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On the role of boundary conditions in the construction of fundamental solutions for Maxwell's equations on spacetimes with timelike boundary

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"*The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen.* "

Paul A.M. Dirac

Abstract

Studiamo le equazioni di Maxwell su spazitempi globalmente iperbolici con bordo di tipo tempo, dapprima in termini del tensore di Faraday F in presenza di un'interfaccia tra due mezzi e successivamente in termini del potenziale vettore A, ridefinendo opportunamente il significato di invarianza di gauge. Con metodi di analisi funzionale e teoria degli operatori su spazi di Hilbert, costruiamo le soluzioni fondamentali per una particolare classe di condizioni al contorno in uno spaziotempo ultrastatico per l'operatore delle onde. Basandoci su di esse, descriviamo lo spazio delle soluzioni delle equazioni di Maxwell. In vista di possibili applicazioni alla teoria quantistica dei campi algebrica, costruiamo infine l'algebra delle osservabili associata, studiandone le proprietà.

We study Maxwell's equations on globally hyperbolic spacetimes with timelike boundary, at first formulated in terms of the electromagnetic tensor F in presence of an interface between two media and secondly formulated in terms of the vector potential A, redefining properly the notion of gauge invariance. With methods from functional analysis and operator theory on Hilbert spaces, we construct fundamental solutions for a particular class of boundary conditions in an ultrastatic background for the wave operator. Relying on their existence, we characterize the space of solutions of Maxwell's equations. In view of possible applications in algebraic quantum field theory, we build the algebra of observables associated, outlining its properties.

Contents

To my family

Introduction

Algebraic quantum field theory is a mathematically rigorous framework that allows to quantize classical fields. It does not give a quantum theory from first principles, but it is a technique that, whenever one is able to construct a classical relativistic field theory, permits to characterize the observables and the states of the associated quantum system. Moreover, this framework is suitable for a generalization to curved backgrounds. When dealing with quantum field theories, a formulation in terms of a fixed Hilbert space is inadequate, since on the one hand the system possesses an infinite number of degrees of freedom and on the other hand on curved spacetimes there is no Poincaré invariance to rely on. Hence, in the algebraic formulation, a physical system is described no more by self-adjoint operators on a Hilbert space, but by self-adjoint elements of an abstract $*$-algebra called the *algebra of observables*, which encodes the collection of physical quantities that can be measured on the system.

Our aim in this thesis is to build the algebra of observables for a free electromagnetic system in a suitable class of curved spacetimes with boundary. This is based on the construction of the classical Maxwell field in terms of differential k-forms, proving the existence of distinguished advanced and retarded fundamental solutions for the Maxwell operator under suitable boundary conditions.

A fundamental solution or Green operator G for a differential operator P is an *inverse* of P with prescribed support properties. It allows to solve the equation $Pu = f$ for any compactly supported source f. More precisely, fundamental solutions for P acting on sections of a vector bundle E are defined as $\mathrm{G} : \Gamma_c(E) \to \Gamma(E)$ satisfying

$$P \circ \mathrm{G} = \mathrm{G} \circ P = \mathrm{id}|_{\Gamma_c(E)} \,.$$

We plan to construct explicitly, for each choice of boundary conditions, two operators $\mathrm{G}^\pm$ for Maxwell operator, called *advanced* and *retarded* fundamental solutions, such that $\mathrm{supp}\,(\mathrm{G}^\pm(f)) \subseteq J^\pm(\mathrm{supp}(f))$ for any f, where the symbols $J^\pm$ denote the causal future $(+)$ and the causal past $(-)$. These conditions ensures a *finite speed of propagation*.
Analyzing the support properties of the Green operators gives us information about the propagation of initial data. Moreover fundamental solutions are important for the construction of (causal) propagators, that allow to implement quantum commutation relations at the level of the algebra of observables for a quantum field theory.

We will construct such operators choosing physically meaningful classes of boundary conditions, namely those that ensure that the flux of physically relevant quantities, such as those built from the stress-energy tensor, through the boundary is vanishing. This, combined with the requirement of unitary evolution, translates mathematically in the condition of self-adjointness of the operators involved.

From a geometric point of view, we will focus on globally hyperbolic spacetimes with timelike boundary (M, g), which have been introduced in a very recent paper by Aké, Flores and Sanchez – *cf.* [AFS18]. They are the natural class of spacetimes where boundary conditions can be assigned. In the *ultrastatic* case, when there exists a global irrotational timelike Killing field, M splits smoothly as a product $\mathbb{R} \times \Sigma$, where the metric admits a decomposition $g = -\mathrm{d}t^2 + h$, where Σ is a Riemannian manifold with boundary such that the Cauchy surface $\{t\} \times \Sigma$ has Riemannian metric h for any time t.

We focus initially on homogeneous Maxwell equations for the Faraday 2-form F.
Maxwell's equations read

$$\begin{cases} \mathrm{d}F = 0 \\ \delta F = 0, \end{cases} \tag{1}$$

where d, δ are the exterior derivative and the codifferential, respectively, while $F \in \Omega^2(M)$ is the Faraday 2-form, which in a static case has the following decomposition in terms of electric and magnetic time-dependent differential forms $E \in C^\infty(\mathbb{R}, \Omega^1(\Sigma))$ and $B \in C^\infty(\mathbb{R}, \Omega^2(\Sigma))$:

$$F = B + \mathrm{d}t \wedge E. \tag{2}$$

In the second chapter of this thesis we will be able to construct fundamental solutions to Maxwell's equations for F with prescribed conditions at an interface between two media, which will be regarded as two different globally hyperbolic spacetimes with timelike boundary.

Otherwise, Maxwell's equations can be written in terms of a generic vector potential $A \in \Omega^k(M)$, $0 < k < \dim M$ that is locally defined as a primitive of $F \in \Omega^{k+1}(M)$, in other words $F = \mathrm{d}A$ locally. In this case the equations of motion are $\delta\mathrm{d}A = 0$ and we have to take into account the *gauge invariance* of the theory and the interplay between the gauge freedom and the choice of boundary conditions on a globally hyperbolic spacetime with timelike boundary.
At first, we will prove the existence of fundamental solutions for the D'alembert–de Rham wave operator $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ in this framework for a certain class of boundary conditions in static spacetimes. Usually, in solving Maxwell's equations in a spacetime with no boundary, one works in the so called *Lorenz gauge* $\delta A = 0$ so to recast the problem into an hyperbolic form $\Box A = 0$. We shall employ the same technique, which is available only for two restricted classes of boundary conditions for A that we named $\delta\mathrm{d}$-tangential (vanishing tangential component) and $\delta\mathrm{d}$-normal (vanishing normal derivative). Relying on these results we will be able to characterize the space

of solutions of Maxwell's equations, but it will become clear that two distinct notions of gauge invariance have to be defined for each of the aforementioned boundary conditions.

In conclusion, as we mentioned earlier, we construct the algebra of observables for Maxwell's equations for the vector potential A under those boundary conditions. We will prove that the algebras so constructed are physically sound: they are *optimal*. This means that they contain enough and no more elements to distinguish between different configurations of the field.
Furthermore, we will show that, in analogy with the case without boundary, the algebra possesses a non-trivial centre: This topological obstruction is a feature which is common in Abelian gauge theories such as electromagnetism and it translates the impossibility to interpret such models as *locally covariant quantum field theories*. In fact, electromagnetism is not a local theory: It possesses non-local observables which measure the electric flux through surfaces that include monopoles. – *cf.* [SDH14].

The thesis is organized as follows.
The first chapter is devoted to establishing the geometric and analytic framework in which we will work. In particular, we define globally hyperbolic spacetimes with timelike boundary and we recall the notion of differential forms. Subsequently, we give an account of Sobolev spaces for Riemannian manifolds of bounded geometry. Then we recall the definition of fundamental solutions (or Green operators) and we give some example. In conclusion we state Maxwell's equations and we outline the problems that we tackle in the following chapters.

In the second chapter we analyze Maxwell's equations for the field strength F in a spacetime with a codimension 1 interface $Z \subset \Sigma$ between two media, regarded as manifolds with timelike boundary. We will separate the equations in a non-dynamical part, that will be treated using the so-called Hodge decomposition (*cf.* [Sch95]) and a dynamical part, whose fundamental solutions are constructed using a technique from functional analysis, namely that of *Lagrangian subspaces*. These allow to discriminate physically sound interface conditions. In the end, we give an account of the possible extensions of the results to the construction of the algebra of quantum observables for F.

The third and final chapter starts with the proof of the existence of advanced and retarded fundamental solutions for $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ operating on k-forms in ultrastatic spacetimes with timelike boundary using a functional analysis technique called *boundary triples*, under certain classes of boundary conditions. We apply our results to identify the space of solutions of $\delta\mathrm{d}A = 0$ under the aforementioned $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions. In addition we distinguish two different notions of gauge invariance and we show that within the two gauge equivalence classes is always possible to find a representative that abides Lorenz gauge. We give an account of the notion of algebra of observables within the framework of algebraic quantum field theory and we identify, for the two boundary conditions, the optimal algebras.

Part of the content of this thesis has appeared as an independent publication, available on the ArXiv: [DDL19].

Chapter 1

Geometric preliminaries

In this chapter, we begin by recalling the basic definitions in order to fix the geometric setting in which we will work.
Globally hyperbolic spacetimes (M, g) are used in context of geometric analysis and mathematical relativity because in them there exists a smooth and spacelike Cauchy hypersurface Σ and that ensures the well-posedness of the Cauchy problem. Moreover, as shown by Bernal and Sánchez [BS05, Th. 1.1], in such spacetimes there exists a splitting for the full spacetime M as an orthogonal product $\mathbb{R} \times \Sigma$. These results corroborate the idea that in globally hyperbolic spacetimes one can preserve the notion of a global passing of time. In a globally hyperbolic spacetime, the entire future and past history of the universe can be predicted from conditions imposed at a fixed instant represented by the hypersurface Σ.

1.1 Globally Hyperbolic spacetimes with timelike boundary

The main goal of this section is to analyse the main properties of globally hyperbolic spacetimes and to generalise them to a natural class of spacetimes where boundary values problems can be formulated. This class is that of globally hyperbolic spacetimes with timelike boundary.While, in the case of $\partial M = \emptyset$ global hyperbolicity is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18].

Manifolds with boundary. From now on M will denote a smooth connected oriented manifold of dimension $m > 1$ with boundary. M is then locally diffeomorphic to open subsets of the closed half space of $\mathbb{R}^n$. We will assume that the boundary ∂M, which is the set of points for which all neighbourhoods are diffeomorphic to the closed half space of $\mathbb{R}^n$, is smooth and, for simplicity, connected. A point $p \in M$ such that there exists an open neighbourhood U containing p diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $n - 1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$.
In addition we endow M with a smooth Lorentzian metric g of signature $(-, +, ..., +)$ so that

$\iota^* g$ identifies a Lorentzian metric on ∂M and we require (M, g) to be time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M, g) has a *timelike boundary*.

For any $p \in M$, we denote by $J^+(p)$ the set of all points that can be reached by future-directed causal smooth curves emanating from p. For any subset $A \subset M$ we set $J^+(A) := \bigcup_{p\in A} J^+(p)$. If A is closed so is $J_+(A)$. We denote by $I^+(p)$ the set of all points in M that can be reached by future-directed timelike curves emanating from p. The set $I^+(p)$ is the interior of $J^+(p)$; in particular, it is an open subset of M. Interchanging the roles of future and past, we similarly define $J^-(p)$, $J^-(A)$, $I^-(p)$, see

Definition 1.1.1.

- *A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.*
- *A spacetime with timelike boundary is* causal *if it possesses no closed, causal curve.*
- *A causal spacetime with timelike boundary M such that for all $p, q \in M$ $J^+(p) \cap J^-(q)$ is compact is called* globally hyperbolic.

These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 1.1.2. *Let (M, g) be a spacetime of dimension m. Then*

1. *(M, g) is a globally hyperbolic spacetime with timelike boundary if and only if it possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve,*
2. *if (M, g) is globally hyperbolic, then it is isometric to $\mathbb{R} \times \Sigma$ endowed with the metric*

$$g = -\beta d\tau^2 + h_\tau, \tag{1.1}$$

where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1]*, whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a smooth Cauchy surface for (M, g).*

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M, g), we work directly with (1.1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M.

[1] Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

Definition 1.1.3. *A spacetime with boundary* (M, g) *is* static *if it possesses a nowhere vanishing irrotational timelike Killing vector field* $\chi \in \Gamma(TM)$ *whose restriction to* ∂M *is tangent to the boundary, i.e.* $g_p(\chi, \nu) = 0$ *for all* $p \in \partial M$ *where* ν *is the unit vector, normal to the boundary at* p.

Remark 1.1.4. A spacetime with boundary (M, g) is *stationary* if we do not require neither the Killing vector χ nor its restriction to the boundary to be irrotational.

Locally, every stationary or static Lorentzian manifold looks like the corresponding standard one with metric (1.1) with $\chi = \partial_\tau$. Hence the static property translates into the request that both β and h_τ are independent from τ.

Definition 1.1.5. *We call* standard static *a static spacetime with timelike boundary* (M, g) *isometric to* $(\mathbb{R} \times \Sigma, -\beta\, \mathrm{d}t^2 + h)$, *where* Σ *is a Riemannian manifold with boundary endowed with a metric* h *and* $\beta \in C^\infty(\Sigma, (0, \infty))$.

Corollary 1.1.6. *(see [DDF19, Cor. 2]) Let* (M, g) *be a standard static spacetime with timelike boundary. Then also* ∂M *is a standard static spacetime (with empty boundary), endowed with the induced metric.*

Example 1.1.7. We consider some examples of globally hyperbolic spacetimes without boundary ($\partial M = \emptyset$).

- The Minkowski spacetime $\mathbb{M}^m = (\mathbb{R}^m, \eta)$ is static and globally hyperbolic. Every spacelike hyperplane (of co-dimension 1) is a Cauchy hypersurface. We have $\mathbb{M}^m = \mathbb{R} \times \Sigma$ with $\Sigma = \mathbb{R}^{m-1}$, endowed with the time-independent Euclidean metric.

- Let Σ be a Riemannian manifold with time independent metric h and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times \Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\, h$, called *cosmological spacetime*, is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold, see [BGP07, Lem A.5.14]. This applies in particular if (Σ, h) is compact.

- The interior and exterior *Schwarzschild spacetimes*, that represent non-rotating black holes of mass $\mathrm{m} > 0$ are globally hyperbolic. Denoting S^2 the 2-dimensional sphere embedded in $\mathbb{R}^3$, we set
$$M_{\text{ext}} := \mathbb{R} \times (2\mathrm{m}, +\infty) \times S^2,$$
$$M_{\text{int}} := \mathbb{R} \times (0, 2\mathrm{m}) \times S^2.$$
The metric is given by
$$g = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\, g_{S^2},$$

where $f(r) = 1 - \frac{2\mathrm{m}}{r}$, while $g_{S^2} = r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2$ is the metric in polar coordinates on the sphere. In particular, the exterior Schwarzschild spacetime is *static* and we have $M_{\text{ext}} = \mathbb{R}\times\Sigma$ with $\Sigma = (2\mathrm{m}, +\infty)\times S^2$, $\beta = f$ and $h = \frac{1}{f(r)}\mathrm{d}r^2 + r^2\,g_{S^2}$.

■

Example 1.1.8. Now we consider some examples of globally hyperbolic spacetimes with timelike boundary in which the boundary is not empty.

- The half Minkowski spacetime $\mathbb{M}^m_+ = (\mathbb{R}^{m-1}\times[0, +\infty), \eta)$ is static and globally hyperbolic. Every spacelike half-hyperplane (of co-dimendion 1) is a Cauchy hypersurface. We have $\mathbb{M}^m = \mathbb{R}\times\Sigma$ with $\Sigma = \mathbb{R}^{m-2}\times[0, +\infty)$, endowed with the time-independent Euclidean metric.

- Let Σ be a Riemannian manifold with boundary with time independent metric h and let $I\subset\mathbb{R}$ an interval. Let $f: I\to\mathbb{R}$ be a smooth positive function. The manifold $M = I\times\Sigma$ with the metric $g = -\mathrm{d}t^2 + f^2(t)\,h$ is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold with boundary.

■

A particular role will be played by the support of the functions that we consider. In the following definition we introduce the different possibilities that we will consider - *cf.* [Bär15].

Definition 1.1.9. *Let (M, g) be a Lorentzian spacetime with timelike boundary and let $E\to M$ be a finite rank vector bundle on M. We denote with*

1. *$\Gamma_{\mathrm{c}}(E)$ the space of smooth sections of E with compact support in M while with $\Gamma_{\mathrm{cc}}(E)\subset\Gamma_{\mathrm{c}}(M)$ the collection of smooth and compactly supported sections f of E such that $\mathrm{supp}(f)\cap\partial M = \emptyset$.*

2. *$\Gamma_{\mathrm{spc}}(E)$ (resp. $\Gamma_{\mathrm{sfc}}(E)$) the space of strictly past compact (resp. strictly future compact) sections of E, that is the collection of $f\in\Gamma(E)$ such that there exists a compact set $K\subseteq M$ for which $J^+(\mathrm{supp}(f))\subseteq J^+(K)$ (resp. $J^-(\mathrm{supp}(f))\subseteq J^-(K)$), where $J^\pm$ denotes the causal future and the causal past in M. Notice that $\Gamma_{\mathrm{sfc}}(E)\cap\Gamma_{\mathrm{spc}}(E) = \Gamma_{\mathrm{c}}(E)$.*

3. *$\Gamma_{\mathrm{pc}}(E)$ (resp. $\Gamma_{\mathrm{fc}}(E)$) denotes the space of future compact (resp. past compact) sections of E, that is, $f\in\Gamma(E)$ for which $\mathrm{supp}(f)\cap J^-(K)$ (resp. $\mathrm{supp}(f)\cap J^+(K)$) is compact for all compact $K\subset M$.*

4. *$\Gamma_{\mathrm{tc}}(E) := \Gamma_{\mathrm{fc}}(E)\cap\Gamma_{\mathrm{pc}}(E)$, the space of timelike compact sections.*

5. *$\Gamma_{\mathrm{sc}}(E) := \Gamma_{\mathrm{sfc}}(E)\cap\Gamma_{\mathrm{spc}}(E)$, the space of spacelike compact sections.*

1.2 Differential forms and operators on manifolds with boundary

To treat Maxwell's equations properly and to be able to generalise them, we will use the language of differential forms. In this section (M, g) will denote a generic oriented pseudo-Riemannian manifold with boundary with signature $(-, +, \ldots, +)$ or $(+, \ldots, +)$. In the former case, when the manifold is Lorentzian, it is understood that the boundary is timelike in the sense of Definition 1.1.1. We present the following definitions in such a general framework since we will work both on spacetimes (M, g) with timelike boundary and on their Cauchy hypersurfaces (Σ, h), which are Riemannian manifolds with boundary on account of Theorem 1.1.2.

On top of a pseudo-Riemannian Hausdorff, connected, oriented and paracompact manifold (M, g) with boundary we consider the spaces of complex valued k-forms $\Omega^k(M)$, with $k \in \mathbb{N} \cup \{0\}$, as smooth sections of $\Lambda^k T^*M$. Since (M, g) is oriented, we can identify a unique, metric-induced, Hodge operator $\star : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \mathrm{d}\mu_g$, where $\wedge$ is the exterior product of forms and $\mathrm{d}\mu_g$ the metric induced volume form. We endow $\Omega^k(M)$ with the standard, metric induced, pairing

$$(\alpha, \beta) := \int_M \overline{\alpha} \wedge \star\beta \,, \tag{1.2}$$

Remark 1.2.1. In case $E = \Lambda^k T^*M$, the spaces with support properties defined in Definition 1.1.9 will be denoted respectively by the following spaces of k-forms: $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{cc}}(M)$, $\Omega^k_{\mathrm{spc/sfc}}(M)$, $\Omega^k_{\mathrm{pc/fc}}(M)$, $\Omega^k_{\mathrm{tc/sc}}(M)$. If the regularity required for any of these spaces is different than smoothness, it will be denoted putting it in front of the space. For example, the space of square integrable k-forms will be indicated with $\mathrm{L}^2\Omega^k(M)$.

We indicate the exterior derivative with $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. A differential form α is called closed when $\mathrm{d}\alpha = 0$ and exact when $\alpha = \mathrm{d}\beta$ for some differential form β. Since M is endowed with a pseudo-Riemannian metric it holds that, when acting on smooth k-forms, $\star^{-1} = (-1)^{k(m-k)+\sigma_M}\star$, where σ_M is the signature of g. Combining these data we define the *codifferential* operator $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta \doteq \star^{-1} \circ \mathrm{d} \circ \star$.
To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its projection along the direction normal to ∂M.

Remark 1.2.2. Since we will be considering not only forms lying in $\Omega^k(M)$, $k \in \mathbb{N} \cup \{0\}$, but also those in $\Omega^k(\partial M)$, we shall distinguish the operators acting on this space with a subscript $_\partial$, *e.g.* d_∂, $\star_\partial$, δ_∂ or $(,)_\partial$.

Definition 1.2.3. *Let (M, g_M) be a smooth Lorentzian manifold and let $\iota_N : N \to M$ be a codimension 1 smoothly embedded submanifold of M with induced metric $g_N := \iota_N^* g_M$. We* *define the tangential and normal components relative to N as*

$$\mathrm{t}_N\colon \Omega^k(M) \to \Omega^k(N)\,, \qquad \omega \mapsto \mathrm{t}_N\omega := \iota_N^*\omega\,, \tag{1.3a}$$

$$\mathrm{n}_N\colon \Omega^k(M) \to \Omega^{k-1}(N)\,, \qquad \omega \mapsto \mathrm{n}_N\omega := \star_N^{-1}\mathrm{t}_N \star_M \omega\,, \tag{1.3b}$$

where $\star_M, \star_N$ denote the Hodge dual over M, N respectively. In particular, for all $k \in \mathbb{N}\cup\{0\}$ we define

$$\Omega^k_{\mathrm{t}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}_N\omega = 0\}\,, \qquad \Omega^k_{\mathrm{n}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}_N\omega = 0\}\,. \tag{1.4}$$

Similarly we will use the symbols $\Omega^k_{\mathrm{c},\mathrm{t_N}}(M)$ and $\Omega^k_{\mathrm{c},\mathrm{n_N}}(M)$ when we consider only smooth, compactly supported k-forms.

Remark 1.2.4. In this paper the rôle of N will be played often by ∂M. In this case, we shall drop the subscript form Equation (1.3), namely $\mathrm{t} \equiv \mathrm{t}_{\partial M}$ and $\mathrm{n} \equiv \mathrm{n}_{\partial M}$.

Remark 1.2.5. With reference to Definition 1.2.3, observe that the following linear map is surjective:

$$\Omega^k(M) \ni \omega \to (\mathrm{n}\omega, \mathrm{t}\omega, \mathrm{t}\delta\omega, \mathrm{n}\mathrm{d}\omega) \in \Omega^{k-1}(\partial M)\times\Omega^k(\partial M)\times\Omega^{k-1}(\partial M)\times\Omega^k(\partial M)\,.$$

Remark 1.2.6. The normal map $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ can be equivalently read as the restriction to ∂M of the contraction $\nu \lrcorner \omega$ between $\omega \in \Omega^k(M)$ and the vector field $\nu \in \Gamma(TM)|_{\partial M}$ which corresponds pointwisely to the outward pointing unit vector, normal to ∂M.

