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On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

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"*The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen.* "

Paul A.M. Dirac

UNIVERSITY OF PAVIA

Abstract

Department of Physics

Master Degree

On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

by Rubens Longhi

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

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Introduction

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Chapter 1

Geometric preliminaries

In this chapter, we begin by recalling the basic definitions in order to fix the geometric setting in which we will work.
Globally hyperbolic spacetimes (M, g) are used in context of geometric analysis and mathematical relativity because in them there exists a smooth and spacelike Cauchy hypersurface Σ and that ensures the well-posedness of the Cauchy problem. Moreover, as shown by Bernal and Sánchez [BS05, Th. 1.1], in such spacetimes there exists a splitting for the full spacetime M as an orthogonal product $\mathbb{R} \times \Sigma$. These results corroborate the idea that in globally hyperbolic spacetimes one can preserve the notion of a global passing of time. In a globally hyperbolic spacetime, the entire future and past history of the universe can be predicted from conditions imposed at a fixed instant represented by the hypersurface Σ.

1.1 Globally Hyperbolic spacetimes with timelike boundary

The main goal of this section is to analyse the main properties of globally hyperbolic spacetimes and to generalise them to a natural class of spacetimes where boundary values problems can be formulated. This class is that of globally hyperbolic spacetimes with timelike boundary.While, in the case of $\partial M = \emptyset$ global hyperbolicity is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18].

Manifolds with boundary. From now on M will denote a smooth connected oriented manifold of dimension $m > 1$ with boundary. M is then locally diffeomorphic to open subsets of the closed half space of $\mathbb{R}^n$. We will assume that the boundary ∂M, which is the set of points for which all neighbourhoods are diffeomorphic to the closed half space of $\mathbb{R}^n$, is smooth and, for simplicity, connected. A point $p \in M$ such that there exists an open neighbourhood U containing p diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$. In addition we endow M with a smooth Lorentzian metric g of signature $(-, +, ..., +)$ so that

$\iota^* g$ identifies a Lorentzian metric on ∂M and we require (M, g) to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M, g) has a *timelike boundary*.

For any $p \in M$, we denote by $J^+(p)$ the set of all points that can be reached by future-directed causal smooth curves emanating from p. For any subset $A \subset M$ we set $J^+(A) := \bigcup_{p\in A} J^+(p)$. If A is closed so is $J_+(A)$. We denote by $I^+(p)$ the set of all points in M that can be reached by future-directed timelike curves emanating from p. The set $I^+(p)$ is the interior of $J^+(p)$; in particular, it is an open subset of M. Interchanging the roles of future and past, we similarly define $J^-(p)$, $J^-(A)$, $I^-(p)$, see

Definition 1.1.1.

- *A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.*
- *A spacetime with timelike boundary is* causal *if it possesses no closed, causal curve.*
- *A causal spacetime with timelike boundary M such that for all $p, q \in M$ $J^+(p) \cap J^-(q)$ is compact is called* ***globally hyperbolic****.*

These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 1.1.2. *Let (M, g) be a spacetime of dimension m. Then*

1. *(M, g) is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve,*
2. *if (M, g) is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the metric*

$$g = -\beta d\tau^2 + h_\tau, \qquad (1.1)$$?eq:line_element?

where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1]*, whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a smooth Cauchy surface for (M, g).*

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M, g), we work directly with (1.1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M.

[1]Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

Definition 1.1.3. *A spacetime with boundary (M, g) is* static *if it possesses a nowhere vanishing irrotational timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to ∂M is tangent to the boundary, i.e. $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where ν is the unit vector, normal to the boundary at p.*

Remark 1.1.4. A spacetime with boundary (M, g) is *stationary* if we do not require neither the Killing vector χ nor its restriction to the boundary to be irrotational.

Locally, every stationary or static Lorentzian manifold looks like the corresponding standard one with metric (1.1) with $\chi = \partial_\tau$. Hence the static property translates into the request that both β and h_τ are independent from τ.

Definition 1.1.5. *We call* standard static *a static spacetime with timelike boundary (M, g) isometric to $(\mathbb{R} \times \Sigma, -\beta\, \mathrm{d}t^2 + h)$, where Σ is a Riemannian manifold with boundary endowed with a metric h and $\beta \in C^\infty(\Sigma, (0, \infty))$.*

Corollary 1.1.6. *(see [DDF19, Cor. 2]) Let (M, g) be a standard static spacetime with timelike boundary. Then also ∂M is a standard static spacetime (with empty boundary), endowed with the induced metric.*

Example 1.1.7. We consider some examples of globally hyperbolic spacetimes without boundary ($\partial M = \emptyset$).

- The Minkowski spacetime $\mathbb{M}^m = (\mathbb{R}^m, \eta)$ is static and globally hyperbolic. Every spacelike hyperplane (of co-dimension 1) is a Cauchy hypersurface. We have $\mathbb{M}^m = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-1}$, endowed with the time-independent Euclidean metric.

- Let Σ be a Riemannian manifold with time independent metric h and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$, called **cosmological spacetime**, is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold, see [BGP07, Lem A.5.14]. This applies in particular if (Σ, h) is compact.

- The interior and exterior **Schwarzschild spacetimes**, that represent non-rotating black holes of mass $\mathrm{m} > 0$ are globally hyperbolic. Denoting S^2 the 2-dimensional sphere embedded in $\mathbb{R}^3$, we set
$$M_{\text{ext}} := \mathbb{R} \times (2\mathrm{m}, +\infty) \times S^2,$$
$$M_{\text{int}} := \mathbb{R} \times (0, 2\mathrm{m}) \times S^2.$$
The metric is given by
$$g = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2},$$

where $f(r) = 1 - \frac{2\mathrm{m}}{r}$, while $g_{S^2} = r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2$ is the polar coordinates metric on the sphere. In particular, the exterior Schwarzschild spacetime is *static* and we have $M_{\text{ext}} = \mathbb{R}\times\Sigma$ with $\Sigma = (2\mathrm{m}, +\infty)\times S^2$, $\beta = f$ and $h = \frac{1}{f(r)}\mathrm{d}r^2 + r^2\,g_{S^2}$.

■

Example 1.1.8. Now we consider some examples of globally hyperbolic spacetimes with timelike boundary in which the boundary is not empty.

- The half Minkowski spacetime $\mathbb{M}^m = (\mathbb{R}^{m-1}\times[0,+\infty),\eta)$ is static and globally hyperbolic. Every spacelike half-hyperplane (of co-dimendion 1) is a Cauchy hypersurface. We have $\mathbb{M}^m = \mathbb{R}\times\Sigma$ with $\Sigma = \mathbb{R}^{m-2}\times[0,+\infty)$, endowed with the time-independent Euclidean metric.
- Let Σ be a Riemannian manifold with boundary with time independent metric h and let $I\subset\mathbb{R}$ an interval. Let $f: I\to\mathbb{R}$ be a smooth positive function. The manifold $M = I\times\Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\,h$ is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold with boundary.

■

A particular role will be played by the support of the functions that we consider. In the following definition we introduce the different possibilities that we will consider - *cf.* [Bär15].

Definition 1.1.9. *Let (M,g) be a Lorentzian spacetime with timelike boundary and let $E\to M$ be a finite rank vector bundle on M. We denote with*

1. *$C^\infty_{\mathrm{c}}(M,E)$ the space of smooth sections of E with compact support in M while with $C^\infty_{\mathrm{cc}}(M,E)\subset C^\infty_{\mathrm{c}}(M)$ the collection of smooth and compactly supported sections f of E such that* $\mathrm{supp}(f)\cap\partial M = \emptyset$*.*
2. *$C^\infty_{\mathrm{spc}}(M,E)$ (resp. $C^\infty_{\mathrm{sfc}}(M,E)$) the space of strictly past compact (resp. strictly future compact) sections of E, that is the collection of $f\in C^\infty(M,E)$ such that there exists a compact set $K\subseteq M$ for which $J^+(\mathrm{supp}(f))\subseteq J^+(K)$ (resp. $J^-(\mathrm{supp}(f))\subseteq J^-(K)$), where $J^\pm$ denotes the causal future and the causal past in M. Notice that $C^\infty_{\mathrm{sfc}}(M,E)\cap C^\infty_{\mathrm{spc}}(M,E) = C^\infty_{\mathrm{c}}(M,E)$.*
3. *$C^\infty_{\mathrm{pc}}(M,E)$ (resp. $C^\infty_{\mathrm{fc}}(M,E)$) denotes the space of future compact (resp. past compact) sections of E, that is, $f\in C^\infty(M,E)$ for which $\mathrm{supp}(f)\cap J^-(K)$ (resp. $\mathrm{supp}(f)\cap J^+(K)$) is compact for all compact $K\subset M$.*
4. *$C^\infty_{\mathrm{tc}}(M,E) := C^\infty_{\mathrm{fc}}(M,E)\cap C^\infty_{\mathrm{pc}}(M,E)$, the space of timelike compact sections.*
5. *$C^\infty_{\mathrm{sc}}(M,E) := C^\infty_{\mathrm{sfc}}(M,E)\cap C^\infty_{\mathrm{spc}}(M,E)$, the space of spacelike compact sections.*

1.2 Differential forms and operators on manifolds with boundary

To treat Maxwell equations properly and to be able to generalise them, we will use the language of differential forms. In this section (M, g) will denote a generic oriented pseudo-Riemannian manifold with boundary with signature $(-, +, \ldots, +)$ or $(+, \ldots, +)$. In the former case, when the manifold is Lorentzian, it is understood that the boundary is timelike in the sense of Definition 1.1.1. We present the following definitions in such a general framework since we will work both on spacetimes (M, g) with timelike boundary and on their Cauchy hypersurfaces (Σ, h), which are Riemannian manifolds with boundary on account of Theorem 1.1.2.

On top of a pseudo-Riemannian Hausdorff, connected, oriented and paracompact manifold (M, g) with boundary we consider the spaces of complex valued k-forms $\Omega^k(M)$, with $k \in \mathbb{N} \cup \{0\}$, as smooth sections of $\Lambda^k T^*M$. Since (M, g) is oriented, we can identify a unique, metric-induced, Hodge operator $\star : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \mathrm{d}\mu_g$, where $\wedge$ is the exterior product of forms and $\mathrm{d}\mu_g$ the metric induced volume form. We endow $\Omega^k(M)$ with the standard, metric induced, pairing

$$(\alpha, \beta) := \int_M \overline{\alpha} \wedge \star\beta \,, \tag{1.2}$$ {?}

Remark 1.2.1. In case $E = \Lambda^k T^*M$, the spaces with support properties defined in Definition 1.1.9 will be denoted respectively by the following spaces of k-forms: $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{cc}}(M)$, $\Omega^k_{\mathrm{spc/sfc}}(M)$, $\Omega^k_{\mathrm{pc/fc}}(M)$, $\Omega^k_{\mathrm{tc/sc}}(M)$. If the regularity required for any of these spaces is different than smoothness, it will be denoted putting it in front of the space. For example, the space of square integrable k-forms will be indicated with $\mathrm{L}^2\Omega^k(M)$.

We indicate the exterior derivative with $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. A differential form α is called closed when $\mathrm{d}\alpha = 0$ and exact when $\alpha = \mathrm{d}\beta$ for some differential form β. Since M is endowed with a pseudo-Riemannian metric it holds that, when acting on smooth k-forms, $\star^{-1} = (-1)^{k(m-k)+\sigma_M} \star$, where σ_M is the signature of g. Combining these data we define the *codifferential* operator $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta \doteq \star^{-1} \circ \mathrm{d} \circ \star$.
To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its projection along the direction normal to ∂M.

Remark 1.2.2. Since we will be considering not only forms lying in $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, but also those in $\Omega^k(\partial M)$, we shall distinguish the operators acting on this space with a subscript $_\partial$, *e.g.* d_∂, $\star_\partial$, δ_∂ or $(,)_\partial$.

Definition 1.2.3. *Let (M, g_M) be a smooth Lorentzian manifold and let $\iota_N \colon N \to M$ be a codimension 1 smoothly embedded submanifold of M with induced metric $g_N := \iota_N^* g_M$. We define the tangential and normal components relative to N as*

$$\mathrm{t}_N \colon \Omega^k(M) \to \Omega^k(N)\,, \qquad \omega \mapsto \mathrm{t}_N\omega := \iota_N^*\omega\,, \tag{1.3a} $$ {?}

$$\mathrm{n}_N \colon \Omega^k(M) \to \Omega^{k-1}(N)\,, \qquad \omega \mapsto \mathrm{n}_N\omega := \star_N^{-1}\mathrm{t}_N \star_M \omega\,, \tag{1.3b} $$ {?}

where $\star_M, \star_N$ denote the Hodge dual over M, N respectively. In particular, for all $k \in \mathbb{N} \cup \{0\}$ we define

$$\Omega^k_{\mathrm{t}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}_N\omega = 0\}\,, \qquad \Omega^k_{\mathrm{n}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}_N\omega = 0\}\,. \tag{1.4}$$?Eqn: k-forms wit

Similarly we will use the symbols $\Omega^k_{\mathrm{c,t_N}}(M)$ and $\Omega^k_{\mathrm{c,n_N}}(M)$ when we consider only smooth, compactly supported k-forms.

Remark 1.2.4. In this paper the rôle of N will be played often by ∂M. In this case, we shall drop the subscript form Equation (1.3), namely $\mathrm{t} \equiv \mathrm{t}_{\partial M}$ and $\mathrm{n} \equiv \mathrm{n}_{\partial M}$.

Remark 1.2.5. With reference to Definition 1.2.3, observe that the following linear map is surjective:

$$\Omega^k(M) \ni \omega \to (\mathrm{n}\omega, \mathrm{t}\omega, \mathrm{t}\delta\omega, \mathrm{n}\mathrm{d}\omega) \in \Omega^{k-1}(\partial M) \times \Omega^k(\partial M) \times \Omega^{k-1}(\partial M) \times \Omega^k(\partial M)\,.$$

Remark 1.2.6. The normal map $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ can be equivalently read as the restriction to ∂M of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(M)$ and the vector field $\nu \in \Gamma(TM)|_{\partial M}$ which corresponds pointwisely to the outward pointing unit vector, normal to ∂M.

As last step, we observe that (1.3) together with (1.4) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$\star\,\delta = (-1)^k \mathrm{d}\star\,, \qquad \delta\star = (-1)^{k+1}\star\mathrm{d}\,, \tag{1.5a}$$?Eqn: relations-b

$$\star_\partial\,\mathrm{n} = \mathrm{t}\star\,, \qquad \star_\partial\mathrm{t} = (-1)^k\mathrm{n}\star\,, \qquad \mathrm{d}_\partial\mathrm{t} = \mathrm{t}\mathrm{d}\,, \qquad \delta_\partial\mathrm{n} = -\mathrm{n}\delta\,. \tag{1.5b}$$?Eqn: relations-b

A notable consequence of (1.5b) is that, while on manifolds with empty boundary, the operators d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. Indeed, a direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial, \tag{1.6}$$?Eqn: boundary te

for all $\alpha \in \Omega^k_{\mathrm{c}}(M), \beta \in \Omega^{k+1}_{\mathrm{c}}(M)$ such that $\mathrm{supp}\,\alpha \cap \mathrm{supp}\,\mathrm{d}\beta$ and $\mathrm{supp}\,\alpha \cap \mathrm{supp}\,\delta\beta$ are compact and where the pairing in the right-hand side is the one associated to forms living on ∂M.

1.3 Bounded Geometry and associated functional spaces

We introduce both the geometric setting and the Sobolev functional spaces that are extensively used in 2. We will follow mainly the discussion of [DDF19] and [GS13].

Definition 1.3.1. *A Riemannian manifold (Σ, h) with empty boundary is called of* bounded geometry *if the injectivity radius*[2] $r_{inj}(\Sigma) > 0$ *and if* $T\Sigma$ *is of* totally bounded curvature, *that is* $\|\nabla^k R\|_{L^\infty(M)} < \infty$ *for all* $k \in \mathbb{N} \cup \{0\}$, R *being the scalar curvature and* ∇ *the Levi-Civita connection associated with* h.

In view of its definition, the injectivity radius of a manifold with non-empty boundary vanishes, hence we must regard $\partial\Sigma$ as a submanifold of an extension with empty boundary of the Riemannian manifold Σ. This requires a notion of bounded geometry for a generic submanifold.

Definition 1.3.2. *Let (Σ, h) be a Riemannian manifold of bounded geometry and let $(Y, \iota_Y^* h)$ be a co-dimension k closed, embedded, smooth submanifold with an inward pointing, unit normal vector field ν . We say that $(Y, \iota_Y^* h)$ is a bounded geometry submanifold if the following holds:*

- *the second fundamental form K_Y of Y in Σ together with all its covariant derivatives on Y is bounded,*
- *there exists $\varepsilon > 0$ such that the map $\varphi : Y \times (-\varepsilon, \varepsilon) \to \Sigma$ defined as $\phi(p, z) = \exp_p(z\nu|_p)$ is injective, where $\exp_p$ is the exponential map of Σ at p.*

We are now ready to give the definition in case the boundary is non-empty.

Definition 1.3.3. *An n-dimensional Riemannian manifold (Σ, h) with boundary is of* bounded geometry *if there exists an n-dimensional Riemannian manifold $(\widehat{\Sigma}, \widehat{h})$ (with empty boundary) of bounded geometry such that $\Sigma \subset \widehat{\Sigma}$, $h = \widehat{h}|_\Sigma$ and $(\partial\Sigma, \iota_{\partial\Sigma}^* \widehat{h})$ is a bounded geometry submanifold of $\widehat{\Sigma}$.*

We remark that all Riemannian manifolds with compact boundary meet the requirements of the former Definition. At the same time one can also consider non-compact boundaries such as the n-dimensional half-space $\mathbb{R}^n_+ = [0, +\infty) \times \mathbb{R}^{n-1}$ endowed with the standard Euclidean metric. To conclude, we study the interplay between the notion of Riemannian manifold with boundary and of bounded geometry and that of standard static Lorentzian manifold with timelike boundary, *cf.* Definition 1.1.5.

Proposition 1.3.4. *(*cf. *[DDF19, Prop. 9])*
Let (Σ, h) be a Riemannian manifold with boundary and of bounded geometry and let $(\widehat{\Sigma}, \widehat{h})$ be the empty-boundary extension of bounded geometry as in Definition 1.3.1. Then

[2] The injectivity radius $r_{inj}(p)$ at a point p of a Riemannian manifold is the largest radius for which the exponential map at p is a diffeomorphism. The injectivity radius of a Riemannian manifold is $r_{inj}(\Sigma) = \inf_{p\in\Sigma} r_{inj}(p)$.

1. *Every $\beta \in C^\infty(\Sigma, (0, +\infty))$ identifies an isometry class of standard static Lorentzian manifolds with timelike boundary,*

2. *if in addition there exists $\widehat{\beta} \in C^\infty(\widehat{\Sigma}, (0, +\infty))$ such that $\widehat{\beta}|_\Sigma = \beta$ and $\widehat{h}/\widehat{\beta}$ identifies a complete Riemannian metric on $\widehat{\Sigma}$ then each representative (M, g) of the isometry class is a submanifold with boundary of a standard static globally hyperbolic spacetime $(\widehat{M}, \widehat{g})$.*

A manifold (M, g) that satisfies the first condition will be called static Lorentzian spacetime with timelike boundary and of bounded geometry

1.3.1 Sobolev spaces

We consider a finite rank complex vector bundle $E \to \Sigma$ endowed with a fiberwise Hermitian product $\langle\, , \rangle_E$ and a product preserving connection ∇ built out of h.

Definition 1.3.5. *We say that a section $u \in \Gamma(E)$ is measurable if the function*

$$\Sigma \ni x \mapsto \langle u(x), u(x)\rangle_E,$$

is measurable with respect to the measure $\mathrm{d}\mu_h$ and we denote the space of equivalence classes of almost everywhere equal measurable sections of E with $\Gamma_{me}(E)$.
Moreover, a measurable section $u \in \Gamma_{me}(E)$ is in $u \in \mathrm{L}^p(E)$ if the function $\Sigma \ni x \mapsto \langle u(x), u(x)\rangle_E^p$ is integrable.

Definition 1.3.6. *For all $\ell \in \mathbb{N} \cup \{0\}$ and $p \in (1, \infty)$, we define the Sobolev spaces*

$$\mathrm{H}^\ell_p\Gamma(E) = \left\{ u \in \Gamma_{me}(E) \,|\, \nabla^j u \in \mathrm{L}^p(E \otimes T^*\Sigma^{\otimes j}),\, j \leq \ell \right\}. \tag{1.7}$$?Eq: Sobolev spac

In case $p = 2$ we denote the Sobolev spaces as $\mathrm{H}^\ell\Gamma(E) := \mathrm{H}^\ell_2\Gamma(E)$.

Whenever $E = \Lambda^k T^*\Sigma$, i.e. $\Gamma(E)$ is the space of differential k-forms, we will use the notation $\Omega^k(\Sigma) := \Gamma(\Lambda^k T^*\Sigma)$ (according to the definitions in Section 1.2) and $\mathrm{H}^\ell\Omega^k(\Sigma) := \mathrm{H}^\ell\Gamma(\Lambda^k T^*\Sigma)$.

Remark 1.3.7. The space $\mathrm{H}^\ell\Gamma(E)$ is an Hilbert space endowed with the norm

$$\|u\|^2_{\mathrm{H}^\ell\Gamma(E)} = \sum_{j=0}^{\ell} \|\nabla^j u\|^2_{\mathrm{L}^2(E\otimes T^*\Sigma^{\otimes j})}. \tag{1.8}$$ {?}

The theory of these space has been thoroughly studied in the literature and for the case in hand we refer mainly to [GS13].

Remark 1.3.8. The space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ can otherwise be defined as the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ (see Section 1.2) with respect to the pairing $(\ ,\ )_\Sigma$ between k-forms

$$(\alpha,\beta)_\Sigma := \int_\Sigma \overline{\alpha}\wedge\star_\Sigma\beta \qquad \alpha,\beta\in\Omega^k_{\mathrm{c}}(\Sigma)\,, \tag{1.9}$$ {?}

where $\star_\Sigma$ is the Hodge dual on Σ.

