Introduction to Quantum Backflow

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- In classical physics: $P(t)$ is always decreasing in time.
- In quantum physics: not necessarily.

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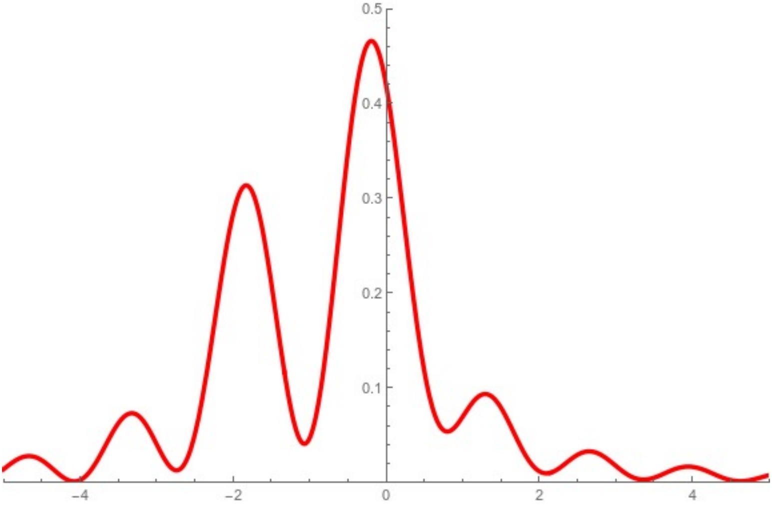

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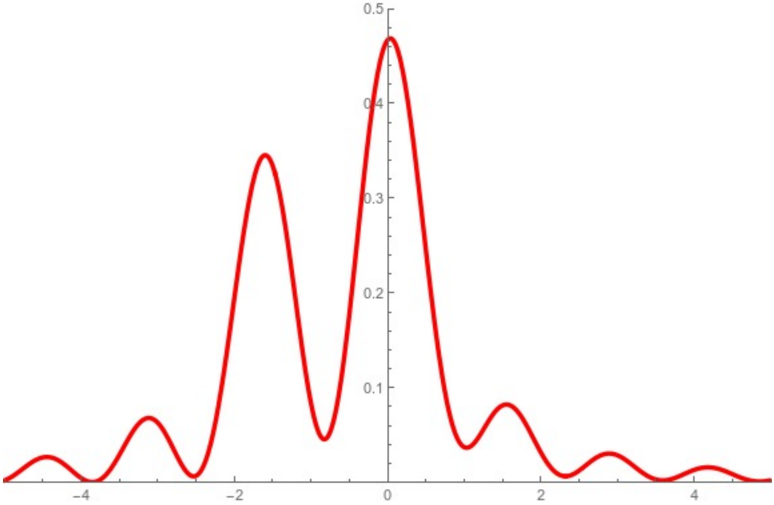

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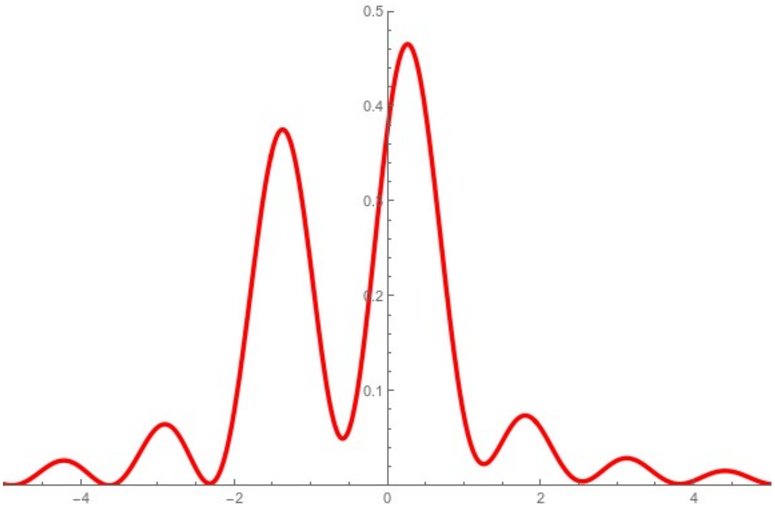