As last step, we observe that (1.3) together with (1.4) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N}\cup\{0\}$.

$$\star\,\delta = (-1)^k \mathrm{d}\star\,, \qquad \delta\star = (-1)^{k+1}\star\mathrm{d}\,, \tag{1.5a}$$

$$\star_\partial\,\mathrm{n} = \mathrm{t}\star\,, \quad \star_\partial\mathrm{t} = (-1)^k\mathrm{n}\star\,, \quad \mathrm{d}_\partial\mathrm{t} = \mathrm{t}\mathrm{d}\,, \quad \delta_\partial\mathrm{n} = -\mathrm{n}\delta\,. \tag{1.5b}$$

A notable consequence of (1.5b) is that, while on manifolds with empty boundary, the operators d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. Indeed, a direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial, \tag{1.6}$$

for all $\alpha \in \Omega^k(M), \beta \in \Omega^{k+1}(M)$ such that $\mathrm{supp}\,\alpha \cap \mathrm{supp}\,\mathrm{d}\beta$ and $\mathrm{supp}\,\alpha\cap\mathrm{supp}\,\delta\beta$ are compact and where the pairing in the right-hand side is the one associated to forms living on ∂M.

1.3 Bounded Geometry and associated functional spaces

We introduce both the geometric setting and the Sobolev functional spaces which will play akey role in 2 and . We will follow mainly the discussion of [DDF19] and [GS13].

Definition 1.3.1. *A Riemannian manifold (Σ, h) with empty boundary is called of* bounded geometry *if the injectivity radius*[2] $r_{inj}(\Sigma) > 0$ *and if* $T\Sigma$ *is of* totally bounded curvature, *that is* $\|\nabla^k R\|_{L^\infty(M)} < \infty$ *for all* $k \in \mathbb{N} \cup \{0\}$, R *being the scalar curvature and* ∇ *the Levi-Civita connection associated with* h.

In view of its definition, the injectivity radius of a manifold with non-empty boundary vanishes. Hence we must regard $\partial\Sigma$ as a submanifold of an extension with empty boundary of the Riemannian manifold Σ. This requires a notion of bounded geometry for a generic submanifold.

Definition 1.3.2. *Let (Σ, h) be a Riemannian manifold of bounded geometry and let $(Y, \iota_Y^* h)$ be a co-dimension k closed, embedded, smooth submanifold with an inward pointing, unit normal vector field ν, where $\iota_Y : Y \to \Sigma$ is the immersion map. We say that $(Y, \iota_Y^* h)$ is a bounded geometry submanifold if the following holds:*

- *the second fundamental form K_Y of Y in Σ together with all its covariant derivatives on Y is bounded,*
- *there exists $\varepsilon > 0$ such that the map $\varphi : Y \times (-\varepsilon, \varepsilon) \to \Sigma$ defined as $\phi(p, z) = \exp_p(z\nu|_p)$ is injective, where $\exp_p$ is the exponential map of Σ at p.*

We are ready to give a definition in case the boundary is non-empty.

Definition 1.3.3. *An n-dimensional Riemannian manifold (Σ, h) with non-empty boundary is of* bounded geometry *if there exists an n-dimensional Riemannian manifold $(\widehat{\Sigma}, \widehat{h})$ (with empty boundary) of bounded geometry such that $\Sigma \subset \widehat{\Sigma}$, $h = \widehat{h}|_\Sigma$ and $(\partial\Sigma, \iota_{\partial\Sigma}^* \widehat{h})$ is submanifold of $\widehat{\Sigma}$ of bounded geometry.*

We remark that all Riemannian manifolds with compact boundary meet the requirements of the former Definition. At the same time one can also consider non-compact boundaries such as the n-dimensional half-space $\mathbb{R}^n_+ = [0, +\infty) \times \mathbb{R}^{n-1}$ endowed with the standard Euclidean metric. To conclude, we study the interplay between the notion of Riemannian manifold with boundary and of bounded geometry and that of standard static Lorentzian manifold with timelike boundary, *cf.* Definition 1.1.5.

Proposition 1.3.4. *(*cf. *[DDF19, Prop. 9])*
Let (Σ, h) be a Riemannian manifold with boundary and of bounded geometry and let $(\widehat{\Sigma}, \widehat{h})$ be the empty-boundary extension of bounded geometry as in Definition 1.3.1. Then

[2] The injectivity radius $r_{\text{inj}}(p)$ at a point p of a Riemannian manifold is the largest radius for which the exponential map at p is a diffeomorphism. The injectivity radius of a Riemannian manifold is $r_{\text{inj}}(\Sigma) = \inf_{p\in\Sigma} r_{\text{inj}}(p)$.

1. *Every* $\beta \in C^\infty(\Sigma, (0, +\infty))$ *identifies an isometry class of standard static Lorentzian manifolds with timelike boundary* (cf. *Definition 1.1.5*),

2. *if in addition there exists* $\widehat{\beta} \in C^\infty(\widehat{\Sigma}, (0, +\infty))$ *such that* $\widehat{\beta}|_\Sigma = \beta$ *and* $\widehat{h}/\widehat{\beta}$ *identifies a complete Riemannian metric on* $\widehat{\Sigma}$ *then each representative* (M, g) *of the isometry class is a submanifold with boundary of a standard static globally hyperbolic spacetime* $(\widehat{M}, \widehat{g})$.

A manifold (M, g) *that satisfies the first condition will be called* static Lorentzian spacetime with timelike boundary and of bounded geometry

1.3.1 Sobolev spaces

We consider a finite rank complex vector bundle $E \to \Sigma$ endowed with a fiberwise Hermitian product $\langle\, ,\, \rangle_E$ and a product preserving connection ∇ built out of h.

Definition 1.3.5. *We say that a section* $u \in \Gamma(E)$ *is measurable if the function*

$$\Sigma \ni x \mapsto \langle u(x), u(x)\rangle_E,$$

is measurable with respect to the measure $\mathrm{d}\mu_h$ *and we denote the space of equivalence classes of almost everywhere equal measurable sections of* E *with* $\Gamma_{me}(E)$.
Moreover, a measurable section $u \in \Gamma_{me}(E)$ *lies in* $u \in \mathrm{L}^p(E)$ *if the function* $\Sigma \ni x \mapsto \langle u(x), u(x)\rangle_E^p$ *is integrable.*

Definition 1.3.6. *For all* $\ell \in \mathbb{N} \cup \{0\}$ *and* $p \in (1, \infty)$, *we define the Sobolev spaces*

$$\mathrm{H}^\ell_p\Gamma(E) = \left\{ u \in \Gamma_{me}(E) \,|\, \nabla^j u \in \mathrm{L}^p(E \otimes T^*\Sigma^{\otimes j}),\, j \leq \ell \right\}. \tag{1.7}$$

In case $p = 2$ *we denote the Sobolev spaces as* $\mathrm{H}^\ell\Gamma(E) := \mathrm{H}^\ell_2\Gamma(E)$.

Whenever $E = \Lambda^k T^*\Sigma$, i.e. $\Gamma(E)$ is the space of differential k-forms, we will use the notation $\Omega^k(\Sigma) := \Gamma(\Lambda^k T^*\Sigma)$ (in agreement with the definitions in Section 1.2) and $\mathrm{H}^\ell\Omega^k(\Sigma) := \mathrm{H}^\ell\Gamma(\Lambda^k T^*\Sigma)$.

Remark 1.3.7. The space $\mathrm{H}^\ell\Gamma(E)$ is an Hilbert space if endowed with the norm

$$\|u\|^2_{\mathrm{H}^\ell\Gamma(E)} = \sum_{j=0}^{\ell} \|\nabla^j u\|^2_{\mathrm{L}^2(E\otimes T^*\Sigma^{\otimes j})}. \tag{1.8}$$

The theory of these space has been thoroughly studied in the literature and for the case in hand we refer mainly to [GS13].

Remark 1.3.8. The space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ can be defined otherwise as the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ (see Section 1.2) with respect to the pairing $(\ ,\ )_\Sigma$ between k-forms

$$(\alpha, \beta)_\Sigma := \int_\Sigma \overline{\alpha} \wedge \star_\Sigma \beta \qquad \alpha, \beta \in \Omega^k_{\mathrm{c}}(\Sigma)\,, \tag{1.9}$$

where $\star_\Sigma$ is the Hodge dual on Σ.

Whenever a boundary is present, one can introduce the subspace $\mathrm{H}^\ell_0\Gamma(E) \subset \mathrm{H}^\ell\Gamma(E)$ defined as the completion of $\Gamma_{\mathrm{cc}}(E)$ (the space of compactly supported sections in the interior of M) with respect to the $\mathrm{H}^\ell\Gamma(E)$-norm. Whenever Σ is metric complete (for example, if Σ is a Riemannian manifold of bounded geometry, in particular if $\Sigma = \mathbb{R}^n$) the two spaces coincide: $\mathrm{H}^\ell_0\Gamma(E) = \mathrm{H}^\ell\Gamma(E)$.

1.3.2 Restrictions and trace maps for differential forms

Using *uniformly locally finite trivializations*, one can define, following [GS13, Def. 11], the real-exponent Sobolev spaces $\mathrm{H}^s_p\Gamma(E)$, with $s \in \mathbb{R}$.

Proposition 1.3.9. *Let (Σ, h) be a Riemannian manifold of bounded geometry with boundary. Then for every $\ell \geq \frac{1}{2}$ there exists a continuous surjective map*

$$\mathrm{res}_\ell : \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma), \tag{1.10}$$

that extends the restriction on $\Omega^k_{\mathrm{c}}(\Sigma)$, i.e. $\mathrm{res}_\ell\, \alpha = \alpha|_{\partial\Sigma}$ if $\alpha \in \Omega^k_{\mathrm{c}}(\Sigma)$.

Remark 1.3.10. In particular, according to [Geo79, p. 171] and [Wec04, Sec. 2], the tangential and normal maps defined in Definition 1.2.3 can be extended to continuous surjective maps

$$\mathrm{t} \oplus \mathrm{n} \colon \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \oplus \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma) \qquad \forall \ell \geq \frac{1}{2}\,. \tag{1.11}$$

1.4 Green operators

In this section we will follow mainly [Bär15]. Let $E_1, E_2 \to M$ be vector bundles over a globally hyperbolic spacetime (M, g) with $\partial M = \emptyset$. Let $P : \Gamma(M, E_1) \to \Gamma(M, E_2)$ be a linear differential operator.

Definition 1.4.1. *An* advanced Green operator *of P, or* advanced fundamental solution *for P, is a linear map $G^+ : \Gamma_{\mathrm{c}}(E_2) \to \Gamma(E_1)$ such that*

(i) $G^+ P = \mathrm{Id}_{\Gamma_{\mathrm{c}}(E_1)}$,

(ii) $PG^+ = \mathrm{Id}_{\Gamma_{\mathrm{c}}(E_2)}$,

(iii) $\mathrm{supp}(G^+ f) \subset J^+(\mathrm{supp}\, f)$, *for all $f \in \Gamma_{\mathrm{c}}(E_2)$.*

Analogously, a linear map $G^- : \Gamma_c(E_2) \to \Gamma(E_1)$ *is called a* retarded Green operator *of* P, *or* retarded fundamental solution *for* P *if* (i) *and* (ii) *hold, while it also holds*

(iii') $\operatorname{supp}(G^- f) \subset J^-(\operatorname{supp} f)$, *for all* $f \in \Gamma_c(E_2)$.

Definition 1.4.2. *The operator* P *is called* Green hyperbolic *if* P *and* P^t *possess advanced and retarded Green operator, where* $P^t : \Gamma(E_2^*) \to \Gamma(E_1^*)$, *known as the* formal dual *of* P, *is the unique linear differential operator such that*

$$(\varphi, Pf)_M = (P^t\varphi, f)_M, \quad \textit{i.e.} \quad \int_M \langle \varphi, Pf\rangle \, \mathrm{d}\mu_g = \int_M \left\langle P^t\varphi, f\right\rangle \mathrm{d}\mu_g, \tag{1.12}$$

for all $f \in \Gamma(E_1)$ *and* $\varphi \in \Gamma(E_2^*)$ *such that* $\operatorname{supp} f \cap \operatorname{supp} \varphi$ *is compact.*

Remark 1.4.3. If (M, g) has empty boundary, the Green operators of a Green hyperbolic operator P are unique, see [Bär15, Cor. 3.12]. If the spacetime has a boundary, the differential operators must be given together with boundary conditions. These conditions are encoded in the domain of the operator, that is replaced by the subset $\Gamma_{\text{b.c.}}(E_1) \subset \Gamma(E_1)$ of sections that satisfies the boundary conditions. Hence, in the case of non-empty boundary, the codomain $\Gamma(E_1)$ of G must be replaced, in Definitions 1.4.1 and 1.4.2, with the corresponding subspace $\Gamma_{\text{b.c.}}(E_1)$.

Example 1.4.4. An important example of Green-hyperbolic operators are the *wave operators*, or the *normally hyperbolic operators*. Locally they are of the form

$$P = g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a^j(x)\frac{\partial}{\partial x^j} + b(x), \tag{1.13}$$

where g^{ij} denote the components of the inverse metric tensor, while a_j and b are smooth functions of x. Physically relevant examples of such operators are the *d'Alembert wave operator* acting on scalars ($E_1 = E_1 = \mathbb{R}$) $P = \Box$ and the Klein-Gordon operator $P = \Box + m^2$, $m > 0$. Moreover, in case $E_1 = E_2 = \Lambda^k T^*M$, we have the *d'Alembert-De Rham-Beltrami operator* $P = \Box_k = \mathrm{d}\delta + \delta\mathrm{d}$ acting on k-forms as well as the *Proca operator* $P = \delta\mathrm{d}_k + m^2$ (for further discussions on the Proca field see [FP03]).
It is shown in [BGP07, Cor. 3.4.3] that if (M, g) is a globally hyperbolic spacetime with empty boundary, wave operators as well as their formal duals (since they are wave operator themselves) have retarded and advanced Green operators. Hence, they are Green hyperbolic. ■

Definition 1.4.5. *The operator* $G := G^+ - G^- : \Gamma_c(E_2) \to \Gamma(E_1)$ *is called the* causal propagator *or* advanced minus retarded *Green operator.*

Remark 1.4.6. Recalling Definition 1.1.9 and the support properties of $G^\pm$ in Definition 1.4.1, we see that Green operators of P are in fact linear maps between the following spaces:

$$G^+ : \Gamma_{\mathrm{c}}(E_2) \to \Gamma_{\mathrm{spc}}(E_1), \tag{1.14}$$
$$G^- : \Gamma_{\mathrm{c}}(E_2) \to \Gamma_{\mathrm{sfc}}(E_1), \tag{1.15}$$
$$G : \Gamma_{\mathrm{c}}(E_2) \to \Gamma_{\mathrm{sc}}(E_1). \tag{1.16}$$

Moreover, as shown in [Bär15, Thm. 3.8, Cor. 3.10, 3.11], there are unique continuous linear extensions of $G^\pm$:

$$\overline{G}_+ : \Gamma_{\mathrm{pc}}(E_2) \to \Gamma_{\mathrm{pc}}(E_1) \quad \text{and} \quad \overline{G}_- : \Gamma_{\mathrm{fc}}(E_2) \to \Gamma_{\mathrm{fc}}(E_1), \tag{1.17}$$
$$\widetilde{G}_+ : \Gamma_{\mathrm{spc}}(E_2) \to \Gamma_{\mathrm{spc}}(E_1) \quad \text{and} \quad \widetilde{G}_- : \Gamma_{\mathrm{sfc}}(E_2) \to \Gamma_{\mathrm{sfc}}(E_1). \tag{1.18}$$

Proposition 1.4.7 (see Cor. 3.9, [Bär15])**.** *Let P be a Green hyperbolic operator. Then there are no nontrivial solutions $u \in \Gamma(E_1)$ of $Pu = 0$ with past-compact or future-compact support. In other words if u has past-compact or future-compact support, $Pu = 0$ implies $u = 0$. Moreover, for any $f \in \Gamma_{\mathrm{pc}}(E_2)$ or $f \in \Gamma_{\mathrm{fc}}(E_2)$ there exists a unique $u \in \Gamma(E_1)$ solving $Pu = f$ and such that $\operatorname{supp}(u) \subset J^+(\operatorname{supp} f)$ or $\operatorname{supp}(u) \subset J^-(\operatorname{supp} f)$, respectively.*

Remark 1.4.8. The solutions $u^\pm$ of the equation $Pu = f$ with different support properties discussed in the former Proposition are given explicitly by $u^\pm = G^\pm(f)$. Hence u^+ is the unique solution to the following initial value problem:

$$\begin{cases} Pu = f \text{ in } M, \ f \in \Gamma_{\mathrm{pc}}(E_2), \\ u|_\Sigma = 0, \end{cases} \tag{1.19}$$

where $\Sigma \overset{\iota}{\hookrightarrow} M$ is any Cauchy surface that lies in the past of $\operatorname{supp} f$, i.e. $\iota(\Sigma) \subset J^-(\operatorname{supp} f)$. Analogously u^- is the unique solution with vanishing final data on any Cauchy surface in the future of $f \in \Gamma_{\mathrm{fc}}(E_2)$.

This discussion extends to the case of a spacetime with non-empty timelike boundary, particularly, Proposition 1.4.7 extends, provided the existence of Green operators for a specified boundary condition. In this case, for example $u^+ = G^+_{\mathrm{b.c.}}(f)$ is the solution to the initial data/boundary value problem

$$\begin{cases} Pu = f \text{ in } M, \ f \in \Gamma_{\mathrm{pc}}(E_2), \\ \text{boundary conditions on } \partial M, \\ u|_\Sigma = 0, \end{cases} \tag{1.20}$$

where, as before, Σ is any Cauchy surface such that $\Sigma \subset J^-(\operatorname{supp} f)$.

The following is an important theorem that will be generalized in case of non-empty timelike boundary. (see [BG12, Thm. 3.5])

Theorem 1.4.9. *Let G be the causal propagator of a Green-hyperbolic operator P on a spacetime with empty boundary. Then the following is an exact sequence:*

$$0 \longrightarrow \Gamma_{\mathrm{c}}(E_1) \xrightarrow{P} \Gamma_{\mathrm{c}}(E_2) \xrightarrow{G} \Gamma_{\mathrm{sc}}(E_1) \xrightarrow{P} \Gamma_{\mathrm{sc}}(E_2) \longrightarrow 0. \tag{1.21}$$

In the case of non-empty boundary, the existence of Green operators and all their properties must be proven for any suitable class of boundary conditions, and that will be the main focus of Chapters 2 and 3 when P is Maxwell operator.

Example 1.4.10. (Wave operator on $\mathbb{R} \times \mathbb{R}_+$)
We consider the problem of the existence and the construction of advanced and retarded Green operators of $\Box = -\partial_t^2 + \partial_x^2$ on $M = \mathbb{R} \times \mathbb{R}_+ \ni (t,x)$. Clearly M is a globally hyperbolic spacetime with timelike boundary, endowed with the usual Minkowski metric $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2$. The boundary is the set $\{(t,0),\, t \in \mathbb{R}\}$. Given some initial condition, the differential equation $\Box u = f$, with $f \in C^\infty(M)$, is well posed (i.e. there exists a unique solution) provided one requires u to satisfy some suitable boundary conditions. We construct explicitly the Green operators for $\Box$ on M with Dirichlet and Neumann boundary conditions using the Green operators for $\Box$ on $(\mathbb{R}^2, \eta)$, whose existence is well known. We recall that, for a scalar function u, Dirichlet, Neumann and Robin boundary conditions are obtained by imposing, respectively,

$$u|_{\partial M} = 0; \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} = 0; \quad u|_{\partial M} = f \left.\frac{\partial u}{\partial \nu}\right|_{\partial M}, \text{ for } f \in C^\infty(\partial M)\,,$$

ν being the vector field normal to ∂M.
Consequently, we define $\Box_D : C^\infty_D(M) \to C^\infty(M)$ and $\Box_N : C^\infty_N(M) \to C^\infty(M)$, with $C^\infty_D(M) := \{u \in C^\infty(M) \,|\, u|_{x=0} = 0\} = \Omega^0_{\mathrm{t}}(M)$ and $C^\infty_N(M) := \{u \in C^\infty(M) \,|\, \partial_x u|_{x=0} = 0\}$. The problem is to find the following advanced and retarded Green operators

$$G^\pm_D : C^\infty_{\mathrm{c}}(M) \to C^\infty_D(M), \quad G^\pm_N : C^\infty_{\mathrm{c}}(M) \to C^\infty_N(M). \tag{1.22}$$

As stated in [Bär15, Ex. 3.4], advanced and retarded Green operators for $\Box$ on $\mathbb{R}^2$ exist and have the following explicit expression

$$G^\pm(f)(t,x) = -\frac{1}{2} \int_{J^\mp_{\mathbb{R}^2}(t,x)} f(s,y)\, \mathrm{d}s\, \mathrm{d}y. \tag{1.23}$$

This expression entails that the integral kernel of $G^\pm$ (also known as Green function or fundamental solution) is $-\frac{1}{2}$ times the characteristic function of $\{(t,x,s,y) \in \mathbb{R}^4 \,|\, (s,y) \in J^\mp(t,x)\}$. The ansatz, based on the method of images ([Jac99, p. 480]), is that the Dirichlet and Neumann

Green operators will be respectively of the form

$$
\begin{aligned}
G_D^{\pm}(f)(t,x) &= G^{\pm}(f)(t,x) - G^{\pm}(f)(t,-x) = \\
&= -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y - \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M,\\
G_D^{\pm}(f)(t,x) &= G^{\pm}(f)(t,x) + G^{\pm}(f)(t,-x) = \\
&= -\frac{1}{2}\left[\int_{J^{\mp}_{\mathbb{R}^2}(t,x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y + \int_{J^{\mp}_{\mathbb{R}^2}(t,-x)} f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right], \text{ for } (t,x)\in M.
\end{aligned}
$$

It is a straightforward calculation to verify $G^{\pm}_{D/N}(f) \in C^{\infty}_{D/N}(M)$ (i.e. $G^{\pm}_D(f)(t,x)|_{x=0} = 0$ and $\partial_x G^{\pm}_D(f)(t,x)|_{x=0} = 0$), in addition the support properties still hold.
Focusing on the Dirichlet Green operators, they are constructed by imagining to extend the manifold M by reflection to be the entire $\mathbb{R}^2$ and, to enforce $G^{\pm}_D(f)$ to vanish on $x = 0$, add a negative reflected source $-f(t,-x)$. This gives the desired result. ■

1.5 Maxwell's equations for k-forms with empty boundary

We focus our attention on an m-dimensional spacetime (M, g) with empty boundary. Classically, electromagnetism is the theory of electric and magnetic fields E, B encoded in the Faraday 2-form F. The equations for $F \in \Omega^2(M)$ read

$$
\begin{aligned}
\mathrm{d}F &= 0,\\
\delta F &= -J,
\end{aligned}
\tag{1.24}
$$

where J is the co-exact current 1-form, which encodes the current conservation laws. Indeed, if M is static with $M = \mathbb{R}\times\Sigma$, the decomposition $F = B + \mathrm{d}t \wedge E$ holds, where $E \in C^{\infty}(\mathbb{R}, \Omega^1(\Sigma))$ and $B \in C^{\infty}(\mathbb{R}, \Omega^2(\Sigma))$, in agreement with the fact that the magnetic field B is usually referred to as a *pseudo-vector*.
The first equation imposes a geometric constraint: it ensures that the 2-form F is closed. Hence, in virtue of Poincaré lemma, whenever the second de Rham cohomology group $H^2(M)$ (see A.1) is trivial, there exists a global 1-form A such that $F = \mathrm{d}A$. One can object that the choice of $A \in \Omega^1(M)$ is not unique. Indeed if we assume M to be with empty boundary, the configuration $A' := A + \mathrm{d}\chi$, $\chi \in \Omega^0(M)$ is equivalent to A since it gives rise to the same Faraday tensor F. This freedom in the choice of A is extensively used and it is called *gauge freedom* or gauge invariance. In this case A, A' are said to be gauge-equivalent.
Thanks to gauge invariance we can therefore first write Maxwell's equations for A as $\delta\mathrm{d}A = -J$. Subsequently, taken any fixed $A \in \Omega^1(M)$, and imposing the so-called *Lorenz gauge*, one can substitute the problem $\delta\mathrm{d}A = -J$ with the following hyperbolic system of equations

$$
\begin{cases}
\Box A = -J,\\
\delta A = 0.
\end{cases}
\tag{1.25}
$$

where $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ is the wave operator. Moreover the second equation can be seen as a constraint called the *Lorenz gauge condition*. This system can be obtained by requiring a 1-form A', gauge-equivalent to A, to satisfy the Lorenz gauge condition $\delta A' = 0$. This is always possible in a globally hyperbolic spacetime with empty boundary since the equation $\Box\chi = \delta A$ has always at least a solution $\chi \in \Omega^0(M)$ for any fixed $A \in \Omega^1(M)$.

