Università degli Studi di Pavia
*Dipartimento di Fisica*
Corso di laurea in Fisica

# On the fundamental solutions for wave-like equations on curved backgrounds

Relazione per la laurea di:
**Rubens Longhi**

Supervisore:
**Dott. Claudio Dappiaggi**

Co-supervisore:
**Dott. Nicoló Drago**

Anno Accademico 2016/2017

*A mamma e papà*

**Abstract**

La tesi si propone di analizzare la risolubilitá di equazioni differenziali alle derivate parziali di tipo ondulatorio e le loro proprietá tramite la costruzione di soluzioni fondamentali distribuzionali. Dapprima si affronterá il caso nello spazio di Minkowski $n$-dimensionale e poi si trasporteranno, per quanto possibile, le principali proprietá delle soluzioni su particolari varietá curve di interesse per le loro applicazioni fisiche.

The aim of the thesis is to analyse the solvability of wave-like partial differential equations and their properties via the construction of distributional fundamental solutions. Initially will be explored the $n$-dimensional Minkowski case and then the main properties of the solutions will be translated, when possible, on particular manifolds, interesting for their physical applications.

# Contents

# Introduction

Wave-like equations are a class of differential equations which rule the dynamics of many physical processes, from electromagnetism to quantum field theory. Such equations contain a differential operator $P$, which is a generalization of the d'Alembert wave operator

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2},$$

that describes the propagation of a wave with velocity $c \equiv 1$ in an $n$-dimensional spacetime. In this thesis will be presented a constructive method of fundamental solutions to solve wave-like differential equations, firstly on a flat background and then on suitable manifolds of physical interest, where the standard methods used in the Minkowskian scenario are not applicable.

Fundamental solutions are used to solve inhomogeneous equations of the form $Pf = \psi$, where $\psi$ is a generic source. They can be thought as the distributional solutions to a differential equation in which the source is a point-like instantaneous perturbation, namely the Dirac delta. To be more precise, if $P$ is a differential operator, a fundamental solution $u_x$ for $P$ at $x$ is a distributional solution to the equation

$$Pu_x = \delta_x.$$

Once a fundamental solution $u_x$ is found, it is easy to find the desired solution $f$ of the equation $Pf = \psi$ and deduce the properties of the solution.

In a flat spacetime, when dealing with wave operator $\Box$, it turns out that the fundamental solutions can be constructed by applying the methods of the Fourier transform to the underlying partial differential equation. Moreover, the translational symmetry of the background allows the solution for $Pf = \psi$ to be given simply by the convolution $u_0 * \psi$, namely it suffices to solve the problem at any but fixed point of the background, which can thus be chosen without loss of generality to be. In particular we will describe two fundamental solutions at 0, denoted with $G_+$ and $G_-$, the *retarded* and the *advanced* one, that propagate the source of the equation respectively in the causal future and in the causal past, in accordance with the causality principle.
When dealing with wave-like operators on generic Lorentzian manifolds, there is no symmetry available. Hence, in order to find fundamental solutions it is necessary to follow another path. In Chapter **??**, following the construction of [2] and [5], we define on Minkowski spacetime a family of distributions, dubbed *Riesz distributions* $R_\pm(\alpha)$, supported respectively on the future and on the past of 0, depending on a complex parameter $\alpha$, when Re $\alpha > n$. Subsequently through analytic continuation the family $R_\pm(\alpha)$ is extended to the whole complex plane and some important properties are deduced. It is noteworthy that for the specific value of $\alpha = 2$, one finds the aforementioned advanced and retarded fundamental solutions. In addition their support properties can be inferred for any dimension of the underlying flat background, hence proving the so-called *Huygens' principle* (see Section **??** and Theorem **??**).
Within the framework of flat space, we will address the Cauchy problem for $\Box f = \psi$, with initial data to be set on a particular class of hypersurfaces: Cauchy hypersurfaces (see Definition **??**). These surfaces are strongly related to the causality principle. In fact any point of a Cauchy hypersurface is not in the past nor in the future of any other point of the surface, i.e. the initial data cannot influence each other. In view of these considerations, the existence and uniqueness of the solution will be shown.

Then we will focus on the problem of analyzing the solutions of wave-like operators on a suitable class of Lorentzian manifolds. We give examples of such operators on backgrounds of physical interest, such as the cosmological and the Schwartzschild spacetimes. Then we will try to use Riesz distributions, which are well-suited for generalization, to solve the problem locally. To tackle this problem our strategy is the following: At any but fixed point $x \in \mathrm{M}$ we can consider the tangent space and a coordinate system such that the metric reads as the Minkowski one at $x$. On $T_x\mathrm{M}$ one can define the Riesz distributions as in the flat scenario. Via the exponential map these can be pulled-back to a local neighbourhood of $x \in \mathrm{M}$. While this operations preserves most of the

desired properties, it fails to yield fundamental solutions for the operator of interest, neither locally. However, if one combines in a proper manner certain Riesz distributions to form a formal series, one can find local approximate fundamental solutions for a generic wave-like operator that converge to the true one in a suitable way.
The Cauchy problem will be addressed locally, proving that the local solution exists and it is unique, while the support properties, that we found in the flat case, survive on generic manifolds. To find global solutions, we will have to restrict to a particular class of manifolds, the *globally hyperbolic* spacetimes, in order to ensure the existence of a Cauchy hypersurface (see Theorem **??**) where to set initial data.

A synopsis of the thesis is the following:
In the first chapter, it will be presented an overview of the main mathematical notions needed in order to set the discussion on a curved background. The main topics will be *differentiable manifolds*, *tangent space*, *Lorentzian manifolds*, *causality and global hyperbolicity*, *hyperbolic differential operators* as well as the *theory of integration* on manifolds.

In the second chapter, it will be discussed the concept of fundamental solutions and we will focus on the particular case of the d'Alembert wave operator in Minkowski spacetime. Two approaches will be followed: the first relies on the theory of Fourier transform while the second is based on the identification of a particular class of distributions, the *Riesz distributions*, that allow to construct the sought fundamental solutions for any spacetime dimension. At last, an overview of the Cauchy problem for the wave operator will be discussed in the simplest cases.

In the third chapter, we will give some examples of Lorentzian manifolds of physical interest, whose associated, metric induced, wave-like operator is such that the associated fundamental solutions cannot be constructed with the same methods used in Minkowski spacetime. To avoid this hurdle, the first step will be to extend of Riesz distributions to Lorentzian manifolds to be able to identify fundamental solutions of a wave-like operator, first locally and then globally provided that the underlying manifolds abides to suitable, physically and structurally motivated constraints. The Cauchy problem will be solved both in a local and a global setting, and it will give information on the regularity as well as on the support of the solutions.

# Geometric preliminaries 1

In this chapter, we begin by recalling the basic definitions in order to fix the geometric setting in which we will work.
Globally hyperbolic spacetimes $(M, g)$ are used in context of geometric analysis and mathematical relativity because in them there exists a smooth and spacelike Cauchy hypersurface $\Sigma$ and that ensures the well-posedness of the Cauchy problem. Moreover, as shown by Bernal and Sánchez (**?** , Th. 1.1), in such spacetimes there exists a splitting for the full spacetime $M$ as an orthogonal product $\mathbb{R} \times \Sigma$. These results corroborate the idea that in globally hyperbolic spacetimes one can preserve the notion of a global passing of time. In a globally hyperbolic spacetime, the entire future and past history of the universe can be predicted from conditions imposed at a fixed instant represented by the hypersurface $\Sigma$.

## 1.1 Globally Hyperbolic spacetimes with timelike boundary

The main goal of this section is to analyse the main properties of globally hyperbolic spacetimes and to generalise them to a natural class of spacetimes where boundary values problems can be formulated. This class is that of globally hyperbolic spacetimes with timelike boundary.While, in the case of $\partial M = \emptyset$ global hyperbolicity is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in (**?** ).

**Manifolds with boundary.** From now on $M$ will denote a smooth manifold with boundary with dimension $m > 1$. $M$ is then locally diffeomorphic to open subsets of the closed half space of $\mathbb{R}^n$. We will assume that the boundary $\partial M$ is smooth and, for simplicity, connected. A point $p \in M$ such that there exists an open neighbourhood $U$ containing $p$, diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$.
In addition we endow $M$ with a smooth Lorentzian metric $g$ of signature $(-,+,...,+)$ so that $\iota^* g$ identifies a Lorentzian metric on $\partial M$ and we require $(M,g)$ to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that $(M,g)$ has a *timelike boundary.*

**1.1.1 Definition.**

- *A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.*

- *A spacetime with timelike boundary is* causal *if it possesses no closed, causal curve,*

- *A causal spacetime with timelike boundary $M$ such that for all $p, q \in M$ $J_+(p) \cap J_-(q)$ is compact is called* ***globally hyperbolic****.*

These conditions entail the following consequences, see (**?** , Th. 1.1 & 3.14):

**1.1.2 Theorem.** *Let $(M,g)$ be a spacetime of dimension $m$. Then*

1. *$(M,g)$ is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of $M$ which is intersected only once by every inextensible timelike curve,*

2. *if $(M,g)$ is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the metric*
$$g = -\beta d\tau^2 + h_\tau, \tag{1.1}$$

*where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function*[1]*, whose gradient is tangent to $\partial M$, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a Cauchy surface for $(M, g)$.*

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary $(M, g)$, we work directly with (**??**) and we shall refer to $\tau$ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of $M$.