Whenever a boundary is present, one can introduce the subspace $\mathrm{H}^\ell_0\Gamma(E)\subset\mathrm{H}^\ell\Gamma(E)$ defined as the completion of $\Gamma_{\mathrm{c}}(E)$ (the space of compactly supported sections of E) with respect to the $\mathrm{H}^\ell\Gamma(E)$-norm. Whenever Σ is metric complete (for example, if Σ is a Riemannian manifold of bounded geometry, in particular if $\Sigma=\mathbb{R}^n$) the two spaces coincide: $\mathrm{H}^\ell_0\Gamma(E)=\mathrm{H}^\ell\Gamma(E)$.

1.3.2 Restrictions and trace maps for differential forms

Using *uniformly locally finite trivializations*, one can define, following [GS13, Def. 11], the real-exponent Sobolev spaces $\mathrm{H}^s_p\Gamma(E)$, with $s\in\mathbb{R}$.

Proposition 1.3.9. *Let (Σ,h) be a Riemannian manifold of bounded geometry with boundary. Then for every $\ell\geq\frac{1}{2}$ there exists a continuous surjective map*

$$\mathrm{res}_\ell:\mathrm{H}^\ell\Omega^k(\Sigma)\to\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma), \tag{1.10}$$ {?}

that extends the restriction on $\Omega^k_{\mathrm{c}}(\Sigma)$, i.e. $\mathrm{res}_\ell\,\alpha=\alpha|_{\partial\Sigma}$ if $\alpha\in\Omega^k_{\mathrm{c}}(\Sigma)$.

Remark 1.3.10. In particular, according to [Geo79, p. 171] and [Wec04, Sec. 2], the tangential and normal maps defined in Definition 1.2.3 can be extended to continuous surjective maps

$$\mathrm{t}\oplus\mathrm{n}\colon\mathrm{H}^\ell\Omega^k(\Sigma)\to\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma)\oplus\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma)\qquad\forall\ell\geq\frac{1}{2}\,. \tag{1.11}$$?Eqn: Sobolev tangent

1.4 Green operators

In this section we will follow mainly [Bär15]. Let $E_1,E_2\to M$ be vector bundles over a globally hyperbolic spacetime with $\partial M=\emptyset$. Let $P:C^\infty(M,E_1)\to C^\infty(M,E_2)$ be a linear differential operator.

Definition 1.4.1. *An* advanced Green operator *of P, or* advanced fundamental solution *for P, is a linear map $G^+:C^\infty_{\mathrm{c}}(M,E_2)\to C^\infty(M,E_1)$ such that*

(i) $G^+P=\mathrm{Id}_{C^\infty_{\mathrm{c}}(M,E_1)}$,

(ii) $PG^+=\mathrm{Id}_{C^\infty_{\mathrm{c}}(M,E_2)}$,

(iii) $\mathrm{supp}(G^+f)\subset J^+(\mathrm{supp}\,f)$, *for all* $f\in C^\infty_{\mathrm{c}}(M,E_2)$.

Analogously, a linear map $G^- : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty(M, E_1)$ *is called a* retarded Green operator *of* P*, or* retarded fundamental solution *for* P *if* (i) *and* (ii) *hold, while it also holds*

(iii') $\operatorname{supp}(G^- f) \subset J^-(\operatorname{supp} f)$*, for all* $f \in C^\infty_{\mathrm{c}}(M, E_2)$.

Definition 1.4.2. *The operator* P *is called* Green hyperbolic *if* P *and* P^{t} *possess advanced and retarded Green operator, where* $P^{\mathrm{t}} : C^\infty(M, E_2^*) \to C^\infty(M, E_1^*)$*, known as the* formal dual *of* P*, is the unique linear differential operator such that*

$$(\varphi, Pf)_M = (P^{\mathrm{t}}\varphi, f)_M, \quad \textit{i.e.} \quad \int_M \langle \varphi, Pf \rangle \, \mathrm{d}\mu_g = \int_M \left\langle P^{\mathrm{t}}\varphi, f \right\rangle \mathrm{d}\mu_g, \tag{1.12}$$ {?}

for all $f \in C^\infty(M, E_1)$ *and* $\varphi \in C^\infty(M, E_2^*)$ *such that* $\operatorname{supp} f \cap \operatorname{supp} \varphi$ *is compact.*

Remark 1.4.3. If (M, g) has empty boundary, the Green operators of a Green hyperbolic operator P are unique, see [Bär15, Cor. 3.12]. If the spacetime has a boundary, the differential operators must be given together with boundary conditions. These conditions are encoded in the domain of the operator, that is replaced by the subset $C^\infty_{\mathrm{b.c.}}(M, E_1) \subset C^\infty(M, E_1)$ of sections that satisfies the boundary conditions. Hence, in the case of non-empty boundary, the codomain $C^\infty(M, E_1)$ of G must be replaced, in Definitions 1.4.1 and 1.4.2, with the corresponding subspace $C^\infty_{\mathrm{b.c.}}(M, E_1)$.

Example 1.4.4. An important example of Green-hyperbolic operators are the *wave operators*, or the *normally hyperbolic operators*. Locally they are of the form

$$P = g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + a^j(x) \frac{\partial}{\partial x^j} + b(x), \tag{1.13}$$ {?}

where g^{ij} denote the components of the inverse metric tensor, while a_j and b are smooth functions of x. Physically relevant examples of such operators are the *d'Alembert wave operator* acting on scalars ($E_1 = E_1 = \mathbb{R}$) $P = \Box$ and the Klein-Gordon operator $P = \Box + m^2$, $m > 0$. Moreover, in case $E_1 = E_2 = \Lambda^k T^* M$, we have the *d'Alembert-De Rham-Beltrami operator* $P = \Box_k = \mathrm{d}\delta + \delta\mathrm{d}$ acting on k-forms as well as the *Proca operator* $P = \delta\mathrm{d}_k + m^2$ (for further discussions on the Proca field see [FP03]).
It is shown in [BGP07, Cor. 3.4.3] that (M, g) is a globally hyperbolic spacetime with empty boundary, wave operators as well as their formal duals (since they are wave operator themselves) have retarded and advanced Green operators. Hence, they are Green hyperbolic. ■

Definition 1.4.5. *The operator* $G := G^+ - G^- : C^\infty_{\mathrm{c}}(M, E_2) \to C^\infty(M, E_1)$ *is called the* causal propagator *or* advanced minus retarded *Green operator.*

Remark 1.4.6. Recalling Definition 1.1.9 and the support properties of $G^{\pm}$ in Definition 1.4.1, we see that Green operators of P are in fact linear maps between the following spaces:

$$G^{+} : C^{\infty}_{\mathrm{c}}(M, E_2) \to C^{\infty}_{\mathrm{spc}}(M, E_1), \tag{1.14}$$
$$G^{-} : C^{\infty}_{\mathrm{c}}(M, E_2) \to C^{\infty}_{\mathrm{sfc}}(M, E_1), \tag{1.15}$$
$$G : C^{\infty}_{\mathrm{c}}(M, E_2) \to C^{\infty}_{\mathrm{sc}}(M, E_1). \tag{1.16}$$

Moreover, as shown in [Bär15, Thm. 3.8, Cor. 3.10, 3.11], there are unique continuous linear extensions of $G^{\pm}$:

$$\overline{G}_{+} : C^{\infty}_{\mathrm{pc}}(M, E_2) \to C^{\infty}_{\mathrm{pc}}(M, E_1) \quad \text{and} \quad \overline{G}_{-} : C^{\infty}_{\mathrm{fc}}(M, E_2) \to C^{\infty}_{\mathrm{fc}}(M, E_1), \tag{1.17}$$
$$\widetilde{G}_{+} : C^{\infty}_{\mathrm{spc}}(M, E_2) \to C^{\infty}_{\mathrm{spc}}(M, E_1) \quad \text{and} \quad \widetilde{G}_{-} : C^{\infty}_{\mathrm{sfc}}(M, E_2) \to C^{\infty}_{\mathrm{sfc}}(M, E_1). \tag{1.18}$$

Proposition 1.4.7 (see Cor. 3.9, [Bär15])**.** *Let P be a Green hyperbolic operator. Then there are no nontrivial solutions $u \in C^{\infty}(M, E_1)$ of $Pu = 0$ with past-compact or future-compact support. In other words if u has past-compact or future-compact support, $Pu = 0$ implies $u = 0$. Moreover, for any $f \in C^{\infty}_{\mathrm{pc}}(M, E_2)$ or $f \in C^{\infty}_{\mathrm{fc}}(M, E_2)$ there exists a unique $u \in C^{\infty}(M, E_1)$ solving $Pu = f$ and such that $\operatorname{supp}(u) \subset J^{+}(\operatorname{supp} f)$ or $\operatorname{supp}(u) \subset J^{-}(\operatorname{supp} f)$, respectively.*

Remark 1.4.8. The solutions $u^{\pm}$ of the equation $Pu = f$ with different support properties discussed in the former Proposition are given explicitly by $u^{\pm} = G^{\pm}(f)$. Hence u^{+} is the unique solution to the following initial value problem:

$$\begin{cases} Pu = f \text{ in } M, \ f \in C^{\infty}_{\mathrm{pc}}(M, E_2), \\ u|_{\Sigma} = 0, \end{cases} \tag{1.19}$$

where $\Sigma \overset{\iota}{\hookrightarrow} M$ is any Cauchy surface that lies in the past of $\operatorname{supp} f$, i.e. $\iota(\Sigma) \subset J^{-}(\operatorname{supp} f)$. Analogously u^{-} is the unique solution with vanishing final data on any Cauchy surface in the future of $f \in C^{\infty}_{\mathrm{fc}}(M, E_2)$.

This discussion extends to the case of a spacetime with non-empty timelike boundary, particularly, Proposition 1.4.7 extends, provided the existence of Green operators for a specified boundary condition. In this case, for example $u^{+} = G^{+}_{\mathrm{b.c.}}(f)$ is the solution to the initial data/boundary value problem

$$\begin{cases} Pu = f \text{ in } M, \ f \in C^{\infty}_{\mathrm{pc}}(M, E_2), \\ \text{boundary conditions on } \partial M, \\ u|_{\Sigma} = 0, \end{cases} \tag{1.20}$$

where, as before, Σ is any Cauchy surface such that $\Sigma \subset J^{-}(\operatorname{supp} f)$.

The following is an important theorem that will be generalized in case of non-empty timelike boundary. (see [BG12, Thm. 3.5])

Theorem 1.4.9. *Let G be the causal propagator of a Green-hyperbolic operator P on a spacetime with empty boundary. Then the following is an exact sequence:*

$$0 \longrightarrow C^\infty_c(M,E_1) \xrightarrow{P} C^\infty_c(M,E_2) \xrightarrow{G} C^\infty_{sc}(M,E_1) \xrightarrow{P} C^\infty_{sc}(M,E_2) \longrightarrow 0. \quad (1.21)\,\{?\}$$

In the case of non-empty boundary, the existence of Green operators and all their properties must be proven for any suitable class of boundary conditions, and that will be the main focus of Chapters 2 and 3 when P is Maxwell operator.

Example 1.4.10. (Wave operator on $\mathbb{R}\times\mathbb{R}_+$)
We consider the problem of the existence and the construction of advanced and retarded Green operators of $\Box = -\partial_t^2 + \partial_x^2$ on $M = \mathbb{R}\times\mathbb{R}_+ \ni (t,x)$. Clearly M is a globally hyperbolic spacetime with timelike boundary, endowed with the usual Minkowski metric $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2$. The boundary is the set $\{(t,0),\, t\in\mathbb{R}\}$. Given some initial condition, the differential equation $\Box u = f$, with $f \in C^\infty(M)$, is well posed (i.e. there exists a unique solution) provided one requires u to satisfy some suitable boundary conditions. We construct explicitly the Green operators for $\Box$ on M with Dirichlet and Neumann boundary conditions using the Green operators for $\Box$ on $(\mathbb{R}^2,\eta)$, whose existence is well known. We recall that, for a scalar function u, Dirichlet, Neumann and Robin boundary conditions are obtained by imposing, respectively,

$$u|_{\partial M} = 0; \quad \frac{\partial u}{\partial \nu}\Big|_{\partial M} = 0; \quad u|_{\partial M} = f\frac{\partial u}{\partial \nu}\Big|_{\partial M}, \text{ for } f\in C^\infty(\partial M)\,,$$

ν being the vector field normal to ∂M.
Consequently, we define $\Box_D : C^\infty_D(M) \to C^\infty(M)$ and $\Box_N : C^\infty_N(M) \to C^\infty(M)$, with $C^\infty_D(M) := \{u \in C^\infty(M)\,|\, u|_{x=0} = 0\} = \Omega^0_{\mathrm{t}}(M)$ and $C^\infty_N(M) := \{u\in C^\infty(M)\,|\,\partial_x u|_{x=0} = 0\}$. The problem is to find the following advanced and retarded Green operators

$$G^\pm_D : C^\infty_c(M) \to C^\infty_D(M), \quad G^\pm_N : C^\infty_c(M) \to C^\infty_N(M). \quad (1.22)\,\{?\}$$

As stated in [Bär15, Ex. 3.4], advanced and retarded Green operators for $\Box$ on $\mathbb{R}^2$ exist and have the following explicit expression

$$G^\pm(f)(t,x) = -\frac{1}{2}\int_{J^\mp_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y. \quad (1.23)\,\{?\}$$

This expression entails that the integral kernel of $G^\pm$ (also known as Green function or fundamental solution) is $-\frac{1}{2}$ times the characteristic function of $\{(t,x,s,y)\in\mathbb{R}^4\,|\,(s,y)\in J^\mp(t,x)\}$. The ansatz, based on the method of images ([Jac99, p. 480]), is that the Dirichlet and Neumann

Green operators will be respectively of the form

$$
\begin{aligned}
G^{\pm}_{D}(f)(t,x) =& \, G^{\pm}(f)(t,x) - G^{\pm}(f)(t,-x) = \\
=& -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y - \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M, \\
G^{\pm}_{D}(f)(t,x) =& \, G^{\pm}(f)(t,x) + G^{\pm}(f)(t,-x) = \\
=& -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y + \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M.
\end{aligned}
$$

It is a straightforward calculation to verify $G^{\pm}_{D/N}(f) \in C^{\infty}_{D/N}(M)$ (i.e. $G^{\pm}_{D}(f)(t,x)|_{x=0} = 0$ and $\partial_x G^{\pm}_{D}(f)(t,x)|_{x=0} = 0$), in addition the support properties still hold.
Focusing on the Dirichlet Green operators, they are constructed by imagining to extend the manifold M by reflection to be the entire $\mathbb{R}^2$ and, to enforce $G^{\pm}_{D}(f)$ to vanish on $x = 0$, add a negative reflected source $-f(t,-x)$. This gives the desired result. ■

1.5 Maxwell equations for k-forms with empty boundary

We focus our attention on a m-dimensional spacetime (M,g) with empty boundary. Classically, electromagnetism is the theory of electric and magnetic fields E, B encoded in the Faraday 2-form F. The equations for $F \in \Omega^2(M)$ read

$$
\begin{aligned}
\mathrm{d}F &= 0, \\
\delta F &= -J,
\end{aligned} \tag{1.24}
$$
?Eqn: Maxwellsequatic

where J is the co-exact current 1-form, which encodes the current conservation laws. Indeed, if M is static with $M = \mathbb{R}\times\Sigma$, the decomposition $F = B + \mathrm{d}t\wedge E$ holds, where $E \in C^{\infty}(\mathbb{R},\Omega^1(\Sigma))$ and $B \in C^{\infty}(\mathbb{R},\Omega^2(\Sigma))$, in agreement with the fact that the magnetic field B is usually referred to as a *pseudo-vector*.
The first equation imposes a geometric constraint: it ensures that the 2-form F is closed. Hence, in virtue of Poincaré lemma, whenever the second de Rham cohomology group $H^2(M)$ (see A.1) is trivial, there exists a global 1-form A such that $F = \mathrm{d}A$. One can object that the choice of $A \in \Omega^1(M)$ is not unique. Indeed if we assume M to be globally hyperbolic with empty boundary, the configuration $A' := A + \mathrm{d}\chi$, $\chi \in \Omega^0(M)$ is equivalent to A since it gives rise to the same Faraday tensor F. This freedom in the choice of A is extensively used and it is called **gauge freedom** or gauge invariance. In this case A, A' are said to be gauge-equivalent.
Thanks to gauge invariance we can therefore first write Maxwell equations for A as $\delta\mathrm{d}A = -J$. Subsequently, taken any fixed $A \in \Omega^1(M)$, and imposing the so-called **Lorenz gauge**, one can

substitute the problem $\delta \mathrm{d}A = -J$ with the following hyperbolic system of equations

$$\begin{cases} \Box A = -J, \\ \delta A = 0. \end{cases} \tag{1.25}$$ {?}

where $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ is the wave operator. Moreover the second equation can be seen as a constraint called the *Lorenz gauge condition*. This system can be obtained by requiring a 1-form A', gauge-equivalent to A, to satisfy the Lorenz gauge condition $\delta A' = 0$. This is always possible in a globally hyperbolic spacetime with empty boundary since the equation $\Box\chi = \delta A$ has always at least a solution $\chi \in \Omega^0(M)$ for any fixed $A \in \Omega^1(M)$.

One could argue that the most general possible gauge transformation between A and A' is of the form $A' = A + \omega$ for a closed form $\omega \in \Omega^1(M)$. That is certainly true in the sense that the equations of motion (1.24) are unchanged by this transformation. Anyway we will refer to gauge-invariance exclusively in the sense previously defined since electromagnetism can be seen as an abelian **gauge theory** with structure group $U(1)$. In this framework, the classical vector potential A is a principal connection on a principal $U(1)$-bundle E over M (for more details see [Nak90, Ch. 10]). Then we identify (this choice is non-unique) the connection A with a 1-form $A \in \Omega^1(M)$. Locally, this principal connection can be expressed as an $U(1)$-valued operator $D = \mathrm{d} + A$ and the Faraday field can be recovered as the curvature of this connection: $F = D \circ D$. A gauge transformation for A in this context is of the form

$$A' = g^{-1}Ag - ig^{-1}\mathrm{d}g, \tag{1.26}$$ {?}

for any $g \in C^\infty(M, U(1))$. If now we express $g = e^{i\chi}$, for $\chi \in \Omega^0(M)$, we recover the transformation $A' := A + \mathrm{d}\chi$.

Remark 1.5.1. In the homogeneous case ($J = 0$), one can generalize the Maxwell field to be $F \in \Omega^k(M)$, imposing $\mathrm{d}F = 0$ and $\delta F = 0$ and the equation for $A \in \Omega^{k-1}(M)$ becomes $\delta\mathrm{d}A = 0$. In this case gauge freedom is understood as a transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-2}(M)$. It is worth noticing that in case $k = 0$ and $k = m$, the equations $\delta F = 0$ and $\mathrm{d}F = 0$ become, respectively, trivial.

From a physical point of view, one wonders whether it is A or it is F the observable field of the dynamical system. Hence one can regard electromagnetism as a theory for $F \in \Omega^2(M)$ or as a theory for a non-unique $A \in \Omega^1(M)$ wondering whether the initial and boundary value problem for Maxwell equations is well-posed in both cases. The former case will be covered in Chapter 2 and the latter in Chapter 3.

In many, but not all, practical physical situations, the triviality of $H^2(M)$ ensures that description of electromagnetism in terms of F or of A is completely indistinguishable. There is in fact one particular physical effect that enlightens the true nature of electromagnetism as a theory for

the potential 1-form A: this is the so-called *Aharonov-Bohm effect*. To discuss this effect we refer mostly to [SDH14, Ex. 3.1]. Consider indeed as a globally hyperbolic spacetime M the Cauchy development in the 4-dimensional Minkowski spacetime $\mathbb{M}^4$ of the time-fixed hypersurface $\{0\} \times \mathbb{R}^3$ with a cylinder surrounding the z-axis (which is given in cylindrical coordinates (t, r, φ, z) by $r \leq 1$) removed. The removed cylinder represents an infinitely long coil with a current running through it whose magnetic flux Φ gives rise outside the coil to a vanishing Faraday tensor F but also to a non-vanishing vector potential given in very good approximation by $A_\Phi = \frac{\Phi}{2\pi}\mathrm{d}\varphi$.
In the Aharonov-Bohm experiment one sends quantum particles from one side to the other of the coil and measures a quantum phase shift proportional to the integral of A_Φ around a circular path that embraces the cylinder (see [Pes89] for an experimental description). This setup shows that even if the Faraday tensor F vanishes outside the coil, there still is a measurable physical effect which depends on the vector potential A_Φ, which appears to be the true observable field.
In particular this effect happens because A_Φ is closed, but not exact in M. Moreover A_Φ is not gauge equivalent to 0. From a topological point of view this corresponds to the fact that the first de Rham cohomology group $H^1(M) \neq \{0\}$. Indeed $H^1(M)$ is spanned just by the vector potential $\mathrm{d}\varphi$. Whenever $H^1(M)$ is trivial, the two descriptions with F and A are indistinguishable. For further discussions, see [Ben+14].

On the other hand, the formulation in terms of the field strength F has its advantages. Indeed in the Lagrangian formulation of the field theory, the Maxwell action

$$\mathcal{S}_{EM} = -\frac{1}{4}(F, F) = -\frac{1}{4}\int_M F \wedge \star F, \tag{1.27}$$ {?}

is invariant under a conformal scaling $g \mapsto \Omega^2 g$ of the metric. This consideration is useful when the underlying spacetime possesses a conformal boundary, such as in asymptotically flat spacetimes and AdS spacetimes. AdS are spacetimes with conformal timelike boundary, together with a metric which is conformally related to that being under our exam. The study of quantum field theories and boundary conditions on AdS spacetime is motivated by the long-term ambition to understand in rigorous mathematical terms the AdS/CFT conjecture.

The next chapters will be devoted to tackle the problem of well-posedness of electromagnetism equations when the spacetime has non-empty timelike boundary.

Chapter 2

Maxwell equations for F with interface conditions

As outlined in Section 1.5, the form of Maxwell equations allows us to use both F and A as variables with which we can describe electromagnetic phenomena. Whenever the second cohomology group $H^2(M)$ is trivial, the two theories are equivalent, since $F = \mathrm{d}A$.
In this chapter, we regard $F \in \Omega^2(M)$ as the physical dynamical variable which describes electromagnetism. This is not always true, whenever the first cohomology group with integer coefficients is non-trivial, as previously discussed in Section 1.5. The aim of this chapter is to present a technique which allows to characterize, in a class of manifolds with the presence of an interface between two media, the existence of fundamental solutions for Maxwell equations, written in terms of the Faraday form $F \in \Omega^2(M)$. The presence of an interface on the one hand generalizes the idea of a timelike boundary, allowing to recover the geometric setting outlined in Chapter 1 if one side of the interface is a perfect insulator. On the other hand, in order to make use of geometric techniques such as Hodge decomposition, we will have to make several geometric assumptions which ensure global hyperbolicity, but unfortunately they lead to a loss in generality.

