UNIVERSITÀ DEGLI STUDI DI PAVIA

DIPARTIMENTO DI FISICA

Corso di Laurea Magistrale in Scienze Fisiche

On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

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“*The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen.* ”

Paul A.M. Dirac

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Chapter 1

Geometric preliminaries

In this chapter, we begin by recalling the basic definitions in order to fix the geometric setting in which we will work.
Globally hyperbolic spacetimes (M, g) are used in context of geometric analysis and mathematical relativity because in them there exists a smooth and spacelike Cauchy hypersurface Σ and that ensures the well-posedness of the Cauchy problem. Moreover, as shown by Bernal and Sánchez [BS05, Th. 1.1], in such spacetimes there exists a splitting for the full spacetime M as an orthogonal product $\mathbb{R} \times \Sigma$. These results corroborate the idea that in globally hyperbolic spacetimes one can preserve the notion of a global passing of time. In a globally hyperbolic spacetime, the entire future and past history of the universe can be predicted from conditions imposed at a fixed instant represented by the hypersurface Σ.

1.1 Globally Hyperbolic spacetimes with timelike boundary

The main goal of this section is to analyse the main properties of globally hyperbolic spacetimes and to generalise them to a natural class of spacetimes where boundary values problems can be formulated. This class is that of globally hyperbolic spacetimes with timelike boundary.While, in the case of $\partial M = \emptyset$ global hyperbolicity is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18].

Manifolds with boundary. From now on M will denote a smooth manifold with boundary with dimension $m > 1$. M is then locally diffeomorphic to open subsets of the closed half space of $\mathbb{R}^n$. We will assume that the boundary ∂M is smooth and, for simplicity, connected. A point $p \in M$ such that there exists an open neighbourhood U containing p, diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$. In addition we endow M with a smooth Lorentzian metric g of signature $(-,+,...,+)$ so that $\iota^* g$ identifies a Lorentzian metric on ∂M and we require (M,g) to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M,g) has a *timelike boundary*.

Definition 1:

- *A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.*
- *A spacetime with timelike boundary is* causal *if it possesses no closed, causal curve,*
- *A causal spacetime with timelike boundary M such that for all $p,q \in M$ $J_+(p) \cap J_-(q)$ is compact is called* ***globally hyperbolic****.*

These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 2: *Let (M,g) be a spacetime of dimension m. Then*

1. *(M,g) is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve,*
2. *if (M,g) is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the line-element*
$$ds^2 = -\beta d\tau^2 + h_\tau, \tag{1.1}$$
where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1], whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0,\infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a Cauchy surface for (M,g).

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M,g), we work directly with (1.1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M. In addition we shall say that (M,g) is *static* if it possesses a timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to ∂M is tangent to the boundary, *i.e.* $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where ν is the unit vector, normal to the boundary at p. With reference to (1.1) this translates simply into the request that both β and h_τ are independent from τ.

Example 3: *We first consider some examples of globally hyperbolic spacetimes without boundary ($\partial M = \emptyset$).*

[1] Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

- *The Minkowski spacetime $M = (\mathbb{R}^m, \eta)$ is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-1}$, endowed with the time-independent Euclidean metric.*

- *Let Σ be a Riemannian manifold with time independent metric h and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$, called* ***cosmological spacetime****, is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold, see [BGP15, Lem A.5.14]. This applies in particular if (Σ, h) is compact.*

- *The interior and exterior* ***Schwarzschild spacetimes****, that represent non-rotating black holes of mass $m > 0$ are globally hyperbolic. Denoting S^2 the 2-dimensional sphere embedded in $\mathbb{R}^3$, we set*

$$M_{ext} := \mathbb{R} \times (2m, +\infty) \times S^2,$$

$$M_{int} := \mathbb{R} \times (0, 2m) \times S^2.$$

The metric is given by

$$g = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2},$$

where $f(r) = 1 - \frac{2m}{r}$, while $g_{S^2} = r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2$ is the polar coordinates metric on the sphere. For the exterior Schwarzschild spacetime we have $M_{ext} = \mathbb{R} \times \Sigma$ with $\Sigma = (2m, +\infty) \times S^2$, $\beta = f$ and $h = \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2}$.

