\,,$$
$$\mathrm{dom}(\delta_{\Sigma,[\mathrm{n}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\ \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ [\mathrm{n}\omega] = 0\} \qquad \delta_{\Sigma,[\mathrm{n}]}\omega := \delta_\Sigma\omega\,.$$

This time $\mathrm{d}^*_{\Sigma,[\mathrm{t}]} = \delta_{\Sigma,[\mathrm{n}]}$ as well as $\delta^*_{\Sigma,[\mathrm{n}]} = \mathrm{d}_{\Sigma,[\mathrm{t}]}$ so that in particular $\ker\mathrm{d}^*_{\Sigma,[\mathrm{t}]} \cap \ker\delta_{\Sigma,[\mathrm{n}]} = \mathcal{H}^k(\Sigma)$.

2.2.4 Constraints Maxwell equations

The Hodge decomposition with interface proved in Theorem 2.2.12 can be exploited to formulate the correct generalization of the non-dynamical Maxwell equations (2.3b) as follows.

We interpret the constraint $\operatorname{div} E = \delta_\Sigma E = 0$ (and analogously $\operatorname{div} B = 0$) in a weak sense. Recalling Stokes' theorem in Equation 1.6, we can write formally:

$$(\mathrm{d}_\Sigma\psi, E)_{\Sigma_\pm} = (\psi, \delta_\Sigma E)_{\Sigma_\pm} + (\mathrm{t}\psi, \mathrm{n}E)_{\partial\Sigma_\pm}\,, \quad \text{for } \psi \in \mathrm{H}^1\Omega^0(\Sigma). \tag{2.35}$$

By a formal manipulation one obtains, if $\operatorname{supp}\psi \cap Z \neq \emptyset$,

$$\begin{aligned}(\mathrm{d}_\Sigma\psi, E)_\Sigma =&(\mathrm{d}_\Sigma\psi, E)_{\Sigma_+} + (\mathrm{d}_\Sigma\psi, E)_{\Sigma_-} = \\ =&(\psi, \delta_\Sigma E)_{\Sigma_+} + (\mathrm{t}\psi, \mathrm{n}_+ E)_Z + (\psi, \delta_\Sigma E)_{\Sigma_-} + (-1)(\mathrm{t}\psi, \mathrm{n}_- E)_Z = \\ =&(\psi, \delta_\Sigma E)_\Sigma + (\mathrm{t}\psi, [\mathrm{n}E])_Z.\end{aligned} \tag{2.36}$$

Definition 2.2.14. *We say that $E \in \mathrm{H}^1\Omega^1(\Sigma_Z)$ satisfies $\delta_\Sigma E = 0$ weakly if both therms of the right hand side of Equation (2.36) vanish for any $\psi \in \mathrm{H}^1\Omega^0(\Sigma) \equiv \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$, i.e.*

$$(\mathrm{d}_\Sigma\psi, E)_\Sigma = 0\,, \text{ for any } \psi \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.37}$$

In view of the previous definition and in what follows we will substitute equations (2.3b) with the requirement

$$E, B \perp \mathrm{d}_\Sigma \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.38}$$

Notice that, because of Equation (2.36), this entails $\delta_\Sigma E = \delta_\Sigma B = 0$ pointwise in $\Sigma_\pm$ as well as $[\mathrm{n}E] = [\mathrm{n}B] = 0$. Configurations of the electric field E in the presence of a charge density ρ on $\Sigma_\pm$ and a surface charge density σ over Z are described by expanding $E = \mathrm{d}_\Sigma\alpha + \delta_\Sigma\beta + \kappa$ and demanding $\alpha \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$ to satisfy

$$(\mathrm{d}_\Sigma\varphi, \mathrm{d}_\Sigma\alpha)_\Sigma = (\varphi, \rho)_\Sigma + (\mathrm{t}\varphi, \sigma)_Z \qquad \forall\varphi \in C^\infty_{\mathrm{c}}(\Sigma)\,.$$

This provides a weak formulation for the electrostatic boundary problem. For sufficiently regular α this is equivalent to the Poisson problem $\Delta_\Sigma\alpha = \rho$, $[\mathrm{n}\mathrm{d}_\Sigma\alpha] = \sigma$, recovering the classical equations outlined in [Jac99, Sec. I.5].

