University of Pavia
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TESI TRIENNALE

On the fundamental solutions for wave-like equations on curved backgrounds

$$\Box u = \delta$$

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Preface

To do

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Abstract

To do

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Mathematical Tools 1

1.1 An overview of Differential Geometry

We shall begin with a recall of very well known definitions in order to introduce the basic geometrical objects which are used in the text.

A **manifold** is essentially a space that is locally similar to $\mathbb{R}^n$. To define it we use the concepts of **topological space** and of **homeomorphism**.

1.1.1 Definition (**Topological Space**). *A set* X *together with a family* $\mathcal{T}$ *(**topology**) of subset of* X *is called a **topological space** if the following are satisfied:*

a. $\emptyset, \mathrm{X} \in \mathcal{T}$,

b. *for all* U *and* $V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$,

c. *for any index set* A, *if* $U_i \in \mathcal{T}$ *for all* $i \in A$, $\bigcup_{i \in A} U_i \in \mathcal{T}$.

A set in $\mathcal{T}$ *is called **open**. If a point* p *is in an open* U*, we call* U *a **neighborhood** of* p.

1.1.2 Definition (**Continuity and homomorphism**). *Let* X *and* Y *be two topological spaces. A function* $f : \mathrm{X} \to \mathrm{Y}$ *is **continuous** if for any open set* U *of* Y*, the preimage* $f^{-1}(U)$ *is an open set of* X.

A continuous and bijective map $\varphi : \mathrm{X} \to \mathrm{Y}$ is an ***homomorphism*** *if $\varphi^{-1} : \mathrm{Y} \to \mathrm{X}$ is also continuous.*

As in vectorial space, we can talk about **basis** of topological spaces. A subset $\mathcal{B} \subset \mathcal{T}$ is a basis if any open can be expressed as union of elements of $\mathcal{B}$. A topology is **Haussdorf** if for any two distinct points $p, q \in \mathrm{X}$, there exist two open neighborhoods U of p and V of q such that $U \cap V = \emptyset$.

A topological space X is called **compact** if each of its open covers has a finite subcover, i.e. for any collection $\{U_i\}_{i \in A}$, (where A is a set of indexes) such that

$$\mathrm{X} \subseteq \bigcup_{i \in A} U_i,$$

there is a finite subset A' of A such that

$$\mathrm{X} \subseteq \bigcup_{i \in A'} U_i.$$

We are now ready to introduce the concept of **manifold**.

1.1.3 Definition. *An n-dimensional* ***manifold*** *M is a topological Haussdorf space (with a countable basis) that is locally homeomorphic to $\mathbb{R}^n$, i.e. for every $p \in \mathrm{M}$ there exists an open neighbourhood U of p and a homeomorphism*

$$\varphi : U \to \varphi(U)$$

such that $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

Such a homeomorphism is called a **(local) coordinate chart** of M. An **atlas** of M is a family $\{U_i, \varphi_i\}_{i \in A}$ of local charts together with an open covering of M, i.e. $\bigcup_{i \in A} U_i = \mathrm{M}$.

1.1.4 Definition. *A* ***differentiable atlas*** *of a manifold M is an atlas $\{U_i, \varphi_i\}_{i \in A}$ such that the function*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

called ***chart transition****, is differentiable (of class C^∞) for any $i, j \in A$ such that $U_i \cap U_j \neq \emptyset$.*

With this definition, such chart transitions are **diffeomorphisms** because you can always interchange the indexes i and j.

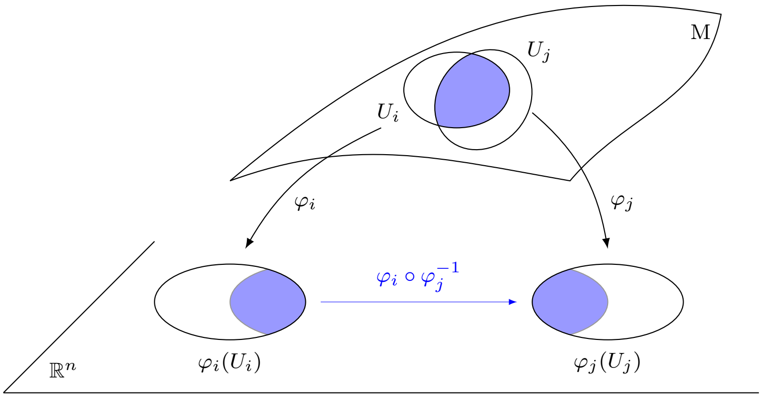


FIGURE 1.1: A differentiable atlas on a manifold M.

We are only interested in **differentiable** (or **smooth**) **manifolds**, which are manifolds together with a maximal differentiable atlas. Here maximality of the atlas means that if φ is a chart of M and $\{U_i, \varphi_i\}_{i\in A}$ is a differentiable atlas, then φ itself also belongs to $\{U_i, \varphi_i\}_{i\in A}$. We call a differentiable manifold with an atlas for which all chart transitions have positive jacobian determinant an **orientable manifold**.

1.1.5 Convention. *For now on the word* ***manifold*** *will always mean* ***differentiable manifold****.*

1.1.6 Definition (**Submanifold**)**.** *Let $n \leq m$. An n-dimensional* ***submanifold*** N *of an m-dimensional manifold* M *is a nonempty subset* N *of* M *such that for every point $q \in \mathrm{N}$ there exists a local chart $\{U, \varphi\}$ of* M *about q with*

$$\varphi(U \cap \mathrm{N}) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^m.$$

In case $n = m - 1$ we call N *an* ***hypersurface*** *of* M*.*

1.1.7 Example. If M and N are manifolds, the cartesian product $\mathrm{M} \times \mathrm{N}$ is also a manifold. If $\{U_i, \varphi_i\}_{i\in A}$ is an differentiable atlas for M and $\{V_j, \psi_j\}_{j\in B}$ ia an atlas for N, then $\big\{U_i \times V_j, (\varphi_i, \psi_j)\big\}_{(i,j)\in A\times B}$ is a differentiable atlas for $\mathrm{M} \times \mathrm{N}$. □

As in the euclidean case, we have the concept of **differentiable** map between manifolds:

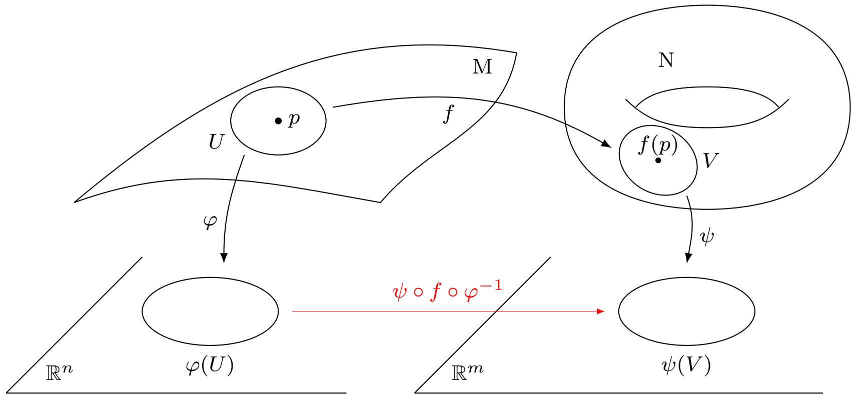


FIGURE 1.2: The concept of differentiable map f between manifolds M and N.

1.1.8 Definition. *A continuous map $f : \mathrm{M} \to \mathrm{N}$ between manifolds* M *and* N *is* ***differentiable*** *in $p \in \mathrm{M}$ if there exist local charts $\{U, \varphi\}$ and $\{V, \psi\}$ about p in* M *and about $f(p)$ in* N *respectively, such that $f(U) \subset V$ and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is differentiable (of class C^∞) at $\varphi(p)$. The function f is said to be differentiable on M *if it is differentiable at every point of* M.

The space of differentiable functions between two manifold is denoted by $C^\infty(\mathrm{M}, \mathrm{N})$, and if $\mathrm{N} = \mathbb{R}$ we denote it by $C^\infty(\mathrm{M})$.

We now introduce the **tangent space** of a point of a manifold. It may be thought as a local approximation of the $\mathbb{R}^n$ structure that lies on the manifold. We will try to construct it using the derivatives of curves which passes through the point, because when working with manifolds we have to think in a coordinate independent way.