Example 4: *Now we consider some examples of globally hyperbolic spacetimes in which the boundary is not empty.*

- *The Half Minkowski spacetime $M = (\mathbb{R}^{m-1} \times [0, +\infty), \eta)$ is globally hyperbolic. Every spacelike half-hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-2} \times [0, +\infty)$, endowed with the time-independent Euclidean metric.*

On top of a Lorentzian spacetime (M, g) with timelike boundary we consider $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, the space of real valued smooth k-forms endowed with the standard, metric induced, pairing $(,) : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}$. A particular role will be played by the support of the forms that we consider. In the following definition we introduce the different possibilities that we will consider, which are a generalization of the counterpart used for scalar fields which correspond in our scenario to $k = 0$, *cf.* [Bär15].

Definition 5: *Let (M, g) be a Lorentzian spacetime with timelike boundary. We denote with*

1. $\Omega^k_{\mathrm{c}}(M)$ *the space of smooth* k*-forms with compact support in* M *while with* $\Omega^k_{\mathrm{cc}}(M) \subset \Omega^k_{\mathrm{c}}(M)$ *the collection of smooth and compactly supported* k*-forms* ω *such that supp*$(\omega) \cap \partial M = \emptyset$*.*

2. $\Omega^k_{\mathrm{spc}}(M)$ *(resp.* $\Omega^k_{\mathrm{sfc}}(M)$*) the space of strictly past compact (resp. strictly future compact)* k*-forms, that is the collection of* $\omega \in \Omega^k(M)$ *such that there exists a compact set* $K \subseteq M$ *for which* $J^+(\mathrm{supp}(\omega)) \subseteq J^+(K)$ *(resp.* $J^-(\mathrm{supp}(\omega)) \subseteq J^-(K)$*), where* $J^\pm$ *denotes the causal future and the causal past in* M*. Notice that* $\Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M) = \Omega^k_{\mathrm{c}}(M)$*.*

3. $\Omega^k_{\mathrm{pc}}(M)$ *(resp.* $\Omega^k_{\mathrm{fc}}(M)$*) denotes the space of future compact (resp. past compact)* k*-forms, that is,* $\omega \in \Omega^k(M)$ *for which* $\mathrm{supp}(\omega) \cap J^-(K)$ *(resp.* $\mathrm{supp}(\omega) \cap J^+(K)$*) is compact for all compact* $K \subset M$*.*

4. $\Omega^k_{\mathrm{tc}}(M) := \Omega^k_{\mathrm{fc}}(M) \cap \Omega^k_{\mathrm{pc}}(M)$*, the space of timelike compact* k*-forms.*

5. $\Omega^k_{\mathrm{sc}}(M) := \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M)$*, the space of spacelike compact* k*-forms.*

We indicate with $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative and, being (M, g) oriented, we can identify a unique, metric-induced, Hodge operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge *\beta = (\alpha, \beta)\mu_g$, where $\wedge$ is the exterior product of forms and μ_g the metric induced volume form. Since M is endowed with a Riemannian metric it holds that, when acting on smooth k-forms, $*^{-1} = (-1)^{k(m-k)}*$. Combining these data first we define the *codifferential* operator $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta \doteq *^{-1} \circ \mathrm{d} \circ *$. Secondly we introduce the *D'Alembert-de Rham* wave operator $\Box_k : \Omega^k(M) \to \Omega^k(M)$ such that $\Box_k \doteq \mathrm{d}\delta + \delta\mathrm{d}$, as well as the *Maxwell* operator $\mathcal{M}_k : \Omega^k(M) \to \Omega^k(M)$ such that $\mathcal{M}_k \doteq \delta\mathrm{d}$. The subscript k is here introduced to make explicit on which space of k-forms the operator is acting. Observe, furthermore, that $\Box_k$ differs by the more commonly used D'Alembert wave operator acting on k-forms by 0-order term built out of the metric and whose explicit form depends on the value of k, see for example [Pfe09, Sec. II].

To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its derivative along the direction normal to ∂M.