**1.1.3 Definition.** *A spacetime $(M, g)$ is* static *if it possesses a timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to $\partial M$ is tangent to the boundary, i.e. $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where $\nu$ is the unit vector, normal to the boundary at $p$.*

With reference to (**??**) this translates simply into the request that both $\beta$ and $h_\tau$ are independent from $\tau$.

**1.1.4 Example.** We first consider some examples of globally hyperbolic spacetimes without boundary ($\partial M = \emptyset$).

- The Minkowski spacetime $M = (\mathbb{R}^m, \eta)$ is stati and globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-1}$, endowed with the time-independent Euclidean metric.

- Let $\Sigma$ be a Riemannian manifold with time independent metric $h$ and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$, called **cosmological spacetime**, is globally hyperbolic if and only if $(\Sigma, h)$ is a complete Riemannian manifold, see (**?** , Lem A.5.14). This applies in particular if $(\Sigma, h)$ is compact.

- The interior and exterior **Schwarzschild spacetimes**, that represent non-rotating black holes of mass m $> 0$ are static and globally hyperbolic. Denoting $S^2$ the 2-dimensional sphere embedded in $\mathbb{R}^3$, we set

$$M_{\text{ext}} := \mathbb{R} \times (2\mathrm{m}, +\infty) \times S^2,$$

[1]Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

$$M_{\text{int}} := \mathbb{R} \times (0, 2\mathrm{m}) \times S^2.$$

The metric is given by

$$g = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2},$$

where $f(r) = 1 - \frac{2\mathrm{m}}{r}$, while $g_{S^2} = r^2\,\mathrm{d}\vartheta^2 + r^2 \sin^2\vartheta \mathrm{d}\varphi^2$ is the polar coordinates metric on the sphere. For the exterior Schwarzschild spacetime we have $M_{\text{ext}} = \mathbb{R} \times \Sigma$ with $\Sigma = (2\mathrm{m}, +\infty) \times S^2$, $\beta = f$ and $h = \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2}$.

**1.1.5 Example.** Now we consider some examples of globally hyperbolic spacetimes with timelike boundary in which the boundary is not empty.

- The Half Minkowski spacetime $M = (\mathbb{R}^{m-1} \times [0, +\infty), \eta)$ is static and globally hyperbolic. Every spacelike half-hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-2} \times [0, +\infty)$, endowed with the time-independent Euclidean metric.

- Let $\Sigma$ be a Riemannian manifold with boundary with time independent metric $h$ and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$ is globally hyperbolic if and only if $(\Sigma, h)$ is a complete Riemannian manifold with boundary.

A particular role will be played by the support of the functions that we consider. In the following definition we introduce the different possibilities that we will consider - *cf.* (**?** ).

**1.1.6 Definition.** *Let $(M, g)$ be a Lorentzian spacetime with timelike boundary. We denote with*

1. *$C^\infty_{\mathrm{c}}(M)$ the space of smooth functions with compact support in $M$ while with $C^\infty_{\mathrm{cc}}(M) \subset C^\infty_{\mathrm{c}}(M)$ the collection of smooth and compactly supported functions $f$ such that supp$(f) \cap \partial M = \emptyset$.*

2. *$C^\infty_{\mathrm{spc}}(M)$ (resp. $C^\infty_{\mathrm{sfc}}(M)$) the space of strictly past compact (resp. strictly future compact) functions, that is the collection of $f \in C^\infty(M)$ such that there exists a compact set $K \subseteq M$ for which $J^+(\text{supp}(f)) \subseteq J^+(K)$ (resp. $J^-(\text{supp}(f)) \subseteq J^-(K)$), where $J^\pm$ denotes the causal future and the causal past in $M$. Notice that $C^\infty_{\mathrm{sfc}}(M) \cap C^\infty_{\mathrm{spc}}(M) = C^\infty_{\mathrm{c}}(M)$.*

3. *$C^\infty_{\mathrm{pc}}(M)$ (resp. $C^\infty_{\mathrm{fc}}(M)$) denotes the space of future compact (resp. past compact) functions, that is, $f \in C^\infty(M)$ for which supp$(f) \cap J^-(K)$ (resp. supp$(f) \cap J^+(K)$) is compact for all compact $K \subset M$.*

4. *$C^\infty_{\mathrm{tc}}(M) := C^\infty_{\mathrm{fc}}(M) \cap C^\infty_{\mathrm{pc}}(M)$, the space of timelike compact functions.*

5. *$C^\infty_{\mathrm{sc}}(M) := C^\infty_{\mathrm{sfc}}(M) \cap C^\infty_{\mathrm{spc}}(M)$, the space of spacelike compact functions.*

## 1.2 Differential forms and operators on manifolds with boundary

To treat Maxwell equations properly and to be able to generalise them, we will use the language of differential forms. In this section $(E, g)$ will denote a generic oriented pseudo-Riemannian manifold with boundary with signature $(-, +, \ldots, +, +)$ or $(+, +, \ldots, +, +)$. In the former case, when the manifold is Lorentzian, it is understood that the boundary is timelike in the sense of Definition **??**. We present the following definitions in such a general framework since we will work both on spacetimes $(M, g)$ with timelike boundary and on their Cauchy hypersurfaces $(\Sigma, h)$, which are Riemannian manifolds with boundary.

On top of a pseudo-Riemannian Hausdorff, connected, oriented and para-compact manifold $(E, g)$ with boundary we consider the spaces of complex valued $k$-forms $\Omega^k(E)$ as smooth sections of $\wedge^k T^*E$. Since $(E, g)$ is oriented, we can identify a unique, metric-induced, Hodge operator $* : \Omega^k(E) \to \Omega^{m-k}(E)$, $m = \dim E$ such that, for all $\alpha, \beta \in \Omega^k(E)$, $\alpha \wedge *\beta = (\alpha, \beta)\mathrm{d}\mu_g$, where $\wedge$ is the exterior product of forms and $\mathrm{d}\mu_g$ the metric induced volume form. We endow $\Omega^k(E)$ with the standard, metric induced, pairing

$$(\alpha, \beta) := \int_E \overline{\alpha} \wedge *\beta \,, \tag{1.2}$$

**1.2.1 Remark.** One can easily generalize of the spaces defined for scalar fields in Definition **??** respectively to the following spaces of $k$-forms: $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{cc}}(M)$, $\Omega^k_{\mathrm{spc/sfc}}(M)$, $\Omega^k_{\mathrm{pc/fc}}(M)$, $\Omega^k_{\mathrm{tc/sc}}(M)$.

We indicate the exterior derivative with $\mathrm{d} : \Omega^k(E) \to \Omega^{k+1}(E)$. A differential form $\alpha$ is called closed when $\mathrm{d}\alpha = 0$ and exact when $\alpha = \mathrm{d}\beta$ for some differential form $\beta$. Since $E$ is endowed with a pseudo-Riemannian metric it holds that, when acting on smooth $k$-forms, $*^{-1} = (-1)^{k(m-k)}*$. Combining these data we define the *codifferential* operator $\delta : \Omega^{k+1}(E) \to \Omega^k(E)$ as $\delta \doteq *^{-1} \circ \mathrm{d} \circ *$.

To conclude the section, we focus on the boundary $\partial E$ and on the interplay with $k$-forms lying in $\Omega^k(E)$. The first step consists of defining two notable maps. These relate $k$-forms defined on the whole $E$ with suitable counterparts living on $\partial E$ and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its projection along the direction normal to $\partial E$.

**1.2.2 Remark.** Since we will be considering not only form lying in $\Omega^k(E)$, $k \in \mathbb{N} \cup \{0\}$, but also those in $\Omega^k(\partial E)$, we shall distinguish the operators acting on this space with a subscript $_\partial$, e.g. $\mathrm{d}_\partial$, $*_\partial$, $\delta_\partial$ or $(,)_\partial$.

**1.2.3 Definition.** *Let $(E, g)$ be a pseudo-Riemannian manifold with boundary together with the embedding map $\iota_\partial : \partial E \hookrightarrow E$. We call* tangential *and* normal *maps*

$$\mathrm{t} \colon \Omega^k(E) \to \Omega^k(\partial E) \qquad \omega \mapsto \mathrm{t}\omega \doteq \iota_\partial^* \omega \tag{1.3a}$$

$$\mathrm{n} \colon \Omega^k(E) \to \Omega^{k-1}(\partial E) \qquad \omega \mapsto \mathrm{n}\omega \doteq *_\partial^{-1} \circ \mathrm{t} \circ *_E \,, \tag{1.3b}$$

*In particular, for all $k \in \mathbb{N} \cup \{0\}$ we define*

$$\Omega^k_{\mathrm{t}}(E) := \{\omega \in \Omega^k(E) \mid \mathrm{t}\omega = 0\} \,, \qquad \Omega^k_{\mathrm{n}}(E) := \{\omega \in \Omega^k(E) \mid \mathrm{n}\omega = 0\} \,. \tag{1.4}$$

**1.2.4 Remark.** The normal map $\mathrm{n} : \Omega^k(E) \to \Omega^{k-1}(\partial E)$ can be equivalently read as the restriction to $\partial E$ of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(E)$ and the vector field $\nu \in \Gamma(TE)|_{\partial E}$ which corresponds pointwisely to the unit vector, normal to $\partial E$.