2.3 Dynamical equations: lagrangian subspaces

In this section we will treat the dynamical equations (2.3a). They can be written as follows in a Schrödinger-like fashion as a complex evolution equation and solved by imposing suitable interface conditions on Z.

$$i\partial_t\psi = H\psi \qquad \psi := \begin{bmatrix} E \\ B \end{bmatrix}, \qquad H := \begin{bmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{bmatrix}, \tag{2.39}$$

Here we adopt the convention of [Bär19] according to which

$$\operatorname{curl} := i *_\Sigma \mathrm{d}_\Sigma \quad \text{if } \dim\Sigma = 1 \mod 4\,, \quad \operatorname{curl} := *_\Sigma \mathrm{d}_\Sigma \quad \text{if } \dim\Sigma = 3 \mod 4\,. \tag{2.40}$$

With this convention curl is formally selfadjoint on $C^\infty_{\mathrm{c}}\Omega^1(\Sigma)$.

As outlined before in Section 2.2 we wish to consider equation (2.39) on Σ_Z, allowing for jump discontinuities across the interface Z. For that we regard H as a densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)^{\times 2} = \mathrm{L}^2\Omega^1(\Sigma_Z)^{\times 2}$ (see Equation (2.20)) with domain

$$\operatorname{dom}(H) := C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_+) \oplus C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_-)\,, \tag{2.41}$$

where $C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_\pm)$ denotes the subspace of $C^\infty_{\mathrm{c}}\Omega^1(\Sigma_\pm)$ with support in $\Sigma_\pm \setminus \partial\Sigma_\pm$.

In solving Maxwell equations, we require to be in an isolated system, so that the flux of relevant physical quantities, built from the stress-energy tensor, is zero through the interface. To meet this requirement we need to look for symmetric extensions $\widehat{H}$ of H, in other words

$$(\widehat{H}\psi_1, \psi_2)_\Sigma - (\psi_1, \widehat{H}\psi_2)_\Sigma = \text{ vanishing interface terms} \quad \forall \psi_1, \psi_2 \in \operatorname{dom}(\widehat{H}) \subseteq \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}. \tag{2.42}$$

Moreover, since H has the role of a quantum mechanical time evolution operator, we want the extension to be self-adjoint.

Proposition 2.3.1. *Let $u, v \in C^\infty_{\mathrm{c}}(\Sigma_Z)$, then a Green formula holds*

$$(\operatorname{curl} u, v)_\Sigma - (u, \operatorname{curl} v)_\Sigma = (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z\,, \tag{2.43}$$

where $\gamma_0 u := \frac{1}{\sqrt{2}} *_Z [\mathrm{t}u]$ and $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$. Moreover, the operator H, defined in (2.39) is symmetric on its domain (see Equation (2.41)), since for any $\psi_1, \psi_2 \in C^\infty_{\mathrm{c}}\Omega^1(\Sigma)^{\times 2}$ it holds

$$(H\psi_1, \psi_2)_\Sigma - (\psi_1, H\psi_2)_\Sigma = (\Gamma_1\psi_1, \Gamma_0\psi_2)_Z - (\Gamma_0\psi_1, \Gamma_1\psi_2)_Z, \tag{2.44}$$

where $\Gamma_0\psi = [i\gamma_1 B, \gamma_1 E]$ and $\Gamma_1\psi = [\gamma_0 E, i\gamma_0 B]$.

Hence, the operator H symmetric and hence closable (*cf.* [Mor18, Thm. 5.10]), its adjoint H^* being defined by

$$\operatorname{dom}(H^*) = \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\ H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} \qquad H^*\psi := H\psi\,. \tag{2.45}$$

Equation (2.39) is solved by selecting a self-adjoint extension of H. Now we introduce a technique which allows us to parametrize the self-adjoint extensions of H by Lagrangian subspaces of a suitable complex symplectic space – *cf.* [EM99, EM03, EM05]. The aim is to construct the Green operators for Equation (2.39) together with an interface condition and this technique, even if it does not give a complete characterization of self-adjoint extensions in terms of boundary conditions, allows us to check whether a chosen interface condition admits Green operators or not.