2.1 Geometrical setup

The physical and practical situation we want to approach is that of a manifold split into two parts, filled with two media, each of them with different electromagnetic properties. The two media will be separated by an hypersurface, on which our aim will be that of putting *jump conditions*.
We consider a globally hyperbolic, standard static Lorentzian manifold (M, g) with *empty boundary*, such that M can be decomposed as $\mathbb{R} \times \Sigma$, where the Cauchy hypersurface (Σ, h) is assumed to be a complete, connected, odd-dimensional, *closed* Riemannian manifold. Under these conditions, Σ is of *bounded geometry* (see 1.3.1). In this chapter, we denote with d_M, δ_M the differential and co-differential over M, while d, δ denote those over Σ.

Maxwell equations read

$$\mathrm{d}_M F = 0\,, \quad \delta_M F = 0\,, \quad F \in \Omega^2(M). \tag{2.1}$$

?Eqn: covariant M

Given the decomposition $M \simeq \mathbb{R} \times \Sigma$ and recalling Theorem 1.1.2, let us indicate with $\iota_t \colon \Sigma \to M$ the smooth one-parameter group of embedding maps which realizes Σ at time t as $\iota_t\Sigma = \{t\} \times \Sigma \doteq \Sigma_t$. Then, it holds the diffeomorphism $\Sigma_t \simeq \Sigma_{t'} \simeq \Sigma$ for all $t, t' \in \mathbb{R}$. Hence, for all $\omega \in \Omega^k(M)$ and $t \in \mathbb{R}$, $\omega|_{\Sigma_t} \in \Gamma(\iota_t^* \Lambda^k T^* M)$, where $\iota_t^*(\Lambda^k T^* M)$ denotes the pull-back bundle over $\Sigma_t \simeq \Sigma$ built out of $\Lambda^k T^* M$ via ι_t – *cf.* [Hus66]. Moreover, recalling Definition 1.2.3, it holds that $\omega|_{\Sigma_t}$ can be further decomposed as

$$\omega|_{\Sigma_t} := (\star_{\Sigma_t}^{-1} \iota_t^* \star_M)\omega \wedge \mathrm{d}t + \iota_t^*\omega = \mathrm{n}_{\Sigma_t}\omega \wedge \mathrm{d}t + \mathrm{t}_{\Sigma_t}\omega\,.$$

where $\mathrm{t}_{\Sigma_t}\omega \in \Omega^k(\Sigma_t)$ while $\mathrm{n}_{\Sigma_t}\omega \in \Omega^{k-1}(\Sigma_t)$ – *cf.* Definition 1.2.3. With the identification $\Sigma_t \simeq \Sigma_{t'}$ the decomposition induces the isomorphisms

$$\begin{aligned}
\Gamma(\iota_\tau^* \Lambda^k T^* M) &\simeq \Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma) \\
\omega &\to (\omega_0 \oplus \omega_1)\,, && (2.2) \\
\Omega^k(M) &\simeq C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k(\Sigma)) \\
\omega &\to (\tau \mapsto \mathrm{n}_{\Sigma_\tau}\omega) \oplus (\tau \mapsto \mathrm{t}_{\Sigma_\tau}\omega)\,. && (2.3)
\end{aligned}$$

{?} {?}

In this way, we have rewritten any differential form as a pair of differential form valued functions of time.
To recover the electric and magnetic components of F, we simply define $E \doteq -\mathrm{t}_{\Sigma_t} F$ and $\star_\Sigma B = \mathrm{n}_{\Sigma_t} F$, such that

$$F = \star_\Sigma B + \mathrm{d}t \wedge E\,, \tag{2.4}$$

?Eqn: electric ar

where now $E, B \in C^\infty(\mathbb{R}, \Omega^1(\Sigma))$ while $\star_\Sigma$ is the Hodge dual on Σ. Maxwell equations reduce to

$$\partial_t E - \mathrm{curl} B = 0\,, \qquad \partial_t B + \mathrm{curl} E = 0\,, \tag{2.5a}$$

$$\mathrm{div}(E) = \mathrm{div}(B) = 0\,, \tag{2.5b}$$

?Eqn: dynamical M ?Eqn: non-dynamic

where $\mathrm{div} = \delta$ is the co-differential on Σ, while curl is defined in Equation (2.41) – in particular $\mathrm{curl} = \star_\Sigma \mathrm{d}$ if $\dim \Sigma = 3$.
To model the presence of an interface that divides M in two distinct regions, we consider Z a codimension 1 smooth embedded hypersurface of Σ.

In this setting we consider Maxwell equations with Z-interface boundary conditions, that is we allow discontinuities to occur on $\mathbb{R} \times Z$. Hence, we split $\Sigma = \Sigma_+ \cup \Sigma_-$, such that

$$\Sigma_Z := \Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-, \tag{2.6}$$?Eqn: Sigma Z splitti

and we refer to Σ_- (*resp.* Σ_+) as the left (*resp.* right) component of Σ. Moreover, $\Sigma_\pm$ are compact manifolds with boundary $\partial\Sigma_\pm = Z$, and the orientation on Z induced by Σ_+ is the opposite of the one induced by Σ_-. Hence, the manifolds $(\mathbb{R} \times \Sigma_\pm, g = -\mathrm{d}t^2 + h)$ are *globally hyperbolic spacetimes with timelike boundary* (see 1.1.1), which is $\mathbb{R} \times Z$.

Whenever the interface $Z \neq \emptyset$ the system (2.5) has to be modified, in particular the non-dynamical equations (2.5b) involving the divergence operator div have to be suitably interpreted – *cf.* Subsection 2.2.4. In particular one expects that the condition $\operatorname{div}(E) = \operatorname{div}(B) = 0$ should be read at a distributional level, leading to a constraint on the values at Z of the normal component of E. In addition, the dynamical equations (2.5a) have to be combined with boundary conditions at the interface Z – *cf.* [Jac99, Sec. I.5].
In what follows we will state the precise meaning of the problem (2.5) with interface Z with the help of Hodge theory and of Lagrangian subspaces [EM99; EM03; EM05].

2.2 Constraint equations: Hodge theory with interface

In this section we present a Hodge decomposition a the closed Riemannian manifold (Σ, h) with interface Z. This generalizes the known results on classical Hodge decomposition on manifolds possibly with non-empty boundary [Ama17; AM04; Gaf55; Gro+91; Kod49; Li09; Sch95; Sco95; AS+00].

Hodge theory is a generalization of Helmholtz decomposition. The latter was formulated as a splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the *Hodge decomposition*. The idea behind Helmholtz decomposition is that any vector field $\mathbf{F}$ in $\mathbb{R}^3$ can be read as a sum of an irrotational field $\mathbf{U}$, i.e. such that $\operatorname{curl}\mathbf{U} = \mathrm{d}\mathbf{U} = 0$, and a solenoidal field $\mathbf{V}$, i.e. such that $\operatorname{div}\mathbf{V} = \delta\mathbf{V} = 0$. In other words, for $\mathbf{F} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$, one can write

$$\mathbf{F} = -\nabla\Phi + \operatorname{curl}\mathbf{A}, \tag{2.7}$$ {?}

where we used the fact that in $\mathbb{R}^3$ $\operatorname{curl}\mathbf{U} = 0$ implies $\mathbf{U} = -\nabla\Phi$, since $\mathbb{R}^3$ is simply connected.

Remark 2.2.1. With reference to Definition 1.3.5 and Remark 1.3.8, in what follows $\mathrm{L}^2\Omega^k(\Sigma)$ will denote the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ (see Section 1.2) with respect to the pairing $(\ ,\ )_\Sigma$ between k-forms

$$(\alpha, \beta)_\Sigma := \int_\Sigma \overline{\alpha} \wedge \star_\Sigma \beta \qquad \alpha, \beta \in \Omega^k_{\mathrm{c}}(\Sigma)\,, \tag{2.8}$$?Eqn: L2-scalar produ

where $\star_\Sigma$ is the Hodge dual on Σ.

Remark 2.2.2. With a slight abuse of notation we denote still with d and δ the extension to the space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ of the action of the differential and of the codifferential on $\Omega^k_{\mathrm{c}}(\Sigma)$.

Remark 2.2.3. In agreement with Remark 1.2.1, we denote with $\Omega^k_{\mathrm{c}}(\Sigma)$ the space of smooth and compactly supported k-forms. Moreover, since Σ is of bounded geometry, for $\ell \geq \frac{1}{2}$, we use the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$ and $\mathrm{H}^\ell_0\Omega^k(\Sigma)$ of k-forms as defined in Definition 1.3.6 and in Subsection 1.3.2.
If Σ is compact, $\Omega^k_{\mathrm{c}}(\Sigma)$ coincides with the space of smooth k-forms $\Omega^k(\Sigma)$, but we will still use $\Omega^k_{\mathrm{c}}(\Sigma)$ in view of possible generalizations. In addition, we remark that $\mathrm{H}^{-\ell}\Omega^k(\Sigma) = \mathrm{H}^\ell_0\Omega^k(\Sigma)^*$, where $*$ indicates the dual with respect to the scalar product $(\, ,\,)_\Sigma$.

The Hodge theorem for a closed manifold Σ states that there is an L^2-orthogonal decomposition

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}\mathrm{H}^1\Omega^{k-1}(\Sigma) \oplus \delta\mathrm{H}^1\Omega^{k+1}(\Sigma) \oplus \ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}\,, \tag{2.9}$$

?Eqn: Hodge decom

where $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ is the Laplace operator and $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ denotes the space of *harmonic forms*. If Σ has an empty boundary, the space of harmonic forms coincides with that of *harmonic fields*, $\ker(\delta)_{\mathrm{H}^1\Omega^k(\Sigma)} \cap \ker(\mathrm{d})_{\mathrm{H}^1\Omega^k(\Sigma)}$ (see [Kod49] and [Sch95]). The last result can be stated as follows and it is very easy to prove.

Proposition 2.2.4. *Let $\alpha \in \mathrm{H}^2\Omega^k(\Sigma)$, where Σ is a closed manifold. Then $\Delta\alpha = 0$ if and only if $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$.*

Proof. If $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$, $\Delta\alpha = 0$. On the other hand if $\Delta\alpha = 0$,

$$0 = (\Delta\alpha, \alpha)_\Sigma = ((\mathrm{d}\delta + \delta\mathrm{d})\alpha, \alpha)_\Sigma = (\mathrm{d}\delta\alpha, \alpha)_\Sigma + (\delta\mathrm{d}\alpha, \alpha)_\Sigma = \tag{2.10}$$
{?}

$$= (\delta\alpha, \delta\alpha)_\Sigma + (\mathrm{d}\alpha, \mathrm{d}\alpha)_\Sigma = \|\delta\alpha\|^2 + \|\mathrm{d}\alpha\|^2. \tag{2.11}$$
{?}

So both $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$. ■

2.2.1 Hodge decomposition on compact manifold with non-empty boundary

For a compact manifold Σ with non-empty boundary $\partial\Sigma$ the decomposition (2.9) requires a slight adjustment and harmonic forms do not coincide with harmonic fields anymore. Because of boundary terms, $\ker\Delta$ no longer coincides with the closed and co-closed forms. It turns out that every harmonic field is a harmonic form, but the converse is false. To show this, consider the following example.

Example 2.2.5. Let U be a bounded subset of $\mathbb{R}^2$, endowed with the standard Euclidean metric. On U, the 1-form $\omega = x\,dy$ is harmonic, since its second derivatives vanish, but $\omega \notin \ker\mathrm{d}$, since

$$\mathrm{d}(x\,\mathrm{d}y) = \partial_x\, x\,\mathrm{d}x \wedge \mathrm{d}y + \partial_y\, x\,\mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y.$$

ω is though in $\ker\delta$ as $\star\,\mathrm{d}\star(x\,\mathrm{d}y) = \star\,\mathrm{d}(x\,\mathrm{d}x) = 0$. ■

Definition 2.2.6. *We call $\mathcal{H}^k(\Sigma)$ the L^2-closure of the space of harmonic fields*

$$\mathcal{H}^k(\Sigma) = \overline{\{\omega \in \mathrm{H}^1\Omega^k(\Sigma) |\ \mathrm{d}\omega = 0\,,\ \delta\omega = 0\}}\,. \tag{2.12}$$?Eqn: harmonic fields

With a slight abuse of notation, we will refer to the elements of $\mathcal{H}^k(\Sigma)$ as harmonic fields

In fact, the space of harmonic fields is infinite dimensional and the spaces $\mathrm{dH}^1\Omega^{k-1}(\Sigma)$, $\delta H^1\Omega^{k+1}(\Sigma)$, $\mathcal{H}^k(\Sigma)$ are not orthogonal unless suitable boundary conditions are imposed. Therefore, one has to give a precise meaning to the boundary value of a differential form. Since differential forms are not scalar quantities, one can define a normal and a tangential projection along the boundary.

Remark 2.2.7. We recall that the tangential and normal traces t and n of a differential form are defined according to Definition 1.2.3 and are extended as in Subsection 1.3.2 to continuous surjective maps as in Equation (1.11), that we recall for completeness:

$$\mathrm{t}\oplus\mathrm{n}\colon \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \oplus \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \qquad \forall \ell \geq \frac{1}{2}\,. \tag{2.13}$$ {?}

Next, we present the Hodge decomposition for compact manifolds with boundary, a proof of which can be found at [Sch95, Thm. 2.4.2].

Theorem 2.2.8. *Let (Σ, h) be a compact, connected, Riemannian manifold with non-empty boundary*

1. *For all $\omega \in \Omega^{k-1}_{\mathrm{c}}(\Sigma)$ and $\eta \in \Omega^k_{\mathrm{c}}(\Sigma)$ it holds*

$$(\mathrm{d}\omega, \eta)_\Sigma - (\omega, \delta\eta)_\Sigma = (\mathrm{t}\omega, \mathrm{n}\eta)_{\partial\Sigma}\,, \tag{2.14}$$?Eqn: boundary terms?

   where $(\ ,\ )_\Sigma$ has been defined in Equation (2.8) while $(\ ,\ )_{\partial\Sigma}$ is defined similarly. Equation (2.14) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma)$.

2. *The Hilbert space $\mathrm{L}^2\Omega^k(\Sigma)$ of square integrable k-forms splits in the L^2-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{dH}^1\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.15}$$?Eqn: Hodge decomposi

   where $\mathcal{H}^k(\Sigma)$ is defined per Equation (2.12) while, in view of Equation (1.4)

$$\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma) |\ \mathrm{t}\alpha = 0\}\,, \tag{2.16}$$?Eqn: Dirichlet and N

$$\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma) |\ \mathrm{n}\beta = 0\}\,. \tag{2.17}$$ {?}

Remark 2.2.9. The previous decomposition generalizes to Sobolev spaces, in particular for all $\ell \in \mathbb{N} \cup \{0\}$ we have

$$\mathrm{H}^\ell\Omega^k(\Sigma) = \mathrm{d}\mathrm{H}^{\ell+1}\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta\mathrm{H}^{\ell+1}\Omega^{k+1}_{\mathrm{n}} \oplus \mathrm{H}^\ell\mathcal{H}^k(\Sigma)\,, \tag{2.18}$$

?Eqn: Hodge decom

where $\mathrm{H}^\ell\mathcal{H}^k(\Sigma) = \mathcal{H}^k(\Sigma) \cap \mathrm{H}^\ell\Omega^k(\Sigma)$, since $\mathrm{H}^\ell\Omega^k(\Sigma) \hookrightarrow \mathrm{L}^2\Omega^k(\Sigma)$.

2.2.2 Hodge decomposition for compact manifold with interface

In this section we generalize Theorem 2.2.8 to the case of a closed Riemannian manifold Σ together with an interface Z. As starting point, we need to distinguish between regular k-forms which are defined on the whole manifold, and hence continuous, and pairs of forms which are regular separately on the two sides $\Sigma_\pm$ and are allowed to be discontinuous on Z.

Definition 2.2.10. *We call*

$$\Omega^k(\Sigma_Z) := \Omega^k(\Sigma_+) \oplus \Omega^k(\Sigma_-)\,, \tag{2.19}$$

?Eqn: splitting c

where it is understood that the pair $\omega+\eta \in \Omega^k(\Sigma_+)\oplus\Omega^k(\Sigma_-)$ *identifies an element* $\alpha \in \Omega^k(\Sigma_Z)$ *such that* $\alpha|_{\Sigma_+} = \omega$ *and* $\alpha|_{\Sigma_-} = \eta$.

Following the previous definition,

$$\Omega^k_{\mathrm{c}}(\Sigma_Z) = \Omega^k_{\mathrm{c}}(\Sigma_+) \oplus \Omega^k_{\mathrm{c}}(\Sigma_-)\,. \tag{2.20}$$

{?}

This implies $\omega \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ if and only if ω is a smooth k-form in Σ_Z and $\mathrm{supp}_\Sigma\,\omega := \overline{\{x \in \Sigma_Z \,|\, \omega(x) \neq 0\}}^{\Sigma}$ is compact. Hence, forms in $\Omega^k_{\mathrm{c}}(\Sigma_Z)$ have support overlapping with the interface, where they are allowed to be discontinuous.

Observe that Theorem 2.2.8 applies to both $\mathrm{L}^2\Omega^k(\Sigma_\pm)$. In addition, since Z has zero measure the space of square integrable k-forms splits as

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{L}^2\Omega^k(\Sigma_Z) = \mathrm{L}^2\Omega^k(\Sigma_+) \oplus \mathrm{L}^2\Omega^k(\Sigma_-)\,. \tag{2.21}$$

?Eqn: splitting c

We expect that a counterpart of (2.15) holds true, though $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\mathrm{H}^1\Omega^{k-1}_{\mathrm{n}}(\Sigma)$ ought to be replaced by suitable jump conditions across Z. To this end, notice that the splitting (2.21) does not generalize to the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$, in particular

$$\mathrm{H}^\ell\Omega^k(\Sigma) \hookrightarrow \mathrm{H}^\ell\Omega^k(\Sigma_Z) = \mathrm{H}^\ell\Omega^k(\Sigma_+) \oplus \mathrm{H}^\ell\Omega^k(\Sigma_-)\,, \tag{2.22}$$

{?}

is a proper inclusion. Indeed, consider any regular form ω in Σ_Z which has $[\mathrm{t}\omega] \neq 0$. In this case ω can not have square integrable (weak) derivatives, since a non-vanishing jump gives rise to a distributional derivative which is proportional to the Dirac delta.

Definition 2.2.11. *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface $Z \hookrightarrow \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = Z$ such that $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. For $\omega \in \Omega^k(\Sigma_Z)$ we define the tangential jump $[\mathrm{t}\omega] \in \Omega^k(Z)$ and normal jump $[\mathrm{n}\omega] \in \Omega^{k-1}(Z)$ across Z by*

$$[\mathrm{t}\omega] := \mathrm{t}_+\omega - \mathrm{t}_-\omega\,, \qquad [\mathrm{n}\omega] := \mathrm{n}_+\omega - \mathrm{n}_-\omega\,, \tag{2.23}$$
?Eqn: tangential and

where $\mathrm{t}_\pm, \mathrm{n}_\pm$ denote the tangential and normal map on $\Sigma_\pm$ as per Definition 1.2.3.

Remark 2.2.12. The tangential and normal traces $\mathrm{t}_\pm, \mathrm{n}_\pm$ as well as the tangential and normal jump extend by continuity on $\mathrm{H}^1\Omega^k(\Sigma_Z)$ and are surjective if the codomain is $\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(Z)$ - *cf.* Remark 2.2.7. As a consequence of Definition 2.2.11 it holds that

$$\mathrm{H}^1\Omega^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma_Z) |\; [\mathrm{t}\omega] = 0\,,\; [\mathrm{n}\omega] = 0\}\,. \tag{2.24}$$
{?}

An analogous equality does not hold for $\Omega^k(\Sigma)$ because it would require traces of higher order derivatives to match at Z.

Theorem 2.2.13. *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface Z. Moreover let $(\Sigma_\pm, h_\pm)$ be the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = Z$ such that $\Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-$.*

1. *For all $\omega \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_Z)$ and $\eta \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ it holds*

$$(\mathrm{d}\omega, \eta)_Z - (\omega, \delta\eta)_Z = ([\mathrm{t}\omega], \mathrm{n}_+\eta)_Z - (\mathrm{t}_-\omega, [\mathrm{n}\eta])_Z\,, \tag{2.25}$$
?Eqn: boundary terms

where $(\;,\;)_Z$ is the scalar product between forms on Z – cf. Equation (2.8) – while $\mathrm{t}_\pm, \mathrm{n}_\pm$ are the tangential and normal maps on $\Sigma_\pm$ as per Definition 1.2.3. Equation (2.25) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for all $\ell \geq 1$.