One could argue that the most general possible gauge transformation between A and A' is of the form $A' = A + \omega$ for a closed form $\omega \in \Omega^1(M)$. That is certainly true in the sense that the equations of motion (1.24) are unchanged by this transformation. Anyway we will refer to gauge-invariance exclusively in the sense previously defined since electromagnetism can be seen as an Abelian *gauge theory* with structure group $U(1)$. In this framework, the classical vector potential A is a principal connection on a principal $U(1)$-bundle E over M (for more details see [Nak90, Ch. 10]). Then we identify (this choice is non-unique) the connection A with a 1-form $A \in \Omega^1(M)$. Locally, this principal connection can be expressed as an $U(1)$-valued operator $D = \mathrm{d} + A$ and the Faraday field can be recovered as the curvature of this connection: $F = D \circ D$. A gauge transformation for A in this context is of the form

$$A' = g^{-1}Ag - ig^{-1}\mathrm{d}g, \tag{1.26}$$

for any $g \in C^\infty(M, U(1))$. The group of gauge transformations certainly includes $\mathrm{d}\Omega^0(M)$, since if we write $g = e^{i\chi}$, for $\chi \in \Omega^0(M)$, we recover $A' := A + \mathrm{d}\chi$.

Remark 1.5.1. In the homogeneous case ($J = 0$), one can generalize the Maxwell field to $F \in \Omega^k(M)$, imposing $\mathrm{d}F = 0$ and $\delta F = 0$ and the equation for $A \in \Omega^{k-1}(M)$ becomes $\delta\mathrm{d}A = 0$. In this case gauge freedom is understood as a transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-2}(M)$. It is worth noticing that in case $k = 0$ and $k = m$, the equations $\delta F = 0$ and $\mathrm{d}F = 0$ become, respectively, trivial.

From a physical point of view, one wonders whether it is A or it is F the observable field of the dynamical system. Hence one can regard electromagnetism as a theory for $F \in \Omega^2(M)$ or as a theory for a non-unique $A \in \Omega^1(M)$ wondering whether the initial and boundary value problem for Maxwell's equations is well-posed in both cases. The former will be covered in Chapter 2 and the latter in Chapter 3.

In many, but not all, practical physical situations, the triviality of $H^2(M)$ ensures that the description of electromagnetism in terms of F or of A is completely indistinguishable. There is in fact one particular physical effect that enlightens the true nature of electromagnetism as a theory for the potential 1-form A: this is the so-called *Aharonov-Bohm effect*. To discuss this effect we refer mostly to [SDH14, Ex. 3.1]. Consider indeed as a globally hyperbolic spacetime M the Cauchy development in the 4-dimensional Minkowski spacetime $\mathbb{M}^4$ of the time-fixed hypersurface $\{0\} \times \mathbb{R}^3$ with a cylinder surrounding the z-axis (which is given in cylindrical coordinates (t, r, φ, z) by $r \leq 1$) removed. The cylinder represents an infinitely long coil with a current running through it whose magnetic flux Φ gives rise outside the coil to a vanishing Faraday tensor

F but also to a non-vanishing vector potential which reads approximately $A_\Phi = \frac{\Phi}{2\pi}\mathrm{d}\varphi$.
In the Aharonov-Bohm experiment one sends quantum particles from one side to the other of the coil and measures a quantum phase shift proportional to the integral of A_Φ around a circular path that embraces the cylinder (see [Pes89] for an experimental description). This setup shows that, even if the Faraday tensor F vanishes outside the coil, there is still a measurable physical effect which depends on the vector potential A_Φ, which appears to be the true observable field. In particular this effect happens because A_Φ is closed, but not exact in M. Moreover A_Φ is not gauge equivalent to 0. From a topological point of view this corresponds to the fact that the first de Rham cohomology group with integer coefficients $H^1(M,\mathbb{Z}) \neq \{0\}$. Indeed $H^1(M)$ is spanned just by the vector potential $\mathrm{d}\varphi$. Whenever $H^1(M,\mathbb{Z})$ is trivial, the two descriptions with F and A are indistinguishable. For further discussions, see [BHS14].

At the same time, the formulation in terms of the field strength F has its advantages. Indeed in the Lagrangian formulation of the field theory, the Maxwell action

$$\mathcal{S}_{EM} = -\frac{1}{4}(F,F) = -\frac{1}{4}\int_M F \wedge \star F, \tag{1.27}$$

is invariant under a conformal scaling $g \mapsto \Omega^2 g$ of the metric. This consideration is useful when the underlying spacetime possesses a conformal boundary, such as asymptotically flat and AdS spacetimes. AdS are particular spacetimes with conformal timelike boundary. The study of quantum field theories and boundary conditions on AdS spacetime is motivated by the long-term ambition to understand in rigorous mathematical terms the AdS/CFT conjecture.

The next chapters will be devoted to tackling the problem of well-posedness of electromagnetism equations when the spacetime has non-empty timelike boundary.

Chapter 2

Maxwell's equations for the field strength and interface conditions

As outlined in Section 1.5, the form of Maxwell's equations allows us to use both F and A as variables with which we can describe electromagnetic phenomena. Whenever the second cohomology group $H^2(M)$ is trivial, the two theories are equivalent, since $F = \mathrm{d}A$.
In this chapter, we regard $F \in \Omega^2(M)$ as the physical dynamical variable which describes electromagnetism. This is not always true, whenever the first cohomology group with integer coefficients is non-trivial, as previously discussed in Section 1.5. The aim of this chapter is to present a technique which allows to characterize, in a class of manifolds with the presence of an interface between two media, the existence of fundamental solutions for Maxwell's equations, written in terms of the Faraday form $F \in \Omega^2(M)$. The presence of an interface on the one hand generalizes the idea of a timelike boundary, allowing to recover the geometric setting outlined in Chapter 1 if one side of the interface is a perfect insulator. On the other hand, in order to make use of geometric techniques such as Hodge decomposition, we will have to make several geometric assumptions which ensure global hyperbolicity, but unfortunately they lead to a loss in generality.

2.1 Geometrical setup

The physical and practical situation we want to approach is that of a manifold split into two parts, filled with two media, each of them with different electromagnetic properties. The two media will be separated by an hypersurface, on which our aim will be that of putting *jump conditions*.
We consider a globally hyperbolic, standard static Lorentzian manifold (M, g) with *empty boundary*, such that M can be decomposed as $\mathbb{R}\times\Sigma$, where the Cauchy hypersurface (Σ, h) is assumed to be a complete, connected, odd-dimensional, *closed* Riemannian manifold. Under these conditions, Σ is of *bounded geometry* (see 1.3.1). In this chapter, we denote with d_M, δ_M the differential and co-differential over M, while d, δ denote those over Σ.
Maxwell's equations read

$$\mathrm{d}_M F = 0\,, \quad \delta_M F = 0\,, \quad F \in \Omega^2(M). \tag{2.1}$$

Given the decomposition $M \simeq \mathbb{R} \times \Sigma$ and recalling Theorem 1.1.2, let us indicate with $\iota_t \colon \Sigma \to M$ the smooth one-parameter group of embedding maps which realizes Σ at time t as $\iota_t \Sigma = \{t\} \times \Sigma \doteq \Sigma_t$. Then, it holds the diffeomorphism $\Sigma_t \simeq \Sigma_{t'} \simeq \Sigma$ for all $t, t' \in \mathbb{R}$. Hence, for all $\omega \in \Omega^k(M)$ and $t \in \mathbb{R}$, $\omega|_{\Sigma_t} \in \Gamma(\iota_t^* \Lambda^k T^* M)$, where $\iota_t^*(\Lambda^k T^* M)$ denotes the pull-back bundle over $\Sigma_t \simeq \Sigma$ built out of $\Lambda^k T^* M$ via ι_t – *cf.* [Hus66]. Moreover, recalling Definition 1.2.3, it holds that $\omega|_{\Sigma_t}$ can be further decomposed as

$$\omega|_{\Sigma_t} := (\star_{\Sigma_t}^{-1} \iota_t^* \star_M)\omega \wedge \mathrm{d}t + \iota_t^* \omega = \mathrm{n}_{\Sigma_t}\omega \wedge \mathrm{d}t + \mathrm{t}_{\Sigma_t}\omega \,.$$

where $\mathrm{t}_{\Sigma_t}\omega \in \Omega^k(\Sigma_t)$ while $\mathrm{n}_{\Sigma_t}\omega \in \Omega^{k-1}(\Sigma_t)$ – *cf.* Definition 1.2.3. With the identification $\Sigma_t \simeq \Sigma_{t'}$ the decomposition induces the isomorphisms

$$\begin{aligned} \Gamma(\iota_\tau^* \Lambda^k T^* M) &\simeq \Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma) \\ \omega &\to (\omega_0 \oplus \omega_1)\,, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \Omega^k(M) &\simeq C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k(\Sigma)) \\ \omega &\to (\tau \mapsto \mathrm{n}_{\Sigma_\tau}\omega) \oplus (\tau \mapsto \mathrm{t}_{\Sigma_\tau}\omega)\,. \end{aligned} \tag{2.3}$$

In this way, we have rewritten any differential form as a pair of differential form valued functions of time.

To recover the electric and magnetic components of F, we simply define $E \doteq -\mathrm{t}_{\Sigma_t} F$ and $\star_\Sigma B = \mathrm{n}_{\Sigma_t} F$, such that

$$F = \star_\Sigma B + \mathrm{d}t \wedge E\,, \tag{2.4}$$

where now $E, B \in C^\infty(\mathbb{R}, \Omega^1(\Sigma))$ while $\star_\Sigma$ is the Hodge dual on Σ. Maxwell's equations reduce to

$$\partial_t E - \mathrm{curl} B = 0\,, \qquad \partial_t B + \mathrm{curl} E = 0\,, \tag{2.5a}$$

$$\mathrm{div}(E) = \mathrm{div}(B) = 0\,, \tag{2.5b}$$

where $\mathrm{div} = \delta$ is the co-differential on Σ, while curl is defined in Equation (2.41) – in particular $\mathrm{curl} = \star_\Sigma \mathrm{d}$ if $\dim \Sigma = 3$.

To model the presence of an interface that divides M in two distinct regions, we consider Z a codimension 1 smooth embedded hypersurface of Σ.

In this setting we consider Maxwell's equations with Z-interface boundary conditions, that is we allow discontinuities to occur on $\mathbb{R} \times Z$. Hence, we split $\Sigma = \Sigma_+ \cup \Sigma_-$, such that

$$\Sigma_Z := \Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-\,, \tag{2.6}$$

and we refer to Σ_- (*resp.* Σ_+) as the left (*resp.* right) component of Σ. Moreover, $\Sigma_\pm$ are compact manifolds with boundary $\partial\Sigma_\pm = Z$, and the orientation on Z induced by Σ_+ is the opposite of the one induced by Σ_-. Hence, the manifolds $(\mathbb{R} \times \Sigma_\pm, g = -\mathrm{d}t^2 + h)$ are *globally* *hyperbolic spacetimes with timelike boundary* (see 1.1.1), which is $\mathbb{R} \times Z$.

Whenever the interface $Z \neq \emptyset$ the system (2.5) has to be modified, in particular the non-dynamical equations (2.5b) involving the divergence operator div have to be suitably interpreted – *cf.* Subsection 2.2.4. In particular one expects that the condition $\operatorname{div}(E) = \operatorname{div}(B) = 0$ should be read at a distributional level, leading to a constraint on the values at Z of the normal component of E. In addition, the dynamical equations (2.5a) have to be combined with boundary conditions at the interface Z – *cf.* [Jac99, Sec. I.5].
In what follows we will state the precise meaning of the problem (2.5) with interface Z with the help of Hodge theory and of Lagrangian subspaces [EM99; EM03; EM05].

2.2 Constraint equations: Hodge theory with interface

In this section we present a Hodge decomposition a the closed Riemannian manifold (Σ, h) with interface Z. This generalizes the known results on classical Hodge decomposition on manifolds possibly with non-empty boundary [Ama17; AM04; Gaf55; Gro+91; Kod49; Li09; Sch95; Sco95; AS+00].

Hodge theory is a generalization of Helmholtz decomposition. The latter was formulated as a splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the *Hodge decomposition*. The idea behind Helmholtz decomposition is that any vector field $\mathbf{F}$ in $\mathbb{R}^3$ can be read as a sum of an irrotational field $\mathbf{U}$, i.e. such that $\operatorname{curl} \mathbf{U} = \mathrm{d}\mathbf{U} = 0$, and a solenoidal field $\mathbf{V}$, i.e. such that $\operatorname{div} \mathbf{V} = \delta \mathbf{V} = 0$. In other words, for $\mathbf{F} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$, one can write

$$\mathbf{F} = -\nabla\Phi + \operatorname{curl} \mathbf{A}, \tag{2.7}$$

where we used the fact that in $\mathbb{R}^3$ $\operatorname{curl} \mathbf{U} = 0$ implies $\mathbf{U} = -\nabla\Phi$, since $\mathbb{R}^3$ is simply connected.

Remark 2.2.1. With reference to Definition 1.3.5 and Remark 1.3.8, in what follows $\mathrm{L}^2\Omega^k(\Sigma)$ will denote the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ (see Section 1.2) with respect to the pairing $(\ ,\ )_\Sigma$ between k-forms

$$(\alpha, \beta)_\Sigma := \int_\Sigma \overline{\alpha} \wedge \star_\Sigma \beta \qquad \alpha, \beta \in \Omega^k_{\mathrm{c}}(\Sigma)\,, \tag{2.8}$$

where $\star_\Sigma$ is the Hodge dual on Σ.

Remark 2.2.2. With a slight abuse of notation we denote still with d and δ the extension to the space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ of the action of the differential and of the codifferential on $\Omega^k_{\mathrm{c}}(\Sigma)$.

Remark 2.2.3. In agreement with Remark 1.2.1, we denote with $\Omega^k_{\mathrm{c}}(\Sigma)$ the space of smooth and compactly supported k-forms. Moreover, since Σ is of bounded geometry, for $\ell \geq \frac{1}{2}$, we

use the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$ and $\mathrm{H}_0^\ell\Omega^k(\Sigma)$ of k-forms as defined in Definition 1.3.6 and in Subsection 1.3.2.
If Σ is compact, $\Omega_c^k(\Sigma)$ coincides with the space of smooth k-forms $\Omega^k(\Sigma)$, but we will still use $\Omega_c^k(\Sigma)$ in view of possible generalizations. In addition, we remark that $\mathrm{H}^{-\ell}\Omega^k(\Sigma) = \mathrm{H}_0^\ell\Omega^k(\Sigma)^*$, where $*$ indicates the dual with respect to the scalar product $(\,,\,)_\Sigma$.

The Hodge theorem for a closed manifold Σ states that there is an L^2-orthogonal decomposition

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}\mathrm{H}^1\Omega^{k-1}(\Sigma) \oplus \delta\mathrm{H}^1\Omega^{k+1}(\Sigma) \oplus \ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}\,, \tag{2.9}$$

where $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ is the Laplace operator and $\ker(\Delta)_{\mathrm{H}^1\Omega^k(\Sigma)}$ denotes the space of *harmonic forms*. If Σ has an empty boundary, the space of harmonic forms coincides with that of *harmonic fields*, $\ker(\delta)_{\mathrm{H}^1\Omega^k(\Sigma)} \cap \ker(\mathrm{d})_{\mathrm{H}^1\Omega^k(\Sigma)}$ (see [Kod49] and [Sch95]). The last result can be stated as follows and it is very easy to prove.

Proposition 2.2.4. *Let $\alpha \in \mathrm{H}^2\Omega^k(\Sigma)$, where Σ is a closed manifold. Then $\Delta\alpha = 0$ if and only if $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$.*

Proof. If $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$, $\Delta\alpha = 0$. On the other hand if $\Delta\alpha = 0$,

$$0 = (\Delta\alpha, \alpha)_\Sigma = ((\mathrm{d}\delta + \delta\mathrm{d})\alpha, \alpha)_\Sigma = (\mathrm{d}\delta\alpha, \alpha)_\Sigma + (\delta\mathrm{d}\alpha, \alpha)_\Sigma = \tag{2.10}$$
$$= (\delta\alpha, \delta\alpha)_\Sigma + (\mathrm{d}\alpha, \mathrm{d}\alpha)_\Sigma = \|\delta\alpha\|^2 + \|\mathrm{d}\alpha\|^2. \tag{2.11}$$

So both $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$. ■

2.2.1 Hodge decomposition on compact manifold with non-empty boundary

For a compact manifold Σ with non-empty boundary $\partial\Sigma$ the decomposition (2.9) requires a slight adjustment and harmonic forms do not coincide with harmonic fields anymore. Because of boundary terms, $\ker\Delta$ no longer coincides with the closed and co-closed forms. It turns out that every harmonic field is a harmonic form, but the converse is false. To show this, consider the following example.

Example 2.2.5. Let U be a bounded subset of $\mathbb{R}^2$, endowed with the standard Euclidean metric. On U, the 1-form $\omega = x\,dy$ is harmonic, since its second derivatives vanish, but $\omega \notin \ker\mathrm{d}$, since

$$\mathrm{d}(x\,\mathrm{d}y) = \partial_x x\,\mathrm{d}x \wedge \mathrm{d}y + \partial_y x\,\mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y.$$

ω is though in $\ker\delta$ as $\star\,\mathrm{d}\star(x\,\mathrm{d}y) = \star\,\mathrm{d}(x\,\mathrm{d}x) = 0$. ■

Definition 2.2.6. *We call $\mathcal{H}^k(\Sigma)$ the L^2-closure of the space of harmonic fields*

$$\mathcal{H}^k(\Sigma) = \overline{\{\omega \in \mathrm{H}^1\Omega^k(\Sigma)|\ \mathrm{d}\omega = 0\,,\ \delta\omega = 0\}}\,. \tag{2.12}$$

With a slight abuse of notation, we will refer to the elements of $\mathcal{H}^k(\Sigma)$ as harmonic fields

In fact, the space of harmonic fields is infinite dimensional and the spaces $\mathrm{dH}^1\Omega^{k-1}(\Sigma)$, $\delta H^1\Omega^{k+1}(\Sigma)$, $\mathcal{H}^k(\Sigma)$ are not orthogonal unless suitable boundary conditions are imposed. Therefore, one has to give a precise meaning to the boundary value of a differential form. Since differential forms are not scalar quantities, one can define a normal and a tangential projection along the boundary.

Remark 2.2.7. We recall that the tangential and normal traces t and n of a differential form are defined according to Definition 1.2.3 and are extended as in Subsection 1.3.2 to continuous surjective maps as in Equation (1.11), that we recall for completeness:

$$\mathrm{t}\oplus\mathrm{n}\colon \mathrm{H}^\ell\Omega^k(\Sigma)\to\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma)\oplus\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma)\qquad\forall\ell\geq\frac{1}{2}\,. \tag{2.13}$$

Next, we present the Hodge decomposition for compact manifolds with boundary, a proof of which can be found at [Sch95, Thm. 2.4.2].

Theorem 2.2.8. *Let (Σ, h) be a compact, connected, Riemannian manifold with non-empty boundary*

1. *For all $\omega\in\Omega^{k-1}_{\mathrm{c}}(\Sigma)$ and $\eta\in\Omega^k_{\mathrm{c}}(\Sigma)$ it holds*

$$(\mathrm{d}\omega,\eta)_\Sigma-(\omega,\delta\eta)_\Sigma=(\mathrm{t}\omega,\mathrm{n}\eta)_{\partial\Sigma}\,, \tag{2.14}$$

   where $(\;,\;)_\Sigma$ has been defined in Equation (2.8) while $(\;,\;)_{\partial\Sigma}$ is defined similarly. Equation (2.14) still holds true for $\omega\in\mathrm{H}^\ell\Omega^{k-1}(\Sigma)$ and $\eta\in\mathrm{H}^\ell\Omega^k(\Sigma)$.

2. *The Hilbert space $\mathrm{L}^2\Omega^k(\Sigma)$ of square integrable k-forms splits in the L^2-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma)=\mathrm{dH}^1\Omega^k_{\mathrm{t}}(\Sigma)\oplus\delta\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma)\oplus\mathcal{H}^k(\Sigma)\,, \tag{2.15}$$

   where $\mathcal{H}^k(\Sigma)$ is defined per Equation (2.12) while, in view of Equation (1.4)

$$\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma):=\{\alpha\in\mathrm{H}^1\Omega^{k-1}(\Sigma)|\ \mathrm{t}\alpha=0\}\,, \tag{2.16}$$
$$\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma):=\{\beta\in\mathrm{H}^1\Omega^{k+1}(\Sigma)|\ \mathrm{n}\beta=0\}\,. \tag{2.17}$$

Remark 2.2.9. The previous decomposition generalizes to Sobolev spaces, in particular for all $\ell\in\mathbb{N}\cup\{0\}$ we have

$$\mathrm{H}^\ell\Omega^k(\Sigma)=\mathrm{dH}^{\ell+1}\Omega^k_{\mathrm{t}}(\Sigma)\oplus\delta\mathrm{H}^{\ell+1}\Omega^{k+1}_{\mathrm{n}}\oplus\mathrm{H}^\ell\mathcal{H}^k(\Sigma)\,, \tag{2.18}$$

where $\mathrm{H}^\ell\mathcal{H}^k(\Sigma)=\mathcal{H}^k(\Sigma)\cap\mathrm{H}^\ell\Omega^k(\Sigma)$, since $\mathrm{H}^\ell\Omega^k(\Sigma)\hookrightarrow\mathrm{L}^2\Omega^k(\Sigma)$.

2.2.2 Hodge decomposition for compact manifold with interface

In this section we generalize Theorem 2.2.8 to the case of a closed Riemannian manifold Σ together with an interface Z. As starting point, we need to distinguish between regular k-forms which are defined on the whole manifold, and hence continuous, and pairs of forms which are regular separately on the two sides $\Sigma_\pm$ and are allowed to be discontinuous on Z.

Definition 2.2.10. *We call*

$$\Omega^k(\Sigma_Z) := \Omega^k(\Sigma_+) \oplus \Omega^k(\Sigma_-)\,, \tag{2.19}$$

where it is understood that the pair $\omega+\eta \in \Omega^k(\Sigma_+)\oplus\Omega^k(\Sigma_-)$ *identifies an element* $\alpha \in \Omega^k(\Sigma_Z)$ *such that* $\alpha|_{\Sigma_+} = \omega$ *and* $\alpha|_{\Sigma_-} = \eta$.

Following the previous definition,

$$\Omega^k_{\mathrm{c}}(\Sigma_Z) = \Omega^k_{\mathrm{c}}(\Sigma_+) \oplus \Omega^k_{\mathrm{c}}(\Sigma_-)\,. \tag{2.20}$$

This implies $\omega \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ if and only if ω is a smooth k-form in Σ_Z and $\mathrm{supp}_\Sigma\,\omega := \overline{\{x \in \Sigma_Z \,|\, \omega(x) \neq 0\}}^\Sigma$ is compact. Hence, forms in $\Omega^k_{\mathrm{c}}(\Sigma_Z)$ have support overlapping with the interface, where they are allowed to be discontinuous.

