On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

Claudio Dappiaggi[1,2,3,a], Nicolò Drago[4,5,b] and Rubens Longhi[1,c]

[1] Dipartimento di Fisica – Università di Pavia, Via Bassi 6, I-27100 Pavia, Italy.

[2] INFN, Sezione di Pavia – Via Bassi 6, I-27100 Pavia, Italy.

[3] Istituto Nazionale di Alta Matematica – Sezione di Pavia, Via Ferrata, 5, 27100 Pavia, Italy.

[4] Dipartimento di Matematica – Università di Trento, via Sommarive 15, I-38123 Povo (Trento), Italy.

[5] INFN, TIFPA – via Sommarive 15, I-38123 Povo (Trento), Italy.

[a] claudio.dappiaggi@unipv.it , [b] nicolo.drago89@gmail.com , [c] rubens.longhi01@universitadipavia.it

June 29, 2019

Abstract

To be filled

Keywords: to be filled

MSC 2010: 81T20, 81T05

1 Introduction

To be filled

2 Geometric Data

In this subsection, our goal is to fix notations and conventions, as well as to summarize the main geometric data, which play a key role in our analysis. Following the standard definition, see for example [Lee00, Ch. 1], M indicates a smooth, second-countable, connected, oriented manifold of dimension $n > 1$, with smooth boundary ∂M, assumed for simplicity to be connected. A point $p \in M$ such that there exists an open neighbourhood U containing p, diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $Int(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$.

In addition we endow M with a smooth Lorentzian metric g of signature $(-, +, ..., +)$ so that $\iota^* g$ identifies a Lorentzian metric on ∂M and we require (M, g) to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M, g) has a *timelike boundary.*

Since we will be interested particularly in the construction of advanced and retarded fundamental solutions for normally hyperbolic operators, we focus our attention on a specific class of Lorentzian manifolds with timelike boundary, namely those which are globally hyperbolic. While, in the case of $\partial M = \emptyset$ this is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18]. Summarizing part of their constructions and results, we say that a time-oriented, Lorentzian manifold with timelike boundary (M, g) is *causal* if it possesses no closed, causal curve, while it is *globally hyperbolic* if it is causal and, for all $p, q \in M$, $J^+(p) \cap J^-(q)$ is either empty or compact. These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 1: *Let (M, g) be a time-oriented of dimension n. Then*

1. *(M, g) is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve,*

2. *if (M, g) is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the line-element*

$$ds^2 = -\beta d\tau^2 + h_\tau, \tag{1}$$

where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1], whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a Cauchy surface for (M, g).

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M, g), we work directly with (1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M. In addition we shall say that (M, g) is *static* if it possesses a timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to ∂M is tangent to the boundary, *i.e.* $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where ν is the unit vector, normal to the boundary at p. With reference to (1) this translates simply into the request that both β and h_τ are independent from τ.

On top of a Lorentzian spacetime (M, g) with timelike boundary we consider $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, the space of real valued smooth k-forms endowed with the standard, metric induced, pairing $(,) : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}$. A particular role will be played by the support of the forms that we consider. In the following definition we introduce the different possibilities that we will consider, which are a generalization of the counterpart used for scalar fields which correspond in our scenario to $k = 0$, *cf.*m [Bär15].

Definition 2: *Let (M, g) be a Lorentzian spacetime with timelike boundary. We denote with*

1. *$\Omega^k_{\mathrm{c}}(M)$ the space of smooth k-forms with compact support in M while with $\Omega^k_{\mathrm{cc}}(M) \subset \Omega^k_{\mathrm{c}}(M)$ the collection of smooth and compactly supported k-forms ω such that $supp(\omega) \cap \partial M = \emptyset$.*

2. *$\Omega^k_{\mathrm{spc}}(M)$ (resp. $\Omega^k_{\mathrm{sfc}}(M)$) the space of strictly past compact (resp. strictly future compact) k-forms, that is the collection of $\omega \in \Omega^k(M)$ such that there exists a compact set $K \subseteq M$ for which $J^+(supp(\omega)) \subseteq J^+(K)$ (resp. $J^-(supp(\omega)) \subseteq J^-(K)$), where $J^\pm$ denotes the causal future and the causal past in M. Notice that $\Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M) = \Omega^k_{\mathrm{c}}(M)$.*

3. *$\Omega^k_{\mathrm{pc}}(M)$ (resp. $\Omega^k_{\mathrm{fc}}(M)$) denotes the space of future compact (resp. past compact) k-forms, that is, $\omega \in \Omega^k(M)$ for which $\mathrm{supp}(\omega) \cap J^-(K)$ (resp. $\mathrm{supp}(\omega) \cap J^+(K)$) is compact for all compact $K \subset M$.*

[1] Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

4. $\Omega^k_{\mathrm{tc}}(M) := \Omega^k_{\mathrm{fc}}(M) \cap \Omega^k_{\mathrm{pc}}(M)$, *the space of timelike compact k-forms.*

5. $\Omega^k_{\mathrm{sc}}(M) := \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M)$, *the space of spacelike compact k-forms.*

We indicate with $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative and, being (M, g) oriented, we can identify a unique, metric-induced, Hodge operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge *\beta = (\alpha, \beta)\mu_g$, where $\wedge$ is the exterior product of forms and μ_g the metric induced volume form. Since M is endowed with a Riemannian metric it holds that, when acting on smooth k-forms, $*^{-1} = (-1)^{k(m-k)}*$. Combining these data first we define the *codifferential* operator $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta \doteq *^{-1} \circ d \circ *$. Secondly we introduce the *D'Alembert-de Rham* wave operator $\Box_k : \Omega^k(M) \to \Omega^k(M)$ such that $\Box_k \doteq d\delta + \delta d$, as well as the *Maxwell* operator $\mathcal{M}_k : \Omega^k(M) \to \Omega^k(M)$ such that $\mathcal{M}_k \doteq \delta d$. The subscript k is here introduced to make explicit on which space of k-forms the operator is acting. Observe, furthermore, that $\Box_k$ differs by the more commonly used D'Alembert wave operator acting on k-forms by 0-order term built out of the metric and whose explicit form depends on the value of k, see for example [Pfe09, Sec. II].

To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its derivative along the direction normal to ∂M.

Remark 3: *Since we will be considering not only form lying in $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, but also those in $\Omega^k(\partial M)$, we shall distinguish the operators acting on this space with a subscript $_\partial$, e.g. d_∂, $*_\partial$, δ_∂ or $(,)_\partial$.*

Definition 4: *Let (M, g) be a Lorentzian spacetime with timelike boundary together with the embedding map $\iota_{\partial M} : \partial M \hookrightarrow M$. We call* tangential *and* normal *maps*

$$\mathrm{t} \colon \Omega^k(M) \to \Omega^k(\partial M) \qquad \omega \mapsto \mathrm{t}\omega \doteq \iota^*_{\partial M}\omega \tag{2a}$$

$$\mathrm{n} \colon \Omega^k(M) \to \Omega^{k-1}(\partial M) \qquad \omega \mapsto \mathrm{n}\omega \doteq *^{-1}_\partial \circ \mathrm{t} \circ *_M\,, \tag{2b}$$

In particular, for all $k \in \mathbb{N} \cup \{0\}$ we define

$$\Omega^k_{\mathrm{t}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\}\,, \qquad \Omega^k_{\mathrm{n}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\}\,. \tag{3}$$

Remark 5: *The normal map $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ can be equivalently read as the restriction to ∂M of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(M)$ and the vector field $\nu \in \Gamma(TM)|_{\partial M}$ which corresponds pointwisely to the unit vector, normal to ∂M.*

As last step, we observe that (2) together with (3) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$* \,\delta = (-1)^k \mathrm{d}*\,, \quad \delta * = (-1)^{k+1} * \mathrm{d}\,, \tag{4a}$$

$$*_\partial \,\mathrm{n} = \mathrm{t}*\,, \quad *_\partial \mathrm{t} = (-1)^k \mathrm{n}*\,, \quad \mathrm{d}_\partial \mathrm{t} = \mathrm{t}\mathrm{d}\,, \quad \delta_\partial \mathrm{n} = \mathrm{n}\delta\,. \tag{4b}$$

A notable consequence of (4b) is that, while on globally hyperbolic spacetimes with empty boundary, the operators d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. A direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial \qquad \forall \alpha \in \Omega^k_{\mathrm{c}}(M)\,,\ \forall \beta \in \Omega^{k+1}_{\mathrm{c}}(M)\,, \tag{5}$$

where the pairing in the right-hand side is the one associated to forms living on ∂M.

3 Maxwell Equations and Boundary Conditions

In this section we analyze the space of solutions of the Maxwell equations for arbitrary k-forms on a globally hyperbolic spacetime with timelike boundary (M, g). We proceed in two separate steps. First we focus our attention on the D'Alembert - de Rham wave operator $\Box_k = \delta d + d\delta$ acting on $\Omega^k(M)$. We identify a large class of boundary conditions which correspond to imposing that the underlying system is closed (*i.e.* the symplectic flux across ∂M vanishes) and we characterize the kernel of the operator in terms of its advanced and retarded fundamental solutions. These are assumed to exist and, following the same strategy employed in [DDF19] for the scalar wave equation, we prove that this is indeed the case whenever (M, g) is a static spacetime.

In the second part of the section we focus instead on the Maxwell operator $\mathcal{M}_k$. In order to characterize its kernel we will need to discuss the interplay between the choice of a boundary condition and that of a gauge fixing. This represent the core of this part of our work.

3.1 On the D'Alembert - de Rham wave operator

Consider the operator $\Box_k : \Omega^k(M) \to \Omega^k(M)$, where (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = n \geq 2$. Then, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, the following Green's formula holds true:

$$(\Box_k\alpha, \beta) - (\alpha, \Box_k\beta) = (\mathrm{t}\delta\alpha, \mathrm{n}\beta)_\partial - (\mathrm{n}\alpha, \mathrm{t}\delta\beta)_\partial - (\mathrm{n}d\alpha, \mathrm{t}\beta)_\partial + (\mathrm{t}\alpha, \mathrm{n}d\beta)_\partial\,, \tag{6}$$

where t, n are the maps defined in (2), while $(,)$ is the standard, metric induced pairing between k-forms. In view of Definition 4, it descends that the right-hand side of (6) vanishes automatically if we restrict our attention to $\Omega_{cc}(M)$, but boundary conditions ought to be imposed for the same property to hold true on a larger set of k-forms. From a physical viewpoint this requirement is tantamount to imposing that the system described by k-forms obeying the D'Alembert - de Rham wave equation is closed.

