Gauge of Thrones

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Contents

1 Introduction

Section 2 deals with the construction of the algebra of observable associated with the Faraday tensor F in the presence of an interface Z. For that we need Hodge decomposition, boundary triples out of which we define an exact sequence – *cf.* proposition 16.

Section **??** introduces the $*$-algebra of the vector potential A subjected to Dirichlet boundary conditions.

2 Maxwell equations with interface conditions

2.1 Geometrical set-up

We consider an ultrastatic Lorentzian manifolds (M, g) with closed Cauchy surface Σ.
(We should look for references where the (weak) Hodge decomposition is established for non-compact Riemannian manifold with boudary. If this is the case and if the results presented below remain valid with the weak Hodge decomposition, we will drop the closedness assumption.)
Actually we assume that $M = \mathbb{R} \times \Sigma$ and $g = -\mathrm{d}t^2 + h$ where (Σ, h) is a complete, connected, odd-dimensional, closed Riemannian manifold. We also let Z be an codimension one smooth embedded hypersurface of Σ. We denote with d, δ the differential and co-differential over M, while $\mathrm{d}_\Sigma, \delta_\Sigma$ denote the differential and co-differential over Σ.

In this setting we wish to consider Maxwell equations with Z-interface boundary conditions. This means that we will consider Maxwell equations on $M \setminus (\mathbb{R}\times Z)$, allowing for jump discontinuities to occur on $\mathbb{R}\times Z$. In order to enlighten the differences with the standard case, we briefly review the case $Z=\emptyset$. In this standard situation Maxwell equations reduce to the system of PDE

$$\mathrm{d}F = 0\,, \qquad \delta F = 0\,, \tag{1}$$

where $F \in \Omega^2(M)$ is the Faraday tensor.
(We should decide whether we want to consider k-Maxwell equations or not. In the former case the curl **operator acts as** $\mathrm{curl}\colon \Omega^k(\Sigma)\to\Omega^k(\Sigma)$ **where** $\dim\Sigma = 2k+1$**.)**
The geometrical assumptions on M in order to split F into electric and magnetic components

$$F = *_\Sigma B + \mathrm{d}t\wedge E\,, \tag{2}$$

where $E, B\in\Omega^1(\Sigma)$ while $*_\Sigma$ is the Hodge dual of Σ. Maxwell equations are then reduced to

$$\partial_t E - \mathrm{curl} B = 0\,, \qquad \partial_t B + \mathrm{curl} E = 0\,, \tag{3a}$$
$$\mathrm{div}(E) = \mathrm{div}(B) = 0\,, \tag{3b}$$

where $\mathrm{div} = \delta_\Sigma$ is the co-differential of Σ, while curl is defined in equation (29) – in particular $\mathrm{curl} = *_\Sigma \mathrm{d}_\Sigma$ if $\dim\Sigma = 3 \mod 4$.

Whenever $Z\neq\emptyset$ the system (3) has to be modified, in particular the non-dynamical equations (3b) involving the divergence operator div have to be suitably interpreted – *cf.* remark 11. In particular one expects that the condition $\mathrm{div}(E) = \mathrm{div}(B) = 0$ should be interpreted weakly, leading to a constraint on the normal jump of E across Z. Moreover, the dynamical equations (3a) have to be combined with boundary conditions at the interface Z – *cf.* **. . . .**

In what follows we will state the precise meaning of the problem (3) with interface Z with the help of Hodge theory and Lagrangian subspaces [15, 16, 17]. We also characterize the space of solutions of the problem (3) in term of an exact sequence discussing its applications to AQFT – *cf.* **. . . .**

2.2 Non-dynamical equations: Hodge theory with interface

In this section we present a Hodge decomposition for the closed Riemannian manifold (Σ, h) with interface Z. This generalizes the known results on classical Hodge decomposition on manifolds with possible non-empty boundary [3, 4, 18, 21, 25, 26, 31, 30, 33].

In what follows $\mathrm{L}^2\Omega^k(\Sigma)$ will denote the space of sections of $\wedge^k T^*\Sigma$ which are square integrable with respect to the pairing induced by the metric h

$$(\alpha,\beta)_\Sigma := \int_\Sigma \overline{\alpha}\wedge *_\Sigma\beta\,, \tag{4}$$

where $*_\Sigma$ is the Hodge dual. Similarly we shall denote with $C^\infty_{\mathrm{c}}\Omega^k(\Sigma)$ the space of smooth and compactly supported k-forms, while $\mathrm{H}^\ell\Omega^k(\Sigma)$ will denote k-forms with weak L^2-derivatives up to order $\ell\in\mathbb{N}\cup\{0\}$ with respect to one (hence all) connection over Σ – as usual we also set $\mathrm{H}^{-\ell}\Omega^k(\Sigma) := \mathrm{H}^\ell\Omega^k(\Sigma)^*$.
(If Σ is not compact we only have that only $\mathrm{H}^\ell_{\mathrm{loc}}(\Sigma)$ is independent from the choice of the connection. If we assume (Σ, h) to be of bounded geometry the ambiguity disappears because of the results of [14].)
Moreover, $\mathrm{d}_\Sigma, \delta_\Sigma$ denoted the differential and co-differential over Σ respectively.

2.2.1 Hodge decomposition on compact manifold with non-empty boundary

The Hodge theorem for a closed manifold Σ states that there is an L^2-orthogonal decomposition

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}(\Sigma) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}(\Sigma) \oplus \mathcal{H}^k(\Sigma)\,, \tag{5}$$

where $\mathcal{H}^k(\Sigma)$ denotes the space of harmonic forms

$$\mathcal{H}^k(\Sigma) := \{\omega \in \mathrm{H}^1\Omega^k(\Sigma)|\ \mathrm{d}_\Sigma\omega = 0\,,\ \delta_\Sigma\omega = 0\}\,. \tag{6}$$

For a compact manifold Σ with non-empty boundary $\partial\Sigma$ the decomposition (5) requires a slight adjustment. Indeed the spaces $\mathrm{d}_\Sigma \mathrm{H}^1\Omega^{k-1}(\Sigma)$, $\delta_\Sigma H^1\Omega^{k+1}(\Sigma)$, $\mathcal{H}^k(\Sigma)$ are not orthogonal unless suitable boundary conditions are imposed. This requires to introduce the notion of tangential and normal components of a k-form which we will recall below – *cf.* [5, Sec. 2.4] or [19, p.167].