Definition 2.3.2. *Let S be a complex vector space and let $\sigma\colon \mathsf{S}\times\mathsf{S}\to\mathbb{C}$ a sesquilinear map. The pair (S,σ) is called complex symplectic space if σ is non-degenerate – i.e. $\sigma(x,y)=0$ for all $y\in\mathsf{S}$ implies $x=0$ – and $\sigma(x,y)=-\overline{\sigma(y,x)}$ for all $x,y\in\mathsf{S}$. A subspace $L\subseteq\mathsf{S}$ is called Lagrangian subspace if $L=L^\perp:=\{x\in\mathsf{S}|\ \sigma(x,y)=0\ \forall y\in L\}$.*

For convenience, we summarize the major results in the following theorem:

Theorem 2.3.3 ([EM99])**.** *Let H a separable Hilbert space and let $A\colon\ \mathrm{dom}(A)\subseteq\mathsf{H}\to\mathsf{H}$ be a densely defined, symmetric operator. Then the bilinear map*

$$\sigma(x,y):=(A^*x,y)-(x,A^*y)\,,\qquad \forall x,y\in\mathrm{dom}(A^*)\,,\tag{2.46}$$

satisfies $\sigma(x,y)=-\overline{\sigma(y,x)}$. It also descends to the quotient space $\mathsf{S}_A:=\mathrm{dom}(A^*)/\,\mathrm{dom}(A)$ and the pair (S_A,σ) is a complex symplectic space as per definition 2.3.2. Moreover, for all Lagrangian subspace $L\subseteq\mathsf{S}_A$ – cf. Definition 2.3.2 – the operator

$$A_L:=A^*|_{L+\mathrm{dom}(A)}\,,\tag{2.47}$$

defines a self-adjoint extension of A – here $L+\mathrm{dom}(A)$ denotes the pre-image of L with respect to the projection $\mathrm{dom}(A^*)\to\mathsf{S}_A$. Finally the map

$$\{\text{Lagrangian subspaces } L \text{ of } \mathsf{S}_A\}\ni L\mapsto A_L\in\{\text{self-adjoint extensions of } A\}\,,\tag{2.48}$$

is one-to-one.

Example 2.3.4. As a concrete example of Theorem 2.3.3 we discuss the case of the self-adjoint extensions of the curl operator on a closed manifold Σ with interface Z. For simplicity we assume that $\dim\Sigma=2k+1$ with $\dim\Sigma=3\mod 4$, while curl is defined according to (2.40). We consider the operator curl_Z defined by

$$\mathrm{dom}(\mathrm{curl}_Z):=\overline{C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_Z)}^{\|\|_{\mathrm{curl}}}\,,\qquad \mathrm{curl}_Z\,u:=\mathrm{curl}\,u\,.\tag{2.49}$$

Notice that $C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_Z)=C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_+)\oplus C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_-)$. The adjoint curl_Z^* of curl_Z is

$$\mathrm{dom}(\mathrm{curl}_Z^*)=\mathrm{dom}(\mathrm{curl}_+)\oplus\mathrm{dom}(\mathrm{curl}_-)\,,\tag{2.50}$$
$$\mathrm{dom}(\mathrm{curl}_\pm):=\{u_\pm\in\mathrm{L}^2\Omega^k(\Sigma_\pm)|\ \mathrm{curl}_\pm u_\pm\in\mathrm{L}^2\Omega^k(\Sigma_\pm)\}\,,\quad \mathrm{curl}_\pm u:=\mathrm{curl}\,u\,.\tag{2.51}$$

Since complex conjugation commutes with curl, it follows from Von Neumann's criterion [Mor18, Thm. 5.43] that curl_Z admits self-adjoints extensions. We now provide a description of the complex symplectic space $\mathsf{S}_{\mathrm{curl}_Z}:=(\mathrm{dom}(\mathrm{curl}_Z^*)/\,\mathrm{dom}(\mathrm{curl}_Z),\sigma_Z)$ whose Lagrangian subspaces allows to characterize all self-adjoint extensions of curl_Z. According to theorem 2.3.3 the symplectic structure σ_Z on the vector space $\mathsf{S}_{\mathrm{curl}_Z}$ is defined by

$$\sigma_Z(u,v) := (\mathrm{curl}_Z^* u, v) - (u, \mathrm{curl}_Z^* v)\,, \qquad \forall u,v \in \mathrm{dom}(\mathrm{curl}_Z^*)\,. \tag{2.52}$$