As last step, we observe that (**??**) together with (**??**) entail the following series of identities on $\Omega^k(E)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$* \, \delta = (-1)^k \mathrm{d} * \,, \quad \delta * = (-1)^{k+1} * \mathrm{d} \,, \tag{1.5a}$$

$$*_\partial \, \mathrm{n} = \mathrm{t} * \,, \quad *_\partial \mathrm{t} = (-1)^k \mathrm{n} * \,, \quad \mathrm{d}_\partial \mathrm{t} = \mathrm{t}\mathrm{d} \,, \quad \delta_\partial \mathrm{n} = \mathrm{n}\delta \,. \tag{1.5b}$$

A notable consequence of (**??**) is that, while on manifolds with empty boundary, the operators d and $\delta$ are one the formal adjoint of the other, in the case in hand, the situation is different. Indeed, a direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial \qquad \forall \alpha \in \Omega^k_{\mathrm{c}}(E) \,, \, \forall \beta \in \Omega^{k+1}_{\mathrm{c}}(E) \,, \tag{1.6}$$

where the pairing in the right-hand side is the one associated to forms living on $\partial E$.

## 1.3 Maxwell equations for $k$-forms

We now focus our attention on a $m$-dimensional spacetime $(M, g)$. As usual, the electromagnetic field will be regarded as a 2-form $F$, called Faraday field, and the potential as a 1-form $A$ such that, locally, holds $F = \mathrm{d}A$. This is permitted by the first Maxwell equation, namely

$$\mathrm{d}F = 0, \tag{1.7}$$

that ensures the Faraday form $F$ being closed, hence locally exact. The equation $F = \mathrm{d}A$ holds globally whenever one can rely on the Poincaré lemma, which cannot be always applied since it fails to hold true if the second cohomology group $H^2(M)$ is not trivial.

From a physical point of view, one wonders whether it is $A$ or it is $F$ the observable field of the dynamical system, because $F$ encodes the usual electric and magnetic fields $E, B \in \Omega^1(\Sigma)$ (if $M$ is static with $M = \mathbb{R} \times \Sigma$, holds the decomposition $F = *_\Sigma B + \mathrm{d}t \wedge E$). Moreover, one can object that the choice of $A \in \Omega^1(M)$ is not unique. Indeed if for a moment one assumes, for simplicity, $M$ to be globally hyperbolic with empty boundary, the configuration $A' := A + \mathrm{d}\chi$, $\chi \in \Omega^0(M)$ is equivalent to $A$ since it gives rise to the same Faraday field $F$. This freedom in the choice of $A$ is extensively used and it is called **gauge freedom**.

To recap, one can regard the electromagnetism as a theory for $F \in \Omega^2(M)$ or as a theory for a non-unique $A \in \Omega^1(M)$ and wonder if the initial and boundary value problem for Maxwell equations is well-posed in both cases. The former case for $F$ will be covered in Chapter **??** and the latter for $A$ in Chapter **??**.

The second Maxwell equations tells us the dynamics of the electromagnetic field. If $J$ denotes the co-closed current 1-form ($\delta J = 0$), which encodes the charge density and the electric current, the second Maxwell equation can be written as

$$\delta F = -J, \tag{1.8}$$

and the corresponding

In the homogeneous case ($J = 0$), one can generalise the Maxwell field to be $F \in \Omega^k(M)$, imposing $\mathrm{d}F = 0$ and $\delta F = 0$ and the equation for $A \in \Omega^{k-1}(M)$ becomes $\delta \mathrm{d}A = 0$. In this case gauge freedom is understood as a transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-2}(M)$.

# Maxwell equations with interface conditions 2

As outlined in Section **??**, the very nature of Maxwell equations allows us to use both $F$ and $A$ as variables with which to describe electromagnetic phenomena. Whenever the second cohomology group $H^2(M)$ is trivial, the two theories are equivalent, since $F = \mathrm{d}A$.

In this chapter, we regard $F \in \Omega^2(M)$ as the true physical dynamical variable which describes electromagnetism. The aim of this chapter is to present a technique which allows to characterize, in a class of manifolds with the presence of an interface between two media, the existence of fundamental solutions for Maxwell equations, written in terms of the Faraday form $F \in \Omega^2(M)$. The presence of an interface on the one hand generalizes the idea of the presence of a timelike boundary, allowing to recover the geometric setting outlined in Chapter **??** in case on one side of the interface lies a perfect insulator. On the other hand, in order to make use of geometric techniques such as Hodge decomposition, we will have to make several geometric assumptions which ensure global hyperbolicity, but unfortunately are way less general.

## 2.1 Geometrical set-up

The physical and practical situation we want to approach is that of a manifold split into two parts, filled with two media, each of them with different electromagnetic properties. The two media will be separated by an hypersurface, on which our aim will be that of putting *jump conditions*.

We consider a static Lorentzian manifold $(M, g)$ with **empty boundary**, such that $M$ can be decomposed as $\mathbb{R}\times\Sigma$, where the Cauchy hypersurface $(\Sigma, h)$ is assumed to be a complete, connected, odd-dimensional, **closed** Riemannian manifold. The assumptions we made so far on imply that $(M, g)$ is a globally hyperbolic spacetime without boundary.

Maxwell equations, recalling formulas (**??**) and (**??**), for $F \in \Omega^2(M)$ are simply

$$\mathrm{d}F = 0\,, \qquad \delta F = 0\,, \tag{2.1}$$

The geometrical assumptions on $M$ permit us to split $F$ into electric and magnetic components

$$F = *_\Sigma B + \mathrm{d}t \wedge E\,, \tag{2.2}$$

where $E, B \in \Omega^1(\Sigma)$ while $*_\Sigma$ is the Hodge dual of $\Sigma$.
Maxwell equations are then reduced to

$$\partial_t E - \mathrm{curl} B = 0\,, \qquad \partial_t B + \mathrm{curl} E = 0\,, \tag{2.3a}$$

$$\mathrm{div}(E) = \mathrm{div}(B) = 0\,, \tag{2.3b}$$

where $\mathrm{div} = \delta_\Sigma$ is the co-differential of $\Sigma$, while curl is defined in equation (**??**) – in particular $\mathrm{curl} = *_\Sigma \mathrm{d}_\Sigma$ if $\dim\Sigma = 3 \mod 4$.
To model the presence of an interface that divides $M$ in two distinct regions, we also let $Z$ be a codimension 1 smooth embedded hypersurface of $\Sigma$.

We denote with $\mathrm{d}, \delta$ the differential and co-differential over $M$, while $\mathrm{d}_\Sigma, \delta_\Sigma$ denote the differential and co-differential over $\Sigma$.

In this setting we would like to consider Maxwell equations with $Z$-interface boundary conditions. This means that we will consider Maxwell equations on $M \setminus (\mathbb{R} \times Z)$, allowing for jump discontinuities to occur on $\mathbb{R} \times Z$.
(We should decide whether we want to consider $k$-Maxwell equations or not. In the former case the curl operator acts as $\mathrm{curl}\colon \Omega^k(\Sigma) \to \Omega^k(\Sigma)$ where $\dim\Sigma = 2k + 1$.)

Whenever $Z \neq \emptyset$ the system (**??**) has to be modified, in particular the non-dynamical equations (**??**) involving the divergence operator div have to be suitably interpreted – *cf.* remark **??**. In particular one expects that the condition $\mathrm{div}(E) = \mathrm{div}(B) = 0$ should be interpreted weakly, leading to a constraint on the normal jump of $E$ across $Z$. Moreover, the dynamical equations (**??**) have to be combined with boundary conditions at the interface $Z$ – *cf.* (**?** , Sec. I.5).

In what follows we will state the precise meaning of the problem (**??**) with interface $Z$ with the help of Hodge theory and Lagrangian subspaces (**? ? ?** ).

## 2.2 Non-dynamical equations: Hodge theory with interface

In this section we present a Hodge decomposition for the closed Riemannian manifold $(\Sigma, h)$ with interface $Z$. This generalizes the known results on classical Hodge decomposition on manifolds with possible non-empty boundary (**?** **?** **?** **?** **?** **?** **?** **?** **?** **?** ).

Hodge theory comes as a generalization of Helmholtz decomposition. He first formulated a result on the splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the *Hodge decomposition.* The idea behind Helmholtz decomposition is that any vector field in $\mathbb{R}^3$ can be decomposed as a sum of an irrotational field, i.e. $\mathrm{curl} = \mathrm{d}_\Sigma = 0$, and a solenoidal field, i.e. $\mathrm{div} = \delta_\Sigma = 0$. In other words, for $\mathbf{F} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$, one can write

$$\mathbf{F} = -\nabla\Phi + \mathrm{curl}\,\mathbf{A}. \tag{2.4}$$

In what follows $\mathrm{L}^2\Omega^k(\Sigma)$ will denote the space of sections of $\wedge^k T^*\Sigma$ which are square integrable with respect to the pairing induced by the metric $h$

$$(\alpha, \beta)_\Sigma := \int_\Sigma \overline{\alpha} \wedge *_\Sigma \beta\,, \tag{2.5}$$

where $*_\Sigma$ is the Hodge dual. Similarly we shall denote with $C^\infty_\mathrm{c}\Omega^k(\Sigma)$ the space of smooth and compactly supported $k$-forms, while $\mathrm{H}^\ell\Omega^k(\Sigma)$ will denote $k$-forms with weak $\mathrm{L}^2$-derivatives up to order $\ell \in \mathbb{N} \cup \{0\}$ with respect to one (hence all) connection over $\Sigma$ – as usual we also set $\mathrm{H}^{-\ell}\Omega^k(\Sigma) := \mathrm{H}^\ell\Omega^k(\Sigma)^*$. (If $\Sigma$ is not compact we only have that only $\mathrm{H}^\ell_{\mathrm{loc}}(\Sigma)$ is independent from the choice of the connection. If we assume $(\Sigma, h)$ to be of bounded geometry the ambiguity disappears because of the results of (**?** ).)