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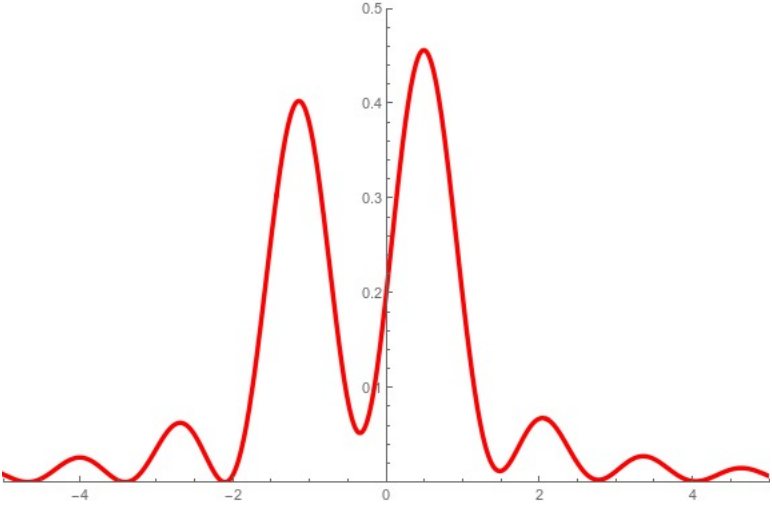

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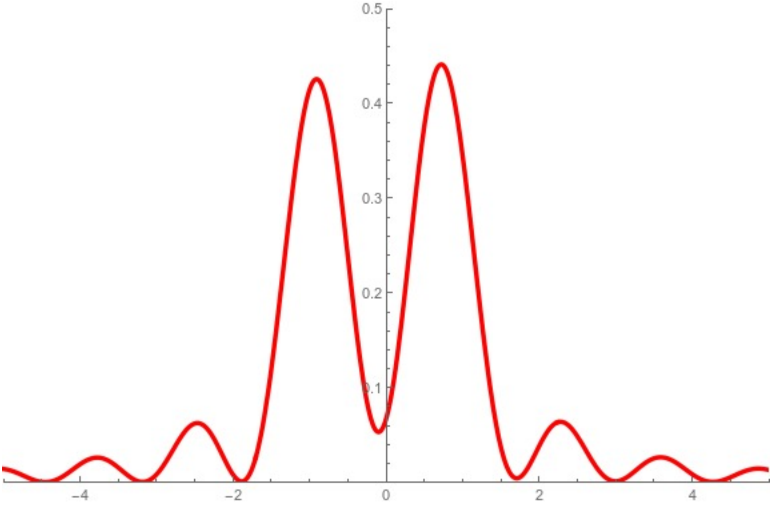

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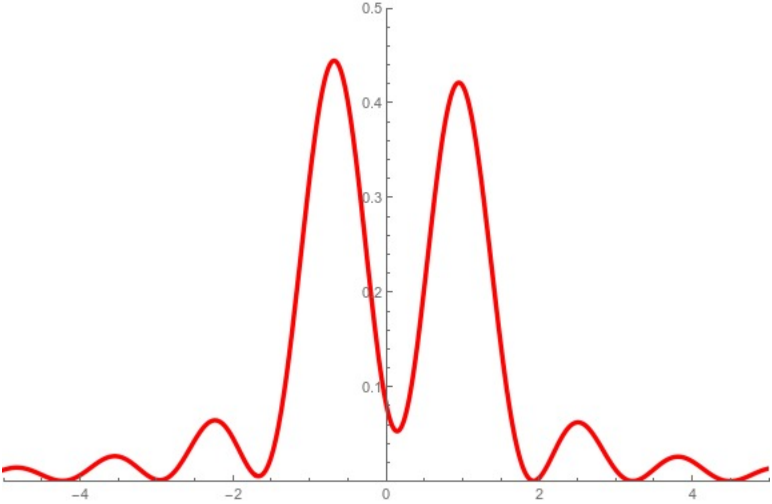

