University of Pavia
Department of Physics

TESI TRIENNALE

On the fundamental solutions for wave-like equations on curved backgrounds

$$\Box u = \delta$$

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July 2017

Preface

To do

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July 2017

Abstract

To do

Contents

Introduction 0

This is meant to be an introduction to my thesis but I do not know what to say.

Mathematical Tools 1

1.1 An overview of Differential Geometry

We begin by recalling of very well known definitions in order to introduce the basic geometrical objects which are used in the text.

A **manifold** is, heuristically speaking, a space that is locally similar to $\mathbb{R}^n$. To define it we use the concepts of **topological space** and of **homeomorphism**.

1.1.1 Definition (**Topological Space**). *A set* X *together with a family* $\mathcal{T}$ *(**topology**) of subsets of* X *is called a **topological space** if the following conditions are met:*

a. $\emptyset, \mathrm{X} \in \mathcal{T}$,

b. *for all* U *and* $V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$,

c. *for any index set* A*, if* $U_i \in \mathcal{T}$ *for all* $i \in A$, $\bigcup_{i \in A} U_i \in \mathcal{T}$.

An element of $\mathcal{T}$ *is called **open set**. If a point* p *is in an open set* U*, we call* U *a **neighborhood** of* p*.* ■

1.1.2 Definition (**Continuity and homomorphism**). *Let* X *and* Y *be two topological spaces. A function* $f : \mathrm{X} \to \mathrm{Y}$ *is **continuous** if for any open set* U

of Y*, the preimage* $f^{-1}(U)$ *is an open set of* X*.*
A continuous and bijective map $\varphi : \mathrm{X} \to \mathrm{Y}$ *is an* ***homomorphism*** *if* $\varphi^{-1} : \mathrm{Y} \to \mathrm{X}$ *is also continuous.* ■

As for vector spaces, we can talk of a **basis** for topological space. A subset $\mathcal{B} \subset \mathcal{T}$ is a basis if any open set can be expressed as union of elements of $\mathcal{B}$. A topology is **Hausdorff** if, for any two distinct points $p, q \in \mathrm{X}$, there exist two open neighborhoods U of p and V of q such that $U \cap V = \emptyset$.

A topological space X is called **compact** if each of its open covers has a finite subcover, i.e. for any collection $\{U_i\}_{i\in A}$, (where A is a set of indexes) such that

$$\mathrm{X} \subseteq \bigcup_{i\in A} U_i,$$

there is a finite subset A' of A such that

$$\mathrm{X} \subseteq \bigcup_{i\in A'} U_i.$$

We are now ready to introduce the concept of **manifold**.

1.1.3 Definition. *An n-dimensional topological* ***manifold*** M *is a topological Hausdorff space (with a countable basis) that is locally homeomorphic to* $\mathbb{R}^n$*, i.e. for every* $p \in \mathrm{M}$ *there exists an open neighbourhood* U *of* p *and a homeomorphism*

$$\varphi : U \to \varphi(U),$$

such that $\varphi(U)$ *is an open subset of* $\mathbb{R}^n$*.* ■

Such homeomorphism is called a **(local) chart** of M. An **atlas** of M is a family $\{U_i, \varphi_i\}_{i\in A}$ of local charts together with an open covering of M, i.e. $\bigcup_{i\in A} U_i = \mathrm{M}$.

1.1.4 Definition. *A* ***differentiable atlas*** *of a manifold* M *is an atlas* $\{U_i, \varphi_i\}_{i\in A}$ *such that the functions*

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

are differentiable (of class C^∞*) for any* $i, j \in A$ *such that* $U_i \cap U_j \neq \emptyset$*. Each* φ_{ij} *is called* ***transition function****.* ■

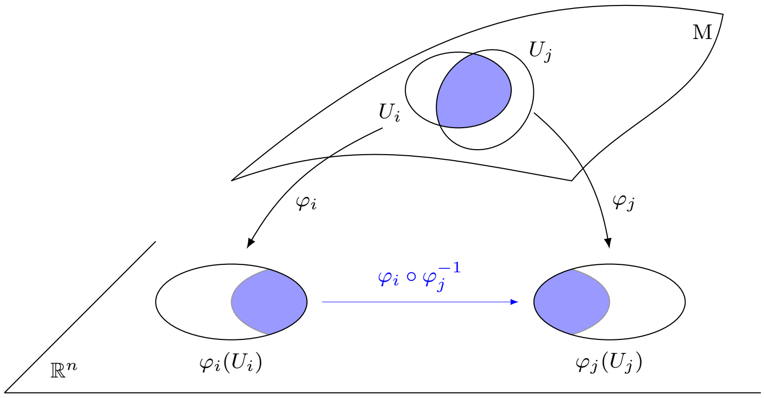


FIGURE 1.1: A differentiable atlas on a manifold M.

With this definition, each φ_{ij} is a **diffeomorphism** because one can always interchange the indexes i and j.

We are only interested in **differentiable** (or **smooth**) **manifolds**, endowed with a maximal differentiable atlas. Here maximality of the atlas means that, if φ is a chart of M and $\{U_i, \varphi_i\}_{i\in A}$ is a differentiable atlas, then φ belongs to $\{U_i, \varphi_i\}_{i\in A}$. We call a differentiable manifold with an atlas for which all chart transitions have positive Jacobian determinant an **orientable manifold**.

1.1.5 Remark. For now on, the word **manifold** will always mean **differentiable manifold** and to indicate them it will be used the letters M or N. ■

1.1.6 Definition (**Submanifold**)**.** *Let $n \leq m$. An n-dimensional **submanifold** N of M is a nonempty subset N of M such that, for every point $q \in$ N, there exists a local chart $\{U, \varphi\}$ of M about q such that*

$$\varphi(U \cap \mathrm{N}) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^m.$$

*If $n = m - 1$ we call N an **hypersurface** of M.* ■

1.1.7 Example. If M and N are manifolds, the Cartesian product M × N is endowable with canonical structure of a manifold. If $\{U_i, \varphi_i\}_{i\in A}$ is an differentiable atlas for M and $\{V_j, \psi_j\}_{j\in B}$ ia an atlas for N, then $\big\{U_i \times V_j, (\varphi_i, \psi_j)\big\}_{(i,j)\in A\times B}$ is a differentiable atlas for M × N. ■

As in the Euclidean case, one can introduce the notion of **differentiable** map between manifolds:

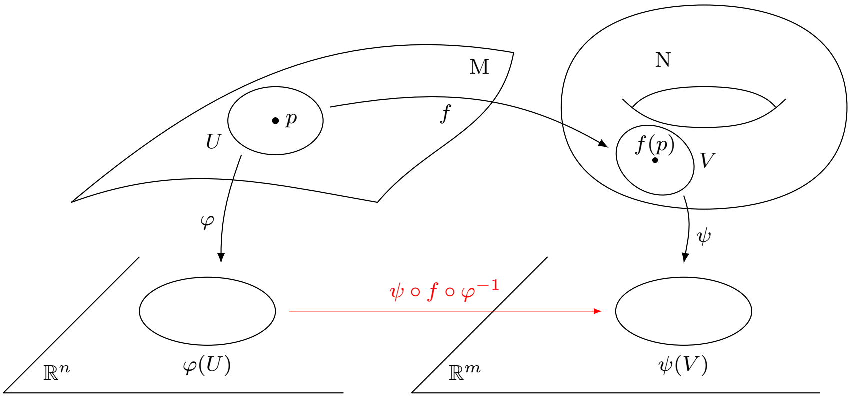


Figure 1.2: The notion of differentiable map f between manifolds M and N.

1.1.8 Definition. *A continuous map* $f : \mathrm{M} \to \mathrm{N}$ *between two manifolds* M *and* N *is* ***differentiable*** *at* $p \in \mathrm{M}$ *if there exist local charts* $\{U, \varphi\}$ *and* $\{V, \psi\}$ *about* p *in* M *and about* $f(p)$ *in* N *respectively, such that* $f(U) \subset V$ *and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V),$$

is differentiable (of class C^∞*) at* $\varphi(p)$*. The function* f *is said to be differentiable on* M *if it is differentiable at every point of* M. ■

The space of differentiable functions between two manifolds is denoted by $C^\infty(\mathrm{M}, \mathrm{N})$, and if $\mathrm{N} = \mathbb{C}$ simply by $C^\infty(\mathrm{M})$. If $\psi \circ f \circ \varphi^{-1}$ is of class C^k, we say $f \in C^k(\mathrm{M}, \mathrm{N})$.

We introduce the **tangent space** of a point of a manifold. It will be constructed using the derivatives of curves which pass through the point. DA AMPLIARE

1.1.9 Definition (**Tangent space**)**.** *Let* $p \in \mathrm{M}$ *and let* I *be an interval containing* 0*. We indicate* $\mathcal{C}_p = \{c \in C^\infty(I, \mathrm{M}) |\ c(0) = p\}$ *the set of differentiable curves passing through* p*.*
We consider the equivalence relation $(\sim)$*, according to which two curves* $c_1, c_2 \in \mathcal{C}_p$ *are equivalent if there exists a local chart* φ *about* p *such that* $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$*.*
The ***tangent space*** *of* M *at* p *is the set* $\mathrm{T}_p\mathrm{M} := \mathcal{C}_p / \sim$ *.* ■

One can check that the definition of the equivalence relation does not depend on the choice of local chart. In fact, if $\{U, \varphi\}$ and $\{V, \psi\}$ are local charts at p,

$$(\varphi \circ c)'(0) = (\varphi \circ \psi^{-1} \circ \psi \circ c)'(0) = D(\varphi \circ \psi^{-1})(\psi(p)) \cdot (\psi \circ c)'(0),$$

where $D(\varphi \circ \psi^{-1})(\psi(p))$ stands for the Jacobian of the transition function calculated at $\psi(p)$. It holds that $(\varphi \circ c_1)'(0)$ and $(\varphi \circ c_2)'(0)$ coincide if and only if $(\psi \circ c_1)'(0)$ and $(\psi \circ c_2)'(0)$ coincide.

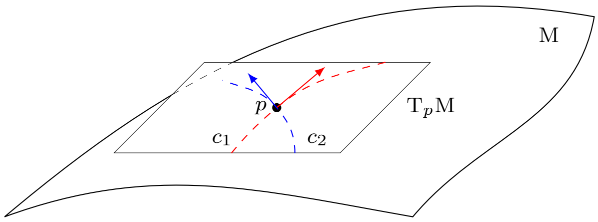


FIGURE 1.3: Tangent space $\mathrm{T}_p\mathrm{M}$ where $c_1 \nsim c_2$.

It can be proven that (for a fixed atlas φ about p) the following map is a linear isomorphism between $\mathrm{T}_p\mathrm{M}$ and $\mathbb{R}^n$:

$$\begin{aligned} \Theta_\varphi : \ \mathrm{T}_p\mathrm{M} &\to \mathbb{R}^n, \\ [c] &\mapsto (\varphi \circ c)'(0). \end{aligned}$$

Hence we can think of $\mathrm{T}_p\mathrm{M}$ as being a copy of $\mathbb{R}^n$ attached to the point p on the manifold.
For reasons that we will make clear later, we denote the basis of $\mathrm{T}_p\mathrm{M}$ as

$$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}.$$

Collecting all tangent spaces, one builds the **tangent bundle** of a manifold M, defined as:

$$\mathrm{TM} = \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M} = \bigcup_{p \in \mathrm{M}} \{p\} \times \mathrm{T}_p\mathrm{M}.$$

1.1.10 Definition. *Let $f : \mathrm{M} \to \mathrm{N}$ be a differentiable map and let $p \in \mathrm{M}$. The **differential** of f at p is the linear map*

$$\mathrm{d}_p f : \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{N}, \quad [c] \mapsto [f \circ c] \ \cong (f \circ c)'(0).$$

*The **differential** of f is the map $\mathrm{d}f : \mathrm{TM} \to \mathrm{TN}$ such that $\mathrm{d}f|_{\mathrm{T}_p\mathrm{M}} = \mathrm{d}_p f$.* ■

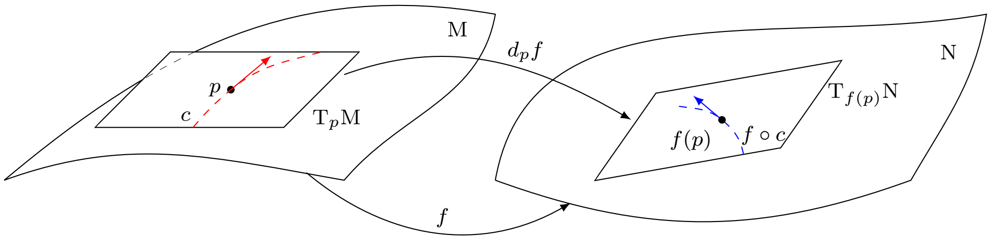


FIGURE 1.4: A scheme of a differential map.

Given M and a chart $\{U, \varphi\}$ near p, fix $X = [c] \in \mathrm{T}_p\mathrm{M}$. If we identify $\mathrm{T}_p\mathbb{R} \cong \mathbb{R}$, we can interpret the differential $\mathrm{d}_p f(X)$ of a function $f \in C^\infty(\mathrm{M})$ at a point p as the **derivative** in the direction of X:

$$\partial_X f(p) := \mathrm{d}_p f(X).$$

A functional which is linear and follows Leibniz rule, such as $\partial_X : C^\infty(\mathrm{M}) \to \mathbb{R}$, is called a **derivation**. The set of all derivations at p is denoted as Der_p and it is a vector space. The map $X \in \mathrm{T}_p\mathrm{M} \mapsto \partial_X$ is an isomorphism between $\mathrm{T}_p\mathrm{M}$ and Der_p.
Define:

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(\mathrm{M}) \to \mathbb{R}, \quad f \mapsto \left.\frac{\partial f}{\partial x^i}\right|_p = \partial_X f(p),$$

where $X = [c]$ and $c(t) = \varphi^{-1}(\varphi(p) + te_i)$ (e_i is the i-th canonical basis vector). Note that, from the definition of the differential, it holds

$$\partial_X f(p) = (f \circ c)'(0) = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(p)},$$

which shows that the object we defined can be seen as a partial derivative in the usual sense.
It can be shown that the set of derivations $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ forms a basis for Der_p and, due to the isomorphism, for $\mathrm{T}_p\mathrm{M}$, X can be expressed as

$$X = X^i \frac{\partial}{\partial x^i},$$

where Einstein summation has been employed.
Observe that linearity entails that

$$\partial_X f(p) = \mathrm{d}_p f(X) = X^i \mathrm{d}_p f\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = X^i \left.\frac{\partial f}{\partial x^i}\right|_p.$$

$$\begin{tikzcd} & \mathrm{T}_p\mathrm{M} \arrow[dl, "X\mapsto\partial_X"'] \arrow[dr, "\Theta_\varphi"] & \\ \mathrm{Der}_p & & \mathbb{R}^n \end{tikzcd}$$

FIGURE 1.5: Isomorphism relations for the tangent space.

1.1.11 Definition. *Let* M *be a manifold, we define a projection map* $\pi : \mathrm{TM} \to \mathrm{M}$ *such that* $\pi(\mathrm{T}_p\mathrm{M}) = p$*, and we call a* ***section*** *in the tangent bundle a map* $s : \mathrm{M} \to \mathrm{TM}$ *such that* $\pi \circ s = \mathrm{id}_{\mathrm{M}}$ *.* ■

The dual space of the tangent space $\mathrm{T}_p\mathrm{M}$ is called the **cotangent space**, denoted with $\mathrm{T}_p^*\mathrm{M}$, which has a canonical basis denoted with $\{\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p\}$. The elements of such basis act on any element of the tangent space basis at p as follows:

$$\mathrm{d}x^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{ij}.$$

Similarly is defined the **cotangent bundle** $\mathrm{T}^*\mathrm{M}$ as the disjoint union of cotangent spaces.

1.1.12 Definition. *Sections in the tangent bundle, denoted by* $C^\infty(\mathrm{M}, \mathrm{TM})$*, are called* ***vector fields****, whereby sections in the cotangent bundle are called* 1***-forms****.* ■

Vector fields are locally expressed in terms of linear combinations of

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\} =: \{\partial_1, \ldots, \partial_n\},$$

where $\partial_i = \partial_i|_p$ at any point p, whereas 1-forms are expressed as linear combinations of

$$\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\},$$

where $\mathrm{d}x^i$ is the 1-form that acts at p as $\mathrm{d}x^i|_p$.

1.1.13 Definition. *We define the derivative in the direction of* X *as an operator* $\partial_X : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ *such that*

$$\partial_X f = \mathrm{d}f(X),$$

for any vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$*.* ■

It follows immediately that Leibniz's rule holds: $\partial_X(f \cdot g) = g\,\partial_X f + f\,\partial_X g$, and again holds the useful formula

$$\partial_X f = \mathrm{d}f(X) = X^i \mathrm{d}f(\partial_i) = X^i \frac{\partial f}{\partial x^i}.$$

1.1.14 Observation. Given two vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$, there is a unique vector field $[X, Y] \in C^\infty(\mathrm{M}, \mathrm{TM})$ such that

$$\partial_{[X,Y]} f = \partial_X \partial_Y - \partial_Y \partial_X f$$

for all $f \in C^\infty(\mathrm{M})$. The map $[\cdot, \cdot] : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) \to C^\infty(\mathrm{M}, \mathrm{TM})$ is called the **Lie bracket**, it is bilinear, skew-symmetric and satisfies the *Jacobi identity*: for any $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

■

1.1.15 Definition. *An **affine connection** or **covariant derivative** on a manifold* M *is a bilinear map*

$$\begin{aligned} \nabla : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) &\to C^\infty(\mathrm{M}, \mathrm{TM}) \\ (X, Y) &\mapsto \nabla_X Y\,, \end{aligned}$$

such that for all smooth functions $f \in C^\infty(\mathrm{M})$ and all vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$:

- $\nabla_{fX} Y = f \nabla_X Y$*, i.e., ∇ is $C^\infty(\mathrm{M})-$linear in the first variable;*
- $\nabla_X f = \partial_X f$*;*
- $\nabla_X (fY) = \partial_X f + f \nabla_X Y$*, i.e., ∇ satisfies the* Leibniz rule *in the second variable.*

■

The covariant derivative on the direction of the basis vector fields $\{\partial_1, \ldots, \partial_n\}$ is indicated

$$\nabla_j := \nabla_{\partial_j}.$$

We are now ready to introduce metric structures on manifolds.

1.2 Lorentzian Manifolds

We start in the simple case of **Minkowski spacetime**.