Lemma 6: *Let $f, f' \in C^\infty(\partial M)$ and let*

$$\Omega^k_{f,f'}(M) \doteq \{\omega \in \Omega^k(M) \mid t\omega = fnd\omega \quad \text{and} \quad t\delta\omega = f'n\omega\}. \tag{7}$$

Then, $\forall \alpha, \beta \in \Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, it holds

$$(\Box_k\alpha, \beta) - (\alpha, \Box_k\beta) = 0.$$

Proof. This is a direct consequence of (6) together with the property that, for every $f \in C^\infty(\partial M)$ and for every $\alpha \in \Omega^k(\partial M)$, $*_\partial(f\alpha) = f(*_\partial\alpha)$. In addition observe that the assumption on the support of α and β descend also to the forms present in each of the pairing in the right hand side of (6). $\Box$

Remark 7: *Observe that the boundary conditions necessary to let the right hand side of* (6) *are always since the first two terms involve $(k-1)$-forms and the last two k-forms. The only exception is the case $k = 0$ when* (6) *reduces to the case studied in [DDF19]. In addition, in analogy to the terminology used in the scalar scenario, we shall say that* (7) *implements boundary conditions of Robin type. It is important to stress that these are not the largest class of boundary conditions which make the right hand side* (6) *vanish. As a matter of fact one can think of additional possibilities similar to the so-called Wentzell boundary conditions, which were considered in the scalar scenario, see e.g. [DDF19, DFJ18, Za15].*

Lemma (6) individuates therefore a class of boundary conditions which make the operator $\Box_k$ formally self-adjoint. In between all these possibilities we highlight in the following definition four notable extremal cases, which are of particular interest to our analysis.

Definition 8: *Let (M,g) be a globally hyperbolic spacetime with timelike boundary and let $k\in\mathbb{N}$. We call*

- *space of k-forms with* Dirichlet *boundary condition*
$$\Omega^k_D(M)\doteq\{\omega\in\Omega^k(M)\mid t\omega=0\text{ and }n\omega=0\},$$
- *space of k-forms with* Neumann *boundary condition*
$$\Omega^k_N(M)\doteq\{\omega\in\Omega^k(M)\mid nd\omega=0\text{ and }t\delta\omega=0\},$$
- *space of k-forms with* tangential *boundary condition*
$$\Omega^k_T(M)\doteq\{\omega\in\Omega^k(M)\mid t\omega=0\text{ and }t\delta\omega=0\},$$
- *space of k-forms with* normal *boundary condition*
$$\Omega^k_\perp(M)\doteq\{\omega\in\Omega^k(M)\mid n\omega=0\text{ and }nd\omega=0\}.$$

Whenever the domain of the operator $\Box_k$ is restricted to one of these space we shall indicate it with symbol $\Box_{k,\sharp}$ where $\sharp\in\{D,N,T,\perp\}$.

Remark 9: *Observe that, in the previous definition, we have excluded the case $k=0$ since, in such case, only two possibilities survive, namely*
$$\Omega^0_D(M)\doteq\{\omega\in C^\infty(M)\mid t\omega=\omega|_{\partial M}=0\},\quad \Omega^0_N(M)\doteq\{\omega\in C^\infty(M)\mid n\omega=\nu(\omega)|_{\partial M}=0\},$$
where, for all $p\in\partial M$ ν coincides with the unit vector, normal to the boundary. These two options coincide with the standard Dirichlet and Neumann boundary conditions for scalar functions.

Remark 10: *It is interesting to observe that different boundary conditions can be related via the action of the Hodge operator. In particular, using Equation (4) and (7), one can infer that, for any $f,f'\in C^\infty(\partial M)$ which are nowhere vanishing, it holds that*
$$*\Omega^k_{f,f'}(M)=\Omega^{n-k}_{\tilde f,\tilde f'}(M),$$
where $\tilde f=\frac{(-1)^k}{f'}$ and $\tilde f'=\frac{(-1)^k}{f}$. At the same time, with reference, to the space of k-forms in Definition 8 it holds
$$*\Omega^k_D(M)=\Omega^{m-k}_D(M),\quad *\Omega^k_N(M)=\Omega^{m-k}_N(M),\quad *\Omega^k_T(M)=\Omega^{m-k}_\perp(M)\tag{8}$$

In the following we shall make a key assumption on the existence of distinguished fundamental solutions for the operator $\Box_k$. Subsequently we shall prove that such hypothesis holds true whenever the underlying globally hyperbolic spacetime with timelike boundary is static. Recalling both Definition 2 and Equation (7) we require the following:

Assumption 11: *For all $f,f'\in C^\infty(\partial M)$ and for all $k\in\mathbb{N}\cup\{0\}$, there exist advanced $(-)$ and retarded $(+)$ fundamental solutions for the d'Alembert-de Rham wave operator $\Box_k$, $G^\pm_{f,f'}:\Omega^k_{\mathrm{c}}(M)\to\Omega^k_{\mathrm{sc,f,f'}}(M)\doteq\Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_{f,f'}(M)$ such that*
$$\Box_k\circ G^\pm_{f,f'}=\mathrm{Id}_{\Omega^k_{\mathrm{c}}(M)}\,,\qquad G^\pm_{f,f'}\circ\Box_{k,sc,f,f'}=\mathrm{Id}_{\Omega^k_{f,f'}(M)}\,,\qquad \mathrm{supp}(G^\pm_{f,f'}\omega)\subseteq J^\pm(\mathrm{supp}(\omega))\,,\tag{9}$$
for all $\omega\in\Omega^k_{\mathrm{c}}(M)$ where $J^\pm$ denote the causal future and past and where $\Box_{k,sc,f,f'}$ indicates that the domain of $\Box_k$ is restricted to $\Omega^k_{\mathrm{sc,f,f'}}(M)$.

Remark 12: *Notice that domain of $G^{\pm}_{f,f'}$ is not restricted to $\Omega^k_{f,f'}(M)\cap\Omega^k_c(M)$. Furthermore the second identity in (9) cannot be extended to $G^{\pm}_{f,f'}\circ\Box_k=\mathrm{Id}_{\Omega^k_\sharp(M)}$ since it would entail $G_{\mathrm{f,f'}}\Box_k\omega=\omega$ for all $\omega\in\Omega^k_{\mathrm{c}}(M)$. Yet the left hand side also entails that $\omega\in\Omega^k_{sc,f,f'}$, which is manifestly a contradiction.*

Corollary 13: *Under the same hypothesis of Assumption 11, if the fundamental solutions $G^{\pm}_{f,f'}$ exist, they are unique.*

Proof. Suppose that, beside $G^-_{f,f'}$, there exists a second map $\widetilde{G}^-_{f,f'}\colon\Omega^k_{\mathrm{c}}(M)\to\Omega^k_{\mathrm{sc}}(M)$ enjoying the properties of Equation (9). Then, for any but fixed $\alpha\in\Omega^k_c(M)$ it holds

$$(\alpha,G^+_{f,f'}\beta)=(\Box_k G^-_{f,f'}\alpha,G^+_{f,f'}\beta)=(G^-_{f,f'}\alpha,\Box_k G^+_{f,f'}\beta)=(G^-_{f,f'}\alpha,\beta),\quad\forall\beta\in\Omega^k_c(M)$$

where we used both the support properties of the fundamental solutions and Lemma 6 which guarantees that $\Box_k$ is formally self-adjoint on $\Omega^k_{f,f'}(M)$. Similarly, replacing $G^-_{f,f'}$ with $\widetilde{G}^-_{f,f'}$, it holds $(\alpha,\widetilde{G}^+_{f,f'}\beta)=(\widetilde{G}^-_{f,f'}\alpha,\beta)$. It descends that $((\widetilde{G}^-_{f,f'}-G^-_{f,f'})\alpha,\beta)=0$, which entails $\widetilde{G}^-_{f,f'}\alpha=G^-_{f,f'}\alpha$ being the pairing between $\Omega^k(M)$ and $\Omega^k_c(M)$ separating. A similar result holds for the retarded fundamental solution. □

This corollary can be also read as a consequence of the property that, for all $\omega\in\Omega^k_{\mathrm{c}}(M)$, $G^{\pm}_{f,f'}\omega\in\Omega^k_{\mathrm{sc}}(M)$ can be characterized as the unique solution to the Cauchy problem

$$\Box\psi=\omega\,,\qquad \mathrm{supp}(\psi)\cap M\setminus J^{\pm}(\mathrm{supp}(\omega))=\emptyset\,,\qquad \psi\in\Omega^k_{f,f'}(M)\,. \tag{10}$$

Remark 14: *Observe that both Assumption 11 and Corollary 13 have been stated for $\Omega^k_{f,f'}(M)$. Per direct inspection one can infer that, mutatis mutandis, they hold true for $\Omega^k_\sharp(M)$ with $\sharp=\{N,D,T,\perp\}$, cf. Definition 8. Henceforth we shall only be working with $\Omega^k_{f,f'}$ but, unless stated otherwise, all results hold true also for $\Omega^k_\sharp(M)$. In particular in this case the associated fundamental solutions will be indicated as $G^{\pm}_\sharp:\Omega^k_c(M)\to\Omega^k_{sc}(M)\cap\Omega^k_\sharp(M)$.*

Remark 15: *For all $f,f'\in C^\infty(\partial M)$ the fundamental solutions $G^+_{f,f'}$ (resp. $G^-_{f,f'}$) can be extended to a linear operator $G^+_{f,f'}\colon\Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_{f,f'}(M)$ (resp. $G^-_{f,f'}\colon\Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_{f,f'}(M)$) – cf. [Bär15, Thm. 3.8]. As a consequence the problem $\Box_k\psi=\omega$ with $\omega\in\Omega^k(M)$ always admits a solution lying in $\Omega^k_{f,f'}(M)$. As a matter of facts, consider any smooth function $\eta\equiv\eta(\tau)$, where $\tau\in\mathbb{R}$, cf. Equation (1), such that $\eta(\tau)=1$ for all $\tau>\tau_1$ and $\eta(\tau)=0$ for all $\tau<\tau_0$. Then calling $\omega^+\doteq\eta\omega$ and $\omega^-=(1-\eta)\omega$, it holds $\omega^+\in\Omega^k_{\mathrm{pc}}(M)$ while $\omega^-\in\Omega^k_{\mathrm{fc}}(M)$. Hence $\psi=G^+_{f,f'}\omega^++G^-_{f,f'}\omega^-\in\Omega^k_{f,f'}(M)$ is the sought solution.*

We prove the main result of this section, which characterizes the kernel of $\Box_k$ on the space of smooth k-forms with prescribed boundary condition.

Proposition 16: *Whenever the Assumption 11 is fulfilled, then, for all $f,f'\in C^\infty(\partial M)$, setting $G_{f,f'}\doteq G^+_{f,f'}-G^-_{f,f'}:\Omega^k_c(M)\to\Omega^k_{sc,f,f'}(M)$, the following statements hold true:*

1. *letting $\tilde{f}=\frac{(-1)^k}{f'}$ and $\tilde{f}'=\frac{(-1)^k}{f}$,*

$$*\circ G^{\pm}_{f,f'}=G^{\pm}_{\tilde{f},\tilde{f}'}\circ *\,. \tag{11}$$

2. *for all $\alpha,\beta\in\Omega^k_{\mathrm{c}}(M)$ it holds*

$$(\alpha,G^{\pm}_{f,f'}\beta)=(G^{\mp}_{f,f'}\alpha,\beta)\,. \tag{12}$$

3. *the interplay between $G_{f,f'}$ and $\Box_k$ is encoded in the exact sequence:*

$$0 \to \Omega^k_{\mathrm{c},f,f'}(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{c}}(M) \xrightarrow{G_{f,f'}} \Omega^k_{sc,f,f'}(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{sc}}(M) \to 0\,, \tag{13}$$

where $\Omega^k_{\mathrm{c},f,f'}(M) \doteq \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{f,f'}(M)$

Proof. We prove the different items separately. Starting from *1.*, observe that $*\Box_k = \Box_{n-k}*$. Together with Remark 10, this entails that, for all $\alpha \in \Omega^k_c(M)$,

$$\Box_{n-k} * G^{\pm}_{f,f'}\alpha = \Box_{n-k} G^{\pm}_{\tilde{f},\tilde{f}'} * \alpha = *\alpha.$$

The uniqueness of the fundamental solutions as per Corollary 13 entails (11).