In particular for $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ and $v \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ we have

$$\sigma(u,v) = (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z = \tag{2.53}$$
$$= \sum_\pm \pm \int_Z \overline{\mathrm{t}_\pm u} \wedge \mathrm{t}_\pm v = \sum_\pm \mp\, {}_{-\frac{1}{2}}\langle \mathrm{t}_\mp u, *_Z \mathrm{t}_\mp v\rangle_{\frac{1}{2}}\,, \tag{2.54}$$

where we recall $\gamma_0 u := \frac{1}{\sqrt{2}} *_Z [\mathrm{t}u]$, $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$ as defined in Proposition and where ${}_{-\frac{1}{2}}\langle\ ,\ \rangle_{\frac{1}{2}}$ denotes the pairing between $\mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ and $\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$. In particular this shows that $\mathrm{t}_\pm u \in \mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ for all $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ – *cf.* [AV96, BCS02, Geo79, Paq82, Wec04] for more details on the trace space associated with the curl-operator on a manifold with boundary. **dovrei scrivere di più.**

According to theorem 2.3.3 all self-adjoint extensions of curl_Z are in one-to-one correspondence to the Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$. Unfortunately a complete characterization of all Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$ is not at disposal. We content ourself to present a family of Lagrangian subspaces – a generalization of the results presented in [HKT12] may provide other examples. For $\theta \in \mathbb{R}$ let

$$L_\theta := \{u \in \mathrm{dom}(\mathrm{curl}_Z^*) |\ \mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u\}\,, \tag{2.55}$$

where $\mathrm{t}_\pm$ denote the tangential traces – *cf.* definition 2.2.10, remark 2.2.5 and equation (2.53). To show that L_θ are Lagrangian subspaces let $u,v \in L_\theta$ and let $v_n \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ be such that $\|v - v_n\|_{\mathrm{curl}} \to 0$. In particular $\|(\mathrm{t}_+ - e^{i\theta}\mathrm{t}_-)v_n\|_{\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)} \to 0$ so that

$$\sigma_Z(u,v) = \lim_n \sigma_Z(u,v_n) = -\lim_n {}_{-\frac{1}{2}}\langle \mathrm{t}_+ u, *_Z(\mathrm{t}_+ v_n - e^{i\theta}\mathrm{t}_- v_n)\rangle_{\frac{1}{2}} = 0\,. \tag{2.56}$$

It follows that $L_\theta \subseteq L_\theta^\perp$. Conversely if $u \in L_\theta^\perp$ let consider $v \in L_\theta$. Since $u \in L_\theta^\perp$ we find

$$0 = \sigma_Z(u,v) = -\,{}_{-\frac{1}{2}}\langle \mathrm{t}_+ u - e^{i\theta}\mathrm{t}_- u, *_Z \mathrm{t}_+ v\rangle_{\frac{1}{2}}\,.$$

Since $\mathrm{t}_+ \colon \mathrm{H}^1\Omega^k(\Sigma_Z) \to \mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$ is surjective, it follows that $\mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u$.

Notice that the self-adjoint extension obtained for $\theta = 0$ coincides with the closure of curl on $C^\infty_{\mathrm{c}}(\Sigma)$ which is known to be self-adjoint by [Bär19, Lem. 2.6]. Indeed, since $[\mathrm{t}]$ is continuous we have $\mathrm{dom}(\overline{\mathrm{curl}}) \subseteq L_0$ so that curl_{Z,L_0} is a self-adjoint extension of $\overline{\mathrm{curl}}$. Since the latter operator is already self-adjoint we have equality among the two.

Appendix A

Poincaré-Lefschetz duality for manifold with boundary

In this section we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non empty boundary. A reader interested in more details can refer to [BT82, Sch95].

For the purpose of this section M refers to a smooth, oriented manifold of dimension $\dim M = d$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : M \to \partial M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work only with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_c(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\mathrm{Ker}(d_k)}{\mathrm{Im}(d_{k-1})},$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (1.4) and (1.5b) entail that we can define the $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_t(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M;\delta)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M;\delta) \doteq \frac{\mathrm{Ker}(\delta_k)}{\mathrm{Im}(\delta_{k+1})}.$$

Equations (1.4) and (1.5b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M;\delta)$, the subcomplex of $\Omega^\bullet(M;\delta)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$. The associated homology groups will be denoted as $H_{k,n}(M;\delta)$, $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition, the Hodge operator induces an isomorphism $H^k(M) \simeq H_{d-k}(M;\delta)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [*\alpha] \in H_{d-k}(M;\delta)$. Similarly, on account of Equation (1.5b), it holds $H^k_t(M) \simeq H_{d-k,n}(M;\delta)$.