1.2.1 Definition. *Let V be an n-dimensional vector space. A **Lorentzian scalar product** is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ with signature $(- + \cdots +)$, i.e. such that one can find a basis $\{e_1, \ldots, e_n\}$ such that*

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_i, e_j \rangle = \delta_{ij} \ \text{ if } i, j > 1.$$

■

The **Minkowski product** $\langle x, y \rangle_0$, defined by the formula

$$\langle x, y \rangle_0 = \eta_{ik} x^i y^k = -x_1 \, y_1 + x_2 \, y_2 + \cdots + x_n \, y_n,$$

with $\eta := \mathrm{diag}(-1, 1, \ldots, 1, 1)$ is the simplest example of Lorentzian scalar product on $\mathbb{R}^n$. The n-dimensional Minkowski space, denoted by $\mathbb{M}^n$ is simply $\mathbb{R}^n$ equipped with Minkowski product.

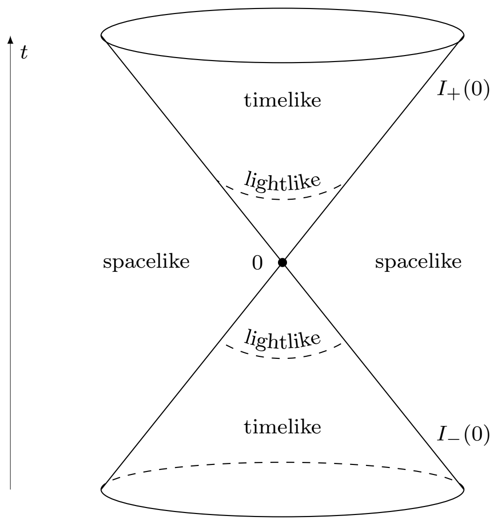


Figure 1.6: Minkowski time orientation.

1.2.2 Definition. *We call the **negative squared length** of a vector $X \in V$*

$$\gamma(X) = - \|X\|^2 := - \langle X, X \rangle .$$

A vector $X \in V \setminus \{0\}$ is called

- ***timelike*** *if* $\gamma(X) > 0$,
- ***lightlike*** *if* $\gamma(X) = 0$,
- ***spacelike*** *if* $\gamma(X) < 0$ *or* $X = 0$,
- ***causal*** *if it is either timelike or lightlike.*

■

This definition will mostly be used for tangent vectors, in case V is the tangent space of a Lorentzian manifold at a point.
For $n \geq 2$ the set of timelike vectors $I(0)$ consists of two connected components. A **time orientation** is the choice of one of these two components, that we call $I_+(0)$.

1.2.3 Definition. *We call*

- $J_+(0) := \overline{I_+(0)}$ *(elements are called* ***future-directed****),*
- $C_+(0) := \partial I_+(0)$ *(upper* ***light cone****),*
- $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$ *(elements are called* ***past-directed****),*
- $C_-(0) := -C_+(0)$ *(lower* ***light cone****).*

■

1.2.4 Definition. *A* ***metric*** g *on a manifold* M *is the assignment of a scalar product at each tangent space*

$$g : \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$$

which depends smoothly on the base point p*. We call it a* ***Riemannian metric*** *if the scalar product is pointwise positive definite, and a* ***Lorentzian metric*** *if it is a Lorentzian scalar product.*
A pair (M, g)*, where* M *is a manifold and* g *is a Lorentzian (Riemannian) metric is called a* ***Lorentzian (Riemannian) manifold****.* ■

The request of smooth dependence on p may be specified as follows: given any chart $\{U, \varphi = (x^1, \ldots, x^n)\}$ about p, the functions $g_{ij} : \varphi(U) \to \mathbb{R}$ defined by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, for any $i, j = 1, \ldots, n$ should be differentiable. With respect to these coordinates one writes

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j \equiv \sum_{i,j} g_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j .$$

The scalar product of two tangent vectors $v, w \in \mathrm{T}_p\mathrm{M}$, with coordinate chart $\varphi = (x^1, \dots x^n)$, such that $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$ is

$$\langle v, w \rangle = g_{ij}(\varphi(p)) v^i w^j.$$

When the choice of the chart is clear we will often write, with abuse of notation $g_{ij}(p) := g_{ij}(\varphi(p))$. We will indicate $(g^{ij})_{i,j=1,\dots,n} :=)^{-1}$.

The **negative squared length** of a tangent vector X at $p \in \mathrm{M}$ generalizes naturally as follows:

$$\gamma(X) = -\langle X, X \rangle. \tag{1.1}$$

From now on M will always indicate a **Lorentzian manifold**.

1.2.5 Definition. *A vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$ is called timelike, spacelike, lightlike or causal, if $X(p)$ is timelike, spacelike, lightlike or causal, respectively, at every point $p \in \mathrm{M}$.*

A differentiable curve $c : I \to \mathrm{M}$ is called timelike, lightlike, spacelike, causal, future-directed or past-directed if $[c](t) \in \mathrm{T}_{c(t)}\mathrm{M}$ is, for all $t \in I$, timelike, lightlike, spacelike, causal, future-directed or past-directed, respectively.

*A Lorentzian manifold M is called **time-oriented** if there exists a nowhere vanishing timelike vector field on M. If a manifold is time-oriented, we refer to it as **spacetime**.* ■

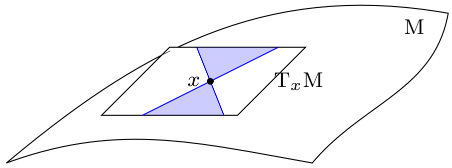
The **causality relations** on M are defined as follows. Let $p, q \in \mathrm{M}$,

- $p \ll q$ iff there exists a future-directed timelike curve from p to q,
- $p < q$ iff there is a future-directed causal curve from p to q,
- $p \leq q$ iff $p < q$ or $p = q$.

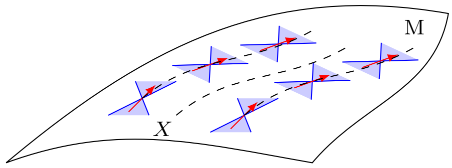
The causality relation "$<$" is a strict weak ordering and the relation "$\leq$" makes the manifold a partially ordered set.

1.2.6 Definition. *The **chronological future** of a point $x \in \mathrm{M}$ is the set $I_+^M(x)$ of points that can be reached by future-directed timelike curves, i.e.*

$$I_+^M(x) = \{y \in \mathrm{M} \mid x < y\}.$$



(a) A time-oriented tangent space.



(b) A time-oriented manifold together with field lines of a timelike vector field X.

FIGURE 1.7: Time orientations.

The ***causal future*** $J_+^M(x)$ *of a point* $x \in \mathrm{M}$ *is the set of points that can be reached by future-directed causal curves from* x*, i.e.,*

$$J_+^M(x) = \{y \in \mathrm{M} \mid x \leq y\}.$$

Given a subset $A \subset \mathrm{M}$ *the* ***chronological future*** *and the* ***causal future*** *of* A *are respectively*

$$I_+^M(A) = \bigcup_{x \in A} I_+^M(x), \qquad J_+^M(A) = \bigcup_{x \in A} J_+^M(x).$$

In a similar way, one defines the ***chronological*** *and* ***causal pasts*** *of a point* x *of a subset* $A \subset \mathrm{M}$ *by replacing future-directed curves with past directed curves. They are denoted by* $I_-^M(x), I_-^M(A), J_-^M(x)$*, and* $J_-^M(A)$*, respectively. We will also use the notation* $J^M(A) := J_-^M(A) \cup J_+^M(A)$*.* ■

Any subset Ω of a spacetime M is a spacetime itself, if one restricts the metric to Ω. So $I_\pm^\Omega(x)$ and $J_\pm^\Omega(x)$ are well defined.

1.2.7 Definition. *A subset* $\Omega \subset \mathrm{M}$ *of a spacetime is called* ***causally compatible*** *if for any point* $x \in \Omega$ *holds*

$$J_\pm^\Omega(x) = J_\pm^\mathrm{M}(x) \cap \Omega,$$

where it can be noted that the inclusion "$\subset$*" always holds.* ■

The condition we defined means that taken two points in Ω that can be joined by a causal curve in M, there also exists a causal curve connecting them inside Ω.

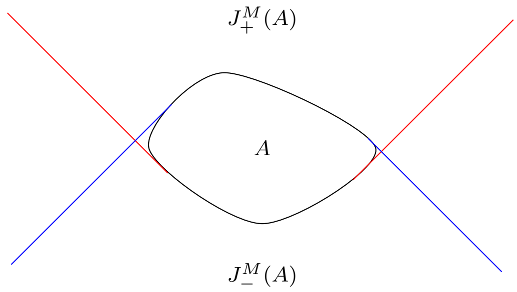


FIGURE 1.8: Causal future J_+^M and causal past J_-^M of a subset $A \subset \mathrm{M}$.

We now can introduce the concept of **geodesics** and **exponential map**.

1.2.8 Definition. *Let $c : [a,b] \to \mathrm{M}$ be a curve on a Lorentzian manifold* M. *The length $L[c]$ is defined by (with Einstein summation convention)*

$$L[c] = \int_a^b \sqrt{|g([c](t),[c](t))|}\, dt = \int_a^b \sqrt{\left| g_{ik}(c(t)) \frac{\mathrm{d}x^i}{\mathrm{dt}} \frac{\mathrm{d}x^k}{\mathrm{dt}} \right|}\, dt,$$

where $x^i(t) := (\varphi \circ c)^i(t)$ are the coordinates of the point $c(t)$ in a chart φ. Given $p, q \in \mathrm{M}$, if $p \leq q$ we define the ***time-separation*** *between p and q as*

$$\tau(p,q) = \sup\{L[c] \mid c \text{ is a future directed causal curve from } p \text{ to } q\},$$

and 0 otherwise.
A ***geodesic*** *between two points $p, q \in \mathrm{M}$ such that $p \leq q$, if it exists, is a curve c such that $L[c] = \tau(p,q)$, i.e. the curve of maximum time-separation.* ■

The request on the geodesics implies that (in variational sense) $\delta L[c] = 0$. It can be demonstrated that the stationary problem for the functional $L[c]$ is equivalent to $\delta E[c] = 0$ for the functional, called **energy**, defined by

$$E[c] = \frac{1}{2} \int_a^b |g([c](t),[c](t))|\, dt.$$

Since the Euler-Lagrange equations for a functional $I[c] = \int_a^b f(t, c(t), [c](t))\, dt$ read

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial f}{\partial \dot{x}^i} \right) - \frac{\partial f}{\partial x^i} = 0,$$

being $c = (x^1, \dots, x^n)$, then, in our case, setting $f(t, c, [c]) = g([c], [c])$:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0.$$

Here $\Gamma^i_{jk} \in C^\infty(U \subset \mathrm{M})$ are the **Christoffel symbols**, defined in the chart $\varphi = (\xi^1, \dots, \xi^n)$ as

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lj}}{\partial \xi^k} + \frac{\partial g_{lk}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l} \right).$$

1.2.9 Definition. *A connection ∇ on a manifold* M *with a metric g is said to be a **metric connection** if for all $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds the following* Leibniz rule*:*

$$\partial_X\, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The unique metric connection which is also torsion-free, i.e.,

$$T := \nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

*is called the **Levi-Civita connection**.* ■

Another way to define **geodesics** is to say a geodesic between two points p, q on a manifold M with the Levi-Civita connection ∇ is the curve c which links p and q such that parallel transport along it preserves the tangent vector to the curve, i.e.

$$\nabla_{[c](t)}[c](t) = 0 \quad \text{for all } t \in [a, b]. \tag{1.2}$$

More precisely, in order to define the covariant derivative of $[c]$ it is necessary first to extend $[c]$ to a smooth vector field in an open set containing the image of the curve, but it can be shown that the derivative is independent of the choice of the extension.

1.2.10 Observation. We can express the Christoffel symbols in terms of the Levi-Civita connection:

$$\nabla_j \partial_k = \Gamma^i_{jk} \partial_i \tag{1.3}$$

in a local chart $\varphi = (x^1, \dots, x^n)$. ■

1.2.11 Proposition. *Let ∇ be a connection over a manifold* M *and $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$ be vector fields. It holds*

$$\nabla_X Y = \left(X^j \partial_j Y^k + X^j Y^i \Gamma^k_{ij} \right) \partial_k,$$

in particular

$$(\nabla_j Y)^i = \partial_j Y^i + Y^i \Gamma^k_{ij}.$$

■

Proof. From Definition (1.1.15) holds:

$$\nabla_X Y = \nabla_{X^j e_j} Y^i e_i = X^j \partial_j Y^i e_i = X^j Y^i \nabla_j e_i + X^j e_i \partial_j Y^i =$$

$$= X^j Y^i \Gamma^k_{ij} e_k + (X^j \partial_j Y^k) e_k.$$

1.2.12 Proposition. *Let us consider $p \in \mathrm{M}$ and a tangent vector $\xi \in \mathrm{T}_p\mathrm{M}$. Then there exists $\varepsilon > 0$ and precisely one geodesic*

$$c_\xi : [0, \varepsilon] \to \mathrm{M},$$

such that $c_\xi(0) = p$ and $\dot{c}_\xi(0) = \xi$. ■

1.2.13 Definition. *In the conditions of the proposition above, if we put*

$$\mathcal{D}_p = \{\xi \in \mathrm{T}_p\mathrm{M} \mid c_\xi \text{ is defined on } [0,1]\} \subset \mathrm{T}_p\mathrm{M},$$

*the **exponential map** at point p is defined as $\exp_p : \mathcal{D}_p \to \mathrm{M}$ such that $\exp_p(\xi) = c_\xi(1)$.*
*The local coordinates defined by the chart $\{U := \exp_p(\mathcal{D}_p), \exp_p^{-1}\}$ are called **normal coordinates** centered at p.* ■

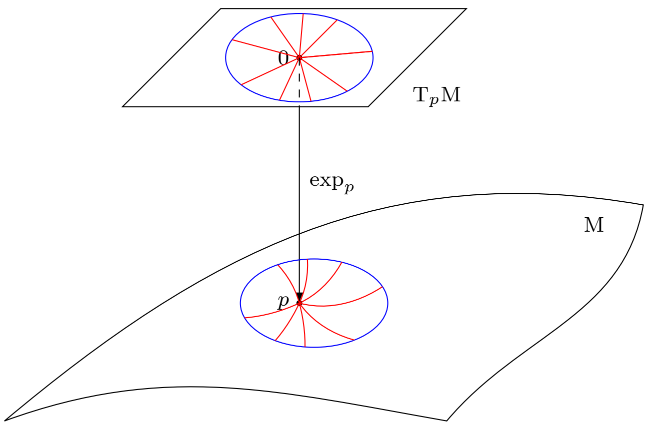


FIGURE 1.9: The exponential maps from the tangent space to the manifold.

1.2.14 Proposition. *Given normal coordinates centered at $p \in \mathrm{M}$, it holds*

$$g_{ij}(p) := g_{ij}(\exp_p(0)) = \eta_{ij},$$

$$\Gamma^i_{jk} = 0,$$

for all indexes i, j, k. ■

We are now ready to talk about **geodesically starshaped** sets.

1.2.15 Definition. *An open subset $\Omega \subset \mathrm{M}$ is called **geodesically starshaped** with respect to a point $p \in \mathrm{M}$ if there exists an open subset $\Omega' \subset \mathrm{T}_p\mathrm{M}$, starshaped with respect to 0, such that the exponential map*

$$\exp_p|_{\Omega'} : \Omega' \to \Omega,$$

*is a diffeomorphism. If Ω is geodesically starshaped with respect to all of its points, one calls it **convex**.* ■

1.2.16 Proposition. *Under the conditions of the last definition, let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to point $p \in \mathrm{M}$. Then one has*

$$I^{\Omega}_{\pm}(p) = \exp_p(I_{\pm}(0) \cap \Omega'),$$

$$J^{\Omega}_{\pm}(p) = \exp_p(J_{\pm}(0) \cap \Omega').$$

■

We continue our preparations with another function that we are going to need in Chapter (3).

1.2.17 Definition. *Let $\Omega \subset \mathrm{M}$ be open and geodesically starshaped with respect to $x \in \Omega$. We define*

$$\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R},$$

where $\gamma : \mathrm{T}_x\mathrm{M} \to \mathbb{R}$ is defined in Equation (1.1). ■

1.2.1 Causality and Global Hyperbolicity

Now we introduce causal domains, because they will appear in the theory of wave equations. The local construction of fundamental solutions is always possible on causal domains, provided they are small enough.

1.2.18 Definition. *A domain* $\Omega \subset \mathrm{M}$ *is called* ***causal*** *if its closure* $\overline{\Omega}$ *is contained in a convex domain* Ω' *and for any* $p, q \in \overline{\Omega}$ $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ *is compact and contained in* $\overline{\Omega}$.
A subset $A \subset \mathrm{M}$ *is called* ***past-compact*** *(respectively* ***future-compact****) if, for all* $p \in \mathrm{M}$, $A \cap J_-^{\mathrm{M}}(p)$ *(respectively* $A \cap J_+^{\mathrm{M}}(p)$*) is compact.* ■

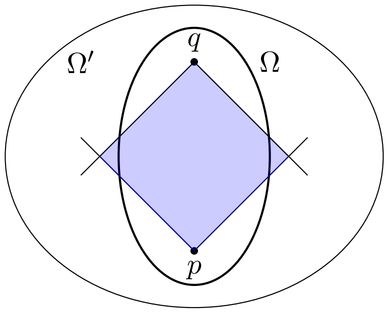


FIGURE 1.10: Convex, but non causal, domain

We can notice that, if we look at compact spacetimes, something physically unsound occurs:

1.2.19 Proposition. *If a spacetime* M *is compact, there exists a closed timelike curve in* M. ■

In a few words, there are manifolds, such as compact spacetimes, where there are timelike loops that can produce science fictional paradoxes. To avoid such unphysical and unrealistic things we require suitable causality conditions:

1.2.20 Definition. *A spacetime satisfies the* ***causality condition*** *if it does not contain any closed causal curve. A spacetime* M *satisfies the* ***strong causality condition*** *if there are no almost closed causal curves, i.e. if for any* $p \in \mathrm{M}$ *there exists a neighborhood* U *of* p *such that there exists no timelike curve that passes through* U *more than once.* ■

It is clear that the strong causality condition implies the causality condition.