Definition 1: *Let (Σ, h) be a compact, connected, Riemanniann manifold with non-empty boundary $\partial\Sigma \overset{\iota_{\partial\Sigma}}{\hookrightarrow} \Sigma$. For all $\omega \in C^\infty\Omega^k(\Sigma)$ we define the tangential component $\mathrm{t}\omega \in C^\infty\Omega^k(\partial\Sigma)$ of ω by*

$$\mathrm{t}\omega := \iota_{\partial\Sigma}\omega\,. \tag{7}$$

Similarly the normal component $\mathrm{n}\omega \in \Omega^{k-1}(\partial\Sigma)$ of ω is defined by

$$\mathrm{n}\omega := *_{\partial\Sigma}^{-1} \circ \mathrm{t} *_\Sigma\ . \tag{8}$$

Remark 2: *Notice that the following relations descend from definition 1:*

$$*_\Sigma \circ \mathrm{t} = \mathrm{n} \circ *_\Sigma\,, \qquad \mathrm{d}_{\partial\Sigma} \circ \mathrm{t} = \mathrm{t} \circ \mathrm{d}_\Sigma\,, \qquad \delta_{\partial\Sigma} \circ \mathrm{n} = -\mathrm{n} \circ \delta_\Sigma\,. \tag{9}$$

(Some author – p.e. [31] – defines $\mathrm{n}\omega := \omega|_{\partial\Sigma} - \mathrm{t}\omega$. This however does not define an element in $\Omega^{k.1}(\partial\Sigma)$, actually one has $\mathrm{n}\omega \in \Omega^k(\Sigma)|_{\partial\Sigma}$.)

Remark 3: *According to [19, p. 171] the tangential and normal maps can be extended to continuous surjective maps*

$$\mathrm{t} \oplus \mathrm{n} \colon \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \oplus \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \qquad \forall \ell \geq \frac{1}{2}\,. \tag{10}$$

We can now recall the Hodge decomposition for compact manifolds with boundary [31, Thm. 2.4.2].

Theorem 4 ([31])**:** *Let (Σ, h) be a compact, connected, Riemannian manifold with non-empty boundary $\partial\Sigma \overset{\iota_{\partial\Sigma}}{\hookrightarrow} \Sigma$.*

1. *For all $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma)$ it holds*

$$(\mathrm{d}_\Sigma\omega, \eta)_\Sigma - (\omega, \delta_\Sigma\eta)_\Sigma = (\mathrm{t}\omega, \mathrm{n}\eta)_{\partial\Sigma}\,, \tag{11}$$

where $(\ ,\ )_\Sigma$ has been defined in equation (4) while $(\ ,\ )_{\partial\Sigma}$ is defined similarly. Equation (11) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma)$ – cf. remark 3.

2. *There is an orthogonal decomposition*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^1\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) \oplus \mathcal{H}^k(\Sigma)\,, \tag{12}$$

where $\mathcal{H}^k(\Sigma)$ is defined as per equation (6) and

$$\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma)|\ \mathrm{t}\alpha = 0\}\,, \qquad \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma)|\ \mathrm{n}\beta = 0\}\,. \tag{13}$$

Remark 5: *The previous decomposition generalizes to Sobolev spaces, in particular for all $\ell \in \mathbb{N} \cup \{0\}$ we have*

$$\mathrm{H}^\ell\Omega^k(\Sigma) = \mathrm{d}_\Sigma \mathrm{H}^{\ell+1}\Omega^k_{\mathrm{t}}(\Sigma) \oplus \delta_\Sigma \mathrm{H}^{\ell+1}\Omega^{k+1}_{\mathrm{n}} \oplus \{\omega \in \mathrm{H}^{\ell+1}\Omega^k(\Sigma)|\ \mathrm{d}_\Sigma\omega = 0\,,\ \delta_\Sigma\omega = 0\}\,. \tag{14}$$

2.2.2 Hodge decomposition for compact manifold with interface

We would like to consider a decomposition similar to the one of theorem 4 for the case of a closed Riemannian manifold Σ with interface Z. In this setting we split $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$ and we refer to Σ_- (*resp.* Σ_+) as the left (*resp.* right) component of Σ. Notice that $\Sigma_\pm$ are compact manifolds with boundary $\partial\Sigma_\pm = \pm Z$ – notice that the orientation on Z induced by Σ_+ is the opposite of the one induced by Σ_-. Therefore theorem 4 applies to $\mathrm{L}^2\Omega^k(\Sigma_\pm)$.

Since Z has zero measure the space of square integrable k-forms splits as

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{L}^2\Omega^k(\Sigma_Z) = \mathrm{L}^2\Omega^k(\Sigma_+) \oplus \mathrm{L}^2\Omega^k(\Sigma_-)\,. \tag{15}$$

We expect a Z-relative Hodge decomposition as in (12) to hold true in this situation, where the boundary conditions of the spaces $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\mathrm{H}^1\Omega^{k-1}_{\mathrm{n}}(\Sigma)$ should be replaced by appropriate jump conditions across Z. For that, notice that the splitting (15) does not generalize to the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$, in particular

$$\mathrm{H}^\ell\Omega^k(\Sigma) \subset \mathrm{H}^\ell\Omega^k(\Sigma_Z) = \mathrm{H}^\ell\Omega^k(\Sigma_+) \oplus \mathrm{H}^\ell\Omega^k(\Sigma_-)\,, \tag{16}$$

is a proper inclusion.

(Is it clear that the objects associated with Σ_Z are automatically a direct sum of objects associated with $\Sigma_\pm$?)

Definition 6: *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. For $\omega \in C^\infty\Omega^k(\Sigma_Z)$ we define the tangential jump $[\mathrm{t}\omega] \in C^\infty\Omega^k(Z)$ and normal jump $[\mathrm{n}\omega] \in C^\infty\Omega^{k-1}(Z)$ across Z by*

$$[\mathrm{t}\omega] := \mathrm{t}_+\omega - \mathrm{t}_-\omega\,, \qquad [\mathrm{n}\omega] := \mathrm{n}_+\omega - \mathrm{n}_-\omega\,, \tag{17}$$

where $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ denote the tangential and normal map on $\Sigma_\pm$ as per definition 1.