2. *The Hilbert space $\mathrm{L}^2\Omega^k(\Sigma)$ of square integrable k-forms splits into the L^2-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.26}$$
?Eqn: Hodge decomposi

where $\mathcal{H}^k(\Sigma)$ is defined per Equation (2.12), while

$$\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma_Z) |\; [\mathrm{t}\alpha] = 0\}\,, \tag{2.27}$$
?Eqn: Dirichlet and N

$$\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma_Z) |\; [\mathrm{n}\beta] = 0\}\,. \tag{2.28}$$
{?}

Proof. Equation (2.25) is an immediate consequence of (2.14). In particular for $\omega \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_Z)$ and $\eta \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ we decompose $\omega = \omega_+ + \omega_-$ and $\eta = \eta_+ + \eta_-$ where $\omega_\pm \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_\pm)$ and

$\eta_\pm \in \Omega^k_{\mathrm{c}}(\Sigma_\pm)$. (Notice that we have $\mathrm{t}_\pm\omega = \mathrm{t}_\pm\omega_\pm$.) Applying Equation (2.14) it holds

$$\begin{aligned}(\mathrm{d}\omega,\eta) - (\omega,\delta\eta) &= \sum_\pm \left((\mathrm{d}\omega_\pm,\eta_\pm) - (\omega_\pm,\delta\eta_\pm)\right) = \int_Z \mathrm{t}_+\overline{\omega}\wedge\star_Z\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega}\wedge\star_Z\mathrm{n}_-\eta \\ &= \int_Z [\mathrm{t}\overline{\omega}]\wedge\star_Z\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega}\wedge\star_Z[\mathrm{n}\beta]\,.\end{aligned}$$

A density argument leads to the same identity for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for $\ell \geq 1$. We prove the splitting (2.26). The spaces $\mathrm{dH}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)$, $\delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$, $\mathcal{H}^k(\Sigma)$ are orthogonal because of Equation (2.25). Let ω be in the orthogonal complement of $\mathrm{dH}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)\oplus\delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. We wish to show that $\omega \in \mathcal{H}^k(\Sigma)$. We split $\omega = \omega_+ + \omega_-$ with $\omega_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)$, we apply Theorem 2.2.8 to each component so that

$$\omega = \sum_\pm \left(\mathrm{d}\alpha_\pm + \delta\beta_\pm + \kappa_\pm\right),$$

where $\alpha_\pm \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_\pm)$, $\beta_\pm \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_\pm)$ and $\kappa_\pm \in \mathcal{H}^k(\Sigma_\pm)$. Let $\hat{\alpha} \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_+)$: This identifies an element of $\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ by considering its extension to zero on Σ_-. Since $\omega \in \left[\mathrm{dH}^1\Omega_{[\mathrm{t}]}(\Sigma_Z)\right]^\perp$ we have $0 = (\mathrm{d}\hat{\alpha},\omega) = (\mathrm{d}\hat{\alpha},\mathrm{d}\alpha_+)$, thus $\mathrm{d}\alpha_+ = 0$ by the arbitrariness of $\hat{\alpha}$. With a similar argument we have $\alpha_- = 0$ as well as $\beta_\pm = 0$.
Therefore $\omega \in \mathcal{H}^k(\Sigma_Z)$. In order to prove that $\omega \in \mathcal{H}^k(\Sigma)$ we need to show that $[\mathrm{t}\omega] = 0$ as well as $[\mathrm{n}\omega] = 0$ – *cf.* Remark 2.2.12. This is a consequence of $\omega \in \left[\mathrm{dH}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)\oplus\delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)\right]^\perp$. Indeed, let $\alpha \in \mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$. Applying Equation (2.25) we find

$$0 = (\mathrm{d}\alpha,\omega) = -\int_Z \mathrm{t}_-\overline{\alpha}\wedge\star_Z[\mathrm{n}\omega]\,. \tag{2.29}$$ {?}

The arbitrariness of $\mathrm{t}_-\alpha$, t_- being surjective, implies $[\mathrm{n}\omega] = 0$. Similarly $[\mathrm{t}\omega] = 0$ follows by $\omega \perp \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. ■

Remark 2.2.14. The harmonic part of decomposition (2.26) contains harmonic k-forms which are continuous across the interface Z – *cf.* Remark 2.2.12. One can also consider a decomposition which allows for a discontinuous harmonic component. In particular it can be shown that

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{dH}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)\oplus\delta\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)\oplus\mathcal{H}^k(\Sigma_Z)\,,$$

where now $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)$ is the subspace of $\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ made of $(k-1)$-forms α such that $\mathrm{t}_\pm\omega = 0$ and similarly $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)$ if and only if $\beta \in \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$ and $\mathrm{n}_\pm\beta = 0$.

2.2.3 Further perspectives on Hodge decomposition

The results of Theorem 2.2.8 can be generalized. In 1949, Kodaira (see [Kod49]) proved a *weak* L^2 orthogonal decomposition, for any (non-compact) Riemannian manifold (M, g) with no boundary, of the form

$$\mathrm{L}^2\Omega^k(M) = \overline{\mathrm{d}\Omega_{\mathrm{c}}^{k-1}(M)} \oplus \overline{\delta\Omega_{\mathrm{c}}^{k+1}(M)} \oplus \mathcal{H}^k(M)\,. \tag{2.30}$$ {?}

Gromov, in [Gro+91], proved that under the assumption that the Laplacian has a spectral gap in $\mathrm{L}^2\Omega^k(M)$, i.e. there is no spectrum of Δ in an open interval $(0, \eta)$, with $\eta > 0$, the following strong L^2-orthogonal decomposition holds for any (non-compact) Riemannian manifold (M, g) with empty boundary:

$$\mathrm{L}^2\Omega^k(M) = \mathrm{d}\mathrm{H}^1\Omega^{k-1}(M) \oplus \delta\mathrm{H}^1\Omega^{k+1}(M) \oplus \mathcal{H}^k(M)\,. \tag{2.31}$$ {?}

For the case $\partial M \neq \emptyset$, the paper by Amar, [Ama17], recovers a strong L^p decomposition for complete non-compact manifolds, while both [Li09] and [AS+00] prove the strong L^p decomposition within the framework of weighted Sobolev spaces. [Sco95] discusses instead a strong L^p-decomposition on compact manifolds. Finally, using weighted Sobolev spaces, Schwartz [Sch95] extends to the Hodge decomposition on non-compact manifolds with non-empty boundary whenever M is the complement of an open bounded domain in $\mathbb{R}^n$.
The papers by [AM04; Gaf55] are devoted to developing the Hodge decomposition from the point of view of the theory of Hilbert space, thus arriving at it without the use of differential equation theory as in [Sch95]. For the case of a non-compact Riemannian manifold Σ one may follow the results of [AM04] in order to achieve the following weak-Hodge decomposition – *cf.* Equation (2.15). We consider the operators $\mathrm{d}_{\mathrm{t}}, \delta_{\mathrm{n}}$

$$\mathrm{dom}(\mathrm{d}_{\mathrm{t}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ \mathrm{t}\omega = 0\} \qquad \mathrm{d}_{\mathrm{t}}\omega := \mathrm{d}\omega\,, \tag{2.32}$$?Eqn: Dirichlet diffe

$$\mathrm{dom}(\delta_{\mathrm{n}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ \mathrm{n}\omega = 0\} \qquad \delta_{\mathrm{n}}\omega := \delta\omega\,. \tag{2.33}$$?Eqn: Neumann codiffe

Notice that d_{t} as well as δ_{n} are nihilpotent because of the relations (1.5). These operators are closed and from Equation (2.14) it follows that their adjoints are:

$$\begin{aligned} \mathrm{dom}(\mathrm{d}) &:= \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\}\,, & \delta_{\mathrm{n}}^* &= \mathrm{d}\,,\\ \mathrm{dom}(\delta) &:= \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\}\,, & \mathrm{d}_{\mathrm{t}}^* &= \delta\,. \end{aligned}$$

It follows that $(\overline{\mathrm{Ran}(\mathrm{d}_{\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\mathrm{n}})})^{\perp} = \overline{\ker(\mathrm{d}) \cap \ker\delta} = \mathcal{H}^k(\Sigma)$ so that

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\mathrm{n}})} \oplus \mathcal{H}^k(\Sigma)\,. \tag{2.34}$$?Eqn: weak-Hodge deco

Following the same steps of the proof of Theorem 2.2.13 it descends that a similar weak-Hodge decomposition holds for the case of non-compact Riemannian manifolds Σ with interface Z:

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{[\mathrm{t}]})} \oplus \overline{\mathrm{Ran}(\delta_{[\mathrm{n}]})} \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.35}$$

?Eqn: weak-Hodge

where $\mathrm{d}_{[\mathrm{t}]}, \delta_{[\mathrm{n}]}$ are

$$\begin{aligned} &\mathrm{dom}(\mathrm{d}_{[\mathrm{t}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma)|\ \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ [\mathrm{t}\omega] = 0\} \qquad &&\mathrm{d}_{[\mathrm{t}]}\omega := \mathrm{d}\omega\,,\\ &\mathrm{dom}(\delta_{[\mathrm{n}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma)|\ \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ [\mathrm{n}\omega] = 0\} \qquad &&\delta_{[\mathrm{n}]}\omega := \delta\omega\,. \end{aligned}$$

It holds $\mathrm{d}_{[\mathrm{t}]}^* = \delta_{[\mathrm{n}]}$ as well as $\delta_{[\mathrm{n}]}^* = \mathrm{d}_{[\mathrm{t}]}$ so that in particular $\ker \mathrm{d}_{[\mathrm{t}]}^* \cap \ker \delta_{[\mathrm{n}]} = \mathcal{H}^k(\Sigma)$.

2.2.4 Non-dynamical Maxwell equations

The Hodge decomposition with interface proved in Theorem 2.2.13 can be exploited to formulate the correct generalization of the non-dynamical components of Maxwell equations (2.5b) as follows.

We interpret the constraint $\operatorname{div} E = \delta E = 0$ (and analogously $\operatorname{div} B = 0$) in a distributional sense. Recalling Stokes' theorem in Equation 1.6, we can write formally:

$$(\mathrm{d}\psi, E)_{\Sigma_\pm} = (\psi, \delta E)_{\Sigma_\pm} + (\mathrm{t}\psi, \mathrm{n}E)_{\partial\Sigma_\pm}\,, \quad \text{for } \psi \in \mathrm{H}^1\Omega^0(\Sigma). \tag{2.36}$$

{?}

By a formal manipulation one obtains that, if $\operatorname{supp}\psi \cap Z \neq \emptyset$,

$$\begin{aligned} (\mathrm{d}\psi, E)_\Sigma =& (\mathrm{d}\psi, E)_{\Sigma_+} + (\mathrm{d}\psi, E)_{\Sigma_-} =\\ =& (\psi, \delta E)_{\Sigma_+} + (\mathrm{t}\psi, \mathrm{n}_+ E)_Z + (\psi, \delta E)_{\Sigma_-} - (\mathrm{t}\psi, \mathrm{n}_- E)_Z =\\ =& (\psi, \delta E)_\Sigma + (\mathrm{t}\psi, [\mathrm{n}E])_Z. \end{aligned} \tag{2.37}$$

?Eqn: formal mani

Definition 2.2.15. *We say that $E \in \mathrm{H}^1\Omega^1(\Sigma_Z)$ satisfies $\delta E = 0$ weakly if both terms of the right hand side of Equation (2.37) vanish for any $\psi \in \mathrm{H}^1\Omega^0(\Sigma) \equiv \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$, i.e.*

$$(\mathrm{d}\psi, E)_\Sigma = 0\,, \text{ for any } \psi \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.38}$$

{?}

In view of the previous definition, in what follows we will replace equations (2.5b) with the requirement

$$E, B \perp \mathrm{d}\mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.39}$$

?Eqn: non-dynamic

Notice that, because of Equation (2.37), this entails $\delta E = \delta B = 0$ pointwisely in $\Sigma_\pm$ as well as $[\mathrm{n}E] = [\mathrm{n}B] = 0$. Configurations of the electric field E in presence of a charge density ρ on $\Sigma_\pm$ and a surface charge density σ over Z are described by expanding $E = \mathrm{d}\alpha + \delta\beta + \kappa$ and demanding $\alpha \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$ to satisfy

$$(\mathrm{d}\varphi, \mathrm{d}\alpha)_\Sigma = (\varphi, \rho)_\Sigma + (\mathrm{t}\varphi, \sigma)_Z \qquad \forall \varphi \in C^\infty_{\mathrm{c}}(\Sigma)\,.$$

This provides a weak formulation for the electrostatic boundary problem. For sufficiently regular α this is equivalent to the Poisson problem $\Delta_\Sigma \alpha = \rho$, $[\mathrm{n}\mathrm{d}\alpha] = \sigma$, recovering the classical equations outlined in [Jac99, Sec. I.5].

2.3 Dynamical equations: Lagrangian subspaces

In this section we will discuss the dynamical equations (2.5a). They can be written in a Schrödinger-like form as a complex evolution equation, solutions can be found imposing suitable interface conditions on Z.

Definition 2.3.1. *We call* first order Maxwell equations *the following system of partial differential equations:*

$$i\partial_t \psi = H\psi \qquad \psi := \begin{bmatrix} E \\ B \end{bmatrix}, \qquad H := \begin{bmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{bmatrix}, \tag{2.40}$$?Eqn: dynamical eqns

where H will be called the first order Maxwell operator, *or simply the* Maxwell operator. *Here we adopt the convention of [Bär19] according to which*

$$\mathrm{curl} := i \star_\Sigma \mathrm{d} \quad \textit{if } \dim\Sigma = 1 \mod 4\,, \quad \mathrm{curl} := \star_\Sigma \mathrm{d} \quad \textit{if } \dim\Sigma = 3 \mod 4\,. \tag{2.41}$$?Eqn: curl convention

With this convention curl *is a formally a selfadjoint operator on $\Omega^1_{\mathrm{c}}(\Sigma)$.*

As outlined in Section 2.2 we consider Equation (2.40) on Σ_Z, allowing for jump discontinuities across the interface Z. To this end we regard H as a densely defined operator on the Cartesian product

$$\mathrm{L}^2\Omega^1(\Sigma) \times \mathrm{L}^2\Omega^1(\Sigma) =: \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} = \mathrm{L}^2\Omega^1(\Sigma_Z)^{\times 2} \tag{2.42}$$?Eqn: L^2 with H?

(the former equality follows from Equation (2.21)) with domain

$$\mathrm{dom}(H) := \Omega^1_{\mathrm{cc}}(\Sigma_+)^{\times 2} \oplus \Omega^1_{\mathrm{cc}}(\Sigma_-)^{\times 2}\,, \tag{2.43}$$?Eqn: curl-Hamiltonia

where $\Omega^1_{\mathrm{cc}}(\Sigma_\pm)$ denotes the subspace of $\Omega^1_{\mathrm{c}}(\Sigma_\pm)$ with support in $\Sigma_\pm \setminus \partial\Sigma_\pm$.

In solving Maxwell equations, we require the underlying system to be isolated, so that the flux of relevant physical quantities, such as those built from the stress-energy tensor, is zero through the interface. To translate mathematically this requirement we need to look for symmetric extensions

$\widehat{H}$ of H, in other words

$$(\widehat{H}\psi_1, \psi_2)_\Sigma - (\psi_1, \widehat{H}\psi_2)_\Sigma = \text{ vanishing interface terms } \quad \forall \psi_1, \psi_2 \in \mathrm{dom}(\widehat{H}) \subseteq \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}. \tag{2.44}$$
{?}

Moreover, we require the extensions of H to be self-adjoint so that the spectral resolution of the operator has only real eigenvalues. This prevents the fundamental solutions of $\widehat{H}$ to have exponentially increasing modes, which would result to an unstable physical system.

Proposition 2.3.2. *Let $u, v \in \Omega^k_\mathrm{c}(\Sigma_Z)$, then a Green formula holds*

$$(\mathrm{curl}\, u, v)_\Sigma - (u, \mathrm{curl}\, v)_\Sigma = (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z\,, \tag{2.45}$$
?Eqn: Green curl?

where $\gamma_0 u := \frac{1}{\sqrt{2}} \star [\mathrm{t}u]$, $\star$ is the Hodge dual operator on Z and $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$. Moreover, the operator H, defined in (2.40) is symmetric on its domain (see Equation (2.43)), since for any $\psi_1, \psi_2 \in \Omega^1_\mathrm{c}(\Sigma)^{\times 2}$ it holds

$$(H\psi_1, \psi_2)_\Sigma - (\psi_1, H\psi_2)_\Sigma = (\Gamma_1\psi_1, \Gamma_0\psi_2)_Z - (\Gamma_0\psi_1, \Gamma_1\psi_2)_Z, \tag{2.46}$$
?Eqn: Green H?

where $\psi = [E, B]$ and $\Gamma_0\psi = [i\gamma_1 B, \gamma_1 E]$, $\Gamma_1\psi = [\gamma_0 E, i\gamma_0 B]$.

The former Proposition entails that the operator H is symmetric and hence closable (*cf.* [Mor18, Thm. 5.10]), its adjoint H^* being defined on

$$\mathrm{dom}(H^*) = \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\ H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} \qquad H^*\psi := H\psi\,. \tag{2.47}$$
?Eqn: adjoint cur

Equation (2.40) is solved by selecting a self-adjoint extension of H. We outline a technique which allows us to parametrize the self-adjoint extensions of H by Lagrangian subspaces of a suitable complex symplectic space – *cf.* [EM99; EM03; EM05]. The aim is to construct the Green operators for Equation (2.40) together with an interface condition. This technique, even if it does not give a complete characterization of self-adjoint extensions in terms of boundary conditions, allows us to check whether a chosen interface condition admits Green operators or not.

Definition 2.3.3. *Let S be a complex vector space and let $\sigma\colon \mathsf{S}\times\mathsf{S} \to \mathbb{C}$ be a sesquilinear map. The pair (S, σ) is called complex symplectic space if σ is non-degenerate – i.e. $\sigma(x, y) = 0$ for all $y \in \mathsf{S}$ implies $x = 0$ – and $\sigma(x, y) = -\overline{\sigma(y, x)}$ for all $x, y \in \mathsf{S}$. A subspace $L \subseteq \mathsf{S}$ is called Lagrangian subspace if $L = L^\perp := \{x \in \mathsf{S} |\ \sigma(x, y) = 0\ \forall y \in L\}$.*

For convenience, we summarize the main results in the following theorem:

Theorem 2.3.4 ([EM99]). *Let H be a separable Hilbert space and let $A\colon \operatorname{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ be a densely defined, symmetric operator. Then, the bilinear map*

$$\sigma(x,y) := (A^*x, y) - (x, A^*y)\,, \qquad \forall x, y \in \operatorname{dom}(A^*)\,, \tag{2.48}$$?Eqn: symplectic form

satisfies $\sigma(x,y) = -\overline{\sigma(y,x)}$. The symplectic form σ descends to the quotient space $\mathsf{S}_A := \operatorname{dom}(A^*)/\operatorname{dom}(A)$ and the pair (S_A, σ) is a complex symplectic space as per Definition 2.3.3. Moreover, for all Lagrangian subspaces $L \subseteq \mathsf{S}_A$ – cf. Definition 2.3.3 – the operator

$$A_L := A^*|_{L+\operatorname{dom}(A)}\,, \tag{2.49}$$?Eqn: Lagrangian self

defines a self-adjoint extension of A, where $L+\operatorname{dom}(A)$ denotes the pre-image of L with respect to the projection $\operatorname{dom}(A^*) \to \mathsf{S}_A$. Finally the map

$$\{\textit{Lagrangian subspaces } L \textit{ of } \mathsf{S}_A\} \ni L \mapsto A_L \in \{\textit{self-adjoint extensions of } A\}\,, \tag{2.50}$$ {?}

is one-to-one.

Remark 2.3.5. The symplectic form σ associated to the operator A on H is called *symplectic flux*. The physically motivated requirement of closedness of the extensions of A is translated into imposing the symplectic flux to vanish.

Example 2.3.6. As a concrete example of Theorem 2.3.4 we discuss the case of the self-adjoint extensions of the curl operator on a closed manifold Σ with interface Z. For simplicity we assume that $\dim\Sigma = 2k+1$ with $\dim\Sigma = 3 \mod 4$, while curl is defined according to (2.41). We consider the operator curl_Z defined by

$$\operatorname{dom}(\operatorname{curl}_Z) := \overline{\Omega^k_{\mathrm{cc}}(\Sigma_Z)}^{\|\|_{\mathrm{curl}}}\,, \qquad \operatorname{curl}_Z u := \operatorname{curl} u\,. \tag{2.51}$$?Eqn: Z-curl operator

Notice that $\Omega^k_{\mathrm{cc}}(\Sigma_Z) = \Omega^k_{\mathrm{cc}}(\Sigma_+) \oplus \Omega^k_{\mathrm{cc}}(\Sigma_-)$. The adjoint curl^*_Z of curl_Z is defined on

$$\operatorname{dom}(\operatorname{curl}^*_Z) = \operatorname{dom}(\operatorname{curl}_+) \oplus \operatorname{dom}(\operatorname{curl}_-)\,, \tag{2.52}$$?Eqn: adjoint of Z-cu

$$\operatorname{dom}(\operatorname{curl}_\pm) := \{u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)|\ \operatorname{curl}_\pm u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)\}\,, \quad \operatorname{curl}_\pm u := \operatorname{curl} u\,. \tag{2.53}$$ {?}

Since complex conjugation commutes with curl, it follows from Von Neumann's criterion [Mor18, Thm. 5.43] that curl_Z admits self-adjoints extensions. We give a description of the complex symplectic space $\mathsf{S}_{\operatorname{curl}_Z} := (\operatorname{dom}(\operatorname{curl}^*_Z)/\operatorname{dom}(\operatorname{curl}_Z), \sigma_{\mathrm{curl}})$ whose Lagrangian subspaces allow to characterize all self-adjoint extensions of curl_Z. According to Theorem 2.3.4 the symplectic structure σ_{curl} on the vector space $\mathsf{S}_{\operatorname{curl}_Z}$ is defined by

$$\sigma_{\mathrm{curl}}(u,v) := (\operatorname{curl}^*_Z u, v) - (u, \operatorname{curl}^*_Z v)\,, \qquad \forall u, v \in \operatorname{dom}(\operatorname{curl}^*_Z)\,. \tag{2.54}$$?Eqn: presymplectic s