### 2.2.1 Hodge decomposition on compact manifold with non-empty boundary

The Hodge theorem for a closed manifold $\Sigma$ states that there is an $\mathrm{L}^2$-orthogonal decomposition

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}(\Sigma) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}(\Sigma) \oplus \ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}\,, \tag{2.6}$$

where $\Delta = \mathrm{d}_\Sigma\delta_\Sigma + \delta_\Sigma\mathrm{d}_\Sigma$ is the Laplacian and $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ denotes the space of *harmonic forms.* If $\Sigma$ has no an empty boundary, the space of harmonic forms $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ coincides with that of **harmonic fields**, defined (following (**?** ) and (**?** )) as

$$\mathcal{H}^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma) |\ \mathrm{d}_\Sigma\omega = 0\,,\ \delta_\Sigma\omega = 0\} = \ker(\delta_\Sigma)_{\mathrm{H}^1\Omega^k(\Sigma)} \cap \ker(\mathrm{d}_\Sigma)_{\mathrm{H}^1\Omega^k(\Sigma)}\,. \tag{2.7}$$

The last result can be stated as follows and it is very easy to prove.

**2.2.1 Proposition.** *Let $\alpha \in \mathrm{H}^1\Omega^k(\Sigma)$, where $\Sigma$ is a closed manifold. Then $\Delta\alpha = 0$ if and only if $\mathrm{d}_\Sigma\alpha = 0$ and $\delta_\Sigma\alpha = 0$.*

*Proof.* Clearly if $\mathrm{d}_\Sigma\alpha = 0$ and $\delta_\Sigma\alpha = 0$, $\Delta\alpha = 0$. On the other hand if $\Delta\alpha = 0$,

$$0 = (\Delta\alpha, \alpha)_\Sigma = ((\mathrm{d}_\Sigma\delta_\Sigma + \delta_\Sigma\mathrm{d}_\Sigma)\alpha, \alpha)_\Sigma = (\mathrm{d}_\Sigma\delta_\Sigma\alpha, \alpha)_\Sigma + (\delta_\Sigma\mathrm{d}_\Sigma\alpha, \alpha)_\Sigma = \tag{2.8}$$

$$= (\delta_\Sigma\alpha, \delta_\Sigma\alpha)_\Sigma + (\mathrm{d}_\Sigma\alpha, \mathrm{d}_\Sigma\alpha)_\Sigma = ||\delta_\Sigma\alpha||^2 + ||\mathrm{d}_\Sigma\alpha||^2. \tag{2.9}$$

So both $\mathrm{d}_\Sigma\alpha = 0$ and $\delta_\Sigma\alpha = 0$. ■ For a compact manifold $\Sigma$ with non-empty boundary $\partial\Sigma$ the decomposition (**??**) requires a slight adjustment and harmonic forms do not coincide with harmonic fields anymore. Because of boundary terms, $\ker\Delta$ no longer coincides with the closed and co-closed forms. It is clear that every harmonic field is a harmonic form, but the converse is false. To show this, consider the following example.

**2.2.2 Example.** Let $U$ a bounded subset of $\mathbb{R}^2$, endowed with the standard euclidean metric. On $U$, the 1-form $\omega = x\,dy$ is clearly harmonic, since its second derivatives vanish, but it is not in $\ker\mathrm{d}$ as

$$\mathrm{d}(x\,\mathrm{d}y) = \partial_x\, x\,\mathrm{d}x \wedge \mathrm{d}y + \partial_y\, x\,\mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y.$$

$\omega$ is though in $\ker\delta$ as $*\mathrm{d}*(x\,\mathrm{d}y) = *\mathrm{d}(x\,\mathrm{d}x) = 0$.

In fact, the space $\mathcal{H}^k(\Sigma)$ of harmonic fields is infinite dimensional and so is much too big to represent the cohomology, and to recover the Hodge isomorphism one has to impose boundary conditions. Indeed the spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^{k-1}(\Sigma)$, $\delta_\Sigma H^1\Omega^{k+1}(\Sigma)$, $\mathcal{H}^k(\Sigma)$ are not orthogonal unless suitable boundary conditions are imposed.

**2.2.3 Remark.** According to (? , p. 171), the tangential and normal maps defined in Definition ?? can be extended to continuous surjective maps

$$\mathrm{t} \oplus \mathrm{n} \colon \mathrm{H}^{\ell}\Omega^{k}(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^{k}(\partial\Sigma) \oplus \mathrm{H}^{\ell-\frac{1}{2}}\Omega^{k}(\partial\Sigma) \qquad \forall \ell \geq \frac{1}{2}\,. \tag{2.10}$$

We can now recall the Hodge decomposition for compact manifolds with boundary (? , Thm. 2.4.2).

**2.2.4 Theorem.** *Let $(\Sigma, h)$ be a compact, connected, Riemannian manifold with non-empty boundary $\partial\Sigma \overset{\iota_{\partial\Sigma}}{\hookrightarrow} \Sigma$.*

1. *For all $\omega \in C^{\infty}_{\mathrm{c}}\Omega^{k-1}(\Sigma)$ and $\eta \in C^{\infty}_{\mathrm{c}}\Omega^{k}(\Sigma)$ it holds*

$$(\mathrm{d}_{\Sigma}\omega, \eta)_{\Sigma} - (\omega, \delta_{\Sigma}\eta)_{\Sigma} = (\mathrm{t}\omega, \mathrm{n}\eta)_{\partial\Sigma}\,, \tag{2.11}$$

   *where $(\ ,\ )_{\Sigma}$ has been defined in equation (??) while $(\ ,\ )_{\partial\Sigma}$ is defined similarly. Equation (??) still holds true for $\omega \in \mathrm{H}^{\ell}\Omega^{k-1}(\Sigma)$ and $\eta \in \mathrm{H}^{\ell}\Omega^{k}(\Sigma)$ – cf. remark ??.*

2. *There is an orthogonal decomposition*

$$\mathrm{L}^{2}\Omega^{k}(\Sigma) = \mathrm{d}_{\Sigma}\mathrm{H}^{1}\Omega^{k}_{\mathrm{t}}(\Sigma) \oplus \delta_{\Sigma}\mathrm{H}^{1}\Omega^{k+1}_{\mathrm{n}}(\Sigma) \oplus \mathrm{L}^{2}\mathcal{H}^{k}(\Sigma)\,, \tag{2.12}$$

   *where $\mathrm{L}^{2}\mathcal{H}^{k}(\Sigma)$ is the closure with respect to the $\mathrm{L}^{2}$ norm of the space of harmonic fields, as defined per equation (??) and*

$$\mathrm{H}^{1}\Omega^{k-1}_{\mathrm{t}}(\Sigma) := \{\alpha \in \mathrm{H}^{1}\Omega^{k-1}(\Sigma) |\ \mathrm{t}\alpha = 0\}\,, \tag{2.13}$$

$$\mathrm{H}^{1}\Omega^{k+1}_{\mathrm{n}}(\Sigma) := \{\beta \in \mathrm{H}^{1}\Omega^{k+1}(\Sigma) |\ \mathrm{n}\beta = 0\}\,, \tag{2.14}$$

   *following the definitions of Equation (??).*

**2.2.5 Remark.** The previous decomposition generalizes to Sobolev spaces, in particular for all $\ell \in \mathbb{N} \cup \{0\}$ we have

$$\mathrm{H}^{\ell}\Omega^{k}(\Sigma) = \mathrm{d}_{\Sigma}\mathrm{H}^{\ell+1}\Omega^{k}_{\mathrm{t}}(\Sigma) \oplus \delta_{\Sigma}\mathrm{H}^{\ell+1}\Omega^{k+1}_{\mathrm{n}} \oplus \mathrm{H}^{\ell}\mathcal{H}^{k}(\Sigma)\,, \tag{2.15}$$

where $\mathrm{H}^{\ell}\mathcal{H}^{k}(\Sigma) = \mathrm{L}^{2}\mathcal{H}^{k}(\Sigma) \cap \mathrm{H}^{\ell}\Omega^{k}(\Sigma)$, since $\mathrm{H}^{\ell}\Omega^{k}(\Sigma) \hookrightarrow \mathrm{L}^{2}\Omega^{k}(\Sigma)$.

#### 2.2.1.1 Hodge decomposition for compact manifold with interface

We would like to consider a decomposition similar to the one of theorem ?? for the case of a closed Riemannian manifold $\Sigma$ with interface $Z$. In this setting we split $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$ and we refer to $\Sigma_-$ (*resp.* $\Sigma_+$) as the left (*resp.* right) component of $\Sigma$. Notice that $\Sigma_\pm$ are compact manifolds with boundary $\partial\Sigma_\pm = \pm Z$ – notice that the orientation on $Z$ induced by $\Sigma_+$ is the opposite of the one induced by $\Sigma_-$. Therefore theorem **??** applies to $\mathrm{L}^2\Omega^k(\Sigma_\pm)$.