Remark 7: *It is an immediate consequence of definition 6 that*

$$\mathrm{H}^1\Omega^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma_Z)|\ [\mathrm{t}\omega] = 0\,,\ [\mathrm{n}\omega] = 0\}\,. \tag{18}$$

The same equality does not hold for $C^\infty\Omega^k(\Sigma)$ because higher order traces have to match at Z.

Theorem 8: *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$.*

1. *For all $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ it holds*

$$(\mathrm{d}_\Sigma\omega, \eta)_Z - (\omega, \delta_\Sigma\eta)_Z = ([\mathrm{t}\omega], \mathrm{n}_+\eta)_Z - (\mathrm{t}_-\omega, [\mathrm{n}\eta])_Z\,, \tag{19}$$

   where $(\ ,\ )_Z$ is the scalar product between forms on Z – cf. equation (4) – while $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ are the tangential and normal maps on $\Sigma_\pm$ as per definition 1. Equation (11) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma)$ for all $\ell \geq 1$.

2. *There is an orthogonal decomposition*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma)\,, \tag{20}$$

   where we defined $\mathcal{H}^k(\Sigma)$ as per equation (6) and

$$\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma_Z)|\ [\mathrm{t}\alpha] = 0\} \qquad \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma_Z)|\ [\mathrm{n}\beta] = 0\}\,. \tag{21}$$

Proof. Equation (19) is an immediate consequence of (11). In particular for $\omega \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_Z)$ and $\eta \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)$ we decompose $\omega = \omega_+ + \omega_-$ and $\eta = \eta_+ + \eta_-$ where $\omega_\pm \in C^\infty_{\mathrm{c}}\Omega^{k-1}(\Sigma_\pm)$ and $\eta_\pm \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_\pm)$. (Notice that with this notation we have $\mathrm{t}_\pm\omega = \mathrm{t}_\pm\omega_\pm$.) Applying equation (11) we have

$$\begin{aligned}(\mathrm{d}_\Sigma\omega, \eta) - (\omega, \delta_\Sigma\eta) &= \sum_\pm \big((\mathrm{d}_\Sigma\omega_\pm, \eta_\pm) - (\omega_\pm, \delta_\Sigma\eta_\pm)\big) = \int_Z \mathrm{t}_+\overline{\omega} \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *_\Sigma\mathrm{n}_-\eta \\ &= \int_Z [\mathrm{t}\overline{\omega}] \wedge *_\Sigma\mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge *[\mathrm{n}\beta]\,.\end{aligned}$$

A density argument leads to the same identity for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for $\ell \geq 1$.
We now prove the splitting (20). The spaces $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)$, $\delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$, $\mathcal{H}^k(\Sigma)$ are orthogonal because of equation (19). Let now ω be in the orthogonal complement of $\mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. We wish to show that $\omega \in \mathcal{H}^k(\Sigma)$. We split $\omega = \omega_+ + \omega_-$ with $\omega_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)$ and apply theorem 4 to each component so that

$$\omega = \sum_\pm \left(\mathrm{d}_\Sigma\alpha_\pm + \delta_\Sigma\beta_\pm + \kappa_\pm\right),$$

where $\alpha_\pm \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_\pm)$, $\beta_\pm \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_\pm)$ and $\kappa_\pm \in \mathcal{H}^k(\Sigma_\pm)$. Let now be $\hat{\alpha} \in \mathrm{H}^1\Omega^{k-1}(\Sigma_+)$: this defines an element in $\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ by considering its extension by zero on Σ_-. Since $\omega \perp \mathrm{d}_\Sigma\mathrm{H}^1\Omega_{[\mathrm{t}]}(\Sigma_Z)$ we have $0 = (\mathrm{d}_\Sigma\hat{\alpha}, \omega) = (\mathrm{d}_\Sigma\hat{\alpha}, \mathrm{d}_\Sigma\alpha_+)$, thus $\mathrm{d}_\Sigma\alpha_+ = 0$ by the arbitrariness of $\hat{\alpha}$. With a similar argument we have $\alpha_- = 0$ as well as $\beta_\pm = 0$.
Therefore $\omega \in \mathcal{H}^k(\Sigma_Z)$. In order to prove that $\omega \in \mathcal{H}^k(\Sigma)$ we need to show that $[\mathrm{t}\omega] = 0$ as well as $[\mathrm{n}\omega] = 0$ – *cf.* remark 7. This is a consequence of $\omega \perp \mathrm{d}_\Sigma\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. Indeed, let $\alpha \in \mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$: applying equation (19) we find

$$0 = (\mathrm{d}_\Sigma\alpha, \omega) = -\int_Z \mathrm{t}_-\overline{\alpha} \wedge *[\mathrm{n}\omega]\,. \tag{22}$$

The arbitrariness of $\mathrm{t}_-\alpha$ implies $[\mathrm{n}\omega] = 0$. Similarly $[\mathrm{t}\omega] = 0$ follows by $\omega \perp \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. □

Remark 9: *The harmonic part of decomposition* (20) *contains harmonic k-forms which are continuous across the interface Z – cf. remark 7. One can also consider a decomposition which allows for a discontinuous harmonic component: in particular it can be shown that*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}_\Sigma\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z) \oplus \delta_\Sigma\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma_Z)\,,$$

where now $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)$ *is the subspace of* $\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ *made of* $(k-1)$*-forms* α *such that* $\mathrm{t}_\pm\omega = 0$ *and similarly* $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)$ *if and only if* $\beta \in \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$ *and* $\mathrm{n}_\pm\beta = 0$.