Observe that Theorem 2.2.8 applies to both $\mathrm{L}^2\Omega^k(\Sigma_\pm)$. In addition, since Z has zero measure the space of square integrable k-forms splits as

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{L}^2\Omega^k(\Sigma_Z) = \mathrm{L}^2\Omega^k(\Sigma_+) \oplus \mathrm{L}^2\Omega^k(\Sigma_-)\,. \tag{2.21}$$

We expect that a counterpart of (2.15) holds true, though $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma)$, $\mathrm{H}^1\Omega^{k-1}_{\mathrm{n}}(\Sigma)$ ought to be replaced by suitable jump conditions across Z. To this end, notice that the splitting (2.21) does not generalize to the Sobolev spaces $\mathrm{H}^\ell\Omega^k(\Sigma)$, in particular

$$\mathrm{H}^\ell\Omega^k(\Sigma) \hookrightarrow \mathrm{H}^\ell\Omega^k(\Sigma_Z) = \mathrm{H}^\ell\Omega^k(\Sigma_+) \oplus \mathrm{H}^\ell\Omega^k(\Sigma_-)\,, \tag{2.22}$$

is a proper inclusion. Indeed, consider any regular form ω in Σ_Z which has $[\mathrm{t}\omega] \neq 0$. In this case ω can not have square integrable (weak) derivatives, since a non-vanishing jump gives rise to a distributional derivative which is proportional to the Dirac delta.

Definition 2.2.11. *Let* (Σ, h) *be an oriented, compact, Riemanniann manifold with interface* $Z \hookrightarrow \Sigma$. *Moreover let* $(\Sigma_\pm, h_\pm)$ *the oriented, compact Riemannian manifolds with boundary* $\partial\Sigma_\pm = Z$ *such that* $\Sigma_Z := \Sigma \setminus Z = \Sigma_+ \cup \Sigma_-$. *For* $\omega \in \Omega^k(\Sigma_Z)$ *we define the tangential jump* $[\mathrm{t}\omega] \in \Omega^k(Z)$ *and normal jump* $[\mathrm{n}\omega] \in \Omega^{k-1}(Z)$ *across* Z *by*

$$[\mathrm{t}\omega] := \mathrm{t}_+\omega - \mathrm{t}_-\omega\,, \qquad [\mathrm{n}\omega] := \mathrm{n}_+\omega - \mathrm{n}_-\omega\,, \tag{2.23}$$

where $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ *denote the tangential and normal map on* $\Sigma_\pm$ *as per Definition 1.2.3.*

Remark 2.2.12. The tangential and normal traces $\mathrm{t}_\pm$, $\mathrm{n}_\pm$ as well as the tangential and normal jump extend by continuity on $\mathrm{H}^1\Omega^k(\Sigma_Z)$ and are surjective if the codomain is $\mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(Z)$ - *cf.*

Remark 2.2.7. As a consequence of Definition 2.2.11 it holds that

$$\mathrm{H}^1\Omega^k(\Sigma) = \{\omega \in \mathrm{H}^1\Omega^k(\Sigma_Z)|\ [\mathrm{t}\omega] = 0\,,\ [\mathrm{n}\omega] = 0\}\,. \tag{2.24}$$

An analogous equality does not hold for $\Omega^k(\Sigma)$ because it would require traces of higher order derivatives to match at Z.

Theorem 2.2.13. *Let (Σ, h) be an oriented, compact, Riemanniann manifold with interface Z. Moreover let $(\Sigma_\pm, h_\pm)$ be the oriented, compact Riemannian manifolds with boundary $\partial\Sigma_\pm = Z$ such that $\Sigma \setminus Z = \mathring{\Sigma}_+ \cup \mathring{\Sigma}_-$.*

1. *For all $\omega \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_Z)$ and $\eta \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ it holds*

$$(\mathrm{d}\omega, \eta)_Z - (\omega, \delta\eta)_Z = ([\mathrm{t}\omega], \mathrm{n}_+\eta)_Z - (\mathrm{t}_-\omega, [\mathrm{n}\eta])_Z\,, \tag{2.25}$$

where $(\ ,\ )_Z$ is the scalar product between forms on Z – cf. Equation (2.8) – while $\mathrm{t}_\pm, \mathrm{n}_\pm$ are the tangential and normal maps on $\Sigma_\pm$ as per Definition 1.2.3. Equation (2.25) still holds true for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for all $\ell \geq 1$.

2. *The Hilbert space $\mathrm{L}^2\Omega^k(\Sigma)$ of square integrable k-forms splits into the L^2-orthogonal direct sum*

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{d}\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.26}$$

where $\mathcal{H}^k(\Sigma)$ is defined per Equation (2.12), while

$$\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z) := \{\alpha \in \mathrm{H}^1\Omega^{k-1}(\Sigma_Z)|\ [\mathrm{t}\alpha] = 0\}\,, \tag{2.27}$$

$$\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z) := \{\beta \in \mathrm{H}^1\Omega^{k+1}(\Sigma_Z)|\ [\mathrm{n}\beta] = 0\}\,. \tag{2.28}$$

Proof. Equation (2.25) is an immediate consequence of (2.14). In particular for $\omega \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_Z)$ and $\eta \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$ we decompose $\omega = \omega_+ + \omega_-$ and $\eta = \eta_+ + \eta_-$ where $\omega_\pm \in \Omega^{k-1}_{\mathrm{c}}(\Sigma_\pm)$ and $\eta_\pm \in \Omega^k_{\mathrm{c}}(\Sigma_\pm)$. (Notice that we have $\mathrm{t}_\pm\omega = \mathrm{t}_\pm\omega_\pm$.) Applying Equation (2.14) it holds

$$\begin{aligned}(\mathrm{d}\omega, \eta) - (\omega, \delta\eta) &= \sum_\pm \left((\mathrm{d}\omega_\pm, \eta_\pm) - (\omega_\pm, \delta\eta_\pm)\right) = \int_Z \mathrm{t}_+\overline{\omega} \wedge \star_Z \mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge \star_Z \mathrm{n}_-\eta \\ &= \int_Z [\mathrm{t}\overline{\omega}] \wedge \star_Z \mathrm{n}_+\eta - \int_Z \mathrm{t}_-\overline{\omega} \wedge \star_Z [\mathrm{n}\beta]\,.\end{aligned}$$

A density argument leads to the same identity for $\omega \in \mathrm{H}^\ell\Omega^{k-1}(\Sigma_Z)$ and $\eta \in \mathrm{H}^\ell\Omega^k(\Sigma_Z)$ for $\ell \geq 1$. We prove the splitting (2.26). The spaces $\mathrm{d}\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z)$, $\delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$, $\mathcal{H}^k(\Sigma)$ are orthogonal because of Equation (2.25). Let ω be in the orthogonal complement of $\mathrm{d}\mathrm{H}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. We wish to show that $\omega \in \mathcal{H}^k(\Sigma)$. We split $\omega = \omega_+ + \omega_-$ with $\omega_\pm \in$

$\mathrm{L}^2\Omega^k(\Sigma_\pm)$, we apply Theorem 2.2.8 to each component so that

$$\omega = \sum_\pm \left(\mathrm{d}\alpha_\pm + \delta\beta_\pm + \kappa_\pm\right),$$

where $\alpha_\pm \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_\pm)$, $\beta_\pm \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_\pm)$ and $\kappa_\pm \in \mathcal{H}^k(\Sigma_\pm)$. Let $\hat{\alpha} \in \mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_+)$: This identifies an element of $\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ by considering its extension to zero on Σ_-. Since $\omega \in \left[\mathrm{dH}^1\Omega_{[\mathrm{t}]}(\Sigma_Z)\right]^\perp$ we have $0 = (\mathrm{d}\hat{\alpha}, \omega) = (\mathrm{d}\hat{\alpha}, \mathrm{d}\alpha_+)$, thus $\mathrm{d}\alpha_+ = 0$ by the arbitrariness of $\hat{\alpha}$. With a similar argument we have $\alpha_- = 0$ as well as $\beta_\pm = 0$.
Therefore $\omega \in \mathcal{H}^k(\Sigma_Z)$. In order to prove that $\omega \in \mathcal{H}^k(\Sigma)$ we need to show that $[\mathrm{t}\omega] = 0$ as well as $[\mathrm{n}\omega] = 0$ – *cf.* Remark 2.2.12. This is a consequence of $\omega \in \left[\mathrm{dH}^1\Omega^k_{[\mathrm{t}]}(\Sigma_Z) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)\right]^\perp$. Indeed, let $\alpha \in \mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$. Applying Equation (2.25) we find

$$0 = (\mathrm{d}\alpha, \omega) = -\int_Z \mathrm{t}_-\overline{\alpha} \wedge \star_Z[\mathrm{n}\omega]\,. \tag{2.29}$$

The arbitrariness of $\mathrm{t}_-\alpha$, t_- being surjective, implies $[\mathrm{n}\omega] = 0$. Similarly $[\mathrm{t}\omega] = 0$ follows by $\omega \perp \delta\mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$. ■

Remark 2.2.14. The harmonic part of decomposition (2.26) contains harmonic k-forms which are continuous across the interface Z – *cf.* Remark 2.2.12. One can also consider a decomposition which allows for a discontinuous harmonic component. In particular it can be shown that

$$\mathrm{L}^2\Omega^k(\Sigma) = \mathrm{dH}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z) \oplus \delta\mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z) \oplus \mathcal{H}^k(\Sigma_Z)\,,$$

where now $\mathrm{H}^1\Omega^{k-1}_{\mathrm{t}}(\Sigma_Z)$ is the subspace of $\mathrm{H}^1\Omega^{k-1}_{[\mathrm{t}]}(\Sigma_Z)$ made of $(k-1)$-forms α such that $\mathrm{t}_\pm\omega = 0$ and similarly $\beta \in \mathrm{H}^1\Omega^{k+1}_{\mathrm{n}}(\Sigma_Z)$ if and only if $\beta \in \mathrm{H}^1\Omega^{k+1}_{[\mathrm{n}]}(\Sigma_Z)$ and $\mathrm{n}_\pm\beta = 0$.

2.2.3 Further perspectives on Hodge decomposition

The results of Theorem 2.2.8 can be generalized. In 1949, Kodaira (see [Kod49]) proved a *weak* L^2 orthogonal decomposition, for any (non-compact) Riemannian manifold (M, g) with no boundary, of the form

$$\mathrm{L}^2\Omega^k(M) = \overline{\mathrm{d}\Omega^{k-1}_{\mathrm{c}}(M)} \oplus \overline{\delta\Omega^{k+1}_{\mathrm{c}}(M)} \oplus \mathcal{H}^k(M)\,. \tag{2.30}$$

Gromov, in [Gro+91], proved that under the assumption that the Laplacian has a spectral gap in $\mathrm{L}^2\Omega^k(M)$, i.e. there is no spectrum of Δ in an open interval $(0, \eta)$, with $\eta > 0$, the following strong L^2-orthogonal decomposition holds for any (non-compact) Riemannian manifold (M, g) with empty boundary:

$$\mathrm{L}^2\Omega^k(M) = \mathrm{dH}^1\Omega^{k-1}(M) \oplus \delta\mathrm{H}^1\Omega^{k+1}(M) \oplus \mathcal{H}^k(M)\,. \tag{2.31}$$

For the case $\partial M \neq \emptyset$, the paper by Amar, [Ama17], recovers a strong L^p decomposition for complete non-compact manifolds, while both [Li09] and [AS+00] prove the strong L^p decomposition within the framework of weighted Sobolev spaces. [Sco95] discusses instead a strong L^p-decomposition on compact manifolds. Finally, using weighted Sobolev spaces, Schwartz [Sch95] extends to the Hodge decomposition on non-compact manifolds with non-empty boundary whenever M is the complement of an open bounded domain in $\mathbb{R}^n$.
The papers by [AM04; Gaf55] are devoted to developing the Hodge decomposition from the point of view of the theory of Hilbert space, thus arriving at it without the use of differential equation theory as in [Sch95]. For the case of a non-compact Riemannian manifold Σ one may follow the results of [AM04] in order to achieve the following weak-Hodge decomposition – *cf.* Equation (2.15). We consider the operators $\mathrm{d_t}, \delta_\mathrm{n}$

$$\mathrm{dom}(\mathrm{d_t}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ \mathrm{t}\omega = 0\} \qquad \mathrm{d_t}\omega := \mathrm{d}\omega\,, \tag{2.32}$$
$$\mathrm{dom}(\delta_\mathrm{n}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ \mathrm{n}\omega = 0\} \qquad \delta_\mathrm{n}\omega := \delta\omega\,. \tag{2.33}$$

Notice that $\mathrm{d_t}$ as well as δ_n are nihilpotent because of the relations (1.5). These operators are closed and from Equation (2.14) it follows that their adjoints are:

$$\mathrm{dom}(\mathrm{d}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\}\,, \qquad \delta_\mathrm{n}^* = \mathrm{d}\,,$$
$$\mathrm{dom}(\delta) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\}\,, \qquad \mathrm{d}_\mathrm{t}^* = \delta\,.$$

It follows that $(\overline{\mathrm{Ran}(\mathrm{d_t})} \oplus \overline{\mathrm{Ran}(\delta_\mathrm{n})})^\perp = \overline{\ker(\mathrm{d}) \cap \ker\delta} = \mathcal{H}^k(\Sigma)$ so that

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d_t})} \oplus \overline{\mathrm{Ran}(\delta_\mathrm{n})} \oplus \mathcal{H}^k(\Sigma)\,. \tag{2.34}$$

Following the same steps of the proof of Theorem 2.2.13 it descends that a similar weak-Hodge decomposition holds for the case of non-compact Riemannian manifolds Σ with interface Z:

$$\mathrm{L}^2\Omega^k(\Sigma) = \overline{\mathrm{Ran}(\mathrm{d_{[t]}})} \oplus \overline{\mathrm{Ran}(\delta_\mathrm{[n]})} \oplus \mathcal{H}^k(\Sigma)\,, \tag{2.35}$$

where $\mathrm{d_{[t]}}, \delta_\mathrm{[n]}$ are

$$\mathrm{dom}(\mathrm{d_{[t]}}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \mathrm{d}\omega \in \mathrm{L}^2\Omega^{k+1}(\Sigma)\,,\ [\mathrm{t}\omega] = 0\} \qquad \mathrm{d_{[t]}}\omega := \mathrm{d}\omega\,,$$
$$\mathrm{dom}(\delta_\mathrm{[n]}) := \{\omega \in \mathrm{L}^2\Omega^k(\Sigma) |\, \delta\omega \in \mathrm{L}^2\Omega^{k-1}(\Sigma)\,,\ [\mathrm{n}\omega] = 0\} \qquad \delta_\mathrm{[n]}\omega := \delta\omega\,.$$

It holds $\mathrm{d}_\mathrm{[t]}^* = \delta_\mathrm{[n]}$ as well as $\delta_\mathrm{[n]}^* = \mathrm{d_{[t]}}$ so that in particular $\ker \mathrm{d}_\mathrm{[t]}^* \cap \ker \delta_\mathrm{[n]} = \mathcal{H}^k(\Sigma)$.

2.2.4 Non-dynamical Maxwell's equations

The Hodge decomposition with interface proved in Theorem 2.2.13 can be exploited to formulate the correct generalization of the non-dynamical components of Maxwell's equations (2.5b) as follows.

We interpret the constraint $\operatorname{div} E = \delta E = 0$ (and analogously $\operatorname{div} B = 0$) in a distributional sense. Recalling Stokes' theorem in Equation 1.6, we can write formally:

$$(\mathrm{d}\psi, E)_{\Sigma_\pm} = (\psi, \delta E)_{\Sigma_\pm} + (\mathrm{t}\psi, \mathrm{n}E)_{\partial\Sigma_\pm}\,, \qquad \text{for } \psi \in \mathrm{H}^1\Omega^0(\Sigma). \tag{2.36}$$

By a formal manipulation one obtains that, if $\operatorname{supp}\psi \cap Z \neq \emptyset$,

$$\begin{aligned}(\mathrm{d}\psi, E)_\Sigma =&(\mathrm{d}\psi, E)_{\Sigma_+} + (\mathrm{d}\psi, E)_{\Sigma_-} = \\ =&(\psi, \delta E)_{\Sigma_+} + (\mathrm{t}\psi, \mathrm{n}_+ E)_Z + (\psi, \delta E)_{\Sigma_-} - (\mathrm{t}\psi, \mathrm{n}_- E)_Z = \\ =&(\psi, \delta E)_\Sigma + (\mathrm{t}\psi, [\mathrm{n}E])_Z .\end{aligned} \tag{2.37}$$

Definition 2.2.15. *We say that $E \in \mathrm{H}^1\Omega^1(\Sigma_Z)$ satisfies $\delta E = 0$ weakly if both terms of the right hand side of Equation (2.37) vanish for any $\psi \in \mathrm{H}^1\Omega^0(\Sigma) \equiv \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$, i.e.*

$$(\mathrm{d}\psi, E)_\Sigma = 0\,, \text{ for any } \psi \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.38}$$

In view of the previous definition, in what follows we will replace equations (2.5b) with the requirement

$$E, B \perp \mathrm{d}\mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)\,. \tag{2.39}$$

Notice that, because of Equation (2.37), this entails $\delta E = \delta B = 0$ pointwisely in $\Sigma_\pm$ as well as $[\mathrm{n}E] = [\mathrm{n}B] = 0$. Configurations of the electric field E in presence of a charge density ρ on $\Sigma_\pm$ and a surface charge density σ over Z are described by expanding $E = \mathrm{d}\alpha + \delta\beta + \kappa$ and demanding $\alpha \in \mathrm{H}^1\Omega^0_{[\mathrm{t}]}(\Sigma_Z)$ to satisfy

$$(\mathrm{d}\varphi, \mathrm{d}\alpha)_\Sigma = (\varphi, \rho)_\Sigma + (\mathrm{t}\varphi, \sigma)_Z \qquad \forall \varphi \in C^\infty_{\mathrm{c}}(\Sigma)\,.$$

This provides a weak formulation for the electrostatic boundary problem. For sufficiently regular α this is equivalent to the Poisson problem $\Delta_\Sigma \alpha = \rho$, $[\mathrm{n}\mathrm{d}\alpha] = \sigma$, recovering the classical equations outlined in [Jac99, Sec. I.5].

2.3 Dynamical equations: Lagrangian subspaces

In this section we will discuss the dynamical equations (2.5a). They can be written in a Schrödinger-like form as a complex evolution equation, solutions can be found imposing suitable interface conditions on Z.

Definition 2.3.1. *We call* first order Maxwell's equations *the following system of partial differential equations:*

$$i\partial_t\psi = H\psi \qquad \psi := \begin{bmatrix} E \\ B \end{bmatrix}, \qquad H := \begin{bmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{bmatrix}, \tag{2.40}$$

where H will be called the first order Maxwell operator, *or simply the* Maxwell operator. *Here we adopt the convention of [Bär19] according to which*

$$\mathrm{curl} := i \star_\Sigma \mathrm{d} \quad \text{if } \dim\Sigma = 1 \mod 4\,, \quad \mathrm{curl} := \star_\Sigma \mathrm{d} \quad \text{if } \dim\Sigma = 3 \mod 4\,. \tag{2.41}$$

With this convention curl *is a formally a selfadjoint operator on $\Omega^1_{\mathrm{c}}(\Sigma)$.*

As outlined in Section 2.2 we consider Equation (2.40) on Σ_Z, allowing for jump discontinuities across the interface Z. To this end we regard H as a densely defined operator on the Cartesian product

$$\mathrm{L}^2\Omega^1(\Sigma) \times \mathrm{L}^2\Omega^1(\Sigma) =: \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} = \mathrm{L}^2\Omega^1(\Sigma_Z)^{\times 2} \tag{2.42}$$

(the former equality follows from Equation (2.21)) with domain

$$\mathrm{dom}(H) := \Omega^1_{\mathrm{cc}}(\Sigma_+)^{\times 2} \oplus \Omega^1_{\mathrm{cc}}(\Sigma_-)^{\times 2}\,, \tag{2.43}$$

where $\Omega^1_{\mathrm{cc}}(\Sigma_\pm)$ denotes the subspace of $\Omega^1_{\mathrm{c}}(\Sigma_\pm)$ with support in $\Sigma_\pm \setminus \partial\Sigma_\pm$.

In solving Maxwell's equations, we require the underlying system to be isolated, so that the flux of relevant physical quantities, such as those built from the stress-energy tensor, is zero through the interface. To translate mathematically this requirement we need to look for symmetric extensions $\widehat{H}$ of H, in other words

$$(\widehat{H}\psi_1, \psi_2)_\Sigma - (\psi_1, \widehat{H}\psi_2)_\Sigma = \text{ vanishing interface terms} \quad \forall \psi_1, \psi_2 \in \mathrm{dom}(\widehat{H}) \subseteq \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}. \tag{2.44}$$

Moreover, we require the extensions of H to be self-adjoint so that the spectral resolution of the operator has only real eigenvalues. This prevents the fundamental solutions of $\widehat{H}$ to have exponentially increasing modes, which would result to an unstable physical system.

Proposition 2.3.2. *Let $u, v \in \Omega^k_{\mathrm{c}}(\Sigma_Z)$, then a Green formula holds*

$$(\mathrm{curl}\, u, v)_\Sigma - (u, \mathrm{curl}\, v)_\Sigma = (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z\,, \tag{2.45}$$

where $\gamma_0 u := \frac{1}{\sqrt{2}} \star [\mathrm{t}u]$, $\star$ is the Hodge dual operator on Z and $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$. Moreover, the operator H, defined in (2.40) is symmetric on its domain (see Equation (2.43)), since for any $\psi_1, \psi_2 \in \Omega^1_{\mathrm{c}}(\Sigma)^{\times 2}$ it holds

$$(H\psi_1, \psi_2)_\Sigma - (\psi_1, H\psi_2)_\Sigma = (\Gamma_1\psi_1, \Gamma_0\psi_2)_Z - (\Gamma_0\psi_1, \Gamma_1\psi_2)_Z, \tag{2.46}$$

where $\psi = [E, B]$ *and* $\Gamma_0\psi = [i\gamma_1 B, \gamma_1 E]$, $\Gamma_1\psi = [\gamma_0 E, i\gamma_0 B]$.

The former Proposition entails that the operator H is symmetric and hence closable (*cf.* [Mor18, Thm. 5.10]), its adjoint H^* being defined on

$$\mathrm{dom}(H^*) = \{\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2} |\ H\psi \in \mathrm{L}^2\Omega^1(\Sigma)^{\times 2}\} \qquad H^*\psi := H\psi\,. \tag{2.47}$$

Equation (2.40) is solved by selecting a self-adjoint extension of H. We outline a technique which allows us to parametrize the self-adjoint extensions of H by Lagrangian subspaces of a suitable complex symplectic space – *cf.* [EM99; EM03; EM05]. The aim is to construct the Green operators for Equation (2.40) together with an interface condition. This technique, even if it does not give a complete characterization of self-adjoint extensions in terms of boundary conditions, allows us to check whether a chosen interface condition admits Green operators or not.

Definition 2.3.3. *Let* S *be a complex vector space and let* $\sigma\colon \mathsf{S}\times\mathsf{S}\to\mathbb{C}$ *be a sesquilinear map. The pair* (S,σ) *is called complex symplectic space if* σ *is non-degenerate – i.e.* $\sigma(x,y)=0$ *for all* $y\in\mathsf{S}$ *implies* $x=0$ *– and* $\sigma(x,y) = -\overline{\sigma(y,x)}$ *for all* $x,y\in\mathsf{S}$*. A subspace* $L\subseteq\mathsf{S}$ *is called Lagrangian subspace if* $L = L^\perp := \{x\in\mathsf{S}|\ \sigma(x,y)=0\ \forall y\in L\}$.