1.1.9 Definition (**Tangent space**)**.** *Let* M *be a manifold and $p \in \mathrm{M}$. We indicate $\mathcal{C}_p = \{c : I \to \mathrm{M} \text{ diff.} \mid 0 \in I \ \wedge \ c(0) = p\}$ the set of differentiable curves passing through p.*
We declare equivalent ($\sim$) two curves $c_1, c_2 \in \mathcal{C}_p$ if there exist a local chart φ about p such that $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$.
The ***tangent space*** *of* M *at p is the set*

$$\mathrm{T}_p\mathrm{M} := \mathcal{C}_p / \sim$$

i.e. the quotient of $\mathcal{C}_p$ with the relation of equivalence just defined.

We will refer at the equivalence classes of curves just with the dotted letter $\dot{c}$ (instead of using the more correct notation $[c]$), to indicate we must think of it as being a sort of **velocity**.

One checks that the definition of the equivalence relation does not depend on the choice of local chart. In fact, if $\{U, \varphi\}$ and $\{V, \psi\}$ are local chart at p,

$$(\varphi \circ c)'(0) = (\varphi \circ \psi^{-1} \circ \psi \circ c)'(0) = D(\varphi \circ \psi^{-1})(\psi(p)) \cdot (\psi \circ c)'(0)$$

if $D(\varphi \circ \psi^{-1})(\psi(p))$ stands for the jacobian matrix of the chart transition calculated at $\psi(p)$. Then it is clear that $(\varphi \circ c_1)'(0)$ and $(\varphi \circ c_2)'(0)$ coincide iff $(\psi \circ c_1)'(0)$ and $(\psi \circ c_2)'(0)$ coincide.

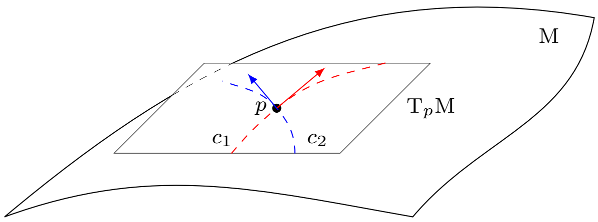


FIGURE 1.3: Tangent space $\mathrm{T}_p\mathrm{M}$ where $c_1 \sim c_2$.

It can be proven that (for a fixed atlas φ about p) the following map is a linear isomorphism between $\mathrm{T}_p\mathrm{M}$ and $\mathbb{R}^n$:

$$\begin{aligned} \Theta_\varphi : \ \mathrm{T}_p\mathrm{M} &\to \mathbb{R}^n, \\ \dot{c} &\mapsto (\varphi \circ c)'(0). \end{aligned}$$

Hence we can think about $\mathrm{T}_p\mathrm{M}$ as being a copy of $\mathbb{R}^n$ attached to the point p on the manifold.
For reasons that we will see later, we denote the basis of $\mathrm{T}_p\mathrm{M}$ as

$$\left\{ \frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p) \right\}.$$

Collecting all the tangent spaces we have the **tangent bundle** of a manifold M, defined as the disjoint union of them:

$$\mathrm{TM} = \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M} = \bigcup_{p \in \mathrm{M}} \{p\} \times \mathrm{T}_p\mathrm{M}.$$

1.1.10 Definition. *Let $f : \mathrm{M} \to \mathrm{N}$ be a differentiable map between manifolds and let $p \in \mathrm{M}$. The **differential** of f at p is the linear map*

$$\mathrm{d}_p f : \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{N}, \quad \dot{c} \mapsto (f \circ c)^{\cdot} \cong (f \circ c)'(0).$$

*The **differential** of f is the map $\mathrm{d}f : \mathrm{TM} \to \mathrm{TN}$ such that $\mathrm{d}f|_{\mathrm{T}_p\mathrm{M}} = \mathrm{d}_p f$.*

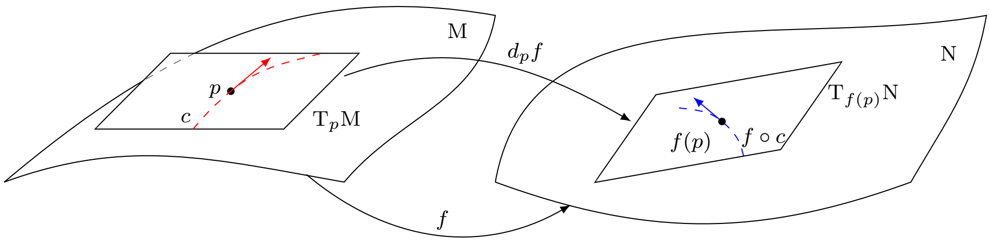


FIGURE 1.4: A scheme of differential map.

Given a differentiable manifold M and some chart $\{U, \varphi = (x^1, \dots, x^n)\}$ near p, we set $X = \dot{c} \in \mathrm{T}_p\mathrm{M}$. If we identify $\mathrm{T}_p\mathbb{R} \cong \mathbb{R}$, we can interpret the differential $\mathrm{d}_p f(X)$ of a function $f \in C^\infty(\mathrm{M})$ at a point p as the **derivative** in the direction of X:

$$\partial_X f(p) := \mathrm{d}_p f(X).$$

A functional which is linear and follows Leibniz's rule, such as $\partial_X : C^\infty(\mathrm{M}) \to \mathbb{R}$, is called a **derivation**. The set of all derivations at p is denoted as Der_p and it is a vector space. It is clear that the map $X \in \mathrm{T}_p\mathrm{M} \mapsto \partial_X$ is an isomorphism between $\mathrm{T}_p\mathrm{M}$ and Der_p.
We define the map:

$$\frac{\partial}{\partial x^i}(p) : C^\infty(\mathrm{M}) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x^i}(p) = \partial_X f(p),$$

where $X = \dot{c}$ and $c(t) = \varphi^{-1}(\varphi(p) + te_i)$ (e_i is the basis vector).
Note that, from the definition of the differential, it holds

$$\partial_X f(p) = (f \circ c)'(0) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)),$$

which shows that the object we defined can be seen as a partial derivative in the usual sense.
It can be shown that the set of derivations $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ form a basis for

$$\begin{tikzcd}
 & \mathrm{T}_p\mathrm{M} \arrow[dl, "X\mapsto\partial_X"'] \arrow[dr, "\Theta_p"] & \\
\mathrm{Der}_p & & \mathbb{R}^n
\end{tikzcd}$$

FIGURE 1.5: Isomorphism relations for the tangent space.

Der_p and, due to the isomorphism, for $\mathrm{T}_p\mathrm{M}$. It is now clear why the generic tangent vector X can be expressed as

$$X = X^i \frac{\partial}{\partial x^i},$$

with Einstein convention on the sum.
From linearity we have the useful formula

$$\partial_X f(p) = \mathrm{d}_p f(X) = X^i \mathrm{d}_p f\left(\frac{\partial}{\partial x^i}(p)\right) = X^i \frac{\partial f}{\partial x^i}(p).$$

1.1.11 Definition. *Let* M *be a manifold, we define a projection map* $\pi : \mathrm{TM} \to \mathrm{M}$ *such that* $\pi(\mathrm{T}_p\mathrm{M}) = p$, *and we call a* ***section*** *in the tangent bundle a map* $s : \mathrm{M} \to \mathrm{TM}$ *such that*

$$\pi \circ s = \mathrm{id}_{\mathrm{M}}\,.$$

The dual space of the tangent space $\mathrm{T}_p\mathrm{M}$ *is called the* ***cotangent space****, denoted with* $\mathrm{T}_p^*\mathrm{M}$*, which has a basis denoted with* $\{\mathrm{d}x^1(p), \dots, \mathrm{d}x^n(p)\}$*. Similarly is defined the* ***cotangent bundle*** $\mathrm{T}^*\mathrm{M}$ *as the disjoint union of cotangent spaces.*
Sections in the tangent bundle, denoted by $C^\infty(\mathrm{M}, \mathrm{TM})$*, are called* ***vector fields*** *and sections in the cotangent bundle are called* 1***-forms****.*

Vector fields are expressed in terms of combinations of

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\} =: \{\partial_1, \dots, \partial_n\}\,,$$

whereas 1-forms are expressed as combination of

$$\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}.$$

1.1.12 Definition. *We define the derivative in the direction of X as an operator $\partial_X : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ such that*

$$\partial_X f = \mathrm{d}f(X),$$

for any vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$.