In particular for $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ and $v \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ we have

$$\begin{aligned}\sigma_{\mathrm{curl}}(u,v) &= (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z = \\ &= \sum_{\pm} \pm \int_Z \overline{\mathrm{t}_\pm u} \wedge \mathrm{t}_\pm v = \sum_{\pm} \mp \,_{-\frac{1}{2}}\langle \mathrm{t}_\mp u, \star_Z \mathrm{t}_\mp v\rangle_{\frac{1}{2}}\,, \end{aligned} \tag{2.55}$$?Eqn: simplified

where $\gamma_0 u := \frac{1}{\sqrt{2}} \star [\mathrm{t}u]$, $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$ as in Proposition 2.3.2 and where $_{-\frac{1}{2}}\langle\ ,\ \rangle_{\frac{1}{2}}$ denotes the pairing between $\mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ and $\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$. In particular this shows that $\mathrm{t}_\pm u \in \mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ for all $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ – *cf.* [AV96; BCS02; Geo79; Paq82; Wec04] for more details on the trace space associated with the curl-operator on a manifold with boundary.
According to Theorem 2.3.4 all self-adjoint extensions of curl_Z are in one-to-one correspondence to the Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$. Unfortunately a complete characterization of all Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$ is not available. For our purposes, it suffices to give a family of Lagrangian subspaces – a generalization of the results presented in [HKT12] may provide other examples. For $\theta \in \mathbb{R}$ let

$$L_\theta := \{u \in \mathrm{dom}(\mathrm{curl}_Z^*) |\ \mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u\}\,, \tag{2.56}$$ {?}

where $\mathrm{t}_\pm$ denote the tangential traces – *cf.* Definition 2.2.11, Remark 2.2.7 and Equation (2.55).
To show that L_θ are Lagrangian subspaces let $u, v \in L_\theta$ and let $v_n \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ be such that $\|v - v_n\|_{\mathrm{curl}} \to 0$. In particular $\|(\mathrm{t}_+ - e^{i\theta}\mathrm{t}_-)v_n\|_{\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)} \to 0$ so that

$$\sigma_{\mathrm{curl}}(u,v) = \lim_{n\to\infty} \sigma_{\mathrm{curl}}(u, v_n) = -\lim_{n\to\infty} \,_{-\frac{1}{2}}\langle \mathrm{t}_+ u, \star(\mathrm{t}_+ v_n - e^{i\theta}\mathrm{t}_- v_n)\rangle_{\frac{1}{2}} = 0\,. \tag{2.57}$$ {?}

It follows that $L_\theta \subseteq L_\theta^\perp$. Conversely if $u \in L_\theta^\perp$ let us consider $v \in L_\theta$. Since $u \in L_\theta^\perp$ we find

$$0 = \sigma_{\mathrm{curl}}(u,v) = -\,_{-\frac{1}{2}}\langle \mathrm{t}_+ u - e^{i\theta}\mathrm{t}_- u, \star \mathrm{t}_+ v\rangle_{\frac{1}{2}}\,.$$

Since $\mathrm{t}_+ \colon \mathrm{H}^1\Omega^k(\Sigma_Z) \to \mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$ is surjective, it follows that $\mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u$.
Notice that the self-adjoint extension obtained for $\theta = 0$ coincides with the closure of curl on $\Omega^k_{\mathrm{c}}(\Sigma)$ which is known to be self-adjoint by [Bär19, Lem. 2.6]. Indeed, since $[\mathrm{t}]$ is continuous we have $\mathrm{dom}(\overline{\mathrm{curl}}) \subseteq L_0$ so that curl_{Z,L_0} is a self-adjoint extension of $\overline{\mathrm{curl}}$. Since this last operator is already self-adjoint, the two coincide. ■

Example 2.3.7. We provide a concrete example of 2.3.4 in the case we are mostly interested in: Maxwell equations in the Schrödinger-like form as in Equation (2.40). According to Theorem 2.3.4, the operator H has an associated symplectic space $\mathsf{S}_H := (\mathrm{dom}(H^*)/\mathrm{dom}(H), \sigma_H)$, where

$$\sigma_H(\psi_1,\psi_2) = (H^*\psi_1, \psi_2) - (\psi_1, H^*\psi_2),\ \forall \psi_1,\psi_2 \in \mathrm{dom}(H^*). \tag{2.58}$$ {?}

In particular, if $\psi_1 \in \mathrm{dom}(H^*)$ and $\psi_2 \in \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2}$ and denoting ψ as the couple $[E, B]$, we can write

$$\begin{aligned}\sigma_H(\psi_1, \psi_2) =& -i\sigma_{\mathrm{curl}}(B_1, E_2) + i\sigma_{\mathrm{curl}}(E_1, B_2) = \\ =& i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\, \mathrm{t}_+\psi_2\rangle_{\frac{1}{2}} - {}_{-\frac{1}{2}}\langle \mathrm{t}_-\psi_1, \star S\, \mathrm{t}_-\psi_2\rangle_{\frac{1}{2}}\right],\end{aligned} \tag{2.59}$$ {?}

where σ_{curl} is defined in Equations (2.54), (2.55), ${}_{-\frac{1}{2}}\langle\, ,\, \rangle_{\frac{1}{2}}$ is the pairing between $\mathrm{H}^{-\frac{1}{2}}$ and $\mathrm{H}^{\frac{1}{2}}$,

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SO}(2),$$

$\star$ is the Hodge operator on Z and, as usual, $\mathrm{t}_\pm$ denote the tangential traces – *cf.* Definition 2.2.11. We give a family of Lagrangian subspaces which encode the following class of interface conditions. For $\mathrm{U} \in \mathrm{SO}(2)$, let us define the space

$$L_{\mathrm{U}} := \{\psi \in \mathrm{dom}(H^*) \,|\, \mathrm{t}_+\psi = \mathrm{U}\mathrm{t}_-\psi\}. \tag{2.60}$$ {?}

To show that L_{U} are Lagrangian subspaces we mimic the technique used in the former Example 2.3.6. Let $\psi_1 = [E_1, B_1], \psi_2 = [E_2, B_2] \in L_{\mathrm{U}}$ and let $\psi_n = [E_n, B_n] \in \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2}$ such that $\|E_2 - E_n\|_{\mathrm{curl}} \to 0$ and $\|B_2 - B_n\|_{\mathrm{curl}} \to 0$.
In particular it holds that $\|(\mathrm{t}_+ - \mathrm{U}\mathrm{t}_-)\psi_n\|_{\mathrm{H}^{\frac{1}{2}}\Omega^1(\Sigma_Z)^{\times 2}} \to 0$. Hence $L_{\mathrm{U}} \subseteq L_{\mathrm{U}}^{\perp}$ follows from

$$\begin{aligned}\sigma_H(\psi_1, \psi_2) &= \lim_{n\to\infty} \sigma_H(\psi_1, \psi_n) = \\ &= \lim_{n\to\infty} i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\, \mathrm{t}_+\psi_n\rangle_{\frac{1}{2}} - {}_{-\frac{1}{2}}\langle \mathrm{U}^{-1}\mathrm{t}_+\psi_1, \star S\, \mathrm{t}_-\psi_n\rangle_{\frac{1}{2}}\right] = \\ &= \lim_{n\to\infty} i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\,(\mathrm{t}_+\psi_n - \mathrm{U}\mathrm{t}_-\psi_n)\rangle_{\frac{1}{2}}\right] = 0.\end{aligned} \tag{2.61}$$ {?}

Conversely if $\psi_1 \in L_{\mathrm{U}}^{\perp}$ let us consider $v \in L_{\mathrm{U}}$. Hence, we find

$$0 = \sigma_H(\psi_1, \psi_2) = i\left[{}_{-\frac{1}{2}}\langle (\mathrm{t}_+\psi_1 - \mathrm{U}\mathrm{t}_-\psi_1), \star S\, \mathrm{t}_+\psi_2\rangle_{\frac{1}{2}}\right].$$

Since $\mathrm{t}_+ \colon \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2} \to \mathrm{H}^{\frac{1}{2}}\Omega^1(Z)^{\times 2}$ is surjective, it follows that $\mathrm{t}_+\psi_1 = \mathrm{U}\mathrm{t}_-\psi_1$.

Following slavishly the passages of Example 2.3.6, one can also show that the following family of subspaces of S_H, that can be expressed in terms of interface conditions, are Lagrangian and hence, give rise to a self-adjoint extensions of H:

$$L_\theta := \{u \in \mathrm{dom}(H^*) |\ \mathrm{t}_+\psi = e^{i\theta}\mathrm{t}_-\psi\}. \tag{2.62}$$ {?}

■

We conclude this section by introducing an exact sequence which provides a complete description of the solution space of the Maxwell equations (2.5a) with interface Z.

Theorem 2.3.8. *Let H be the densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)^{\times 2}$ with domain defined by (2.43) and let H^* be its adjoint, defined as in (2.47). Let $L \subset \mathsf{S}_H = (\mathrm{dom}(H^*)/\mathrm{dom}\, H), \sigma_H)$ be a Lagrangian subspace in the sense of Definition 2.3.3 and consider the self-adjoint extension H_L as per Theorem 2.3.4. Furthermore, let $\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2} := \bigcap_{k\geq 0}\mathrm{dom}(H_L^k)$ and let $G_L^\pm$ be the operators $G_L^\pm\colon C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \to C^\infty(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})$ completely determined in terms of the bidistributions $\mathcal{G}_L^+ = \theta(t-t')\mathcal{G}_L$ and $\mathcal{G}_L^- = -\theta(t'-t)\mathcal{G}_L$, with*

$$\mathcal{G}_L(\psi_1, \psi_2) = \int_{\mathbb{R}^2} \left(\psi_1(t)\middle| e^{-i(t-t')H_L}\psi_2(t')\right) \mathrm{d}t\,\mathrm{d}t' \quad \forall \psi_1, \psi_2 \in C^\infty_{\mathrm{c}}(\mathbb{R}, \Omega^1_c(\Sigma)^{\times 2}). \tag{2.63}$$

?Eqn: Fundamental

The the operator G_L^+ (resp. G_L^-) is an advanced (resp. retarded) solution of $i\partial_t - H_L$, that is, it holds

$$(i\partial_t - H_L)\circ G_L^\pm = \mathrm{Id}_{C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})}\,, \tag{2.64}$$

{?}

$$G_L^\pm \circ (i\partial_t - H_L) = \mathrm{Id}_{C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})}\,. \tag{2.65}$$

{?}

Moreover, let $G_L := G_L^+ - G_L^-$. Then the following is a short exact sequence

$$\begin{aligned} 0 \to C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) &\xrightarrow{i\partial_t - H_L} C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \\ &\xrightarrow{G_L} C^\infty(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \xrightarrow{i\partial_t - H_L} C^\infty(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \to 0\,. \end{aligned} \tag{2.66}$$

?Eqn: exact seque

Proof. Most of it is an analogue of [DDF19, Thm. 30- Prop. 36]. We observe that the function $\sigma(H_L) \ni \lambda \mapsto e^{-i\lambda\tau}$ is smooth and bounded for all $\tau \in \mathbb{R}$. Hence, for any $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}, \Omega^1_c(\Sigma_Z)^{\times 2})$, $G_L^\pm\psi \in C^\infty(\mathbb{R}, \mathrm{dom}\, H_L)$. We have, for all $k \in \mathbb{N}\cup\{0\}$ and $t \in \mathbb{R}$

$$(1 + H_L)^k[G_L^\pm\psi](t) = G_L^\pm[(1 + H_L)^k\psi](t) = G_L^\pm[(1 + H)^k\psi](t),$$

which is an element of $\mathrm{L}^2\Omega(\Sigma)^{\times 2}$, since $(1 + H)\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}, \Omega^1_{\mathrm{c}}(\Sigma))$. It follows that $G_L^\pm\psi(t) \in \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}$ and Equation (2.63) holds true.
It remains to prove the finite speed of propagation, which follows from [HR00; MM13]. In particular, the hypotheses of [MM13, Thm. 1.1] are met since H_L is self-adjoint and from a straightforward computation it holds

$$\|[\eta I, H_L]\psi\| \leq \|\nabla\eta\|_\infty\|\psi\| \quad \forall\psi \in \mathrm{dom}\, H_L,\ \eta \in \mathrm{Lip}(\Sigma)\cap C^1(\Sigma).$$

Hence, [MM13, Thm. 1.1] ensures that the propagation speed of the one-parameter group e^{itH_L} is finite and smaller than 1 in the sense that

$$\mathrm{supp}(e^{-itH_L}\psi) \subset J^+(\mathrm{supp}\,\psi),\ t \geq 0,$$

where the brackets [,] denote the commutator.
The second part of the statement regarding the exact sequence follows imitating slavishly the standard arguments of [BGP07, Th. 3.4.7] [DDF19, Prop. 36]. ■

Notice that the exact sequence (2.66) implies that the space of smooth solution of the dynamical equations (2.5a) is isomorphic as a vector space to the image of G_L.

2.4 Perspectives on algebraic quantization of the field strength F

Having the causal propagator G in hand, for a choice of boundary conditions, the following step would be that of developing a quantization scheme for the field strength $F \in \Omega^2(M)$ on an arbitrary four dimensional globally hyperbolic spacetime with timelike boundary (M, g) within the framework of the algebraic formulation of quantum field theory.

In the case of empty boundary, the construction is obtained in [DL12]. They prove in particular that the commutator between two generators of the algebra of observables of F is given by the *Lichnerowicz propagator* regardless of the chosen spacetime. Moreover, they prove the existence of a non trivial centre for the field algebra whenever the second de Rham cohomology group of the manifold is non trivial.
To be more clear they initially prove the existence of Green operators for the wave operator $\Box = \delta_M \mathrm{d}_M + \mathrm{d}_M \delta_M$ in a globally hyperbolic spacetime with empty boundary. Then, they use the fact that F itself satisfies a wave equation (since $F \in \ker \delta_M \cap \ker \mathrm{d}_M$) to entail the existence of the causal propagator $G_\Box$ that commutes with δ_M and they identify the space $\mathrm{Sol}(M)$ of solutions of Maxwell equations as the 2-forms $F \in \Omega^2(M)$ such that

$$F = G_\Box(\delta_M \alpha + \mathrm{d}_M \beta),\ \alpha \in \Omega^3_{\mathrm{c}}(M) \cap \ker \mathrm{d}_M, \beta \in \Omega^1_{\mathrm{c}}(M) \cap \ker \delta_M. \qquad (2.67)\ \{?\}$$

Hence, they construct the field algebra as an the associative, unital $*$-algebra: the universal tensor algebra generated by elements of the form $\mathbf{F}(\omega)$, $\omega \in \Omega^2_{\mathrm{c}}(M)$ with componentwise addition, componentwise multiplication with a scalar, componentwise antilinear involution $*$ and multiplication induced by the algebraic tensor product $\otimes$, while the $*$-operation is the one induced from complex conjugation. In addition they impose the Maxwell equations at a dual level and implement the Canonical Commutation Relations requiring

$$[\mathbf{F}(\omega), \mathbf{F}(\omega')] = i\mathsf{G}(\omega, \omega')\mathcal{I},$$

where $\mathcal{I}$ is the identity and $\mathsf{G}(\omega, \omega') \doteq \int_M G_\Box(\delta\omega) \wedge \star\delta\omega'$, which is called *Lichnerowicz propagator*.

In addition, they show that the field algebra, in general, possesses a non trivial centre. This feature, thoroughly studied in [Ben+14; BDS14; SDH14], is in common with Abelian gauge theories and will be discussed in the case of the vector potential A in the next chapter. Indeed, from a physical point of view, the existence of a non trivial centre leads to an obstruction in the interpretation of the model in terms of locally covariant quantum field theories.

In the case of non-empty boundary or an interface, we could not rely on the existence of Green operators for $\Box$ such that they commute with δ_M, hence we had to prove in the previous sections the existence of distinguished advanced and retarded Green operators, and consequently of a causal propagator G for Maxwell equations under suitable boundary conditions. The next step would be that of following [DL12] once again and construct the field algebra for a manifold with timelike boundary or with interface. The passages would be identical, but now, since we have an exact sequence for the causal propagator of Maxwell equations for F, the space of solutions Sol will be characterized as the 2-forms F such that $F = G(\eta)$. Moreover, the Canonical Commutation Relations will in fact be implemented as follows:

$$[\mathbf{F}(\omega), \mathbf{F}(\omega')] = i\widetilde{\mathsf{G}}(\omega, \omega')\mathbb{I},$$

with $\widetilde{\mathsf{G}}(\omega, \omega') = \int_M G(\omega) \wedge \star\omega$.

In the next chapter, we focus our efforts on the construction of Green operators for $\Box$ acting on k-forms on a special class of spacetimes with timelike boundary. Subsequently, relying on the existence of such operators, we will construct the algebra of observable for Maxwell equations for the vector potential A.

Chapter 3

Maxwell Equations and Boundary Conditions

In this Chapter we analyze the space of solutions of Maxwell equations for the vector potential A, regarded as system of equations for k-forms in a globally hyperbolic spacetime with timelike boundary (M, g). As discussed in Section 1.5, in the case with vanishing currents and empty boundary one imposes the Lorenz gauge condition to translate the problem $\delta \mathrm{d} A = 0$ into an hyperbolic problem that involves a wave equation for A, where a D'Alembert - de Rham operator $\Box_k = \delta\mathrm{d} + \mathrm{d}\delta$ acting on $\Omega^k(M)$ appears. In our non-empty boundary case, at first we identify a set of boundary conditions for the D'Alembert - de Rham operator that ensure he closedness of the underlying physical system, in analogy with the case for F described in Section 2.3. In the following, we prove the existence of advanced and retarded Green operators for $\Box_k$ in a special case: When the underlying globally hyperbolic spacetime with timelike boundary is *static* and of *bounded geometry*, using the technique of boundary triples as already done in the scalar case in [DDF19].

In the second part of the Chapter we focus precisely on the Maxwell operator $\delta\mathrm{d}\colon \Omega^k(M) \to \Omega^k(M)$. Among the boundary conditions for $\delta\mathrm{d}$, we dwell on two particular cases, namely $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions, which will provide two different notions of gauge equivalence. We prove that in both cases, every class of gauge-equivalent solutions admits a representative satisfying the Lorentz gauge. We use this property and the analysis of the operator $\Box_k$ to construct and to classify the space of gauge equivalence classes of solutions of the Maxwell's equations with the prescribed boundary conditions.
As a last step, we construct the associated unital $*$-algebras of observables proving in particular that, as in the case of empty boundary, they possess a non-trivial center.

3.1 On the D'Alembert–de Rham wave operator

Consider the operator $\Box : \Omega^k(M) \to \Omega^k(M)$, where (M, g) is a m-dimensional globally hyperbolic spacetime with timelike boundary (see Definition 1.1.1) with $m \geq 2$. We denote with

d, δ the differential and the codifferential operators on the spacetime M. Then, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, manipulating Equation (1.6), one can obtain the following Green formula:

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = (\mathrm{t}\delta\alpha, \mathrm{n}\beta)_\partial - (\mathrm{n}\alpha, \mathrm{t}\delta\beta)_\partial - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_\partial + (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_\partial \,, \tag{3.1}$$

?Eqn: boundary te

where t, n are the maps introduced in Definition 1.2.3, while $(,)$ and $(,)_\partial$ are the standard, metric induced pairing between k-forms respectively on M and on ∂M. As an immediate consequence, if $\alpha, \beta \in \Omega^k_{\mathrm{cc}}(M)$, *i.e.* their support does not intersect the boundary, the right-hand side of (3.1) vanishes automatically. In other words, $\Box$ is formally self-adjoint.
Clearly, $\Omega^k_{\mathrm{cc}}(M)$ is a rather restrictive set of k-forms, since forms in that space do not have any interplay with the boundary. A larger set, but not the largest[1], can be that of forms whose support intersect the boundary, but which have boundary conditions such that the scalar products appearing in Equation (3.1) cancel one another. Indeed, we define

$$\Omega^k_{f,f'}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\,,\ \mathrm{t}\delta\omega = f'\mathrm{n}\omega,\ f, f' \in C^\infty(\partial M)\}\,. \tag{3.2}$$

?Eqn: f,f' bounda

Noting that for every $f \in C^\infty(\partial M)$ and for every $\alpha \in \Omega^k(\partial M)$, $\star_\partial(f\alpha) = f(\star_\partial\alpha)$, one can straightforwardly infer the following:

Lemma 3.1.1. *If $\alpha, \beta \in \Omega^k_{f,f'}(M)$, $0 \leq k \leq n = \dim M$ are such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, then it holds*

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = 0\,,$$

in other words, $\Box$ is formally self-adjoint.

It is important to notice that, whenever $k = 0$, the second condition in Equation (3.2) is trivially satisfied, since $\delta\omega = \mathrm{t}\delta\omega = \mathrm{n}\omega = 0$, for $\omega \in \Omega^0(M)$, but $\mathrm{n}\omega = 0$ automatically in the scalar case, that was extensively studied in [DDF19]. Similarly, in the case $k = m$ the first condition becomes empty, since $\mathrm{d}\eta = \mathrm{n}\mathrm{d}\eta = \mathrm{t}\eta = 0$ if $\eta \in \Omega^m(M)$.

Equation (3.2) individuates therefore a class of boundary conditions which makes the operator $\Box$ formally self-adjoint. We would like to generalize the standard Dirichlet, Neumann and Robin boundary conditions to forms of higher degree. We recall that, as already mentioned in Example 1.4.10, for a scalar function u, Dirichlet, Neumann and Robin boundary conditions are obtained by imposing, respectively,

$$\mathrm{t}u|_{\partial M} = 0; \quad \mathrm{n}\mathrm{d}\, u = \frac{\partial u}{\partial \nu}\Big|_{\partial M} = 0; \quad u|_{\partial M} = f\frac{\partial u}{\partial \nu}\Big|_{\partial M}, \text{ for } f \in C^\infty(\partial M)\,,$$

[1]One can think of other possibilities such as Wentzell boundary conditions, which were studied in the scalar scenario in [DDF19; DFJA18; Zah18].

ν being the vector field normal to ∂M.
With non-scalar functions, we will have there are several other possibilities, which are obtained fixing a particular choice for $f, f' \in C^\infty(\partial M)$ in Equation (3.2). In between all these possibilities we highlight those which are of particular interest since we will be able to prove, at least in the static case, that these cases admit Green operators.

Definition 3.1.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $f \in C^\infty(\partial M)$. We call*

1. *space of k-forms with* Dirichlet *boundary condition*

$$\Omega^k_{\mathrm{D}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{n}\omega = 0\}\,, \tag{3.3}$$

?Eqn: Dirichlet k-for

2. *space of k-forms with* $\Box$-tangential *boundary condition*

$$\Omega^k_{\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{t}\delta\omega = 0\}\,, \tag{3.4}$$

?Eqn: Box-tangential

3. *space of k-forms with* $\Box$-normal *boundary condition*

$$\Omega^k_{\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\ ,\ \mathrm{n}\mathrm{d}\omega = 0\}\,. \tag{3.5}$$

?Eqn: Box-normal k-fo

4. *space of k-forms with* Robin $\Box$-tangential *boundary condition*

$$\Omega^k_{f_\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\delta\omega = f\mathrm{n}\omega\ ,\ \mathrm{t}\omega = 0\}\,, \tag{3.6}$$

?Eqn: Robin Box-tange

5. *space of k-forms with* Robin $\Box$-normal *boundary condition*

$$\Omega^k_{f_\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\ ,\ \mathrm{n}\omega = 0\}\,, \tag{3.7}$$

?Eqn: Robin Box-norma

Whenever the operator $\Box$ is restricted to act on one of these space we denote it with symbol $\Box_\sharp$ where $\sharp \in \{\mathrm{D}, \parallel, \perp, f_\parallel, f_\perp\}$.