2. Equation (12) is a consequence of the following chain of identities valid for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$

$$(\alpha, G^{\pm}_{\tilde{f},\tilde{f}'}\beta) = (\Box_k G^{\mp}_{\tilde{f},\tilde{f}'}\alpha, G^{\pm}_{\tilde{f},\tilde{f}'}\beta) = (G^{\mp}_{\tilde{f},\tilde{f}'}\alpha, \Box_k G^{\pm}_{\tilde{f},\tilde{f}'}\beta) = (G^{\mp}_{\tilde{f},\tilde{f}'}\alpha, \beta)\,,$$

where we used both the support properties of the fundamental solutions and Lemma 6.

3. The exactness of the series is proven using the properties already established for the fundamental solutions $G^{\pm}_{f,f'}$. The left exactness of the sequence is a consequence of the second identity in Equation (9) which ensures that $\Box_k\alpha = 0$, $\alpha \in \Omega^k_{\mathrm{c},f,f'}(M)$, entails $\alpha = 0$. In order to prove that $\ker(G_{f,f'}) = \Box_k[\Omega^k_{c,f,f'}]$, it suffices to observe that, if $\beta \in \Omega^k_c(M)$ is such that $G_{f,f'}(\beta) = 0$, then $G^+_{f,f'}(\beta) = G^-_{f,f'}(\beta)$. Hence, in view of the support properties of the fundamental solutions $G^+_{f,f'}(\beta) \in \Omega^k_c(M) \cap \Omega^k_{sc,f,f'}(M)$ and $\beta = \Box_k(G^+_{f,f'}(\beta))$. Subsequently we need to verify that $\ker \Box_k = G_{f,f'}[\Omega^k_c(M)]$. This is a standard argument, namely, considering any smooth function $\eta \equiv \chi(\tau)$ such that $\eta = 1$ for $\tau \geq \tau_1$ and $\eta = 0$ for $\tau \leq \tau_0$ as well as any $\omega \in \Omega^k_{sc}(M)$ such that $\Box_k\omega = 0$, let $\omega_\eta \doteq \Box_k\eta\omega \in \Omega_c(M)$. A direct computation shows that $G_{f,f'}(\omega_\eta) = \omega$. To conclude we need to establish the right exactness of the sequence. Consider any $\alpha \in \Omega^k_{sc}(M)$ and the equation $\Box_k\omega = \alpha$. Consider the function $\eta(\tau)$ as above and let $\omega \doteq G^+_{f,f'}(\chi\alpha) + G^-_{f,f'}((1-\eta)\alpha)$. In view of Remark 15 and of the support properties of the fundamental solutions, $\omega \in \Omega^k_{sc,f,f'}(M)$ and $\Box_k\omega = \alpha$. $\Box$

It is worth focusing specifically on the boundary conditions individuated in Definition 8 since it is possible to improve slightly the results of Proposition 16.

Lemma 17: *Under the hypotheses of Assumption 11 and of Remark 14, it holds that*

1. *the fundamental solutions obey the following duality relations*

$$* \circ G^{\pm}_{\mathrm{T}} = G^{\pm}_{\perp} \circ * \qquad * \circ G^{\pm}_{\mathrm{D}} = G^{\pm}_{\mathrm{D}} \circ * \qquad * G^{\pm}_{\mathrm{N}} = G^{\pm}_{\mathrm{N}} *\,. \tag{14}$$

In addition

$$G^{\pm}_{\mathrm{N}} \circ d = d \circ G^{\pm}_{\mathrm{N}} \text{ on } \Omega^k_{\mathrm{t}}(M) \cap \Omega^k_{pc/fc}(M)\,, \quad G^{\pm}_{\mathrm{T}} \circ \mathrm{d} = \mathrm{d} \circ G^{\pm}_{\mathrm{T}} \text{ on } \Omega^k_{\mathrm{t}}(M) \cap \Omega^k_{pc/fc}(M)\,, \tag{15}$$

$$G^{\pm}_{\mathrm{N}} \circ \delta = \delta \circ G^{\pm}_{\mathrm{N}} \text{ on } \Omega^k_{\mathrm{n}}(M) \cap \Omega^k_{pc/fc}(M)\,, \quad G^{\pm}_{\perp/T} \circ \delta = \delta \circ G^{\pm}_{\perp/T} \text{ on } \Omega^k_{\mathrm{n}}(M) \cap \Omega^k_{pc/fc}(M)\,. \tag{16}$$

2. *for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$ it holds*

$$(\alpha, G^{\pm}_{\sharp}\beta) = (G^{\mp}_{\sharp}\alpha, \beta)\,, \tag{17}$$

where $\sharp \in \{D, N, T, \perp\}$.

3. *calling $G_\sharp \doteq G^+_\sharp - G^-_\sharp$, the following is a short exact sequence:*

$$0 \to \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{c}}(M) \xrightarrow{G_\sharp} \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{sc}}(M) \to 0\,. \tag{18}$$

Proof. The second and the third item can be proven slavishly as in Proposition 16 and hence we omit it. The same applies to Equation (14), taking into account Equation (8). With reference to (15) and (16) their proof is mutatis mutandis the same. Hence we shall only focus on the first identity

For every $\alpha \in \Omega^k_c(M) \cap \Omega^k_t(M)$, $G^\pm_{\mathrm{N}}(d\alpha)$ and $dG^\pm_{\mathrm{N}}(\alpha)$ lie both in $\Omega^k_N(M)$. In particular, using Equation (4b), $t\delta dG^\pm_{\mathrm{N}}(\alpha) = t(\Box_k - d\delta)G^\pm_{\mathrm{N}}(\alpha) = t\alpha = 0$ while the second boundary condition is automatically satisfied since $d^2 = 0$. Hence, considering $\beta = G^\pm_{\mathrm{N}}(d\alpha) - dG^\pm_{\mathrm{N}}(\alpha)$, it holds that $\Box_k\beta = 0$ and $\beta \in \Omega^k_{\mathrm{N}} \cap \Omega^k_{pc/fc}(M)$. In view of Remark 15, this entails $\beta = 0$. $\Box$

Remark 18: *Following the same reasoning as in [Bär15] and as in Remark 14 together with minor adaption of the proofs of [DDF19], one may extend both $G_{f,f'}$ and $G_\sharp$ to operators $G_{f,f'}\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k(M)$ and $G_\sharp\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k(M)$ for all $f, f' \in C^\infty(\partial M)$ and for all $\sharp \in \{D, N, T, \perp\}$. As a consequence the exact sequences of Proposition 16 and of Lemma 17 generalize as*

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_\sharp(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_\sharp} \Omega^k_\sharp(M) \xrightarrow{\Box_k} \Omega^k(M) \to 0\,. \tag{19}$$

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_{f,f'}(M) \xrightarrow{\Box_k} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_\sharp} \Omega^k_{f,f'}(M) \xrightarrow{\Box_k} \Omega^k(M) \to 0\,. \tag{20}$$

We conclude with a corollary to Lemma 17 which shows that, when considering the difference between the advances and retarded fundamental solutions, the support restrictions present in Equation (15) and in Equation (16) disappear.

Corollary 19: *Under the hypotheses of Assumption 11 and of Remark 14, it holds that*

$$G_{\mathrm{N}} \circ d = d \circ G_{\mathrm{N}} \text{ on } \Omega^k_{tc}(M)\,, \quad G_{\mathrm{T}} \circ \mathrm{d} = \mathrm{d} \circ G_{\mathrm{T}} \text{ on } \Omega^k_{tc}(M)\,, \tag{21}$$

$$G_{\mathrm{N}} \circ \delta = \delta \circ G_{\mathrm{N}} \text{ on } \Omega^k_{tc}(M)\,, \quad G_{\perp/T} \circ \delta = \delta \circ G_{\perp/T} \text{ on } \Omega^k_{tc}(M)\,. \tag{22}$$

Proof. In all cases the reasoning is the same as in the proof of Equation (15) and in Equation (16). Focusing for simplicity on the first identity of Equation (21), the only additional necessary information comes from $t\delta dG^\pm_{\mathrm{N}}(\alpha) = t(\Box_k - d\delta)G^\pm_{\mathrm{N}}(\alpha) = t\alpha$, for all $\alpha \in \Omega^k_{tc}(M)$. This entails that, being $G_{\mathrm{N}} = G^+_{\mathrm{N}} - G^-_{\mathrm{N}}$, $\mathrm{t}\delta dG_{\mathrm{N}}(\alpha) = 0$. $\Box$

3.2 On the Maxwell operator

In this section we focus our attention on the Maxwell operator $\mathcal{M}_k \doteq \delta d : \Omega^k(M) \to \Omega^k(M)$ studying its kernel in connection both to the D'Alembert - de Rham wave operator $\Box_k$ and to the identification of suitable boundary conditions. We shall keep the assumption that (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $n \geq 2$. In addition we impose that $0 < k < n$ since, if $k = n$, than the Maxwell operator becomes trivial, while, if $k = 0$, $\mathcal{M}_0 = \Box_0$. Hence this case falls in the one studied in the preceding section and in [DDF19].

In complete analogy to the analysis of $\Box_k$, we observe that, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, the following Green's formula holds true:

$$(\delta \mathrm{d}\alpha, \beta) - (\alpha, \delta \mathrm{d}\beta) = (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_\partial - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_\partial\,. \tag{23}$$

In the same spirit of Lemma 6, the operator $\mathcal{M}_k$ becomes formally self-adjoint if we restrict its domain to

$$\Omega^k_f(M) \doteq \{\omega \in \Omega^k(M) \mid t\alpha = f n d\alpha\}, \tag{24}$$

where $f \in C^\infty(\partial M)$ is arbitrary but fixed. Also in this scenario it is convenient to disentangle two distinguished classes of boundary conditions. Observe that one of the possibilities has been already introduced in (3), but we feel necessary to repeat it, so to emphasize the connection with the Green's formula (23).

Definition 20: *Let (M,g) be a globally hyperbolic spacetime with timelike boundary and let $0 < k < n$, $n = \dim M$. We call*

- *space of k-forms with tangential boundary condition*

$$\Omega^k_{\mathrm{t}}(M) := \{\omega \in \Omega^k(M) |\ \mathrm{t}\omega = 0\}\,. \tag{25}$$

- *space of k-forms with d-normal boundary condition*

$$\Omega^k_{\mathrm{dn}}(M) := \{\omega \in \Omega^k(M) |\ \mathrm{n}d\omega = 0\}\,. \tag{26}$$

In the following our first goal is to characterize the kernel of the Maxwell operator with a prescribed boundary condition, *cf.* Equation (24). To this end we need to focus on the *gauge invariance* of the underlying theory. In the case in hand this translates in the following characterization.