Remark 6: *Since we will be considering not only form lying in* $\Omega^k(M)$*,* $k \in \mathbb{N} \cup \{0\}$*, but also those in* $\Omega^k(\partial M)$*, we shall distinguish the operators acting on this space with a subscript* ∂*, e.g.* d_∂*,* $*_\partial$*,* δ_∂ *or* $(,)_\partial$*.*

Definition 7: *Let* (M, g) *be a Lorentzian spacetime with timelike boundary together with the embedding map* $\iota_{\partial M} : \partial M \hookrightarrow M$*. We call* tangential *and* normal *maps*

$$\mathrm{t} \colon \Omega^k(M) \to \Omega^k(\partial M) \qquad \omega \mapsto \mathrm{t}\omega \doteq \iota^*_{\partial M}\omega \tag{1.2a}$$

$$\mathrm{n}\colon \Omega^k(M) \to \Omega^{k-1}(\partial M) \qquad \omega \mapsto \mathrm{n}\omega \doteq *_\partial^{-1} \circ \mathrm{t} \circ *_M \,, \tag{1.2b}$$

In particular, for all $k \in \mathbb{N} \cup \{0\}$ we define

$$\Omega^k_{\mathrm{t}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\}\,, \qquad \Omega^k_{\mathrm{n}}(M) := \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\}\,. \tag{1.3}$$

Remark 8: *The normal map $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ can be equivalently read as the restriction to ∂M of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(M)$ and the vector field $\nu \in \Gamma(TM)|_{\partial M}$ which corresponds pointwisely to the unit vector, normal to ∂M.*

As last step, we observe that (1.2) together with (1.3) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$* \, \delta = (-1)^k \mathrm{d} * \,, \quad \delta * = (-1)^{k+1} * \mathrm{d}\,, \tag{1.4a}$$

$$*_\partial \, \mathrm{n} = \mathrm{t} * \,, \quad *_\partial \mathrm{t} = (-1)^k \mathrm{n} * \,, \quad \mathrm{d}_\partial \mathrm{t} = \mathrm{t}\mathrm{d}\,, \quad \delta_\partial \mathrm{n} = \mathrm{n}\delta\,. \tag{1.4b}$$

A notable consequence of (1.4b) is that, while on globally hyperbolic spacetimes with empty boundary, the operators d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. A direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial \qquad \forall \alpha \in \Omega^k_{\mathrm{c}}(M)\,,\ \forall \beta \in \Omega^{k+1}_{\mathrm{c}}(M)\,, \tag{1.5}$$

where the pairing in the right-hand side is the one associated to forms living on ∂M.

1.2 Poincaré-Lefschetz duality for manifold with boundary

In this section we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non empty boundary. A reader interested in more details can refer to [BT82, Sch95].

For the purpose of this section M refers to a smooth, oriented manifold of dimension $\dim M = d$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : M \to \partial M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work only with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_c(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\mathrm{Ker}(d_k)}{\mathrm{Im}(d_{k-1})},$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (1.3) and (1.4b) entail that we can define the $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_t(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M;\delta)$ which in degree $k \in \mathbb{N}\cup\{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M;\delta) \doteq \frac{\mathrm{Ker}(\delta_k)}{\mathrm{Im}(\delta_{k+1})}.$$

Equations (1.3) and (1.4b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M;\delta)$, the subcomplex of $\Omega^\bullet(M;\delta)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$. The associated homology groups will be denoted as $H_{k,n}(M;\delta)$, $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition, the Hodge operator induces an isomorphism $H^k(M) \simeq H_{d-k}(M;\delta)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [*\alpha] \in H_{d-k}(M;\delta)$. Similarly, on account of Equation (1.4b), it holds $H^k_t(M) \simeq H_{d-k,n}(M;\delta)$.

As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT82]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to

$$\Omega^k(M, \partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M;\partial M) \to \Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta) \in \Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta) = (\mathrm{d}\omega, \iota^*_{\partial M}\omega - \mathrm{d}_\partial\theta)\,. \tag{1.6}$$

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M;\partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M;\partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will be always clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT82, Prop. 6.49]:

Proposition 9: *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M;\partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\iota_{\partial M,*}} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M;\partial M) \to \ldots, \tag{1.7}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and $\iota_{\partial M,*}$ indicate the natural counterpart of the maps π_1, π_2 and $\iota_{\partial M}$ at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition 10: *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between $H^k_t(M)$ and $H^k(M, \partial M)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Consider $\omega \in \Omega^k_t(M) \cap \ker(d)$ and let $(\omega, 0) \in \Omega^k(M; \partial M)$, $k \in \mathbb{N} \cup \{0\}$. Equation (1.6) entails

$$\underline{\mathrm{d}}_k(\omega, 0) = (\mathrm{d}\omega, \iota^*_{\partial M}\omega) = (\mathrm{d}\omega, \mathrm{t}\omega) = (0, 0)\,,$$

where we used (1.2a) in the second equality. At the same time, if $\omega = \mathrm{d}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $\underline{\mathrm{d}}_{k-1}(\beta, 0) = (\mathrm{d}\beta, 0)$. Hence the embedding $\omega \mapsto (\omega, 0)$ identifies an injective map $\rho : H^k_t(M) \to H^k(M; \partial M)$ such that $\rho([\omega]) \doteq [(\omega, 0)]$.