Remark 10: *The results of theorem 4 generalize in several directions [3, 4, 18, 21, 25, 26, 31, 30, 33]. For the case of a non-compact Riemannian manifold* Σ *one may follow the results of [4] in order to achieve the following weak-Hodge decomposition – cf. equation* (12)*. We consider the operators* $\mathrm{d}_{\Sigma,\mathrm{t}}, \delta_{\Sigma,\mathrm{n}}$ *defined by*

$$\mathrm{dom}(\mathrm{d}_{\Sigma,\mathrm{t}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ \mathrm{t}\omega = 0\} \qquad \mathrm{d}_{\Sigma,\mathrm{t}}\omega := \mathrm{d}_\Sigma\omega\,, \tag{23}$$

$$\mathrm{dom}(\delta_{\Sigma,\mathrm{n}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ \mathrm{n}\omega = 0\} \qquad \delta_{\Sigma,\mathrm{n}}\omega := \delta_\Sigma\omega\,. \tag{24}$$

Notice that $\mathrm{d}_{\Sigma,\mathrm{t}}$ *as well as* $\delta_{\Sigma,\mathrm{n}}$ *are nihilpotent because of relations* (9)*. These operators are closed and from equation* (11) *it follows that their adjoints are the following:*

$$\mathrm{dom}(\mathrm{d}_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\}\,, \qquad \delta^*_{\Sigma,\mathrm{n}} = \mathrm{d}_\Sigma\,,$$
$$\mathrm{dom}(\delta_\Sigma) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\}\,, \qquad \mathrm{d}^*_{\Sigma,\mathrm{t}} = \delta_\Sigma\,.$$

It then follows immediately that $(\overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})})^\perp = \ker(\mathrm{d}) \cap \ker\delta = \mathcal{H}^k(\Sigma)$ *so that*

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,\mathrm{t}})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,\mathrm{n}})} \oplus \mathcal{H}^k(\Sigma)\,. \tag{25}$$

Following the same steps of proof of theorem 8 it follows that a similar weak-Hodge decomposition holds for the case of non-compact Riemannian manifolds Σ *with interface* Z*, actually*

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d}_{\Sigma,[\mathrm{t}]})} \oplus \overline{\mathrm{Ran}(\delta_{\Sigma,[\mathrm{n}]})} \oplus \mathcal{H}^k(\Sigma)\,, \tag{26}$$

where $\mathrm{d}_{\Sigma,[\mathrm{t}]}, \delta_{\Sigma,[\mathrm{n}]}$ *are defined by*

$$\mathrm{dom}(\mathrm{d}_{\Sigma,[\mathrm{t}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}_\Sigma\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ [\mathrm{t}\omega] = 0\} \qquad \mathrm{d}_{\Sigma,[\mathrm{t}]}\omega := \mathrm{d}_\Sigma\omega\,,$$
$$\mathrm{dom}(\delta_{\Sigma,[\mathrm{n}]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta_\Sigma\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ [\mathrm{n}\omega] = 0\} \qquad \delta_{\Sigma,[\mathrm{n}]}\omega := \delta_\Sigma\omega\,.$$

This time $\mathrm{d}^*_{\Sigma,[\mathrm{t}]} = \delta_{\Sigma,[\mathrm{n}]}$ *as well as* $\delta^*_{\Sigma,[\mathrm{n}]} = \mathrm{d}_{\Sigma,[\mathrm{t}]}$ *so that in particular* $\ker \mathrm{d}^*_{\Sigma,[\mathrm{t}]} \cap \ker\delta_{\Sigma,[\mathrm{n}]} = \mathcal{H}^k(\Sigma)$*.*
(The notation is sloppy, in principle $\mathrm{d}_{\Sigma,\mathrm{t}}, \delta_{\Sigma,\mathrm{n}}$ ***depend on*** k***.)***

Remark 11 (Non-dynamical Maxwell equations)**:** *The Hodge decomposition with interface proved in theorem 8 can be exploited to formulate the correct generalization of the non-dynamical equations* (3b)*. Actually in what follows we will substitute equations* (3b) *with the requirement*

$$E, B \perp \mathrm{d}_\Sigma \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{27}$$

Notice that this entails $\delta_\Sigma E = \delta_\Sigma B = 0$ *as well as* $[\mathrm{n}E] = [\mathrm{n}B] = 0$*. Configurations of the electric field* E *in the presence of a charge density* ρ *on* $\Sigma_\pm$ *and a surface charge density* ρ_Z *over* Z *are described by expanding* $E = \mathrm{d}_\Sigma\alpha + \delta_\Sigma\beta + \kappa$ *and demanding* $\alpha \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$ *to satisfy*

$$(\mathrm{d}_\Sigma\varphi, \mathrm{d}_\Sigma\alpha)_\Sigma = (\varphi, \rho)_\Sigma + (\varphi, \rho)_Z \qquad \forall\varphi \in C^\infty_{\mathrm{c}}(\Sigma)\,.$$

For sufficiently regular α *this is equivalent to the Poisson problem* $\Delta_\Sigma\alpha = \rho$, $[\mathrm{n}\mathrm{d}_\Sigma\alpha] = \rho_Z$*.*

2.3 Dynamical equations: boundary triples

In this section we will deal with the dynamical equations (3a). These can be easily regarded as a Schrödinger equation and solved by imposing suitable boundary conditions on Z. Actually we can rewrite equations (3a) in Schrödinger form

$$i\partial_t\psi = H\psi \qquad \psi := \begin{bmatrix} E \\ B \end{bmatrix}, \qquad H := \begin{bmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{bmatrix}, \tag{28}$$

Here we adopt the convention of [7] according to which

$$\mathrm{curl} := i *_\Sigma \mathrm{d}_\Sigma \quad \text{if } \dim\Sigma = 1 \mod 4\,, \qquad \mathrm{curl} := *_\Sigma\mathrm{d}_\Sigma \quad \text{if } \dim\Sigma = 3 \mod 4\,. \tag{29}$$

Which this convention curl is formally selfadjoint on $C^\infty_{\mathrm{c}} H^1\Omega^1(\Sigma)$.