1.2.21 Definition. *A spacetime* M *that satisfies the strong causality condition and such that for all* $p, q \in \mathrm{M}$ $J_+^{\mathrm{M}}(p) \cap J_-^{\mathrm{M}}(q)$ *is compact is called* ***globally hyperbolic****.* ■

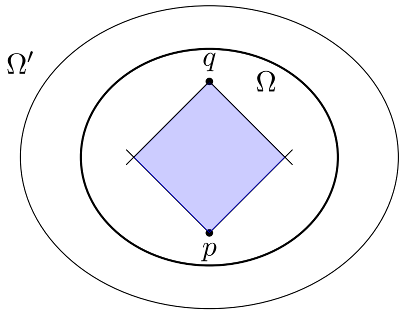
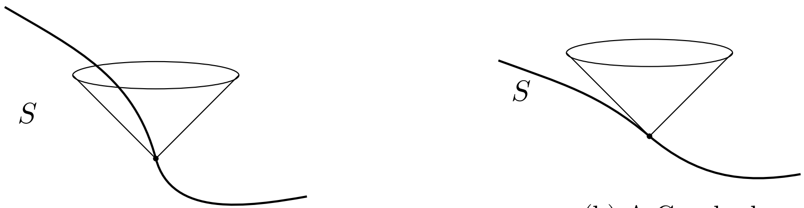


Figure 1.11: Causal domain



(a) A non-Cauchy hypersurface

(b) A Cauchy hypersurface

Figure 1.12: Hypersurfaces.

It can be demonstrated that in globally hyperbolic manifolds for any $p \in \mathrm{M}$ and any compact set $K \subset \mathrm{M}$ the sets $J^{\mathrm{M}}_{\pm}(p)$ and $J^{\mathrm{M}}_{\pm}(K)$ are closed.

1.2.22 Definition. *A subset S of a connected time-oriented Lorentzian manifold M is a* ***Cauchy hypersurface*** *if each inextensible timelike curve (i.e. no reparametrisation of the curve can be continuously extended) in M meets S at exactly one point.* ■

In other words, no point of a Cauchy hypersurface is in the light cone of another point of the surface.

1.2.23 Theorem. *Let M be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

- M *is globally hyperbolic.*
- *There exists a Cauchy hypersurface in* M.
- M *is isometric to $\mathbb{R} \times S$ with metric $g = -\beta \mathrm{d}t^2 + b_t$, where β is a smooth positive function, b_t is a Riemannian metric on S depending smoothly*

on t and each $\{t\} \times S$ is a smooth Cauchy hypersurface in M.

In such case there exists a smooth function $h : \mathrm{M} \to \mathbb{R}$ whose gradient is past-directed timelike at every point and all of whose level sets are Cauchy hypersurfaces. ■

1.3 Operators and integration on manifolds

We call $C_0^\infty(\mathrm{M})$ the set of C^∞ functions on a manifold with compact support. The integral map is defined as the unique map

$$\int_{\mathrm{M}} \cdot \,\mathrm{d}\mu : C_0^\infty(\mathrm{M}) \to \mathbb{C},$$

such that it is linear and for any local chart $\{U, \varphi\}$ and for any $f \in C_0^\infty(U)$ holds

$$\int_{\mathrm{M}} f \,\mathrm{d}\mu = \int_{\varphi(U)} (f \circ \varphi^{-1})(x)\, \mu_x \,\mathrm{d}x,$$

where we define

$$\mu_x := |\det g(x)|^{1/2}. \tag{1.4}$$

In this section we introduce the **generalized d'Alembert** operators, whose general form in local coordinates is given by

$$P = -g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a_j(x)\frac{\partial}{\partial x^j} + b(x).$$

The d'Alembert operator $\Box$ is defined for smooth functions f as

$$\Box f = -\operatorname{div} \operatorname{grad} f,$$

where $\operatorname{grad} f$ is a vector field defined by the requirement that the formula

$$\langle \operatorname{grad} f, X\rangle = \partial_X f$$

holds for any vector field X. At the same time div is defined as follows

1.3.1 Definition. *The **divergence** of a vector field $Z = Z^i\partial_i$ is defined as*

$$\operatorname{div} Z = \sum_j (\nabla_j Z)_j = \partial_j Z^j + \Gamma^i_{ij} Z^j.$$

■

1.3.2 Proposition. *The following formula holds:*

$$\operatorname{div} Z = \mu_x^{-1} \frac{\partial}{\partial x^j} \left(\mu_x Z^j\right), \tag{1.5}$$

and the definition of divergence is consistent with that of integral. ■

Proof. Let $h \in C_0^\infty(\mathrm{M})$; using integration by parts

$$\int_{\mathrm{M}} h \cdot \operatorname{div}(Z) \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mu_x \, \mathrm{d}x.$$

Now integrating by parts again in the chart it holds:

$$-\int_{\mathrm{M}} \partial_j h \; Z^j \mu_x \, \mathrm{d}x = \int_{\mathrm{M}} h \; \partial_j(\mu_x \, Z^j) \, \mathrm{d}x = \int_{\mathrm{M}} h \; \mu_x^{-1} \partial_j(\mu_x \; Z^j) \, \mathrm{d}\mu.$$

Since this is true for any function h, the formula is proven. ■

From the definition of gradient one can show

$$g_{ij} (\operatorname{grad} f)^i X^j = \partial_X f = X^j \frac{\partial f}{\partial x^j}, \quad \operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \partial_j.$$

Hence it holds

$$\Box f = -\mu_x^{-1} \frac{\partial}{\partial x^j} \left(\mu_x \, g^{ij} \frac{\partial f}{\partial x^i}\right).$$

In Minkowski spacetime, where $g = \eta$,

$$\Box f = -\frac{\partial}{\partial x^j} \left(\eta^{jj} \frac{\partial f}{\partial x^j}\right) = \frac{\partial^2 f}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial x_i^2} = -\partial^i \partial_i f.$$

Fundamental solutions in Minkowski spacetime 2

In this chapter, we illustrate the concept of fundamental solutions and their use in solving initial value problems. We focus on the case of the d'Alembert wave operator $\Box$ built on top of the n-dimensional Minkowski spacetime $\mathbb{M}^n$. Two different approaches will be followed. The first relies on the Fourier transform. It is useful to build explicit formulas for the fundamental solutions in the lower dimensional cases and to show the existence of such solutions in the general case. The second approach, via Riesz distributions, is useful for its generality and because it will be used in the next chapter to construct fundamental solutions on suitable Lorentzian manifolds, although it is more abstract and does not lead to explicit formulas.
We will find two independent fundamental solutions, the *retarded* and the *advanced* one, that preserve the causal structure of the spacetime, i.e. that propagate the source of the equation respectively in the future cone and in the past cone, in accordance with the causality principle.

2.1 Fundamental solutions

2.1.1 Definition. *Let P be a differential operator on a manifold* M *and* $x_0 \in \mathrm{M}$. *A* ***fundamental solution*** *for P at x_0 is a distribution $u_{x_0} \in \mathcal{D}'(\mathrm{M})$ such that*

$$Pu_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac delta distribution in x_0, i.e. $(\delta_{x_0}, f) = f(x_0)$ for all $f \in \mathcal{D}(\mathrm{M})$. ∎

A fundamental solution $u_x \in \mathcal{D}'(\mathrm{M})$, with $x \in \mathrm{M}$ defines a distributional kernel $u \in \mathcal{D}'(\mathrm{M} \times \mathrm{M})$??????

Under the assumptions of the previous definition, the distribution $F_\psi(x) = (u_x, \psi)$, where u_x is a fundamental solution of P at x with a continuous dependence on x (i.e. the function $x \mapsto (u_x, \varphi)$ is continuous for all $\varphi \in \mathcal{D}(\mathrm{M})$) and $\psi \in \mathcal{D}'(\mathrm{M})$, is a solution for the differential equation

$$PF_\psi = \psi.$$

This comes applying the operator P on F_ψ, for which one obtains

$$PF_\psi = (u_x, P^*\psi) = (Pu_x, \psi) = (\delta_x, \psi) = \psi(x),$$

where P^* stands for the formal adjoint of P.

2.2 The d'Alembert wave operator in Minkowski

In order to be more concrete, we begin by computing the fundamental solution of $\Box$ in $\mathbb{M}^n$ with $2 \leq n \leq 4$.
As we recalled in the previous chapter, the d'Alembert wave operator is defined in $\mathbb{M}^n$, with respect to the variables $x = (t, \mathbf{x})$, as

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} = -\partial^i \partial_i.$$

We will find two fundamental solutions $G^+_{x_0}, G^-_{x_0} \in \mathcal{D}'(\mathbb{M}^n)$ for $\Box$ at $x_0 \in \mathbb{M}^n$ with the following properties

$$\mathrm{supp}(G^+_{x_0}) \subset J_+(x_0), \qquad \mathrm{supp}(G^-_{x_0}) \subset J_-(x_0). \tag{2.1}$$

Such solutions will be called respectively **retarded** (G_+) and **advanced** (G_-) fundamental solutions.
In order to find the fundamental solutions at x_0, the following proposition guarantees it suffices to solve the problem $\Box u_0 = \delta_0$.

2.2.1 Proposition. *Let $x_0 \in \mathbb{M}^n$ and T_{x_0} be the translation operator as in Definition A.0.3. Then $[\Box, T_{x_0}] = 0$ (i.e. $\Box$ and T_{x_0} commute) and a fundamental solution for $\Box$ at x_0 is*

$$u_{x_0} = T_{x_0} u_0,$$

where u_0 is a fundamental solution at 0. ∎

Proof. Let $\varphi \in \mathcal{D}(\mathbb{M}^n)$, then

$$(\Box T_{x_0} u, \varphi(x)) = (T_{x_0} u, \Box\varphi(x)) = (u, \Box\varphi(x + x_0)) =$$

$$= (\Box u, \varphi(x + x_0)) = (T_{x_0} \Box u, \varphi(x)) \,,$$

where it was used the fact that $\Box$ is formally self-adjoint and invariant under translations when acting on smooth functions. Hence, it holds

$$\Box \left(T_{x_0} u_0\right) = T_{x_0} \left(\Box u_0\right) = T_{x_0} \delta_0 = \delta_{x_0},$$

because $\Box$ and T_{x_0} commute, so $T_{x_0} u_0$ is a fundamental solution at x_0. ■

2.2.2 Proposition. *Let $\psi \in \mathcal{D}(\mathbb{M}^n)$. Then*

$$F_\psi = u_0 * \psi \in C^\infty(\mathbb{M}^n) \tag{2.2}$$

is a (smooth) solution for the differential equation $PF_\psi = \psi$ (here $*$ denotes convolution). ■

Proof. Since $u_x = T_x u_0$ and $F_\psi = (u_x, \psi) = (T_x u_0, \psi)$ is a solution to the equation, the thesis follows immediately noting that, by definition, $(T_x u_0, \psi) = (u_0 * \psi)(x)$. The smoothness of the solution follows from Theorem A.0.7. ■

2.3 The Fourier transform approach

In order to employ the theory of Fourier transforms, it is necessary to work with distributions in $\mathcal{S}'(\mathbb{M}^n)$.
We shall begin with a lemma that helps in the computations:

2.3.1 Lemma. *For $u \in \mathcal{S}'(\mathbb{M}^n)$ if we let $x = (t, \mathbf{x}) = (t, x_1, \ldots, x_{n-1})$ and $k = (\omega, \mathbf{k}) = (\omega, k_1, \ldots, k_{n-1})$, it holds*

$$\widehat{\Box u}(k) = \|k\|^2 \, \widehat{u} = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k). \tag{2.3}$$

■

Proof. For any test function $f \in \mathcal{S}(\mathbb{M}^n)$, and for any $u \in \mathcal{S}'(\mathbb{M}^n)$ $(\Box u, f) = (u, \Box f)$, from which it descends

$$(\Box u, e^{-i\langle k,x\rangle_0}) = \left(u, \Box e^{-i\langle k,x\rangle_0}\right) = \left(u, \Box e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\right) =$$

$$= \left(u, (|\mathbf{k}|^2 - \omega^2) e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\left(u, e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k).$$

■

We start by transforming the equation:

$$\widehat{\Box u}(k) = \widehat{\delta}(k) \Rightarrow (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k) = 1 \tag{2.4}$$

The difference of two solutions $\widehat{u} \in \mathcal{S}'(\mathbb{M}^n)$, is a solution $\widehat{v}$ to the equation

$$(|\mathbf{k}|^2 - \omega^2)\widehat{v}(k) = 0. \tag{2.5}$$

This equation can be seen as the Fourier transform of the correspondent homogeneous equation $\Box v = 0$. Hence $\widehat{u} + \widehat{v}$ is the transform of the sum of a particular solution for $\Box$ and a solution of the homogeneous equation, then it is a solution for Equation (2.4).

If we concentrate on the solutions for Equation (2.5), it is easy to see with a direct computation that any distribution of the form

$$\widehat{v}(k) = A(k)\delta(|\mathbf{k}|^2 - \omega^2),$$

where $A(k)$ is any suitably regular function of k, solves the equation, because the Dirac delta is supported on $\{k|\,|\mathbf{k}|^2 - \omega^2 = 0\}$. Any solution to the homogeneous wave equation can be obtained by the inverse transform of $\widehat{v}$:

$$v(x) = (2\pi)^{-n}\widehat{\widehat{v}}(-x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0} A(k)\delta(|\mathbf{k}|^2 - \omega^2)\,\mathrm{d}k.$$

Making use of formula (A.1), the last expression becomes

$$\begin{aligned} v(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0}\frac{A(k)}{2|\mathbf{k}|}\left[\delta(|\mathbf{k}| - \omega) + \delta(|\mathbf{k}| + \omega)\right]\,\mathrm{d}k = \\ &= (2\pi)^{-n}\int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{e^{i|\mathbf{k}|t}A(k)|_{|\mathbf{k}|=\omega} + e^{-i|\mathbf{k}|t}A(k)|_{|\mathbf{k}|=-\omega}}{|\mathbf{k}|}. \end{aligned} \tag{2.6}$$

To solve for $\widehat{u}$ it is tempting to write

$$\widehat{u}(k) = \frac{1}{|\mathbf{k}|^2 - \omega^2} = \frac{1}{(|\mathbf{k}| - \omega)(|\mathbf{k}| + \omega)},$$

which is ill-defined as a distribution wherever $\langle k, k\rangle_0 = 0$, i.e. on the light-cone of the Fourier space. Hence, we will define $\widehat{u}$ as limit of a sequence of distributions depending on the parameter ε

$$\widehat{u}_\varepsilon = \frac{1}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} = \frac{1}{(|\mathbf{k}| \mp i\varepsilon - \omega)(|\mathbf{k}| \pm i\varepsilon + \omega)}$$

promoting ω to a complex variable and taking the limit for $\varepsilon \to 0^+$ after performing the inverse transform. The choice of the signs in such expressions leads to different fundamental solutions.

2.4 Fundamental solutions via Fourier transform

The distributions G_+ and G_- in $\mathcal{S}'(\mathbb{M}^n)$, defined respectively as the weak limit for $\varepsilon \to 0^+$ of

$$G_\varepsilon^+(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega + i\varepsilon)^2}\, \mathrm{d}k, \tag{2.7}$$

$$G_\varepsilon^-(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega - i\varepsilon)^2}\, \mathrm{d}k, \tag{2.8}$$

are respectively a **retarded** and an **advanced** fundamental solutions at $x_0 = 0$ for the d'Alembert wave operator.
The aim is to prove that $\mathrm{supp}(G_+) \subset J_+(0)$ and $\mathrm{supp}(G_-) \subset J_-(0)$, and we proceed firstly by calculating the explicit formula for $2 \leq n \leq 4$ and then discuss the general case via Riesz distributions.

We will show that, if we denote with $d := n-1$ the spatial dimensions, the retarded fundamental solutions $G^\pm_{(d)}$ turn out to be

$$G^+_{(1)}(t,x) = \frac{\Theta(t-|x|)}{2},$$

$$G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi} \frac{\Theta(t^2 - |\mathbf{x}|^2)}{\sqrt{t^2 - |\mathbf{x}|^2}},$$

$$G^+_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{4\pi} \frac{\delta(t - |\mathbf{x}|)}{|\mathbf{x}|}.$$

We compute $G_\pm$ as a limit of the inverse of the Fourier transform:

$$G_\varepsilon^\pm(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2}\, \mathrm{d}k$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2}\, \mathrm{d}\omega. \tag{2.9}$$

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Computing the complex integrals

In order to calculate the inner integral in the former expression,

$$\tilde{G}_{\varepsilon}^{\pm}(t, \mathbf{k}) := \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}\omega,$$

we make use of techniques of complex analysis as follows.
Denote

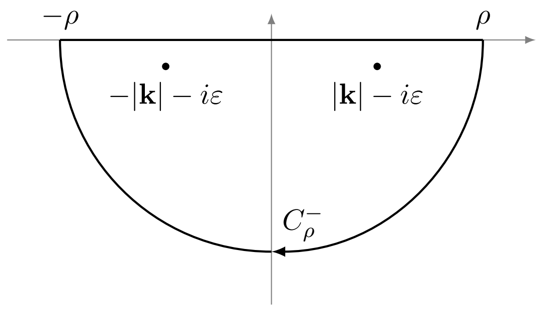

FIGURE 2.1: The circuit to compute $\tilde{G}_{\varepsilon}^{+}(t, \mathbf{k})$ for $t > 0$.

- C_ρ^+ the upper half-circle of radius ρ centered at $\omega = 0$ which has Im $(\omega) > 0$, oriented counter-clockwise;
- C_ρ^- the lower half-circle of radius ρ centered at $\omega = 0$ which has Im $(\omega) < 0$, oriented clockwise;
- $[-\rho, \rho]$ the interval of the real line connecting $-\rho$ and ρ, oriented from left to right.

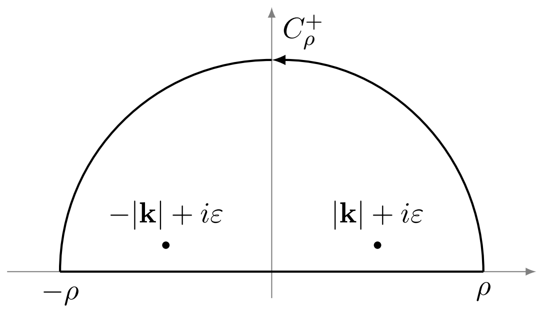

FIGURE 2.2: The circuit to compute $\tilde{G}_{\varepsilon}^{-}(t, \mathbf{k})$ for $t < 0$.