As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT82]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k\in\mathbb{N}\cup\{0\}$ corresponds to

$$\Omega^k(M,\partial M)\doteq\Omega^k(M)\oplus\Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k:\Omega^k(M;\partial M)\to\Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta)\in\Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta)=(\mathrm{d}\omega,\iota^*_{\partial M}\omega-\mathrm{d}_\partial\theta)\,. \tag{A.1}$$

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1:\Omega^k(M;\partial M)\to\Omega^k(M)$ and $\pi_2:\Omega^k(M;\partial M)\to\Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will be always clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT82, Prop. 6.49]:

Proposition A.0.1. *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots\to H^k(M;\partial M)\xrightarrow{\pi_{1,*}}H^k(M)\xrightarrow{\iota_{\partial M,*}}H^k(\partial M)\xrightarrow{\pi_{2,*}}H^{k+1}(M;\partial M)\to\ldots, \tag{A.2}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and $\iota_{\partial M,*}$ indicate the natural counterpart of the maps π_1, π_2 and $\iota_{\partial M}$ at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition A.0.2. *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between $H^k_t(M)$ and $H^k(M,\partial M)$ for all $k\in\mathbb{N}\cup\{0\}$.*

Proof. Consider $\omega\in\Omega^k_t(M)\cap\ker(d)$ and let $(\omega,0)\in\Omega^k(M;\partial M)$, $k\in\mathbb{N}\cup\{0\}$. Equation (A.1) entails

$$\underline{\mathrm{d}}_k(\omega,0)=(\mathrm{d}\omega,\iota^*_{\partial M}\omega)=(\mathrm{d}\omega,\mathrm{t}\omega)=(0,0)\,,$$

where we used (1.3a) in the second equality. At the same time, if $\omega=\mathrm{d}\beta$ with $\beta\in\Omega^{k-1}_{\mathrm{t}}(M)$, then $\underline{\mathrm{d}}_{k-1}(\beta,0)=(\mathrm{d}\beta,0)$. Hence the embedding $\omega\mapsto(\omega,0)$ identifies an injective map $\rho:H^k_t(M)\to H^k(M;\partial M)$ such that $\rho([\omega])\doteq[(\omega,0)]$.

To conclude, we need to prove that ρ is surjective. Let thus $[(\omega',\theta)]\in H^k(M;\partial M)$. It holds that $d\omega'=0$ and $\iota^*_{\partial M}\omega'-d_\partial\theta=t(\omega')-d_\partial\theta=0$. Recalling that $t:\Omega^k(M)\to\Omega^k(\partial M)$ is surjective for all values of $k\in\mathbb{N}\cup\{0\}$, there must exist $\eta\in\Omega^{k-1}(M)$ such that $t(\eta)=\theta$.

Let $\omega \doteq \omega' - d\eta$. On account of (1.5b) $\omega \in \Omega^k_t(M) \cap \ker(d)$ and $(\omega, 0)$ is a representative if $[(\omega', \theta)]$ which entails the conclusion sought. □

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in-hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau80]:

Theorem A.0.3. *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N} \cup \{0\}$*

$$H^k(M; \partial M) \simeq H^{n-k}_c(M; \partial M)^*,$$

where $n = \dim M$ and where on the right hand side we consider the dual of the $(n-k)$-th cohomology group built out compactly supported forms.

The proof proceeds in some steps. Let $\iota : \partial M \to M$ be the immersion map. First of all we have to check that the spaces are finite-dimensional and that the pairing $\langle\, ,\, \rangle : H^{n-k}(M) \otimes H^k_c(M, \partial M)$ defined by

$$\langle \alpha, (\omega, \theta) \rangle := \int_M \alpha \wedge \omega + \int_{\partial M} \iota^* \alpha \wedge \theta \qquad \forall \alpha \in H^{n-k}(M) \text{ and } (\omega, \theta) \in H^k_c(M, \partial M)\,, \tag{A.3}$$

is non-degenerate, equivalently the map $\alpha \to \langle \alpha, \cdot \rangle$ should be an isomorphism.

Since a manifold M with boundary is locally homeomorphic to $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ we need Poincaré lemmas for $\mathbb{R}^n_+$.