As in section 2.2 we wish to consider equation (28) on Σ_Z, allowing for jump discontinuities across the interface Z. For that we regard H as a densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)^{\times 2} = \mathrm{L}^2\Omega^1(\Sigma_Z)^{\times 2}$ with domain

$$\mathrm{dom}(H) := C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_+) \oplus C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_-)\,, \tag{30}$$

where $C^\infty_{\mathrm{cc}}\Omega^1(\Sigma_\pm)$ denotes the subspace of $C^\infty_{\mathrm{c}}\Omega^1(\Sigma_\pm)$ with support in $\Sigma_\pm \setminus \partial\Sigma_\pm$. The operator H is closable and symmetric, its adjoint H^* being defined by

$$\mathrm{dom}(H^*) = \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\, H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} \qquad H^*\psi := H\psi\,. \tag{31}$$

Equation (28) is solved by selecting a self-adjoint extension of H. The latter can be parametrized by Lagrangian subspaces of a suitable complex symplectic space – [15, 16, 17].

Definition 12: *Let S be a complex vector space and let $\sigma\colon \mathsf{S}\times\mathsf{S} \to \mathbb{C}$ a bilinear map. The pair (S,σ) is called complex symplectic space if σ is non-degenerate – i.e. $\sigma(x,y) = 0$ for all $y \in \mathsf{S}$ implies $x = 0$ – and $\sigma(x,y) = -\overline{\sigma(y,x)}$ for all $x,y \in \mathsf{S}$. A subspace $L \subseteq \mathsf{S}$ is called Lagrangian subspace if $L = L^\perp := \{x \in \mathsf{S} |\, \sigma(x,y) = 0\ \forall y \in L\}$.*

For convenience we summarize the major results in the following theorem:

Theorem 13 ([15])**:** *Let H a separable Hilbert space and let $A\colon \mathrm{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ be a symmetric operator. Then the bilinear map*

$$\sigma(x,y) := (A^*x, y) - (x, A^*y)\,, \qquad \forall x,y \in \mathrm{dom}(A^*)\,, \tag{32}$$

satisfies $\sigma(x,y) = -\overline{\sigma(y,x)}$. It also descends to the quotient space $\mathsf{S}_A := \mathrm{dom}(A^*)/\,\mathrm{dom}(A)$ and the pair (S_A,σ) is a complex symplectic space as per definition 12. Moreover, for all Lagrangian subspace $L \subseteq \mathsf{S}_A$ – cf. definition 12 – the operator

$$A_L := A^*|_{L+\mathrm{dom}(A)}\,, \tag{33}$$

defines a self-adjoint extension of A – here $L + \mathrm{dom}(A)$ denotes the pre-image of L with respect to the projection $\mathrm{dom}(A^*) \to \mathsf{S}_A$. Finally the map

$$\{\textit{Lagrangian subspaces } L \textit{ of } \mathsf{S}_A\} \ni L \mapsto A_L \in \{\textit{self-adjoint extensions of } A\}\,, \tag{34}$$

is one-to-one.

Example 14: *As a concrete example of theorem 13 we discuss the case of the self-adjoint extensions of the* curl *operator on a closed manifold Σ with interface Z. For simplicity we assume that $\dim\Sigma = 2k+1$ with $\dim\Sigma = 3 \mod 4$, while* curl *is defined according to (29). We consider the operator curl_Z defined by*

$$\mathrm{dom}(\mathrm{curl}_Z) := \overline{C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z)}^{\|\|_{\mathrm{curl}}}\,, \qquad \mathrm{curl}_Z\, u := \mathrm{curl}\, u\,. \tag{35}$$

Notice that $C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z) = C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_+) \oplus C^\infty_{\mathrm{cc}}\Omega^k(\Sigma_-)$. The adjoint curl_Z^* of curl_Z is

$$\mathrm{dom}(\mathrm{curl}_Z^*) = \mathrm{dom}(\mathrm{curl}_+) \oplus \mathrm{dom}(\mathrm{curl}_-)\,, \tag{36}$$

$$\mathrm{dom}(\mathrm{curl}_\pm) := \{u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm) |\ \mathrm{curl}_\pm u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)\}\,, \qquad \mathrm{curl}_Z\, u := \mathrm{curl}\, u\,. \tag{37}$$

Since complex conjugation commutes with curl*, it follows from Von Neumann's criterion [28, Thm. 5.43] that* curl_Z *admits self-adjoints extensions. We now provide a description of the complex symplectic space* $\mathsf{S}_{\mathrm{curl}_Z} := (\mathrm{dom}(\mathrm{curl}_Z^*)/\,\mathrm{dom}(\mathrm{curl}_Z), \sigma_Z)$ *whose Lagrangian subspaces allows to characterize all self-adjoint extensions of* curl_Z*. According to theorem 13 the symplectic structure* σ_Z *on the vector space* $\mathsf{S}_{\mathrm{curl}_Z}$ *is defined by*

$$\sigma_Z(u,v) := (\mathrm{curl}_Z^*\, u, v) - (u, \mathrm{curl}_Z^*\, v)\,, \qquad \forall u,v \in \mathrm{dom}(\mathrm{curl}_Z^*)\,. \tag{38}$$

In particular for $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ *and* $v \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ *we have*

$$\sigma(u,v) = \sum_{\pm} \pm \int_Z \overline{\mathrm{t}_\pm u} \wedge \mathrm{t}_\pm v = \sum_{\pm} \mp {}_{-\frac{1}{2}}\langle \mathrm{t}_\mp u, *_Z \mathrm{t}_\mp v\rangle_{\frac{1}{2}} \tag{39}$$

$$= (\gamma_1 u, \gamma_0 v) - (\gamma_0 u, \gamma_1 v)\,, \qquad \gamma_0 u := \frac{1}{\sqrt{2}} *_Z [\mathrm{t}u]\,, \qquad \gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)\,., \tag{40}$$

where ${}_{-\frac{1}{2}}\langle\ ,\ \rangle_{\frac{1}{2}}$ *denotes the pairing between* $\mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ *and* $\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$*. In particular this shows that* $\mathrm{t}_\pm u \in \mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ *for all* $u \in \mathrm{dom}(\mathrm{curl}_Z^*)$ *– cf. [2, 12, 19, 29, 32] for more details on the trace space associated with the* curl*-operator on a manifold with boundary.* ***Provide more details.***