For convenience, we summarize the main results in the following theorem:

Theorem 2.3.4 ([EM99])**.** *Let* H *be a separable Hilbert space and let* $A\colon\ \mathrm{dom}(A)\subseteq\mathsf{H}\to\mathsf{H}$ *be a densely defined, symmetric operator. Then, the bilinear map*

$$\sigma(x,y) := (A^*x,y) - (x,A^*y)\,, \qquad \forall x,y\in\mathrm{dom}(A^*)\,, \tag{2.48}$$

satisfies $\sigma(x,y) = -\overline{\sigma(y,x)}$*. The symplectic form* σ *descends to the quotient space* $\mathsf{S}_A := \mathrm{dom}(A^*)/\,\mathrm{dom}(A)$ *and the pair* (S_A,σ) *is a complex symplectic space as per Definition 2.3.3. Moreover, for all Lagrangian subspaces* $L\subseteq\mathsf{S}_A$ *– cf. Definition 2.3.3 – the operator*

$$A_L := A^*|_{L+\mathrm{dom}(A)}\,, \tag{2.49}$$

defines a self-adjoint extension of A*, where* $L+\mathrm{dom}(A)$ *denotes the pre-image of* L *with respect to the projection* $\mathrm{dom}(A^*)\to\mathsf{S}_A$*. Finally the map*

$$\{\textit{Lagrangian subspaces } L \textit{ of } \mathsf{S}_A\} \ni L\mapsto A_L\in\{\textit{self-adjoint extensions of } A\}\,, \tag{2.50}$$

is one-to-one.

Remark 2.3.5. The symplectic form σ associated to the operator A on H is called *symplectic flux*. The physically motivated requirement of closedness of the extensions of A is translated into imposing the symplectic flux to vanish.

Example 2.3.6. As a concrete example of Theorem 2.3.4 we discuss the case of the self-adjoint extensions of the curl operator on a closed manifold Σ with interface Z. For simplicity we assume that $\dim\Sigma = 2k+1$ with $\dim\Sigma = 3 \mod 4$, while curl is defined according to (2.41). We consider the operator curl_Z defined by

$$\mathrm{dom}(\mathrm{curl}_Z) := \overline{\Omega^k_{\mathrm{cc}}(\Sigma_Z)}^{\|\|_{\mathrm{curl}}}, \qquad \mathrm{curl}_Z\, u := \mathrm{curl}\, u\,. \tag{2.51}$$

Notice that $\Omega^k_{\mathrm{cc}}(\Sigma_Z) = \Omega^k_{\mathrm{cc}}(\Sigma_+) \oplus \Omega^k_{\mathrm{cc}}(\Sigma_-)$. The adjoint curl^*_Z of curl_Z is defined on

$$\mathrm{dom}(\mathrm{curl}^*_Z) = \mathrm{dom}(\mathrm{curl}_+) \oplus \mathrm{dom}(\mathrm{curl}_-)\,, \tag{2.52}$$
$$\mathrm{dom}(\mathrm{curl}_\pm) := \{u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)|\ \mathrm{curl}_\pm\, u_\pm \in \mathrm{L}^2\Omega^k(\Sigma_\pm)\}\,, \quad \mathrm{curl}_\pm\, u := \mathrm{curl}\, u\,. \tag{2.53}$$

Since complex conjugation commutes with curl, it follows from Von Neumann's criterion [Mor18, Thm. 5.43] that curl_Z admits self-adjoints extensions. We give a description of the complex symplectic space $\mathsf{S}_{\mathrm{curl}_Z} := (\mathrm{dom}(\mathrm{curl}^*_Z)/\,\mathrm{dom}(\mathrm{curl}_Z), \sigma_{\mathrm{curl}})$ whose Lagrangian subspaces allow to characterize all self-adjoint extensions of curl_Z. According to Theorem 2.3.4 the symplectic structure σ_{curl} on the vector space $\mathsf{S}_{\mathrm{curl}_Z}$ is defined by

$$\sigma_{\mathrm{curl}}(u,v) := (\mathrm{curl}^*_Z\, u, v) - (u, \mathrm{curl}^*_Z\, v)\,, \qquad \forall u,v \in \mathrm{dom}(\mathrm{curl}^*_Z)\,. \tag{2.54}$$

In particular for $u \in \mathrm{dom}(\mathrm{curl}^*_Z)$ and $v \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ we have

$$\begin{aligned}\sigma_{\mathrm{curl}}(u,v) &= (\gamma_1 u, \gamma_0 v)_Z - (\gamma_0 u, \gamma_1 v)_Z = \\ &= \sum_\pm \pm \int_Z \overline{\mathrm{t}_\pm u} \wedge \mathrm{t}_\pm v = \sum_\pm \mp\, {}_{-\frac{1}{2}}\langle \mathrm{t}_\mp u, \star_Z \mathrm{t}_\mp v\rangle_{\frac{1}{2}}\,,\end{aligned} \tag{2.55}$$

where $\gamma_0 u := \frac{1}{\sqrt{2}} \star [\mathrm{t}u]$, $\gamma_1 u := \frac{1}{\sqrt{2}}(\mathrm{t}_+ u + \mathrm{t}_- u)$ as in Proposition 2.3.2 and where ${}_{-\frac{1}{2}}\langle\ ,\ \rangle_{\frac{1}{2}}$ denotes the pairing between $\mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ and $\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$. In particular this shows that $\mathrm{t}_\pm u \in \mathrm{H}^{-\frac{1}{2}}\Omega^k(Z)$ for all $u \in \mathrm{dom}(\mathrm{curl}^*_Z)$ – *cf.* [AV96; BCS02; Geo79; Paq82; Wec04] for more details on the trace space associated with the curl-operator on a manifold with boundary.
According to Theorem 2.3.4 all self-adjoint extensions of curl_Z are in one-to-one correspondence to the Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$. Unfortunately a complete characterization of all Lagrangian subspaces of $\mathsf{S}_{\mathrm{curl}_Z}$ is not available. For our purposes, it suffices to give a family of Lagrangian subspaces – a generalization of the results presented in [HKT12] may provide other examples. For $\theta \in \mathbb{R}$ let

$$L_\theta := \{u \in \mathrm{dom}(\mathrm{curl}^*_Z)|\ \mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u\}\,, \tag{2.56}$$

where $\mathrm{t}_\pm$ denote the tangential traces – *cf.* Definition 2.2.11, Remark 2.2.7 and Equation (2.55). To show that L_θ are Lagrangian subspaces let $u, v \in L_\theta$ and let $v_n \in \mathrm{H}^1\Omega^k(\Sigma_Z)$ be such that

$\|v - v_n\|_{\mathrm{curl}} \to 0$. In particular $\|(\mathrm{t}_+ - e^{i\theta}\mathrm{t}_-)v_n\|_{\mathrm{H}^{\frac{1}{2}}\Omega^k(Z)} \to 0$ so that

$$\sigma_{\mathrm{curl}}(u, v) = \lim_{n\to\infty} \sigma_{\mathrm{curl}}(u, v_n) = - \lim_{n\to\infty} {}_{-\frac{1}{2}}\langle \mathrm{t}_+ u, \star(\mathrm{t}_+ v_n - e^{i\theta}\mathrm{t}_- v_n)\rangle_{\frac{1}{2}} = 0\,. \tag{2.57}$$

It follows that $L_\theta \subseteq L_\theta^\perp$. Conversely if $u \in L_\theta^\perp$ let us consider $v \in L_\theta$. Since $u \in L_\theta^\perp$ we find

$$0 = \sigma_{\mathrm{curl}}(u, v) = - {}_{-\frac{1}{2}}\langle \mathrm{t}_+ u - e^{i\theta}\mathrm{t}_- u, \star \mathrm{t}_+ v\rangle_{\frac{1}{2}}\,.$$

Since $\mathrm{t}_+ \colon \mathrm{H}^1\Omega^k(\Sigma_Z) \to \mathrm{H}^{\frac{1}{2}}\Omega^k(Z)$ is surjective, it follows that $\mathrm{t}_+ u = e^{i\theta}\mathrm{t}_- u$.
Notice that the self-adjoint extension obtained for $\theta = 0$ coincides with the closure of curl on $\Omega^k_{\mathrm{c}}(\Sigma)$ which is known to be self-adjoint by [Bär19, Lem. 2.6]. Indeed, since $[\mathrm{t}]$ is continuous we have $\mathrm{dom}(\overline{\mathrm{curl}}) \subseteq L_0$ so that curl_{Z,L_0} is a self-adjoint extension of $\overline{\mathrm{curl}}$. Since this last operator is already self-adjoint, the two coincide. ■

Example 2.3.7. We provide a concrete example of 2.3.4 in the case we are mostly interested in: Maxwell's equations in the Schrödinger-like form as in Equation (2.40). According to Theorem 2.3.4, the operator H has an associated symplectic space $\mathsf{S}_H := (\mathrm{dom}(H^*)/\,\mathrm{dom}(H), \sigma_H)$, where

$$\sigma_H(\psi_1, \psi_2) = (H^*\psi_1, \psi_2) - (\psi_1, H^*\psi_2),\ \forall \psi_1, \psi_2 \in \mathrm{dom}(H^*). \tag{2.58}$$

In particular, if $\psi_1 \in \mathrm{dom}(H^*)$ and $\psi_2 \in \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2}$ and denoting ψ as the couple $[E, B]$, we can write

$$\begin{aligned}\sigma_H(\psi_1, \psi_2) =& - i\sigma_{\mathrm{curl}}(B_1, E_2) + i\sigma_{\mathrm{curl}}(E_1, B_2) = \\ =& i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\,\mathrm{t}_+\psi_2\rangle_{\frac{1}{2}} - {}_{-\frac{1}{2}}\langle \mathrm{t}_-\psi_1, \star S\,\mathrm{t}_-\psi_2\rangle_{\frac{1}{2}}\right],\end{aligned} \tag{2.59}$$

where σ_{curl} is defined in Equations (2.54), (2.55), ${}_{-\frac{1}{2}}\langle\,,\,\rangle_{\frac{1}{2}}$ is the pairing between $\mathrm{H}^{-\frac{1}{2}}$ and $\mathrm{H}^{\frac{1}{2}}$,

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SO}(2),$$

$\star$ is the Hodge operator on Z and, as usual, $\mathrm{t}_\pm$ denote the tangential traces – *cf.* Definition 2.2.11. We give a family of Lagrangian subspaces which encode the following class of interface conditions. For $\mathrm{U} \in \mathrm{SO}(2)$, let us define the space

$$L_{\mathrm{U}} := \{\psi \in \mathrm{dom}(H^*)\,|\,\mathrm{t}_+\psi = \mathrm{U}\mathrm{t}_-\psi\}. \tag{2.60}$$

To show that L_{U} are Lagrangian subspaces we mimic the technique used in the former Example 2.3.6. Let $\psi_1 = [\mathscr{E}_1, \mathscr{B}_1], \psi_2 = [\mathscr{E}_2, \mathscr{B}_2] \in L_{\mathrm{U}}$ and let $\phi_n = [E_n, B_n] \in \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2}$ for $n \in \mathbb{N}$ such that $\|E_2 - E_n\|_{\mathrm{curl}} \to 0$ and $\|B_2 - B_n\|_{\mathrm{curl}} \to 0$.

In particular it holds that $\|(\mathrm{t}_+ - \mathrm{U}\mathrm{t}_-)\psi_n\|_{\mathrm{H}^{\frac{1}{2}}\Omega^1(\Sigma_Z)^{\times 2}} \to 0$. Hence $L_{\mathrm{U}} \subseteq L_{\mathrm{U}}^{\perp}$ follows from

$$\begin{aligned}\sigma_H(\psi_1,\psi_2) &= \lim_{n\to\infty}\sigma_H(\psi_1,\psi_n) = \\ &= \lim_{n\to\infty} i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\,\mathrm{t}_+\psi_n\rangle_{\frac{1}{2}} - {}_{-\frac{1}{2}}\langle \mathrm{U}^{-1}\mathrm{t}_+\psi_1, \star S\,\mathrm{t}_-\psi_n\rangle_{\frac{1}{2}}\right] = \\ &= \lim_{n\to\infty} i\left[{}_{-\frac{1}{2}}\langle \mathrm{t}_+\psi_1, \star S\,(\mathrm{t}_+\psi_n - \mathrm{U}\mathrm{t}_-\psi_n)\rangle_{\frac{1}{2}}\right] = 0.\end{aligned} \tag{2.61}$$

Conversely if $\psi_1 \in L_{\mathrm{U}}^{\perp}$ let us consider $v \in L_{\mathrm{U}}$. Hence, we find

$$0 = \sigma_H(\psi_1,\psi_2) = i\left[{}_{-\frac{1}{2}}\langle (\mathrm{t}_+\psi_1 - \mathrm{U}\mathrm{t}_-\psi_1), \star S\,\mathrm{t}_+\psi_2\rangle_{\frac{1}{2}}\right].$$

Since $\mathrm{t}_+\colon \mathrm{H}^1\Omega^1(\Sigma_Z)^{\times 2} \to \mathrm{H}^{\frac{1}{2}}\Omega^1(Z)^{\times 2}$ is surjective, it follows that $\mathrm{t}_+\psi_1 = \mathrm{U}\mathrm{t}_-\psi_1$.

Following slavishly the passages of Example 2.3.6, one can also show that the following family of subspaces of S_H, that can be expressed in terms of interface conditions, are Lagrangian and hence, give rise to a self-adjoint extensions of H:

$$L_\theta := \{u \in \mathrm{dom}(H^*)|\ \mathrm{t}_+\psi = e^{i\theta}\mathrm{t}_-\psi\}\,. \tag{2.62}$$

■

We conclude this section by introducing an exact sequence which provides a complete description of the solution space of the Maxwell's equations (2.5a) with interface Z.

Theorem 2.3.8. *Let H be the densely defined operator on $\mathrm{L}^2\Omega^1(\Sigma)^{\times 2}$ with domain defined by (2.43) and let H^* be its adjoint, defined as in (2.47). Let $L \subset \mathsf{S}_H = (\mathrm{dom}(H^*)/\mathrm{dom}\,H), \sigma_H)$ be a Lagrangian subspace in the sense of Definition 2.3.3 and consider the self-adjoint extension H_L as per Theorem 2.3.4. Furthermore, let $\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2} := \bigcap_{k\geq 0}\mathrm{dom}(H_L^k)$ and let $G_L^\pm$ be the operators $G_L^\pm\colon C^\infty_{\mathrm{tc}}(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \to C^\infty(\mathbb{R}, \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})$ completely determined in terms of the bidistributions $\mathcal{G}_L^+ = \theta(t-t')\mathcal{G}_L$ and $\mathcal{G}_L^- = -\theta(t'-t)\mathcal{G}_L$, with*

$$\mathcal{G}_L(\psi_1,\psi_2) = \int_{\mathbb{R}^2}\left(\psi_1(t)\middle|e^{-i(t-t')H_L}\psi_2(t')\right)\mathrm{d}t\,\mathrm{d}t' \quad \forall\psi_1,\psi_2 \in C^\infty_{\mathrm{c}}(\mathbb{R},\Omega^1_c(\Sigma)^{\times 2}). \tag{2.63}$$

The the operator G_L^+ (resp. G_L^-) is an advanced (resp. retarded) solution of $i\partial_t - H_L$, that is, it holds

$$(i\partial_t - H_L)\circ G_L^\pm = \mathrm{Id}_{C^\infty_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})}\,, \tag{2.64}$$

$$G_L^\pm\circ(i\partial_t - H_L) = \mathrm{Id}_{C^\infty_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2})}\,. \tag{2.65}$$

Moreover, let $G_L := G_L^+ - G_L^-$. Then the following is a short exact sequence

$$0 \to C^\infty_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \xrightarrow{i\partial_t - H_L} C^\infty_{\mathrm{tc}}(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \\ \xrightarrow{G_L} C^\infty(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \xrightarrow{i\partial_t - H_L} C^\infty(\mathbb{R},\mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}) \to 0\,. \tag{2.66}$$

Proof. Most of it is an analogue of [DDF19, Thm. 30- Prop. 36]. We observe that the function $\sigma(H_L) \ni \lambda \mapsto e^{-i\lambda\tau}$ is smooth and bounded for all $\tau \in \mathbb{R}$. Hence, for any $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R},\Omega^1_c(\Sigma_Z)^{\times 2})$, $G_L^\pm\psi \in C^\infty(\mathbb{R},\mathrm{dom}\,H_L)$. We have, for all $k \in \mathbb{N}\cup\{0\}$ and $t \in \mathbb{R}$

$$(1+H_L)^k[G_L^\pm\psi](t) = G_L^\pm[(1+H_L)^k\psi](t) = G_L^\pm[(1+H)^k\psi](t),$$

which is an element of $\mathrm{L}^2\Omega(\Sigma)^{\times 2}$, since $(1+H)\psi \in C^\infty_{\mathrm{c}}(\mathbb{R},\Omega^1_{\mathrm{c}}(\Sigma))$. It follows that $G_L^\pm\psi(t) \in \mathrm{H}^\infty_L\Omega^1(\Sigma_Z)^{\times 2}$ and Equation (2.63) holds true.
It remains to prove the finite speed of propagation, which follows from [HR00; MM13]. In particular, the hypotheses of [MM13, Thm. 1.1] are met since H_L is self-adjoint and from a straightforward computation it holds

$$\|[\eta I, H_L]\psi\| \leq \|\nabla\eta\|_\infty\|\psi\| \quad \forall\psi \in \mathrm{dom}\,H_L,\ \eta \in \mathrm{Lip}(\Sigma)\cap C^1(\Sigma).$$

Hence, [MM13, Thm. 1.1] ensures that the propagation speed of the one-parameter group e^{itH_L} is finite and smaller than 1 in the sense that

$$\mathrm{supp}(e^{-itH_L}\psi) \subset J^+(\mathrm{supp}\,\psi),\ t \geq 0,$$

where the brackets [,] denote the commutator.
The second part of the statement regarding the exact sequence follows imitating slavishly the standard arguments of [BGP07, Th. 3.4.7] [DDF19, Prop. 36]. ■

Notice that the exact sequence (2.66) implies that the space of smooth solution of the dynamical equations (2.5a) is isomorphic as a vector space to the image of G_L.

2.4 Perspectives on algebraic quantization of the field strength F

Having the causal propagator G in hand, for a choice of boundary conditions, the following step would be that of developing a quantization scheme for the field strength $F \in \Omega^2(M)$ on an arbitrary four dimensional globally hyperbolic spacetime with timelike boundary (M,g) within the framework of the algebraic formulation of quantum field theory.

In the case of empty boundary, the construction is obtained in [DL12]. They prove in particular that the commutator between two generators of the algebra of observables of F is given by the *Lichnerowicz propagator* regardless of the chosen spacetime. Moreover, they prove the existence

of a non trivial centre for the field algebra whenever the second de Rham cohomology group of the manifold is non trivial.
To be more clear they initially prove the existence of Green operators for the wave operator $\Box = \delta_M \mathrm{d}_M + \mathrm{d}_M \delta_M$ in a globally hyperbolic spacetime with empty boundary. Then, they use the fact that F itself satisfies a wave equation (since $F \in \ker \delta_M \cap \ker \mathrm{d}_M$) to entail that $F = G_\Box \omega$ with $\omega \in \Omega_{[t]}c^2(M)$, Eventually, exploiting the fact that $G_\Box$ commutes with δ_M and they identify the space $\mathrm{Sol}(M)$ of solutions of Maxwell's equations as the 2-forms $F \in \Omega^2(M)$ such that

$$F = G_\Box(\delta_M \alpha + \mathrm{d}_M \beta),\ \alpha \in \Omega^3_{\mathrm{c}}(M) \cap \ker \mathrm{d}_M, \beta \in \Omega^1_{\mathrm{c}}(M) \cap \ker \delta_M. \tag{2.67}$$

Hence, they construct the field algebra as an the associative, unital $*$-algebra: the universal tensor algebra generated by elements of the form $\mathbf{F}(\omega)$, $\omega \in \Omega^2_{\mathrm{c}}(M)$ with componentwise addition, componentwise multiplication with a scalar, componentwise antilinear involution $*$ and multiplication induced by the algebraic tensor product $\otimes$, while the $*$-operation is the one induced from complex conjugation. In addition they impose the Maxwell's equations at a dual level and implement the canonical commutation relations (CCR) requiring

$$[\mathbf{F}(\omega), \mathbf{F}(\omega')] = i\mathsf{G}(\omega, \omega')\mathcal{I},$$

where $\mathcal{I}$ is the identity and $\mathsf{G}(\omega, \omega') \doteq \int_M G_\Box(\delta\omega) \wedge \star\delta\omega'$, which is called *Lichnerowicz propagator*.

In addition, they show that the field algebra, in general, possesses a non trivial centre. This feature, thoroughly studied in [BHS14; BDS14; SDH14], is in common with Abelian gauge theories and will be discussed in the case of the vector potential A in the next chapter. Indeed, from a physical point of view, the existence of a non trivial centre leads to an obstruction in the interpretation of the model in terms of locally covariant quantum field theories.

In the case of non-empty boundary or an interface, we could not rely on the existence of Green operators for $\Box$ such that they commute with δ_M, hence we had to prove in the previous sections the existence of distinguished advanced and retarded Green operators, and consequently of a causal propagator G for Maxwell's equations under suitable boundary conditions. The next step would be that of following [DL12] once again and construct the field algebra for a manifold with timelike boundary or with interface. The passages would be identical, but now, since we have an exact sequence for the causal propagator of Maxwell's equations for F, the space of solutions Sol will be characterized as the 2-forms F such that $F = G(\eta)$. Moreover, the canonical commutation relations (CCR) will in fact be implemented as follows:

$$[\mathbf{F}(\omega), \mathbf{F}(\omega')] = i\widetilde{\mathsf{G}}(\omega, \omega')\mathcal{I},$$

with $\widetilde{\mathsf{G}}(\omega, \omega') = \int_M G(\omega) \wedge \star\omega$.

In the next chapter, we focus our efforts on the construction of Green operators for $\Box$ acting on k-forms on a special class of spacetimes with timelike boundary. Subsequently, relying on the existence of such operators, we will construct the algebra of observable for Maxwell's equations for the vector potential A.

Chapter 3

Maxwell's Equations for the vector potential and Boundary Conditions

In this Chapter we analyze the space of solutions of Maxwell's equations for the vector potential A, regarded as a system of equations for k-forms in a globally hyperbolic spacetime with timelike boundary (M, g) – *cf.* Definition 1.1.1. As discussed in Section 1.5, in the case with vanishing currents and empty boundary one can impose the Lorenz gauge condition to translate $\delta \mathrm{d} A = 0$ into an hyperbolic problem that involves a wave equation for A, where a D'Alembert - de Rham operator $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ acting on $\Omega^k(M)$ appears. In the case in hand, at first we identify a set of boundary conditions for the D'Alembert - de Rham operator that ensures the closedness of the underlying physical system, in analogy with the case for F described in Section 2.3. In the following, we prove the existence of advanced and retarded Green operators for $\Box_k$ in a special case: When the underlying globally hyperbolic spacetime with timelike boundary is *static* and of *bounded geometry*, using the technique of boundary triples as already done in the scalar case in [DDF19].

In the second part of the Chapter we focus precisely on the Maxwell operator $\delta\mathrm{d}\colon \Omega^k(M) \to \Omega^k(M)$. Among the boundary conditions for $\delta\mathrm{d}$, we dwell in two particular cases, namely the $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions, which will provide two different notions of gauge equivalence. We prove that in both cases, every class of gauge-equivalent solutions admits a representative satisfying the Lorentz gauge. We use this property and the analysis of the operator $\Box_k$ to construct and to classify the space of gauge equivalence classes of solutions of the Maxwell's equations with the prescribed boundary conditions.
As a last step, we construct the associated unital $*$-algebras of observables proving in particular that, as in the case of empty boundary, they possess a non-trivial center.