Definition 21: *Let (M,g) be a globally hyperbolic spacetime with timelike boundary and let $\mathcal{M}_k$ be the Maxwell operators acting on $\Omega^k(M)$, $0 < k < \dim M$. We say that*

- *$A \in \Omega^k_f(M) \cap \ker(\mathcal{M}_k)$, $f \in C^\infty(\partial M)$, is* gauge equivalent *to $A' \in \Omega^k_f(M)$ if there exists $\chi \in \Omega^{k-1}(M)$ such that $A' = A + d\chi$ and $t\chi = 0$.*
- *$A \in \Omega^k_{dn}(M) \cap \ker(\mathcal{M}_k)$, is* gauge equivalent *to $A' \in \Omega^k_{dn}(M)$ if there exists $\chi \in \Omega^{k-1}(M)$ such that $A' = A + d\chi$.*

Observe that the boundary condition in Equation (26) leads to a different notion of gauge equivalence since the boundary condition is giving no additional constraint on account of the fact that $ndA' = ndA$ for every choice of $\chi \in \Omega_{k-1}(M)$. In the following proposition we characterize the space of equivalence classes of solution to the Maxwell equation up to gauge transformations. To this end we recall that we are still working in the framework of Assumption 11.

Proposition 22: *Let (M,g) be a globally hyperbolic spacetime with timelike boundary and let $f \in C^\infty(\partial M)$. It holds that*

1. *for all $A \in \Omega^k_f(M)$, such that $\mathcal{M}_k(A) = 0$, there exists $\chi \in \Omega^0_t(M)$ for which $A' = A + d\chi \in \Omega^k_f(M)$ and*
$$\begin{cases} \Box_k A' = 0 \\ \delta A' = 0 \\ tA' + fndA' = 0 \end{cases}, \tag{27}$$
2. *if, in addition, $A \in \Omega^k_{\mathrm{sc}}(M)$, then χ can be fixed so that also $A' \in \Omega^k_{\mathrm{sc}}(M)$.*

The same statement holds true mutatis mutandis for $A \in \Omega^k_{dn}(M)$.

Proof. We focus only on the first point, since the second is a direct consequence of the first one and of the exact sequence (13). Let $A \in \Omega^k_f(M)$ be as per hypothesis. Consider any $\chi \in \Omega^{k-1}(M)$ such that

$$\Box_{k-1}\chi = -\delta A, \quad \delta\chi = 0, \quad t\chi = 0. \tag{28}$$

In view of Assumption 11 and of Remark 15, we can fix $\chi = G_T(\delta A)$, since the constraint $\delta\chi = 0$ entails $t\delta\chi = 0$. On account of Corollary 19, $\delta\chi = \delta(G_T(\delta A)) = G_T(\delta^2 A) = 0$. In addition since $t\chi = 0$ then, Equation (4b) yields $d_\partial t\chi = td\chi = 0$. Hence A' is gauge equivalent to A as per Definition 21. $\Box$

A direct inspection of (28) unveils that choosing a solution to this equation does not fix completely the gauge and a residual freedom is left. This amount either to

$$\mathcal{G}_f(M) \doteq \{\chi' \in \Omega^{k-1}(M) \mid \Box_{k-1}\chi = 0,\ \delta\chi = 0 \text{ and } dt\chi = 0\},$$

whenever $f \in C^\infty(\partial M)$ or, in the case of a d-normal boundary condition, to

$$\mathcal{G}_{dn} \doteq \{\chi' \in \Omega^{k-1}(M) \mid \Box_{k-1}\chi = 0 \text{ and } \delta\chi = 0\}.$$

To better codify the results of the preceding discussion, it is also convenient to introduce the following linear spaces:

$$\mathcal{S}_{k,f}(M) \doteq \{A \in \Omega^k(M) \mid \mathcal{M}_k(A) = 0 \text{ and } tA = fndA\}, \tag{29}$$

$$\mathcal{S}^\Box_{k,f}(M) \doteq \{A \in \Omega^k(M) \mid \Box_k(A) = 0,\ \delta A = 0 \text{ and } tA = fndA\}, \tag{30}$$

where $f \in C^\infty(\partial M)$ and where we adopt the convention that

$$\mathcal{S}_{k,\infty}(M) \doteq \{A \in \Omega^k(M) \mid \mathcal{M}_k(A) = 0 \text{ and } ndA = 0\}, \tag{31}$$

$$\mathcal{S}^\Box_{k,\infty}(M) \doteq \{A \in \Omega^k(M) \mid \Box_k(A) = 0,\ \delta A = 0 \text{ and } ndA = 0\}. \tag{32}$$

Hence Proposition 22 can be summarized as stating the existence of the following isomorphisms:

$$\mathcal{S}_{\mathcal{G}_f,k}(M) \doteq \frac{\mathcal{S}_{k,f}(M)}{d\Omega^{k-1}_t(M)} \simeq \frac{\mathcal{S}^\Box_{k,f}(M)}{\mathcal{G}_f(M)} \quad \text{and} \quad \mathcal{S}_{\mathcal{G}_{dn},k}(M) \doteq \frac{\mathcal{S}_{k,\infty}(M)}{d\Omega^{k-1}(M)} \simeq \frac{\mathcal{S}^\Box_{k,\infty}(M)}{\mathcal{G}_{dn}(M)}, \tag{33}$$

where the special case of $f = 0$ corresponds to considering tangential boundary conditions. The spaces $\mathcal{S}^{sc}_{\mathcal{G}_f,k}(M)$ and $\mathcal{S}^{sc}_{\mathcal{G}_{dn},k}(M)$, where the superscript *sc* entails that we consider only those equivalence classes admitting a representative which is spacelike compact, can be endowed with a presymplectic form – *cf.* [HS13, Prop. 5.1].

Proposition 23: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $f \in C^\infty(\partial M)$. Then, for any k such that $0 < k < \dim M$, the following map $\sigma_{k,f} : \mathcal{S}^{sc}_{\mathcal{G}_f,k}(M) \times \mathcal{S}^{sc}_{\mathcal{G}_f,k}(M) \to \mathbb{R}$ is a presymplectic form:*

$$\sigma_{k,f}([A_1], [A_2]) = (\delta \mathrm{d} A_1^+, A_2)\,, \quad \forall [A_1], [A_2] \in \mathcal{S}^{sc}_{\mathcal{G}_f,k}(M) \tag{34}$$

where $(,)$ is the standard pairing between k-forms, while $A_1^+ \doteq \eta(\tau)A_1$, with $\eta \equiv \eta(\tau)$ a smooth function such that $\eta = 1$ for all $\tau > \tau_1$ while $\eta = 0$ when $\tau < \tau_0$, $\tau_0 < \tau_1$ being arbitrary. The similar result holds for $\mathcal{S}^{sc}_{\mathcal{G}_{dn},k}(M)$ and we denote the associated presymplectic form $\sigma_{k,dn}$.

Proof. To start with we observe that the right hand side of (34) is finite since A_2 is a spacelike compact k-form while $\delta \mathrm{d}A_1^+$ is compactly supported on account of A_1 being on-shell. Secondly the pairing does not depend on the choice of representative for any equivalence class in $\mathcal{S}^{sc}_{\mathcal{G}_f,k}(M)$. As a matter of fact, consider $[A_2] \in \mathcal{S}^{sc}_{\mathcal{G}_f,k}(M)$ and let $A_2, A_2 + d\chi$ be two representatives. A direct calculations shows that

$$(\delta \mathrm{d}A_1^+, d\chi) = (\delta^2 \mathrm{d}A_1^+, \chi) = 0,$$

where we used that $\delta \mathrm{d}A_1^+$ is compactly supported. As last consistency check we need to show that (34) is independent from the choice of $\eta(\tau)$. Fix therefore a second cut off function $\widetilde{\eta}(\tau)$ and let $[A_1], [A_2] \in \mathcal{S}^{sc}_{\mathcal{G}_f,k}(M)$. It holds that

$$(\delta \mathrm{d}(A_1^+ - \widetilde{A}_1^+), A_2) = (A_1^+ - \widetilde{A}_1^+, \delta dA_2) = 0,$$

where $\widetilde{A}_1^+ \doteq \widetilde{\eta} A_1$ and where we used implicitly that $A_1^+ - \widetilde{A}_1^+$ is compactly supported. To conclude the proof we need to show that Equation (34) identifies a presymplectic form. While the pairing between k-forms is bilinear per construction, $\sigma_{k,f}$ is antisymmetric on account of the following chain of identities:

$$\sigma_{k,f}([A_1], [A_2]) = (\delta dA_1^+, A_2) = -(\delta dA_1^-, A_2^+) = (A_1^-, \delta dA_2^+) = (A_1, \delta dA_2^+) = -\sigma_{k,f}([A_1], [A_2]),$$

where $A_1^- \doteq (1-\eta)A_1$. Observe that, in the second equality, we used the equations of motion and that, switching from A_1^+ to A_1^- has been necessary to ensure that $\mathrm{supp}(A_1^-) \cap \mathrm{supp}(A_2^+)$ is compact. The proof for $\mathcal{S}^{sc}_{\mathcal{G}_{dn,k}(M)}$ is identical, hence we omit it. □

Remark 24: *Following [HS13, Cor. 5.3] – cf. also Proposition 33, $\sigma_{k,f}$ does not define in general a symplectic form on the space of spacelike compact solutions of the Maxwell equation. The obstruction can be ascribed to the structural properties of the gauge transformations, in particular that $d[\Omega^{k-1}_t(M) \cap \Omega^{k-1}_{sc}(M)] \subset d\Omega^{k-1}_t(M) \cap \Omega^k_{sc}(M)$.*

Working either with (30) or with (32) leads to the natural question whether it is possible to give an equivalent representation of these spaces in terms of compactly supported k-forms. In the case of $f = 0$ (tangential boundary conditions) and of (32), using Assumption 11, the following proposition holds true:

Lemma 25: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then both $\mathcal{S}^{\Box}_{k,0}$ and $\mathcal{S}^{\Box}_{k,\infty}$ are isomorphic to $\frac{\widetilde{\Omega}^k_{tc}(M)}{\Box[\Omega^k_{tc}(M)]}$ where*

$$\widetilde{\Omega}^k_{tc}(M) \doteq \{\omega \in \Omega^k_{tc}(M) \mid \delta\omega = \delta d\lambda,\ \lambda \in \Omega^k_{tc}(M)\},$$

and where the isomorphism is implemented respectively by G_T and G_N. The same statement holds true for spacelike compact solutions replacing timelike compact with compact test forms.