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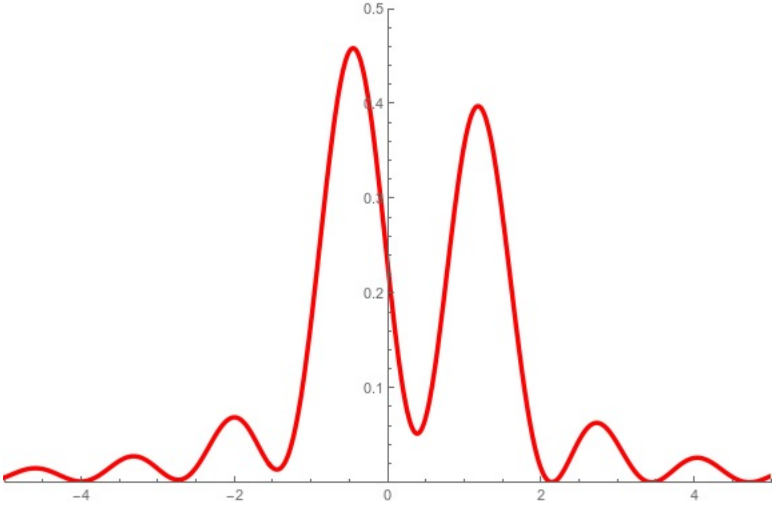

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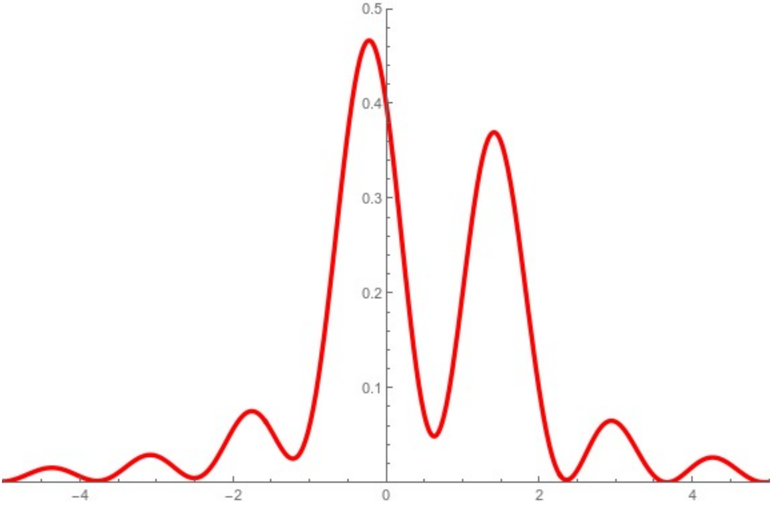

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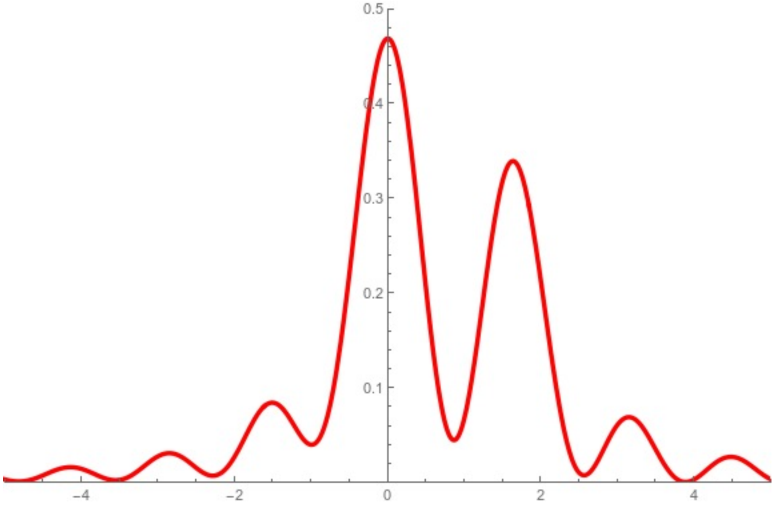