The singularities are

$$\text{for } \tilde{G}^+_\varepsilon : \quad \omega = \pm|\mathbf{k}| - i\varepsilon,$$

$$\text{for } \tilde{G}^-_\varepsilon : \quad \omega = \pm|\mathbf{k}| + i\varepsilon.$$

Hence we have

$$\text{for } t<0 \quad \tilde{G}^+_\varepsilon(t,\mathbf{k}) = \lim_{\rho\to\infty}\int_{C^+_\rho+[-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2-(\omega+i\varepsilon)^2}\,\mathrm{d}\omega = 2\pi i\sum \mathrm{Res} = 0,$$

where the sum is extended to the singularities in the upper half-plane. The expression vanishes because we choose the circuit such that the integral on C^+_ρ vanishes in virtue of Jordan's lemma and there are no singularities in the region bounded by the circuit.
For the same reasons

$$\text{for } t>0,\ \tilde{G}^-_\varepsilon(t,\mathbf{k}) = \lim_{\rho\to\infty}\int_{C^-_\rho+[-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2-(\omega-i\varepsilon)^2}\,\mathrm{d}\omega = -2\pi i\sum \mathrm{Res} = 0,$$

where the sum is extended th the singularities of the function in the lower half-plane and the minus sign arises because of the clockwise circuit.
The non-zero integrals are

$$\tilde{G}^+_\varepsilon(t,\mathbf{k}),\ \text{for } t>0, \qquad \tilde{G}^-_\varepsilon(t,\mathbf{k}),\ \text{for } t<0.$$

The first is computed via the lower circuit in Figure (2.1): $C^-_\rho + [-\rho,\rho]$, the second via the upper counterpart in Figure (2.2): $C^+_\rho + [-\rho,\rho]$, in order to get rid of contributes from the half-circles.

The results are

$$\text{for } t>0 \quad \tilde{G}^+_\varepsilon(t,\mathbf{k}) = 2\pi i e^{-\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = 2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t},$$

$$\text{for } t<0 \quad \tilde{G}^-_\varepsilon(t,\mathbf{k}) = -2\pi i e^{\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = -2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\varepsilon|\mathbf{k}|t}.$$

Summing up everything in one formula:

$$\tilde{G}^\pm_\varepsilon(t,\mathbf{k}) = \pm 2\pi\Theta(\pm t)\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\mp\varepsilon|\mathbf{k}|t}, \tag{2.10}$$

where Θ is the Heaviside step-function.
It can be noticed that

$$\tilde{G}^{+}_{\varepsilon}(t,\mathbf{k}) = \tilde{G}^{-}_{\varepsilon}(-t,\mathbf{k}),$$

because of the parity of sine function. So, we can deduce that the **advanced** solution can be calculated from the retarded one via time inversion:

$$G_{-}(t,\mathbf{x}) = G_{+}(-t,\mathbf{x}). \tag{2.11}$$

We can now show that the support $G_{\pm}$ is included in $J_{+}(0) \cup J_{-}(0)$.

2.4.1 Proposition. *If $x \in \mathbb{M}^n$ satisfies $\gamma(x) < 0$ (where γ is defined in Definition 1.2.2), $G_{\pm}(x) = 0$.* ■

Proof. Consider a reference frame R in which $x = (t,\mathbf{x})$ and suppose for now $t \geq 0$. It descends $G_{+}(x) = 0$. Since $G_{\pm}$ are manifestly Lorentz invariant and $\gamma(x) < 0$, one can find a reference frame R' in which $x = (t',\mathbf{x}')$ and $t' < 0$, so that $G_{-}(x) = 0$. The converse can be treated similarly. ■

We can focus once more on equation (2.9) to show explicit solutions for spatial dimensions d ranging from 1 to 3:

$$G^{+}_{(d)}(x) = \frac{1}{(2\pi)^{d+1}} \lim_{\varepsilon\to 0^{+}} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{G}^{+}_{\varepsilon}(t,\mathbf{k}) =$$
$$= \lim_{\varepsilon\to 0^{+}} \frac{\Theta(t)}{(2\pi)^{d}} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} e^{-\varepsilon|\mathbf{k}|t}.$$

Dimension $n = 1+1$ - wave on a line

The integral we have to make in the $1+1$-dimensional case is:

$$G^{+}_{(1)}(t,x) = \frac{\Theta(t)}{2\pi} \lim_{\varepsilon\to 0^{+}} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x} \frac{\sin kt}{k} e^{-\varepsilon kt}.$$

It holds

$$\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x+i\varepsilon t)} \frac{\sin kt}{k} \overset{k\to k'=kt}{=} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x/t+i\varepsilon)} \frac{\sin k}{k} \underset{\varepsilon\to 0^{+}}{\longrightarrow}$$
$$\longrightarrow \pi\chi_{[-1,1]}\left(\frac{x}{t}\right) = \pi\chi_{[-t,t]}(x),$$

where

$$\chi_{[a,b]}(z) = \begin{cases} 1, & \text{if } z \in [a,b] \\ 0, & \text{otherwise,} \end{cases}$$

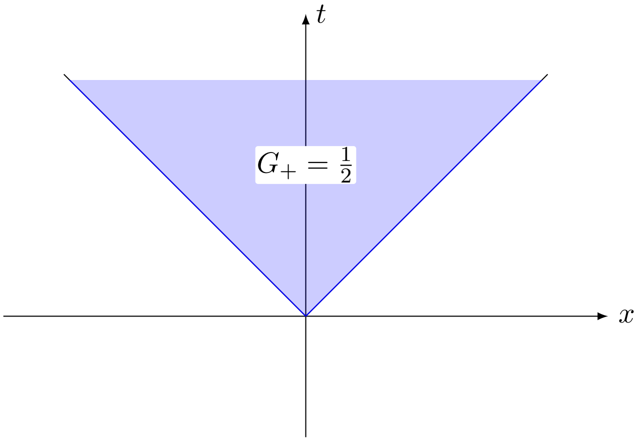

FIGURE 2.3: The support of G_+ in 1+1 dimensional case.

Finally the integrals become

$$G^+_{(1)}(t,x) = \frac{\Theta(t)}{2}\chi_{[-t,t]}(x) = \frac{\Theta(t-|x|)}{2}.$$

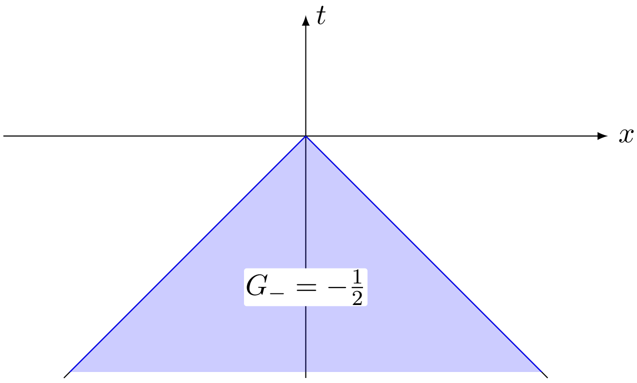

FIGURE 2.4: The support of G_- in 1+1 dimensional case.

From FIGUREs (2.3) and (2.4) one can infer that the fundamental solutions are supported respectively on $J_+(0)$ and $J_-(0)$.

Dimension $n = 1+2$ - wave on a surface

The integral is two dimensional:

$$G^{+}_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{(2\pi)^2} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} e^{-\varepsilon|\mathbf{k}|t}.$$

To evaluate it we switch to polar coordinates $\mathbf{k} = (k\cos\varphi, k\sin\varphi)$. With the integral measure

$$\mathrm{d}\mathbf{k} = k\,\mathrm{d}k\,\mathrm{d}\varphi,$$

the integral becomes ($x := |\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}k\, e^{ikx\cos\varphi} \sin kt\, e^{-\varepsilon kt}.$$

It holds that

$$\int_0^{\infty} \mathrm{d}k\, e^{ik(y+i\varepsilon)} \sin kt = \frac{1}{2i}\left[\int_0^{+\infty} e^{ik(y+i\varepsilon+t)}\,\mathrm{d}k + \int_0^{+\infty} e^{ik(y+i\varepsilon-t)}\,\mathrm{d}k\right] =$$
$$= \frac{1}{2}\left[I_\varepsilon(y+t) + I_\varepsilon(y-t)\right],$$

where we set $I_\varepsilon(y) := \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k$. Since

$$I_\varepsilon(y) = \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k = \frac{1}{y+i\varepsilon},$$

the integral which needs to be evaluated is

$$\frac{1}{2}\int_0^{2\pi} \left[\frac{1}{x\cos\varphi + t + i\varepsilon} + \frac{1}{x\cos\varphi - t + i\varepsilon}\right] \mathrm{d}\varphi.$$

Such integral has a counterpart over a unit circle in the complex plane with the substitutions $\mathrm{d}\varphi = -i\mathrm{d}z/z$ and $\cos\varphi = (z+z^{-1})/2$. Hence, using Cauchy residue theorem

$$\int_0^{2\pi} \frac{1}{x\cos\varphi \pm t + i\varepsilon}\,\mathrm{d}\varphi = -2i \oint \frac{\mathrm{d}z}{xz^2 + 2(\pm t + i\varepsilon) + x} = \frac{2\pi}{\sqrt{(t \mp i\varepsilon)^2 - x^2}}.$$

Putting everything together in the weak limit $\varepsilon \to 0$ it holds

$$G^{+}_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi} \frac{\Theta(t^2 - |\mathbf{x}|^2)}{\sqrt{t^2 - |\mathbf{x}|^2}} = \frac{\Theta(t)}{2\pi} \frac{\Theta(\gamma(x))}{\sqrt{\gamma(x)}}, \tag{2.12}$$

where $\Theta(t^2 - |\mathbf{x}|^2)$ stems from Proposition 2.4.1. As a by-product, $\mathrm{supp}(G_\pm) \subseteq J_\pm(0)$.

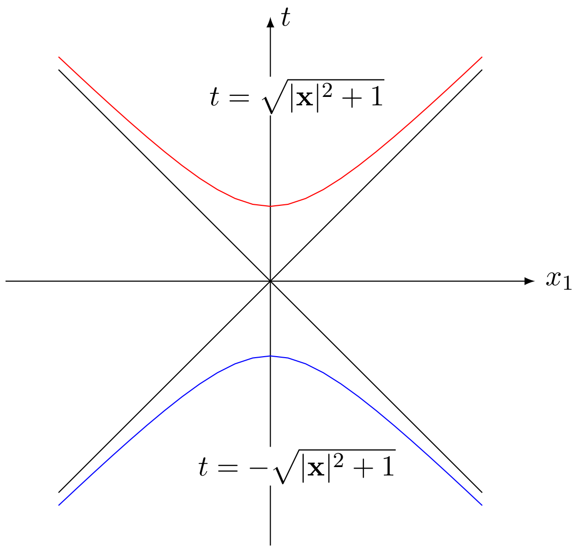

FIGURE 2.5: The level set $G_{\pm}(\mathbf{x}, t) = 1$ in the 1+2 dimensional case, plotted for one spatial axis.

Dimension $n = 1+3$ - spherical wave

The three-dimensional integral is

$$G^{+}_{(3)}(t, \mathbf{x}) = \frac{\Theta(t)}{(2\pi)^3} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} e^{-\varepsilon|\mathbf{k}|t}.$$

Again, we make a change of coordinate, switching to the spherical ones: $\mathbf{k} = (k\sin\vartheta\cos\varphi, k\sin\vartheta\sin\varphi, k\cos\vartheta)$. The integral measure reads

$$\mathrm{d}\mathbf{k} = k^2 \sin\vartheta\, \mathrm{d}k\, \mathrm{d}\vartheta\, \mathrm{d}\varphi$$

and the integral to calculate is ($x := |\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}k\, k \sin kt \int_{-1}^{1} e^{ikx\cos\vartheta} \mathrm{d}(\cos\vartheta) = \frac{4\pi}{x+i\varepsilon} \int_0^{+\infty} \sin kt \sin k(x+i\varepsilon)\, \mathrm{d}k.$$

Hence we can write using the exponential function

$$\sin kt \sin kx = \frac{1}{4}\left\{\left[e^{ik(x+i\varepsilon-t)} + e^{-ik(x+i\varepsilon-t)}\right] - \left[e^{ik(x+i\varepsilon+t)} + e^{-ik(x+i\varepsilon+t)}\right]\right\},$$

and with the change of variables $k \leftrightarrow -k$ it holds

$$\frac{4\pi}{x} \int_0^{+\infty} \sin kt \sin kx\, \mathrm{d}k = \frac{2\pi^2}{x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x+i\varepsilon-t)} - e^{ik(x+i\varepsilon+t)}\, \mathrm{d}k \underset{\varepsilon \to 0^+}{\longrightarrow}$$

$$\longrightarrow \frac{2\pi^2}{x}\left[\delta(t-x)-\delta(t+x)\right].$$

To find the correct retarded and advanced fundamental solutions we notice that the second term, $\delta(t+x)$, vanishes for G_+ because $x>0$ and $t>0$. Conversely the first term $\delta(t-x)$ vanishes when computing G_-. In view of these considerations, the general formula for the $1+3$-case becomes

$$G^+_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{4\pi}\frac{\delta(t-|\mathbf{x}|)}{|\mathbf{x}|}. \tag{2.13}$$

One can verify that $G^\pm_{(3)}$ vanish outside the support of the delta distribution. Hence they are supported respectively on the upper and on the lower light cones $C_+(0)$ and $C_-(0)$. This is a particularity of the odd spatial dimensions, as we shall prove in the next section. This feature is known as the **Huygens' principle**. It states that in general, we have for spatial dimensions $d \neq 1$

$$\operatorname{supp} G^\pm_{(d)} = J_\pm(0) \quad \text{for } d \text{ even},$$

$$\operatorname{supp} G^\pm_{(d)} = C_\pm(0) \quad \text{for } d \text{ odd}.$$

Physically, we can see δ_0 as a point source at 0 of a signal that propagates with constant speed. Inside the future light cone the solution is zero, so if d is even, the wave propagates strictly on the cone. In case d is odd, the signal of a point source propagates also inside the light cone. For an observer, the wave is noticeable not only at a single moment but still after the signal has arrived. An example of such waves are the 2-dimensional ones like water waves.

The method of *descent*

The lower dimensional advanced and retarded distributions can be directly deduced from the $d=3$ case as we shall see. In general if we know the explicit solution to the d-dimensional case we can find the $d-1$-dimensional counterpart with the formula

$$G^\pm_{(d-1)}(t,x_1,\ldots,x_{d-1}) = \int_{-\infty}^{\infty} G^\pm_{(d)}(t,x_1,\ldots,x_d)\,\mathrm{d}x_d.$$

This technique is called *method of descent*. The last assertion can be proven taking the fundamental solution equation

$$\Box_d\, G_{(d)}(t,x_1,\ldots,x_d) = \delta(t)\delta(x_1)\cdots\delta(x_d),$$

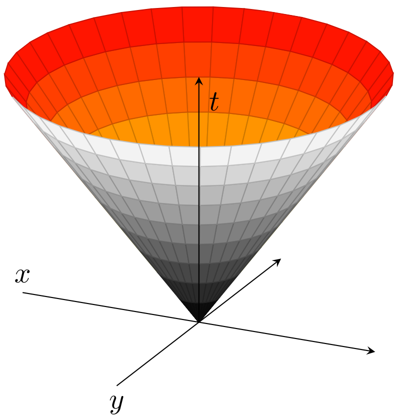

FIGURE 2.6: The support of G_+ in 1+3 dimensional case, i.e. the upper light cone $C_+(0)$, plotted for two spatial axis.

where $\square_d$ stands for $(\partial_t^2 - \partial_1^2 - \cdots - \partial_d^2)$. Integrating $G_{(d)}$ on the last variable against a test-function $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\int \square_d \, G_{(d)} \, \varphi \, \mathrm{d}x_d = \square_{(d-1)} \int G_{(d)} \, \varphi \, \mathrm{d}x_d - \int \partial_d^2 G_{(d)} \, \varphi \, \mathrm{d}x_d =$$

$$= \delta(t)\delta(x_1) \cdots \delta(x_{d-1}) \int \delta(x_d)\varphi \, \mathrm{d}x_d.$$

By letting the test-function become a sequence of cut-off functions covering the real axis, the formula

$$\square_{(d-1)} \int G_{(d)} \, \mathrm{d}x_d = \delta(t)\delta(x_1) \cdots \delta(x_{d-1})$$

is proven.
To calculate the $d = 2$ fundamental solution from the $d = 3$ case, according to (2.13) and using Equation (A.1) we can write for the retarded fundamental solution

$$\Theta(t)\frac{\delta(t - |\mathbf{x}|)}{2|x|} = \Theta(t)\delta(t^2 - |\mathbf{x}|^2) = \Theta(t)\delta(t^2 - x_1^2 - x_2^2 - x_3^2) =$$

$$= \Theta(t)\Theta(t^2 - x_1^2 - x_2^2)\frac{\delta(x_3 - \sqrt{t^2 - x_1^2 - x_2^2}) + \delta(x_3 + \sqrt{t^2 - x_1^2 - x_2^2})}{2\sqrt{t^2 - x_1^2 - x_2^2}},$$

where we insert the Heaviside step function to take into account Proposition 2.4.1. Hence, integrating over the third variable yields

$$G^+_{(2)}(t, x_1, x_2) = \frac{\Theta(t)}{2\pi} \frac{\Theta(t^2 - x_1^2 - x_2^2)}{\sqrt{t^2 - x_1^2 - x_2^2}},$$

which is identical to Equation (2.12) as we expected. Similarly, the expression for the case $d = 1$ can be derived again.

If we now apply to the general expression for $G_{(d)}$ the Fourier transform in the last variable, similarly we obtain

$$\int_{-\infty}^{+\infty} \partial_d^2 G_{(d)}(t, x_1, \ldots, x_d) e^{-imx_d} \, \mathrm{d}x_d =$$

$$= -m^2 \int_{-\infty}^{+\infty} G_{(d)}(t, x_1, \ldots, x_d) e^{-imx_d} \, \mathrm{d}x_d =: -m^2 \widehat{G}_{d-1}(t, x_1, \ldots, x_{d-1}, m).$$

Hence, the Fourier transform on the last variable $\widehat{G}_{d-1}(t, x_1, \ldots, x_{d-1}, m)$ is, for a fixed $m \in \mathbb{R}$, a fundamental solution for the $d-1$ dimensional **Klein-Gordon** operator

$$\Box + m^2,$$

that describes the motion of spinless particles with mass m.

2.5 The Riesz distributions

To discuss explicit and useful formulas for the fundamental solution in the general n-dimensional case, the approach we adopted in the last section is not very effective. We outline a method devised by M. Riesz in the first half of the 20th century in order to find solutions to a certain class of differential equations.