Since $Z$ has zero measure the space of square integrable $k$-forms splits as

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{L}^2\Omega^k(\Sigma_Z) = \mathrm{L}^2\Omega^k(\Sigma_+) \oplus \mathrm{L}^2\Omega^k(\Sigma_-)\,. \tag{2.16}$$

We expect a $Z$-relative Hodge decomposition as in (**??**) to hold true in this situation, where the boundary conditions of the spaces $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\mathrm{H}^1\Omega^{k-1}_{\mathrm{n}}(\Sigma)$ should be replaced by appropriate jump conditions across $Z$. For that, notice that the splitting (**??**) does not generalize to the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$, in particular

$$\mathrm{H}^\ell\Omega^k(\Sigma) \subset \mathrm{H}^\ell\Omega^k(\Sigma_Z) = \mathrm{H}^\ell\Omega^k(\Sigma_+) \oplus \mathrm{H}^\ell\Omega^k(\Sigma_-)\,, \tag{2.17}$$

is a proper inclusion.
(Is it clear that the objects associated with $\Sigma_Z$ are automatically a direct sum of objects associated with $\Sigma_\pm$?)

**2.2.6 Definition.** *Let $(\Sigma, h)$ be an oriented, compact, Riemanniann manifold with interface $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. For $\omega \in C^\infty\Omega^k(\Sigma_Z)$ we define the tangential jump $[\mathrm{t}\omega] \in C^\infty\Omega^k(Z)$ and normal jump $[\mathrm{n}\omega] \in C^\infty\Omega^{k-1}(Z)$ across $Z$ by*

$$[\mathrm{t}\omega] := \mathrm{t}_+\omega - \mathrm{t}_-\omega\,, \qquad [\mathrm{n}\omega] := \mathrm{n}_+\omega - \mathrm{n}_-\omega\,, \tag{2.18}$$

*where $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ denote the tangential and normal map on $\Sigma_\pm$ as per definition* **??**.

**2.2.7 Remark.** It is an immediate consequence of definition **??** that

$$\mathrm{H}^1\Omega^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma_Z)|\ [\mathrm{t}\omega] = 0\,,\ [\mathrm{n}\omega] = 0\}\,. \tag{2.19}$$

The same equality does not hold for $C^\infty\Omega^k(\Sigma)$ because higher order traces have to match at $Z$.

**2.2.8 Theorem.** *Let $(\Sigma, h)$ be an oriented, compact, Riemanniann manifold with interface $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$.*

1. *For all* $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ *and* $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ *it holds*

$$(\mathrm{d}_\Sigma\omega, \eta)_Z - (\omega, \delta_\Sigma\eta)_Z = ([\mathrm{t}\omega], \mathrm{n}_+\eta)_Z - (\mathrm{t}_-\omega, [\mathrm{n}\eta])_Z \,, \tag{2.20}$$

*where* $(\ ,\ )_Z$ *is the scalar product between forms on* $Z$ *– cf. equation* (**??**) *– while* $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ *are the tangential and normal maps on* $\Sigma_\pm$ *as per definition* **??**. *Equation* (**??**) *still holds true for* $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ *and* $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma)$ *for all* $\ell \geq 1$.

2. *There is an orthogonal decomposition*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma) \,, \tag{2.21}$$

*where we defined* $\mathcal{H}^k(\Sigma)$ *as per equation* (**??**) *and*

$$\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma_Z) |\ [\mathrm{t}\alpha] = 0\} \,, \tag{2.22}$$

$$\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma_Z) |\ [\mathrm{n}\beta] = 0\} \,. \tag{2.23}$$

*Proof.* Equation (**??**) is an immediate consequence of (**??**). In particular for $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ we decompose $\omega = \omega_+ + \omega_-$ and $\eta = \eta_+ + \eta_-$ where $\omega_\pm \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_\pm)$ and $\eta_\pm \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_\pm)$. (Notice that with this notation we have $\mathrm{t}_\pm\omega = \mathrm{t}_\pm\omega_\pm$.) Applying equation (**??**) we have

$$\begin{aligned}(\mathrm{d}_\Sigma\omega, \eta) - (\omega, \delta_\Sigma\eta) &= \sum_\pm \big((\mathrm{d}_\Sigma\omega_\pm, \eta_\pm) - (\omega_\pm, \delta_\Sigma\eta_\pm)\big) = \int_Z \mathrm{t}_+\overline{\omega} \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *_\Sigma\mathrm{n}_-\eta \\ &= \int_Z [\mathrm{t}\overline{\omega}] \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *[\mathrm{n}\beta] \,.\end{aligned}$$

A density argument leads to the same identity for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for $\ell \geq 1$.
We now prove the splitting (**??**). The spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)$, $\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$, $\mathcal{H}^k(\Sigma)$ are orthogonal because of equation (**??**). Let now $\omega$ be in the orthogonal complement of $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. We wish to show that $\omega \in \mathcal{H}^k(\Sigma)$. We split $\omega = \omega_+ + \omega_-$ with $\omega_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)$ and apply theorem **??** to each component so that

$$\omega = \sum_\pm \big(\mathrm{d}_\Sigma\alpha_\pm + \delta_\Sigma\beta_\pm + \kappa_\pm\big) \,,$$

where $\alpha_\pm \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_\pm)$, $\beta_\pm \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_\pm)$ and $\kappa_\pm \in \mathcal{H}^k(\Sigma_\pm)$. Let now be $\widehat{\alpha} \in \mathrm{H}^1\Omega^{k-1}(\Sigma_+)$: this defines an element in $\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ by considering its

extension by zero on $\Sigma_-$. Since $\omega \perp \mathrm{d}_\Sigma \mathrm{H}^1\Omega_{[\mathrm{t}]}(\Sigma_Z)$ we have $0 = (\mathrm{d}_\Sigma\widehat{\alpha}, \omega) = (\mathrm{d}_\Sigma\widehat{\alpha}, \mathrm{d}_\Sigma\alpha_+)$, thus $\mathrm{d}_\Sigma\alpha_+ = 0$ by the arbitrariness of $\widehat{\alpha}$. With a similar argument we have $\alpha_- = 0$ as well as $\beta_\pm = 0$.
Therefore $\omega \in \mathcal{H}^k(\Sigma_Z)$. In order to prove that $\omega \in \mathcal{H}^k(\Sigma)$ we need to show that $[\mathrm{t}\omega] = 0$ as well as $[\mathrm{n}\omega] = 0$ – *cf.* remark **??**. This is a consequence of $\omega \perp \mathrm{d}_\Sigma \mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. Indeed, let $\alpha \in \mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$: applying equation (**??**) we find

$$0 = (\mathrm{d}_\Sigma\alpha, \omega) = -\int_Z \mathrm{t}_-\overline{\alpha} \wedge *[\mathrm{n}\omega]\,. \tag{2.24}$$

The arbitrariness of $\mathrm{t}_-\alpha$ implies $[\mathrm{n}\omega] = 0$. Similarly $[\mathrm{t}\omega] = 0$ follows by $\omega \perp \delta_\Sigma \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. ■

**2.2.9 Remark.** The harmonic part of decomposition (**??**) contains harmonic $k$-forms which are continuous across the interface $Z$ – *cf.* remark **??**. One can also consider a decomposition which allows for a discontinuous harmonic component: in particular it can be shown that

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma_Z)\,,$$

where now $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)$ is the subspace of $\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ made of $(k-1)$-forms $\alpha$ such that $\mathrm{t}_\pm\omega = 0$ and similarly $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)$ if and only if $\beta \in \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$ and $\mathrm{n}_\pm\beta = 0$.

**2.2.10 Remark.** The results of theorem **??** generalize in several directions (**? ? ? ? ? ? ? ? ? ?** ). For the case of a non-compact Riemannian manifold $\Sigma$ one may follow the results of (**?** ) in order to achieve the following weak-Hodge decomposition – *cf.* equation (**??**). We consider the operators $\mathrm{d}_{\Sigma,\mathrm{t}}, \delta_{\Sigma,\mathrm{n}}$ defined by

$$\mathrm{dom}(\mathrm{d}_{\Sigma,\mathrm{t}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ \mathrm{t}\omega = 0\} \qquad \mathrm{d}_{\Sigma,\mathrm{t}}\omega := \mathrm{d}_\Sigma\omega\,, \tag{2.25}$$

$$\mathrm{dom}(\delta_{\Sigma,\mathrm{n}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ \mathrm{n}\omega = 0\} \qquad \delta_{\Sigma,\mathrm{n}}\omega := \delta_\Sigma\omega\,. \tag{2.26}$$

Notice that $\mathrm{d}_{\Sigma,\mathrm{t}}$ as well as $\delta_{\Sigma,\mathrm{n}}$ are nihilpotent because of relations (**??**). These operators are closed and from equation (**??**) it follows that their adjoints are the following:

$$\mathrm{dom}(\mathrm{d}_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\}\,, \qquad \delta^*_{\Sigma,\mathrm{n}} = \mathrm{d}_\Sigma\,,$$
$$\mathrm{dom}(\delta_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\}\,, \qquad \mathrm{d}^*_{\Sigma,\mathrm{t}} = \delta_\Sigma\,.$$

It then follows immediately that $(\overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})})^{\perp} = \ker(\mathrm{d}) \cap \ker \delta = \mathcal{H}^k(\Sigma)$ so that

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})} \oplus \mathcal{H}^k(\Sigma)\,. \tag{2.27}$$

Following the same steps of proof of theorem **??** it follows that a similar weak-Hodge decomposition holds for the case of non-compact Riemannian manifolds $\Sigma$ with interface $Z$, actually

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,[\mathrm{t}]})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,[\mathrm{n}]})} \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.28}$$

where $\mathrm{d}_{\Sigma,[\mathrm{t}]}, \delta_{\Sigma,[\mathrm{n}]}$ are defined by

$$\begin{aligned}
\mathrm{dom}(\mathrm{d}_{\Sigma,[\mathrm{t}]}) &:= \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ [\mathrm{t}\omega] = 0\} \qquad \mathrm{d}_{\Sigma,[\mathrm{t}]}\omega := \mathrm{d}_\Sigma\omega\,,\\
\mathrm{dom}(\delta_{\Sigma,[\mathrm{n}]}) &:= \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ [\mathrm{n}\omega] = 0\} \qquad \delta_{\Sigma,[\mathrm{n}]}\omega := \delta_\Sigma\omega\,.
\end{aligned}$$

This time $\mathrm{d}^*_{\Sigma,[\mathrm{t}]} = \delta_{\Sigma,[\mathrm{n}]}$ as well as $\delta^*_{\Sigma,[\mathrm{n}]} = \mathrm{d}_{\Sigma,[\mathrm{t}]}$ so that in particular $\ker \mathrm{d}^*_{\Sigma,[\mathrm{t}]} \cap \ker \delta_{\Sigma,[\mathrm{n}]} = \mathcal{H}^k(\Sigma)$.
(The notation is sloppy, in principle $\mathrm{d}_{\Sigma,\mathrm{t}}, \delta_{\Sigma,\mathrm{n}}$ depend on $k$.)