Lemma A.0.4 (Poincaré lemmas for half spaces)**.** *Let $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ and $k \geq 0$. Then*

$$H^k(\mathbb{R}^n_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \{0\} & \text{otherwise} \end{cases} \tag{A.4}$$

$$H^k_c(\mathbb{R}^n_+, \partial \mathbb{R}^n_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = n \\ \{0\} & \text{otherwise} \end{cases} \tag{A.5}$$

Proof. The proof for the case $n = 1$, i.e. $\mathbb{R}_+ = [0, +\infty)$ is straightforward and the n-dimensional generalisation is obtained as in ([BT82, Sec. 4]). □

Lemma A.0.5 (Mayer-Vietoris sequences)**.** *Let M be an orientable manifold with boundary ∂M, suppose $M = U \cup V$ with U, V open and denote $\partial M_A := \partial M \cap A$. Then the following* *are exact sequences:*

$$\cdots \to H^k(M,\partial M) \to H^k(U,\partial M_U) \oplus H^k(V,\partial M_V) \to H^k(U\cap V,\partial M_{U\cap V}) \to H^{k+1}(M,\partial M) \to \cdots \tag{A.6}$$

$$\cdots \leftarrow H_c^k(M,\partial M) \leftarrow H_c^k(U,\partial M_U) \oplus H^k(V,\partial M_V) \leftarrow H_c^k(U\cap V,\partial M_{U\cap V}) \leftarrow H_c^{k+1}(M,\partial M) \leftarrow \cdots \tag{A.7}$$

Proof. We will only prove the non-compact cohomology line.
We have the following Mayer-Vietoris short exact sequences for M and ∂M:

$$0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U)\oplus\Omega^k(V) \longrightarrow \Omega^k(U\cap V) \longrightarrow 0$$
$$0 \to \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M_U)\oplus\Omega^{k-1}(\partial M_V) \to \Omega^{k-1}(\partial M_{U\cap V}) \to 0.$$

Hence applying the direct sum between the two sequences we obtain

$$0 \longrightarrow \Omega^k(M,\partial M) \longrightarrow \Omega^k(U,\partial M_U)\oplus\Omega^k(V,\partial M_V) \longrightarrow \Omega^k(U\cap V,\partial M_{U\cap V}) \longrightarrow 0.$$

The last row induces the desired long sequence because of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(M,\partial M) \arrow[r] \arrow[d, "d"] & \Omega^k(U,\partial M_U)\oplus\Omega^k(V,\partial M_V) \arrow[r] \arrow[d, "\mathrm{d}:=d\oplus d"] & \Omega^k(U\cap V,\partial M_{U\cap V}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(M,\partial M) \arrow[r] & \Omega^{k+1}(U,\partial M_U)\oplus\Omega^{k+1}(V,\partial M_V) \arrow[r] & \Omega^{k+1}(U\cap V,\partial M_{U\cap V}) \arrow[r] & 0
\end{tikzcd} \tag{A.8}$$

following the arguments in [BT82], section 2. Fix a closed form $\omega \in \Omega^k(U\cap V,\partial M_{U\cap V})$, since the first row is exact there exists a unique $\xi \in \Omega^{k+1}(M,\partial M)$ which is mapped to ω. Now, since $\mathrm{d}\omega = 0$ and the diagram is commutative $\mathrm{d}\xi$ is mapped to 0. Hence from the exactness of the second row there exists χ which is mapped to $\mathrm{d}\xi$ and it easy to see χ is closed. □

Lemma A.0.6. *If the manifold with boundary M has a* finite good cover *(see [BT82, Sec. 5]) then its (relative) cohomology and (relative) compact cohomology is finite dimensional.*

Proof. The proof is based on the existence of a Mayer-Vietoris sequence in any of the desired cases (proved in the previous proposition) and follows the outline of [BT82, Prop. 5.3.1]. □

Lemma A.0.7 (Five lemma)**.** *Given the commutative diagram with exact rows*

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "r"] & E \arrow[r] \arrow[d, "s"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots
\end{tikzcd} \tag{A.9}$$

if f, g, h, s are isomorphism, then so is r.

Lemma A.0.8. *Suppose $M = U \cup V$ with U, V open. The pairing (A.3) induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is commutative:*

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-k}(M) \arrow[r] \arrow[d] & H^{n-k}(U) \oplus H^{n-k}(V) \arrow[r] \arrow[d] & H^{n-k+1}(M) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^k(M, \partial M)^* \arrow[r] & H^k(U, \partial M_U)^* \oplus H^k(V, \partial M_V)^* \arrow[r] & H^{k-1}(M)^* \arrow[r] & \cdots \end{tikzcd} \tag{A.10}$$

Proof. The proof follows that of [BT82, Lem. 5.6]. □

Now we are ready to prove the main theorem of this section:

Proof of Poincaré-Lefschetz Duality. Follow the argument given in [BT82, Sec. 5]. By the Five lemma if Poincaré-Lefschetz duality holds for U, V and $U \cap V$, then it holds for $U \cup V$. Then it is sufficient to proceed by induction on the cardinality of a finite good cover. □

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

List of Symbols

a	distance	m
P	power	W ($\mathrm{J\,s^{-1}}$)
ω	angular frequency	rad

Bibliography

[AFS18] Aké L., Flores J.L. , Sanchez M., *Structure of globally hyperbolic spacetimes with timelike boundary*, arXiv:1808.04412 [gr-qc].