2.5.1 Definition. *For $\alpha \in \mathbb{C}$ with Re $\alpha > n$ let $R_\pm(\alpha)$ be the complex-valued continuous functions defined for any $x \in \mathbb{M}^n$ by*

$$R_\pm(\alpha)(x) := \begin{cases} C(\alpha, n)\gamma(x)^{\frac{\alpha-n}{2}} & \text{if } x \in J_\pm(0) \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where γ is defined in Definition 1.2.2 and

$$C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2} + 1)},$$

and $z \mapsto \Gamma(z)$ is the Gamma function. ■

2.5.2 Remark. The functions $R_\pm(\alpha)$ are continuous because $\gamma = -\langle \cdot, \cdot \rangle_0$ vanishes on the boundary of $J_\pm(0)$ and the exponent $(\alpha - n)/2$ is assumed to have positive real part. If we increase the real part of the exponent then higher derivatives of the function vanishes at the boundary and the functions become more regular. As a matter of facts $R_\pm(\alpha) \in C^k(\mathbb{M}^n)$ whenever Re $\alpha > n + 2k$. ■

Now we discuss the first properties of $R_\pm(\alpha)$.

2.5.3 Proposition. *For all $\alpha \in \mathbb{C}$ with Re $\alpha > n$ it holds*

(1) $\gamma\, R_\pm(\alpha) = \alpha(\alpha - n + 2) R_\pm(\alpha + 2)$;

(2) $(\operatorname{grad} \gamma)\, R_\pm(\alpha) = 2\alpha \operatorname{grad} R_\pm(\alpha + 2)$;

(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.

Moreover, the map

$$\begin{aligned} \mathbb{X}_n &\to \mathcal{D}'(\mathbb{M}^n) \\ \alpha &\mapsto R_\pm(\alpha) \end{aligned}$$

(where $\mathbb{X}_n := \{\alpha \in \mathbb{C} \,|\, \text{Re } \alpha > n\}$) extends uniquely the whole complex plain as a holomorphic family of distributions, i.e. for each test-function $\varphi \in \mathcal{D}(\mathbb{M}^n)$, the function $\alpha \mapsto (R_\pm(\alpha), \varphi)$ is holomorphic. ■

Proof. To prove (1), we evaluate $\gamma\, R_\pm(\alpha)$ inside $J_\pm(0)$, because both sides of the equation vanish outside. By definition one has

$$\gamma\, R_\pm(\alpha) = C(\alpha, n)\gamma(x)^{\frac{\alpha+2-n}{2}} = \frac{C(\alpha, n)}{C(\alpha + 2, n)} R_\pm(\alpha + 2),$$

and, in virtue of the fact that $z\Gamma(z - 1) = \Gamma(z)$,

$$\frac{C(\alpha, n)}{C(\alpha + 2, n)} = \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{(\frac{\alpha}{2} - 1)!(\frac{\alpha-n}{2})!} \frac{(\frac{\alpha+2}{2} - 1)!(\frac{\alpha+2-n}{2})!}{2^{1-\alpha-2}\pi^{\frac{2-n}{n}}} =$$

$$= 4\frac{\alpha}{2}\frac{\alpha + 2 - n}{2} = \alpha(\alpha - n + 2).$$

For the second identity we evaluate $\partial_i\gamma \cdot R_\pm(\alpha)$. In view of Remark 2.5.2, $R_\pm(\alpha+2) \in C^1(\mathbb{M}^n)$ For any φ integrating by parts yields:

$$\begin{aligned}
\partial_i\gamma \cdot (R_\pm(\alpha), \varphi) &= C(\alpha, n) \int_{J_\pm} \gamma(x)^{\frac{\alpha-n}{2}} \partial_i\gamma(x)\varphi(x)\,\mathrm{d}x \\
&= \frac{2C(\alpha,n)}{\alpha+2-n} \int_{J_\pm} \partial_i(\gamma(x))^{\frac{\alpha-n+2}{2}} \varphi(x)\,\mathrm{d}x \\
&= -2C(\alpha+2, n) \int_{J_\pm} \gamma(x)^{\frac{\alpha-n+2}{2}} \partial_i\varphi(x)\,\mathrm{d}x \\
&= -2\alpha(R_\pm(\alpha+2), \partial_i\varphi) \\
&= 2\alpha(\partial_i R_\pm(\alpha), \varphi).
\end{aligned}$$

To prove the third formula, from (2) we have

$$\begin{aligned}
\partial_i^2 R_\pm(\alpha+2) &= \partial_i \left(\frac{1}{2\alpha} \partial_i\gamma \cdot R_\pm(\alpha) \right) \\
&= \frac{1}{2\alpha}\partial_i^2\gamma \cdot R_\pm(\alpha) + \frac{1}{4\alpha(\alpha-2)}(\partial_i\gamma)^2 \frac{(\alpha-2)(\alpha-n)}{\gamma} \cdot R_\pm(\alpha) \\
&= \left(\frac{1}{2\alpha}\partial_i^2\gamma + \frac{\alpha-n}{4\alpha} \cdot \frac{(\partial_i\gamma)^2}{\gamma} \right) \cdot R_\pm(\alpha).
\end{aligned}$$

Applying $\Box$ we find

$$\Box R_\pm(\alpha+2) = \left(\frac{n}{\alpha} + \frac{\alpha-n}{4\alpha} \cdot \frac{4\gamma}{\gamma} \right) R_\pm(\alpha) = R_\pm(\alpha),$$

as claimed.
The last identity allows us to extend $R_\pm(\alpha)$ for every $\alpha \in \mathbb{C}$. For Re $\alpha > n-2$ we set

$$\widetilde{R}_\pm(\alpha) := \Box R_\pm(\alpha+2),$$

and the extension is holomorphic on $\mathbb{X}_{n-2}$. Now, proceeding by induction over n one can extend the function over the whole complex plane. ■

2.5.4 Definition. *The distributions $R_+(\alpha)$ and $R_-(\alpha)$ defined in the last proposition are called respectively the* ***retarded*** *and* ***advanced*** *Riesz distributions for the parameter $\alpha \in \mathbb{C}$.* ■

The Riesz distributions do not have an immediate explicit formula, but next lemma shows a more comfortable way to evaluate them when the test-function has a particular form.

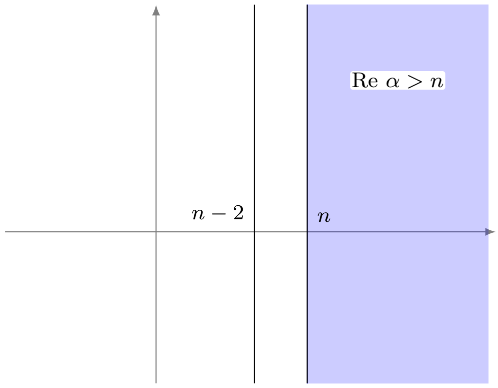

FIGURE 2.7: Iterative extension of $R_\pm(\alpha)$ on $\mathbb{C}$.

2.5.5 Lemma. *Denote $x = (t, \mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^{n-1}$. Let $f \in \mathcal{D}(\mathbb{R})$ and let $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$ be such that $\varphi(x) := f(t)\psi(\mathbf{x}) \in \mathcal{D}(\mathbb{M}^n)$ and $\varphi(x) = f(t)$ on $J_+(0)$. If Re $\alpha > 1$ it holds*

$$(R_\pm(\alpha), \varphi) = \frac{1}{(\alpha-1)!}\int_0^{+\infty} t^{\alpha-1} f(t)\,\mathrm{d}t.$$

■

To link Riesz distributions to fundamental solutions the following facts are noteworthy.

2.5.6 Proposition. *The Riesz distributions satisfy*

(1) for any $\alpha \in \mathbb{C}$, supp $R_\pm(\alpha) \subset J_\pm(0)$

(2) $R_\pm(0) = \delta_0$

*(3) $\Box R_\pm(2) = \delta_0$, in particular $R_+(2)$ and $R_-(2)$ are respectively a **retarded** and an **advanced** fundamental solution for $\Box$ at 0.*

■

Proof. The first assertion descends from the definition of Riesz distributions. To prove (2) fix $K \subset \mathbb{M}^n$ compact subset. Let $\sigma_K \in \mathcal{D}(\mathbb{M}^n)$ such that $\sigma_K|_K = 1$. For any $\varphi \in \mathcal{D}(\mathbb{M}^n)$ with supp $\varphi \subset K$ one finds suitable smooth functions φ_j such that

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x^j \varphi_j(x),$$

then it holds

$$\begin{aligned}
(R_\pm(0),\varphi) &= (R_\pm(0),\sigma_K\varphi)\\
&= \left(R_\pm(0), \varphi(0) + \sum_{j=1}^{n} x^j\varphi_j(x)\right)\\
&= \varphi(0)\,\overbrace{(R_\pm(0),\sigma_K)}^{=:c_K} + \sum_{j=1}^{n}\left(\overbrace{x^j R_\pm(0)}^{=0}, \sigma_K\varphi_j\right)\\
&= c_K\varphi(0),
\end{aligned}$$

where $x^j R_\pm(0)$ vanishes because of Equation (1) in Proposition 2.5.3 one can show that c_K does not depend on the choice of K since for $K' \supset K$ and $\operatorname{supp}\varphi \subset K \subset K'$,

$$c'_K\varphi(0) = (R_+(0),\varphi) = c_K\varphi(0).$$

It descends $c_K = c'_K =: c$. To show $c = 1$, concentrating on the case of a retarded distribution, using test-functions as in Lemma 2.5.5,

$$\begin{aligned}
c\cdot\varphi(0) &= (R_+(0),\varphi)\\
&= (\Box R_+(2),\varphi) = (R_+(2),\Box\varphi)\\
&= \int_0^{+\infty} t\, f''(t)\,\mathrm{d}t = -\int_0^{+\infty} f'(t)\,\mathrm{d}t\\
&= f(0) = \varphi(0),
\end{aligned}$$

which concludes the proof.
The third assertion is obtained considering (1) and making use of Equation (3) in Proposition 2.5.3. ■

2.5.7 Remark. We will prove in Section 2.6 that the retarded and the advanced fundamental solutions are **unique**. Hence, we have

$$G_\pm = R_\pm(2).$$

■

2.5.8 Remark. As one expects, if $\alpha \in \mathbb{R}$, then $(R_\pm(\alpha),\varphi)$ is real for any $\varphi \in \mathcal{D}(\mathbb{M}^n,\mathbb{R})$ i.e. $R_\pm(\alpha)$ is a real-valued distribution. ■

We are now ready to prove **Huygens' principle**, that we already mentioned before. One can restate it as follows.

2.5.9 Theorem (Huygens' principle). *If $n \geq 4$ is even, supp $G_\pm = C_\pm(0)$. If $n \geq 3$ is odd, supp $G_\pm = J_\pm(0)$.* ■

To prove it we work with Riesz distributions and we need the following lemma

2.5.10 Lemma. *The following holds:*

(1) for every $\alpha \in \mathbb{C}\backslash(\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$ we have supp $R_\pm(\alpha) = J_\pm(0)$;

(2) for $n \geq 3$ and $\alpha = n-2, n-4, \dots, 1$ if n is odd or $\alpha = n-2, n-4, \dots, 2$ if n is even, we have supp $R_\pm(\alpha) = C_\pm(0)$.

■

Proof of Theorem 2.5.9. The fundamental solutions are $G^\pm = R_\pm(2)$, so $\alpha = 2$. Since $2 = (n-2)+(4-n)$, if $n \geq 4$ and n is even, $2 \in \{n-2, n-4, \dots\}$; conversely, if n is odd 2 is not in $\{n-2, n-4, \dots, 1\}$. So the theorem follows from the last lemma. ■

2.6 General solution and Cauchy problem

We found the fundamental solution for the d'Alembert wave operator for the point $x_0 = 0$. To find the generic solution at a point $y \in \mathbb{M}^n$, as we have seen in Proposition 2.2.2, it suffices to write

$$G_y^+(x) = T_y G_+ = G_+(x-y).$$

Hence, to find the retarded solution $u_+(x)$ to the wave equation $\Box u = \psi$, where ψ is a distribution, we simply evaluate the convolution $G_+ * \psi$. The general solution is obtained by adding the solutions of the homogeneous equation as in Formula (2.6).

Now we discuss the uniqueness of the distributional solution and for its regularity we refer to Proposition 2.2.2.
To begin, we shall prove the following

2.6.1 Theorem. *Let $\psi \in \mathcal{D}'(\mathbb{M}^n)$ such that $\psi(t, \mathbf{x}) = 0$ if $t < 0$. Then ψ and G_+ can be convoluted and $u_+ = G_+ * \psi$ is the unique solution to the wave equation with source ψ such that $u_+(t, \mathbf{x}) = 0$ for $t < 0$.*

■

Proof. At fixed x, the distribution $G_+(x-y)\psi(y)$ has compact support in the variable y, so G_+ and ψ can be convoluted. Since $\operatorname{supp} G_+ \subset J_+(0)$, $u_+ = 0$ for $t < 0$.
The solution is unique because if there were another u solving the equation and satisfying the requested conditions, then $\phi := u_+ - u$ would be a solution to the homogeneous equation, i.e. $\Box\phi = 0$, and ϕ could be convoluted with G_+:

$$\phi = \phi * \delta = \phi * \Box G_+ = \Box \phi * G_+ = 0,$$

hence $u = u_+$. ■

Remaining in the Minkowski case, since $\mathbb{M}^n$ is a globally hyperbolic manifold, we can find smooth Cauchy hypersurfaces, where we can assign initial values.
Physically, it is clear why these can only be specified on a spacelike surface. A wave cannot travel from one point of the initial value surface to another and change the initial conditions.

2.6.2 Definition. *Let S be a smooth Cauchy hypersurface of $\mathbb{M}^n$ with a timelike unit normal vector field $\nu : S \to \mathrm{T}\mathbb{M}^n$.*
*Given a triple $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$, we call a **classical Cauchy problem** for $\Box$ on S the system of equations*

$$\begin{cases} \Box u = \psi \\ \\ u|_S = u_0 \\ \\ \partial_\nu u|_S = u_1. \end{cases} \tag{2.15}$$

*In case the triple is taken in $\mathcal{D}'(\mathbb{M}^n) \oplus \mathcal{D}'(S) \oplus \mathcal{D}'(S)$, i.e. the data are distributions, the problem is called **generalized Cauchy problem**.* ■

For simplicity, we concentrate on the case where S is the hyperplane

$$S_0 := \{(t, \mathbf{x}) \in \mathrm{M} \,|\, t = 0\},$$

and $\partial_\nu = \partial_t$. The general case will be addressed later in Section 3.2, when we will discuss of Cauchy problem on manifolds. To solve the initial value problem, we start with a lemma

2.6.3 Lemma. *Suppose u is a solution for the Cauchy problem on S_0. If we set*

$$\tilde{u}(t,\mathbf{x}) := \begin{cases} u(t,\mathbf{x}) & \textit{if} \quad t > 0 \\ 0 & \textit{otherwise}, \end{cases}$$

$$\tilde{\psi}(t,\mathbf{x}) := \begin{cases} \psi(t,\mathbf{x}) & \textit{if} \quad t > 0 \\ 0 & \textit{otherwise}, \end{cases}$$

it holds

$$\Box\, \tilde{u}(t,\mathbf{x}) = \tilde{\psi}(t,\mathbf{x}) + u_0(\mathbf{x})\delta'(t) + u_1(\mathbf{x})\delta(t). \tag{2.16}$$

■

Proof. Let $\varphi \in \mathcal{D}'(\mathbb{M}^n)$, then

$$(\Box\tilde{u}, \varphi) = (\tilde{u}, \Box\varphi) = \int_0^\infty \mathrm{d}t \int u\,\Box\varphi\,\mathrm{d}\mathbf{x} =$$

integrating by parts in the time variable

$$= \int_0^\infty \mathrm{d}t \int \Box u\,\varphi\,\mathrm{d}\mathbf{x} + \int \partial_t u(0,\mathbf{x})\,\varphi(0,\mathbf{x}) - u(0,\mathbf{x})\,\partial_t\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} =$$

$$= \int \tilde{\psi} + u_1\delta(t) + u_0\delta'(t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}.$$

■

2.6.4 Lemma. *If $u_\pm \in C^\infty(\mathbb{M}^n)$ solves the Cauchy problem on S_0 and $\mathrm{supp}\, u_\pm \subset J_\pm(K)$, where $K := \mathrm{supp}\, u_0 \cup \mathrm{supp}\, u_1 \cup \mathrm{supp}\, \psi$, then it holds*

$$u_\pm = G_\pm * (\psi + u_0 \otimes \delta' + u_1 \otimes \delta). \tag{2.17}$$

■

This leads to the following Corollary, that will be addressed in the general setting in Section 3.2.

2.6.5 Corollary. *If $u \in C^\infty(\mathbb{M}^n)$ solves the Cauchy problem on S_0 with $Pu = 0$ (i.e. $\psi = 0$), then*

$$\mathrm{supp}\, u \subset J_+(K) \cup J_-(K),$$

where $K := \mathrm{supp}\, u_0 \cup \mathrm{supp}\, u_1$, and a solution is given by

$$u = G * (u_0 \otimes \delta' + u_1 \otimes \delta) \tag{2.18}$$

for all $\varphi \in \mathcal{D}(\mathbb{M}^n)$, where $G = G_+ - G_-$. ■

Now follows the general result on S_0.

2.6.6 Theorem. *For each* $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S_0) \oplus \mathcal{D}(S_0)$, *there exists a unique solution* u *to the Cauchy problem on* S_0. *Furthermore* $\operatorname{supp} u \subset J_+(K) \cup J_-(K)$, *where* $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$. ■

Proof. We will only prove the existence. To build it with initial values u_0, u_1 we need the sum of a particular solution and the solution for the homogeneous equation.
We start by noting that a particular solution of $\Box\widetilde{\phi} = \psi$ induces initial values on S_0 given by $\widetilde{u}_0(\mathbf{x}) = G_+ * \psi(0, \mathbf{x})$ and $\widetilde{u}_1(\mathbf{x}) = \partial_t G_+ * \psi(0, \mathbf{x})$. Hence, for the homogeneous equation with initial values u_0, u_1, we apply Corollary 2.6.5 with initial values $u_0 - \widetilde{u}_0$ and $u_1 - \widetilde{u}_0$ and find $\widehat{\phi}$.
The general solution will be $\phi = \widehat{\phi} + \widetilde{\phi}$. ■

The solution, if we concentrate on the classical Cauchy problem, is smooth (Proposition 2.2.2) and depends continuously on the initial data, hence the map that gives the solution u can be seen as a **linear continuous operator**

$$\begin{aligned} \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S) &\to C^\infty(\mathbb{M}^n) \\ (\psi, u_0, u_1) &\mapsto u. \end{aligned}$$

Fundamental solutions on manifolds 3

In this chapter, the Riesz distributions will be transported on suitable domains of Lorentzian manifolds to construct local fundamental solutions and it will be discussed the local and global solvability of the Cauchy problem.