**2.2.11 Remark** (Non-dynamical Maxwell equations)**.** The Hodge decomposition with interface proved in theorem **??** can be exploited to formulate the correct generalization of the non-dynamical equations (**??**). Actually in what follows we will substitute equations (**??**) with the requirement

$$E, B \perp \mathrm{d}_\Sigma \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.29}$$

Notice that this entails $\delta_\Sigma E = \delta_\Sigma B = 0$ as well as $[\mathrm{n}E] = [\mathrm{n}B] = 0$. Configurations of the electric field $E$ in the presence of a charge density $\rho$ on $\Sigma_\pm$ and a surface charge density $\rho_Z$ over $Z$ are described by expanding $E = \mathrm{d}_\Sigma\alpha + \delta_\Sigma\beta + \kappa$ and demanding $\alpha \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$ to satisfy

$$(\mathrm{d}_\Sigma\varphi, \mathrm{d}_\Sigma\alpha)_\Sigma = (\varphi, \rho)_\Sigma + (\varphi, \rho)_Z \qquad \forall \varphi \in C^\infty_{\mathrm{c}}(\Sigma)\,.$$

For sufficiently regular $\alpha$ this is equivalent to the Poisson problem $\Delta_\Sigma\alpha = \rho$, $[\mathrm{n}\mathrm{d}_\Sigma\alpha] = \rho_Z$, recovering the classical equations outlined in (**?** , Sec. I.5).

### 2.2.2 Dynamical equations: lagrangian subspaces

In this section we will deal with the dynamical equations (**??**). These can be easily regarded as a Schrödinger equation and solved by imposing suitable boundary conditions on $Z$. Actually we can rewrite equations (**??**) in

Schrödinger form

$$i\partial_t \psi = H\psi \qquad \psi := \begin{bmatrix} E \\ B \end{bmatrix}, \qquad H := \begin{bmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{bmatrix}, \tag{2.30}$$

Here we adopt the convention of (? ) according to which

$$\mathrm{curl} := i *_\Sigma \mathrm{d}_\Sigma \quad \text{if } \dim \Sigma = 1 \mod 4\,, \qquad \mathrm{curl} := *_\Sigma \mathrm{d}_\Sigma \quad \text{if } \dim \Sigma = 3 \mod 4\,. \tag{2.31}$$

Which this convention curl is formally selfadjoint on $C^\infty_{\mathrm{c}} H^1\Omega^1(\Sigma)$.

As in section **??** we wish to consider equation (**??**) on $\Sigma_Z$, allowing for jump discontinuities across the interface $Z$. For that we regard $H$ as a densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)^{\times 2} = \mathrm{L}^2\Omega^1(\Sigma_Z)^{\times 2}$ with domain

$$\mathrm{dom}(H) := C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_+) \oplus C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_-)\,, \tag{2.32}$$

where $C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_\pm)$ denotes the subspace of $C^\infty_{\mathrm{c}}\Omega^1(\Sigma_\pm)$ with support in $\Sigma_\pm \setminus \partial\Sigma_\pm$. The operator $H$ is closable and symmetric, its adjoint $H^*$ being defined by

$$\mathrm{dom}(H^*) = \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\ H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} \qquad H^*\psi := H\psi\,. \tag{2.33}$$

Equation (**??**) is solved by selecting a self-adjoint extension of $H$. The latter can be parametrized by Lagrangian subspaces of a suitable complex symplectic space – (? ? ? ).

**2.2.12 Definition.** *Let* $\mathsf{S}$ *be a complex vector space and let* $\sigma\colon \mathsf{S}\times\mathsf{S} \to \mathbb{C}$ *a bilinear map. The pair* $(\mathsf{S},\sigma)$ *is called complex symplectic space if* $\sigma$ *is non-degenerate – i.e.* $\sigma(x,y) = 0$ *for all* $y \in \mathsf{S}$ *implies* $x = 0$ *– and* $\sigma(x,y) = -\overline{\sigma(y,x)}$ *for all* $x,y \in \mathsf{S}$*. A subspace* $L \subseteq \mathsf{S}$ *is called Lagrangian subspace if* $L = L^\perp := \{x \in \mathsf{S} |\ \sigma(x,y) = 0\ \forall y \in L\}$*.*

For convenience we summarize the major results in the following theorem:

**2.2.13 Theorem** ((? ))**.** *Let* $\mathsf{H}$ *a separable Hilbert space and let* $A\colon \mathrm{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ *be a symmetric operator. Then the bilinear map*

$$\sigma(x,y) := (A^*x, y) - (x, A^*y)\,, \qquad \forall x,y \in \mathrm{dom}(A^*)\,, \tag{2.34}$$

*satisfies* $\sigma(x,y) = -\overline{\sigma(y,x)}$*. It also descends to the quotient space* $\mathsf{S}_A := \mathrm{dom}(A^*)/\mathrm{dom}(A)$ *and the pair* $(\mathsf{S}_A,\sigma)$ *is a complex symplectic space as per* *definition* **??**. *Moreover, for all Lagrangian subspace $L \subseteq \mathsf{S}_A$ – cf. definition* **??** *– the operator*

$$A_L := A^*|_{L+\mathrm{dom}(A)}\,, \tag{2.35}$$

*defines a self-adjoint extension of $A$ – here $L + \mathrm{dom}(A)$ denotes the pre-image of $L$ with respect to the projection $\mathrm{dom}(A^*) \to \mathsf{S}_A$. Finally the map*

$$\{\textit{Lagrangian subspaces } L \textit{ of } \mathsf{S}_A\} \ni L \mapsto A_L \in \{\textit{self-adjoint extensions of } A\}\,, \tag{2.36}$$

*is one-to-one.*

**2.2.14 Example.** As a concrete example of theorem **??** we discuss the case of the self-adjoint extensions of the curl operator on a closed manifold $\Sigma$ with interface $Z$. For simplicity we assume that $\dim\Sigma = 2k+1$ with $\dim\Sigma = 3$ mod 4, while curl is defined according to (**??**). We consider the operator $\mathrm{curl}_Z$ defined by

$$\mathrm{dom}(\mathrm{curl}_Z) := \overline{C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)}^{\|\|_{\mathrm{curl}}}\,,\qquad \mathrm{curl}_Z\, u := \mathrm{curl}\, u\,. \tag{2.37}$$

Notice that $C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z) = C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_+) \oplus C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_-)$. The adjoint $\mathrm{curl}_Z^*$ of $\mathrm{curl}_Z$ is

$$\mathrm{dom}(\mathrm{curl}_Z^*) = \mathrm{dom}(\mathrm{curl}_+) \oplus \mathrm{dom}(\mathrm{curl}_-)\,, \tag{2.38}$$

$$\mathrm{dom}(\mathrm{curl}_\pm) := \{u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)|\ \mathrm{curl}_\pm\, u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)\}\,,\qquad \mathrm{curl}_Z\, u := \mathrm{curl}\, u\,. \tag{2.39}$$

Since complex conjugation commutes with curl, it follows from Von Neumann's criterion (**?** , Thm. 5.43) that $\mathrm{curl}_Z$ admits self-adjoints extensions. We now provide a description of the complex symplectic space $\mathsf{S}_{\mathrm{curl}_Z} := (\mathrm{dom}(\mathrm{curl}_Z^*)/\,\mathrm{dom}(\mathrm{curl}_Z), \sigma_Z)$ whose Lagrangian subspaces allows to characterize all self-adjoint extensions of $\mathrm{curl}_Z$. According to theorem **??** the symplectic structure $\sigma_Z$ on the vector space $\mathsf{S}_{\mathrm{curl}_Z}$ is defined by

$$\sigma_Z(u,v) := (\mathrm{curl}_Z^*\, u, v) - (u, \mathrm{curl}_Z^*\, v)\,,\qquad \forall u,v \in \mathrm{dom}(\mathrm{curl}_Z^*)\,. \tag{2.40}$$

In particular for $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ and $v \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ we have

$$\sigma(u,v) = \sum_\pm \pm \int_Z \overline{\mathrm{t}_\pm u} \wedge \mathrm{t}_\pm v = \sum_\pm \mp -\frac{1}{2}\langle \mathrm{t}_\mp u, *_Z \mathrm{t}_\mp v\rangle_{\frac{1}{2}} \tag{2.41}$$

$$= (\gamma_1 u, \gamma_0 v) - (\gamma_0 u, \gamma_1 v)\,,\qquad \gamma_0 u := \frac{1}{\sqrt{2}} *_Z [\mathrm{t}u]\,,\qquad \gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)\,.\,, \tag{2.42}$$

where $-\frac{1}{2}\langle\ ,\ \rangle_{\frac{1}{2}}$ denotes the pairing between $\mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ and $\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$. In particular this shows that $\mathrm{t}_\pm u \in \mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ for all $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ – *cf.* (**? ? ? ? ?** ) for more details on the trace space associated with the curl-operator on a manifold with boundary. Provide more details.