[AV96] Alonso A., Valli A., *Some remark on the characterization of the space of tangential trace of $H(\mathrm{rot}, \Omega)$ and the construction of an extension operator*, Operator, Manuscripta Math., 89 (1996), 159-178.

[Ama17] Amar E., *On the L^r Hodge theory in complete non-compact Riemannian manifolds*, Mathematische Zeitschrift, Springer, 2017, 287, pp.751-795.

[AM04] Axelsson A., McIntosh A. (2004) *Hodge Decompositions on Weakly Lipschitz Domains.* In: Qian T., Hempfling T., McIntosh A., Sommen F. (eds) Advances in Analysis and Geometry. Trends in Mathematics. Birkhäuser, Basel.

[AK12] Kurz S., Auchmann B. *Differential Forms and Boundary Integral Equations for Maxwell-Type Problems*. In: Langer U., Schanz M., Steinbach O., Wendland W. (eds) Fast Boundary Element Methods in Engineering and Industrial Applications. Lecture Notes in Applied and Computational Mechanics, vol 63. Springer, Berlin, Heidelberg (2012).

[BG12] Bär, C., Ginoux, N., *Classical and quantum fields on Lorentzian manifolds*, in: C. Bär et al. (eds.), *Global Differential Geometry*, pp. 359–400, Springer, Berlin (2012)

[BGP15] Bär, C., Ginoux, N., Pfäffle, F. *Wave equations on Lorentzian Manifolds and quantization*, EMS Publishing House, Zurich, 2007.

[Bär15] Bär C., *Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes*, Commun. Math. Phys. (2015) 333: 1585.

[Bär19] Bär C., *The curl operator on odd-dimensional manifolds*, Journal of Mathematical Physics 60, 031501 (2019).

[BL12] Behrndt, J., Langer, M., Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples. In: de Snoo, H.S.V. (ed.) Operator Methods for Boundary Value Problems. London Mathematical Society, Lecture Notes, p. 298. Cambridge University Press, Cambridge (2012)

[BDS14] M. Benini, C. Dappiaggi and A. Schenkel, *"Quantized Abelian principal connections on Lorentzian manifolds,"* Commun. Math. Phys. **330** (2014) 123 [arXiv:1303.2515 [math-ph]].

[Ben16] Benini M., *Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies*, J. Math. Phys. **57** (2016) no.5, 053502

[BS05] Bernal, A. N., Sanchez M., *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes.* Commun. Math. Phys. 257 (2005), 43–50

[BT82] R. Bott, L. W. Tu, *Differential forms in algebraic topology*, Springer, New York, 1982.

[BFV03] Brunetti, R., Fredenhagen, K., Verch, R., *The Generally Covariant Locality Principle ? A New Paradigm for Local Quantum Field Theory*, Commun. Math. Phys. (2003).

[BCS02] Buffa A., Costabel M., Sheen D., *On traces for* $\mathrm{H}(\mathrm{curl}, \Omega)$ *in Lipschitz domains*, J. Math. Anal. Appl. 276 (2002) 845?867.

[DDF19] Dappiaggi, C., Drago, N., Ferreira, H. *"Fundamental solutions for the wave operator on static Lorentzian manifolds with timelike boundary"*, to appear in Lett. Math. Phys. (2019), arXiv:1804.03434 [math-ph].

[DFJ18] C. Dappiaggi, H. R. C. Ferreira and B. A. Juárez-Aubry, *"Mode solutions for a Klein-Gordon field in anti?de Sitter spacetime with dynamical boundary conditions of Wentzell type,"* Phys. Rev. D **97** (2018) no.8, 085022 [arXiv:1802.00283 [hep-th]].

[DS11] C. Dappiaggi and D. Siemssen, *"Hadamard States for the Vector Potential on Asymptotically Flat Spacetimes,"* Rev. Math. Phys. **25** (2013) 1350002 doi:10.1142/S0129055X13500025 [arXiv:1106.5575 [gr-qc]].