3.1 Local fundamental solutions

Riesz distributions have been defined on Minkowski spacetime. They are therefore defined on the tangent spaces at all points of a Lorentzian manifold. We now show how to construct Riesz distributions defined in small open subsets of the Lorentzian manifold. The passage from the tangent space to the manifold will be provided by the exponential map.

3.1.1 Definition. *Let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to some point $x \in \Omega$. The Riesz distribution at $x \in \Omega$ to the parameter $\alpha \in \mathbb{C}$ is defined for every test-function $\varphi \in \mathcal{D}(\Omega \subset \mathrm{M})$ as*

$$\left(R_{\pm}^{\Omega}(\alpha, x), \varphi\right) := \left(R_{\pm}(\alpha), (\mu_x \varphi) \circ \exp_x\right),$$

where $\exp_x : \exp_x^{-1}(\Omega) \subset \mathrm{T}_x\Omega \to \Omega$ is the exponential map at x and $\mu_x : \Omega \to \mathbb{R}$ is defined as in Equation (1.4).
*We call $R_+^{\Omega}(\alpha, x)$ and $R_-^{\Omega}(\alpha, x)$ respectively the **retarded** and the **advanced** Riesz distributions on Ω at x for $\alpha \in \mathbb{C}$.* ■

Note that $\operatorname{supp}(\mu_x\varphi)\circ\exp_x \subset \Omega'$, hence, we can regard it as a test function on $\mathrm{T}_x\Omega$ and apply on it the Riesz distribution.

The distribution on a geodesically starshaped domain maintains some of the properties of the Minkowski flat space (namely the ones expressed in Proposition 2.5.3 and in Proposition 2.5.6), even if with some slight differences.

3.1.2 Theorem. *Let $\Omega \subset \mathrm{M}$ be geodesically starshaped. For all $\alpha \in \mathbb{C}$ and $x \in \Omega$ holds*

(1) If Re $\alpha > n$, then

$$R_\pm^\Omega(\alpha, x) := \begin{cases} C(\alpha, n)\Gamma_x^{\frac{\alpha-n}{2}} & \text{on } J_\pm^\Omega(x) \\ 0 & \text{elsewhere,} \end{cases}$$

where $C(\alpha, n)$ are defined in Definition 2.5.1.

(2) $\operatorname{supp} R_\pm^\Omega(\alpha, x) \subset J_\pm(x)$.

(3) For every fixed test-function φ the map $\alpha \mapsto \big(R_\pm^\Omega(\alpha, x), \varphi\big)$ is holomorphic on $\mathbb{C}$.

(4) $\Gamma_x \cdot R_\pm^\Omega(\alpha, x) = \alpha(\alpha - n + 2)R_\pm^\Omega(\alpha + 2, x)$.

(5) $\operatorname{grad}\Gamma_x \cdot R_\pm^\Omega(\alpha, x) = 2\alpha \operatorname{grad} R_\pm^\Omega(\alpha + 2, x)$.

(6) $R_\pm^\Omega(0, x) = \delta_x$.

(7) $\Box R_\pm^\Omega(\alpha + 2, x) = \left(\dfrac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_\pm^\Omega(\alpha, x)$ for all $\alpha \neq 0$.

■

One may think, in analogy with the Minkowsky case, that properties (2) and (6) expressed in the last Theorem make the Riesz distribution $R_\pm^\Omega(2, x)$ a good candidate to be a fundamental solution for $\Box$ at x, but the situation is more complicated. In fact, one can not compute $\Box R_\pm^\Omega(\alpha + 2, x)$ for $\alpha = 0$ unless $\Box\Gamma_x - 2n$ vanishes identically, which in general is not the case.
It will turn out that $R_\pm^\Omega(2, x)$ does not suffice to construct fundamental solutions. We will also need Riesz distributions $R_\pm^\Omega(2k + 2, x)$ for $k \geq 1$.

To prove the properties of Theorem 3.1.2 we need a Lemma which helps dealing with Γ_x, the function that takes the place of γ in the formulas for the Riesz distributions on a manifold.

3.1.3 Lemma. *Let $\Omega \subset \mathrm{M}$ be open and geodesically starshaped with respect to $x \in \Omega$. Then it holds on Ω that*

(1) $\langle \operatorname{grad} \Gamma_x, \operatorname{grad} \Gamma_x \rangle = -4\Gamma_x$*;*

(2) $\Box \Gamma_x = 2n - \langle \operatorname{grad} \Gamma_x, \operatorname{grad} \ln(\mu_x) \rangle$.

■

Proof. We prove (1) in Minkowski spacetime, where we can identify Γ with $\gamma = (x^1)^2 - (x^2)^2 - \cdots - (x^n)^2$. So

$$\begin{aligned} \operatorname{grad} \gamma &= \eta^{ij} \frac{\partial \gamma}{\partial x^i} \frac{\partial}{\partial x^j} \\ &= -2x^1 \frac{\partial}{\partial x^1} - \cdots \\ &= -2x^i \frac{\partial}{\partial x^i}. \end{aligned} \tag{3.1}$$

Hence the first formula follows by direct computation:

$$\begin{aligned} \langle \operatorname{grad} \gamma, \operatorname{grad} \gamma \rangle &= \left\langle -2x^i \frac{\partial}{\partial x^i}, -2x^i \frac{\partial}{\partial x^i} \right\rangle \\ &= 4((x^1)^2 - (x^2)^2 - \cdots - (x^n)^2) \\ &= -4\gamma. \end{aligned}$$

To prove (3) we notice that Leibniz rule for divergence reads

$$\operatorname{div}(fZ) = f \operatorname{div} Z + \langle \operatorname{grad} f, Z \rangle, \tag{3.2}$$

for any vector field Z and for any smooth function f on M and that Equation (3.1) generalizes in normal coordinates as

$$\operatorname{grad} \Gamma_x = -2x^j \frac{\partial}{\partial x^j}.$$

Using Equation (3.2) with $f = \mu_x^{-1}$ and $Z = \operatorname{grad} \Gamma_x$, we obtain

$$\operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) = \mu_x^{-1} \operatorname{div} \operatorname{grad} \Gamma_x + \langle \operatorname{grad} \mu_x^{-1}, \operatorname{grad} \Gamma_x \rangle.$$

Since

$$\begin{aligned} \Box \Gamma_x &= -\operatorname{div} \operatorname{grad} \Gamma_x \\ &= \mu_x \langle \operatorname{grad} \mu_x^{-1}, \operatorname{grad} \Gamma_x \rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) \\ &= -\langle \operatorname{grad} \ln \mu_x, \operatorname{grad} \Gamma_x \rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x). \end{aligned}$$

It remains to show $\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) = -2n$. Using the definition of divergence and Equation (1.5),

$$\mu_x \operatorname{div}(\mu_x \operatorname{grad} \Gamma_x) = \frac{\partial}{\partial x^j}(\operatorname{grad} \Gamma_x)^j = -2\frac{\partial}{\partial x^j} x^j = -2n.$$

■

Proof of Theorem (3.1.2). Let $\operatorname{Re} \alpha > n$ and $\varphi \in \mathcal{D}(\Omega)$. Then from the definition, (where we denote with X the tangent vectors in $\mathrm{T}_x\Omega$)

$$\begin{aligned}
\left(R_\pm^\Omega(\alpha, x)\varphi\right) &= (R_\pm(\alpha), (\mu_x\varphi) \circ \exp_x) \\
&= C(\alpha, n) \int_{J_\pm(0)} \gamma^{\frac{\alpha-n}{2}}(X) \mu_x \, \varphi(\exp_x(X)) \, \mathrm{d}X \\
&= C(\alpha, n) \int_{J_\pm^\Omega(x)} \Gamma_x^{\frac{\alpha-n}{2}} \varphi(x) \, \mathrm{d}\mu.
\end{aligned}$$

Assertions (2) and (3) follow directly from the corresponding properties of the Riesz distributions on Minkowski spacetime.
By (1) the Equation (4) holds when $\operatorname{Re} \alpha > n$ because of the corresponding Equation in the Minkowski case. By analyticity of the distribution it must hold for all α.
To prove (5), we compute for $\operatorname{Re} \alpha > n$

$$\begin{aligned}
2\alpha \operatorname{grad} R_\pm^\Omega(\alpha+2, x) &= 2\alpha C(\alpha+2, n) \operatorname{grad} \Gamma_x^{\frac{\alpha+2-n}{2}} \\
&= 2\alpha C(\alpha+2, n) \frac{\alpha+2-n}{2} \Gamma_x^{\frac{\alpha-n}{2}} \operatorname{grad} \Gamma_x \\
&= R_\pm^\Omega(\alpha, x) \operatorname{grad} \Gamma_x.
\end{aligned}$$

and again it holds for any α from analyticity.
To prove (6) let $\varphi \in \mathcal{D}(\Omega)$. Then,

$$\begin{aligned}
\left(R_\pm^\Omega(0, x), \varphi\right) &= (R_\pm(0), (\mu_x\varphi) \circ \exp_x) \\
&= (\delta_0, (\mu_x\varphi) \circ \exp_x) \\
&= ((\mu_x\varphi) \circ \exp_x)(0) \\
&= \varphi(x) \\
&= (\delta_x, \varphi).
\end{aligned}$$

To prove (7) we take $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha > n+2$ so that $R_\pm^\Omega(\alpha+2, x)$ is C^2

(analyticity will imply the result all over $\mathbb{C}$):

$$\begin{aligned}
\Box R^{\Omega}_{\pm}(\alpha+2,x) &= -\operatorname{div}\operatorname{grad} R^{\Omega}_{\pm}(\alpha+2,x)\\
&= -\frac{1}{2\alpha}\operatorname{div}\left(R^{\Omega}_{\pm}(\alpha,x)\cdot\operatorname{grad}\Gamma_x\right)\\
&= \frac{1}{2\alpha}\Box\Gamma_x\cdot R^{\Omega}_{\pm}(\alpha,x)-\frac{1}{2\alpha}\langle\operatorname{grad}\Gamma_x,\operatorname{grad}R^{\Omega}_{\pm}(\alpha,x)\rangle\\
&= \frac{1}{2\alpha}\Box\Gamma_x\cdot R^{\Omega}_{\pm}(\alpha,x)-\frac{1}{2\alpha\cdot 2(\alpha-2)}\langle\operatorname{grad}\Gamma_x,\operatorname{grad}\Gamma_x\cdot R^{\Omega}_{\pm}(\alpha-2,x)\rangle,
\end{aligned}$$

and making use of equation (1) in Lemma (3.1.3) we have

$$\begin{aligned}
&= \frac{1}{2\alpha}\Box\Gamma_x\cdot R^{\Omega}_{\pm}(\alpha,x)-\frac{1}{\alpha(\alpha-2)}\Gamma_x R^{\Omega}_{\pm}(\alpha-2,x)\\
&= \frac{1}{2\alpha}\Box\Gamma_x\cdot R^{\Omega}_{\pm}(\alpha,x)-\frac{(\alpha-2)(\alpha-n)}{\alpha(\alpha-2)}R^{\Omega}_{\pm}(\alpha,x)\\
&= \left(\frac{\Box\Gamma_x-2n}{2\alpha}+1\right)R^{\Omega}_{\pm}(\alpha,x).
\end{aligned}$$

■

3.1.1 Formal ansatz

Let Ω be geodesically starshaped with respect to some fixed $x\in\Omega$ so that the Riesz distributions are defined. We look for fundamental solutions $\mathcal{R}_{\pm}$ for a generalized d'Alembert operator P. We make the following formal ansatz:

$$\mathcal{R}_{\pm}(x)=\sum_{k=0}^{\infty}V_x^k\cdot R^{\Omega}_{\pm}(2k+2,x), \tag{3.3}$$

where for each k, V_x^k is a smooth coefficients yet to be found. For $\varphi\in\mathcal{D}(\Omega)$ the function $V_x^k\varphi$ is a test-function and we have $(V_x^k R^{\Omega}_{\pm}(2+2k,x),\varphi)=(R^{\Omega}_{\pm}(2+2k,x),V_x^k\varphi)$. Hence each summand $V_x^k\,R^{\Omega}_{\pm}(2+2k,x)$ is a distribution.

The series above is only formal, but plugging it into the equation

$$P\mathcal{R}_{\pm}(x)=\sum_{k=0}^{\infty}P\left(V_x^k\cdot R^{\Omega}_{\pm}(2k+2,x)\right)=R_{\pm}(0,x)=\delta_x$$

one can translate the condition of $\mathcal{R}_{\pm}(x)$ being a fundamental solution at x into conditions on the V_x^k.
To do this we need a lemma.

3.1.4 Lemma. *Let P be a generalized d'Alembert operator, $X \in C^\infty(\Omega, \mathrm{TM})$ and $f \in C^\infty(\Omega)$. Then it holds*

$$P(fX) = \Box f \cdot X - 2\nabla_{\operatorname{grad} f} X + P(X) \cdot f.$$

■

Using the last lemma with $f = R_\pm^\Omega(2k+2, x)$, $X = V_x^k$ and properties (4), (5) and (7) in Theorem 3.1.2 we compute

$$R_\pm^\Omega(0,x) = \sum_{k=0}^\infty P\left(V_x^k \cdot R_\pm^\Omega(2k+2,x)\right) =$$

$$\begin{aligned}
&= \sum_{k=0}^\infty \left\{V_x^k \cdot \Box R_\pm^\Omega(2+2k,x) - 2\partial_{\operatorname{grad} R_\pm^\Omega(2+2k,x)} V_x^k + + PV_x^k \cdot R_\pm^\Omega(2k+2,x)\right\} \\
&= V_x^0 \cdot \Box R_\pm^\Omega(2,x) - 2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 + \\
&+ \sum_{k=1}^\infty \left\{V_x^k \cdot \left(\frac{\frac{1}{2}\Box\Gamma_x - n}{2k} + 1\right) R_\pm^\Omega(2k,x) - \tfrac{2}{4k}\partial_{\operatorname{grad}\Gamma_x R_\pm^\Omega(2k,x)} V_x^k + PV_x^{k-1} \cdot R_\pm^\Omega(2k,x)\right\} \\
&= V_x^0 \cdot \Box R_\pm^\Omega(2,x) - 2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 + \\
&+ \sum_{k=1}^\infty \tfrac{1}{2k}\left\{V_x^k \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right) - \partial_{\operatorname{grad}\Gamma_x} V_x^k + 2kPV_x^{k-1}\right\} R_\pm^\Omega(2k,x).
\end{aligned}$$

Identifying the coefficients in front of $R_\pm^\Omega(2k,x)$ we get the conditions

$$\text{for } k = 0 \quad 2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 - \Box R_\pm^\Omega(2,x)\, V_x^0 + R_\pm^\Omega(0,x) = 0, \tag{3.4a}$$

$$\text{for } k \geq 1 \quad \partial_{\operatorname{grad}\Gamma_x} V_x^k - \left(\frac{1}{2}\Box\Gamma_x - n + 2k\right) V_x^k = 2k\, PV_x^{k-1}. \tag{3.4b}$$

For the case $k = 0$ one obtains, multiplying Equation (3.4b) by $R_\pm^\Omega(\alpha,x)$

$$\partial_{\operatorname{grad}\Gamma_x R_\pm^\Omega(\alpha,x)} V_x^0 - \left(\frac{1}{2}\Box\Gamma_x - n\right) V_x^0\, R_\pm^\Omega(\alpha,x) = 0$$

and so

$$\partial_{2\alpha \operatorname{grad} R_\pm^\Omega(\alpha+2,x)} V_x^0 - \left(\alpha\Box R_\pm^\Omega(\alpha+2,x) - \alpha R_\pm^\Omega(\alpha,x)\right) V_x^0 = 0.$$

Dividing by α and taking the limit for $\alpha \to 0$ we obtain

$$2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 - \left(\Box R_\pm^\Omega(2,x) - R_\pm^\Omega(0,x)\right) V_x^0 = 0,$$

which is identical to (3.4a) if and only if $V_x^0(x) \equiv 1$.

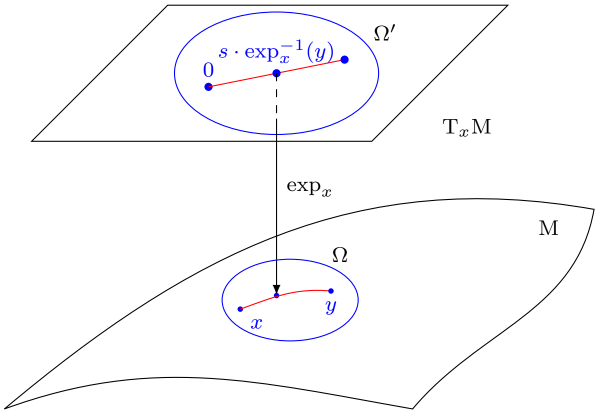

FIGURE 3.1: The action of the map Φ.

3.1.5 Definition. *Let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to $x \in \Omega$. A sequence of* ***Hadamard coefficients*** *for P at $x \in \Omega$ is a sequence $\{V_x^k\}_{k\in\mathbb{N}}$ of $C^\infty(\Omega)$ which fulfills Equation (3.4b) and $V_x^0(x) = 1$ for all $x \in \Omega$.* ■

Hence, to formally give fundamental solutions, the coefficients must satisfy the differential Equation (3.4b), that turns out to be solvable recursively without further assumption.
A simple case in which an explicit formula for the coefficients is achieved is when P has no derivatives of first order.
We define the map $\Phi : \Omega \times [0, 1] \to \Omega$ such that $\Phi(y, s) = \exp_x(s \cdot \exp_x^{-1}(y))$, which is well defined and smooth since Ω is geodesically starshaped. In other words, the function Φ is a parametrization of the geodesic connecting x and another point $y \in \Omega$, as one can see in FIGURE 3.1.