Remark 3.1.3. It is worth to notice the relations between the different classes of boundary conditions under the action of the Hodge dual operator. Using the commutation relations between the Hodge operator and the differential operators mentioned in Equation (1.5) and the definition in (3.2), one obtains that

$$\star\Omega^k_{f,f'}(M) = \Omega^{m-k}_{-f',-f}(M),$$

for any $f, f' \in C^\infty(\partial M)$. At the same time, with reference, to the space of k-forms in Definition 3.1.2 it holds

$$\star\Omega^k_{\mathrm{D}}(M) = \Omega^{m-k}_{\mathrm{D}}(M)\,, \qquad \star\Omega^k_{\parallel}(M) = \Omega^{m-k}_{\perp}(M)\,, \qquad \star\Omega^k_{f_\parallel}(M) = \Omega^{m-k}_{-f_\perp}(M)\,. \tag{3.8}$$

?Eqn: duality between

To recover the standard scalar Dirichlet and Neumann analogues, one can observe that $\mathrm{n}\omega = 0$, whenever $\omega \in \Omega^0(M)$. Hence $\mathrm{t}\omega = \omega|_{\partial M}$ and $\mathrm{n}\mathrm{d}\omega = \partial_\nu \omega|_{\partial M}$, where ν is the outward pointing unit vector normal to the boundary. It follows that $\Omega^0_{\mathrm{D}}(M) = \Omega^0_{\parallel}(M)$ and $\Omega^0_{\perp}(M)$ are, respectively, the spaces of scalar functions (0-forms) that satisfy Dirichlet and Neumann boundary conditions.
Moreover for $f = 0$ we have $\Omega^k_{f_\parallel}(M) = \Omega^k_{\parallel}(M)$ as well as $\Omega^k_{f_\perp}(M) = \Omega^k_{\perp}(M)$.
It is worth to notice that, for a static spacetime (M, g), the boundary conditions 1-3, introduced in Definition 3.1.2, are themselves static, that is they do not depend explicitly on the time coordinate τ. Whenever $f \in C^\infty(\partial M)$ does not depend on the time τ, a similar statement holds true for $f_\perp$, $f_\parallel$ boundary conditions. This will play a key role when we will verify that Assumption 3.1.4 is valid on static spacetime – *cf.* Proposition 3.1.16 in Section 3.1.1.

3.1.1 Existence of Green operators on ultrastatic spacetimes

With reference to Section 1.4, we want to prove the existence of distinguished Green operators for the operator $\Box_\sharp$ for $\sharp \in \{\mathrm{D}, \parallel, \perp, f_\parallel, f_\perp\}$. The existence of Green operators is important since they provide the necessary tools to construct the algebra of the observables for the underlying quantum system.

Recalling that Green operators can be seen as in Remark 1.4.6, in view of Definition 1.1.9 and Definition 3.1.2 we require the following:

Assumption 3.1.4. *For all $f \in C^\infty(\partial M)$ and for all $k \in \mathbb{N} \cup \{0\}$, the d'Alembert-de Rham wave operator $\Box_\sharp$, with $\sharp \in \{\mathrm{D}, \parallel, \perp, f_\parallel, f_\perp\}$, is* Green hyperbolic. *In other words there exist advanced $(+)$ and retarded $(-)$ Green operators for $\Box_\sharp$, $G^\pm_\sharp \colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M) \doteq \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ such that*

$$\Box \circ G^\pm_\sharp = \mathrm{id}_{\Omega^k_{\mathrm{c}}(M)}\,, \qquad G^\pm_\sharp \circ \Box_{\mathrm{c},\sharp} = \mathrm{id}_{\Omega^k_{\mathrm{c},\sharp}(M)}\,, \qquad \mathrm{supp}(G^\pm_\sharp \omega) \subseteq J^\pm(\mathrm{supp}(\omega))\,, \quad (3.9)$$

?Eqn: properties

for all $\omega \in \Omega^k_{\mathrm{c}}(M)$ where $\Box_{\mathrm{c},\sharp}$ indicates that the domain of $\Box$ is restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$.

Remark 3.1.5. Notice that domain of $G^\pm_\sharp$ is not restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$. Furthermore the second identity in (3.9) cannot be extended to $G^\pm_\sharp \circ \Box = \mathrm{id}_{\Omega^k_{\mathrm{c}}(M)}$ since it would entail $G^\pm_\sharp \Box \omega = \omega$ for all $\omega \in \Omega^k_{\mathrm{c}}(M)$. Yet the left hand side also entails that $\omega \in \Omega^k_{\mathrm{c},\sharp}$, which is manifestly a contradiction.

Remark 3.1.6. To ensure the Green hyperbolicity of $\Box_\sharp$ one should also prove the existence of distinguished Green operators for the formal adjoint of $\Box_\sharp$, according to Definition 1.4.2. In view of Lemma 3.1.1 this is not necessary, since $\Box$ is formally self-adjoint.

In this section we prove such existence whenever the underlying globally hyperbolic spacetime with timelike boundary is ultrastatic and $f \in C^\infty(\partial\Sigma)$ - appearing in the boundary conditions $f_\parallel, f_\perp$ - has definite sign.

We mimic the functional analysis technique of *boundary triples* used in the first place in [DDF19] for the scalar case. This method is extensively discussed in [BL12] and here we only recall the main results.
In the following, $\mathcal{H}$ will denote a complex Hilbert space and $S : \mathrm{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is a densely defined closed symmetric operator in $\mathcal{H}$. Since S is densely defined, a notion of adjoint operator S^* is well defined. Once again, we want to meet the physical requirements discussed in Section 2.3 in solving the first order Maxwell equations, hence we impose the underlying system to be isolated, so that the flux of relevant physical quantities, such as those built from the stress-energy tensor, is zero through the boundary. To translate mathematically this requirement we need to look for symmetric extensions of S.
What we are looking for specifically are two surjective maps on the domain of S^* that allow us to write a Green formula similar to that of (2.45) and (2.46), which hold respectively for the domain of curl and of the first order Maxwell operator H, defined in Equation (2.40).

Definition 3.1.7. *A* boundary triple *for the adjoint operator S^* is a triple $(\mathsf{h}, \gamma_0, \gamma_1)$ consisting of a complex separable Hilbert space h and of two linear maps $\gamma_i : \mathrm{dom}(S^*) \to \mathsf{h}$, $i = 0, 1$ such that*

$$(S^* f, f')_{\mathcal{H}} - (f, S^* f')_{\mathcal{H}} = (\gamma_1 f, \gamma_0 f')_{\mathsf{h}} - (\gamma_0 f, \gamma_1 f')_{\mathsf{h}}\,, \quad \forall f, f' \in \mathrm{dom}(S^*)\,,$$

In addition we require the map $\gamma : \mathrm{dom}(S^*) \to \mathsf{h} \times \mathsf{h}$ such that $f \mapsto (\gamma_0(f), \gamma_1(f))$ to be surjective.

Remark 3.1.8. A boundary triple $(\mathsf{h}, \gamma_0, \gamma_1)$ for S^* exists if and only if S admits self-adjoint extensions in $\mathcal{H}$, or in other words the deficiency indexes $n_\pm(S) = \dim\ker(S^* \pm i)$ are equal.We notice that if $S \neq S^*$, then a boundary triple for S^* (if it exists) is not unique.

Boundary triples are a convenient tool to characterize the self-adjoint extensions of a large class of linear operators. The proof of the following proposition can be found in [DM95].

Proposition 3.1.9. *If $\Theta \subseteq \mathsf{h} \times \mathsf{h}$ is a closed, densely defined linear relation, then $S_\Theta \doteq S^*|_{\ker(\gamma_1 - \Theta\gamma_0)}$ is a closed extension of S. In addition the map $\Theta \to S_\Theta$ is one-to-one and $S_\Theta^* = S_{\Theta^*}$. Hence there is a one-to-one correspondence between self-adjoint relations Θ and self-adjoint extensions of S.*

Remark 3.1.10. We recall that, given a relation $\Theta \subseteq \mathsf{h} \times \mathsf{h}$, the adjont relation Θ^* is defined by

$$\Theta^* \doteq \{(y_1, y_2) \in \mathsf{h} \times \mathsf{h} \mid (x_1, y_2)_{\mathsf{h}} = (x_2, y_1)_{\mathsf{h}}\,,\ \forall (x_1, x_2) \in \Theta\}\,. \qquad (3.10)$$?Eqn: adjoint relatic

The relation Θ is self-adjoint if $\Theta = \Theta^*$.

One could argue that the method to parametrize the self-adjoint extensions of a differential operator here discussed is very similar to that of Section 2.3. The main difference with the method of Lagrangian subspaces is that in such framework it is not always clear how to characterize the extensions in terms of boundary conditions. Boundary triples, in fact, offer a solution to this problem. since the relation Θ encodes the choice of a boundary condition in the domain

$$\ker(\gamma_1 - \Theta\gamma_0). \tag{3.11}$$ {?}

Hence to obtain a self-adjoint extension S_Θ it suffices to check the self-adjointness of Θ.

Example 3.1.11. We illustrate, following the discussion of [DDF19, Sec. 2.2], the construction of boundary triples for a differential operator $A = -\Delta + q$ on a Riemannian manifold with boundary (Σ, h) of bounded geometry, where Δ is the Laplace-Beltrami operator and q is a strictly positive bounded function. Since Σ is of bounded geometry, the Laplace-Beltrami operator is uniformly elliptic and the maximal domain in $\mathrm{L}^2(\Sigma)$ on which A can be defined is the Sobolev space $\mathrm{H}^2(\Sigma)$. Hence $A^* : \mathrm{H}^2(\Sigma) \to \mathrm{L}^2(\Sigma)$ defined by $A^* = A$ is the adjoint of A, whose domain is $\mathrm{dom}(A) = \mathrm{H}^2_0(\Sigma)$, *i.e.* the space of functions f that satisfy $\mathrm{t}f = \mathrm{n}\mathrm{d}f|_{\partial\Sigma} = \partial_\nu f|_{\partial\Sigma} = 0$ in the sense of Sobolev space traces. A is then a densely defined closed symmetric operator in $\mathrm{L}^2(\Sigma)$ and the Green identity

$$(A^* f, f')_{\mathrm{L}^2(\Sigma)} - (f, A^* f')_{\mathrm{L}^2(\Sigma)} = (\gamma_1 f, \gamma_0 f')_{\mathrm{L}^2(\partial\Sigma)} - (\gamma_0 f, \gamma_1 f')_{\mathrm{L}^2(\partial\Sigma)}\,, \tag{3.12}$$ {?}

holds for $f, f' \in \mathrm{dom}(A^*) = \mathrm{H}^2(\Sigma)$ and $\gamma_0 f = \mathrm{t}f = f|_{\partial\Sigma}$, $\gamma_1 f = -\mathrm{n}\mathrm{d}f = \partial_\nu f|_{\partial\Sigma}$. The maps γ_0, γ_1 represent the Dirichlet and Neumann boundary conditions, respectively. The Neumann boundary condition is recovered choosing $\Theta = 0$ and the Dirichlet condition represents a kind of degenerate scenario, which has to be included by hand as $\Theta = \infty$. The reason is due to the formulation which we have chosen so to emphasize the connection with the heuristic notion of boundary conditions.

The triple $(\mathrm{L}^2(\partial\Sigma), \gamma_0, \gamma_1)$ is not, however, a boundary triple, since the map $\gamma = (\gamma_0, \gamma_1)$: $\mathrm{dom}(A^*) = \mathrm{H}^2(\Sigma) \to \mathrm{L}^2(\partial\Sigma) \times \mathrm{L}^2(\partial\Sigma)$ is not surjective.
To solve the problem, one must account that the trace maps t and n are surjective on the Sobolev space $H^{1/2}(\partial\Sigma)$, in view of Remark 1.3.10. Hence, since all separable Hilbert spaces are isomorphic, one can introduce the isomorphisms

$$j_\pm : \mathrm{H}^{\pm 1/2}(\partial M) \to \mathrm{L}^2(\partial M), \quad \iota_\pm : \mathrm{H}^{\pm 3/2}(\partial M) \to \mathrm{L}^2(\partial M)\,. \tag{3.13}$$ {?}

As shown in [DDF19, Prop. 24], with the following definitions, $(\mathrm{L}^2(\partial\Sigma), \Gamma_0, \Gamma_1)$ is indeed a boundary triple:

$$\begin{array}{l}\Gamma_0 : \mathrm{H}^2(M) \ni f \mapsto \iota_+\gamma_0 f \in \mathrm{L}^2(\partial M)\\ \Gamma_1 : \mathrm{H}^2(M) \ni f \mapsto j_+\gamma_1 f \in \mathrm{L}^2(\partial M)\end{array}\,. \tag{3.14}$$ {?}

■

Since we have assumed that the underlying spacetime (M, g) is ultrastatic, Equation (1.1) entails that [Pfe09]

$$\Box = \partial_\tau^2 + S,$$

where S is a uniformly elliptic operator whose local form can be found in [Pfe09]. This entails that, in order to construct solution of (3.20), we can follow the rationale outlined in [DDF19].
To this end we start by focusing our attention on S analyzing it within the framework of boundary triples. To apply the framework we need a decomposition that separates the role of space and time in the definition of k-forms, since the technique of boundary triples allows us to deal with operators involving spatial coordinates and that do not evolve in time.

We follow the discussion in Section 2.1 and as a starting point we notice that being (M, g) globally hyperbolic, Theorem 1.1.2 ensures that M is diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ carries the metric h. Let us indicate with $\iota_\tau \colon \Sigma \to M$ the (smooth one-parameter group of) embedding maps which realizes Σ at time τ as $\iota_\tau\Sigma = \{\tau\} \times \Sigma \doteq \Sigma_\tau$. It holds $\Sigma_\tau \simeq \Sigma_{\tau'}$ for all $\tau, \tau' \in \mathbb{R}$. If follows that for all $\omega \in \Omega^k(M)$ and $\tau \in \mathbb{R}$, $\omega|_{\Sigma_\tau} \in C^\infty(M, \iota_\tau^*\Lambda^k T^*M)$, where $\Lambda^k T^*M$ the k-th exterior power of the cotangent bundle over M. Moreover, recalling Section 2.1, we further decompose $\omega|_{\Sigma_\tau}$ as

$$\omega|_{\Sigma_\tau} = \mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{t}_{\Sigma_\tau}\omega\,.$$

where $\mathrm{t}_{\Sigma_\tau}\omega \in \Omega^k(\Sigma_\tau)$ while $\mathrm{n}_{\Sigma_\tau}\omega = (\star_{\Sigma_\tau}^{-1}\iota_\tau^*\star_M)\omega \in \Omega^{k-1}(\Sigma_\tau)$ – *cf.* Definition 1.2.3. With the identification $\Sigma_t \simeq \Sigma_{t'}$ the decomposition induces the isomorphisms

$$\begin{aligned} C^\infty(M, \iota_\tau^*\Lambda^k T^*M) &\simeq \Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma)\\ \omega &\to (\omega_0 \oplus \omega_1)\,, \end{aligned} \tag{3.15}$$ {?}

$$\begin{aligned} \Omega^k(M) &\simeq C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k(\Sigma))\\ \omega &\to (\tau \mapsto \mathrm{n}_{\Sigma_\tau}\omega) \oplus (\tau \mapsto \mathrm{t}_{\Sigma_\tau}\omega)\,. \end{aligned} \tag{3.16}$$?Eqn: identification

Furthermore a direct computation shows that, for all $\omega \in \Omega^k(M)$, it holds that

$$S\omega|_{\Sigma_\tau} = (\Delta_{k-1}\mathrm{n}_{\Sigma_\tau}\omega) \wedge \mathrm{d}\tau + \Delta_k \mathrm{t}_{\Sigma_\tau}\omega\,,$$

where Δ_k is the Laplace-Beltrami operator acting on k-forms, built out of h.
To build the boundary triples as in Definition 3.1.7, we discuss the following construction.

As Hilbert space in which to construct the boundary triples, we use

$$\mathcal{H} \doteq \mathrm{L}^2\Omega^{k-1}(\Sigma) \oplus \mathrm{L}^2\Omega^k(\Sigma)\,,$$

where $\mathrm{L}^2\Omega^k(\Sigma)$ is defined inDefinition 1.3.5 and Remark 1.3.8 with the pairing $(\alpha,\beta)_\Sigma = \int_\Sigma \overline{\alpha} \wedge \star_\Sigma \beta$ for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(\Sigma)$.

Remark 3.1.12. With reference to Remark 2.2.2, we denote still with d_Σ and δ_Σ the extension to the space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ of the action of the differential and of the codifferential on $\Omega^k_{\mathrm{c}}(\Sigma)$.

Moreover, we identify S with $\Delta_{k-1} \oplus \Delta_k$ where Δ_k is the Laplace-Beltrami operator built out of h acting on k-forms. Observe that S can be regarded as an Hermitian and densely defined operator on $\mathrm{H}^2_0(\Lambda^{k-1}T^*\Sigma) \oplus \mathrm{H}^2_0(\Lambda^k T^*\Sigma)$ where $\mathrm{H}^2_0(\Lambda^k T^*\Sigma)$ is the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ with respect to the $\mathrm{H}^2(\Lambda^k T^*\Sigma)$-norm – *cf.* Equation (1.7) with $E \equiv \Lambda^k T^*\Sigma$, where both the inner product and the connection are those induced from the underlying metric h. Hence standard arguments entail that S is a closed symmetric operator on $\mathcal{H}$ whose adjoint S^* is defined on the maximal domain $\mathrm{dom}(S^*) \doteq \{(\omega_0 \oplus \omega_1) \in \mathcal{H} \mid S(\omega_0 \oplus \omega_1) \in \mathcal{H}\}$. In addition $S^*(\omega_0 \oplus \omega_1) = S(\omega_0 \oplus \omega_1)$ for all $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$. Hence, S satisfies the requirements of Definition 3.1.7 and, in view of Remark 3.1.8 S^* admits a boundary triple.

Remark 3.1.13. To realize explicitly the boundary triple, we need the boundary Hilbert space and the two boundary maps γ_0, γ_1. To construct those maps, we recall from Proposition 1.3.9 that there exists a continuous surjective map

$$\mathrm{res}_\ell : \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma), \tag{3.17}$$ {?}

that extends the restriction on $\Omega^k_{\mathrm{c}}(\Sigma)$. Moreover, in terms of the tangential and normal traces $\mathrm{t} = \mathrm{t}_{\partial\Sigma}$ and $\mathrm{n} = \mathrm{n}_{\partial\Sigma}$, defined in Equation 1.11, one can locally write the restriction of any k-form $\omega \in \Omega^k_{\mathrm{c}}(\Sigma)$ to $\partial\Sigma$ as

$$\omega|_{\partial\Sigma} = \mathrm{t}\omega + \mathrm{n}\omega \wedge \mathrm{d}x,$$

where for every $p \in \partial\Sigma$, $\mathrm{d}x$ is the basis element of T^*_pM such that $\mathrm{d}x(\nu_p) = 1$ where ν_p is the outward pointing, unit vector normal to $\partial\Sigma$ at p.

Following the same reasoning of [DDF19] for the scalar case, we construct the following boundary triple for S^*:

Proposition 3.1.14. *Consider S^*, where $S : \mathrm{H}^2_0(\Lambda^{k-1}T^*\Sigma) \oplus \mathrm{H}^2_0(\Lambda^k T^*\Sigma) \to \mathcal{H}$ is defined as $\Delta_{k-1} \oplus \Delta_k$, then the triple $(\mathsf{h}, \gamma_0, \gamma_1)$ is a boundary triple for S^*, where*

- $\mathsf{h} = \mathsf{h}_0 \oplus \mathsf{h}_1$, *with* $\mathsf{h}_0 \doteq \mathrm{L}^2\Omega^{k-1}(\partial\Sigma) \oplus \mathrm{L}^2\Omega^{k-2}(\partial\Sigma)$ *while* $\mathsf{h}_1 = \mathrm{L}^2\Omega^{k-1}(\partial\Sigma) \oplus \mathrm{L}^2\Omega^k(\partial\Sigma)$;

- $\gamma_0 : \mathrm{dom}(S^*) \to \mathsf{h}$ *is such that, for all* $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$,

$$\gamma_0(\omega_0 \oplus \omega_1) = (\mathrm{n}\omega_0 \oplus \mathrm{t}\omega_0) \oplus (\mathrm{n}\omega_1 \oplus \mathrm{t}\omega_1)\,. \tag{3.18}$$?Eq: gamma0?

- $\gamma_1 : \mathrm{dom}(S^*) \to \mathsf{h}$ *is such that, for all* $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$,

$$\gamma_1(\omega_0 \oplus \omega_1) = (\mathrm{t}\delta_\Sigma\omega_0 \oplus \mathrm{n}\mathrm{d}_\Sigma\omega_0) \oplus (\mathrm{t}\delta_\Sigma\omega_1 \oplus \mathrm{n}\mathrm{d}_\Sigma\omega_1)\,. \tag{3.19}$$?Eq: gamma1?