Proof. Mutatis mutandis, the proof is identical in both cases. Hence we focus only on $\mathcal{S}^{\Box}_{k,\infty}(M)$. If we combine (32) together with Assumption 11 and Remark 14, it holds that every $A' \in \Omega^k(M)$ such that $\Box_k A' = 0$ and $ndA' = 0$ can be written as $G_N(\omega)$, where $\omega \in \Omega^k_{tc}(M)$. In view of Lemma 17, the constraint $\delta A' = 0$, implies that $\delta G_N(\omega) = G_N(\delta\omega)0$. Hence $\delta\omega \in \mathrm{Ker}(G_N)$, which entails that there exists $\lambda \in \Omega^k_{tc}(M)$ such that $\delta\omega = \Box\lambda$. By applying δ, one obtains $\Box\delta\lambda = 0$, which implies $\delta\lambda = 0$ and thus $\delta\omega = \delta d\lambda$. We have proven that G_N is a surjective map from $\widetilde{\Omega^k_{tc}}(M)$ to $\mathcal{S}^{\Box}_{k,\infty}$. To conclude the proof it suffices to recall that $\mathrm{Ker}(G_N) = P[\Omega^k_{tc}(M)]$ and that $P[\Omega^k_{tc}(M)] \subset \widetilde{\Omega}^k_{tc}(M)$. □

Remark 26: *The characterization result of Lemma 25 cannot be reproduced for a generic $\mathcal{S}^{\Box}_{k,f}(M)$ since there is no guarantee that a counterpart of Lemma 17 holds true for $G_{f,0}$ which is the causal propagator* *associated to the equation $\Box A = 0$, together with the boundary conditions $tA + fndA = 0$ and $t\delta A = 0$. Although it is easy to construct examples of coclosed solutions of this partial differential equation, the lack of a characterization in terms of test k-forms obstructs the analysis of the algebra of observables for this kind of free field theories.*

3.3 The algebra of observables for $\mathcal{S}_{\mathcal{G}_{k,0}}(M)$ and for $\mathcal{S}_{\mathcal{G}_{k,dn}}(M)$

In this section we would like to introduce the algebras of observables $\mathcal{O}_{\mathrm{gD}}(M)$, $\mathcal{O}_{\mathrm{gN}}(M)$ associated to the solution space $\mathrm{Sol}_{\mathrm{gD}}(M), \mathrm{Sol}_{\mathrm{gN}}(M)$ introduced in definition 20.

We first discuss the case of a globally hyperbolic manifold with $\partial M = \emptyset$ – *cf.* [Ben16, HS13]. In this setting let consider

$$\mathcal{O}(M) := \ker_{\mathrm{c}} \delta / \delta \mathrm{d}\Omega^k_{\mathrm{c}}(M)\,, \tag{35}$$

where $\ker_{\mathrm{c}} \delta$ is defined according to (**??**). As shown in [HS13, Sec. 4] the vector space $\mathcal{O}(M)$ satisfies the following properties:

1. $\mathcal{O}(M)$ is separating and optimal for $\mathrm{Sol}(M) := \ker \delta\mathrm{d}/\mathrm{d}\Omega^{k-1}$ that is the pairing

$$\mathcal{O}(M) \otimes \mathrm{Sol}(M) \ni \widehat{A} \otimes \widehat{\alpha} \mapsto (\alpha, A)\,, \tag{36}$$

   is well-defined – here $A \in \widehat{A}$ and $\alpha \in \widehat{\alpha}$ – and moreover

$$(\widehat{\alpha}, \widehat{A}) = 0 \quad \forall \widehat{\alpha} \in \mathcal{O}(M) \Longrightarrow \widehat{A} = 0 \in \mathrm{Sol}(M)\,, \tag{37}$$

$$(\widehat{\alpha}, \widehat{A}) = 0 \quad \forall \widehat{A} \in \mathrm{Sol}(M) \Longrightarrow \widehat{\alpha} = 0 \in \mathcal{O}(M)\,. \tag{38}$$

2. $\mathcal{O}(M)$ carries an anti-Hermitian form defined by

$$\varsigma \colon \mathcal{O}(M)^{\times 2} \ni (\widehat{\alpha}, \widehat{\beta}) \mapsto (\alpha, G\beta)\,, \tag{39}$$

   where G denotes the causal propagator for the operator $\Box$ while $\alpha \in \widehat{\alpha}$ and $\beta \in \widehat{\beta}$.

3. the causal propagator G induces an isomorphism

$$\mathcal{O}(M) \ni \widehat{\alpha} \mapsto \widehat{G\alpha} \in \mathrm{Sol}^{\mathrm{sc}}(M)\,, \tag{40}$$

   where $\mathrm{Sol}^{\mathrm{sc}}(M) := \ker_{\mathrm{sc}} \delta\mathrm{d}/\mathcal{G}$, being $\mathcal{G} := \mathrm{d}\Omega^k_{\mathrm{sc}}(M)$. Moreover G preserves the anti-Hermitian forms, namely

$$\sigma(\widehat{G\alpha}, \widehat{G\beta}) = \varsigma(\widehat{\alpha}, \widehat{\beta})\,, \qquad \forall \widehat{\alpha}, \widehat{\beta} \in \mathcal{O}(M)\,, \tag{41}$$

   where σ is the anti-Hermitian form on $\mathrm{Sol}^{\mathrm{sc}}(M)$ defined by $\sigma(\widehat{A}_1, \widehat{A}_2) := (\delta\mathrm{d}A_1^+, A_2)$ – *cf.* proposition 23.

In view of (40) the vector space $\mathcal{O}(M)$ can be regarded as an equivalent description of the space $\mathrm{Sol}^{\mathrm{sc}}(M)$. More importantly $\mathcal{O}(M)$ generates the algebra of observables associated with the space of solutions $\mathrm{Sol}(M)$. Indeed let $F_{\widehat{\alpha}}$ denote the linear functional over $\mathrm{Sol}(M)$ associated with $\widehat{\alpha} \in \mathcal{O}(M)$ via the pairing (36). Then the algebra of observables over $\mathrm{Sol}(M)$ is defined as the algebra $\mathcal{A}(M)$ generated by the functionals $F_{\widehat{\alpha}}$. The pairing (36) justifies the interpretation of $\widehat{\alpha} \in \mathcal{O}(M)$ as a local measurement of any configuration $\widehat{A} \in \mathrm{Sol}(M)$. Moreover, equation (37) implies that $\mathcal{O}(M)$ – and thus $\mathcal{A}(M)$ – is capable to distinguish all configurations $\widehat{A} \in \mathrm{Sol}(M)$ while equation (38) shows that the $\mathcal{A}(M)$ is the smallest algebra with property (37).

We would like to find an appropriate equivalent $\mathcal{O}_{\mathrm{g}\sharp}(M)$ for the solution space $\mathrm{Sol}_{\mathrm{g}\sharp}(M)$. As in the case with empty boundary we will look for the optimal algebra which is separating for $\mathrm{Sol}_{\mathrm{g}\sharp}(M)$ – *cf.* definition 20. Since we want the pairing (36) to be well-defined we will consider a suitable quotient of $\Omega^k_{\mathrm{c}}(M)$ with the following properties:

$$0 = (\delta\mathrm{d}A, \alpha) = (A, \delta\mathrm{d}\alpha) + (\mathrm{t}A, \mathrm{n}\mathrm{d}\alpha) - (\mathrm{n}\mathrm{d}A, \mathrm{t}\alpha)\,, \tag{42}$$

for all $A \in \ker\delta\mathrm{d}_{\mathrm{g}\sharp}$, where in the second equality we used (23). Moreover we require

$$0 = (\mathrm{d}\chi, \alpha) = (\chi, \delta\alpha) + (\mathrm{t}\chi, \mathrm{n}\alpha)\,. \tag{43}$$

for all $\mathrm{d}\chi \in \mathcal{G}_{\mathrm{g}\sharp}$. This leads to the following definition

Definition 27: *The algebra of observables $\mathcal{A}_{\mathrm{gD}}(M)$ on the space $\mathrm{Sol}_{\mathrm{gD}}(M)$ is defined by as the algebra generated by the functionals $F_{\widehat{\alpha}}\colon \mathrm{Sol}_{\mathrm{gD}}(M) \ni A \to (\alpha, A) \in \mathbb{C}$ being $A \in \widehat{A}$ and $\alpha \in \widehat{\alpha} \in \mathcal{O}_{\mathrm{gD}}(M)$ where*

$$\mathcal{O}_{\mathrm{gD}}(M) := \ker_{\mathrm{c}}\delta/\delta\mathrm{d}\{\alpha \in \Omega^k_{\mathrm{c}}(M)|\ \mathrm{t}\alpha = 0 \in H^k_{\mathrm{c}}(M, \mathrm{d})\}\,. \tag{44}$$

Similarly the algebra of observables $\mathcal{A}_{\mathrm{gN}}(M)$ on the space $\mathrm{Sol}_{\mathrm{gN}}(M)$ is defined by as the algebra generated by the functionals $F_{\widehat{\alpha}}\colon \mathrm{Sol}_{\mathrm{gN}}(M) \to \mathbb{C}$ being $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gN}}(M)$ where

$$\mathcal{O}_{\mathrm{gN}}(M) := \ker_{\mathrm{c}}\delta_{\mathrm{n}}/\delta\mathrm{d}\big[\Omega^k_{\mathrm{gN}}(M) \cap \Omega^k_{\mathrm{c}}(M)\big]\,. \tag{45}$$

Remark 28: *Notice that the quotient (45) is well-defined since for all $\eta \in \Omega^k_{\mathrm{gN}}(M)$ we have $\mathrm{n}\delta\mathrm{d}\eta = -\delta\mathrm{n}\mathrm{d}\eta = 0$. Moreover (43) holds true for $\widehat{\alpha} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$ so that the pairing with $\mathrm{Sol}_{\mathrm{g}\sharp}(M)$ is well-defined.*

The quotients assure that (42) is satisfied and therefore the pairing with $\mathrm{Sol}_{\mathrm{g}\sharp}(M)$ descends to the quotients. This is clear for $\mathrm{g}\sharp = \mathrm{gN}$ while for $\mathrm{g}\sharp = \mathrm{gD}$ the right-hand side of (42) reduces to

$$(\mathrm{n}\mathrm{d}A, \mathrm{t}\alpha) = (\mathrm{n}\mathrm{d}A, \mathrm{d}\eta) = (\delta\mathrm{n}\mathrm{d}A, \eta) = -(\mathrm{n}\delta\mathrm{d}A, \eta) = 0\,,$$

where we used (5) for ∂M – in this case the right-hand side of (5) vanishes because ∂M has no boundary – as well as (4) and $A \in \mathrm{Sol}_{\mathrm{gN}}(M)$.

Definition 27 provide an algebra $\mathcal{A}_{\mathrm{g}\sharp}(M)$ which satisfies the separability and the optimality conditions (37-38).

Proposition 29: *Let $g\sharp \in \{\mathrm{gD}, \mathrm{gN}\}$ and let $\mathcal{A}_{\mathrm{g}\sharp}(M)$ be the algebra introduced in definition 27. Then the following holds true:*

$$(\widehat{\alpha}, \widehat{A}) = 0 \quad \forall \widehat{A} \in \mathrm{Sol}_{\mathrm{g}\sharp}(M) \Longrightarrow \widehat{\alpha} = 0 \in \mathcal{O}_{\mathrm{g}\sharp}(M)\,, \tag{46}$$

$$(\widehat{\alpha}, \widehat{A}) = 0 \quad \forall \widehat{\alpha} \in \mathcal{O}_{\mathrm{g}\sharp}(M) \Longrightarrow \widehat{A} = 0 \in \mathrm{Sol}_{\mathrm{g}\sharp}(M)\,. \tag{47}$$

Proof: For all $\widehat{A} \in \mathrm{Sol}_{\mathrm{g}\sharp}(M)$ and $\widehat{\alpha} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$ the pairing $(\widehat{\alpha}, \widehat{A}) := (\alpha, A)$, where $\alpha \in \widehat{\alpha}$ and $A \in \widehat{A}$, is well-defined – *cf.* remark 28.