3.1 On the D'Alembert–de Rham wave operator

Consider the operator $\Box : \Omega^k(M) \to \Omega^k(M)$, where (M, g) is a m-dimensional globally hyperbolic spacetime with timelike boundary (see Definition 1.1.1) with $m \geq 2$. We denote with d, δ the differential and the codifferential operators on the spacetime M – *cf.* Section 1.2. Then, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, from Equation (1.6), one can obtain the following Green formula:

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = (\mathrm{t}\delta\alpha, \mathrm{n}\beta)_\partial - (\mathrm{n}\alpha, \mathrm{t}\delta\beta)_\partial - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_\partial + (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_\partial\,, \tag{3.1}$$

where t, n are the maps introduced in Definition 1.2.3, while $(,)$ and $(,)_\partial$ are the standard, metric induced pairing between k-forms respectively on M and on ∂M. As an immediate consequence, if $\alpha, \beta \in \Omega^k_{\mathrm{cc}}(M)$, *i.e.* their support does not intersect the boundary, the right-hand side of (3.1) vanishes automatically. In other words, $\Box$ is formally self-adjoint.
Clearly, $\Omega^k_{\mathrm{cc}}(M)$ is a rather restrictive set of k-forms, since forms in such space do not have any interplay with the boundary. A larger set, but not the largest[1], can be that of forms whose support intersect the boundary, but which have boundary conditions such that the scalar products appearing in Equation (3.1) cancel one another. Indeed, we define

$$\Omega^k_{f,f'}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\,,\ \mathrm{t}\delta\omega = f'\mathrm{n}\omega,\ f, f' \in C^\infty(\partial M)\}\,. \tag{3.2}$$

Noticing that for every $f \in C^\infty(\partial M)$ and for every $\alpha \in \Omega^k(\partial M)$, $\star_\partial(f\alpha) = f(\star_\partial\alpha)$, one can straightforwardly infer the following:

Lemma 3.1.1. *If $\alpha, \beta \in \Omega^k_{f,f'}(M)$, $0 \leq k \leq n = \dim M$ are such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, then it holds*

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = 0\,,$$

in other words, $\Box$ is formally self-adjoint.

It is important to notice that, whenever $k = 0$, the second condition in Equation (3.2) is trivially satisfied, since $\delta\omega = \mathrm{t}\delta\omega = \mathrm{n}\omega = 0$, for $\omega \in \Omega^0(M)$, but $\mathrm{n}\omega = 0$ holds automatically in the scalar case. This scenario was studied extensively in [DDF19]. Similarly, in the case $k = m$ the first condition becomes empty, since $\mathrm{d}\eta = \mathrm{n}\mathrm{d}\eta = \mathrm{t}\eta = 0$ if $\eta \in \Omega^m(M)$.

Equation (3.2) individuates therefore a class of boundary conditions which makes the operator $\Box$ formally self-adjoint. We would like to generalize the standard Dirichlet, Neumann and Robin boundary conditions to forms of higher degree. We recall that, as already mentioned in Example 1.4.10, for a scalar function u, Dirichlet, Neumann and Robin boundary conditions are obtained by imposing, respectively,

$$u|_{\partial M} = 0; \quad \mathrm{n}\mathrm{d}u = \frac{\partial u}{\partial\nu}\Big|_{\partial M} = 0; \quad u|_{\partial M} = f\frac{\partial u}{\partial\nu}\Big|_{\partial M}, \text{ for } f \in C^\infty(\partial M)\,,$$

$\nu \in \Gamma(\iota_\pm^* TM)$ being the outward pointing vector field normal to ∂M, with $\iota_\pm : \partial M \to M$ being the immersion map.
With non-scalar functions, we will have there are several other possibilities, which are obtained

[1] One can think of other possibilities such as Wentzell boundary conditions, which were studied in the scalar scenario in [DDF19; DFJA18; Zah18].

fixing a particular choice for $f, f' \in C^\infty(\partial M)$ in Equation (3.2). In between all these possibilities we highlight those which are of particular interest since we will be able to prove, at least in the static case, that these cases admit Green operators.

Definition 3.1.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $f \in C^\infty(\partial M)$. We call*

1. *space of k-forms with* Dirichlet *boundary condition*

$$\Omega^k_{\mathrm{D}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\,,\ \mathrm{n}\omega = 0\}\,, \tag{3.3}$$

2. *space of k-forms with* $\Box$-tangential *boundary condition*

$$\Omega^k_{\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\,,\ \mathrm{t}\delta\omega = 0\}\,, \tag{3.4}$$

3. *space of k-forms with* $\Box$-normal *boundary condition*

$$\Omega^k_{\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\,,\ \mathrm{n}\mathrm{d}\omega = 0\}\,. \tag{3.5}$$

4. *space of k-forms with* Robin $\Box$-tangential *boundary condition*

$$\Omega^k_{f_\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\delta\omega = f\mathrm{n}\omega\,,\ \mathrm{t}\omega = 0\}\,, \tag{3.6}$$

5. *space of k-forms with* Robin $\Box$-normal *boundary condition*

$$\Omega^k_{f_\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\,,\ \mathrm{n}\omega = 0\}\,, \tag{3.7}$$

Whenever the operator $\Box$ is restricted to act on one of these space we denote it with symbol $\Box_\sharp$ where $\sharp \in \{\mathrm{D}, \parallel, \perp, f_\parallel, f_\perp\}$.

Remark 3.1.3. It is worth highlighting the relations between the different classes of boundary conditions under the action of the Hodge dual operator. Using the commutation relations between the Hodge operator and the differential operators mentioned in Equation (1.5) and the definition in (3.2), one obtains that

$$\star\Omega^k_{f,f'}(M) = \Omega^{m-k}_{-f',-f}(M),$$

for any $f, f' \in C^\infty(\partial M)$. At the same time, with reference, to the space of k-forms in Definition 3.1.2 it holds

$$\star\Omega^k_{\mathrm{D}}(M) = \Omega^{m-k}_{\mathrm{D}}(M)\,, \qquad \star\Omega^k_{\parallel}(M) = \Omega^{m-k}_{\perp}(M)\,, \qquad \star\Omega^k_{f_\parallel}(M) = \Omega^{m-k}_{-f_\perp}(M)\,. \tag{3.8}$$

To recover the standard scalar Dirichlet and Neumann analogues, one can observe that $\mathrm{n}\omega = 0$, whenever $\omega \in \Omega^0(M)$. Hence $\mathrm{t}\omega = \omega|_{\partial M}$ and $\mathrm{n}\mathrm{d}\omega = \partial_\nu\omega|_{\partial M}$, where ν is the outward pointing

unit vector normal to the boundary. It follows that $\Omega^0_{\mathrm{D}}(M) = \Omega^0_{\|}(M)$ and $\Omega^0_{\perp}(M)$ are, respectively, the spaces of scalar functions (0-forms) that satisfy Dirichlet and Neumann boundary conditions. Moreover for $f = 0$ we have $\Omega^k_{f_{\|}}(M) = \Omega^k_{\|}(M)$ as well as $\Omega^k_{f_{\perp}}(M) = \Omega^k_{\perp}(M)$.
It is worth observing that, for a static spacetime (M, g), the boundary conditions 1-3, introduced in Definition 3.1.2, are themselves static, that is they do not depend explicitly on the time coordinate τ. Whenever $f \in C^\infty(\partial M)$ does not depend on τ, a similar statement holds true for $f_\perp$, $f_\|$ boundary conditions. This will play a key role when we will verify that Assumption 3.1.4 is valid on ultrastatic spacetimes – *cf.* Proposition 3.1.16 in the next Section.

3.1.1 Existence of Green operators on ultrastatic spacetimes

With reference to Section 1.4, we want to prove the existence of distinguished Green operators for the operator $\Box_\sharp$ for $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$. This is important since Green operators provide the necessary tools to construct the algebra of the observables for the underlying quantum system.

Recalling Remark 1.4.6, in view of Definition 1.1.9 and Definition 3.1.2 we require the following:

Assumption 3.1.4. *For all $f \in C^\infty(\partial M)$ and for all k such that $0 \leq k \leq m = \dim M$, there exist advanced $(-)$ and retarded $(+)$ fundamental solutions (or Green operators) for the d'Alembert-de Rham wave operator $\Box_\sharp$ where $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$. In other words there exist continuous maps $G^\pm_\sharp \colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M) \doteq \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ such that*

$$\Box \circ G^\pm_\sharp = \mathrm{Id}_{\Omega^k_{\mathrm{c}}(M)}\,, \qquad G^\pm_\sharp \circ \Box_{\mathrm{c},\sharp} = \mathrm{Id}_{\Omega^k_{\mathrm{c},\sharp}(M)}\,, \qquad \mathrm{supp}(G^\pm_\sharp \omega) \subseteq J^\pm(\mathrm{supp}(\omega))\,, \quad (3.9)$$

for all $\omega \in \Omega^k_{\mathrm{c}}(M)$ where $J^\pm$ denote the causal future and past and where $\Box_{\mathrm{c},\sharp}$ indicates that the domain of $\Box$ is restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$.

Remark 3.1.5. Notice that domain of $G^\pm_\sharp$ is not restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$. Furthermore the second identity in (3.9) cannot be extended to $G^\pm_\sharp \circ \Box = \mathrm{id}_{\Omega^k_{\mathrm{c}}(M)}$ since it would entail $G^\pm_\sharp \Box\omega = \omega$ for all $\omega \in \Omega^k_{\mathrm{c}}(M)$. Yet the left hand side also entails that $\omega \in \Omega^k_{\mathrm{c},\sharp}$, which is manifestly a contradiction.

Remark 3.1.6. To ensure Green hyperbolicity of $\Box_\sharp$ one should also prove the existence of distinguished Green operators for the formal adjoint of $\Box_\sharp$, according to Definition 1.4.2. In view of Lemma 3.1.1 this is not necessary, since $\Box$ is formally self-adjoint.

In this section we prove such existence whenever the underlying globally hyperbolic spacetime with timelike boundary is ultrastatic and $f \in C^\infty(\partial\Sigma)$ - appearing in the boundary conditions $f_\|, f_\perp$ - has definite sign.

We mimic the functional analysis technique of *boundary triples* used in the first place in [DDF19] for the scalar case. This method is extensively discussed in [BL12] and here we only recall the main results.

In the following, $\mathcal{H}$ will denote a complex Hilbert space and $S : \mathrm{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is a densely defined closed symmetric operator in $\mathcal{H}$. Since S is densely defined, a notion of adjoint operator S^* is well defined. Once again, we want to meet the physical requirements discussed in Section 2.3 in solving the first order Maxwell's equations, hence we impose the underlying system to be isolated, so that the flux of relevant physical quantities, such as those built from the stress-energy tensor, is zero through the boundary. To translate mathematically this requirement we need to look for symmetric extensions of S.

What we need are two surjective maps on the domain of S^* that allow us to write a Green formula similar to that of (2.45) and (2.46), which hold respectively for the domain of curl and of the first order Maxwell operator H, defined in Equation (2.40).

Definition 3.1.7. *A* boundary triple *for the adjoint operator S^* is a triple $(\mathsf{h}, \gamma_0, \gamma_1)$ consisting of a complex separable Hilbert space h and of two linear maps $\gamma_i : \mathrm{dom}(S^*) \to \mathsf{h}$, $i = 0, 1$ such that*

$$(S^* f, f')_{\mathcal{H}} - (f, S^* f')_{\mathcal{H}} = (\gamma_1 f, \gamma_0 f')_{\mathsf{h}} - (\gamma_0 f, \gamma_1 f')_{\mathsf{h}}\,, \quad \forall f, f' \in \mathrm{dom}(S^*)\,,$$

In addition we require the map $\gamma : \mathrm{dom}(S^*) \to \mathsf{h} \times \mathsf{h}$ such that $f \mapsto (\gamma_0(f), \gamma_1(f))$ to be surjective.

Remark 3.1.8. A boundary triple $(\mathsf{h}, \gamma_0, \gamma_1)$ for S^* exists if and only if S admits self-adjoint extensions in $\mathcal{H}$, or in other words the deficiency indexes $n_\pm(S) = \dim\ker(S^* \pm i)$ are equal.We notice that if $S \neq S^*$, then a boundary triple for S^* (if it exists) is not unique.

Boundary triples are a convenient tool to characterize the self-adjoint extensions of a large class of linear operators. The proof of the following proposition can be found in [DM95].

Proposition 3.1.9. *Let $S : D(S) \subseteq \mathsf{H} \to \mathsf{H}$ be a closed, symmetric operator. If Θ is a closed, densely defined linear relation, then $S_\Theta \doteq S^*|_{\ker(\gamma_1 - \Theta\gamma_0)}$ is a closed extension of S where $\ker(\gamma_1 - \Theta\gamma_0) \doteq \{\psi \in \mathsf{h} \mid (\gamma_0\psi, \gamma_1\psi) \in \Theta\}$. In addition the map $\Theta \to S_\Theta$ is one-to-one and $S^*_\Theta = S_{\Theta^*}$. Hence there is a one-to-one correspondence between self-adjoint relations Θ and self-adjoint extensions of S.*

Remark 3.1.10. We recall that, given a relation $\Theta \subseteq \mathsf{h} \times \mathsf{h}$, the adjont relation Θ^* is defined by

$$\Theta^* \doteq \{(y_1, y_2) \in \mathsf{h} \times \mathsf{h} \mid (x_1, y_2)_{\mathsf{h}} = (x_2, y_1)_{\mathsf{h}}\,,\ \forall (x_1, x_2) \in \Theta\}\,. \tag{3.10}$$

The relation Θ is self-adjoint if $\Theta = \Theta^*$.

One could argue that the method to parametrize the self-adjoint extensions of a differential operator here discussed is very similar to that of Section 2.3. The main difference with the method of Lagrangian subspaces is that in that framework it is not always clear how to characterize the extensions in terms of boundary conditions. Boundary triples, in fact, offer a solution to this problem. since the relation Θ encodes the choice of a boundary condition in the domain

$$\ker(\gamma_1 - \Theta\gamma_0). \tag{3.11}$$

Hence, to obtain a self-adjoint extension S_Θ, it suffices to check the self-adjointness of Θ.

Example 3.1.11. Following the discussion of [DDF19, Sec. 2.2], we illustrate the construction of boundary triples for a differential operator $A = -\Delta + q$ on a Riemannian manifold with boundary (Σ, h) of bounded geometry, where Δ is the Laplace-Beltrami operator built out of h and q is a strictly positive, bounded function. Since Σ is of bounded geometry, the Laplace-Beltrami operator is uniformly elliptic and the maximal domain in $\mathrm{L}^2(\Sigma)$ on which A can be defined is the Sobolev space $\mathrm{H}^2(\Sigma)$. Hence $A^* : \mathrm{H}^2(\Sigma) \to \mathrm{L}^2(\Sigma)$ defined by $A^* = A$ is the adjoint of A, whose domain is $\mathrm{dom}(A) = \mathrm{H}^2_0(\Sigma)$, *i.e.* the space of functions f that satisfy $\mathrm{t}f = \mathrm{n}\mathrm{d}f|_{\partial\Sigma} = \partial_\nu f|_{\partial\Sigma} = 0$ in the sense of Sobolev space traces. A is then a densely defined, closed, symmetric operator in $\mathrm{L}^2(\Sigma)$ and the Green identity

$$(A^* f, f')_{\mathrm{L}^2(\Sigma)} - (f, A^* f')_{\mathrm{L}^2(\Sigma)} = (\gamma_1 f, \gamma_0 f')_{\mathrm{L}^2(\partial\Sigma)} - (\gamma_0 f, \gamma_1 f')_{\mathrm{L}^2(\partial\Sigma)}\,, \tag{3.12}$$

holds for $f, f' \in \mathrm{dom}(A^*) = \mathrm{H}^2(\Sigma)$ and $\gamma_0 f = \mathrm{t}f = f|_{\partial\Sigma}$, $\gamma_1 f = -\mathrm{n}\mathrm{d}f = \partial_\nu f|_{\partial\Sigma}$. The maps γ_0, γ_1 represent the Dirichlet and Neumann boundary conditions, respectively. The Neumann boundary condition is recovered choosing $\Theta = \{(z, 0) \,|\, z \in \mathrm{L}^2(\partial\Sigma)\}$, since equation (3.11) becomes $\ker\gamma_1$. The Dirichlet condition is otherwise recovered choosing $\Theta = \{(0, z) \,|\, z \in \mathrm{L}^2(\partial\Sigma)\}$. The reason is due to the formulation which we have chosen so to emphasize the connection with the heuristic notion of boundary conditions.

The triple $(\mathrm{L}^2(\partial\Sigma), \gamma_0, \gamma_1)$ is not, however, a boundary triple, since the map $\gamma = (\gamma_0, \gamma_1) : \mathrm{dom}(A^*) = \mathrm{H}^2(\Sigma) \to \mathrm{L}^2(\partial\Sigma) \times \mathrm{L}^2(\partial\Sigma)$ is not surjective.
To solve the problem, one must observe that the trace maps t and n are surjective on the Sobolev space $H^{1/2}(\partial\Sigma)$, in view of Remark 1.3.10. Hence, since all separable Hilbert spaces are isomorphic, one can introduce the isomorphisms

$$j_\pm : \mathrm{H}^{\pm 1/2}(\partial M) \to \mathrm{L}^2(\partial M), \quad \iota_\pm : \mathrm{H}^{\pm 3/2}(\partial M) \to \mathrm{L}^2(\partial M)\,. \tag{3.13}$$

As shown in [DDF19, Prop. 24], with the following definitions, $(\mathrm{L}^2(\partial\Sigma), \Gamma_0, \Gamma_1)$ is indeed a boundary triple:

$$\begin{aligned} \Gamma_0 &: \mathrm{H}^2(M) \ni f \mapsto \iota_+ \gamma_0 f \in \mathrm{L}^2(\partial M) \\ \Gamma_1 &: \mathrm{H}^2(M) \ni f \mapsto j_+ \gamma_1 f \in \mathrm{L}^2(\partial M) \end{aligned}\,. \tag{3.14}$$

■

Since we have assumed that the underlying spacetime (M, g) is ultrastatic, Equation (1.1) entails that [Pfe09]

$$\Box = \partial_\tau^2 + S,$$

where S is a uniformly elliptic operator whose local form can be found in [Pfe09]. This entails that, in order to construct solution of (3.20), we can follow the rationale outlined in [DDF19]. To this end we start by focusing our attention on S analyzing it within the framework of boundary triples. We need a decomposition that separates the role of space and time in the definition of k-forms, since the technique of boundary triples allows us to deal with operators involving spatial coordinates and that do not evolve in time.

We follow the discussion in Section 2.1 and as a starting point we notice that being (M, g) globally hyperbolic, Theorem 1.1.2 ensures that M is diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ carries the metric h. Let us indicate with $\iota_\tau \colon \Sigma \to M$ the (smooth one-parameter group of) embedding maps which realizes Σ at time τ as $\iota_\tau \Sigma = \{\tau\} \times \Sigma \doteq \Sigma_\tau$. It holds $\Sigma_\tau \simeq \Sigma_{\tau'}$ for all $\tau, \tau' \in \mathbb{R}$. If follows that, for all $\omega \in \Omega^k(M)$ and $\tau \in \mathbb{R}$, $\omega|_{\Sigma_\tau} \in \Gamma(E\iota_\tau^*\Lambda^k T^*M)$, where $\Lambda^k T^*M$ is the k-th exterior power of the cotangent bundle over M. Moreover, recalling Section 2.1, we further decompose $\omega|_{\Sigma_\tau}$ as

$$\omega|_{\Sigma_\tau} = \omega_0 \wedge \mathrm{d}\tau + \omega_1 \mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{t}_{\Sigma_\tau}\omega\,.$$

where $\mathrm{t}_{\Sigma_\tau}\omega \in \Omega^k(\Sigma_\tau)$ while $\mathrm{n}_{\Sigma_\tau}\omega = (\star_{\Sigma_\tau}^{-1}\iota_\tau^*\star_M)\omega \in \Omega^{k-1}(\Sigma_\tau)$ (*cf.* Definition 1.2.3 and $\star$ denotes the Hodge dual as defined in Section 1.2). With the identification $\Sigma_t \simeq \Sigma_{t'}$ the decomposition induces the isomorphisms

$$\begin{aligned}
\Gamma(\iota_\tau^*\Lambda^k T^*M) &\simeq \Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma) \\
\omega &\to (\omega_0 \oplus \omega_1)\,, && (3.15)\\
\Omega^k(M) &\simeq C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k(\Sigma)) \\
\omega &\to (\tau \mapsto \mathrm{n}_{\Sigma_\tau}\omega) \oplus (\tau \mapsto \mathrm{t}_{\Sigma_\tau}\omega)\,. && (3.16)
\end{aligned}$$

Furthermore a direct computation shows that, for all $\omega \in \Omega^k(M)$, it holds that

$$-S\omega|_{\Sigma_\tau} = (\Delta_{k-1}\mathrm{n}_{\Sigma_\tau}\omega) \wedge \mathrm{d}\tau + \Delta_k \mathrm{t}_{\Sigma_\tau}\omega\,,$$

where Δ_k is the Laplace-Beltrami operator acting on k-forms, built out of h.
To build the boundary triples as in Definition 3.1.7, we discuss the following construction.
As Hilbert space we fix

$$\mathcal{H} \doteq \mathrm{L}^2\Omega^{k-1}(\Sigma) \oplus \mathrm{L}^2\Omega^k(\Sigma)\,,$$

where $\mathrm{L}^2\Omega^k(\Sigma)$ is defined in Definition 1.3.5 and in Remark 1.3.8 with the pairing $(\alpha, \beta)_\Sigma = \int_\Sigma \overline{\alpha} \wedge \star_\Sigma \beta$ for all $\alpha, \beta \in \Omega^k_\mathrm{c}(\Sigma)$.

Remark 3.1.12. With reference to Remark 2.2.2, we denote still with d_Σ and δ_Σ the extension to the space of square-integrable k-forms $\mathrm{L}^2\Omega^k(\Sigma)$ of the action of the differential and of the codifferential on $\Omega^k_{\mathrm{c}}(\Sigma)$.

Moreover, we identify $-S$ with $\Delta_{k-1} \oplus \Delta_k$ where Δ_k is the Laplace-Beltrami operator built out of h acting on k-forms. Observe that S can be regarded as an Hermitian and densely defined operator on $\mathrm{H}^2_0(\Lambda^{k-1}T^*\Sigma) \oplus \mathrm{H}^2_0(\Lambda^k T^*\Sigma)$ where $\mathrm{H}^2_0(\Lambda^k T^*\Sigma)$ is the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ with respect to the $\mathrm{H}^2(\Lambda^k T^*\Sigma)$-norm – *cf.* Equation (1.7). Here $E \equiv \Lambda^k T^*\Sigma$, where both the inner product and the connection are those induced from the underlying metric h. Hence standard arguments entail that S is a closed symmetric operator on $\mathcal{H}$ whose adjoint S^* is defined on the maximal domain $\mathrm{dom}(S^*) \doteq \{(\omega_0 \oplus \omega_1) \in \mathcal{H} \mid S(\omega_0 \oplus \omega_1) \in \mathcal{H}\}$. In addition $S^*(\omega_0 \oplus \omega_1) = S(\omega_0 \oplus \omega_1)$ for all $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$. As a consequence, S satisfies the requirements of Definition 3.1.7 and, in view of Remark 3.1.8 S^* admits a boundary triple.