According to theorem 13 all self-adjoint extensions of curl_Z *are in one-to-one correspondence to the Lagrangian subspaces of* $\mathsf{S}_{\mathrm{curl}_Z}$*. Unfortunately a complete characterization of all Lagrangian subspaces of* $\mathsf{S}_{\mathrm{curl}_Z}$ *is not at disposal. We content ourself to present a family of Lagrangian subspaces – a generalization of the results presented in [24] may provide other examples. For* $\theta \in \mathbb{R}$ *let*

$$L_\theta := \{u \in \mathrm{dom}(\mathrm{curl}_Z^*) |\ \mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u\}\,, \tag{41}$$

where $[\mathrm{t}u] = \mathrm{t}_+ u - \mathrm{t}_- u$ *denotes the tangential jump – cf. definition 6, remark 3 and equation* (39)*. To show that* L_θ *are Lagrangian subspaces let* $u, v \in L_\theta$ *and let* $v_n \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ *be such that* $\|v - v_n\|_{\mathrm{curl}} \to 0$*. In particular* $\|(\mathrm{t}_+ - e^{i\theta}\mathrm{t}_-)v\|_{\mathrm{H}^{\frac{1}{2}}\Omega^k(\Sigma_Z)} \to 0$ *so that*

$$\sigma_Z(u,v) = \lim_n \sigma_Z(u, v_n) = -\lim_n {}_{-\frac{1}{2}}\langle \mathrm{t}_+ u, *_Z(\mathrm{t}_+ v_n - e^{i\theta}\mathrm{t}_- v_n)\rangle_{\frac{1}{2}} = 0\,. \tag{42}$$

It follows that $L_\theta \subseteq L_\theta^\perp$*. Conversely if* $u \in L_\theta^\perp$ *let consider* $v \in C^\infty_{\mathrm{c}}\Omega^k(\Sigma_Z) \cap L_\theta$*. Since* $u \in L_\theta^\perp$ *we find*

$$0 = \sigma_Z(u,v) = -{}_{-\frac{1}{2}}\langle \mathrm{t}_+ u - e^{i\theta}\mathrm{t}_- u, *_Z \mathrm{t}_+ v\rangle_{\frac{1}{2}}\,.$$

Since $\mathrm{t}_+ \colon C^\infty_{\mathrm{c}}\Omega^k(\Sigma_+) \to C^\infty_{\mathrm{c}}\Omega^k(Z)$ *is surjective it follows that* $\mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u$*.*

Notice that the self-adjoint extension obtained for $\theta = 0$ *coincides with the closure of* curl *on* $C^\infty_{\mathrm{c}}(\Sigma)$ *which is known to be self-adjoint by [7, Lem. 2.6].*
(Here we exploited the compactness property. We may deal with non-compact manifolds too because [7, Lem. 2.6] is based on point (i) of [7, Lem. 2.3] which holds true also in this setting.) *Indeed, since* $[\mathrm{t}]$ *is continuous we have* $\mathrm{dom}(\overline{\mathrm{curl}}) \subseteq L_0$ *so that* curl_{Z,L_0} *is a self-adjoint extension of* $\overline{\mathrm{curl}}$*. Since the latter operator is already self-adjoint we have equality among the two.*

We conclude this section by introducing an exact sequence which provides a complete description of the solution space of the Maxwell equations (3a) with interface Z. For that the following standard definition is in order **[Cit.]**

Definition 15: *In the hypothesis of theorem 19 let* $\Theta \colon \mathsf{h} \to \mathsf{h}$ *be a self-adjoint operator and consider the self-adjoint extension* H_Θ *as per theorem 19 and let* $\mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z) := \bigcap_{\ell\geq 1}\mathrm{dom}(H_\Theta^\ell)$*. We define the following subspaces of* $C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$*:*

- *the space* $C^\infty_{\rm sfc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$ *of future-compact functions, made of those* $f \in C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$ *such that* $J^-(x) \cap \mathrm{spt}(f)$ *is compact for all* $x \in M$*. Here* $J^-(x)$ *denotes the causal past of* $x \in M = \mathbb{R}\times\Sigma$ *according to the causal structure induced by* $g = -\mathrm{d}t^2 + h$*;*

- *the space* $C^\infty_{\rm spc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$ *of past-compact functions, made of those* $f \in C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$ *such that* $J^+(x) \cap \mathrm{spt}(f)$ *is compact for all* $x \in M$*. Here* $J^+(x)$ *denotes the causal future of* $x \in M = \mathbb{R}\times\Sigma$ *according to the causal structure induced by* $g = -\mathrm{d}t^2 + h$*;*

- *the space of timelike-compact functions defined by*

$$C^\infty_{\rm tc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) = C^\infty_{\rm sfc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \cap C^\infty_{\rm spc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \,.$$

(Perhaps it is worth to recall somewhere the notion of advanced-retarded fundamental solutions?)

Proposition 16: *Let* (Σ, h) *be a closed manifold with interface* $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$*. Moreover let* $(\Sigma_\pm, h_\pm)$ *the oriented, compact Riemannian manifolds with boundary* $\partial\Sigma_\pm = \pm Z$ *such that* $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$*. Let* H *be the densely defined operator on* $\mathrm{L}^2\Omega^1(\Sigma)$ *with domain defined by* (30) *and let* H^* *be its adjoint, defined as in* (31)*. Let* $(\mathsf{h}, \gamma_0, \gamma_1)$ *be the boundary triple associated with* H^* *as described in proposition 22 and let* $\Theta\colon \mathsf{h} \to \mathsf{h}$ *be a self-adjoint operator and consider the self-adjoint extension* H_Θ *as per theorem 19. Let* $G^\pm_\Theta$ *be the operators* $G^\pm_\Theta\colon C^\infty_{\rm tc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \to C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))$ *defined by*

$$(G^\pm_\Theta\omega)(t) = \int_{\mathbb{R}} \theta(\pm(t-s))e^{-i(t-s)H_\Theta}\omega(s)\mathrm{d}s \,. \tag{43}$$

The the operator G^+_Θ *(resp.* G^-_Θ*) is an advanced (resp. retarded) solution of* $i\partial_t + H_\Theta$*, that is, it holds*

$$(i\partial_t + H_\Theta)\circ G^\pm_\Theta|_{C^\infty_{\rm tc}(\mathbb{R},\mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))} = \mathrm{Id}_{C^\infty_{\rm tc}(\mathbb{R},\mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))} \,, \tag{44}$$

$$G^\pm_\Theta\circ(i\partial_t + H_\Theta)|_{C^\infty_{\rm tc}(\mathbb{R},\mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))} = \mathrm{Id}_{C^\infty_{\rm tc}(\mathbb{R},\mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z))} \,. \tag{45}$$