We now prove the fist statement for gauge Neumann boundary condition. Let as assume $\widehat{A} \in \mathrm{Sol}_{\mathrm{gN}}(M)$ is such that $(\widehat{\alpha}, \widehat{A}) = 0$ for all $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gN}}(M)$. This implies that there exists $A \in \widehat{A}$ such that $(\alpha, A) = 0$ for all $\alpha \in \ker_{\mathrm{c}}\delta_{\mathrm{n}}$. Using equation (5) as well as definition (27) it follows that $\mathrm{d}A = 0$ and that $A = 0 \in H_{\mathrm{c}}(M, \delta_{\mathrm{n}})^*$. By Poincaré duality – *cf.* theorem **??** in appendix B – we have $H^k_{\mathrm{c}}(M, \delta_{\mathrm{n}})^* \simeq H^k(M, \mathrm{d})$, therefore $A = \mathrm{d}\chi \in \mathcal{G}_{\mathrm{gN}}$, that is, $\widehat{A} = 0$.

The proof of the first statement for gauge Dirichlet boundary conditions proceeds in a similar way. In particular let $\widehat{A} \in \mathrm{Sol}_{\mathrm{gD}}(M)$ be such that $(\widehat{\alpha}, \widehat{A}) = 0$ for all $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gD}}(M)$. It follows that there exists $A \in \widehat{A}$ such that $(\alpha, A) = 0$ for all $\alpha \in \ker_{\mathrm{c}} \delta$. This implies that $\mathrm{d}A = 0$ as well as $A = 0 \in H^k_{\mathrm{c}}(M, \delta)^*$. Since $H^k_{\mathrm{c}}(M, \delta)^* \simeq H^k(M, \mathrm{d}_{\mathrm{t}})$ it follows that $A = \mathrm{d}\chi \in \mathcal{G}_{\mathrm{gD}}$, that is, $\widehat{A} = 0$.

The proof of the second statement is identifical for both gauge Dirichlet and gauge Neumann boundary conditions. Let $\widehat{\alpha} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$ be such that $(\widehat{\alpha}, \widehat{A}) = 0$ for all $\widehat{A} \in \mathrm{Sol}_{\mathrm{g}\sharp}(M)$. This implies that there exists $\alpha \in \ker_{\mathrm{c}} \delta$ such that (α, A) for all $A \in \ker \delta \mathrm{d}_{\mathrm{g}\sharp}$. Since for all $\beta \in \Omega^k_{\mathrm{tc}}(M)$ we have $G_\sharp \beta \in \ker \Box_\sharp \cap \ker \delta \subset \ker \delta \mathrm{d}_{\mathrm{g}\sharp}$ it follows that $(\alpha, G_\sharp \beta) = 0$. Using equation (12) and the arbitrariness of $\beta \in \Omega^k_{\mathrm{tc}}(M)$ it follows that $\alpha \in \ker_{\mathrm{c}} G_\sharp$. By proposition 16 it follows that $\alpha = \Box_\sharp \eta$ with $\eta \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M)$. Since $\alpha \in \ker_{\mathrm{c}} \delta$ we have $\Box \delta \eta = 0$ which implies $\delta \eta = 0$ – *cf.* remark **??**. It follows that $\alpha = \delta \mathrm{d}\eta$ with $\eta \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M)$, that is, $\widehat{\alpha} = 0$.

Proposition 29 ensures that the vector space $\mathcal{O}_{\mathrm{g}\sharp}(M)$ and the associated algebra $\mathcal{A}_{\mathrm{g}\sharp}(M)$ are the correct generalizations for the algebra of observables on the solution space $\mathrm{Sol}_{\mathrm{g}\sharp}(M)$ in the case of non-empty boundary ∂M.

We now proceed in showing that the vector space $\mathcal{O}_{\mathrm{g}\sharp}(M)$ introduced in definition 27 carries a natural pre-symplectic structure.

Proposition 30: *Let* $\mathrm{g}\sharp \in \{\mathrm{gD}, \mathrm{gN}\}$. *Then the map* $\varsigma_{\mathrm{g}\sharp} \colon \mathcal{O}_{\mathrm{g}\sharp}(M)^{\times 2} \to \mathbb{C}$ *given by*

$$\varsigma_{\mathrm{g}\sharp}(\widehat{\alpha}, \widehat{\beta}) := (\widehat{\alpha}, G_\sharp \widehat{\beta}) \,, \tag{48}$$

defines a pre-symplectic form – here $\alpha \in \widehat{\alpha}$ *and* $\beta \in \widehat{\beta}$.

Proof. For all $\alpha, \beta \in \ker_{\mathrm{c}} \delta$ the pairing $(\alpha, G_\sharp \beta)$ and anti-Hermitian because of equation (12). Finally we prove that $\varsigma_{\mathrm{g}\sharp}(\widehat{\alpha}, \widehat{\beta}) := (\alpha, G_\sharp \beta)$ is well-defined for $\widehat{\alpha}, \widehat{\beta} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$. This follows from proposition **??** as well as from $\Box_\sharp \circ G_\sharp = 0$: indeed for $\eta \in \Omega^k_{\mathrm{c}}(M)$ we have

$$G_\sharp \delta \mathrm{d}\eta = \delta \mathrm{d} G_\sharp \eta = -\mathrm{d}\delta G_\sharp \eta \,.$$

Therefore for all $\alpha \in \ker_{\mathrm{c}} \delta$ we find $-(\alpha, G_\sharp \delta \mathrm{d}\eta) = (\alpha, \mathrm{d} G_\sharp \delta \eta) = 0$. □

The degeneracy space of $(\mathcal{O}_{\mathrm{g}\sharp}(M), \varsigma_{\mathrm{g}\sharp})$ measures the failure of the assignment $M \to \mathcal{O}_{\mathrm{g}\sharp}(M)$ to satisfy the axioms of a local and covariant theory – *cf.* [BFV03] and definition .... It is therefore worth to investigate sufficient conditions which ensure the degeneracy space to be non-trivial. To this avail we first prove the following result

Proposition 31: *Let* $\sharp \in \{\mathrm{D}, \mathrm{N}\}$. *Then the map*

$$G_\sharp \colon \mathcal{O}_{\mathrm{g}\sharp}(M) \ni \widehat{\alpha} \to \widehat{G_\sharp \alpha} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{g}\sharp}(M) \,, \tag{49}$$

is well-defined. Moreover it preserves the presimplectic structures of $\mathcal{O}_{\mathrm{g}\sharp}(M)$ *and* $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{g}\sharp}(M)$ *that is*

$$\sigma_{\mathrm{g}\sharp}(\widehat{G_\sharp \alpha}, \widehat{G_\sharp \beta}) = \varsigma_{\mathrm{g}\sharp}(\widehat{\alpha}, \widehat{\beta}) \qquad \forall \widehat{\alpha}, \widehat{\beta} \in \mathcal{O}_{\mathrm{g}\sharp}(M) \,. \tag{50}$$

Finally we have that:

1. $G_{\mathrm{D}} \colon \mathcal{O}_{\mathrm{gD}}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$ *is a bijection;*
2. $G_{\mathrm{N}} \colon \mathcal{O}_{\mathrm{gN}}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M)$ *is injective with image*

$$G_{\mathrm{N}} \mathcal{O}_{\mathrm{gN}}(M) = \{\widehat{G_{\mathrm{N}} \alpha} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M) |\, \mathrm{n}\alpha = 0 \in H^k_{\mathrm{c}}(\partial M, \delta)\} \,. \tag{51}$$

Proof. For $\alpha \in \ker_{\mathrm{c}} \delta$ we have $G_\sharp\alpha \in \ker_{\mathrm{sc}} \Box_\sharp \cap \ker_{\mathrm{sc}} \delta \subseteq \ker \delta \mathrm{d}_{\mathrm{g}\sharp}$, therefore $\widehat{G_\sharp\alpha} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{g}\sharp}(M)$. Moreover, for $\eta \in \Omega^k_{\mathrm{c}}(M)$ we have

$$G_\sharp \delta \mathrm{d}\eta = \delta \mathrm{d} G_\sharp \eta = -\mathrm{d}\delta G_\sharp \eta = -\mathrm{d} G_\sharp \delta\eta\,.$$

This implies that $G_{\mathrm{D}}\delta\mathrm{d}\eta \in \mathcal{G}_{\mathrm{gD}}$ while if $\eta \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{gN}}(M)$ we have $G_{\mathrm{N}}\delta\mathrm{d}\eta \in \mathcal{G}_{\mathrm{gN}}$. It follows that $G_\sharp$ induces a well-defined map $G_\sharp \colon \mathcal{O}_{\mathrm{g}\sharp}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{g}\sharp}(M)$.

In order to prove (50) we observe that for all $\alpha \in \ker_{\mathrm{c}} \delta$ we can chose the decomposition of $G_\sharp\alpha = (G_\sharp\alpha)^+ + (G_\sharp\alpha)^-$ – *cf.* proposition 23 – to be $(G_\sharp\alpha)^\pm = G^\pm_\sharp\alpha$. Therefore for all $\widehat{\alpha}, \widehat{\beta} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$ we have

$$\sigma_{\mathrm{g}\sharp}(\widehat{G_\sharp\alpha}, \widehat{G_\sharp\beta}) = (\delta\mathrm{d}G^+_\sharp\alpha, G_\sharp\beta) = (\Box_\sharp G^\pm_\sharp\alpha, G_\sharp\beta) = (\alpha, G_\sharp\beta) = \varsigma_{\mathrm{g}\sharp}(\widehat{\alpha}, \widehat{\beta})\,,$$

where $\alpha \in \widehat{\alpha}$ and $\beta \in \widehat{\beta}$. In the third equality we used $\alpha \in \ker_{\mathrm{c}}\delta$ and equation (16) (*resp.* $\alpha \in \ker_{\mathrm{c}}\delta_{\mathrm{n}}$ and equation (15)) for gauge Dirichlet (*resp.* gauge Neumann) boundary conditions.

We now prove that $G_\sharp$ is injective: let $\widehat{\alpha} \in \mathcal{O}_{\mathrm{g}\sharp}(M)$ be such that $\widehat{G_\sharp\alpha} = 0$. For gauge Dirichlet (*resp.* gauge Neumann) boundary conditions it follows that there exists $\alpha \in \widehat{\alpha}$ with $\alpha \in \ker_{\mathrm{c}}\delta$ (*resp.* $\alpha \in \ker_{\mathrm{c}}\delta_{\mathrm{n}}$) such that $G_\sharp\alpha = \mathrm{d}\chi$, where $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M) \cap \Omega^{k-1}_{\mathrm{t}}(M)$ (*resp.* $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$). From equations (15-16) it follows that $\chi \in \ker_{\mathrm{sc}}\delta\mathrm{d}_{\mathrm{g}\sharp}$. Lemma **??** implies that there exists $\beta \in \ker_{\mathrm{c}}\delta$ such that

$$\chi = G_\sharp\beta + \mathcal{G}_{\mathrm{g}\sharp}\,.$$

It follows that $\alpha - \mathrm{d}\beta \in \ker_{\mathrm{sc}} G_\sharp$ therefore

$$\alpha - \mathrm{d}\beta = \Box_\sharp\eta\,,$$

for $\eta \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M)$ – *cf.* proposition 16. Applying δ to the last identity we find $\Box(\beta + \delta\eta) = 0$ thus $\beta + \delta\eta = 0$ – *cf.* remark **??**. It follows that $\alpha = \delta\mathrm{d}\eta$ with $\eta \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{N}}(M)$, that is, $\widehat{\alpha} = 0$.