It follows immediately that Leibniz's rule holds: $\partial_X(f \cdot g) = g\,\partial_X f + f\,\partial_X g$, and again holds the useful formula

$$\partial_X f = \mathrm{d}f(X) = X^i \mathrm{d}f(\partial_i) = X^i \frac{\partial f}{\partial x^i}.$$

1.1.13 Observation. Given two vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$, there is a unique vector field $[X, Y] \in C^\infty(\mathrm{M}, \mathrm{TM})$ such that

$$\partial_{[X,Y]} f = \partial_X \partial_Y - \partial_Y \partial_X f$$

for all $f \in C^\infty(\mathrm{M})$. The map $[\cdot, \cdot] : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) \to C^\infty(\mathrm{M}, \mathrm{TM})$ is called the **Lie bracket**, it is bilinear, skew-symmetric and satisfies the well known *Jacobi identity.* □

1.1.14 Definition. *An* ***affine connection*** *or* ***covariant derivative*** *on a manifold* M *is a bilinear map*

$$\begin{aligned} \nabla : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) &\to C^\infty(\mathrm{M}, \mathrm{TM}) \\ (X, Y) &\mapsto \nabla_X Y\,, \end{aligned}$$

such that for all smooth functions $f \in C^\infty(\mathrm{M})$ and all vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$:

- $\nabla_{fX} Y = f \nabla_X Y$, *i.e., ∇ is $C^\infty(\mathrm{M})-$linear in the first variable;*
- $\nabla_X(fY) = \partial_X f + f \nabla_X Y$, *i.e., ∇ satisfies* Leibniz rule *in the second variable.*

The covariant derivative on the direction of the basis vector fields $\{\partial_1, \dots, \partial_n\}$ is indicated

$$\nabla_j := \nabla_{\partial_j}.$$

We are now ready to introduce metric structures on manifolds.

1.2 Lorentzian Manifolds

We start in the simple case of **Minkowski spacetime**.

1.2.1 Definition. *Let V be an n-dimensional vector space. A **Lorentzian scalar product** is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ with signature $(- + \cdots +)$, i.e. such that one can find a basis $\{e_1, \ldots, e_n\}$ such that*

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_i, e_j \rangle = 1 \ \text{ if } i = j = 2, \ldots, n \ \text{ and } \langle e_i, e_j \rangle = 0 \text{ otherwise.}$$

The **Minkowski product** $\langle x, y \rangle_0$, defined by the formula (with Einstein convention)

$$\langle x, y \rangle_0 = \eta_{ik} x^i y^k = -x_1\, y_1 + x_2\, y_2 + \cdots + x_n\, y_n,$$

with $\eta := \mathrm{diag}(-1, 1, \ldots, 1, 1)$ is the simplest (and the only) example of Lorentzian scalar product on $\mathbb{R}^n$. The n-dimensional Minkowski space, denoted by M^n is simply $\mathbb{R}^n$ equipped with Minkowski product.

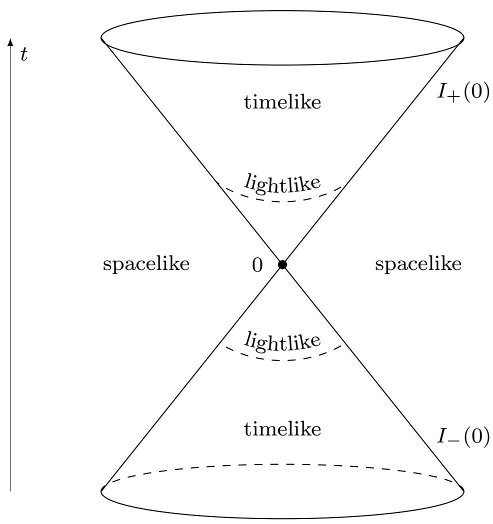


Figure 1.6: Minkowski time orientation.

1.2.2 Definition. *We call the **negative square length** of a vector $X \in V$ the quantity*

$$\gamma(X) = -\|X\|^2 := -\langle X, X \rangle .$$

A vector $X \in V \setminus \{0\}$ is called

- ***timelike*** *if* $\gamma(X) > 0$,
- ***lightlike*** *if* $\gamma(X) = 0$,
- ***spacelike*** *if* $\gamma(X) < 0$ *or* $X = 0$,
- ***causal*** *if it is timelike or lightlike.*

This definition will mostly be used for tangent vectors, in case V is the tangent space of a Lorentzian manifold at some point.

For $n \geq 2$ the set of timelike vectors $I(0)$ consists of two connected components. A **time orientation** is the choice of one of these two components. We name our choice $I_+(0)$ and call its elements **future-directed**.

1.2.3 Definition. *We put*

- $J_+(0) := \overline{I_+(0)}$ *(elements are called* ***future-directed****),*
- $C_+(0) := \partial I_+(0)$ *(upper* ***light cone****),*
- $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$ *(elements are called* ***past-directed****),*
- $C_-(0) := -C_+(0)$ *(lower* ***light cone****).*

1.2.4 Definition. *A* ***metric*** g *on a manifold* M *is given by a scalar product on each tangent space*

$$g : \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$$

which depends smoothly on the base point p. *We call it a* ***Riemannian metric*** *if the scalar product is pointwise positive definite, and* ***Lorentzian metric*** *if it is a Lorentzian scalar product.*
A pair (M, g), *where* M *is a manifold and* g *is a Lorentzian (Riemannian) metric is called a* ***Lorentzian (Riemannian) manifold****.*

The request of smooth dependence on p may be specified as follows: given any chart $\{U, \varphi = (x^1, \ldots, x^n)\}$ about p, the functions $g_{ij} : \varphi(U) \to \mathbb{R}$ defined by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, for any $i, j = 1, \ldots, n$ should be differentiable. W.r.t. these coordinates one writes

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j = \sum_{i,j} g_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j.$$

The scalar product of two tangent vectors $v, w \in \mathrm{T}_p\mathrm{M}$, with coordinate chart $\varphi = (x^1, \dots x^n)$, such that $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$ then is

$$\langle v, w \rangle = g_{ij}(\varphi(p)) v^i w^j.$$

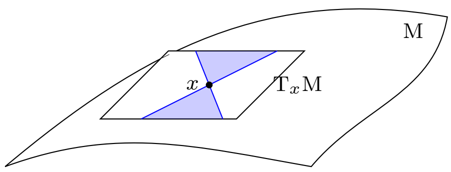
When the choice of the chart is clear we will often write, with abuse of notation $g_{ij}(p) := g_{ij}(\varphi(p))$. We will indicate $g^{ij} := (g^{-1})_{ij}$.

From now on M will always indicate a **Lorentzian manifold**.

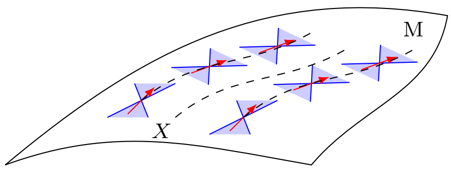
1.2.5 Definition. *A vector field* $X \in C^\infty(\mathrm{M}, \mathrm{TM})$ *is called timelike, spacelike, lightlike or causal, if* $X(p)$ *is timelike, spacelike, lightlike or causal, respectively, at every point* $p \in \mathrm{M}$.

A differentiable curve $c : I \to \mathrm{M}$ *is called timelike, lightlike, spacelike, causal, future-directed or past-directed if* $\dot{c}(t) \in \mathrm{T}_{c(t)}\mathrm{M}$ *is, for all* $t \in I$, *timelike, lightlike, spacelike, causal, future-directed or past-directed, respectively.*

A Lorentzian manifold M *is called* ***time-oriented*** *of there exists a timelike vector field on* M. *If a manifold is time-oriented, we refer to it as* ***spacetime***.



(a) A time-oriented tangent space.



(b) A time-oriented manifold together with field lines of a timelike vector field X.

FIGURE 1.7: Time orientations.

The **causality relations** on M are defined as follows. Let $p, q \in \mathrm{M}$,

- $p \ll q$ iff there is a future-directed timelike curve from p to q,

- $p < q$ iff there is a future-directed causal curve from p to q,
- $p \leq q$ iff $p < q$ or $p = q$.

The causality relation "$<$" is a strict weak ordering and the relation "$\leq$" make the manifold a partially ordered set.