3.1.6 Theorem. *Let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to $x \in \Omega$. Let P be a generalized d'Alembert operator of the form $P = \Box + b$, where $b \in C^\infty(\mathrm{M})$. Then there exist unique Hadamard coefficients V_x^k for P at x given by*

$$V_x^0(y) = \mu_x^{-\frac{1}{2}}(y) \tag{3.5}$$

and for $k \geq 1$

$$V_x^k(y) = -k\mu_x^{-\frac{1}{2}}(y) \int_0^1 \mu_x^{\frac{1}{2}} \left(\Phi(y, s)\right) s^{k-1} \left((PV_x^{k-1})\Phi(y, s)\right) \mathrm{d}s. \tag{3.6}$$

■

3.1.7 Example. Consider the d'Alembert wave operator $\Box$ in Minkowski spacetime $\mathbb{M}^n$. Using Equations (3.5) and (3.6) one can check that the Hadamard coefficients are $V_x^0 = 1$ and $V_x^k = 0$ for all $k \geq 1$, as one expects. ∎

Now we let x vary. If Ω is convex, the Riesz distributions are defined for all $x \in \Omega$. We write $V^k(x, y) := V_x^k(y)$ for the Hadamard coefficients at x. The explicit formulas in Equations (3.5) and (3.6) show that $V^k \in C^\infty(\Omega \times \Omega)$.

3.1.8 Definition. *Let $\Omega \subset \mathrm{M}$ be convex and P a generalized d'Alembert operator. We call*

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} V^k(x, \cdot)\, R_\pm^\Omega(2k+2, x), \tag{3.7}$$

the ***retarded*** *or* ***advanced*** *formal fundamental solutions for P at $x \in \Omega$.* ∎

3.1.9 Remark. The coefficients $V^k \in C^\infty(\Omega \times \Omega)$ can be calculated by the analogous of Equations (3.5) and (3.6):

$$V^0(x, y) = \mu_x^{-\frac{1}{2}}(y),$$

$$V^k(x, y) = -k\mu_x^{-\frac{1}{2}}(y) \int_0^1 \mu_x^{\frac{1}{2}}(\Phi(y, s))\, s^{k-1} \left((P_{(2)} V^{k-1})\Phi(y, s)\right) \mathrm{d}s,$$

where the index "(2)" in $P_{(2)} V^{k-1}$ stands for P acting on the second variable, i.e. on $y \mapsto V^{k-1}(\cdot, y)$. ∎

3.1.2 Approximate fundamental solutions

The series defining $\mathcal{R}_\pm(x)$ may diverge, hence it does not provide any local fundamental solution. The idea is to make the series convergent by keeping the first terms of the formal series and multiplying the higher ones by suitable cut-off functions.
Let Ω' be convex. Fix an integer $N \geq \frac{n}{2}$. Then for all $k \geq N$ the distribution $R_\pm^{\Omega'}(2k+2, x)$ is continuous on Ω'. We split the formal fundamental solutions

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{N-1} V^k(x, \cdot)\, R_\pm^{\Omega'}(2k+2, x) + \sum_{k=N}^{\infty} V^k(x, \cdot)\, R_\pm^{\Omega'}(2k+2, x).$$

3.1.10 Proposition. *Let $\Omega \subset \Omega'$ a relatively compact subset (i.e. its closure $\overline{\Omega}$ is a compact subset). Let $\sigma : \mathbb{R} \to [0, 1]$ be a smooth function with* *supp* $\sigma \subset [-1,1]$ *and* $\sigma = 1$ *on* $[-\frac{1}{2}, \frac{1}{2}]$ *(i.e. a cut-off function). Then there exists a sequence* $\{\varepsilon_k\}_{k\geq N}$ *in* $(0,1]$*, such that for each* $j \geq 0$ *the series*

$$(x,y) \mapsto \sum_{k=N+j}^{\infty} \sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y) R_{\pm}^{\Omega'}(2+2k,x)(y) =$$

$$= \begin{cases} \sum_{k=N+j}^{\infty} C(2+2k,n)\sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y)\Gamma_x(y)^{k+1-\frac{n}{2}} & \text{if} \quad y \in J_{\pm}^{\Omega'}(x) \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

converges in $C^j(\overline{\Omega} \times \overline{\Omega})$*. In particular the series*

$$(x,y) \mapsto \sum_{k=N}^{\infty} \sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y) R_{\pm}^{\Omega'}(2+2k,x)(y)$$

defines a continuous function. ■

Now we present the main tools that we need to approximate the fundamental solutions.

3.1.11 Theorem. *Let* Ω' *be a convex subset of* M*,* P *a generalized d'Alembert operator over* M *and* σ *a cut-off function as the one defined in Proposition 3.1.10. Then for every relatively compact open subset* $\Omega \subset \Omega'$ *there exists a positive sequence* $\{\varepsilon_k\}_{j\geq N}$*, such that for every* $x \in \overline{\Omega}$

$$\widetilde{\mathcal{R}}_{\pm}(x) = \sum_{k=0}^{N-1} V^k(x,\cdot)\, R_{\pm}^{\Omega'}(2k+2,x) + \sum_{k=N}^{\infty} \sigma\left(\tfrac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,\cdot)\, R_{\pm}^{\Omega'}(2k+2,x) \tag{3.9}$$

defines a distribution on Ω *satisfying*

(1) *supp* $\widetilde{\mathcal{R}}_{\pm}(x) \subset J_{\pm}^{\Omega}(x)$*,*

(2) $P_{(2)}\widetilde{\mathcal{R}}_{\pm}(x) = \delta_x + K_{\pm}(x,\cdot)$ *where* $K_{\pm} \in C^{\infty}(\overline{\Omega} \times \overline{\Omega})$*,*

(3) *for every* $\varphi \in \mathcal{D}(\Omega)$*,* $x \mapsto \left(\widetilde{\mathcal{R}}_{\pm}(x), \varphi\right)$ *is smooth on* Ω*.*

■

With a suitable sequence $\{\varepsilon_k\}$ the distributions defined in Equation (3.9) (called approximate **retarded** or **advanced** fundamental solution) are near to the true fundamental solutions in the sense that the difference $P_{(2)}\widetilde{\mathcal{R}}_{\pm}(x) - \delta_x$ is a smooth function.

3.1.3 True fundamental solutions

We now turn the approximate fundamental solution into a true one getting rid of the error terms with methods of functional analysis.
Notice that if a sequence $\{\varepsilon_k\}$ gives an approximate fundamental solution for Ω, the same sequence still provides an approximate fundamental solution for any $\Omega_1 \subset \Omega$.

We see $K_\pm \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ as an integral kernel and define the (bounded) integral operator for any $x \in \Omega$ and for any continuous function u on Ω

$$(\mathcal{K}_\pm u)(x) := \int_\Omega K_\pm(x,y)u(y)\,\mathrm{d}\mu(y). \tag{3.10}$$

Notice that $\mathcal{K}_\pm u \in C^\infty(\Omega)$.
Since the map $\varphi \mapsto (\delta_x, \varphi) : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is indeed the identity operator $\mathbb{1}$, one can rewrite Equation (2) in Theorem 3.1.11 as

$$\left(P_{(2)}\widetilde{\mathcal{R}}_\pm(x), \varphi\right) = \left(\mathbb{1} + \mathcal{K}_\pm\right)\varphi,$$

for all $\varphi \in \mathcal{D}(\Omega)$.
We look for an inverse of $(\mathbb{1} + \mathcal{K}_\pm)$. To see why it is important, set for all $\varphi \in \mathcal{D}(\Omega)$.

$$(G_\pm^\Omega, \varphi) := (\mathbb{1} + \mathcal{K}_\pm)^{-1}\left(y \mapsto \left(\widetilde{\mathcal{R}}_\pm(y), \varphi\right)\right). \tag{3.11}$$

Hence

$$\begin{aligned}
\left(PG_\pm^\Omega(x), \varphi\right) &= \left(G_\pm^\Omega(x), P^*\varphi\right) \\
&= \left[(\mathbb{1} + \mathcal{K}_\pm)^{-1}\left(y \mapsto \left(\widetilde{\mathcal{R}}_\pm(y), P^*\varphi\right)\right)\right](x) \\
&= \left[(\mathbb{1} + \mathcal{K}_\pm)^{-1}\left(\underbrace{y \mapsto \left(P_{(2)}\widetilde{\mathcal{R}}_\pm(y), \varphi\right)}_{(\mathbb{1}+\mathcal{K}_\pm)\,\varphi}\right)\right](x) \\
&= \varphi(x),
\end{aligned}$$

i.e. $PG_\pm^\Omega(x) = \delta_x$. So, if we can find an inverse for $\mathbb{1} + \mathcal{K}_\pm$ and, proven $G_\pm\Omega$ is a well-defined distribution, we get a local **exact** fundamental solution.
The idea is to use the fact that, given a bounded operator A of a Banach space, an operator of the form $(\mathbb{1} + A)$ is invertible if $\|A\| < 1$. In our case this condition can be satisfied on domains with *small* volume. To be more precise we have the following proposition:

3.1.12 Proposition. *Let $\Omega \subset \Omega'$ be a relatively compact and causal set and assume*

$$|\overline{\Omega}| \cdot \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} K_\pm(x,y) < 1, \tag{3.12}$$

where $|\overline{\Omega}|$ is the volume of $\overline{\Omega}$. Then $(\mathbb{1} + \mathcal{K}_\pm) : C^k(\overline{\Omega}) \to C^k(\overline{\Omega})$ is an isomorphism for all $k \in \mathbb{N}$. ∎

The main results are:

3.1.13 Theorem. *Let P be a generalized d'Alembert operator on* M*. Then every point of* M *possesses a relatively compact causal neighborhood Ω such that*

(1) $G^\Omega_\pm(x)$, defined in Equation (3.11)*, are fundamental solutions for P at x over Ω,*

(2) $\operatorname{supp} G^\Omega_\pm(x) \subset J^\Omega_\pm(x)$, i.e. $G^\Omega_\pm(x)$ are a ***retarded*** *and an* ***advanced*** *fundamental solution,*

(3) $\varphi \mapsto (G^\Omega_\pm(x), \varphi)$ is smooth for all $\varphi \in \mathcal{D}(\Omega)$.

∎

3.1.4 Asymptotic behaviour

The formal fundamental solution is asymptotic to the true one, in the sense that the true fundamental solution coincides with the truncated one

$$\mathcal{R}^{N+j}_\pm(x) = \sum_{k=0}^{N-1+j} V^k(x,\cdot)\, R^{\Omega'}_\pm(2+2k, x),$$

up to an error term which is regular on the light cone. More precisely we can say that for all $j \in \mathbb{N}$ the map

$$(x,y) \mapsto (G^\Omega_\pm(x) - \mathcal{R}^{N+j}_\pm(x))(y) \tag{3.13}$$

is in $C^k(\Omega \times \Omega)$. But we can say more, in fact the following holds:

3.1.14 Proposition. *For every $j \in \mathbb{N}$ there exists a constant C_j such that*

$$\sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} (G^\Omega_\pm(x) - \mathcal{R}^{N+j}_\pm(x))(y) \le C_j |\Gamma_x(y)|^j,$$

for all $(x,y) \in \Omega \times \Omega$. ∎

To prove this proposition we need a lemma.

3.1.15 Lemma. *Let $f \in C^{3j+1}(\mathbb{M}^n)$ such that $f = 0$ if $\|x\| < 0$. Then there exists a contiunuous function $h : \mathbb{M}^n \to \mathbb{R}$ such that*

$$f(x) = h(x)\gamma(x)^j,$$

where $\gamma(x) = -\langle x, x\rangle$. ■

Proof of Proposition 3.1.14. Using the properties expressed in Theorem 3.1.2, we find constants C'_k such that

$$\begin{aligned}
&(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+j}_{\pm}(x))(y) = \\
&= (G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y) + \sum_{k=N+j}^{N+3j} V^k(x,y)\, R^{\Omega'}_{\pm}(2+2k,x)(y) \\
&= (G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y) + \\
&+ \sum_{k=N+j}^{N+3j} V^k(x,y)\, C'_k\, \Gamma_x(y)^j\, R^{\Omega'}_{\pm}(2+2(k-j),x)(y).
\end{aligned}$$

The function $h_k(x,y) := C'_k\, V^k(x,y)\, R^{\Omega'}_{\pm}(2+2(k-j),x)(y)$ is continuous since $2+2(k-j) \geq 2+2N \geq 2+n > n$. For what we already said about Equation (3.13), the function $(x,y) \mapsto (G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y)$ is C^{3j+1} and $\operatorname{supp}(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x)) \subset J^{\Omega}_{\pm}(x)$. We apply Lemma 3.1.15 in normal coordinates (note that $\Gamma_x(y) = \gamma(\exp_x^{-1}(y))$) and find a continuous function h such that

$$(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y) = h(x,y)\,\Gamma_x(y)^k.$$

Hence,

$$(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+j}_{\pm}(x))(y) = \left(h(x,y) + \sum_{k=N+j}^{N+3j} h_k(x,y)\right)\Gamma_x(y)^j.$$

If we now set $C_j := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(h(x,y) + \sum_{k=N+j}^{N+3j} h_k(x,y)\right)$ the proposition is demonstrated. ■

3.1.5 Uniqueness and regularity

We want to solve the inhomogeneous equation $Pu = \psi$ for given ψ with small support, *small* in the sense intended in Proposition 3.1.12, i.e. the support Ω is relatively compact and it holds Equation (3.12).

3.1.16 Proposition. *Under the assumption of Proposition 3.1.12, for every $\psi \in \mathcal{D}(\Omega)$ there exists a function $u_\pm \in C^\infty(\Omega)$ such that*

$$\begin{aligned} &Pu_\pm = \psi \\ &\mathit{supp}\, u_\pm \subset J^\Omega_\pm(\mathit{supp}\,\psi). \end{aligned}$$

■

Proof. Put

$$(u_\pm, \varphi) := \int_\Omega (G^\Omega_\pm(x), \varphi)\, \psi(x)\, \mathrm{d}\mu. \tag{3.14}$$

We will not prove Equation (3.14) defines a smooth function. We will only prove $Pu_\pm = \psi$. Let $\varphi \in \mathcal{D}(\Omega)$. It holds

$$\begin{aligned} (Pu_\pm, \varphi) &= (u_\pm, P^*\varphi) \\ &= \int_\Omega (G^\Omega_\pm(x), P^*\varphi)\, \psi(x)\, \mathrm{d}\mu \\ &= \int_\Omega (\underbrace{P_{(2)} G^\Omega_\pm(x)}_{=\delta_x}, \varphi)\, \psi(x)\, \mathrm{d}\mu \\ &= \int_\Omega \varphi(x)\, \psi(x)\, \mathrm{d}\mu = (\psi, \varphi). \end{aligned}$$

To prove the support condition, let $\varphi \in \mathcal{D}(\Omega)$ such that $(u_\pm, \varphi) \neq 0$, then there exists $x \in \Omega$ such that $(G^\Omega_\pm(x), \varphi)\, \psi(x) \neq 0$, which implies $\operatorname{supp} \varphi \cap \operatorname{supp} G^\Omega_\pm(x) \neq \emptyset$ and $x \in \operatorname{supp} \psi$. Hence $\operatorname{supp} \varphi \cap J^\Omega_\pm(x) \neq \emptyset$, i.e. $x \in J^\Omega_\mp(\operatorname{supp} \varphi)$, so that $J^\Omega_\mp(\operatorname{supp} \varphi) \cap \operatorname{supp} \psi$, which is equivalent to the thesis. ■

As we mentioned in Proposition 1.2.19, the topological and geometrical properties of the Lorentzian manifold may be problematic when we are looking for solutions of a differential equation. For example, even if the manifold is not compact, the existence of closed timelike loops can make the problem ill-posed. To avoid these situations and to implement the causality conditions we restrict the discussion to the **globally hyperbolic** setting (see Definition 1.2.21), although some results may be extended to other cases.

In such case the main results are

3.1.17 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M. *Then every solution of the equation $PF = 0$ in $\mathcal{D}'(\mathrm{M})$ with past- or future-compact support (see Definition 1.2.18) vanishes.*

■

Sketch of proof. Take the case F has past-compact support. The thesis is $(F, \varphi) = 0$ for any $\varphi \in \mathcal{D}(\mathrm{M})$. The idea is to solve the inhomogeneous equation

$$\begin{aligned} &P^* u = \varphi \\ &\operatorname{supp} u \subset J_-^{\Omega}(\operatorname{supp} \varphi) \end{aligned}$$

(using Proposition 3.1.16) for any φ such that $\operatorname{supp} \varphi$ is small in the sense expressed in Proposition 3.1.12. If one proves that $\operatorname{supp} F \cap J_-^{\Omega}(\operatorname{supp} \varphi)$ is compact, it holds

$$(F, \varphi) = (F, P^* u) = (PF, u) = 0.$$

The proof of such properties involves the global hyperbolicity of the manifold. ■

Now uniqueness is straightforward:

3.1.18 Corollary. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M *and fix $x \in \mathrm{M}$. Then there exist at most one retarded and at most one advanced fundamental solution for P at x.* ■

Proof. Let G_1 and G_2 be two retarded fundamental solutions at x. Then $G = G_1 - G_2$ is a solution for $PG = 0$. Since G_1 and G_2 are retarded solutions we know that $\operatorname{supp} G \subset \operatorname{supp} G_1 \cup \operatorname{supp} G_2 \subset J_+^{\mathrm{M}}(x)$. On a globally hyperbolic manifold $J_+^{\mathrm{M}}(x)$ is past compact. Then for Theorem 3.1.17 $G = 0$ and hence $G_1 = G_2$. ■

3.2 Local and global Cauchy problem

We now explore the solvability of the Cauchy problem, in analogy with Section 2.6, in order to obtain results needed to prove the existence and uniqueness of the fundamental solutions.

3.2.1 Local solvability

Next we prove existence and uniqueness of solutions to the Cauchy problem on small domains, but before it is useful to show a formula that helps allowing us to control a solution of the problem in terms of its initial data.