According to theorem **??** all self-adjoint extensions of $\mathrm{curl}_Z$ are in one-to-one correspondence to the Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$. Unfortunately a complete characterization of all Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$ is not at disposal. We content ourself to present a family of Lagrangian subspaces – a generalization of the results presented in (**?** ) may provide other examples. For $\vartheta \in \mathbb{R}$ let

$$L_\vartheta := \{u \in \mathrm{dom}(\mathrm{curl}_Z^*)|\ \mathrm{t}_+ u = e^{i\vartheta}\mathrm{t}_- u\}\,, \tag{2.43}$$

where $[\mathrm{t}u] = \mathrm{t}_+u - \mathrm{t}_-u$ denotes the tangential jump – *cf.* definition **??**, remark **??** and equation (**??**). To show that $L_\vartheta$ are Lagrangian subspaces let $u, v \in L_\vartheta$ and let $v_n \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ be such that $\|v - v_n\|_{\mathrm{curl}} \to 0$. In particular $\|(\mathrm{t}_+ - e^{i\vartheta}\mathrm{t}_-)v\|_{\mathrm{H}^{\frac{1}{2}}\Omega^k(\Sigma_Z)} \to 0$ so that

$$\sigma_Z(u, v) = \lim_n \sigma_Z(u, v_n) = -\lim_n -\frac{1}{2}\langle \mathrm{t}_+u, *_Z(\mathrm{t}_+v_n - e^{i\vartheta}\mathrm{t}_-v_n)\rangle_{\frac{1}{2}} = 0\,. \tag{2.44}$$

It follows that $L_\vartheta \subseteq L_\vartheta^\perp$. Conversely if $u \in L_\vartheta^\perp$ let consider $v \in C_c^\infty\Omega^k(\Sigma_Z) \cap L_\vartheta$. Since $u \in L_\vartheta^\perp$ we find

$$0 = \sigma_Z(u, v) = --\frac{1}{2}\langle \mathrm{t}_+u - e^{i\vartheta}\mathrm{t}_-u, *_Z\mathrm{t}_+v\rangle_{\frac{1}{2}}\,.$$

Since $\mathrm{t}_+ \colon C_c^\infty\Omega^k(\Sigma_+) \to C_c^\infty\Omega^k(Z)$ is surjective it follows that $\mathrm{t}_+u = e^{i\vartheta}\mathrm{t}_-u$.

Notice that the self-adjoint extension obtained for $\vartheta = 0$ coincides with the closure of curl on $C_c^\infty(\Sigma)$ which is known to be self-adjoint by (**?** , Lem. 2.6). (Here we exploited the compactness property. We may deal with non-compact manifolds too because (**?** , Lem. 2.6) is based on point *(i)* of (**?** , Lem. 2.3) which holds true also in this setting.) Indeed, since [t] is continuous we have $\mathrm{dom}(\overline{\mathrm{curl}}) \subseteq L_0$ so that $\mathrm{curl}_{Z,L_0}$ is a self-adjoint extension of $\overline{\mathrm{curl}}$. Since the latter operator is already self-adjoint we have equality among the two.

We conclude this section by introducing an exact sequence which provides a complete description of the solution space of the Maxwell equations (**??**) with interface $Z$. For that the following standard definition is in order [Cit.]

**2.2.15 Definition.** *In the hypothesis of theorem* **??** *let* $\Theta \colon \mathsf{h} \to \mathsf{h}$ *be a self-adjoint operator and consider the self-adjoint extension* $H_\Theta$ *as per theorem* **??**

*and let* $\mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z) := \bigcap_{\ell\geq 1}\mathrm{dom}(H^{\ell}_{\Theta})$*. We define the following subspaces of* $C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$*:*

- *the space* $C^{\infty}_{\mathrm{sfc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$ *of future-compact functions, made of those* $f \in C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$ *such that* $J^-(x) \cap \mathrm{spt}(f)$ *is compact for all* $x \in M$*. Here* $J^-(x)$ *denotes the causal past of* $x \in M = \mathbb{R}\times\Sigma$ *according to the causal structure induced by* $g = -\mathrm{d}t^2 + h$*;*
- *the space* $C^{\infty}_{\mathrm{spc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$ *of past-compact functions, made of those* $f \in C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$ *such that* $J^+(x) \cap \mathrm{spt}(f)$ *is compact for all* $x \in M$*. Here* $J^+(x)$ *denotes the causal future of* $x \in M = \mathbb{R}\times\Sigma$ *according to the causal structure induced by* $g = -\mathrm{d}t^2 + h$*;*
- *the space of timelike-compact functions defined by*
$$C^{\infty}_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) = C^{\infty}_{\mathrm{sfc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) \cap C^{\infty}_{\mathrm{spc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))\,.$$

(Perhaps it is worth to recall somewhere the notion of advanced-retarded fundamental solutions?)

**2.2.16 Proposition.** *Let* $(\Sigma, h)$ *be a closed manifold with interface* $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$*. Moreover let* $(\Sigma_{\pm}, h_{\pm})$ *the oriented, compact Riemannian manifolds with boundary* $\partial\Sigma_{\pm} = \pm Z$ *such that* $\Sigma_Z := \Sigma\setminus Z = \Sigma_+ \cup \Sigma_-$*. Let* $H$ *be the densely defined operator on* $\mathrm{L}^2\Omega^1(\Sigma)$ *with domain defined by* (**??**) *and let* $H^*$ *be its adjoint, defined as in* (**??**)*. Let* $(\mathsf{h}, \gamma_0, \gamma_1)$ *be the boundary triple associated with* $H^*$ *as described in proposition* **??** *and let* $\Theta\colon \mathsf{h}\to\mathsf{h}$ *be a self-adjoint operator and consider the self-adjoint extension* $H_{\Theta}$ *as per theorem* **??***. Let* $G^{\pm}_{\Theta}$ *be the operators* $G^{\pm}_{\Theta}\colon C^{\infty}_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) \to C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))$ *defined by*

$$(G^{\pm}_{\Theta}\omega)(t) = \int_{\mathbb{R}} \vartheta(\pm(t-s))e^{-i(t-s)H_{\Theta}}\omega(s)\mathrm{d}s\,. \tag{2.45}$$

*The the operator* $G^{+}_{\Theta}$ *(resp.* $G^{-}_{\Theta}$*) is an advanced (resp. retarded) solution of* $i\partial_t + H_{\Theta}$*, that is, it holds*

$$(i\partial_t + H_{\Theta})\circ G^{\pm}_{\Theta}|_{C^{\infty}_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))} = \mathrm{Id}_{C^{\infty}_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))}\,, \tag{2.46}$$
$$G^{\pm}_{\Theta}\circ(i\partial_t + H_{\Theta})|_{C^{\infty}_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))} = \mathrm{Id}_{C^{\infty}_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))}\,. \tag{2.47}$$

*Moreover, let* $G_{\Theta} := G^{+}_{\Theta} - G^{-}_{\Theta}$*. Then the following is a short exact sequence*

$$\begin{aligned}0 \to C^{\infty}_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) &\overset{i\partial_t + H_{\Theta}}{\to} C^{\infty}_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z))\\ &\overset{G_{\Theta}}{\to} C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) \overset{i\partial_t + H_{\Theta}}{\to} C^{\infty}(\mathbb{R}, \mathrm{H}^{\infty}_{\Theta}\Omega^1(\Sigma_Z)) \to 0\,.\end{aligned} \tag{2.48}$$

*Proof.* Most of it is an analogue of (**?** , Thm. 30- Prop. 36). The finite speed of propagation follows from (**? ?** ). ■

**2.2.17 Remark.** Notice that the exact sequence (**??**) implies that the space of smooth solution of the dynamical equations (**??**) is isomorphic as a vector space to the image of $G_\Theta$.

# List of Figures

# Bibliography


[1] Jost, J. *Riemannian Geometry and Geometric Analysis*, Springer, 2011.

[2] Bär, C., Ginoux, N., Pfäffle, F. *Wave equations on Lorentzian Manifolds and quantization*, EMS Publishing House, Zurich, 2007.

[3] Bär, *Geometric wave equations*, Potsdam, 2017.

[4] Pfäffle,F. *Lorentzian Manifolds*, Lect. Notes Phys. **786**, 2009.

[5] Ginoux, N. *Linear Wave equations*, Lect. Notes Phys. **786**, 2009.