Remark 3.1.13. To realize explicitly the boundary triple, we need boundary Hilbert space and the two boundary maps γ_0, γ_1. To construct them, we recall from Proposition 1.3.9 that there exists a continuous surjective map

$$\mathrm{res}_\ell : \mathrm{H}^\ell\Omega^k(\Sigma) \to \mathrm{H}^{\ell-\frac{1}{2}}\Omega^k(\partial\Sigma), \tag{3.17}$$

that extends the restriction on $\Omega^k_{\mathrm{c}}(\Sigma)$. Moreover, in terms of the tangential and normal traces $\mathrm{t} = \mathrm{t}_{\partial\Sigma}$ and $\mathrm{n} = \mathrm{n}_{\partial\Sigma}$, defined in Equation 1.11, one can locally write the restriction of any k-form $\omega \in \Omega^k_{\mathrm{c}}(\Sigma)$ to $\partial\Sigma$ as

$$\omega|_{\partial\Sigma} = \mathrm{t}\omega + \mathrm{n}\omega \wedge \mathrm{d}x,$$

where for every $p \in \partial\Sigma$, $\mathrm{d}x$ is the basis element of T^*_pM such that $\mathrm{d}x(\nu_p) = 1$ where ν_p is the outward pointing, unit vector normal to $\partial\Sigma$ at p.

Following the same reasoning of [DDF19] for the scalar case, we construct the following boundary triple for S^*:

Proposition 3.1.14. *Consider S^*, where $S : \mathrm{H}^2_0(\Lambda^{k-1}T^*\Sigma) \oplus \mathrm{H}^2_0(\Lambda^k T^*\Sigma) \to \mathcal{H}$ is defined as $(-\Delta_{k-1}) \oplus (-\Delta_k)$, then the triple $(\mathsf{h}, \gamma_0, \gamma_1)$ is a boundary triple for S^*, where*

- $\mathsf{h} = \mathsf{h}_0 \oplus \mathsf{h}_1$, *with* $\mathsf{h}_0 \doteq \mathrm{L}^2\Omega^{k-1}(\partial\Sigma) \oplus \mathrm{L}^2\Omega^{k-2}(\partial\Sigma)$ *while* $\mathsf{h}_1 = \mathrm{L}^2\Omega^{k-1}(\partial\Sigma) \oplus \mathrm{L}^2\Omega^k(\partial\Sigma)$;
- $\gamma_0 : \mathrm{dom}(S^*) \to \mathsf{h}$ *is such that, for all* $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$,

$$\gamma_0(\omega_0 \oplus \omega_1) = (\mathrm{n}\omega_0 \oplus \mathrm{t}\omega_0) \oplus (\mathrm{n}\omega_1 \oplus \mathrm{t}\omega_1)\,. \tag{3.18}$$

- $\gamma_1 : \mathrm{dom}(S^*) \to \mathsf{h}$ *is such that, for all* $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$,

$$\gamma_1(\omega_0 \oplus \omega_1) = (\mathrm{t}\delta_\Sigma\omega_0 \oplus \mathrm{n}\mathrm{d}_\Sigma\omega_0) \oplus (\mathrm{t}\delta_\Sigma\omega_1 \oplus \mathrm{n}\mathrm{d}_\Sigma\omega_1)\,. \tag{3.19}$$

Our goal is to apply these tools to construct advanced and retarded Green operators for the D'Alembert-de Rham wave operator $\Box$ acting on k-forms. In other words, calling as $\Lambda^k T^* \mathring{M}$ the k-th exterior power of the cotangent bundle over $\mathring{M}$, $k \geq 1$, and with $\boxtimes$ the external tensor product, we look for continuous maps $G^\pm : \Omega^k_{\mathrm{c}}(\mathring{M}) \to \Omega^k_{\mathrm{sc}}(\mathring{M})$ such that

$$\Box \circ G^\pm = G^\pm \circ \Box = \mathrm{id}|_{\Omega^k_{\mathrm{c}}(\mathring{M})} ,$$

while $\mathrm{supp}(G^\pm(\omega)) \subseteq J^\pm(\mathrm{supp}(\omega))$ for all $\omega \in \Omega^k_{\mathrm{c}}(\mathring{M})$ – *cf.* Assumption 3.1.4. Working at the level of integral kernels and setting $\mathcal{G}^\pm(\tau - \tau', x, x') = \theta[\pm(\tau - \tau')]\mathcal{G}(\tau - \tau', x, x')$, with $\mathcal{G} \in \mathcal{D}'(\mathring{M} \times \mathring{M}, \Lambda^k T^* \mathring{M} \boxtimes \Lambda^k T^* \mathring{M})$, this amounts to solving the following distributional, initial value problem

$$(\Box \otimes \mathbb{I})\, \mathcal{G} = (\mathbb{I} \otimes \Box)\, \mathcal{G} = 0\,, \qquad \mathcal{G}|_{\tau=\tau'} = 0\,, \qquad \partial_\tau \mathcal{G}|_{\tau=\tau'} = \delta_{\mathrm{diag}(\mathring{M})}\,. \tag{3.20}$$

where $\delta_{\mathrm{diag}(\mathring{M})}$ stands for the Dirac delta bi-distribution on $\mathring{M} \times \mathring{M}$ yielding $\delta_{\mathrm{diag}(\mathring{M})}(\omega_1 \boxtimes \omega_2) = (\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \in \Omega^k_{\mathrm{c}}(\mathring{M})$.
In view of Proposition 3.1.9 we can follow slavishly the proof of [DDF19, Th. 30] to infer the following statement:

Theorem 3.1.15. *Let (M, g) be an ultrastatic and globally hyperbolic spacetime with timelike boundary. Let $(\mathsf{h}, \gamma_0, \gamma_1)$ be the boundary triple built as per Proposition 3.1.14 associated to the operator S^*. Let Θ be a self-adjoint relation on h and let $S_\Theta \doteq S^*|_{\mathrm{dom}(S_\Theta)}$ where $\mathrm{dom}(S_\Theta) = \ker(\gamma_1 - \Theta\gamma_0)$. If the spectrum of S_Θ is bounded from below, then there exists unique advanced and retarded Green's operator $G^\pm_\Theta$ associated to $\partial^2_\tau + S_\Theta$. They are completely determined in terms of the bidistributions $\mathcal{G}^\pm_\Theta = \theta[\pm(\tau - \tau')]\mathcal{G}_\Theta$ where $\mathcal{G}_\Theta \in \mathcal{D}'(\mathring{M} \times \mathring{M}, \Lambda^k T^* \mathring{M} \boxtimes \Lambda^k T^* \mathring{M})$ is such that for $\omega_1, \omega_2 \in \Omega^k_{\mathrm{c}}(\mathring{M})$,*

$$\mathcal{G}_\Theta(\omega_1, \omega_2) = \int_{\mathbb{R}^2} \left(\omega_1|_\Sigma, S_{k,\Theta}^{-\frac{1}{2}} \sin(S_{k,\Theta}^{\frac{1}{2}}(\tau - \tau'))\omega_2|_\Sigma \right)_\Sigma \mathrm{d}\tau \mathrm{d}\tau' ,$$

where $(\ ,\ )_\Sigma$ stands for the pairing between k-forms and where ω_2 identifies an element in $\mathrm{dom}(S_\Theta)$ via the identifications (3.16). Moreover it holds that

$$\gamma_1\left(G^\pm_\Theta \omega\right) = \Theta\, \gamma_0 \left(G^\pm_\Theta \omega\right) , \quad \forall \omega \in \Omega^k_{\mathrm{c}}(\mathring{M})\,. \tag{3.21}$$

The last step consists of proving that the boundary conditions introduced in Definition 3.1.2 fall in the class considered in Theorem 3.1.15. In the following proposition we adopt for simplicity the notation $\mathrm{nd} = \mathrm{n}_{\partial\Sigma}\mathrm{d}_\Sigma$, $\mathrm{t}\delta = \mathrm{t}_{\partial\Sigma}\delta_\Sigma$.

Proposition 3.1.16. *The following relations on* h *are selfadjoint:*

$$\Theta_{\|} \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0\ ;\ 0 \oplus \mathrm{nd}\omega_0 \oplus 0 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\} \tag{3.22}$$

$$\Theta_{\perp} \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1\ ;\ \mathrm{t}\delta\omega_0 \oplus 0 \oplus \mathrm{t}\delta\omega_1 \oplus 0) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\} \tag{3.23}$$

$$\Theta_{f_{\|}} \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0\ ;\ f\mathrm{n}\omega_0 \oplus \mathrm{nd}\omega_0 \oplus f\mathrm{n}\omega_1 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\},\quad f \in C^\infty(\partial\Sigma)\ f \geq 0\,. \tag{3.24}$$

$$\Theta_{f_{\perp}} \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1\ ;\ \mathrm{t}\delta\omega_0 \oplus f\mathrm{t}\omega_0 \oplus \mathrm{t}\delta\omega_1 \oplus f\mathrm{t}\omega_1) \mid \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\},\quad f \in C^\infty(\partial\Sigma)\ f \leq 0\,. \tag{3.25}$$

Moreover the self-adjoint extension $S_{\Theta_\sharp}$ *for* $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$ *abides to the hypotheses of Theorem 3.1.15. The associated propagators* $G_\sharp$, $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$, *obey the boundary conditions as per Definition 3.1.2.*

Proof. With reference to Remark 3.1.10, we recall that a relation $\Theta \subseteq \mathsf{h} \times \mathsf{h}$ is self-adjoint if $\Theta = \Theta^*$. We show that $\Theta_{\|}, \Theta_{\perp}, \Theta_{f_{\|}}, \Theta_{f_{\perp}}$ are self-adjoint relations. Since the proof is very similar in all cases we shall consider only $\Theta_{\|}$. A direct computation shows that $\Theta_{\|} \subseteq \Theta_{\|}^*$. We prove the converse inclusion. Let $\underline{\alpha} := (\alpha_1 \oplus \ldots \alpha_4\, ;\, \alpha_5 \oplus \ldots \alpha_8) \in \Theta_{\|}^*$. Considering Equation (3.10) we find

$$(\mathrm{n}\omega_0, \alpha_5) + (\mathrm{n}\omega_1, \alpha_7) = (\mathrm{nd}\omega_0, \alpha_2) + (\alpha_4, \mathrm{nd}\omega_1, \alpha_4)\,, \qquad \forall \omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)\,. \tag{3.26}$$

Choosing ω_1 and $\mathrm{n}\omega_0 = 0$ – this does not affect the value $\mathrm{nd}\omega_0$ on account of Remark 1.2.5. It follows that $(\alpha_2, \mathrm{nd}\omega_0) = 0$ for all $\omega_0 \in \Omega^{k-1}_{\mathrm{c,n}}(\Sigma)$. Since nd is surjective it descends that $\alpha_2 = 0$. With a similar argument $\alpha_5 = 0$ as well as $\alpha_2 = 0$, $\alpha_4 = 0$. Finally, on account of Remark 1.2.5 there exists $\omega_0 \oplus \omega_1 \in \mathrm{dom}(S^*)$ such that

$$\mathrm{n}\omega_0 = \alpha_1\,, \qquad \mathrm{n}\omega_1 = \alpha_3\,, \qquad \mathrm{nd}\omega_0 = \alpha_6\,, \qquad \mathrm{nd}\omega_1 = \alpha_8\,.$$

It follows that $\alpha \in \Theta_{\|}$, that is, $\Theta_{\|} = \Theta_{\|}^*$.
In addition $S_{\Theta_\sharp}$ is positive definite for $\sharp \in \{\|, \perp, f_{\|}, f_{\perp}\}$. It descends from the following equality, which holds for all $\omega_0 \otimes \omega_1 \in \mathrm{dom}(S^*)$:

$$(\omega_0 \oplus \omega_1, S_{\Theta_\sharp}(\omega_0 \oplus \omega_1))_{\mathcal{H}} = \sum_{j=1}^{2} \left[\|\mathrm{d}\omega_i\|^2 + \|\delta\omega_i\|^2 + (\mathrm{n}\omega_i, \mathrm{t}\delta\omega_i) - (\mathrm{t}\omega_i, \mathrm{nd}\omega_i)\right],$$

where the last two terms are non-negative because of the boundary conditions and of the hypothesis on the sign of f. Therefore we can apply Theorem 3.1.15.
Finally we should prove that the propagators $G^{\pm}_{\Theta_\sharp}$ associated with the relations $\Theta_\sharp$ coincide with the propagators $G^{\pm}_\sharp$ introduced in Assumption 3.1.4. The fulfilment of the appropriate boundary conditions is a consequence of Lemma B.1. ■

Remark 3.1.17. It is worth mentioning that, although we have only considered test sections of compact support in $\mathring{M}$, such assumption can be relaxed allowing the support to intersect ∂M. In order to prove that this operation is legitimate, a rather natural strategy consists of realizing that the boundary conditions here considered fall in the (generalization of those of) Robin type. These were considered in [GW18] for the case of a real scalar field on an asymptotically anti de Sitter spacetime where, in between many results, it was proven the explicit form of the wavefront set of the advanced and retarded Green operators. In particular it was shown that two point lie in the wave front set either if they are connected directly by a light geodesic or by one which is reflected at the boundary. A direct inspection of their approach suggests that the same result holds true if one considers also static globally hyperbolic spacetimes with timelike boundary and vector valued fields. A detailed proof of this statement will be addressed explicitly in a future work.

3.1.2 Properties of Green operators for D'Alembert-de Rham wave operator

For convenience of the reader, we recall the main assumptions on the existence of Green operators for $\Box$, that we have proven for a particular case in the previous section:

Assumption 3.1.4. *For all $f \in C^\infty(\partial M)$ and for all k such that $0 \leq k \leq m = \dim M$, there exist advanced $(-)$ and retarded $(+)$ fundamental solutions for the d'Alembert-de Rham wave operator $\Box_\sharp$ where $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$. In other words there exist continuous maps $G^\pm_\sharp \colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M) \doteq \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ such that*

$$\Box \circ G^\pm_\sharp = \mathrm{id}_{\Omega^k_{\mathrm{c}}(M)}\,, \qquad G^\pm_\sharp \circ \Box_{\mathrm{c},\sharp} = \mathrm{id}_{\Omega^k_{\mathrm{c},\sharp}(M)}\,, \qquad \mathrm{supp}(G^\pm_\sharp \omega) \subseteq J^\pm(\mathrm{supp}(\omega))\,, \tag{3.9}$$

for all $\omega \in \Omega^k_{\mathrm{c}}(M)$ where $\Box_{\mathrm{c},\sharp}$ indicates that the domain of $\Box$ is restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$.

Corollary 3.1.18. *Under the same hypotheses of Assumption 3.1.4, if the Green operators $G^\pm_\sharp$ exist, they are unique.*

Proof. Suppose there exist two maps $G^-_\sharp, \widetilde{G}^-_\sharp \colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M)$ enjoying the properties of Equation (3.9). Then, for any but fixed $\alpha \in \Omega^k_{\mathrm{c}}(M)$ it holds

$$(\alpha, G^+_\sharp \beta) = (\Box G^-_\sharp \alpha, G^+_\sharp \beta) = (G^-_\sharp \alpha, \Box G^+_\sharp \beta) = (G^-_\sharp \alpha, \beta), \quad \forall \beta \in \Omega^k_{\mathrm{c}}(M)\,,$$

where we used both the support properties of the Green operators in (3.9) and Lemma 3.1.1 which guarantees that $\Box$ is formally self-adjoint on $\Omega^k_\sharp(M)$. Similarly, replacing $G^-_\sharp$ with $\widetilde{G}^-_\sharp$, it holds $(\alpha, G^+_\sharp \beta) = (\widetilde{G}^-_\sharp \alpha, \beta)$. It follows that $((\widetilde{G}^-_\sharp - G^-_\sharp)\alpha, \beta) = 0$, which implies $\widetilde{G}^-_\sharp \alpha = G^-_\sharp \alpha$, since the pairing between $\Omega^k(M)$ and $\Omega^k_{\mathrm{c}}(M)$ is separating. A similar result holds for the advanced Green operator. ■

In agreement with Proposition 1.4.7, this corollary can be also read as a consequence of the property that, for all $\omega \in \Omega^k_{\mathrm{c}}(M)$, $G^\pm_\sharp \omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ can be characterized as the unique solution to the Cauchy problem

$$\Box\psi = \omega\,, \qquad \mathrm{supp}(\psi)\cap M\setminus J^{\pm}(\mathrm{supp}(\omega)) = \emptyset\,, \qquad \psi\in\Omega^k_\sharp(M)\,. \tag{3.27}$$

Remark 3.1.19. In view of Remark 1.4.6, the Green operator $G^+_\sharp$ (*resp.* $G^-_\sharp$) can be extended to $G^+_\sharp\colon \Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$ (*resp.* $G^-_\sharp\colon \Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$). As a consequence the problem $\Box\psi=\omega$ with $\omega\in\Omega^k(M)$ always admits a solution lying in $\Omega^k_\sharp(M)$. As a matter of facts, consider any smooth function $\eta\equiv\eta(\tau)$, where $\tau\in\mathbb{R}$, *cf.* Equation (1.1), such that $\eta(\tau)=1$ for all $\tau>\tau_1$ and $\eta(\tau)=0$ for all $\tau<\tau_0$. Then calling $\omega^+\doteq\eta\omega$ and $\omega^-=(1-\eta)\omega$, it holds $\omega^+\in\Omega^k_{\mathrm{pc}}(M)$ while $\omega^-\in\Omega^k_{\mathrm{fc}}(M)$. Hence $\psi=G^+_\sharp\omega^++G^-_\sharp\omega^-\in\Omega^k_\sharp(M)$ is the sought solution.

We now prove some duality relations for $G^\pm_\sharp$, $\sharp\in\{\mathrm{D},\|,\perp,f_\|,f_\perp\}$ and characterization in terms of exact sequence that resumes the content of Assumption 3.1.4 and that mimic the one already present in [DDF19, Prop. 36] for $\Box$ and, for the first order Maxwell operator, in Theorem 2.3.8.

Proposition 3.1.20. *Whenever Assumption 3.1.4 is fulfilled, then, for all $\sharp\in\{\mathrm{D},\|,\perp,f_\|,f_\perp\}$, setting $G_\sharp\doteq G^+_\sharp-G^-_\sharp:\Omega^k_{\mathrm{c}}(M)\to\Omega^k_{\mathrm{sc},\sharp}(M)$, the following statements hold true:*

1. *for all $f\in C^\infty(\partial M)$ the following duality relations hold true:*

$$\star G^\pm_{\mathrm{D}} = G^\pm_{\mathrm{D}}\star\,, \qquad \star G^\pm_{\|} = G^\pm_{\perp}\star\,, \qquad \star G^\pm_{f_\|} = G^\pm_{f_\perp}\star\,. \tag{3.28}$$

2. *for all $\alpha,\beta\in\Omega^k_{\mathrm{c}}(M)$ it holds*

$$(\alpha, G^\pm_\sharp\beta) = (G^\mp_\sharp\alpha,\beta)\,. \tag{3.29}$$

3. *the interplay between $G_\sharp$ and $\Box_\sharp$ is encoded in the short exact sequence:*

$$0\to\Omega^k_{\mathrm{c},\sharp}(M)\xrightarrow{\Box_\sharp}\Omega^k_{\mathrm{c}}(M)\xrightarrow{G_\sharp}\Omega^k_{\mathrm{sc},\sharp}(M)\xrightarrow{\Box_\sharp}\Omega^k_{\mathrm{sc}}(M)\to 0\,, \tag{3.30}$$

where $\Omega^k_{\mathrm{c},\sharp}(M)\doteq\Omega^k_{\mathrm{c}}(M)\cap\Omega^k_\sharp(M)$.

Proof. We prove the different items separately. Starting from *1.*, we observe that $\star\Box=\Box\star$. Together with Remark 3.1.3, this entails that, for all $\alpha\in\Omega^k_{\mathrm{c}}(M)$,

$$\Box\star G^\pm_\sharp\alpha = \star\Box G^\pm_\sharp\star\alpha = \alpha\,.$$

On account of Remark 3.1.3, the uniqueness of the Green operators as per Corollary 3.1.18 entails (3.28).

2. Equation (3.29) is a consequence of the following chain of identities valid for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$

$$(\alpha, G^{\pm}_{\sharp}\beta) = (\Box G^{\mp}_{\sharp}\alpha, G^{\pm}_{\sharp}\beta) = (G^{\mp}_{\sharp}\alpha, \Box G^{\pm}_{\sharp}\beta) = (G^{\mp}_{\sharp}\alpha, \beta)\,,$$

where we used both the support properties of the Green operators and Lemma 3.1.1.

3. The exactness of the series is proven using the properties already established for the Green operators $G^{\pm}_{\sharp}$. The left exactness of the sequence is a consequence of the second identity in Equation (3.9) which ensures that $\Box_{\sharp}\alpha = 0$, $\alpha \in \Omega^k_{\mathrm{c},\sharp}(M)$, entails $\alpha = G^{+}_{\sharp}\Box_{\sharp}\alpha = 0$. In order to prove that $\ker G_{\sharp} = \Box\Omega^k_{\mathrm{c},\sharp}$, we first observe that $G_{\sharp}\Box_{\sharp}\Omega^k_{\mathrm{c},\sharp}(M) = \{0\}$ on account of Equation (3.9). Moreover, if $\beta \in \Omega^k_{\mathrm{c}}(M)$ is such that $G_{\sharp}\beta = 0$, then $G^{+}_{\sharp}\beta = G^{-}_{\sharp}\beta$. Hence, in view of the support properties of the Green operators $G^{+}_{\sharp}\beta \in \Omega^k_{\mathrm{c},\sharp}(M)$ and $\beta = \Box_{\sharp}G^{+}_{\sharp}\beta$. Subsequently we need to verify that $\ker\Box = G_{\sharp}\Omega^k_{\mathrm{c}}(M)$. Once more $\Box_{\sharp}G_{\sharp}\Omega^k_{\mathrm{c}}(M) = \{0\}$ follows from Equation (3.9). Conversely, let $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ be such that $\Box_{\sharp}\omega = 0$. On account of Lemma B.2 we can split $\omega = \omega^{+} + \omega^{-}$ where $\omega^{+} \in \Omega^k_{\mathrm{spc},\sharp}(M)$. Then $\Box_{\sharp}\omega^{+} = -\Box_{\sharp}\omega^{-} \in \Omega^k_{\mathrm{c},\sharp}(M)$ and

$$G_{\sharp}\Box_{\sharp}\omega^{+} = G^{+}_{\sharp}\Box_{\sharp}\omega^{+} + G^{-}_{\sharp}\Box_{\sharp}\omega^{-} = \omega\,.$$

To conclude we need to establish the right exactness of the sequence. Consider any $\alpha \in \Omega^k_{\mathrm{sc}}(M)$ and the equation $\Box_{\sharp}\omega = \alpha$. Consider the function $\eta(\tau)$ as in Remark 3.1.19 and let $\omega \doteq G^{+}_{\sharp}(\eta\alpha) + G^{-}_{\sharp}((1-\eta)\alpha)$. In view of Remark 3.1.19 and of the support properties of the Green operators, $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ and $\Box_{\sharp}\omega = \alpha$. ■

Remark 3.1.21. Following the same reasoning as in [Bär15] together with minor adaptation of the proofs of [DDF19], one may extend $G_{\sharp}$ to an operator $G_{\sharp}\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k_{\sharp}(M)$ for all $\sharp \in \{\mathrm{D}, \|, \perp, f_{\|}, f_{\perp}\}$. As a consequence the exact sequence of Proposition 3.1.20 generalizes as

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_{\sharp}(M) \xrightarrow{\Box_{\sharp}} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_{\sharp}} \Omega^k_{\sharp}(M) \xrightarrow{\Box_{\sharp}} \Omega^k(M) \to 0\,. \tag{3.31}$$

Remark 3.1.22. Proposition 3.1.20 and Remark 3.1.21 ensure that $\ker_{\mathrm{c}}\Box_{\sharp} \subseteq \ker_{\mathrm{tc}}\Box_{\sharp} = \{0\}$. In other words, there are no timelike compact solutions to the equation $\Box\omega = 0$ with $\sharp$-boundary conditions. More generally it can be shown that $\ker_{\mathrm{c}}\Box \subseteq \ker_{\mathrm{tc}}\Box = \{0\}$, namely there are no timelike compact solutions regardless of the boundary condition. This follows by standard arguments using a suitable energy functional defined on the solution space – *cf.* [DDF19, Thm. 30] for the proof for $k = 0$.