To conclude, we need to prove that ρ is surjective. Let thus $[(\omega', \theta)] \in H^k(M; \partial M)$. It holds that $d\omega' = 0$ and $\iota^*_{\partial M}\omega' - d_\partial\theta = t(\omega') - d_\partial\theta = 0$. Recalling that $t : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective for all values of $k \in \mathbb{N} \cup \{0\}$, there must exist $\eta \in \Omega^{k-1}(M)$ such that $t(\eta) = \theta$. Let $\omega \doteq \omega' - d\eta$. On account of (1.4b) $\omega \in \Omega^k_t(M) \cap \ker(d)$ and $(\omega, 0)$ is a representative if $[(\omega', \theta)]$ which entails the conclusion sought. □

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in-hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau80]:

Theorem 11: *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N} \cup \{0\}$*

$$H^k(M; \partial M) \simeq H^{n-k}_c(M; \partial M)^*,$$

where $n = \dim M$ and where on the right hand side we consider the dual of the $(n-k)$-th cohomology group built out compactly supported forms.

The proof proceeds in some steps. Let $\iota : \partial M \to M$ be the immersion map. First of all we have to check that the spaces are finite-dimensional and that the pairing $\langle\,,\,\rangle : H^{n-k}(M) \otimes H^k_{\mathrm{c}}(M, \partial M)$ defined by

$$\langle \alpha, (\omega, \theta)\rangle := \int_M \alpha \wedge \omega + \int_{\partial M} \iota^*\alpha \wedge \theta \qquad \forall \alpha \in H^{n-k}(M) \text{ and } (\omega, \theta) \in H^k_{\mathrm{c}}(M, \partial M)\,, \tag{1.8}$$

is non-degenerate, equivalently the map $\alpha \to \langle \alpha, \cdot\rangle$ should be an isomorphism.

Since a manifold M with boundary is locally homeomorphic to $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ we need Poincaré lemmas for $\mathbb{R}^n_+$.

Lemma 12 (Poincaré lemmas for manifolds with boundary)**:** *Let* $\mathbb{R}^n_+ := \{(x_1,\dots,x_n)\,|\,x_1 \geq 0\}$ *and* $k \geq 0$*. Then*

$$H^k(\mathbb{R}^n_+) \simeq \begin{cases} \mathbb{R} & \textit{if } k = 0 \\ \{0\} & \textit{otherwise} \end{cases} \tag{1.9}$$

$$H^k_c(\mathbb{R}^n_+, \partial\mathbb{R}^n_+) \simeq \begin{cases} \mathbb{R} & \textit{if } k = n \\ \{0\} & \textit{otherwise} \end{cases} \tag{1.10}$$

Proof. The proof for the case $n = 1$, i.e. $\mathbb{R}_+ = [0, +\infty)$ is straightforward and the n-dimensional generalisation is obtained as in ([BT82, Sec. 4]). $\square$

Lemma 13 (Mayer-Vietoris sequences)**:** *Let* M *be an orientable manifold with boundary* ∂M*, suppose* $M = U \cup V$ *with* U, V *open and denote* $\partial M_A := \partial M \cap A$*. Then the following are exact sequences:*

$$\cdots \to H^k(M,\partial M) \to H^k(U,\partial M_U) \oplus H^k(V,\partial M_V) \to H^k(U\cap V,\partial M_{U\cap V}) \to H^{k+1}(M,\partial M) \to \cdots \tag{1.11}$$

$$\cdots \leftarrow H^k_c(M,\partial M) \leftarrow H^k_c(U,\partial M_U) \oplus H^k(V,\partial M_V) \leftarrow H^k_c(U\cap V,\partial M_{U\cap V}) \leftarrow H^{k+1}_c(M,\partial M) \leftarrow \cdots \tag{1.12}$$