1.2.6 Definition. *The* ***chronological future*** *of a point $x \in \mathrm{M}$ is the set $I_+^M(x)$ of points that can be reached by future-directed timelike curves, i.e.*

$$I_+^M(x) = \{y \in \mathrm{M} \mid x < y\}.$$

The ***causal future*** *$J_+^M(x)$ of a point $x \in \mathrm{M}$ is the set of points that can be reached by future-directed causal curves from x, i.e.,*

$$J_+^M(x) = \{y \in \mathrm{M} \mid x \leq y\}.$$

Given a subset $A \subset \mathrm{M}$ the ***chronological future*** and *the* ***causal future*** *of A are respectively*

$$I_+^M(A) = \bigcup_{x \in A} I_+^M(x), \qquad J_+^M(A) = \bigcup_{x \in A} J_+^M(x).$$

In a similar way, one defines the ***chronological*** and ***causal pasts*** *of a point x or a subset $A \subset \mathrm{M}$ by replacing future-directed curves by past directed curves. They are denoted by $I_-^M(x), I_-^M(A), J_-^M(x)$, and $J_-^M(A)$, respectively. We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$.*

1.2.7 Definition. *A connected open subset $\Omega \subset \mathrm{M}$ of a spacetime is called* ***causally compatible*** *if for any point $x \in \Omega$ holds*

$$J_\pm^\Omega(x) = J_\pm^{\mathrm{M}}(x) \cap \Omega,$$

where it is clear that the inclusion "$\subset$" always holds.

We now can introduce the concept of **geodesics** and **exponential map**.

1.2.8 Definition. *Let $c : [a, b] \to \mathrm{M}$ be a curve on a Lorentzian manifold* M. *The length $L[c]$ is defined by (with Einstein convention on sum)*

$$L[c] = \int_a^b \sqrt{|g(\dot{c}(t), \dot{c}(t))|}\, dt = \int_a^b \sqrt{\left| g_{ik}(c(t)) \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} \right|}\, dt,$$

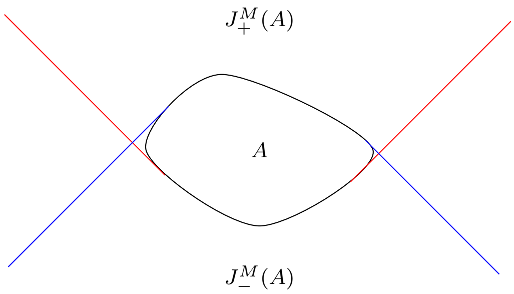


FIGURE 1.8: Causal future J_+^M and causal past J_-^M of a subset $A \subset \mathrm{M}$.

where $x^i(t) := (\varphi \circ c)^i(t)$ are the coordinates of the point $c(t)$ on a chart φ. Given $p, q \in \mathrm{M}$, if $p \leq q$ we define the ***time-separation*** *between p and q as*

$$\tau(p,q) = \sup\{L[c] \mid c \text{ is a future directed causal curve from } p \text{ to } q\}$$

and 0 otherwise.
A ***geodesic*** *between two points $p, q \in \mathrm{M}$ such that $p \leq q$, if it exists, is a curve c such that $L[c] = \tau(p,q)$, i.e. the curve of maximum time-separation.*

The request on the geodesics implies that (in variational sense) $\delta L[c] = 0$. It can be demonstrated that the stationary problem for the functional $L[c]$ is totally equivalent to $\delta E[c] = 0$ for the functional, called **energy**, defined by

$$E[c] = \frac{1}{2}\int_a^b |g(\dot{c}(t), \dot{c}(t))|\, dt.$$

Hence, the Euler-Lagrange equations for the stationarity of a functional $I[c] = \int_a^b f(t, c(t), \dot{c}(t))\, dt$ read

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial f}{\partial \dot{x}^i}\right) - \frac{\partial f}{\partial x^i} = 0,$$

if we still denote $c = (x^1, \dots, x^n)$.
In our case $f(t, c, \dot{c}) = g(\dot{c}, \dot{c})$ and the equations become in a chart φ (with Einstein convention):

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{dt}}\frac{\mathrm{d}x^k}{\mathrm{dt}} = 0,$$

where $\Gamma^i_{jk} \in C^\infty(U \subset \mathrm{M})$ are the **Christoffel symbols**, functions defined in the chart $\varphi = (\xi^1, \dots, \xi^n)$ as

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}\left(\frac{\partial g_{lj}}{\partial \xi^k} + \frac{\partial g_{lk}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l}\right).$$

1.2.9 Definition. *A connection ∇ on a manifold* M *with a metric g is said to be a* ***metric connection*** *if for all $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds the following* Leibniz rule*:*

$$\partial_X\, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The unique metric connection which is also torsion-free, i.e.,

$$T := \nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

is called the ***Levi-Civita connection****.*

Another way to define **geodesics** is to say a geodesic between two points p, q on a manifold M with the Levi-Civita connection ∇ is the curve c which links p and q such that parallel transport along the curve preserves the tangent vector to the curve, i.e.

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0 \quad \text{for all } t \in [a, b]. \tag{1.1}$$

More precisely, in order to define the covariant derivative of $\dot{c}$ it is necessary first to extend $\dot{c}$ to a smooth vector field in an open set containing the image of the curve, but it can be shown that the derivative is independent of the choice of the extension.

1.2.10 Observation. We can express the Christoffel symbols in terms of the Levi-Civita connection:

$$\nabla_j \partial_k = \Gamma^i_{jk} \partial_i \tag{1.2}$$

in a local chart $\varphi = (x^1, \ldots, x^n)$. □

1.2.11 Proposition. *Let ∇ be a connection over a manifold* M *and $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$ be vector fields. It holds*

$$\nabla_X Y = \left(X^j \partial_j Y^k + X^j Y^i \Gamma^k_{ij} \right) \partial_k,$$

in particular

$$(\nabla_j Y)^i = \partial_j Y^i + Y^i \Gamma^k_{ij}$$

□

Proof. From the rules of definition (1.1.14) holds:

$$\nabla_X Y = \nabla_{X^j e_j} Y^i e_i = X^j \partial_j Y^i e_i = X^j Y^i \nabla_j e_i + X^j e_i \partial_j Y^i =$$

$$= X^j Y^i \Gamma^k_{ij} e_k + (X^j \partial_j Y^k) e_k$$

1.2.12 Proposition. *Let us consider $p \in \mathrm{M}$ and a tangent vector $\xi \in \mathrm{T}_p\mathrm{M}$. Then there exists $\varepsilon > 0$ and precisely one geodesic*

$$c_\xi : [0, \varepsilon] \to \mathrm{M}$$

such that $c_\xi(0) = p$ and $\dot{c}_\xi(0) = \xi$. □

1.2.13 Definition. *In the conditions of the proposition above, if we put*

$$\mathcal{D}_p = \{\xi \in \mathrm{T}_p\mathrm{M} \mid c_\xi \text{ is defined on } [0,1]\} \subset \mathrm{T}_p\mathrm{M},$$

*the **exponential map** at point p is defined as $\exp_p : \mathcal{D}_p \to \mathrm{M}$ such that $\exp_p(\xi) = c_\xi(1)$.*
*The local coordinates defined by the chart $\{U := \exp_p(\mathcal{D}_p), \exp_p^{-1}\}$ are called **normal coordinates** centered at p.*

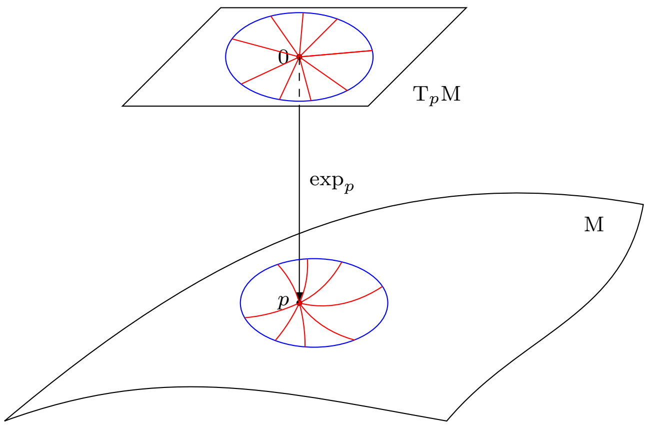


FIGURE 1.9: The exponential maps from the tangent space to the manifold.

1.2.14 Proposition. *Il normal coordinates centered at $p \in \mathrm{M}$, it holds*

$$g_{ij}(p) := g_{ij}(\exp_p(0)) = \eta_{ij},$$

$$\Gamma^i_{jk} = 0$$

for all indices i, j, k. □

We are now ready to talk about **geodesically starshaped** sets.