Lemma **??** ensures that $G_{\mathrm{D}} \colon \mathcal{O}_{\mathrm{gD}}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$ is surjective. We now prove equation (51). The inclusion $\subseteq$ follows from the observation that for all $\alpha \in \widehat{\alpha} \in \mathcal{O}_{\mathrm{gN}}$ we have $\mathrm{n}\alpha = 0$. Conversely let $\widehat{A} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M)$ be such that $G_{\mathrm{N}}\alpha \in \widehat{A}$ where $\alpha \in \ker_{\mathrm{c}}\delta$ and $\mathrm{n}\alpha = \delta\beta$, $\beta \in \Omega^k_{\mathrm{c}}(\partial M)$. Let $\eta \in \Omega^k_{\mathrm{c}}(M)$ se such that $\mathrm{n}\mathrm{d}\eta = \beta$. Proposition **??** implies that

$$G_{\mathrm{N}}\delta\mathrm{d}\eta = \delta\mathrm{d}G_{\mathrm{N}}\eta = -\mathrm{d}G_{\mathrm{N}}\delta\eta \in \mathcal{G}_{\mathrm{gN}}\,,$$

so that $G_{\mathrm{N}}(\alpha + \delta\mathrm{d}\eta) \in \widehat{A}$ and moreover $\mathrm{n}(\alpha + \delta\eta) = \delta\beta - \delta\mathrm{n}\mathrm{d}\eta = 0$. Setting $\alpha_\eta := \alpha + \delta\mathrm{d}\eta$ it follows that $\widehat{\alpha_\eta} \in \mathcal{O}_{\mathrm{gN}}(M)$ is such that $G_{\mathrm{N}}\widehat{\alpha_\eta} = \widehat{A}$. $\Box$

Remark 32: *The obstruction to the surjectivity of G_{N} is cohomological and codified by $H^k_{\mathrm{c}}(\partial M, \delta)$. Notice that for all $\widehat{G_{\mathrm{N}}\alpha} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M)$ with $\alpha \in \ker_{\mathrm{c}}\delta_{\mathrm{n}}$ there are no $\beta \in \ker_{\mathrm{c}}\delta$ with $\mathrm{n}\beta \neq 0 \in H^k_{\mathrm{c}}(\partial M, \delta)$ and such that $G_{\mathrm{N}}\beta \in \widehat{G_{\mathrm{N}}\alpha}$. Indeed this would imply that $G_{\mathrm{N}}(\alpha - \beta) = \mathrm{d}\chi$ with $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$. With the same argument which shows that $G_{\mathrm{N}} \colon \mathcal{O}_{\mathrm{gN}}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M)$ is injective we would deduce that $\alpha - \beta = \delta\eta$, that is, $\beta = 0 \in H^k_{\mathrm{c}}(\partial M, \delta)$, a contradiction.*

We are now in position to state a sufficient condition for the degeneracy of the presymplectic space $(\mathcal{O}_{\mathrm{g}\sharp}(M), \varsigma_{\mathrm{g}\sharp})$.

Proposition 33: *The following holds true:*

1. *If there exists $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that $\mathrm{d}\chi \in \Omega^k_{\mathrm{sc}}(M)$ then the space $(\mathcal{O}_{\mathrm{gD}}(M), \varsigma_{\mathrm{gD}})$ is degenerate, that is, there exists $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gD}}(M)$ such that $\varsigma_{\mathrm{gD}}(\widehat{\alpha}, \widehat{\beta}) = 0$ for all $\widehat{\beta} \in \mathcal{O}_{\mathrm{gD}}(M)$.*

2. *If there exists $\chi \in \Omega^{k-1}(M)$ such that $\mathrm{d}\chi \in \Omega^k_{\mathrm{sc}}(M)$ then the space $(\mathcal{O}_{\mathrm{gN}}(M), \varsigma_{\mathrm{gN}})$ is degenerate, that is, there exists $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gN}}(M)$ such that $\varsigma_{\mathrm{gN}}(\widehat{\alpha}, \widehat{\beta}) = 0$ for all $\widehat{\beta} \in \mathcal{O}_{\mathrm{gN}}(M)$.*

Proof. We first consider the case of gauge Dirichlet boundary conditions. From remark 24 it follows that $\widehat{\mathrm{d}\chi} \neq 0 \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$, however $\varsigma_{\mathrm{gD}}(\widehat{\mathrm{d}\chi}, \widehat{A}) = 0$ for all $\widehat{A} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$. From proposition 31 it follows that there exists $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gD}}(M)$ such that $G_{\mathrm{D}}\widehat{\alpha} = \widehat{\mathrm{d}\chi}$. Moreover for all $\widehat{\beta} \in \mathcal{O}_{\mathrm{gN}}(M)$ we have $\varsigma_{\mathrm{gD}}(\widehat{\alpha}, \widehat{\beta}) = \sigma_{\mathrm{gD}}(\widehat{\mathrm{d}\chi}, G_{\mathrm{D}}\widehat{\beta}) = 0$.

The proof for gauge Neumann boundary conditions proceed in a similar manner. Once again from remark 24 we have $\widehat{\mathrm{d}\chi} \neq 0 \in \mathcal{O}_{\mathrm{gN}}(M)$ with the property that $\sigma_{\mathrm{gN}}(\widehat{\mathrm{d}\chi}, \widehat{A}) = 0$ for all $\widehat{A} \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$. Despite the fact that $G_{\mathrm{N}}\colon \mathcal{O}_{\mathrm{gN}}(M) \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gN}}(M)$ is not surjective, we now show that $\widehat{\mathrm{d}\chi}$ lies in $G_{\mathrm{N}}\mathcal{O}_{\mathrm{gN}}(M)$. This will conclude the proof because G_{N} intertwines the presimplectic structure of $\mathcal{O}_{\mathrm{gN}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{gD}}(M)$.

From Lemma **??** we have that $\mathrm{d}\chi = G_{\mathrm{N}}\alpha + \mathrm{d}\eta$ for $\alpha \in \ker_{\mathrm{c}}\delta$ and $\eta \in \Omega^{k-1}_{\mathrm{c}}(M)$. We redefine $\chi \to \chi - \eta$ so that $\mathrm{d}\chi_\eta = G_{\mathrm{N}}\alpha$ – notice that this does not spoil the property $\chi_\eta \in \Omega^{k-1}_{\mathrm{c}}(M)$, $\mathrm{d}\chi_\eta \in \Omega^k_{\mathrm{sc}}(M)$ and $\mathrm{d}\widehat{(\chi - \eta)} = \widehat{\mathrm{d}\chi}$.

From $\alpha \in \ker_{\mathrm{c}}\delta$ and proposition **??** it follows $\delta\mathrm{d}\chi = 0$. Moreover $\mathrm{n}\mathrm{d}\chi = \mathrm{n}G_{\mathrm{N}}\alpha = 0$ so that $\chi \in \ker\delta\mathrm{d}_{\mathrm{gN}}$. By lemma **??** there exists $\beta \in \ker_{\mathrm{tc}}\delta$ such that $\widehat{\chi} = \widehat{G_{\mathrm{N}}\beta}$. It follows that $\alpha - \mathrm{d}\beta \in \ker_{\mathrm{tc}} G_{\mathrm{N}}$ and thus

$$\alpha - \mathrm{d}\beta = \Box_{\mathrm{N}}\eta\,,$$

for $\eta \in \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_{\mathrm{N}}(M)$ – *cf.* proposition 16 and remark 18. Applying δ to the previous equation we find $\Box(\eta + \mathrm{d}\beta) = 0$ which implies $\eta + \mathrm{d}\beta = 0$ – *cf.* remark **??**. Thus we find $\alpha = \delta\mathrm{d}\eta$ for $\eta \in \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_{\mathrm{N}}(M)$ – notice that $\eta \notin \Omega^k_{\mathrm{c}}(M)$, otherwise $\widehat{\mathrm{d}\chi} = 0$. In particular this implies $\mathrm{n}\alpha = 0$ therefore $\widehat{\alpha} \in \mathcal{O}_{\mathrm{gN}}(M)$ and $G_{\mathrm{N}}\widehat{\alpha} = \widehat{G_{\mathrm{N}}\alpha} = \widehat{\mathrm{d}\chi}$. □

Example 34: *We now provide an example where the hypothesis of proposition 33 are met – cf. [HS13, Ex. 5.7]. Let consider half-Minkowski spacetime $\mathbb{R}^m_+ := \mathbb{R}^{m-1} \times \overline{\mathbb{R}_+}$ with flat metric and let $x \in \mathbb{R}^m_+$. We consider the spacetime $M := \mathbb{R}^m_+ \setminus J(x)$ which is still globally hyperbolic with timelike boundary. Let now $p \in B_1 \subseteq B_2$, where B_1, B_2 are balls in $\mathbb{R}^{m-1}_+$ centred at p.*

We consider a function $\psi \in \Omega^0(M)$ such that $\psi|_{J(B_1\cap M)} = 1$ and $\psi|_{J(B_2\cap M)} = 0$. We also let $\varphi \in \Omega^0_{\mathrm{tc}}(M)$ be a smooth function such that: (a) for all $x \in M$, $\varphi(x)$ depends only on $t(x)$; (b) $\chi := \varphi\psi \in \Omega^0_{\mathrm{tc}}(M)$ is such that $\mathrm{t}\chi = \chi|_{\partial M} = 0$; (c) there exists an interval $I \subset \mathbb{R}$ such that $\varphi|_I = 1$.

In other words φ plays the rôle of a switching cut-off function so that the observable χ is supported for times where $J(B_2 \cap M) \cap \partial M = \emptyset$.