[Eic93] Eichhorn, J, *The manifold structure of maps between open manifolds.* Ann. Global Anal. Geom. 11, 253-300.

[EM99] Everitt W. N. , Markus L., *Complex symplectic geometry with applications to ordinary differential operators*, Journal: Trans. Amer. Math. Soc. 351 (1999), 4905-4945

[EM03] Everitt W., Markus L., *Elliptic Partial Differential Operators and Symplectic Algebra*, no. 770 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence (2003).

[EM05] Everitt W., Markus L., *Complex symplectic spaces and boundary value problems*, Bull. A. Math. Soc 42, 461-500 (2005).

[Gaf55] Gaffney M.P., *Hilbert space methods in the theory of harmonic integrals*, Transactions of the American Mathematical Society Vol. 78, No. 2 (Mar., 1955), pp. 426-444

[Geo79] Georgescu V., *Some boundary value problems for differential forms on compact Riemannian manifolds*, Annali di Matematica (1979) 122: 159.

[GHV72] Greub W., Halperin S., Vanstone R.. *Connections, curvature, and cohomology - Vol.1*, Academic press New York and London (1972).

[Gro91] Gromov M., Kahler hyperbolicity and L2-Hodge theory, J. Differential Geometry 33(1991) 263-292.

[HS13] Hack T.P., Schenkel A., *Linear bosonic and fermionic quantum gauge theories on curved spacetimes*, Gen Relativ Gravit (2013) 45: 877.

[HR00] Higson N., Roe J., *Analytic K-Homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[HKT12] Hiptmair, R., Kotiuga, P.R., Tordeux, S., *Self-adjoint curl operators*, Annali di Matematica (2012) 191: 431.

[Kod49] Kodaira K., *Harmonic fields in Riemannian manifolds - generalized potential theory*, Annals of Mathematics Second Series, Vol. 50, No. 3 (Jul., 1949), pp. 587-665.

[Jac99] Jackson, J. D. *Classical electrodynamics*, 3rd ed. (1999) Wiley.

[Lee00] J.M. Lee *Introduction to Smooth Manifolds*, 2nd ed. (2013) Springer, 706p.

[Li09] Li X-D., *On the strong L^p-Hodge decomposition over complete Riemannian manifolds*, Journal of Functional Analysis Volume 257, Issue 11, 1 December 2009, Pages 3617-3646.

[LM72] Lions, J. L., Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications I*, Grundlehren der mathematischen Wissenschaften 181, Springer Verlag, Berlin.

[Mau80] C. R. F. Maunder, *Algebraic Topology*, (1980) Cambridge University Press, 375p.

[MM13] Mcintosh A., Morris A.J., *Finite propagation speed for I order systems and Huygens' principle for hyperbolic equations*, Proceedings of the American Mathematical Society Vol. 141, No. 10 (OCTOBER 2013), pp. 3515-3527.

[Mor18] Moretti, V., *Spectral Theory and Quantum Mechanics*, 2nd edn, p. 950. Springer, Berlin (2018).

[Nak90] Nakahara, M., *Geometry, Topology and Physics*, Hilger, Bristol (1990)

[Paq82] Paquet, L., *Problèmes mixtes pour le système de Maxwell*, Annales de la Faculté des sciences de Toulouse : Mathématiques, Série 5, Volume 4 (1982) no. 2, pp. 103-141.

[Pfe09] M. J. Pfenning, *"Quantization of the Maxwell field in curved spacetimes of arbitrary dimension,"* Class. Quant. Grav. **26** (2009) 135017 [arXiv:0902.4887 [math-ph]].

[Sco95] Scott C., L^p theory of differential forms on manifolds Transactions of the Ametican Mathematical Society, Volume 347, Number 6, June 1995.

[Sch95] G. Schwarz, *Hodge Decomposition - A Method for Solving Boundary Value Problems*, (1995) Springer, 154p.

[Wec04] Weck N., *Traces of differential forms on Lipschitz boundaries*, Analysis, 24(2), pp. 147-170 (2004).

[Za15] J. Zahn, *"Generalized Wentzell boundary conditions and quantum field theory,"* Annales Henri Poincare **19** (2018) no.1, 163 [arXiv:1512.05512 [math-ph]].

[ZS00] Zulfikar M. A., Stroock D.W., *A Hodge theory for some non-compact manifolds.* J. Differential Geom. 54 (2000), no. 1, 177–225.