Proof. We will only prove the non-compact cohomology line.
We have the following Mayer-Vietoris short exact sequences for M and ∂M:

$$\begin{gathered} 0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U\cap V) \longrightarrow 0 \\ 0 \to \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M_U) \oplus \Omega^{k-1}(\partial M_V) \to \Omega^{k-1}(\partial M_{U\cap V}) \to 0. \end{gathered}$$

Hence applying the direct sum between the two sequences we obtain

$$0 \longrightarrow \Omega^k(M,\partial M) \longrightarrow \Omega^k(U,\partial M_U) \oplus \Omega^k(V,\partial M_V) \longrightarrow \Omega^k(U\cap V,\partial M_{U\cap V}) \longrightarrow 0.$$

The last row induces the desired long sequence because of the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^k(M,\partial M) \arrow[r] \arrow[d, "d"] & \Omega^k(U,\partial M_U) \oplus \Omega^k(V,\partial M_V) \arrow[r] \arrow[d, "\mathrm{d}:=d\oplus d"] & \Omega^k(U\cap V,\partial M_{U\cap V}) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & \Omega^{k+1}(M,\partial M) \arrow[r] & \Omega^{k+1}(U,\partial M_U) \oplus \Omega^{k+1}(V,\partial M_V) \arrow[r] & \Omega^{k+1}(U\cap V,\partial M_{U\cap V}) \arrow[r] & 0 \end{tikzcd} \tag{1.13}$$

following the arguments in [BT82], section 2. Fix a closed form $\omega \in \Omega^k(U\cap V,\partial M_{U\cap V})$, since the first row is exact there exists a unique $\xi \in \Omega^{k+1}(M,\partial M)$ which is mapped to ω. Now, since $\mathrm{d}\omega = 0$ and the diagram is commutative $\mathrm{d}\xi$ is mapped to 0. Hence from the exactness of the second row there exists χ which is mapped to $\mathrm{d}\xi$ and it easy to see χ is closed. $\square$

Lemma 14: *If the manifold with boundary M has a* finite good cover *(see [BT82, Sec. 5]) then its (relative) cohomology and (relative) compact cohomology is finite dimensional.*

Proof. The proof is based on the existence of a Mayer-Vietoris sequence in any of the desired cases (proved in the previous proposition) and follows the outline of [BT82, Prop. 5.3.1]. ☐

Lemma 15 (Five lemma)**:** *Given the commutative diagram with exact rows*

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \\ \cdots & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & \cdots \end{array} \tag{1.14}$$

if f, g, h, s are isomorphism, then so is r.

Lemma 16: *Suppose $M = U \cup V$ with U, V open. The pairing* (1.8) *induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^{n-k}(M) & \longrightarrow & H^{n-k}(U) \oplus H^{n-k}(V) & \longrightarrow & H^{n-k+1}(M) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^k(M, \partial M)^* & \longrightarrow & H^k(U, \partial M_U)^* \oplus H^k(V, \partial M_V)^* & \longrightarrow & H^{k-1}(M)^* & \longrightarrow & \cdots \end{array} \tag{1.15}$$

Proof. The proof follows that of [BT82, Lem. 5.6]. ☐

Now we are ready to prove the main theorem of this section:

Proof of Poincaré-Lefschetz Duality. Follow the argument given in [BT82, Sec. 5]. By the Five lemma if Poincaré-Lefschetz duality holds for U, V and $U \cap V$, then it holds for $U \cup V$. Then it is sufficient to proceed by induction on the cardinality of a finite good cover. ☐

Appendix A

Frequently Asked Questions

A.1 How do I change the colors of links?

The color of links can be changed to your liking using:

`\hypersetup{urlcolor=red}`, or

`\hypersetup{citecolor=green}`, or

`\hypersetup{allcolor=blue}`.

If you want to completely hide the links, you can use:

`\hypersetup{allcolors=.}`, or even better:

`\hypersetup{hidelinks}`.

If you want to have obvious links in the PDF but not the printed text, use:

`\hypersetup{colorlinks=false}`.

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

ll

LAH **L**ist **A**bbreviations **H**ere
WSF **W**hat (it) **S**tands **F**or

lll

a distance
P power ()

ω angular frequency

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