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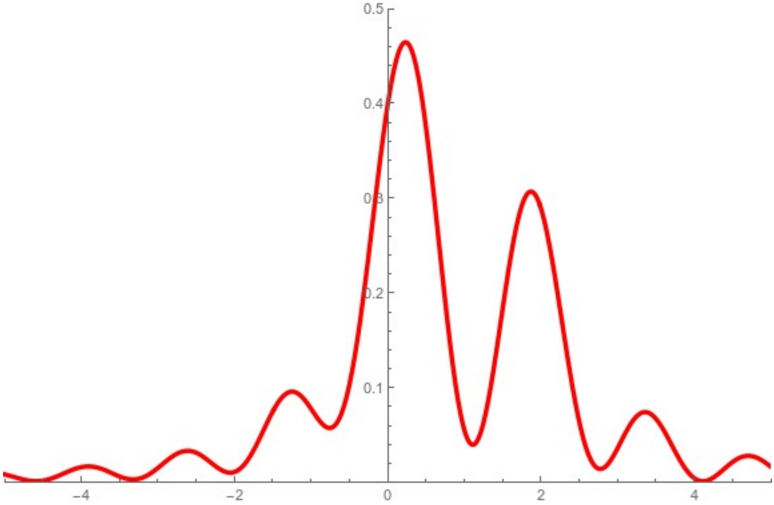

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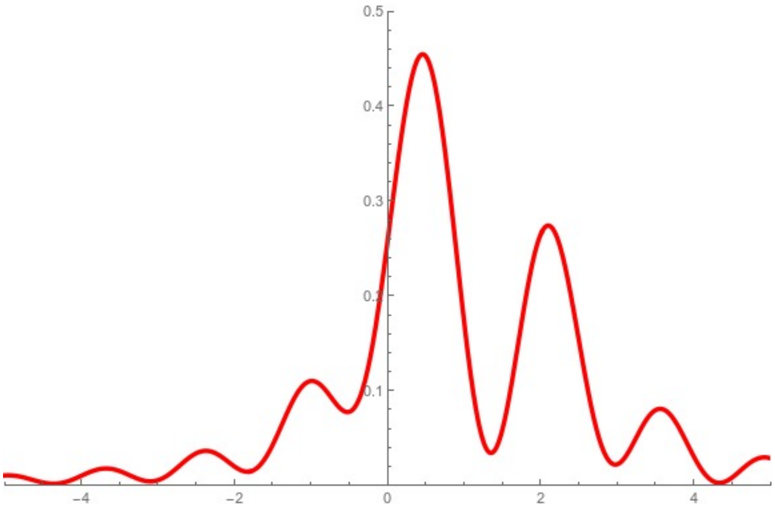

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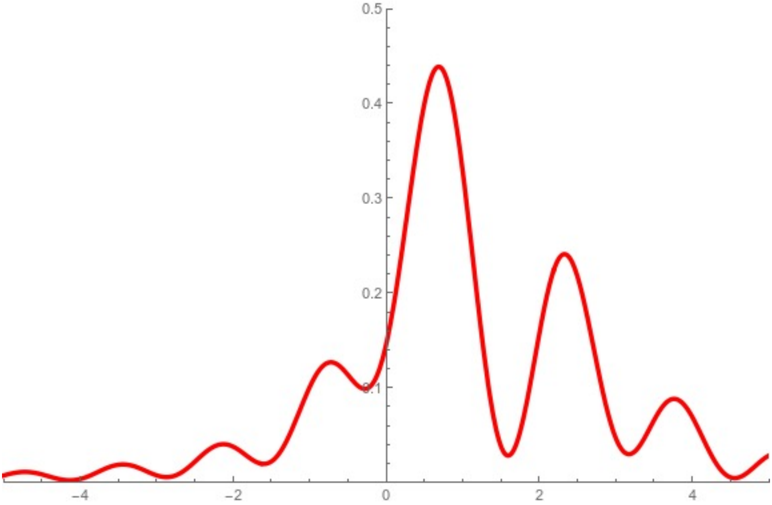

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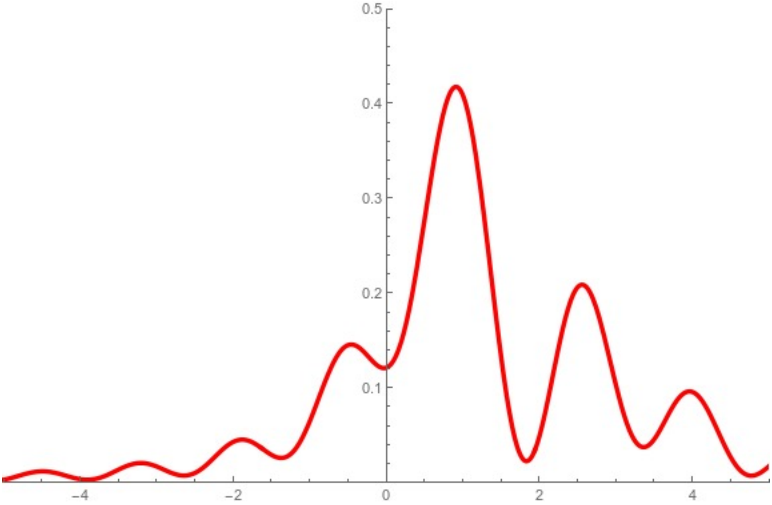