Moreover, let $G_\Theta := G^+_\Theta - G^-_\Theta$*. Then the following is a short exact sequence*

$$\begin{aligned} 0 \to C^\infty_{\rm tc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) &\overset{i\partial_t + H_\Theta}{\to} C^\infty_{\rm tc}(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \\ &\overset{G_\Theta}{\to} C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \overset{i\partial_t + H_\Theta}{\to} C^\infty(\mathbb{R}, \mathrm{H}^\infty_\Theta\Omega^1(\Sigma_Z)) \to 0 \,. \end{aligned} \tag{46}$$

Proof. Most of it is an analogue of [13, Thm. 30- Prop. 36]. The finite speed of propagation follows from [23, 27]. □

Remark 17: *Notice that the exact sequence* (46) *implies that the space of smooth solution of the dynamical equations* (3a) *is isomorphic as a vector space to the image of* G_Θ*.*

2.4 Algebra of observables for the Faraday tensor

Standard construction plus isotony result.

3 Boundary triples

boundary triples, which we recall briefly in the following – *cf.* [8] and references there in.

Definition 18: *Let* H *a separable Hibert space and let* $A\colon \operatorname{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ *be a symmetric operator. A boundary triple for* A^* *is a triple* $(\mathsf{h}, \gamma_0, \gamma_1)$ *made of an Hilbert space* h *and a pair of linear maps* γ_0, γ_1 *such that* $\gamma_0 \oplus \gamma_1\colon \operatorname{dom}(A^*) \to \mathsf{h} \oplus \mathsf{h}$ *is continuous and surjective and moreover*

$$(A^*x, y)_{\mathsf{H}} - (x, A^*y)_{\mathsf{H}} = (\gamma_1 x, \gamma_0 y)_{\mathsf{h}} - (\gamma_0 x, \gamma_1 y)_{\mathsf{h}} \qquad \forall x, y \in \operatorname{dom}(A^*)\,, \tag{47}$$

where $(\ ,\ )_{\mathsf{H}}$, $(\ ,\ )_{\mathsf{h}}$ *denotes the scalar product on* H *and* h *respectively.*

We refer to [8] for a extensive treatment of boundary triples. For the application we have in mind we only need the following properties of boundary triples which we recollect in the following

Theorem 19: *Let* H *a separable Hibert space and let* $A\colon \operatorname{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ *be a symmetric operator. Let* $(\mathsf{h}, \gamma_0, \gamma_1)$ *be a boundary triple for* A^* *as per definition 18 and let* Θ *be a closed relation on* $\mathsf{h} \times \mathsf{h}$.

1. *the operator* $A_\Theta\colon \operatorname{dom}(A_\Theta) \subseteq \mathsf{H} \to \mathsf{H}$ *defined by*

$$\operatorname{dom}(A_\Theta) := \ker(\gamma_1 - \Theta\gamma_1) \qquad A_\Theta := A^*|_{\operatorname{dom}(A_\Theta)}\,, \tag{48}$$

   is a closed extension of A;

2. $A_\Theta^* = A_{\Theta^*}$ *; in particular* A_Θ *is selfadjoint if and only if* Θ *is selfadjoint ;*

3. *the assignment* $\Theta \mapsto A_\Theta$ *leads to a bijective map*

$$\{\text{closed relations } \Theta\} \ni \Theta \mapsto A_\Theta \in \{\text{closed extensions of } A\}\,. \tag{49}$$

We now present a boundary triple for the operator H defined in (30).

Remark 20: *In the following we shall exploit a few basic facts which we shall recall here for convenience. First of all the map* $[\mathrm{t}]\colon \mathrm{H}^\ell\Omega^1(\Sigma_Z) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^1(Z)$ *– cf. remark 3 – extends to a continuous surjection*

$$[\mathrm{t}]\colon \operatorname{dom}(H^*) \to \Omega^1_{\mathrm{Tr}}(Z) := \{\omega \in \mathrm{H}^{-\frac{1}{2}}\Omega^1(Z) |\ \delta_Z\omega \in \mathrm{H}^{-\frac{1}{2}}\Omega^0(Z)\}\,. \tag{50}$$

Details of this construction can be found in [2, 29, 19, 32] for the case of a Riemannian manifold with boundary. The space $\Omega^1_{\mathrm{Tr}}(Z)$ *is an Hilbert space with scalar product*

$$(\omega_1, \omega_2)_{\Omega^1_{\mathrm{Tr}}(Z)} := (\omega_1, \omega_2)_{\mathrm{H}^{-\frac{1}{2}}\Omega^1(Z)} + (\delta_Z\omega_1, \delta_Z\omega_2)_{\mathrm{H}^{-\frac{1}{2}}\Omega^1(Z)}\,.$$

For later convenience we introduce a pair of isomorphisms $\iota_\pm\colon \mathrm{H}^{\pm\frac{1}{2}}\Omega^1(\Sigma_Z) \to \Omega^1_{\mathrm{Tr}}(Z)$ *defined by*

$$(\iota_+\omega_+, \iota_-\omega_-)_{\Omega^1_{\mathrm{Tr}}(Z)} = {}_{\frac{1}{2}}\langle \omega_+, \omega_-\rangle_{-\frac{1}{2}} \qquad \forall \omega_\pm \in \mathrm{H}^{\pm\frac{1}{2}}\Omega^1(\Sigma_Z)\,, \tag{51}$$

where ${}_{\frac{1}{2}}\langle\ ,\ \rangle_{-\frac{1}{2}}$ *denotes the dual pairing between* $\mathrm{H}^{\frac{1}{2}}\Omega^1(\Sigma_Z)$ *and* $\mathrm{H}^{-\frac{1}{2}}\Omega^1(\Sigma_Z)$ *– cf. [8].*