1.2.15 Definition. *An open subset $\Omega \subset \mathrm{M}$ is called* ***geodesically starshaped*** *with respect to a point $p \in \mathrm{M}$ if there exists an open subset $\Omega' \subset \mathrm{T}_p\mathrm{M}$, starshaped with respect to 0, such that the exponential map*

$$\exp_p|_{\Omega'} : \Omega' \to \Omega$$

is a diffeomorphism. If Ω is geodesically starshaped with respect to all of its points, one calls it ***convex****.*

1.2.16 Proposition. *Under the conditions of the last definition, let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to point $p \in \mathrm{M}$. Then one has*

$$I_\pm^\Omega(p) = \exp_p(I_\pm(0) \cap \Omega'),$$

$$J_\pm^\Omega(p) = \exp_p(J_\pm(0) \cap \Omega').$$

□

1.2.1 Causality and Global Hyperbolicity

Now we introduce causal domains, because they will appear in the theory of wave equations: the local construction of fundamental solutions is always possible on causal domains, provided they are small enough.

1.2.17 Definition. *A domain $\Omega \subset \mathrm{M}$ is called* ***causal*** *if its closure $\overline{\Omega}$ is contained in a convex domain Ω' and for any $p, q \in \overline{\Omega}$ $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.*
A subset $A \subset \mathrm{M}$ is called ***past-compact*** *(respectively* ***future-compact****) if, for all $p \in \mathrm{M}$, $A \cap J_-^{\mathrm{M}}(p)$ (respectively $A \cap J_+^{\mathrm{M}}(p)$) is compact.*

We can notice that if we look at compact spacetimes something strange happens:

1.2.18 Proposition. *If a spacetime M is compact, there exists a closed timelike curve in M.* □

In poor words, in a compact spacetimes there are timelike loops that can produce science fictional paradoxes. To avoid such unphysical and unrealistic things we require some causality conditions:

1.2.19 Definition. *A spacetime satisfies the* ***causality condition*** *if it does not contain any closed causal curve. A spacetime M satisfies the* ***strong***

causality condition *if there are no almost closed causal curves, i.e. if for any $p \in \mathrm{M}$ there exists a neighborhood U of p such that there exists no timelike curve that passes through U more than once.*

It is clear that the strong causality condition implies the causality condition.

1.2.20 Definition. *A spacetime M that satisfies the strong causality condition and such that for all $p, q \in \mathrm{M}$ $J_+^{\mathrm{M}}(p) \cap J_-^{\mathrm{M}}(q)$ is compact is called **globally hyperbolic**.*

It can be demonstrated that in globally hyperbolic manifolds for any $p \in \mathrm{M}$ and any compact set $K \subset \mathrm{M}$ the sets $J_\pm^{\mathrm{M}}(p)$ and $J_\pm^{\mathrm{M}}(K)$ are closed.

1.2.21 Definition. *A subset S of a connected time-oriented Lorentzian manifold M is a **Cauchy hypersurface** if each inextendible timelike curve (i.e. no reparametrisation of the curve can be continuously extended) in M meets S at exactly one point.*

1.2.22 Theorem. *Let M be a connected time-oriented Lorentzian manifold. Then M is globally hyperbolic* if and only if *there exists a Cauchy hypersurface in M.*
In such case there exists a smooth function $h : \mathrm{M} \to \mathbb{R}$ whose gradient is past-directed timelike at every point and all of whose level sets are spacelike Cauchy hypersurfaces. $\square$

1.3 Operators and integration on manifolds

We call $\mathcal{D}(\mathrm{M}) := C_0^\infty(\mathrm{M})$ (the set of C^∞ functions on a manifold with compact support) the space of test-functions on M. We define the integral map

$$\int_{\mathrm{M}} \cdot \,\mathrm{d}\mu : \mathcal{D}(\mathrm{M}) \to \mathbb{C}$$

such that for any local chart $\{U, \varphi\}$ and for any $f \in \mathcal{D}(U)$ holds

$$\int_{\mathrm{M}} f \,\mathrm{d}\mu = \int_{\varphi(U)} (f \circ \varphi^{-1})(x)\, \mu_x \,\mathrm{d}^n x,$$

where we define

$$\mu_x := |\det g(x)|^{1/2}. \tag{1.3}$$

In this section we introduce the **generalized d'Alembert** operators. The general form of a generalized d'Alembert operator P is given by

$$P = -g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a_j(x)\frac{\partial}{\partial x^j} + b(x).$$

The d'Alembert operator $\square$ is defined for smooth functions f as

$$\square f = -\operatorname{div}\operatorname{grad} f,$$

where $\operatorname{grad} f$ is a vector field defined by the request that the formula

$$\langle \operatorname{grad} f, X\rangle = \partial_X f$$

holds for any vector field X and div is defined in the following

1.3.1 Definition. *The **divergence** of a vector field $Z = Z^i\partial_i$ is defined as*

$$\operatorname{div} Z = \sum_j (\nabla_j Z)_j = \partial_j Z^j + \Gamma^j_{ij} Z^j.$$

1.3.2 Proposition. *The following formula holds:*

$$\operatorname{div} Z = \mu_x^{-1}\frac{\partial}{\partial x^j}\left(\mu_x Z^j\right)$$

and the definition of divergence is consistent with the definition of integral. $\square$

Proof. Let $h \in \mathcal{D}(\mathrm{M})$, using integration by parts

$$\int_{\mathrm{M}} h \cdot \operatorname{div}(Z)\mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \,\mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \,\mu_x \,\mathrm{d}x.$$

Now integrate by parts again, but this time in the chart geometry:

$$-\int_{\mathrm{M}} \partial_j h \; Z^j \mu_x \,\mathrm{d}x = \int_{\mathrm{M}} h \;\partial_j(\mu_x \, Z^j)\,\mathrm{d}x = \int_{\mathrm{M}} h \;\mu_x^{-1}\partial_j(\mu_x \; Z^j)\,\mathrm{d}\mu.$$

Since this is true for all function h, the formula is proved. $\square$

From the definition of gradient one can simply show

$$g_{ij}(\operatorname{grad} f)^i X^j = \partial_X f = X^j\frac{\partial f}{\partial x^j}, \quad \operatorname{grad} f = g^{ij}\frac{\partial f}{\partial x^i}\partial_j,$$

hence holds

$$\square f = -\mu_x^{-1}\frac{\partial}{\partial x^j}\left(\mu_x \, g^{ij}\frac{\partial f}{\partial x^i}\right).$$

In Minkowski space, where $g = \eta$,

$$\square f = -\frac{\partial}{\partial x^j}\left(\eta^{jj}\frac{\partial f}{\partial x^j}\right) = \frac{\partial^2 f}{\partial t^2} - \sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i^2} = -\partial^i\partial_i f.$$

1.4 Distributions and Fourier Transform

SECTION TO EXTEND!1!!?'?

Firstly, we recall the main concepts of the theory of distributions.

1.4.1 Definition. *The space of distributions over a manifold* M *is defined as*

$$\mathcal{D}'(\mathrm{M}) = \{u : \mathcal{D}(\mathrm{M}) \to \mathbb{C} \textit{ linear and continuous }\}.$$

The distribution $\frac{1}{x}$ is defined as

$$\frac{1}{x} = \mathrm{PV}\left(\frac{1}{x}\right) - i\pi\delta(x). \tag{1.4}$$

A particular useful formula gives

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)] \tag{1.5}$$

Given a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$\widehat{f}(k) = \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle}\,\mathrm{d}x \tag{1.6}$$

We naturally extend the Fourier transform to a unitary map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and to act on tempered distributions in such a way that for $\varphi \in \mathcal{S}'(\mathcal{U})$ holds

$$\widehat{\varphi}(k) = \left(\varphi, e^{-i\langle k,x\rangle}\right).$$

If $\varphi \in \mathcal{E}'(\mathcal{U})$, $\widehat{\varphi}$ results a smooth function which extends to an entire function $\widehat{\varphi}(z)$, $z \in \mathbb{C}$.

The inverse Fourier transform is given as $\check{f} := (2\pi)^{-n}\widehat{f}(-x)$ and it holds that $f = \check{\widehat{f}}$.