Our goal is that of apply those tools to construct advanced and retarded Green operators for the D'Alembert-de Rham wave operator $\Box$ acting on k-forms. In other words, calling as $\Lambda^k T^*\mathring{M}$ the k-th exterior power of the cotangent bundle over $\mathring{M}$, $k \geq 1$, and with $\boxtimes$ the external tensor product, we look for $G^\pm : \Omega^k_{\mathrm{c}}(\mathring{M}) \to \Omega^k_{\mathrm{sc}}(\mathring{M})$ such that

$$\Box \circ G^\pm = G^\pm \circ \Box = \mathrm{id}\,|_{\Omega^k_{\mathrm{c}}(\mathring{M})}\,,$$

while $\mathrm{supp}(G^\pm(\omega)) \subseteq J^\pm(\mathrm{supp}(\omega))$ for all $\omega \in \Omega^k_{\mathrm{c}}(\mathring{M})$ – *cf.* Assumption 3.1.4. Working at the level of integral kernels and setting $\mathcal{G}^\pm(\tau-\tau',x,x') = \theta[\pm(\tau-\tau')]\mathcal{G}(\tau-\tau',x,x')$, with $\mathcal{G} \in \mathcal{D}'(\mathring{M}\times\mathring{M}, \Lambda^kT^*\mathring{M}\boxtimes\Lambda^kT^*\mathring{M})$, this amounts to solving the following distributional, initial value problem

$$(\Box\otimes\mathbb{I})\,\mathcal{G} = (\mathbb{I}\otimes\Box)\,\mathcal{G} = 0\,, \qquad \mathcal{G}|_{\tau=\tau'} = 0\,, \qquad \partial_\tau\mathcal{G}|_{\tau=\tau'} = \delta_{\mathrm{diag}(\mathring{M})}\,. \tag{3.20}$$?Eq: system for G?

where $\delta_{\mathrm{diag}(\mathring{M})}$ stands for the Dirac delta bi-distribution on $\mathring{M}\times\mathring{M}$ yielding $\delta_{\mathrm{diag}(\mathring{M})}(\omega_1\boxtimes\omega_2) = (\omega_1,\omega_2)$ for all $\omega_1,\omega_2 \in \Omega^k_{\mathrm{c}}(\mathring{M})$.
In view of Proposition 3.1.9 we can follow slavishly the proof of [DDF19, Th. 30] to infer the following statement:

Theorem 3.1.15. *Let* (M,g) *be an ultrastatic and globally hyperbolic spacetime with timelike boundary. Let* $(\mathsf{h},\gamma_0,\gamma_1)$ *be the boundary triple built as per Proposition 3.1.14 associated to the operator* S^*. *Let* Θ *be a self-adjoint relation on* h *and let* $S_\Theta \doteq S^*|_{\mathrm{dom}(S_\Theta)}$ *where* $\mathrm{dom}(S_\Theta) = \ker(\gamma_1 - \Theta\gamma_0)$. *If the spectrum of* S_Θ *is bounded from below, then there exists unique advanced and retarded Green's operator* $G^\pm_\Theta$ *associated to* $-\partial^2_\tau + S_\Theta$. *They are completely determined in terms of the bidistributions* $\mathcal{G}^\pm_\Theta = \theta[\pm(\tau-\tau')]\mathcal{G}_\Theta$ *where* $\mathcal{G}_\Theta \in \mathcal{D}'(\mathring{M}\times\mathring{M}, \Lambda^kT^*\mathring{M}\boxtimes\Lambda^kT^*\mathring{M})$ *is such that for* $\omega_1,\omega_2 \in \Omega^k_{\mathrm{c}}(\mathring{M})$,

$$\mathcal{G}_\Theta(\omega_1,\omega_2) = \int_{\mathbb{R}^2}\left(\omega_1|_\Sigma, S^{-\frac{1}{2}}_{k,\Theta}\sin(S^{\frac{1}{2}}_{k,\Theta}(\tau-\tau'))\omega_2|_\Sigma\right)_\Sigma \mathrm{d}\tau\mathrm{d}\tau'\,,$$

where $(\ ,\ )_\Sigma$ *stands for the pairing between* k*-forms and where* ω_2 *identifies an element in* $\mathrm{dom}(S_\Theta)$ *via the identifications* (3.16). *Moreover it holds that*

$$\gamma_1\left(G^\pm_\Theta\omega\right) = \Theta\,\gamma_0\left(G^\pm_\Theta\omega\right)\,, \quad \forall\omega\in\Omega^k_{\mathrm{c}}(\mathring{M})\,. \tag{3.21}$$ {?}

The last step consists of proving that the boundary conditions introduced in Definition 3.1.2 fall in the class considered in Theorem 3.1.15. In the following proposition we adopt for simplicity the notation $\mathrm{nd} = \mathrm{n}_{\partial\Sigma}\mathrm{d}_{\Sigma}$, $\mathrm{t}\delta = \mathrm{t}_{\partial\Sigma}\delta_{\Sigma}$.

Proposition 3.1.16. *The following relations on* h *are selfadjoint:*

$$\Theta_{\|} \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0\,;\; 0 \oplus \mathrm{nd}\omega_0 \oplus 0 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\} \tag{3.22}$$

?Eqn: parallel-re

$$\Theta_{\perp} \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1\,;\; \mathrm{t}\delta\omega_0 \oplus 0 \oplus \mathrm{t}\delta\omega_1 \oplus 0) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\} \tag{3.23}$$

?Eqn: perp-relati

$$\Theta_{f_{\|}} \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0\,;\; f\mathrm{n}\omega_0 \oplus \mathrm{nd}\omega_0 \oplus f\mathrm{n}\omega_1 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\},\quad f \in C^\infty(\partial\Sigma)\; f \geq 0\,. \tag{3.24}$$

?Eqn: f-relation?

$$\Theta_{f_{\perp}} \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1\,;\; \mathrm{t}\delta\omega_0 \oplus f\mathrm{t}\omega_0 \oplus \mathrm{t}\delta\omega_1 \oplus f\mathrm{t}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\},\quad f \in C^\infty(\partial\Sigma)\; f \leq 0\,. \tag{3.25}$$

{?}

Moreover the self-adjoint extension $S_{\Theta_\sharp}$ *for* $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$ *abides to the hypotheses of Theorem 3.1.15. The associated propagators* $G_\sharp$, $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$, *obey the boundary conditions as per Definition 3.1.2.*

Proof. With reference to Remark 3.1.10, we recall that a given a relation $\Theta \subseteq \mathsf{h} \times \mathsf{h}$ is self-adjoint if $\Theta = \Theta^*$. We show that $\Theta_{\|}, \Theta_{\perp}, \Theta_{f_{\|}}, \Theta_{f_{\perp}}$ are self-adjoint relations. Since the proof is very similar we shall consider only the case of $\Theta_{\|}$. A short computation shows that $\Theta_{\|} \subseteq \Theta_{\|}^*$. We prove the converse inclusion. Let $\underline{\alpha} := (\alpha_1 \oplus \dots \alpha_4\,; \alpha_5 \oplus \dots \alpha_8) \in \Theta_{\|}^*$. Considering equation (3.10) we find

$$(\mathrm{n}\omega_0, \alpha_5) + (\mathrm{n}\omega_1, \alpha_7) = (\mathrm{nd}\omega_0, \alpha_2) + (\alpha_4, \mathrm{nd}\omega_1, \alpha_4)\,, \qquad \forall \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\,. \tag{3.26}$$

{?}

Choosing ω_1 and $\mathrm{n}\omega_0 = 0$ – this does not affect the value $\mathrm{nd}\omega_0$ on account of Remark 1.2.5 – it follows that $(\alpha_2, \mathrm{nd}\omega_0) = 0$ for all $\omega_0 \in \Omega^{k-1}_{\mathrm{c,n}}(\Sigma)$. Since nd is surjective it follows that $\alpha_2 = 0$. With a similar argument $\alpha_5 = 0$ as well as $\alpha_2 = 0$, $\alpha_4 = 0$. Finally, on account of Remark 1.2.5 there exists $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$ such that

$$\mathrm{n}\omega_0 = \alpha_1\,, \qquad \mathrm{n}\omega_1 = \alpha_3\,, \qquad \mathrm{nd}\omega_0 = \alpha_6\,, \qquad \mathrm{nd}\omega_1 = \alpha_8\,.$$

It follows that $\alpha \in \Theta_{\|}$, that is, $\Theta_{\|} = \Theta_{\|}^*$.
In addition $S_{\Theta_\sharp}$ is positive definite for $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$. It follows from the following equality, which holds for all $\omega_0 \otimes \omega_1 \in \mathrm{dom}(S^*)$:

$$(\omega_0 \oplus \omega_1, S_{\Theta_\sharp}(\omega_0 \oplus \omega_1))_{\mathcal{H}} = \sum_{j=1}^{2} \left[\|\mathrm{d}\omega_i\|^2 + \|\delta\omega_i\|^2 + (\mathrm{n}\omega_i, \mathrm{t}\delta\omega_i) - (\mathrm{t}\omega_i, \mathrm{nd}\omega_i)\right]\,,$$

where the last two terms are non-negative because of the boundary conditions and of the hypothesis on the sign of f. Therefore we can apply Theorem 3.1.15.

Finally we should prove that the propagators $G^{\pm}_{\Theta_\sharp}$ associated with the relations $\Theta_\sharp$ coincide with the propagators $G^{\pm}_\sharp$ introduced in Assumption 3.1.4. The fulfilment of the appropriate boundary conditions is a consequence of Lemma B.0.1. ■

3.1.2 Properties of Green operators for D'Alembert-de Rham wave operator

For the convenience of the reader, we recall the main assumption on the existence of Green operators for $\Box$, that we have proven for a particular case in the previous section:

Assumption 3.1.5. *For all $f \in C^\infty(\partial M)$ and for all $k \in \mathbb{N} \cup \{0\}$, the d'Alembert-de Rham wave operator $\Box_\sharp$, with $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$, is* Green hyperbolic*. In other words there exist advanced $(+)$ and retarded $(-)$ Green operators for $\Box_\sharp$, $G^{\pm}_\sharp : \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M) \doteq \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ such that*

$$\Box \circ G^{\pm}_\sharp = \mathrm{id}_{\Omega^k_{\mathrm{c}}(M)}\,, \qquad G^{\pm}_\sharp \circ \Box_{\mathrm{c},\sharp} = \mathrm{id}_{\Omega^k_{\mathrm{c},\sharp}(M)}\,, \qquad \mathrm{supp}(G^{\pm}_\sharp\omega) \subseteq J^{\pm}(\mathrm{supp}(\omega))\,, \quad (3.9)\ \{?\}$$

for all $\omega \in \Omega^k_{\mathrm{c}}(M)$ where $\Box_{\mathrm{c},\sharp}$ indicates that the domain of $\Box$ is restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$.

Corollary 3.1.17. *Under the same hypotheses of Assumption 3.1.4, if the Green operators $G^{\pm}_\sharp$ exist, they are unique.*

Proof. Suppose there exist two maps $G^-_\sharp, \widetilde{G}^-_\sharp : \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M)$ enjoying the properties of equation (3.9). Then, for any but fixed $\alpha \in \Omega^k_{\mathrm{c}}(M)$ it holds

$$(\alpha, G^+_\sharp\beta) = (\Box G^-_\sharp\alpha, G^+_\sharp\beta) = (G^-_\sharp\alpha, \Box G^+_\sharp\beta) = (G^-_\sharp\alpha, \beta), \quad \forall\beta \in \Omega^k_{\mathrm{c}}(M)\,,$$

where we used both the support properties of the Green operators in (3.9) and Lemma 3.1.1 which guarantees that $\Box$ is formally self-adjoint on $\Omega^k_\sharp(M)$. Similarly, replacing $G^-_\sharp$ with $\widetilde{G}^-_\sharp$, it holds $(\alpha, G^+_\sharp\beta) = (\widetilde{G}^-_\sharp\alpha, \beta)$. It follows that $((\widetilde{G}^-_\sharp - G^-_\sharp)\alpha, \beta) = 0$, which implies $\widetilde{G}^-_\sharp\alpha = G^-_\sharp\alpha$, since the pairing between $\Omega^k(M)$ and $\Omega^k_{\mathrm{c}}(M)$ is separating. A similar result holds for the advanced Green operator. ■

In agreement with Proposition 1.4.7, this corollary can be also read as a consequence of the property that, for all $\omega \in \Omega^k_{\mathrm{c}}(M)$, $G^{\pm}_\sharp\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ can be characterized as the unique solution to the Cauchy problem

$$\Box\psi = \omega\,, \qquad \mathrm{supp}(\psi) \cap M \setminus J^{\pm}(\mathrm{supp}(\omega)) = \emptyset\,, \qquad \psi \in \Omega^k_\sharp(M)\,. \qquad (3.27)\ \text{?Eqn: Cauchy problem}$$

Remark 3.1.18. In view of Remark 1.4.6, Green operator $G^+_\sharp$ (*resp.* $G^-_\sharp$) can be extended to $G^+_\sharp : \Omega^k_{\mathrm{pc}}(M) \to \Omega^k_{\mathrm{pc}}(M) \cap \Omega^k_\sharp(M)$ (*resp.* $G^-_\sharp : \Omega^k_{\mathrm{pc}}(M) \to \Omega^k_{\mathrm{pc}}(M) \cap \Omega^k_\sharp(M)$). As a consequence the problem $\Box\psi = \omega$ with $\omega \in \Omega^k(M)$ always admits a solution lying in $\Omega^k_\sharp(M)$. As a matter of facts, consider any smooth function $\eta \equiv \eta(\tau)$, where $\tau \in \mathbb{R}$, *cf.* equation (1.1), such that $\eta(\tau) = 1$ for all $\tau > \tau_1$ and $\eta(\tau) = 0$ for all $\tau < \tau_0$. Then calling $\omega^+ \doteq \eta\omega$ and $\omega^- =$

$(1-\eta)\omega$, it holds $\omega^+ \in \Omega^k_{\mathrm{pc}}(M)$ while $\omega^- \in \Omega^k_{\mathrm{fc}}(M)$. Hence $\psi = G^+_\sharp\omega^+ + G^-_\sharp\omega^- \in \Omega^k_\sharp(M)$ is the sought solution.

We prove the main result of this section, which characterizes the kernel of $\Box_\sharp$ on the space of smooth k-forms with prescribed boundary condition $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$.

Proposition 3.1.19. *Whenever Assumption 3.1.4 is fulfilled, then, for all $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$, setting $G_\sharp \doteq G^+_\sharp - G^-_\sharp : \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M)$, the following statements hold true:*

1. *for all $f \in C^\infty(\partial M)$ the following duality relations hold true:*

$$\star G^\pm_{\mathrm{D}} = G^\pm_{\mathrm{D}}\star\,, \qquad \star G^\pm_{\|} = G^\pm_{\perp}\star\,, \qquad \star G^\pm_{f_\|} = G^\pm_{f_\perp}\star\,. \tag{3.28}$$?Eqn: duality bet

2. *for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$ it holds*

$$(\alpha, G^\pm_\sharp\beta) = (G^\mp_\sharp\alpha, \beta)\,. \tag{3.29}$$?Eqn: adjont of p

3. *the interplay between $G_\sharp$ and $\Box_\sharp$ is encoded in the short exact sequence:*

$$0 \to \Omega^k_{\mathrm{c},\sharp}(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{c}}(M) \xrightarrow{G_\sharp} \Omega^k_{\mathrm{sc},\sharp}(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{sc}}(M) \to 0\,, \tag{3.30}$$?Eqn: short exact

where $\Omega^k_{\mathrm{c},\sharp}(M) \doteq \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M)$.

Proof. We prove the different items separately. Starting from *1.*, we observe that $\star\Box = \Box\star$. Together with Remark 3.1.3, this entails that, for all $\alpha \in \Omega^k_{\mathrm{c}}(M)$,

$$\Box \star G^\pm_\sharp\alpha = \star\Box G^\pm_\sharp \star \alpha = \alpha\,.$$

On account of Remark 3.1.3, the uniqueness of the Green operators as per Corollary 3.1.17 entails (3.28).

2. Equation (3.29) is a consequence of the following chain of identities valid for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$

$$(\alpha, G^\pm_\sharp\beta) = (\Box G^\mp_\sharp\alpha, G^\pm_\sharp\beta) = (G^\mp_\sharp\alpha, \Box G^\pm_\sharp\beta) = (G^\mp_\sharp\alpha, \beta)\,,$$

where we used both the support properties of the Green operators and Lemma 3.1.1.

3. The exactness of the series is proven using the properties already established for the Green operators $G^\pm_\sharp$. The left exactness of the sequence is a consequence of the second identity in equation (3.9) which ensures that $\Box_\sharp\alpha = 0$, $\alpha \in \Omega^k_{\mathrm{c},\sharp}(M)$, entails $\alpha = G^+_\sharp\Box_\sharp\alpha = 0$. In order to prove that $\ker G_\sharp = \Box\Omega^k_{\mathrm{c},\sharp}$, we first observe that $G_\sharp\Box_\sharp\Omega^k_{\mathrm{c},\sharp}(M) = \{0\}$ on account of equation (3.9). Moreover, if $\beta \in \Omega^k_{\mathrm{c}}(M)$ is such that $G_\sharp\beta = 0$, then $G^+_\sharp\beta = G^-_\sharp\beta$. Hence, in view of the support properties of the Green operators $G^+_\sharp\beta \in \Omega^k_{\mathrm{c},\sharp}(M)$ and $\beta = \Box_\sharp G^+_\sharp\beta$. Subsequently we need to verify that $\ker \Box = G_\sharp \Omega^k_{\mathrm{c}}(M)$. Once more $\Box_\sharp G_\sharp \Omega^k_{\mathrm{c}}(M) = \{0\}$ follows from equation (3.9). Conversely, let $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ be such that $\Box_\sharp \omega = 0$. On account of Lemma B.0.2 we can split $\omega = \omega^+ + \omega^-$ where $\omega^+ \in \Omega^k_{\mathrm{spc},\sharp}(M)$. Then $\Box_\sharp \omega^+ = -\Box_\sharp \omega^- \in \Omega^k_{\mathrm{c},\sharp}(M)$ and

$$G_\sharp \Box_\sharp \omega^+ = G^+_\sharp \Box_\sharp \omega^+ + G^-_\sharp \Box_\sharp \omega^- = \omega\,.$$

To conclude we need to establish the right exactness of the sequence. Consider any $\alpha \in \Omega^k_{\mathrm{sc}}(M)$ and the equation $\Box_\sharp \omega = \alpha$. Consider the function $\eta(\tau)$ as in Remark 3.1.18 and let $\omega \doteq G^+_\sharp(\eta\alpha) + G^-_\sharp((1-\eta)\alpha)$. In view of Remark 3.1.18 and of the support properties of the Green operators, $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ and $\Box_\sharp \omega = \alpha$. ∎

Remark 3.1.20. Following the same reasoning as in [Bär15] together with minor adaptation of the proofs of [DDF19], one may extend $G_\sharp$ to an operator $G_\sharp\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k_\sharp(M)$ for all $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$. As a consequence the exact sequence of Proposition 3.1.19 generalizes as

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_\sharp(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_\sharp} \Omega^k_\sharp(M) \xrightarrow{\Box_\sharp} \Omega^k(M) \to 0\,. \qquad (3.31)$$
?Eqn: short exact seq

Remark 3.1.21. Proposition 3.1.19 and Remark 3.1.20 ensure that $\ker_{\mathrm{c}} \Box_\sharp \subseteq \ker_{\mathrm{tc}} \Box_\sharp = \{0\}$. In other words, there are no timelike compact solutions to the equation $\Box\omega = 0$ with $\sharp$-boundary conditions. More generally it can be shown that $\ker_{\mathrm{c}} \Box \subseteq \ker_{\mathrm{tc}} \Box = \{0\}$, namely there are no timelike compact solutions regardless of the boundary condition. This follows by standard arguments using a suitable energy functional defined on the solution space – *cf.* [DDF19, Thm. 30] for the proof for $k = 0$.

In view of the applications to the Maxwell operator, it is worth focusing specifically on the boundary conditions $\perp$, $\|$ individuated in Definition 3.1.2 since it is possible to prove a useful relation between the associated propagators and the operators d,δ.

Lemma 3.1.22. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G^\pm_\| \circ \mathrm{d} = \mathrm{d} \circ G^\pm_\| \quad \text{on } \Omega^k_{\mathrm{t}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^\pm_\| \circ \delta = \delta \circ G^\pm_\| \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad (3.32)$$
?Eqn: relations betwe

$$G^\pm_\perp \circ \delta = \delta \circ G^\pm_\perp \quad \text{on } \Omega^k_{\mathrm{n}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^\pm_\perp \circ \mathrm{d} = \mathrm{d} \circ G^\pm_\perp \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,. \qquad (3.33)$$
?Eqn: relations betwe

Proof. From equation (3.28) it follows that equations (3.32-3.33) are dual to each other via the Hodge operator. Hence we shall only focus on equation (3.32).
For every $\alpha \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{t}}(M)$, $G^\pm_\| \mathrm{d}\alpha$ and $\mathrm{d}G^\pm_\| \alpha$ lie both in $\Omega^k_\|(M)$. In particular, using equation (1.5b), $\mathrm{t}\delta\mathrm{d}G^\pm_\|\alpha = \mathrm{t}(\Box_\| - \mathrm{d}\delta)G^\pm_\|(\alpha) = \mathrm{t}\alpha = 0$ while the second boundary condition is automatically satisfied since $\mathrm{t}\mathrm{d}G^\pm_\| = \mathrm{d}\mathrm{t}G^\pm_\| = 0$. Hence, considering $\beta = G^\pm_\|\mathrm{d}\alpha - \mathrm{d}G^\pm_\|\alpha$, it holds that $\Box\beta = 0$ and $\beta \in \Omega^k_\| \cap \Omega^k_{\mathrm{pc/fc}}(M)$. In view of Remark 3.1.18, this entails $\beta = 0$. ∎

We conclude this section with a corollary to Lemma 3.1.22 which shows that, when considering the difference between the advanced and the retarded Green operators, the support restrictions present in equations (3.32-3.33) disappear.