Remark 21: *Let consider the self-adjoint extension* $H_\infty := H^*|_{\operatorname{dom}(H_\infty)}$ *where*

$$\operatorname{dom}(H_\infty) := \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\ H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} = \mathrm{H}^1\Omega^1(\Sigma)^{\times 2}\,, \tag{52}$$

where in the second equality we exploited [31, Thm. 3.1.1]. The self-adjoint extension H_∞ *coincides with the unique self-adjoint extension of the operator* H *defined on the domain* $C^\infty_{\mathrm{c}}\Omega^1(\Sigma)$ *and corresponds to* *the standard Hamiltonian operator for the system* (28) *in the case of empty interface Z. The spectrum of the latter operator on closed manifolds is known [7] in particular $\pm\lambda \in \sigma(H_\infty)$ if and only if $\lambda \in \sigma(\mathrm{curl})$ where* curl *is the (essentially selfadjoint) curl operator* (29) *defined on $C_c^\infty\Omega^k(\Sigma)$. In particular $\sigma(H_\infty)$ has no continuous spectrum and its point spectrum consists of the eigenvalue* 0 *(with infinite multiplicity) and the discrete spectrum (with finite multiplicity) – cf. [7, Thm. 3.1]. Moreover, on account of [13, Lem. 21] – see also [8] – we have that for all $\lambda \in \mathbb{R} \cap \rho(H_\infty)$ it holds*

$$\mathrm{dom}(H^*) = \mathrm{dom}(H_\infty) \dotplus \ker(H^* - \lambda)\,, \tag{53}$$

where $\cdot$ denotes algebraic sum.
(Decomposition* (53) *needs some information about $\sigma(H_\infty)$ – e.g. the fact that $\sigma(H_\infty) \neq \mathbb{R}$ – which are proved in [7] for compact manifolds.)

Proposition 22: *Let (Σ, h) be a closed manifold with interface $Z \overset{\iota_Z}{\hookrightarrow} \Sigma$. Moreover let $(\Sigma_\pm, h_\pm)$ the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = \pm Z$ such that $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. Let H be the densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)$ with domain defined by* (30) *and let H^* be its adjoint, defined as in* (31). *Finally let H_∞ be the self-adjoint extension defined in* (52) *and consider $\lambda \in \mathbb{R}\cap\rho(H_\infty)$. Then the following is a boundary triple for the adjoint operator H^* defined as in* (31)*:*

$$\mathsf{h} := \Omega^1_{\mathrm{Tr}}(Z)^{\times 2} \qquad \gamma_0 \begin{bmatrix} E \\ B \end{bmatrix} := \frac{i}{\sqrt{2}} \begin{bmatrix} \iota_- *_\Sigma [\mathrm{t}B] \\ \iota_- *_\Sigma [\mathrm{t}E] \end{bmatrix}, \qquad \gamma_1 \begin{bmatrix} E \\ B \end{bmatrix} := \frac{i}{\sqrt{2}} \begin{bmatrix} \iota_+ *_\Sigma (\mathrm{t}_- + \mathrm{t}_+) B_\infty \\ \iota_+ *_\Sigma (\mathrm{t}_- + \mathrm{t}_+) E_\infty \end{bmatrix}. \tag{54}$$

Here $E_\infty, B_\infty \in \mathrm{dom}(H_\infty)$ denote the components of E, B according to the decomposition (53) *– cf. remark 21 – while $\iota_\pm$ have been introduced in* (51). *Finally $\mathrm{t}_\pm$ denote the tangential traces $\Sigma_\pm$ as per definitions 1 while $[\mathrm{t}E]$ is the tangential jump as per definition 6.*

Proof. The proof follows [8, Prop. 6.9]. First of all notice that γ_0, γ_1 are well-defined, continuous and surjective – *cf.* remark 20. Moreover, an explicit computation involving equation (11) shows that for all $\omega_1 = \begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \in \mathrm{dom}(H^*)$ and $\eta = \begin{bmatrix} E_2 \\ B_2 \end{bmatrix} \in \mathrm{dom}(H_\infty)$ it holds

$$(H^*\omega, \eta) - (\omega, H^*\eta) = -{}_{-\frac{1}{2}}\left\langle \frac{i}{\sqrt{2}} \begin{bmatrix} \iota_- *_\Sigma [\mathrm{t}B_1] \\ \iota_- *_\Sigma [\mathrm{t}E_1] \end{bmatrix}, \frac{i}{\sqrt{2}} \begin{bmatrix} \iota_+ *_\Sigma (\mathrm{t}_- + \mathrm{t}_+) B_2 \\ \iota_+ *_\Sigma (\mathrm{t}_- + \mathrm{t}_+) E_2 \end{bmatrix} \right\rangle_{\frac{1}{2}} = -(\gamma_0\omega, \gamma_1\eta)_{\Omega^1_{\mathrm{Tr}}(Z)}\,. \tag{55}$$

The latter equality is exploited to prove equation (47). Indeed for $\omega_j = \begin{bmatrix} E_j \\ B_j \end{bmatrix} \in \mathrm{dom}(H^*)$, $j \in \{1, 2\}$, we decompose $\omega_j = \omega_{j,\eta} + \omega_{j,\infty}$ according to decomposition (53) – *cf.* remark 20. Then, exploiting the self-adjointness of H_∞ we have

$$\begin{aligned} (H^*\omega_1, \omega_2) - (\omega_1, H^*\omega_2) &= (H^*\omega_{1,\eta}, \omega_{2,\infty}) + (H^*\omega_{1,\infty}, \omega_{2,\eta}) - (\omega_{1,\eta}, H^*\omega_{2,\infty}) - (\omega_{1,\infty}, H^*\omega_{2,\eta}) \\ &= (\gamma_1\omega_1, \gamma_0\omega_2)_{\Omega^1_{\mathrm{Tr}}(Z)} - (\gamma_0\omega_1, \gamma_1\omega_2)_{\Omega^1_{\mathrm{Tr}}(Z)}\,, \end{aligned}$$

where in the last equality we used equation (55). □

Remark 23: *According to theorem 19, the results of proposition 22 – see also remark 7 – implies that the self-adjoint extension $H^*|_{\ker\gamma_0}$ coincides with the operator H_∞.*
Not really true because of the normal jump!

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