1.4.2 Proposition. *The Fourier transform of the distribution $\delta \in \mathcal{E}'(\mathcal{U})$ is*

$$\widehat{\delta}(k) = 1. \tag{1.7}$$

□

Proof. This is a straightforward computation:

$$\widehat{\delta}(k) = (\delta(x), e^{-i\langle k,x\rangle}) = e^0 = 1.$$

$\square$

1.4.3 Proposition. *For $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ and for any multi-index α hold:*

$$\widehat{\partial^\alpha \varphi}(k) = (ik)^\alpha \widehat{\varphi}(k)$$

$$\widehat{x^\alpha \varphi}(k) = (i\partial)^\alpha \widehat{\varphi}(k).$$

$\square$

Fundamental solutions for wave equation 2

2.1 Fundamental solutions

2.1.1 Definition. *Let P be a differential operator on a manifold* M *and $x \in \mathrm{M}$. A* ***fundamental solution*** *for P at x_0 is a distribution $u \in \mathcal{D}'(\mathrm{M})$ such that*

$$Pu_{x_0} = \delta_{x_0},$$

where δ_{x_0} $\cancel{= \delta(x - x_0)}$ is the Dirac delta distribution in x_0, i.e. $(\delta_{x_0}, f) = f(x_0)$ for all $f \in \mathcal{D}(\mathrm{M})$.

2.1.2 Proposition. *Under the assumptions of the previous definition, the distribution*

$$F(x) = (u_x, \psi) \in \mathcal{D}'(\mathrm{M}),$$

where u_x fundamental solution of P at x and a given $\psi \in \mathcal{D}'(\mathrm{M})$, is a distributional solution for the differential equation

$$PF = \psi.$$

$\square$

Proof. If we apply the operator P on F, we simply get

$$PF = (u_x, P^* \psi) = (Pu_x, \psi) = (\delta_x, \psi) = \psi(x),$$

where P^* stands for the formal adjoint of P. $\square$

2.2 The d'Alembert operator

In order to get things done in a more concrete fashion, we begin approaching the computation of the fundamental solution of $\Box$ via a Fourier transform, concentrating on the lower dimensional Minkowski case.
As we recalled in the previous chapter, the d'Alembert operator is defined in M^n, with the variable $x = (t, \mathbf{x})$, as

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} = -\partial^i \partial_i.$$

We look for fundamental solutions $G^+_{x_0}, G^-_{x_0} \in \mathcal{D}'(\mathrm{M})$ for $\Box$ at $x_0 \in M^n$ such that

$$\operatorname{supp}(G^+_{x_0}) \subset J^M_+(x_0), \qquad \operatorname{supp}(G^-_{x_0}) \subset J^M_-(x_0). \tag{2.1}$$

Such solutions will be called respectively **retarded** (G^+) and **advanced** (G^-) fundamental solutions.
In order to find the fundamental solution at x_0 it suffices to solve the problem $\Box\, u = \delta_0$, and then shift the solution.
We shall begin with a lemma that helps in the computations:

2.2.1 Lemma. *For $u \in \mathcal{S}'(M^n)$ if we write $x = (t, \mathbf{x}) = (t, x_1, \dots, x_{n-1})$ and $k = (\omega, \mathbf{k}) = (\omega, k_1, \dots, k_{n-1})$, it holds*

$$\widehat{\Box\, u}(k) = \|k\|^2\, \widehat{u} = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k). \tag{2.2}$$

$\Box$

Proof. For any test function $f \in \mathcal{S}(M^n)$, and for any $u \in \mathcal{S}'(M^n)$ $(\Box u, f) = (u, \Box f)$, so

$$(\Box u, e^{-i\langle k,x\rangle_0}) = \left(u, \Box e^{-i\langle k,x\rangle_0}\right) = \left(u, \Box e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\right) =$$

$$= \left(u, (|\mathbf{k}|^2 - \omega^2) e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\left(u, e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k).$$

$\Box$

We start transforming the equation:

$$\widehat{\Box u}(k) = \widehat{\delta}(k) \Rightarrow (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k) = 1$$

The set of solutions $\widehat{u}$, which have to be found in $\mathcal{S}'(M^n)$, is the direct sum of two sets

$$\mathrm{S}_0 \oplus \mathrm{S}_1,$$

where S_1 is the set of distributional particular solutions and S_0 is the set of the solutions $\widehat{v}$ of the homogeneous equation $(|\mathbf{k}|^2 - \omega^2)\widehat{v}(k) = 0$.

If we concentrate on the second set it is easy to see with a direct computation that any distribution of the form

$$\widehat{v}(k) = A(k)\delta(|\mathbf{k}|^2 - \omega^2),$$

where $A(k)$ is an arbitrary function of k, belongs to S_0, because the delta is supported exactly where $|\mathbf{k}|^2 - \omega^2$ vanishes. Any solution to the homogeneous wave equation can be obtained by the inverse transform of $\widehat{v}$:

$$v(x) = \check{\widehat{v}}(x) = (2\pi)^{-n}\widehat{\widehat{v}}(-x) = (2\pi)^{-n}\int e^{i\langle k,x\rangle_0} A(k)\delta(|\mathbf{k}|^2 - \omega^2)\,\mathrm{d}k.$$

Making use of the formula (1.5), the former expression becomes

$$\begin{aligned} v(x) = (2\pi)^{-n}\int e^{i\langle k,x\rangle_0}\frac{A(k)}{2|\mathbf{k}|}\left[\delta(|\mathbf{k}|-\omega) + \delta(|\mathbf{k}|+\omega)\right]\,\mathrm{d}k = \\ = (2\pi)^{-n}\int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} A(k)\frac{\cos(|\mathbf{k}|t)}{|\mathbf{k}|}. \end{aligned}$$

To solve for particular solutions we are tempted to write a formula like this:

$$\widehat{u}(k) = \frac{1}{|\mathbf{k}|^2 - \omega^2} = \frac{1}{(|\mathbf{k}| - \omega)(|\mathbf{k}| + \omega)},$$

which is ill-defined where $\langle k, k\rangle_0 = 0$, i.e. on the light-cone of the Fourier space. Hence, we try to define $\widehat{u}$ as limit of something of the form

$$\widehat{u}_\varepsilon = \frac{1}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} = \frac{1}{(|\mathbf{k}| \mp i\varepsilon - \omega)(|\mathbf{k}| \pm i\varepsilon + \omega)}$$

promoting ω to a complex variable and taking the limit for $\varepsilon \to 0$ after performing the inverse transform. The choice of the signs in such expressions must lead to different fundamental solutions.

2.3 Fundamental solutions via Fourier transform

The $\mathcal{S}'(M^n)$ distributions defined as

$$G^+(x) = \frac{1}{(2\pi)^n} \lim_{\varepsilon \to 0^+} \int \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}| - (\omega + i\varepsilon)^2} \, \mathrm{d}k, \tag{2.3}$$

$$G^-(x) = \frac{1}{(2\pi)^n} \lim_{\varepsilon \to 0^+} \int \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}| - (\omega - i\varepsilon)^2} \, \mathrm{d}k, \tag{2.4}$$

are respectively a **retarded** and an **advanced** fundamental solutions at $x_0 = 0$ for the d'Alembert operator.
The aim is to prove that $\mathrm{supp}(G^+) \subset J_+^M(0)$ and $\mathrm{supp}(G^-) \subset J_-^M(0)$, and we proceed firstly by calculating the explicit formula in the $2 \leq n \leq 4$ cases and then discuss the general case via Riesz distributions.

We are authorized to compute $G^\pm$ as a limit of the inverse of the Fourier transform:

$$G^\pm(x) = (2\pi)^{-n} \lim_{\varepsilon \to 0} \int \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}k$$

$$= (2\pi)^{-n} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k} \, e^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}\omega. \tag{2.5}$$

2.3.1 Computing the complex integrals

In order to calculate the inner integral in the former expression,

$$I_\pm(t) := \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}\omega,$$

we can use **Jordan's lemma** on circuits of the complex plane as following.
Now we denote

- C_ρ^+ the upper half-circumference of radius ρ centered at $\omega = 0$ which has Im $(\omega) > 0$, traveled anti-clockwise;
- C_ρ^- the lower half-circumference of radius ρ centered at $\omega = 0$ which has Im $(\omega) < 0$, traveled clockwise;

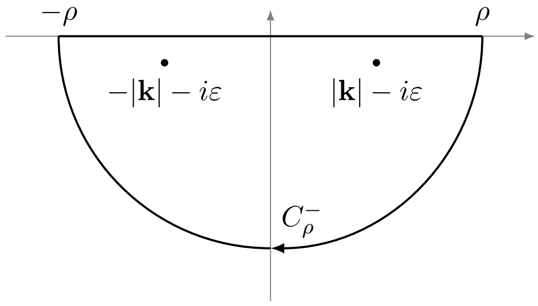


FIGURE 2.1: The circuit to compute $I_+(t)$ for $t > 0$.