Corollary 3.1.23. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G_\sharp \circ \mathrm{d} = \mathrm{d} \circ G_\sharp \quad \text{on } \Omega^k_{\mathrm{tc}}(M)\,, \qquad G_\sharp \circ \delta = \delta \circ G_\sharp \quad \text{on } \Omega^k_{\mathrm{tc}}(M) \qquad \sharp \in \{\|, \perp\}\,. \tag{3.34}$$

?Eqn: relation be

Proof. In all cases the reasoning is similar as in the proof of equation (3.32), but it requires the following characterization of $G_\sharp$. Since $M \simeq \mathbb{R} \times \Sigma$ – *cf.* Theorem 1.1.2 – let $\tau_0 \in \mathbb{R}$ and consider $\alpha_0 \in \Omega^k_{\mathrm{c}}(\Sigma_0)$, where $\Sigma_0 := \{\tau_0\} \times \Sigma$. Setting $\alpha := \alpha_0 \wedge \delta_{\tau_0} \mathrm{d}\tau$ we define a distribution-valued k-form and, following [Bär15, Lem. 4.1., Thm. 4.3], we can consider $G_\sharp \alpha$. It turns out that $G_\sharp \alpha$ is the unique solution to the Cauchy problem

$$\Box \psi = 0\,, \qquad \mathrm{t}_{\Sigma_0} \psi = 0\,, \qquad \mathrm{t}_{\Sigma_0} \mathcal{L}_{\partial_\tau}(\psi) = \alpha_0\,, \qquad \sharp\text{-boundary conditions for } \psi\,, \tag{3.35}$$

?Eqn: Cauchy prob

where $\mathrm{t}_{\Sigma_0} \colon \Omega^k(M) \to \Omega^k(\Sigma_0)$ is defined in (1.3) with $N \equiv \Sigma_0$, while $\mathcal{L}_{\partial_\tau}$ denotes the Lie derivative along the vector field ∂_τ.
With this characterization we can prove equation (3.34). Focusing for simplicity on the first identity of (3.34) for $\sharp = \|$, we need to show that $\mathrm{d}G_\| \alpha$ and $G_\| \mathrm{d}\alpha$ solve the same Cauchy problem (3.35). While the analysis of the equation of motion and of the initial data do not differ from the counterpart on globally hyperbolic spacetimes with empty boundary, the only additional necessary information comes from $\mathrm{t}\delta\mathrm{d}G^\pm_\| \alpha = \mathrm{t}(\Box - \mathrm{d}\delta)G^\pm_\| \alpha = \mathrm{t}\alpha$, for all $\alpha \in \Omega^k_{\mathrm{tc}}(M)$. This entails that, being $G_\| = G^+_\| - G^-_\|$, $\mathrm{t}\delta\mathrm{d}G_\| \alpha = 0$. ■

Appendix A

Poincaré-Lefschetz duality for manifold with boundary

In this appendix we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non-empty boundary. A reader interested in more details can refer to [BT13; Sch95].

For the purpose of this appendix M refers to a smooth, oriented manifold of dimension $\dim M = m$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : \partial M \to M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work also with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_{\mathrm{c}}(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\ker(\mathrm{d}_k \colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d}_{k-1} \colon \Omega^{k-1}(M) \to \Omega^k(M))}, \qquad \text{(A.1) ?Eqn: cohomology?}$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (1.4) and (1.5b) entail that we can define $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_{\mathrm{t}}(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M) \doteq \frac{\ker(\delta_k \colon \Omega^k(M) \to \Omega^{k-1}(M))}{\mathrm{Im}(\delta_{k+1} \colon \Omega^{k+1}(M) \to \Omega^k(M))}.$$

Equations (1.4) and (1.5b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M)$ (*resp.* $\Omega^\bullet_{\mathrm{c}}(M)$, $\Omega^\bullet_{\mathrm{c,n}}(M)$), the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$ (*resp.* $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{c,n}}(M) \subseteq \Omega^k(M)$). The associated homology groups will be denoted as $H_{k,\mathrm{n}}(M)$ (*resp.* $H_{k,\mathrm{c}}(M)$,

$H_{k,\mathrm{c},\mathrm{n}}(M))$, $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition and on account of equation (1.5), the Hodge operator induces an isomorphism $H^k(M) \simeq H_{m-k}(M)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [\star\alpha] \in H_{m-k}(M)$. Similarly, on account of Equation (1.5b), it holds $H^k_{\mathrm{t}}(M) \simeq H_{m-k,\mathrm{n}}(M)$ and $H^k_{\mathrm{c,t}}(M) \simeq H_{m-k,\mathrm{c,n}}(M)$.
As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT13]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to

$$\Omega^k(M;\partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M;\partial M) \to \Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta) \in \Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta) = (\mathrm{d}_k\omega, \mathrm{t}\omega - \mathrm{d}_{k-1}\theta)\,. \tag{A.2}$$?Eq: relative-dif

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M;\partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M;\partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will always be clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT13, Prop. 6.49]:

Proposition A.0.1. *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M;\partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\mathrm{t}_*} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M;\partial M) \to \ldots, \tag{A.3}$$ {?}

where $\pi_{1,*}$, $\pi_{2,*}$ *and* t_* *indicate the natural counterpart of the maps* π_1, π_2 *and* t *at the level of cohomology groups.*

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition A.0.2. *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between* $H^k_{\mathrm{t}}(M)$ *and* $H^k(M;\partial M)$ *for all* $k \in \mathbb{N} \cup \{0\}$.

Proof. Consider $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}$ and let $(\omega, 0) \in \Omega^k(M;\partial M)$, $k \in \mathbb{N} \cup \{0\}$. Equation (A.2) entails

$$\underline{\mathrm{d}}_k(\omega,0) = (\mathrm{d}_k\omega, \mathrm{t}\omega) = (0,0)\,.$$

At the same time, if $\omega = \mathrm{d}_{k-1}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $(\mathrm{d}_{k-1}\beta, 0) = \underline{\mathrm{d}}_{k-1}(\beta,0)$. Hence the embedding $\omega \mapsto (\omega,0)$ identifies a map $\rho : H^k_{\mathrm{t}}(M) \to H^k(M;\partial M)$ such that $\rho([\omega]) \doteq$

$[(\omega,0)]$. To conclude, we need to prove that ρ is surjective and injective. Let thus $[(\omega',\theta)] \in H^k(M;\partial M)$. It holds that $\mathrm{d}_k\omega' = 0$ and $\mathrm{t}\omega' - \mathrm{d}_{k-1}\theta = 0$. Recalling that $\mathrm{t} : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective – *cf.* Remark 1.2.5 – for all values of $k \in \mathbb{N}\cup\{0\}$, there must exists $\eta \in \Omega^{k-1}(M)$ such that $\mathrm{t}\eta = \theta$. Let $\omega \doteq \omega' - \mathrm{d}_{k-1}\eta$. On account of (1.5b) $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}_k$ and $(\omega,0)$ is a representative if $[(\omega',\theta)]$ which entails that ρ is surjective.
Let $[\omega] \in H^k(M)$ be such that $\rho[\omega] = [0] \in H^k(M;\partial M)$. This implies that there exists $\beta \in \Omega^{k-1}(M)$, $\theta \in \Omega^{k-2}(\partial M)$ such that

$$(\omega,0) = \underline{\mathrm{d}}_{k-1}(\beta,\theta) = (\mathrm{d}_{k-1}\beta, \mathrm{t}\beta - \mathrm{d}_{k-2}\theta)\,.$$

Let $\eta \in \Omega^{k-2}(M)$ be such that $\mathrm{t}\eta + \theta = 0$. It follows that

$$(\omega,0) = \underline{\mathrm{d}}_{k-1}\left((\beta,\theta) + \underline{\mathrm{d}}_{k-2}(\eta,0)\right) = \underline{\mathrm{d}}_{k-1}(\beta + \mathrm{d}_{k-2}\eta, 0)\,.$$

This entails that $\omega = \mathrm{d}_{k-1}(\beta + \mathrm{d}_{k-2}\eta)$ where $\mathrm{t}(\beta + \mathrm{d}_{k-2}\eta) = 0$. It follows that $[\omega] = 0$ that is ρ is injective.

■

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau96]:

Theorem A.0.3. *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N}\cup\{0\}$*

$$H^{m-k}(M;\partial M) \simeq H^k_{\mathrm{c}}(M)^*\,, \qquad [\alpha] \to \left(H^k_{\mathrm{c}}(M) \ni [\eta] \mapsto \int_M \overline{\alpha}\wedge\eta \in \mathbb{C}\right). \qquad \text{(A.4)} \{?\}$$

where $m = \dim M$ and where on the right hand side we consider the dual of the $(m-k)$-th cohomology group built out compactly supported forms.

Remark A.0.4. On account of Propositions A.0.2-A.0.3 and of the isomorphisms $H^k_{(\mathrm{c})}(M) \simeq H^{m-k}_{(\mathrm{c})}(M)$ the following are isomorphisms:

$$H^k_{\mathrm{t}}(M) \simeq H^{m-k}_{\mathrm{c}}(M)^* \simeq H_{k,\mathrm{c}}(M)^*\,, \qquad H^k(M) \simeq H_{k,\mathrm{c,n}}(M)^*\,. \qquad \text{(A.5) ?Eqn: relative cohomo}$$

The proof proceeds in some steps. Let $\iota : \partial M \to M$ be the immersion map. First of all we have to check that the spaces are finite-dimensional and that the pairing $\langle\,,\,\rangle : H^{m-k}(M) \otimes H^k_{\mathrm{c}}(M,\partial M)$ defined by

$$\langle\alpha,(\omega,\theta)\rangle := \int_M \alpha\wedge\omega + \int_{\partial M}\iota^*\alpha\wedge\theta \qquad \forall\alpha \in H^{m-k}(M) \text{ and } (\omega,\theta) \in H^k_{\mathrm{c}}(M,\partial M)\,,$$

(A.6) ?eq:dualitypair?

is non-degenerate, equivalently the map $\alpha \to \langle \alpha, \cdot \rangle$ should be an isomorphism.

Since a manifold M with boundary is locally homeomorphic to $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ we need Poincaré lemmas for $\mathbb{R}^m_+$.

Lemma A.0.5 (Poincaré lemmas for half spaces)**.** *Let* $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ *and* $k \geq 0$. *Then*

$$H^k(\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \{0\} & \text{otherwise} \end{cases} \tag{A.7} \ \{?\}$$

$$H^k_c(\mathbb{R}^m_+, \partial\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = n \\ \{0\} & \text{otherwise} \end{cases} \tag{A.8} \ \{?\}$$

Proof. The proof for the case $n = 1$, i.e. $\mathbb{R}_+ = [0, +\infty)$ is straightforward and the n-dimensional generalisation is obtained as in ([BT13, Sec. 4]). ■

Lemma A.0.6 (Mayer-Vietoris sequences)**.** *Let M be an orientable manifold with boundary ∂M, suppose $M = U \cup V$ with U, V open and denote $\partial M_A := \partial M \cap A$. Then the following are exact sequences:*

$$\cdots \to H^k(M, \partial M) \to H^k(U, \partial M_U) \oplus H^k(V, \partial M_V) \to H^k(U \cap V, \partial M_{U\cap V}) \to H^{k+1}(M, \partial M) \to \cdots \tag{A.9} \ \{?\}$$

$$\cdots \leftarrow H^k_c(M, \partial M) \leftarrow H^k_c(U, \partial M_U) \oplus H^k(V, \partial M_V) \leftarrow H^k_c(U \cap V, \partial M_{U\cap V}) \leftarrow H^{k+1}_c(M, \partial M) \leftarrow \cdots \tag{A.10} \ \{?\}$$

Proof. We will only prove the non-compact cohomology line.
We have the following Mayer-Vietoris short exact sequences for M and ∂M:

$$0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V) \longrightarrow 0$$
$$0 \to \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M_U) \oplus \Omega^{k-1}(\partial M_V) \to \Omega^{k-1}(\partial M_{U\cap V}) \to 0.$$

Hence applying the direct sum between the two sequences we obtain

$$0 \longrightarrow \Omega^k(M, \partial M) \longrightarrow \Omega^k(U, \partial M_U) \oplus \Omega^k(V, \partial M_V) \longrightarrow \Omega^k(U \cap V, \partial M_{U\cap V}) \longrightarrow 0.$$

The last row induces the desired long sequence because of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(M,\partial M) \arrow[r] \arrow[d, "d"] & \Omega^k(U,\partial M_U)\oplus\Omega^k(V,\partial M_V) \arrow[r] \arrow[d, "\mathrm{d}:=d\oplus d"] & \Omega^k(U\cap V,\partial M_{U\cap V}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(M,\partial M) \arrow[r] & \Omega^{k+1}(U,\partial M_U)\oplus\Omega^{k+1}(V,\partial M_V) \arrow[r] & \Omega^{k+1}(U\cap V,\partial M_{U\cap V}) \arrow[r] & 0
\end{tikzcd} \tag{A.11}$$

{?}

following the arguments in [BT13], section 2. Fix a closed form $\omega \in \Omega^k(U\cap V,\partial M_{U\cap V})$, since the first row is exact there exists a unique $\xi \in \Omega^{k+1}(M,\partial M)$ which is mapped to ω. Now, since $\mathrm{d}\omega = 0$ and the diagram is commutative $\mathrm{d}\xi$ is mapped to 0. Hence from the exactness of the second row there exists χ which is mapped to $\mathrm{d}\xi$ and it easy to see χ is closed. ■

Lemma A.0.7. *If the manifold with boundary M has a* finite good cover *(see [BT13, Sec. 5]) then its (relative) cohomology and (relative) compact cohomology is finite dimensional.*

Proof. The proof is based on the existence of a Mayer-Vietoris sequence in any of the desired cases (proved in the previous proposition) and follows the outline of [BT13, Prop. 5.3.1]. ■

Lemma A.0.8 (Five lemma)**.** *Given the commutative diagram with exact rows*

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "r"] & E \arrow[r] \arrow[d, "s"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots
\end{tikzcd} \tag{A.12}$$

{?}

if f, g, h, s are isomorphism, then so is r.

Lemma A.0.9. *Suppose $M = U \cup V$ with U, V open. The pairing (A.6) induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^{m-k}(M) \arrow[r] \arrow[d] & H^{m-k}(U)\oplus H^{m-k}(V) \arrow[r] \arrow[d] & H^{m-k+1}(M) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^k(M,\partial M)^* \arrow[r] & H^k(U,\partial M_U)^*\oplus H^k(V,\partial M_V)^* \arrow[r] & H^{k-1}(M)^* \arrow[r] & \cdots
\end{tikzcd} \tag{A.13}$$

{?}

Proof. The proof follows that of [BT13, Lem. 5.6]. ■

Now we are ready to prove the main theorem of this section:

Proof of Poincaré-Lefschetz Duality. Follow the argument given in [BT13, Sec. 5]. By the Five lemma if Poincaré-Lefschetz duality holds for U, V and $U \cap V$, then it holds for $U \cup V$. Then it is sufficient to proceed by induction on the cardinality of a finite good cover. □

Appendix B

An explicit computation

Lemma B.0.1. *Let $M=\mathbb{R}\times\Sigma$ be a globally hyperbolic spacetime – cf. Theorem ??. Moreover, for all $\tau\in\mathbb{R}$, let $\mathrm{t}_{\Sigma_\tau}:\Omega^k(M)\to\Omega^k(\Sigma_\tau)$, $\mathrm{n}_{\Sigma_\tau}:\Omega^k(\Sigma_\tau)\to\Omega^{k-1}(\Sigma)$ be the tangential and normal maps on $\Sigma_\tau\doteq\{\tau\}\times\Sigma$, where $M=\mathbb{R}\times\Sigma$ – cf. Definition 1.2.3. Moreover, let $\mathrm{t}_{\partial\Sigma_\tau}:\Omega^k(\Sigma_\tau)\to\Omega^k(\partial\Sigma_\tau)$ and let $\mathrm{n}_{\partial\Sigma_\tau}:\Omega^k(\Sigma_\tau)\to\Omega^{k-1}(\partial\Sigma_\tau)$ be the tangential and normal maps on $\partial\Sigma_\tau\doteq\{\tau\}\times\partial\Sigma$. Let $f\in C^\infty(\partial\Sigma)$ and set $f_\tau\doteq f|_{\partial\Sigma_\tau}$. Then for $\sharp\in\{\mathrm{D},\|,\perp$ $,f_\|,f_\perp\}$ it holds*

$$\omega\in\Omega^k_\sharp(M)\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\Omega^k_\sharp(\Sigma_\tau)\quad\forall\tau\in\mathbb{R}\,. \tag{B.1}$$?Eqn: equivalence bet

More precisely this entails that

$$\begin{aligned}
\omega\in\ker\mathrm{t}_{\partial M}\cap\ker\mathrm{n}_{\partial M}&\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker\mathrm{n}_{\partial\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker\mathrm{n}_{\partial M}\cap\ker\mathrm{n}_{\partial M}\mathrm{d}&\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{n}_{\partial\Sigma_\tau}\cap\ker\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker\mathrm{t}_{\partial M}\cap\ker\mathrm{t}_{\partial M}\delta&\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker\mathrm{n}_{\partial M}\cap\ker(\mathrm{n}_{\partial M}\mathrm{d}-f\mathrm{t}_{\partial M})&\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{n}_{\partial\Sigma_\tau}\cap\ker(\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}-f_t\mathrm{t}_{\partial\Sigma_\tau})\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker\mathrm{t}_{\partial M}\cap\ker(\mathrm{t}_{\partial M}\delta-f\mathrm{n}_{\partial M})&\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker(\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}-f_t\mathrm{n}_{\partial\Sigma_\tau})\,,\forall t\in\mathbb{R}\,.
\end{aligned}$$

Proof. The equivalence (B.1) is shown for $\perp$-boundary condition. The proof for $\|$-boundary conditions follows per duality – *cf.* (3.1.3) – while the one for D-, $f_\|$-, $f_\perp$-boundary conditions can be carried out in a similar way.
On account of Theorem 1.1.2 we have that for all $\tau\in\mathbb{R}$ we can decompose any $\omega\in\Omega^k(M)$ as follows:

$$\omega|_{\Sigma_\tau}=\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,.$$

Notice that, being the decomposition $M=\mathbb{R}\times\Sigma$ smooth we have that $\tau\to\mathrm{t}_{\Sigma_\tau}\omega\in C^\infty(\mathbb{R},\Omega^k(\Sigma))$ while $\tau\to\mathrm{n}_{\Sigma_\tau}\omega\in C^\infty(\mathbb{R},\Omega^{k-1}(\Sigma))$. Here we have implicitly identified $\Sigma\simeq\Sigma_\tau$.
A similar decomposition holds near the boundary of Σ_τ. Indeed for all $(\tau,p)\in\{\tau\}\times\partial\Sigma$ we consider a neighbourhood of the form $U=[0,\epsilon_\tau)\times U_{\partial\Sigma}$. Let $U_x\doteq\{x\}\times U_{\partial\Sigma}$ for $x\in[0,\epsilon_\tau)$ and let t_{U_x}, n_{U_x} be the corresponding tangential and normal maps – *cf.* Definition 1.2.3. With

this definition we can always split $\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{\Sigma_\tau}\omega$ as follows:

$$\omega|_U = \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \wedge \mathrm{d}x + \mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}x \wedge \mathrm{d}\tau\,. \qquad \text{(B.2) \{?\}}$$

If p ranges on a compact set of $\partial\Sigma$ it follows that $(\tau, x) \to \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}\times[0,\epsilon), \Omega^k(\partial\Sigma))$ and similarly $\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$, $\mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$. Once again we have implicitly identified $U_{\partial\Sigma} \simeq \{x\}\times U_{\partial\Sigma}$.
According to this splitting we have

$$\begin{aligned}
\mathrm{t}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{t}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau = \mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,,\\
\mathrm{n}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau = \mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,.
\end{aligned}$$

It follows that $\mathrm{n}_{\partial M}\omega = 0$ if and only if $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and similarly $\mathrm{t}_{\partial M}\omega = 0$ if and only if $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$. This proves the thesis for Dirichlet boundary conditions. A similar computation leads to

$$\begin{aligned}
\mathrm{n}_{\partial M}\mathrm{d}\omega &= (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + (-1)^{k-1}\partial_\tau\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\\
&\quad + (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} \wedge \mathrm{d}\tau - \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\\
&= (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} \wedge \mathrm{d}\tau\,.
\end{aligned}$$

where the second equality holds true since $\mathrm{n}_{\partial M}\omega = 0$. It follows that $\mathrm{n}_{\partial M}\mathrm{d}\omega = 0$ if and only if $\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} = 0$ and $\partial_x\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} = 0$. When $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ the latter conditions are equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$. ■

Finally, we prove a very useful Lemma.

Lemma B.0.2. *Let $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$, with $f \in C^\infty(\partial M)$. The following statements hold true:*

1. *for all $\omega \in \Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{spc}}(M)\cap\Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{sfc}}(M)\cap\Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

2. *for all $\omega \in \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{fc}}(M)\cap\Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

Proof. We prove the result in the first case, the second one can be proved in complete analogy. Let $\omega \in \Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_\sharp(M)$. Consider Σ_1, Σ_2, two Cauchy surfaces on M – *cf.* [AFS18, Def. 3.10] – such that $J^+(\Sigma_1) \subset J^+(\Sigma_2)$. Moreover, let $\varphi_+ \in \Omega^0_{\mathrm{pc}}(M)$ be such that $\varphi_+|_{J^+(\Sigma_2)} = 1$ and $\varphi_+|_{J^-(\Sigma_1)} = 0$. We define $\varphi_- := 1 - \varphi_+ \in \Omega^0_{\mathrm{fc}}(M)$. Notice that we can always choose φ so that, for all $x \in M$, $\varphi(x)$ depends only on the value $\tau(x)$, where τ is the global time function defined in Theorem 1.1.2. We set $\omega_\pm \doteq \varphi_\pm\omega$ so that $\omega^+ \in \Omega^k_{\mathrm{spc}}(M)\cap\Omega^k_\sharp(M)$ while

$\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$. This is automatic for $\sharp = \mathrm{D}$ on account of the equality

$$\mathrm{t}\omega^\pm = \varphi_\pm \mathrm{t}\omega = 0\,, \qquad \mathrm{n}\omega^\pm = \varphi_\pm \mathrm{n}\omega = 0\,.$$

We now check that $\omega^\pm \in \Omega^k_\sharp(M)$ for $\sharp = \perp$. The proof for the remaining boundary conditions $\perp, f_\parallel, f_\perp$ follows by a similar computation – or by duality *cf.* Remark 3.1.3. It holds

$$\mathrm{n}\omega_\pm = \varphi_\pm|_{\partial M}\mathrm{n}\omega = 0\,, \qquad \mathrm{n}\mathrm{d}\omega_\pm = \mathrm{n}(\mathrm{d}\chi \wedge \omega) = \partial_\tau \chi\, \mathrm{n}_{\partial\Sigma_\tau} \mathrm{t}_{\Sigma_\tau}\omega = 0\,.$$

In the last equality $\mathrm{t}_{\Sigma_\tau} \colon \Omega^k(M) \to \Omega^k(\Sigma_\tau)$ and $\mathrm{n}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ are the maps from Definition 1.2.3 with $N \equiv \Sigma_\tau \doteq \{\tau\} \times \Sigma$, where $M = \mathbb{R} \times \Sigma$. The last identity follows because the condition $\mathrm{n}\omega = 0$ is equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ for all $\tau \in \mathbb{R}$ – *cf.* Lemma B.0.1. ■

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List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

List of Symbols

a	distance	m
P	power	$\mathrm{W}\ (\mathrm{J\,s^{-1}})$
ω	angular frequency	rad

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