3.2.1 Lemma. *Let P be a generalized d'Alembert operator on* M *and let S be a smooth spacelike hypersurface of* M *with a timelike unit normal vector field $\nu : S \to \mathrm{TM}$.*
Let $\Omega \subset \mathrm{M}$ be a small *subset (i.e. such that the hypotheses of Proposition 3.1.12 are satisfied) such that $S \cap \Omega$ is a Cauchy hypersurface of Ω. If $u_\pm \in C^\infty(\Omega)$ such that $\operatorname{supp} u_\pm \subset J_\pm^\Omega(K)$ (where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$) solve the Cauchy problem*

$$\begin{cases} Pu_\pm = \psi \\ \\ u_\pm|_{S\cap\Omega} = u_0 \\ \\ \partial_\nu u_\pm|_{S\cap\Omega} = u_1, \end{cases} \tag{3.15}$$

with $(\psi, u_0, u_1) \in \mathcal{D}(\Omega) \oplus \mathcal{D}(S\cap\Omega) \oplus \mathcal{D}(S\cap\Omega)$, then it holds

$$\begin{aligned} \int_\Omega u_\pm(x)\varphi(x)\,\mathrm{d}\mu = &\int_\Omega \left(G_\pm^\Omega(x), \varphi\right) \psi(x)\,\mathrm{d}\mu + \\ &+ \int_{S\cap\Omega} \left(\partial_\nu (G_\pm^\Omega(x), \varphi) u_0 - (G_\pm^\Omega(x), \varphi) u_1\right)\,\mathrm{dS}, \end{aligned} \tag{3.16}$$

for all $\varphi \in \mathcal{D}(\Omega)$, where dS is the $n-1$ dimensional measure on S. ■

3.2.2 Corollary. *In the conditions of last lemma, if $u \in C^\infty(\Omega)$ solves $Pu = 0$ then*

$$\operatorname{supp} u \subset J_+^\Omega(K) \cup J_-^\Omega(K),$$

where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1$, and

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}\mu = \int_{S\cap\Omega} \left(\partial_\nu (G^\Omega(x), \varphi) u_0 - (G^\Omega(x), \varphi) u_1\right)\,\mathrm{dS}, \tag{3.17}$$

for all $\varphi \in \mathcal{D}(\Omega)$, where $G^\Omega(x) = G_+^\Omega(x) - G_-^\Omega(x)$ and dS is the $n-1$ dimensional measure on S. ■

3.2.3 Theorem. *Under the conditions of Lemma 3.2.1, for each* small *subset $\Omega \subset \mathrm{M}$ (i.e. such that the hypotheses of Proposition 3.1.12 are satisfied)* *such that $S \cap \Omega$ is a Cauchy hypersurface of Ω holds that for all triples $(\psi, u_0, u_1) \in \mathcal{D}(\Omega) \oplus \mathcal{D}(S \cap \Omega) \oplus \mathcal{D}(S \cap \Omega)$ there exists a unique $u \in C^\infty(\Omega)$ with*

$$\begin{cases} Pu = \psi \\ \\ u|_{S\cap\Omega} = u_0 \\ \\ \partial_\nu u|_{S\cap\Omega} = u_1. \end{cases} \tag{3.18}$$

Moreover $\operatorname{supp} u \subset J_+^\Omega(K) \cup J_-^\Omega(K)$, where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$. ■

Sketch of proof. Since causal domains are globally hyperbolic we may apply Theorem 1.2.23 and find an isometry $\Omega = \mathbb{R} \times (S \cap \Omega)$, where the metric takes the form $g = -\beta\, \mathrm{d}t^2 + b_t$. Now we look for a formal solution of the form

$$u(t,x) = \sum_{j=0}^{\infty} t^j\, \widetilde{u}_j(x), \tag{3.19}$$

where $(t,x) \in \mathbb{R} \times (S \cap \Omega)$. On $S \cap \Omega$ holds $\widetilde{u}_0 = u_0$ and $\widetilde{u}_1 = -(\beta)^{\frac{1}{2}} u_1$. The generalized d'Alembert operator can be written in the form $P = \frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y$, where Y contains at most derivatives of order 1 in t.
From equation

$$\psi = Pu = \left(\frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y\right) u = \frac{1}{\beta}\sum_{j=2}^{\infty} j(j-1)t^{j-2}\widetilde{u}_j + Yu, \tag{3.20}$$

for $t = 0$ we have

$$2\beta(0,x)\, \widetilde{u}_2(x) = -Y(\widetilde{u}_0 + t\widetilde{u}_1)(0,x) + \psi(0,x).$$

This relation determines uniquely $\widetilde{u}_2$ from $\widetilde{u}_0$,$\widetilde{u}_1$ and $\psi|_S$. One can hence find recursive relations for $\widetilde{u}_j$ differentiating Equation (3.20) with respect to t and repeating the procedure.
In general the series in Equation (3.19) is non-convergent, but one can find a suitable positive sequence $\{\varepsilon_j\}$ such that the series

$$\widehat{u} := \sum_{j=0}^{\infty} \sigma\left(\frac{t}{\varepsilon_j}\right) t^j \widetilde{u}_j,$$

(where σ is a cut-off function as in Proposition 3.1.10) defines a smooth function on Ω and such that $P\widehat{u} - \psi$ vanishes at least on $S \cap \Omega$. Then Proposition 3.1.16 provides smooth solutions $\widetilde{u}_\pm$ of the inhomogeneous problems

$$\begin{aligned} &P\widetilde{u}_\pm = h_\pm \\ &\operatorname{supp} \widetilde{u}_\pm \subset J^\Omega_\pm(\operatorname{supp} h_\pm), \end{aligned}$$

where $h_\pm|_{J^\Omega_\pm(S\cap\Omega)} := P\widehat{u} - \psi$ and vanishes everywhere else on Ω. One now has to show that $u_\pm := \widehat{u} - \widetilde{u}_\pm$ solves the equation $Pu_\pm = \psi$ on $J^\Omega_\pm(S \cap \Omega)$ and vanishes on $J^\Omega_\mp(S \cap \Omega)$. Then the function

$$u := \begin{cases} u_+ & \text{on} \quad J^\Omega_+(S \cap \Omega) \\ u_- & \text{on} \quad J^\Omega_-(S \cap \Omega) \end{cases}$$

is smooth and solves the Cauchy problem.
Uniqueness follows from Corollary 3.2.2. In fact, if u_1 and u_2 solve the Cauchy problem, for linearity $u_1 - u_2$ solves the problem with null initial data $(\psi, u_0, u_1) \equiv (0, 0, 0)$. Hence $u := u_1 - u_2 = 0$ because of Equation (3.17). The assertion of the support follows from the corresponding one from the homogeneous problem and from Corollary 3.2.2. ■

3.2.2 Global solvability

The main result of this section generalizes what we obtained in Theorem 3.2.3 to the globally hyperbolic case.

3.2.4 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M *and let S be a smooth Cauchy hypersurface of* M *with a timelike unit normal vector field $\nu : S \to \mathrm{TM}$.*
Then it holds that for all triples $(\psi, u_0, u_1) \in \mathcal{D}(\mathrm{M}) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$ there exists a unique $u \in C^\infty(\mathrm{M})$ such that

$$\begin{cases} Pu = \psi \\ \\ u|_S = u_0 \\ \\ \partial_\nu u|_S = u_1. \end{cases} \tag{3.21}$$

Furthermore $\operatorname{supp} u \subset J_+^{\mathrm{M}}(K) \cup J_-^{\mathrm{M}}(K)$, *where* $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$ *and the map*

$$\begin{aligned}\mathcal{D}(\mathrm{M}) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S) &\to C^\infty(\mathrm{M})\\ (\psi, u_0, u_1) &\mapsto u.\end{aligned}$$

is ***linear continuous****.* ■

Sketch of proof. The existence of u is proved in two steps. One construct a solution u in the strip $(-\varepsilon, \varepsilon) \times S$ for some $\varepsilon > 0$ by gluing together local solutions obtained in Theorem 3.2.3, noting that there is only a finite number of them since the supports are compact. Then one extends u in the whole future and past of the strip. For uniqueness, an argument similar to that of Theorem 3.2.3, which made use of Corollary 3.2.2, can be used.
The continuous dependence on the initial data can be proven with methods of functional analysis. ■

3.2.5 Remark. The former result can be extended furtherly. In fact the same thesis holds even if the triple of initial data (ψ, u_0, u_1) is in $C^\infty(\mathrm{M}) \oplus C^\infty(S) \oplus C^\infty(S)$. ■

3.3 Global fundamental solutions

Since uniqueness of retarded and advanced fundamental solution has already been proven in Corollary 3.1.18, it remains to show the global existence.
We condensate the results in the following theorem.

3.3.1 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M*. Then for each $x \in \mathrm{M}$ there exists a unique fundamental solution $G_+(x)$ with past-compact support and a unique one $G_-(x)$ with future-compact support.*
Furthermore, they satisfy

- $\operatorname{supp} G_\pm \subset J_\pm^{\mathrm{M}}(x)$,
- *the maps $x \mapsto (G_\pm(x), \varphi)$ are a smooth function on* M *and satisfy $P^*(G_\pm(\cdot), \varphi) = \varphi$, for every $\varphi \in \mathcal{D}(\mathrm{M})$.*

■

3.3.2 Remark. From the last theorem one can prove that on M globally hyperbolic, the wave-like equation $Pu = \psi$, with $\psi \in \mathcal{D}(\mathrm{M})$ possesses a unique solution $u_\pm$ with $\operatorname{supp} u_+$ (respectively $\operatorname{supp} u_-$) being past (respectively future) compact. ■

Distributions and Fourier Transform A

Firstly, we recall the main concepts of the theory of distributions on manifolds.

For a manifold M we define $\mathcal{D}(\mathrm{M}) := C_0^\infty(\mathrm{M})$ as the space of test-functions on M. We say a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of $C^j(\mathrm{M})$ (with $j \in \mathbb{N} \cup +\infty$) converges to $\varphi \in C^j(\mathrm{M})$ if there exists a compact subset $K \subset \mathrm{M}$ such that $\operatorname{supp}\varphi_k \subset K$ and all the derivatives of φ_k up to the j-th order converge uniformly in K. A linear map $u : \mathcal{D}(\mathrm{M}) \to \mathbb{C}$ is continuous if for all sequences $\{\varphi_k\}_{k\in\mathbb{N}}$ of $\mathcal{D}(\mathrm{M})$ that converge to $\varphi \in \mathcal{D}(\mathrm{M})$, $(u, \varphi_k) \to (u, \varphi)$, where with (u, φ) we denote the map u tested against φ.

A.0.1 Definition. *The space of distributions over* M *is defined as*

$$\mathcal{D}'(\mathrm{M}) = \{u : \mathcal{D}(\mathrm{M}) \to \mathbb{C} \text{ linear and continuous }\}.$$

The support of a distribution is the set $\mathrm{M} \setminus X$, *where* X *is the set of points* $x \in \mathrm{M}$ *such that there exists a neighborhood* U *of* x *such that* $u|_{\mathcal{D}(U)} \neq 0$. *We say* $u \in \mathcal{E}'(\mathrm{M})$ *if* $\operatorname{supp} u$ *is a compact subset of* M. ■

We call $u \in \mathcal{D}'(\mathrm{M})$ the **weak limit** of a sequence of distributions $\{u_i\}_{i\in\mathbb{N}}$ if for all $\varphi \in \mathcal{D}(\mathrm{M})$ holds $\lim_{i\to\infty}(u_i, \varphi) = (u, \varphi)$.

A.0.2 Remark. For any fixed $f \in C^\infty(\mathrm{M})$ the map $\varphi \mapsto \int_\mathrm{M} f(x)\,\varphi(x)\,\mathrm{d}\mu$ defines a distribution on M. We denote this distribution again by f, hence we identify $C^\infty(\mathrm{M})$ as a subset of $\mathcal{D}'(\mathrm{M})$. ■

A.0.3 Definition. *We define the translation operator* $T_{x_0} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ *such that if* $u \in \mathcal{D}'(\mathbb{R}^n)$,

$$(T_{x_0} u, \varphi(x)) = (u, \varphi(x + x_0)),$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$*. We write* $u(x - x_0) := T_{x_0} u$. ■

A.0.4 Example. Given $x \in \mathrm{M}$ the **Dirac delta** δ_x is a distribution defined for $\varphi \in \mathcal{D}'(\mathrm{M})$ by

$$(\delta_x, \varphi) = \varphi(x).$$

If M is isomorphic to $\mathbb{R}^n$, a particularly useful formula gives

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x - a) + \delta(x + a)]. \tag{A.1}$$

■

A.0.5 Definition. *We define the tensor product of two distributions* $u \in \mathcal{D}'(\mathrm{M})$ *and* $v \in \mathcal{D}'(\mathrm{N})$ *as the unique distribution* $u \otimes v \in \mathcal{D}'(\mathrm{M} \times \mathrm{N})$ *such that for any* $g \in \mathcal{D}(\mathrm{M} \times \mathrm{N})$

$$(u \otimes v, g(x, y)) = \big(u, (v, g(x, y))\big) = \big(v, (u, g(x, y))\big).$$

■

Given a differential operator $P : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ there is a unique $P^* : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$, called the **formal adjoint** of P such that for any $\varphi, \psi \in \mathcal{D}(\mathrm{M})$ holds

$$\int_{\mathrm{M}} \psi(P\varphi)\, \mathrm{d}\mu = \int_{\mathrm{M}} (P^*\psi)\varphi\, \mathrm{d}\mu.$$

Any linear differential operator P extends canonically to $P : \mathcal{D}'(\mathrm{M}) \to \mathcal{D}'(\mathrm{M})$ by

$$(Pu, \varphi) = (u, P^*\varphi).$$

In particular, we define the product of a distribution $u \in \mathcal{D}'(\mathrm{M})$ with a function $f \in C^\infty(\mathrm{M})$ as the distribution $f \cdot u$ such that $(f \cdot u, \varphi) = (u, f\varphi)$ for any $\varphi \in \mathcal{D}(\mathrm{M})$.

A.0.6 Definition. *Let* $u \in \mathcal{E}'(\mathbb{R}^n)$ *and* $v \in \mathcal{D}'(\mathbb{R}^n)$*. We define the* ***convolution*** *of* u *and* v *as the unique distribution* $u * v \in \mathcal{D}'(\mathbb{R}^n)$ *such that*

$$(u * v, \varphi) = \big(u \otimes v, \varphi(x + y)\big) = \big(v, (u, \varphi(x + y))\big) = \big(u, (v, \varphi(x + y))\big).$$

■

A.0.7 Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\rho \in \mathcal{D}(\mathbb{R}^n)$. Then $\rho * u \in C^\infty(\mathbb{R}^n)$.* ■

Now we recall the main concepts of Fourier theory on $\mathbb{R}^n$.

Call Schwartz space $\mathcal{S}(\mathbb{R}^n)$ the set of rapidly decreasing functions, i.e. the functions $f : \mathbb{R}^n \in \mathbb{C}$ such that

$$\lim_{|x|\to\infty} x^\alpha \partial^\beta f(x) = 0,$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$. A sequence $\{f_j\}_{j\in\mathbb{N}}$ of rapidly decreasing functions converge to f in $\mathcal{S}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} x^\alpha \partial^\beta (f_j - f)(x) \to 0,$$

as $j \to \infty$.

A.0.8 Definition. *A distribution $u : \mathcal{D}(\mathrm{M}) \to \mathbb{C}$ is called a* ***tempered distribution*** *if for all sequences $\{f_k\}_{k\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}^n)$ that converge to $f \in \mathcal{S}(\mathbb{R}^n)$, $(u, f_k) \to (u, f)$. The set of tempered distribution is denoted with $\mathcal{S}'(\mathbb{R}^n)$.* ■

Given a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$\widehat{f}(k) = \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle} \,\mathrm{d}x \tag{A.2}$$

We naturally extend the Fourier transform to a unitary map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and to a map on tempered distributions in such a way that for $u \in \mathcal{S}'(\mathbb{R}^n)$ holds

$$(\widehat{u}, \varphi) = (u, \widehat{\varphi}),$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $u \in \mathcal{E}'(\mathbb{R}^n)$ holds

$$\widehat{u}(k) = \left(u, e^{-i\langle k,x\rangle}\right),$$

and $\widehat{u}$ results a smooth function which extends to an entire function $\widehat{u}(z)$, $z \in \mathbb{C}$.

The inverse Fourier transform of $f \in L^1(\mathbb{R}^n)$ is given as $\check{f}(x) := (2\pi)^{-n}\widehat{f}(-x)$ and it holds that $f = \check{\widehat{f}}$.

The Fourier transform of the distribution $\delta \in \mathcal{E}'(\mathcal{U})$ is $\widehat{\delta}(k) = 1$. This is a straightforward computation:

$$\widehat{\delta}(k) = (\delta(x), e^{-i\langle k,x\rangle}) = e^0 = 1.$$

Other useful formulas that holds for $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ and for any multi-index α are

- $\widehat{\partial^\alpha \varphi}(k) = (ik)^\alpha \widehat{\varphi}(k)$
- $\widehat{x^\alpha \varphi}(k) = (i\partial)^\alpha \widehat{\varphi}(k)$.

Green's operators B

We want to show the strict correspondence between fundamental solutions on manifolds and the so-called **Green's operators**. They are operators which can be seen as inverses of P when restricted to suitable spaces of functions.

B.0.1 Definition. *Let P be a generalized d'Alembert operator on a spacetime* M*. A retarded **Green's operator** for P on* M *is a linear map*

$$\mathbb{G}_+ : \mathcal{D}(\mathrm{M}) \to C^\infty(\mathrm{M})$$

such that satisfies

(1) $P \circ \mathbb{G}_+ = \mathbb{1}_{\mathcal{D}(\mathrm{M})}$,

(2) $\mathbb{G}_+ \circ P|_{\mathcal{D}(\mathrm{M})} = \mathbb{1}_{\mathcal{D}(\mathrm{M})}$,

1. *supp* $\mathbb{G}_+\varphi \subset J_+^{\mathrm{M}}(\mathit{supp}\,\varphi)$ *for all* $\varphi \in \mathcal{D}(\mathrm{M})$.

*An advanced **Green's operator*** $\mathbb{G}_- : \mathcal{D}(\mathrm{M}) \to C^\infty(\mathrm{M})$ *satisfies* (1) *and* (2) *and supp* $\mathbb{G}_-\varphi \subset J_-^{\mathrm{M}}(\mathit{supp}\,\varphi)$ *for all* $\varphi \in \mathcal{D}(\mathrm{M})$.

■

Next proposition shows that in fact Green's operators and fundamental solutions are two different versions of mainly the same concept.

B.0.2 Proposition. *In the frame of the above definition, retarded (resp. advanced) Green's operators for P stand in one-to-one correspondence with advanced (resp.retarded) fundamental solutions for P^*. More precisely, if $G_\pm(x)$ is a family of retarded or advanced fundamental solutions for the adjoint operator P^* and if*

- $x \mapsto (G_\pm(x), \varphi)$ *is smooth for each test-function* φ,
- $G_\pm(x)$ *satisfies the differential equation* $P(G_\pm(\cdot), \varphi) = \varphi$

, then

$$(\mathbb{G}_\pm \varphi)(x) := (G_\mp(x), \varphi), \tag{B.1}$$

defines retarded or advanced Green's operators for P respectively.
Conversely, for every Green's operators $\mathbb{G}_\pm$ for P, Equation (B.1) *defines fundamental solutions $G_\mp(x)$ such that $x \mapsto (G_\pm(x), \varphi)$ is smooth for each test-function φ and satisfies the differential equation $P(G_\pm(\cdot), \varphi) = \varphi$.* ■

List of Figures

Bibliography

[1] Leslie Lamport, *LaTeX: a document preparation system*, Addison Wesley, Massachusetts, 2nd edition, 1994.