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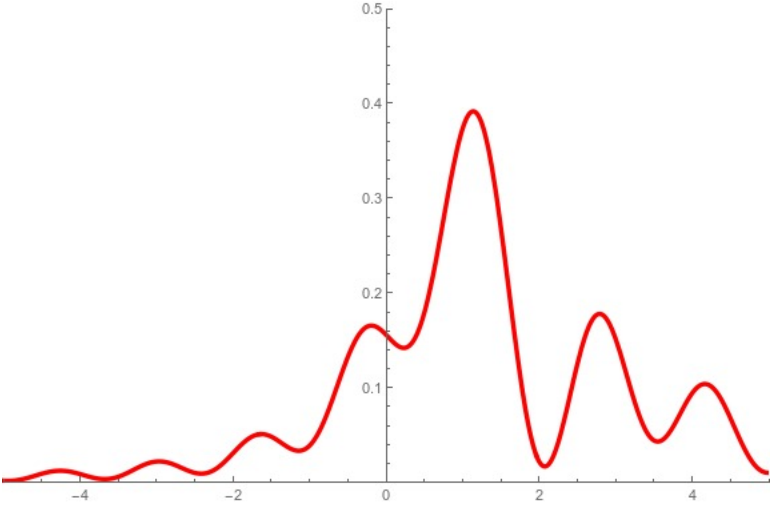

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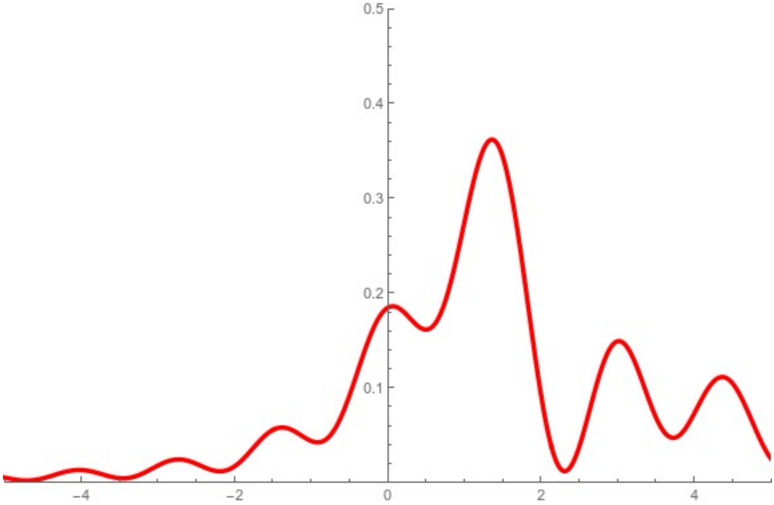

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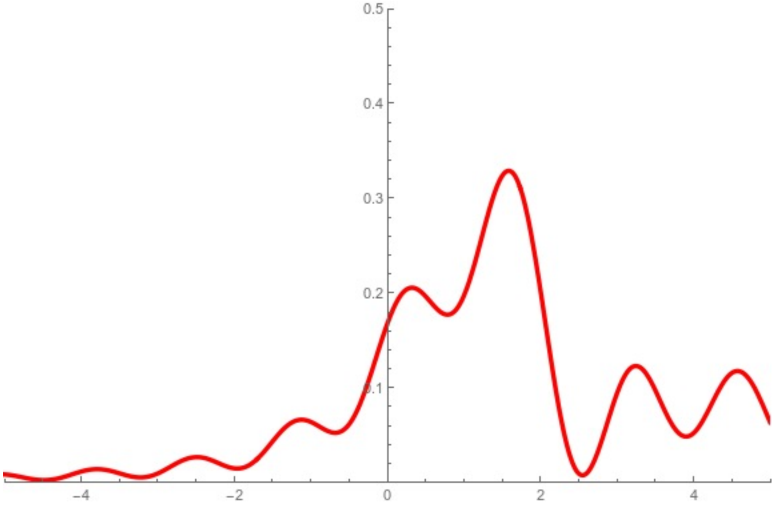