- $[-\rho, \rho]$ the segment of the real line connecting $-\rho$ and ρ, traveled from left to right.

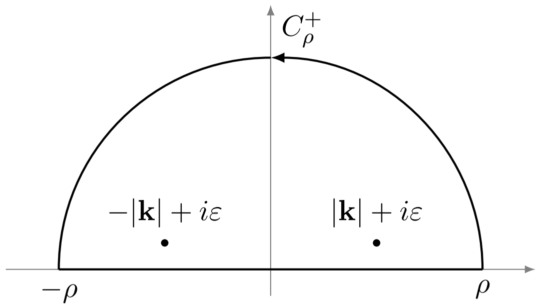


FIGURE 2.2: The circuit to compute $I_-(t)$ for $t < 0$.

The singularities are

$$\text{for } I_+ : \quad \omega = \pm|\mathbf{k}| - i\varepsilon,$$

$$\text{for } I_- : \quad \omega = \pm|\mathbf{k}| + i\varepsilon.$$

Hence we have

$$\text{for } t < 0 \quad I_+(t) = \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_{C_\rho^+ + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega + i\varepsilon)^2}\, \mathrm{d}\omega = 2\pi i \sum_{\mathrm{Im}\, >0} \mathrm{Res} = 0,$$

because the integral on C_ρ^+ vanishes in virtue of Jordan's lemma and there are no singularities in the region bounded by the circuit.

For the same reasons

$$\text{for } t>0,\ I_-(t) = \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_{C_\rho^- + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega - i\varepsilon)^2}\,\mathrm{d}\omega = -2\pi i \sum_{\mathrm{Im}<0} \mathrm{Res} = 0,$$

where the minus sign arises because of the clockwise circuit.
The non-zero integrals are indeed

$$I_+(t),\ \text{for } t>0, \qquad I_-(t),\ \text{for } t<0.$$

The first is computed via the lower circuit in figure (**??**): $C_\rho^- + [-\rho,\rho]$, the second via the upper circuit in figure (**??**): $C_\rho^+ + [-\rho,\rho]$.

The results are

$$\text{for } t>0 \quad I_+(t) = -2\pi i \sum \mathrm{Res} = 2\pi i \left(\frac{e^{-i|\mathbf{k}|t} - e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = 2\pi \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|},$$

$$\text{for } t<0 \quad I_-(t) = -2\pi i \sum \mathrm{Res} = -2\pi i \left(\frac{e^{-i|\mathbf{k}|t} - e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = -2\pi \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}.$$

Summing up everything in one formula:

$$I_\pm(t) = \pm 2\pi \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}\Theta(\pm t), \tag{2.6}$$

where Θ is the Heaviside step-function. Now we come back to equation (2.5) and show explicit solutions for spacial dimensions d 1 to 3:

$$G^\pm_{(d)}(x) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} I_\pm(t) = \pm \frac{\Theta(\pm t)}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}$$

2.3.2 Dimension $n = 1+1$ - wave on a line

The integral we have o perform in the $n = 1+1$ dimensional case is:

$$G^\pm_{(1)}(t,x) = \pm \frac{\Theta(\pm t)}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x} \frac{\sin kt}{k},$$

where we translated the variables as following: $(\mathbf{x}, t) \to (x, t)$ and $(\omega, \mathbf{k}) \to (\omega, k)$.
It is easy to show that

$$\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x} \frac{\sin kt}{k} = \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x/t} \frac{\sin k}{k} = \pi \chi_{[-1,1]}\left(\frac{x}{t}\right) = \pi \chi_{[-t,t]}(x),$$

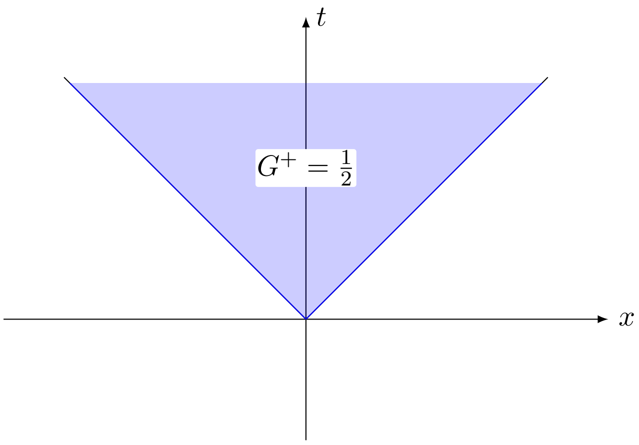


FIGURE 2.3: The support of G^+ in 1+1 dimensional case.

where

$$\chi_{[a,b]}(z) = \begin{cases} 1, & \text{if } z \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

Finally the integrals become

$$G^{\pm}_{(1)}(t,x) = \pm\frac{\Theta(t)}{2}\chi_{[-t,t]}(x) = \pm\frac{\Theta(\pm(t-|x|))}{2}$$

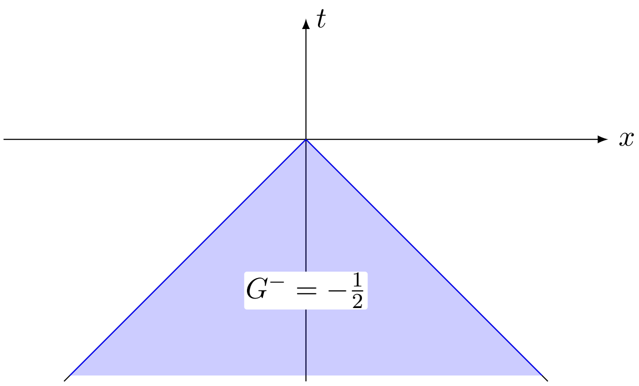


FIGURE 2.4: The support of G^- in 1+1 dimensional case.

From the graphs (**??**) and (**??**) it is clear that the fundamental solutions are supported respectively on $J_+(0)$ and $J_-(0)$.

2.3.3 Dimension $n = 1 + 2$ - wave on a surface

Now the integral is two dimensional:

$$G^{\pm}_{(2)}(t, \mathbf{x}) = \pm \frac{\Theta(\pm t)}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin |\mathbf{k}| t}{|\mathbf{k}|}.$$

To compute it we pass to polar coordinates $\mathbf{k} = (\rho\cos\varphi, \rho\sin\varphi)$, where φ stands for the angle between $\mathbf{x}$ and $\mathbf{k}$. With the substitution

$$\mathrm{d}\mathbf{k} = \rho\, \mathrm{d}\rho\, \mathrm{d}\varphi,$$

the integral becomes (x stands for $|\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}\rho\, e^{i\rho x \cos\varphi} \sin \rho t.$$

It is now easy to show that

$$\int_0^{\infty} \mathrm{d}\rho\, e^{i\rho y} \sin \rho t = \frac{1}{2i} \left[\int_0^{+\infty} e^{i\rho(y+t)}\, \mathrm{d}\rho + \int_0^{+\infty} e^{i\rho(y-t)}\, \mathrm{d}\rho \right] =$$

$$= \frac{1}{2} \left[I(y+t) + I(y-t) \right],$$

where we called $I(y) = \frac{1}{i} \int_0^{+\infty} e^{i\rho y}\, \mathrm{d}\rho$. This integral does not converge, hence needs a regularization.
With an infinitesimally small positive ε we can write[1]

$$I(y) \to \frac{1}{i} \int_0^{+\infty} e^{i\rho(y+i\varepsilon)}\, \mathrm{d}\rho = \frac{1}{y + i\varepsilon}$$

and the integral to calculate is now

$$\frac{1}{2} \int_0^{2\pi} \left[\frac{1}{x\cos\varphi + t + i\varepsilon} + \frac{1}{x\cos\varphi - t + i\varepsilon} \right] \mathrm{d}\varphi.$$

Such integral can be seen as an integral over a unit circle in the complex plane with the usual substitutions $\mathrm{d}\varphi = -i\mathrm{d}z/z$ and $\cos\varphi = (z + z^{-1})/2$. Hence the two parts of the integral become (using the residue theorem)

$$\int_0^{2\pi} \frac{1}{x\cos\varphi \pm t + i\varepsilon}\, \mathrm{d}\varphi = -2i \oint \frac{\mathrm{d}z}{xz^2 + 2(\pm t + i\varepsilon) + x} = \frac{2\pi}{\sqrt{(t \mp i\varepsilon)^2 - x^2}}.$$

[1] The demonstration of this formula is a subtle point DA CONTROLLAREEEE

Putting everything together in the limit $\varepsilon \to 0$

$$G^{\pm}_{(2)}(t, \mathbf{x}) = \pm \frac{\Theta(\pm t)}{2\pi} \frac{\Theta(t^2 - |\mathbf{x}|^2)}{\sqrt{t^2 - |\mathbf{x}|^2}} = \pm \frac{\Theta(\pm t)}{2\pi} \frac{\Theta(\gamma(x))}{\sqrt{\gamma(x)}}, \tag{2.7}$$

where we put the $\Theta(t^2 - |\mathbf{x}|^2)$ to make sure the argument of the square root never becomes negative. For this same reason it is clear that $\operatorname{supp}(G^{\pm}) \subseteq J_{\pm}(0)$.