It then follows that $\mathrm{d}\chi \in \Omega^1_{\mathrm{c}}(M) \subseteq \Omega^1_{\mathrm{sc}}(M)$, however, there is no function $\zeta \in \Omega^k_{\mathrm{c}}(M)$ such that $\mathrm{d}\zeta = \mathrm{d}\chi$. Indeed, let consider the curve $\gamma_t \subseteq M$ parametrized by $(t, x, 0, \dots) \in M$ where $t \in \mathbb{R}$ is such that $\varphi(t) = 1$ while $x \in (x(p), +\infty)$ – $x(p)$ denotes the x-coordinate on p. Integration along γ_t leads to

$$\int_{\gamma_t} \iota^*_{\gamma_t}\mathrm{d}\chi = -1\,, \qquad \int_{\gamma_t} \iota^*_{\gamma_t}\mathrm{d}\zeta = 0\,.$$

3.4 Functorial property of $\mathrm{Sol}_{\mathrm{g}\sharp}$

A An explicit computation

Lemma 35: *Let $\mathrm{t}_\Sigma\colon \Omega^k(M) \to \Omega^k(\Sigma)$, $\mathrm{n}_\Sigma\colon \Omega^k(\Sigma) \to \Omega^{k-1}(\Sigma)$ be the tangential and normal maps on Σ, where $M = \mathbb{R} \times \Sigma$. Moreover, let $\mathrm{t}_{\partial\Sigma}\colon \Omega^k(\Sigma) \to \Omega^k(\partial\Sigma)$, $\mathrm{n}_{\partial\Sigma}\colon \Omega^k(\Sigma) \to \Omega^{k-1}(\partial\Sigma)$ be the tangential* *and normal maps on $\partial\Sigma$. If $\sharp \in \{\mathrm{D},\mathrm{N}\}$ then for all $\omega \in \Omega^k(M)$ it holds*

$$\omega \in \Omega^k_\sharp(M) \Longleftrightarrow \mathrm{t}_\Sigma\omega, \mathrm{n}_\Sigma\omega \in \Omega^k_\sharp(\Sigma)\,. \tag{52}$$

Proof. The equivalence (52) is shown for Neumann boundary condition. The proof for Dirichlet boundary conditions follows by duality – *cf.* (9). Let $\mathrm{t}_{\partial M}\colon \Omega^k(M) \to \Omega^k(\partial M)$, $\mathrm{n}_{\partial M}\colon \Omega^k(M) \to \Omega^{k-1}(\partial M)$ denote the tangential and normal maps on $\partial M = \mathbb{R}\times\partial\Sigma$. Locally we have $\Sigma \simeq \partial\Sigma \times \{x \geq 0\}$, therefore we can decompose $\omega \in \Omega^k(M)$ as

$$\begin{aligned}
\omega &= \omega_\Sigma + \omega_t \wedge \mathrm{d}t = \omega_{\Sigma,\partial\Sigma} + \omega_{\Sigma,x}\wedge \mathrm{d}x + \omega_{t,\partial\Sigma}\wedge \mathrm{d}t + \omega_{t,x}\wedge \mathrm{d}x \wedge \mathrm{d}t\,,\\
\omega &= \omega_{\partial M} + \omega_x \wedge \mathrm{d}x\,,
\end{aligned}$$

where $\mathrm{t}_{\partial M}\omega := \omega_{\partial M}|_{\partial M}$, $\mathrm{n}_{\partial M}\omega := \omega_x|_{\partial M}$ and similarly $\mathrm{t}_\Sigma\omega := \omega_\Sigma|_\Sigma$ and $\mathrm{n}_\Sigma\omega := \omega_t|_\Sigma$. With this notation it follows that

$$\mathrm{t}_{\partial M}\omega = \mathrm{t}_{\partial\Sigma}\mathrm{t}_\Sigma\omega + \mathrm{t}_{\partial\Sigma}\mathrm{n}_\Sigma\omega\wedge \mathrm{d}t\,,\qquad \mathrm{n}_{\partial M}\omega = \mathrm{n}_{\partial\Sigma}\mathrm{t}_\Sigma\omega + \mathrm{n}_{\partial\Sigma}\mathrm{n}_\Sigma\omega\wedge \mathrm{d}t\,.$$

Therefore we find $\mathrm{n}_{\partial M}\omega = 0$ if and only if $\mathrm{n}_{\partial\Sigma}\mathrm{n}_\Sigma\omega = 0$ and $\mathrm{n}_{\partial\Sigma}\mathrm{t}_\Sigma\omega = 0$. A similar computation leads to

$$\begin{aligned}
\mathrm{n}_{\partial M}\mathrm{d}\omega &= (-1)^{k-1}\partial_x\omega_{\Sigma,\partial\Sigma} + \mathrm{d}_{\partial\Sigma}\omega_{\Sigma,x} + (-1)^k\partial_t\omega_{\Sigma,x}\wedge \mathrm{d}t + (-1)^k\partial_x\omega_{t,\partial\Sigma}\wedge \mathrm{d}t + \mathrm{d}_{\partial\Sigma}\omega_{t,x}\\
&= (-1)^{k-1}\partial_x\omega_{\Sigma,\partial\Sigma} + (-1)^k\partial_x\omega_{t,\partial\Sigma}\wedge \mathrm{d}t\,,
\end{aligned}$$

where in the second equality we used the equivalence for $\mathrm{n}_{\partial M}\omega = 0$. It follows that $\mathrm{n}_{\partial M}\mathrm{d}\omega = 0$ if and only if $\partial_x\omega_{\Sigma,\partial\Sigma} = 0$ and $\partial_x\omega_{t,\partial\Sigma} = 0$. When $\mathrm{n}_{\partial\Sigma}\mathrm{t}_\Sigma\omega = 0$ and $\mathrm{n}_{\partial\Sigma}\mathrm{n}_\Sigma\omega = 0$ the latter conditions are equivalent to $\mathrm{n}_{\partial\Sigma}\mathrm{d}_\Sigma\mathrm{n}_\Sigma\omega = 0$ and $\mathrm{n}_{\partial\Sigma}\mathrm{d}_\Sigma\mathrm{t}_\Sigma\omega = 0$. $\square$

B Relative de Rham cohomology

In this appendix we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non empty boundary. A reader interested in more details can refer to [BT82, Sch95].

For the purpose of this section M refers to a smooth, oriented manifold of dimension $\dim M = d$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : M \to \partial M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N}\cup\{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work only with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_c(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{Ker(d_k)}{Im(d_{k-1})},$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (3) and (4b) entail that we can define the $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_t(M)$, $k \in \mathbb{N}\cup\{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M;\delta)$ which in degree $k \in \mathbb{N}\cup\{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M;\delta) \doteq \frac{Ker(\delta_k)}{Im(\delta_{k+1})}.$$

Equations (3) and (4b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M;\delta)$, the subcomplex of $\Omega^\bullet(M;\delta)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$. The associated homology groups will be denoted as $H_{k,n}(M;\delta)$, $k \in \mathbb{N}\cup\{0\}$. Observe that, in view of its definition, the Hodge operator induces an isomorphism $H^k(M) \simeq H_{d-k}(M;\delta)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [*\alpha] \in H_{d-k}(M;\delta)$. Similarly, on account of Equation (4b), it holds $H^k_t(M) \simeq H_{d-k,n}(M;\delta)$.

As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT82]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k \in \mathbb{N}\cup\{0\}$ corresponds to

$$\Omega^k(M,\partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M;\partial M) \to \Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta) \in \Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta) = (\mathrm{d}\omega, \iota^*_{\partial M}\omega - \mathrm{d}_\partial\theta)\,. \tag{53}$$

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M;\partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M;\partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will be always clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT82, Prop. 6.49]:

Proposition 36: *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M;\partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\iota_{\partial M,*}} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M;\partial M) \to \ldots, \tag{54}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and $\iota_{\partial M,*}$ indicate the natural counterpart of the maps π_1, π_2 and $\iota_{\partial M}$ at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition 37: *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between $H^k_t(M)$ and $H^k(M,\partial M)$ for all $k \in \mathbb{N}\cup\{0\}$.*

Proof. Consider $\omega \in \Omega^k_t(M) \cap \ker(d)$ and let $(\omega, 0) \in \Omega^k(M;\partial M)$, $k \in \mathbb{N}\cup\{0\}$. Equation (53) entails

$$\underline{\mathrm{d}}_k(\omega,0) = (\mathrm{d}\omega, \iota^*_{\partial M}\omega) = (\mathrm{d}\omega, \mathrm{t}\omega) = (0,0)\,,$$

where we used (2a) in the second equality. At the same time, if $\omega = \mathrm{d}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $\underline{\mathrm{d}}_{k-1}(\beta,0) = (\mathrm{d}\beta,0)$. Hence the embedding $\omega \mapsto (\omega,0)$ identifies an injective map $\rho : H^k_t(M) \to H^k(M;\partial M)$ such that $\rho([\omega]) \doteq [(\omega,0)]$.

To conclude, we need to prove that ρ is surjective. Let thus $[(\omega',\theta)] \in H^k(M;\partial M)$. It holds that $d\omega' = 0$ and $\iota^*_{\partial M}\omega' - d_\partial\theta = t(\omega') - d_\partial\theta = 0$. Recalling that $t : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective for all values of $k \in \mathbb{N}\cup\{0\}$, there must exist $\eta \in \Omega^{k-1}(M)$ such that $t(\eta) = \theta$. Let $\omega \doteq \omega' - d\eta$. On account of (4b) $\omega \in \Omega^k_t(M) \cap \ker(d)$ and $(\omega,0)$ is a representative if $[(\omega',\theta)]$ which entails the conclusion sought. ☐

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in-hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau80]:

Proposition 38: *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N} \cup \{0\}$*

$$H^k(M; \partial M) \simeq H_c^{n-k}(M; \partial M)^*,$$

where $n = \dim M$ and where on the right hand side we consider the dual of the $(n-k)$-th cohomology group built out compactly supported forms.

Acknowledgements

The work of C. D. was supported by the University of Pavia, while that of N. D. was supported in part by a research fellowship of the University of Trento. We are grateful to Marco Benini, Sonia Mazzucchi, Valter Moretti, Ana Alonso Rodrigez and Alberto Valli for the useful discussions. This work is based partly on the MSc thesis of R. L. .

References

[AFS18] Aké L., Flores J.L. , Sanchez M., *Structure of globally hyperbolic spacetimes with timelike boundary*, arXiv:1808.04412 [gr-qc].

[Bär15] Bär C., *Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes*, Commun. Math. Phys. (2015) 333: 1585.

[BL12] Behrndt, J., Langer, M., Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples. In: de Snoo, H.S.V. (ed.) Operator Methods for Boundary Value Problems. London Mathematical Society, Lecture Notes, p. 298. Cambridge University Press, Cambridge (2012)

[Ben16] Benini M., *Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies*, J. Math. Phys. **57** (2016) no.5, 053502

[BT82] R. Bott, L. W. Tu, *Differential forms in algebraic topology*, Springer, New York, 1982.

[BFV03] Brunetti, R., Fredenhagen, K., Verch, R., *The Generally Covariant Locality Principle ? A New Paradigm for Local Quantum Field Theory*, Commun. Math. Phys. (2003).

[DDF19] Dappiaggi, C., Drago, N., Ferreira, H. *"Fundamental solutions for the wave operator on static Lorentzian manifolds with timelike boundary"*, to appear in Lett. Math. Phys. (2019), arXiv:1804.03434 [math-ph].

[DFJ18] C. Dappiaggi, H. R. C. Ferreira and B. A. Juárez-Aubry, *"Mode solutions for a Klein-Gordon field in anti?de Sitter spacetime with dynamical boundary conditions of Wentzell type,"* Phys. Rev. D **97** (2018) no.8, 085022 [arXiv:1802.00283 [hep-th]].

[HS13] Hack T.P., Schenkel A., *Linear bosonic and fermionic quantum gauge theories on curved spacetimes*, Gen Relativ Gravit (2013) 45: 877.

[Lee00] J.M. Lee *Introduction to Smooth Manifolds*, 2nd ed. (2013) Springer, 706p.

[Mau80] C. R. F. Maunder, *Algebraic Topology*, (1980) Cambridge University Press, 375p.

[Pfe09] M. J. Pfenning, *"Quantization of the Maxwell field in curved spacetimes of arbitrary dimension,"* Class. Quant. Grav. **26** (2009) 135017 [arXiv:0902.4887 [math-ph]].

[Sch95] G. Schwarz, *Hodge Decomposition - A Method for Solving Boundary Value Problems*, (1995) Springer, 154p.

[Za15] J. Zahn, *"Generalized Wentzell boundary conditions and quantum field theory,"* Annales Henri Poincare **19** (2018) no.1, 163 [arXiv:1512.05512 [math-ph]].