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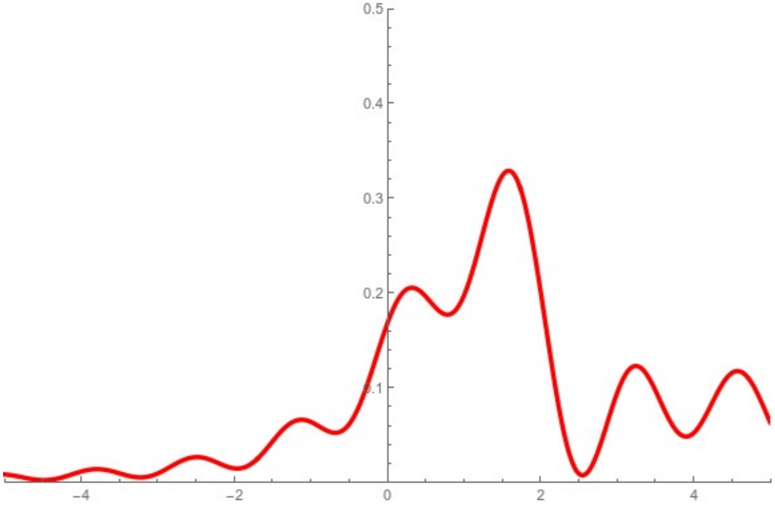

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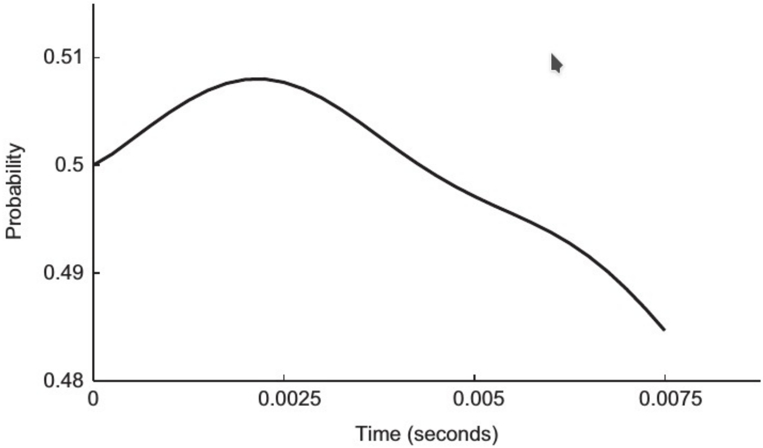

Right-movers

What does it means that a particle “travels to the right”?

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Let $\psi \in L^2(\mathbb{R})$ be a wave-function associated to a single particle. We call ψ a **right-mover** if $\mathrm{supp}\,\widehat{\psi} \in [0, +\infty)$.

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Definition

We call $E_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the operator such that:

$$\mathcal{F}[E_\pm \psi](p) = \vartheta(\pm p)\widehat{\psi}(p) \ \forall \psi \in L^2(\mathbb{R}),$$

where ϑ is the Heaviside function.

E_+ projects into the space of right-movers.

Temporal boundedness of backflow

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Proposition

Let $K : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be the integral operator:

$$(Kf)(u) = -\frac{1}{\pi} \int_0^\infty \frac{\sin(u^2 - v^2)}{u - v} f(v) \, \mathrm{d}v \quad \forall f \in L^2(\mathbb{R}_+).$$

Then K is bounded and self-adjoint, and $\lambda = \sup \sigma(K)$.

In fact, it holds $L(\psi_{-1}) - L(\psi_1) = (\psi | K\psi)$ with $\psi = E_+\psi$.