[6] Alesker, S., Fu, J.H.G., *Integral Geometry and Valuations* Springer, 2014

[7] Günther, Paul, *Huygens' principle and hyperbolic equations* Academic Press, Boston, 1988

[8] Friedlander, F., *The wave equation on a curved space-time*, Cambridge University Press, Cambridge, 1975

[9] Friedlander, F., Joshi, M., *Introduction to the theory of distributions (second edition)*, Cambridge University Press, Cambridge, 1998

[10] Jonsson, T., Yngvason, J., *Waves and distributions* World Scientific, 1995

[11] Hack, T.P., *Cosmological Applications of Algebraic Quantum Field Theory in CurvedSpacetimes*, SpringerBriefs in Mathematical Physics 6, 2015

# Acknowledgements

Ebbene sì, sono riuscito a scrivere correttamente la parola *acknowledgements*, ma questo non implica che io abbia la voglia di scriverli in inglese. Infatti non lo farò.
Solitamente, giunti alla fine del corso di laurea, si ringraziano parenti, amici, professori. Ricordarli e ringraziarli tutti sarebbe impossibile, anche perchè in questi tre anni credo di averne passate talmente tante insieme a tante persone, che la mia povera testolina ha dovuto rimuoverne la maggior parte per fare spazio alle ben meno importanti formule matematiche.

Cerchiamo di mettere ordine. Rubens Longhi approda a Pavia nel lontano 2015, dopo un'estate di delusioni da test d'ingresso falliti. Maledetto sia Edipo. A sostenermi in questo momento difficile c'è la decennale amica Michela, con la quale divido un appartamento per il primo anno di università. Durante quell'anno molti sono i nuovi amici, sebbene il tornare a casa ogni finesettimana mi permette di mantenere vive le amicizie di sempre.
L'inizio non è facile. Ricordo i primi giorni a fisica: io, Pozzo e Mondini emarginati in un angolo, che tentavamo di seguire il Gila senza troppo interagire con il restante centinaio di persone, tra le quali si potevano individuare quegli 8 disagiati del Ghislieri. Mamma mia quanto se la tiravano. Soprattutto un terroncello di nome Angelo e quell'altro torrone di Magoni che, dopo aver affrontato con me i test d'ingresso, mi aveva abbandonato come si fa coi cani in tangenziale. A proposito di Cane, il mio primo ricordo di Amodio Carleo si colloca in una lezione di Algebra lineare durante la quale Carleo Amodio (ma qual'è il nome?) sviene platealmente a seguito di bravate della notte precedente che gli hanno lasciato escoriazioni sulle ginocchia; anche se detto così suona male. (By the way, Pirola andò nel pallone e chiese ad un'aula gremita di fisici e matematici se ci fosse un dottore). Inizio quindi a chiedermi cosa mai faranno in quel collegio per giungere al punto di svenire a lezione. Trovai solo molto più tardi la risposta.
Dopo pochi giorni entro in contatto con le persone che sarebbero stati al centro

del mio primo anno qui: Ale Triple, Monte, Nicole, Bonfo, Vix, Piazza e poi Bressi. I mercoledì sera con loro erano diventati una tappa obbligata della settimana pavese. Benedetto fu l'ingresso gratis al Camillo.
L'amicizia con Ale Triacca si rivela importante: accomunati dal fatto di essere dei morti di figa, durante il primo anno facciamo strage di ragazzine al Camillo. Cioè, lui fa strage e io finisce che bevo e ballo in maniera ridicola (sì, avete presente come ballo).
Il martedì, invece, diventa il giorno del vino di dubbia qualità che Berti portava nel nostro appartamento nell'evidente tentativo di far ubriacare la Michela e poi sedurla. Ma la Michela, almeno ai tempi, non beveva. Inoltre il gruppo di Amaldini fuori sede a Pavia, tra cui Roby, Nicole, Losa e Pozzo si organizza con serate a tema *panterona* (gli interessati sanno di chi sto parlando) nelle quali io e Pozzo diamo del nostro peggio.
Gli esami del primo anno (a parte Chimica, che fa cagare) vanno abbastanza bene da indurmi a fare la follia: spinto anche dai consigli della Marveggio (con la quale solevo andare a bere una birra dopo ogni esame nell'evidente tentativo di sedurla; ma anche lei, ai tempi, non beveva), decido di rifare il test d'ingresso per il Ghislieri.
Stavolta l'esito è positivo: sia benedetto Rotondi che riciclò gli esercizi dell'esame di Meccanica per la prova Iuss. In quel momento tante cose cambiano. Il legame con i miei compagni d'anno collegiali diventa molto forte, anche grazie al kulo, e mi fa quasi dimenticare di essere un anno in ritardo. La mia vita viene completamente assorbita dal collegio, tanto che i periodi che passo a casa iniziano a ridursi davvero al minimo sindacale. Delle amicizie bergamasche si salvano solo quella immortale con mio cugino Daniel e quelle con i miei compagni del liceo, specialmente Pizzo, Giordano, Giulia, Giovanni, Menni e, quando c'è da far serata ignorante, anche Lanceni. Le cene con Berti, non essendo più in appartamento, deficitano della presenza della Michela e quindi lui è costretto a tentare di sedurre me.
La mia vita diventa un tuttuno con quella del mio vicino di stanza Djerme, giovane padawan nell'arte dell'analisi. Il fortissimo legame che si è creato tra di noi mi ha permesso di superare molte delle difficoltà del secondo anno: specialmente il maledetto Elettro 2. Il caffè di Allegria diventa il covo di fisici e matematici come Martina, Zio Sam, Vitto, Mago, Cane, Nino e anche di altri futuri disoccupati come Maria Ganja, Francesca (quella di Nino), Cazzato e Virginia.
Lì scopro che quel terroncello cazzone di Angelo possiede, inspiegabilmente, un cervello e che esso lavora e impara le cose esattamente come lo fa il mio: ovvero completamente a caso, senza una particolare tecnica mnemonica e senza uno studio sistematico. Questo disordine mentale comune ci porta ad

affrontare molti esami insieme e a condividere mille momenti anche sui social, fino al punto da diventare decisamente rompicoglioni (questa cosa mi mancherà parecchio).
I corsi del secondo anno sono decisamente stimolanti, anche e soprattutto perchè non li frequento, dato che chi è in collegio sa bene che nei primi mesi svegliarsi in tempo al mattino non è sempre facile. Per fortuna (o per sfortuna) le esercitazioni di Meccanica Razionale, tenute da tale Claudio Dappiaggi, sono al pomeriggio. Di quelle lezioni mi è rimasta la frase *Non crediate che i vostri professori bevano meno di voi*, la quale mi ha fatto capire di essermi ben instradato nel ruolo del fisico teorico. Al secondo anno incontriamo il Dappia in ben due esami e già lì iniziamo a pensare che ci perseguiti.
Il viaggio al *Cern*, per quanto stimolante, non mi fa recedere dai miei insani propositi teorici, anche se la coppia di prof sperimentaloni Negri+Rebuzzi ci fa vivere emozioni incredibili, come quando la Rebu ha detto a Nino di buttarsi dal trampolino nel lago di Ginevra, e lui ha eseguito.
Pian piano si giunge al terzo anno, che inizia con il prevedibile fervore di noi fagioli per l'arrivo delle nuove matricole. Principalmente il mio interesse è rivolto a torturare le matricole con quesiti impossibili di analisi matematica, ma la situazione cambia decisamente quando, in preda ad un attacco di socializzazione acuta conosco Francesca (non quella di Nino). Le circostanze del nostro primo vero incontro sono decisamente imbarazzanti. La sua cecità la spinge alla cleptomania nei confronti dei miei occhiali, alla quale io reagisco insultando la sua terribile facoltà. Dopo nove mesi insieme, lei ancora mi ruba gli occhiali e io ancora la insulto perchè fa Chimica.
Reincontriamo di nuovo il Dappia, che tiene per la prima volta a noi un corso al terzo anno. Come se non bastasse, ce lo ritroviamo in un corso Iuss e persino in birreria al sabato sera. A questo punto pensiamo al complotto.
Procede tutto bene fino al momento in cui scopro che Nino mi ha battuto sul tempo: ha già chiesto lui la tesi al Dappia. Maledetto. Di reazione, la chiedo anche io al Dappia, che mi affida ad uno dei suoi scagnozzi Nicolò. Al chè, Nino decide di svendersi agli sperimentali, prima rimanendo invischiato in una tesi sperimentale sull'inutile massa del bosone chicchessia e poi spostandosi sulla fenomenologia della forza debole. Perchè Nino è un debole. Fattosta che a fine maggio mi rimangono ancora cinque esami da dare (eh si, quello con la Rimoldi per me è valso doppio) e della tesi ho scritto a malapena il primo capitolo. Ma l'importante, a questo punto, è salvare la faccia. Tutti i miei compagni se ne vanno verso altri, floridi, lidi e io manco sono capace di laurearmi a luglio. All'alba del 20 giugno, con ancora un esame da 12 crediti da dare e a sole tre settimane dalla consegna della tesi, decido di fare la follia. Chissà quante maledizioni mi hanno tirato il Dappia e Nicolò quel giorno. Ma

se sono qui ora, vuol dire che ce l'ho fatta.

Più che ringraziamenti, ho molte scuse da fare. Mi scuso per aver trascurato in questi ultimi tempi tante persone che si meritavano molto di più da parte mia. Ho trascurato la mia famiglia, non tornando a casa quasi mai. Ho trascurato molti di quegli amici che ho citato prima. Ho trascurato la mia salute, mentale e fisica.
Ora è tutto finito, ma se pensate che mi dedicherò a voi fin da subito, vi sbagliate di grosso. Ora ho solo voglia di andare al mare. Ci vediamo a settembre, *bitches*!