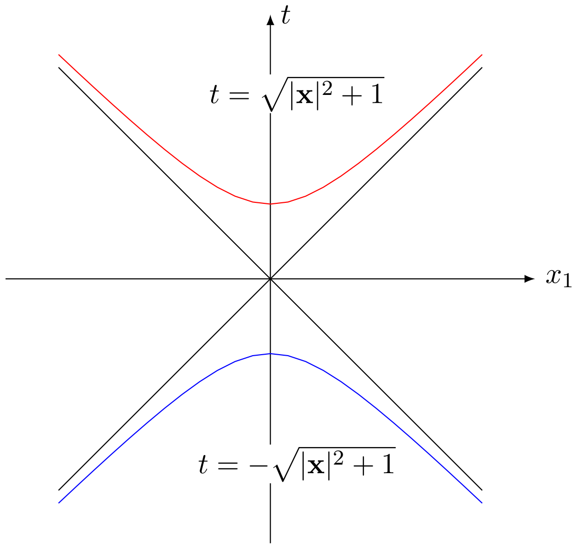


FIGURE 2.5: The level set $G^{\pm}(\mathbf{x}, t) = 1$ in the 1+2 dimensional case, plotted for one spacial axis.

2.3.4 Dimension $n = 1 + 3$ - spherical wave

The three-dimensional integral to perform is

$$G^{\pm}_{(3)}(t, \mathbf{x}) = \pm \frac{\Theta(\pm t)}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}.$$

Again, we make a change of coordinate, passing to the spherical ones: $\mathbf{k} = (k \sin\vartheta \cos\varphi, k \sin\vartheta \sin\varphi, k \cos\vartheta)$. The substitution is

$$\mathrm{d}\mathbf{k} = k^2 \sin\vartheta \, \mathrm{d}k \, \mathrm{d}\vartheta \, \mathrm{d}\varphi$$

and the integral to calculate is (x stands again for $|\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}k\, k \sin kt \int_{-1}^{1} e^{ikx\cos\vartheta} \mathrm{d}(\cos\vartheta) = \frac{4\pi}{x} \int_0^{+\infty} \sin kt \sin kx \, \mathrm{d}k.$$

Hence we can simply write using the exponential function

$$\sin kt \sin kx = \frac{1}{4} \left\{ \left[e^{ik(x-t)} + e^{-ik(x-t)} \right] - \left[e^{ik(x+t)} + e^{-ik(x+t)} \right] \right\}$$

and with a change of variables $k \leftrightarrow -k$ we have

$$\frac{4\pi}{x} \int_0^{+\infty} \sin kt \sin kx \, \mathrm{d}k = \frac{2\pi^2}{x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-t)} - e^{ik(x+t)} \, \mathrm{d}k =$$

$$= \frac{2\pi^2}{x} \left[\delta(t-x) - \delta(t+x) \right].$$

To find the correct retarded and advanced solutions we notice that the second term, $\delta(t+x)$, vanishes for G^+ because $x > 0$ and $t > 0$; conversely the first term $\delta(t-x)$ vanishes when computing G^-. Made these considerations it is easy to write the general formula for the $n = 1{+}3$ case

$$G^{\pm}_{(3)}(t, \mathbf{x}) = \pm \frac{\Theta(\pm t)}{4\pi} \frac{\delta(t \mp |\mathbf{x}|)}{|\mathbf{x}|}. \tag{2.8}$$

Immediately, one can verify that $G^{\pm}$ are zero outside the support of the delta distribution, hence are supported respectively on the upper and the lower light cones $C_+(0)$ and $C_-(0)$. Such supports are included in $J_{\pm}(0)$ as we required, but the unique feature of being supported only on the light cone is a particularity of the even dimensional case and it is known as the **Huygens' principle**. It states that in general, we have for $d \neq 1$

$$\operatorname{supp} G^{\pm}_{(d)} = J_{\pm}(0) \quad \text{for } d \text{ even},$$

$$\operatorname{supp} G^{\pm}_{(d)} = C_{\pm}(0) \quad \text{for } d \text{ odd}.$$

As it can be seen, the one spacial dimensional case is an exception to this rule, as it shares some properties with the even dimensional cases.

2.3.1 Observation. The lower dimensional advanced and retarded distributions can be easily deduced from the $d = 3$ case as we shall see. In

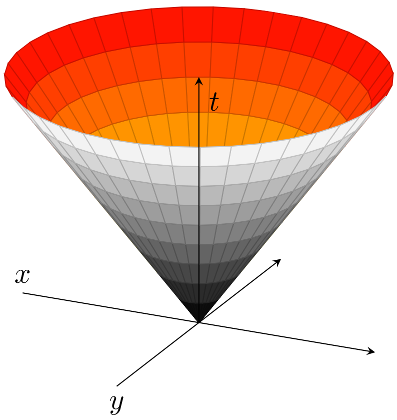


FIGURE 2.6: The support of G^+ in 1+3 dimensional case, i.e. the upper light cone $C_+(0)$, plotted for two spacial axis.

general if we know the explicit solution to the d case we can find the $d-1$ solution with the formula

$$G^{\pm}_{(d-1)}(t, x_1, \ldots, x_{d-1}) = \int_{-\infty}^{\infty} G^{\pm}_{(d)}(t, x_1, \ldots, x_d)\, \mathrm{d}x_d$$

This assertion can be proven by taking the fundamental solution formula

$$\Box_d\, G^{\pm}_{(d)}(t, x_1, \ldots, x_d) = \delta(t)\delta(x_1)\cdots\delta(x_d),$$

where $\Box_d$ stands for $(\partial_t^2 - \partial_1^2 - \cdots - \partial_d^2)$, and integrate it on the last variable with a test-function $\varphi \in \mathcal{D}$:

$$\int \Box_d\, G_{(d)}\, \varphi\, \mathrm{d}x_d = \Box_{d-1} \int G_{(d)}\, \varphi\, \mathrm{d}x_d - \int \partial_d^2 G_{(d)}\, \varphi\, \mathrm{d}x_d =$$

$$= \delta(t)\delta(x_1)\cdots\delta(x_{d-1}) \int \delta(x_d)\varphi\, \mathrm{d}x_d.$$

By letting the test-function become a sequence of cut-off function for the domain, the formula

$$\Box_{d-1} \int G_{(d)}\, \mathrm{d}x_d = \delta(t)\delta(x_1)\cdots\delta(x_{d-1})$$

is proven.
According to (2.8) and using the powerful formula (1.5) we can write for the retarded solution

$$\delta(t^2 - |\mathbf{x}|^2) = \delta(t^2 - x_1^2 - x_2^2 - x_3^2) =$$

$$\Theta(t^2 - x_1^2 - x_2^2)\frac{\delta(x_3 - \sqrt{t^2 - x_1^2 - x_2^2}) + \delta(x_3 + \sqrt{t^2 - x_1^2 - x_2^2})}{2\sqrt{t^2 - x_1^2 - x_2^2}}$$

where we put the Heaviside step function to make sure the argument of the square root never vanishes. Hence, integrating over the third variable makes the deltas disappear with a gain of a factor of 2 and we have

$$G^+_{(2)}(t, x_1, x_2) = \frac{\Theta(t)}{2\pi}\frac{\Theta(t^2 - x_1^2 - x_2^2)}{\sqrt{t^2 - x_1^2 - x_2^2}},$$

which is identical to (2.7) as we expected. □

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