Temporal boundedness of backflow

Theorem (Temporal boundedness of backflow)

Let $\lambda = \sup \sigma(K)$. For any right-mover $\psi \in L^2(\mathbb{R})$ such that $\psi = E_+\psi$ and for any $T > 0$ it holds

$$\int_0^T j_\psi(0,t)\,\mathrm{d}t \geq -\lambda > -\infty.$$

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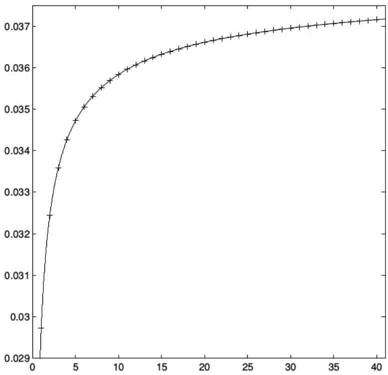
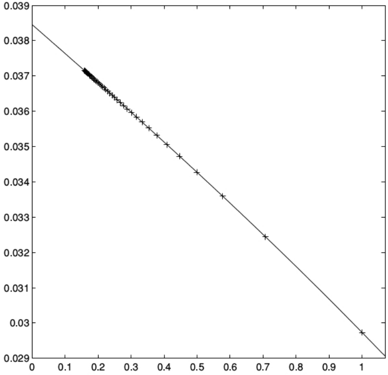
How to evaluate λ? Using numerical methods[1].

[1] M. Penz, G. Grübl, S. Kreidl and P. Wagner, *A new approach to quantum backflow*, J. Phys. A: Math. Gen. **39**, 2005.





$$\lambda \approx 0.0384517$$

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Proposition

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Definition

Let $V \in L^1(\mathbb{R})$ be a potential. We call V as a "short-range" potential (indicated $V \in L^{1+}(\mathbb{R})$) if it satisfies

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Theorem

Let $V \in L^{1+}(\mathbb{R})$. Then

(a) Ω_V *exists.*

(b) $[-\partial_x^2 + 2V(x) - k^2]\psi(x) = 0$ *has unique solutions*

$$\varphi_k(x) = \begin{cases} T_V(k)e^{ikx} + o(1) & \text{for } x \to +\infty \\ e^{ikx} + R_V(k)e^{-ikx} + o(1) & \text{for } x \to -\infty \end{cases}$$

(c) *For any* $\widehat{\psi} \in C_0^\infty(\mathbb{R})$, $(\Omega_V E_+ \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_k(x) \widehat{\psi}(k)\, \mathrm{d}k.$

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Expanding $E_+ \Omega_V^* J(f) \Omega_V E_+$ it holds[3]

$$(\psi | E_+ \Omega_V^* J(f) \Omega_V E_+ \psi) \geq \beta_0(f) - 2\|J(f)(i+P)^{-1}\| [2 + \|P(\Omega_V - T_V)E_+\|].$$

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It descends $\| J(f)(i+P)^{-1} \| \leq \| f \|_\infty + \frac{1}{2} \| f' \|_\infty$.

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Lemma

Let $V \in L^{1+}(\mathbb{R})$. Then, there exists $c_V \in \mathbb{R}$ such that

$$\| P(\Omega_V - T_V) E_+ \| \leq 2 c_V \| V \|_{1+}$$

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Theorem (Boundedness of Backflow in scattering scenarios)

For any potential $V \in L^{1+}(\mathbb{R})$ *and for any non-negative* $f \in \mathcal{S}(\mathbb{R})$, *there exists a constant* $\beta_V(f) \in (-\infty, 0)$ *such that*

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- Reflection does not destroy boundedness of backflow.
- Heuristic explanation: Backflow is a high momentum effect, but for but for high momentum, reflection processes are suppressed sufficiently well.
- What about experimental observations? (Bose-Einstein condensate, Bragg pulse, superposition of different momentum sates...)[4]

[4] M. Palmero, E. Torrontegui, J. G. Muga, and M. Modugno, Phys. Rev. A **87**, 2013.

Conclusions

- Backflow in Free Theory:
  - Independence from time
  - Existence of maximum backflow $\lambda \approx 0.0038$
  - Bound for spatially averaged backflow
- Backflow in Scattering Theory
  - Asymptotic right-movers and Møller operator
  - Existence of backflow in scattering scenarios
  - Boundedness of backflow in scattering scenarios

Appendix: Experimental set-up