In studying Maxwell's equations for A, we will make extensive use of the Green operators for $\Box$ and we will need to intertwine the propagators and the differential operators d, δ. In general this does not happen for an arbitrary boundary condition. Hence we will consider only those $\perp, \|$ individuated in Definition 3.1.2. This will individuate the class of boundary conditions for which standard techniques can be applied in solving Maxwell's equations.

In view of the applications to the Maxwell operator, it is worth focusing specifically on the boundary conditions $\perp$, $\|$ individuated in Definition 3.1.2 since it is possible to prove a useful relation between the associated propagators and the operators d,δ.

Lemma 3.1.23. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G_{\|}^{\pm}\circ\mathrm{d}=\mathrm{d}\circ G_{\|}^{\pm}\qquad\text{on }\Omega^k_{\mathrm{t}}(M)\cap\Omega^k_{\mathrm{pc/fc}}(M)\,,\qquad G_{\|}^{\pm}\circ\delta=\delta\circ G_{\|}^{\pm}\qquad\text{on }\Omega^k_{\mathrm{pc/fc}}(M)\,,\tag{3.32}$$

$$G_{\perp}^{\pm}\circ\delta=\delta\circ G_{\perp}^{\pm}\qquad\text{on }\Omega^k_{\mathrm{n}}(M)\cap\Omega^k_{\mathrm{pc/fc}}(M)\,,\qquad G_{\perp}^{\pm}\circ\mathrm{d}=\mathrm{d}\circ G_{\perp}^{\pm}\qquad\text{on }\Omega^k_{\mathrm{pc/fc}}(M)\,.\tag{3.33}$$

Proof. From Equation (3.28) it follows that equations (3.32-3.33) are dual to each other via the Hodge operator. Hence we shall only focus on Equation (3.32).
For every $\alpha\in\Omega^k_{\mathrm{c}}(M)\cap\Omega^k_{\mathrm{t}}(M)$, $G_{\|}^{\pm}\mathrm{d}\alpha$ and $\mathrm{d}G_{\|}^{\pm}\alpha$ lie both in $\Omega^k_{\|}(M)$. In particular, using Equation (1.5b), $\mathrm{t}\delta\mathrm{d}G_{\|}^{\pm}\alpha=\mathrm{t}(\Box_{\|}-\mathrm{d}\delta)G_{\|}^{\pm}(\alpha)=\mathrm{t}\alpha=0$ while the second boundary condition is automatically satisfied since $\mathrm{t}\mathrm{d}G_{\|}^{\pm}=\mathrm{d}\mathrm{t}G_{\|}^{\pm}=0$. Hence, considering $\beta=G_{\|}^{\pm}\mathrm{d}\alpha-\mathrm{d}G_{\|}^{\pm}\alpha$, it holds that $\Box\beta=0$ and $\beta\in\Omega^k_{\|}\cap\Omega^k_{\mathrm{pc/fc}}(M)$. In view of Remark 3.1.19, this entails $\beta=0$. ■

We conclude this section with a corollary to Lemma 3.1.23 which shows that, when considering the difference between the advanced and the retarded Green operators, the support restrictions present in equations (3.32-3.33) disappear.

Corollary 3.1.24. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G_{\sharp}\circ\mathrm{d}=\mathrm{d}\circ G_{\sharp}\quad\text{on }\Omega^k_{\mathrm{tc}}(M)\,,\qquad G_{\sharp}\circ\delta=\delta\circ G_{\sharp}\quad\text{on }\Omega^k_{\mathrm{tc}}(M)\qquad\sharp\in\{\|,\perp\}\,.\tag{3.34}$$

Proof. In all cases the reasoning is similar as in the proof of Equation (3.32), but it requires the following characterization of $G_{\sharp}$. Since $M\simeq\mathbb{R}\times\Sigma$ – *cf.* Theorem 1.1.2 – let $\tau_0\in\mathbb{R}$ and consider $\alpha_0\in\Omega^k_{\mathrm{c}}(\Sigma_0)$, where $\Sigma_0:=\{\tau_0\}\times\Sigma$. Setting $\alpha:=\alpha_0\wedge\delta_{\tau_0}\mathrm{d}\tau$ we define a distribution-valued k-form and, following [Bär15, Lem. 4.1., Thm. 4.3], we can consider $G_{\sharp}\alpha$. It turns out that $G_{\sharp}\alpha$ is the unique solution to the Cauchy problem

$$\Box\psi=0\,,\qquad\mathrm{t}_{\Sigma_0}\psi=0\,,\qquad\mathrm{t}_{\Sigma_0}\mathcal{L}_{\partial_\tau}(\psi)=\alpha_0\,,\qquad\sharp\text{-boundary conditions for }\psi\,,\tag{3.35}$$

where $\mathrm{t}_{\Sigma_0}\colon\Omega^k(M)\to\Omega^k(\Sigma_0)$ is defined in (1.3) with $N\equiv\Sigma_0$, while $\mathcal{L}_{\partial_\tau}$ denotes the Lie derivative along the vector field ∂_τ.
With this characterization we can prove Equation (3.34). Focusing for simplicity on the first identity of (3.34) for $\sharp=\|$, we need to show that $\mathrm{d}G_{\|}\alpha$ and $G_{\|}\mathrm{d}\alpha$ solve the same Cauchy problem (3.35). While the analysis of the equation of motion and of the initial data does not differ from the counterpart on globally hyperbolic spacetimes with empty boundary, the only additional necessary information comes from $\mathrm{t}\delta\mathrm{d}G_{\|}^{\pm}\alpha=\mathrm{t}(\Box-\mathrm{d}\delta)G_{\|}^{\pm}\alpha=\mathrm{t}\alpha$, for all $\alpha\in\Omega^k_{\mathrm{tc}}(M)$. This entails that, being $G_{\|}=G_{\|}^{+}-G_{\|}^{-}$, $\mathrm{t}\delta\mathrm{d}G_{\|}\alpha=0$. ■

3.2 On the Maxwell operator

Maxwell's equations for the vector potential $A \in \Omega^1(M)$ with vanishing source read $\delta \mathrm{d} A = 0$. Hence, studying the space of solutions amounts to characterizing the kernel of the Maxwell operator $\delta\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$ in connection both to the D'Alembert - de Rham wave operator $\Box$ and to the identification of suitable boundary conditions. We shall keep the assumption that (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = m \geq 2$ – *cf.* Theorem 1.1.2. Notice that, if $k = m$, the Maxwell operator becomes trivial, while, if $k = 0$, it coincides with the D'Alembert - de Rham operator $\Box$. Hence this case falls in the one studied in the preceding section and in [DDF19]. Therefore, unless stated otherwise, henceforth we shall consider only $0 < k < m = \dim M$.

3.2.1 Spaces of solutions for selected boundary conditions

In analogy to the analysis of $\Box$, we observe that, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, from Equation (1.6), one can obtain the following Green formula:

$$(\delta \mathrm{d}\alpha, \beta) - (\alpha, \delta \mathrm{d}\beta) = (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_\partial - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_\partial \,. \tag{3.36}$$

In the same spirit of Lemma 3.1.1, the operator $\delta\mathrm{d}$ becomes formally self-adjoint if we restrict its domain to

$$\Omega^k_f(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\} \,, \tag{3.37}$$

where $f \in C^\infty(\partial M)$ is arbitrary but fixed. In what follows we will consider two particular boundary conditions which are directly related to the $\Box$-tangential and to the $\Box$-normal boundary conditions for the D'Alembert - de Rham operator, labeled $\|, \perp$, respectively – *cf.* Definition 3.1.2. We selected a particular class of boundary conditions from the entire domain (3.37) since in general it is not clear whether there exist intertwining relations between the propagators and the differential operators such as those of Lemma 3.1.23. This is an important obstruction to adapt our analysis to more general cases.

Definition 3.2.1. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $0 < k < \dim M$. We call*

1. *space of k-forms with $\delta\mathrm{d}$-tangential boundary condition, $\Omega^k_{\mathrm{t}}(M)$ as in Equation (1.4) with $N = \partial M$:*
$$\Omega^k_{\mathrm{t}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\} \,. \tag{3.38}$$
2. *space of k-forms with $\delta\mathrm{d}$-normal boundary condition*
$$\Omega^k_{\mathrm{nd}}(M) \doteq \{\omega \in \Omega^k(M) |\ \mathrm{n}\mathrm{d}\omega = 0\} \,. \tag{3.39}$$

In the following our first goal is to characterize the kernel of the Maxwell operator with a prescribed boundary condition, *cf.* Equation (3.37). To this end we need to focus on the *gauge invariance* of the underlying theory.
As we recalled in Section 1.5, gauge freedom is a feature which, in globally hyperbolic spacetimes with empty boundary, ensures that Maxwell's equations can be written as an hyperbolic PDE $\Box A = 0$ together with a constraint on the initial data $\delta A = 0$, which is the Lorenz gauge condition.
In the following, we generalize the concept of gauge invariance when the background has a non-vanishing timelike boundary. It turns out that there exist different notions of gauge invariance that depends on the choice of the boundary condition that we require on the solutions of Maxwell's equations. In particular, if we require a solution to be $A \in \Omega^k_{\mathrm{nd}}(M)$, we notice that the boundary condition $\mathrm{n}\mathrm{d}A = 0$ is invariant under the most general gauge transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-1}(M)$, while for $A \in \Omega^k_{\mathrm{t}}(M)$ the condition $\mathrm{t}A = 0$ is not. For this reason, this scenario is distinguished and we give to the space of solutions in $\Omega^k_{\mathrm{nd}}(M)$ a definition of gauge equivalence that is totally analogous to that with empty boundary. At the same time, for solutions that lie in $\Omega^k_{\mathrm{t}}(M)$, one must restrict the gauge group and our choice is the space of forms with vanishing tangential component. Such choice is fact not unique when working at a level of k-forms.

To avoid this quandary, one should resort to a more geometrical formulation of Maxwell's equations for $A \in \Omega^1(M)$, namely as originating from a theory for the connections of a principal $U(1)$-bundle over the underlying globally hyperbolic spacetime with timelike boundary, *cf.* [BHS14; BDS14] for the case with empty boundary. Following the nomenclature of the cited works and with reference to Equation (1.26), one should focus on the gauge transformations of the form $A \mapsto A' = A + \eta$, where there exists $f \in C^\infty(M, U(1))$ such that $\eta = f^*(\mu_{U(1)})$, with f^* denoting the pull-back of f and $\mu_{U(1)}$ is the Maurer-Cartan form on $U(1)$. Hence, the most general gauge group is the following:

$$B_{U(1)} = \{\eta = f^*(\mu_{U(1)}) \,|\, f \in C^\infty(M, U(1))\}.$$

This space certainly includes $\mathrm{d}\Omega^0(M)$, but it is characterized explicitly in [BHS14] as

$$B_{U(1)} = \{\eta \in \Omega^1_{\mathrm{d}}(M) \,|\, [\eta] \in H^1(M, 2\pi i\mathbb{Z})\},$$

where $H^1(M, 2\pi i\mathbb{Z})$ denotes the first de Rham cohomology group (see Appendix A) with coefficients in $2\pi i\mathbb{Z}$, so that the integral modulo $2\pi i$ is an integer. In this case the gauge group is not a vector space but a free $\mathbb{Z}$-module.
We point out that the analytic description employed in this thesis meets that of a gauge theory if one regards electromagnetism as a connection on a principal bundle with $\mathbb{R}$ as structure group. In that case the gauge group $B_{\mathbb{R}}$ reduces simply to $\mathrm{d}\Omega^0(M)$, that is the description classically used for the gauge group in globally hyperbolic manifolds with empty boundary and that we try

to include in our definitions.

In the case in hand this translates in the following characterization.

Definition 3.2.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $\delta\mathrm{d}$ be the Maxwell operator acting on $\Omega^k(M)$, $0 < k < \dim M$. We say that*

1. *$A \in \Omega^k_{\mathrm{t}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{t}}(M)$ if $A - A' \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$, namely if there exists $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-tangential boundary conditions is denoted by*
$$\mathrm{Sol}_{\mathrm{t}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)}\,. \tag{3.40}$$

2. *$A \in \Omega^k_{\mathrm{nd}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{nd}}(M)$ if there exists $\chi \in \Omega^{k-1}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-normal boundary conditions is denoted by*
$$\mathrm{Sol}_{\mathrm{nd}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}(M)}\,. \tag{3.41}$$

   Similarly the space of spacelike supported solutions with $\delta\mathrm{d}$-tangential (resp. $\delta\mathrm{d}$-normal) boundary conditions are
$$\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t,sc}}(M)}\,, \tag{3.42}$$
$$\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M)}\,. \tag{3.43}$$

The forms that solve the equations of motion, in the sense that they belong to the spaces defined above, are called *on-shell configurations*. On the other hand, the spaces of all possible configurations of the field, respectively $\Omega^k_{\mathrm{t}}(M)$, $\Omega^k_{\mathrm{nd}}(M)$ are called the spaces of *off-shell configurations*.

At this point we ask ourselves whether it is possible to use the Green operators for $\Box$, studied in Section 3.1, to characterize the spaces of solutions in Definition 3.2.2. The answer is positive for the selected boundary conditions since it is possible to find a representative in the gauge equivalence classes $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ (*resp.* $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$) that satisfies the Lorenz gauge $\delta A = 0$ – *cf.* [Ben16, Lem. 7.2].
In addition we provide a connection between $\delta\mathrm{d}$-tangential (*resp.* $\delta\mathrm{d}$-normal) boundary conditions with $\Box$-tangential (*resp.* $\Box$-normal) boundary conditions. These proofs rely heavily on the fact that the propagator G and the operator δ intertwine for our choice of boundary conditions, as shown in Corollary 3.1.24. Recalling Definition 3.1.2 of the $\Box$-tangential boundary condition, the following holds true.

Proposition 3.2.3. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ there exists a representative $A' \in [A]$ such that*

$$\Box_{\parallel} A' = 0\,, \qquad \delta A' = 0\,. \tag{3.44}$$

Moreover, the same result holds true for $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$.

Proof. We focus only on the first assertion, the proof of the second one being similar. Let $A \in [A] \in \mathrm{Sol}_{\mathrm{t}}(M)$, that is, $A \in \Omega^k(M)$, $\delta \mathrm{d} A = 0$ and $\mathrm{t}A = 0$. Consider any $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that

$$\Box\chi = -\delta A, \qquad \delta\chi = 0, \qquad \mathrm{t}\chi = 0\,. \tag{3.45}$$

In view of Assumption 3.1.4 and of Remark 3.1.19, we can fix $\chi = -\sum_{\pm} G^{\pm}_{\parallel}\delta A^{\pm}$, where $A^{\pm}$ is defined as in Remark 3.1.19. Per definition of $G^{\pm}_{\parallel}$, $\mathrm{t}\chi = 0$ while, on account of Corollary 3.1.24, $\delta\chi = -\sum_{\pm} \delta G^{\pm}_{\parallel}\delta A^{\pm} = 0$.
Hence A' is gauge equivalent to A as per Definition 3.2.2. ■

The proof of the analogous result for $\Omega^k_{\mathrm{nd}}(M)$ is slightly different and, thus, we discuss it separately. Recalling Definition 3.1.2 of the $\Box$-normal boundary conditions, the following statement holds true.

Proposition 3.2.4. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ there exists a representative $A' \in [A]$ such that*

$$\Box_{\perp} A' = 0\,, \qquad \delta A' = 0\,. \tag{3.46}$$

Moreover, the same result holds true for $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$.

Proof. As in the previous proposition, we can focus only on the first point. Let A be a representative of $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. Hence $A \in \Omega^k(M)$ so that $\delta\mathrm{d}A = 0$ and $\mathrm{nd}A = 0$. Consider first $\chi_0 \in \Omega^{k-1}(M)$ such that $\mathrm{nd}\chi_0 = -\mathrm{n}A$. The existence is guaranteed since the map nd is surjective – *cf.* Remark 1.2.5. As a consequence we can exploit the residual gauge freedom to select $\chi_1 \in \Omega^{k-1}(M)$ such that

$$\Box\chi_1 = -\delta\widetilde{A}\,, \qquad \delta\chi_1 = 0\,, \qquad \mathrm{nd}\chi_1 = 0 \qquad \mathrm{n}\chi_1 = 0\,, \tag{3.47}$$

where $\widetilde{A} = A + \mathrm{d}\chi_0$. Let $\eta \equiv \eta(\tau)$ be a smooth function such that $\eta = 0$ if $\tau < \tau_0$ while $\eta = 1$ if $\tau > \tau_1$, *cf.* Remark 3.1.19. Since $\mathrm{n}\widetilde{A} = 0$ we can fine tune η in such a way that both $\widetilde{A}^{+} \doteq \eta\widetilde{A}$ and $\widetilde{A}^{-} \doteq (1-\eta)\widetilde{A}$ satisfy $\mathrm{n}\widetilde{A}^{\pm} = 0$. Equation (1.5b) entails that $\mathrm{n}\delta A^{\pm} = -\delta\mathrm{n}A^{\pm} = 0$. Hence we can apply Lemma 3.1.23 and set $\chi_1 = -\sum_{\pm} G^{\pm}_{\perp}\delta\widetilde{A}^{+}$. Calling $A' = A + \mathrm{d}(\chi_0 + \chi_1)$ we obtained the desired result. ■

As it is well known, in globally hyperbolic spacetimes with empty boundary Lorenz gauge leaves a residual freedom in the choice of A'. Classically, as recalled in Section 1.5, if the boundary is

empty, Lorenz gauge is imposed by requiring that, for any but fixed $A \in \Omega^1(M)$, there exists $\chi \in \Omega^0$ such that $\delta A' = \delta(A + \mathrm{d}\chi) = 0$. Since the equation $\delta\mathrm{d}\chi = \Box\chi = -\delta A$ admits solutions, the existence of χ is ensured, but not the uniqueness, since χ is determined modulo solutions of the homogeneous equations $\Box\chi = 0$.
A direct inspection of (3.45) and of (3.46) unveils that even in the case with non-empty boundary a residual freedom is left. This amount either to

$$\mathcal{G}_{\mathrm{t}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0\,,\ \mathrm{t}\chi = 0\}\,,$$

or, in the case of a $\delta\mathrm{d}$-normal boundary condition, to

$$\mathcal{G}_{\mathrm{nd}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0,\ \mathrm{n}\chi = 0\ ,\ \mathrm{nd}\chi = 0\}\,. \tag{3.48}$$

Observe that, in the definition of $\mathcal{G}_{\mathrm{nd}}(M)$, we require χ to be in the kernel of $\delta\mathrm{d}$. Nonetheless since the actual reduced gauge group is $\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)$ we can work with $\chi_0 \in \Omega^{k-1}(M)$ such that $\Box\chi_0 = 0$. As a matter of fact for all $\chi \in \mathcal{G}_{\mathrm{nd}}$ we can set $\chi_0 \doteq \chi + \mathrm{d}\lambda$ where $\lambda \in \Omega^{k-2}(M)$ is such that $\Box\lambda = -\delta\chi$ and $\mathrm{n}\lambda = \mathrm{nd}\lambda = 0$ – *cf.* Proposition 3.2.4. In addition $\mathrm{d}\chi = \mathrm{d}\chi_0$.
To better codify the results of the preceding discussion, it is also convenient to introduce the following linear spaces:

$$\mathcal{S}^{\Box}_{\mathrm{t}}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{t}A = 0\}\,, \tag{3.49}$$
$$\mathcal{S}^{\Box}_{\mathrm{nd}}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{n}A = 0,\ \ \mathrm{nd}A = 0\}\,. \tag{3.50}$$

Hence Propositions 3.2.3-3.2.4 can be summarized as stating the existence of the following isomorphisms:

$$\mathcal{S}_{\mathcal{G}_{\mathrm{t}},k}(M) \doteq \frac{\mathcal{S}^{\Box}_{\mathrm{t}}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{t}}(M)} \simeq \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad \mathcal{S}_{\mathcal{G}_{\mathrm{nd}},k}(M) \doteq \frac{\mathcal{S}^{\Box}_{\mathrm{nd}}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)} \simeq \mathrm{Sol}_{\mathrm{nd}}(M)\,. \tag{3.51}$$

3.3 Introduction to the algebraic formalism

In this section we give an overview on the algebraic approach to quantum field theory, with the aim to associate a unital $*$-algebra both to $\mathrm{Sol}_{\mathrm{t}}(M)$ and to $\mathrm{Sol}_{\mathrm{nd}}(M)$, whose elements are interpreted as the observables of the underlying quantum system. We recall that the corresponding question, when the underlying background (M, g) is globally hyperbolic manifold with $\partial M = \emptyset$ has been thoroughly discussed in the literature – *cf.* [Ben16; DS13; HS13; SDH14].
For the key definitions we follow mainly [KM15].

Definition 3.3.1. *We call $\mathcal{A}$*

- *an associative* algebra *$\mathcal{A}$ is a complex vector space endowed with an associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, distributive with respect to the sum and satisfying $\alpha ab = (\alpha a)b = a(\alpha b)$ if $\alpha \in \mathbb{C}$ and $a, b \in \mathcal{A}$;*

- *a* $*$*-algebra if it is an associative algebra and admits an* involution, *i.e. an anti-linear map,* $a \to a^*$ *which is involutive -* $(a^*)^*$ *- and* $(ab)^* = b^*a^*$, *for any* $a, b \in \mathcal{A}$*;*

- unital *if contains a multiplicative unit* $\mathbb{I} \in \mathcal{A}$.

A set $G \subset \mathcal{A}$ *is said to generate the algebra* $\mathcal{A}$*, and the elements of* G *are said generators of* $\mathcal{A}$*, if each element of* $\mathcal{A}$*is a finite complex linear combination of products (with arbitrary number of factors) of elements of* G*. The centre of the algebra* $\mathcal{A}$ *is the collection of all* $z \in \mathcal{A}$ *commuting with all elements of* $\mathcal{A}$*.*

Traditional quantum mechanics deals with operators on Hilbert spaces. In particular, the observables of a quantum system are self-adjoint operators. Such operators, as it is well known from basic examples such as position and momentum are not bounded by the operator norm, but many of the features of the quantum theory can be understood focusing on an algebra of bounded operators.
This is a special realization of a so-called C^*-algebra. This is a $*$-algebra which is a Banach space with respect to a norm $\| \; \|$ and it satisfies $\|ab\| \leq \|a\| \, \|b\|$ and $\|a^*a\| = \|a\|^2$. This implies $\|a^*\| = \|a\|$ and if a C^*-algebra is unital, $\|\mathbb{I}\| = 1$. A unital $*$-algebra admits a unique norm making it a C^*-algebra.
For the reasons we mentioned, we shall not use C^*-algebras, even if they still have a theoretical interest, but we will focus on the construction of a $*$-algebra.

Definition 3.3.2. *A two-sided ideal of an algebra* $\mathcal{A}$ *is a linear complex subspace* $\mathcal{I} \subset \mathcal{A}$ *such that* $ab \in \mathcal{I}$ *and* $ba \in \mathcal{I}$ *if* $a \in \mathcal{A}$ *and* $b \in \mathcal{I}$*. In a* $*$*-algebra, a two-sided ideal* $\mathcal{I}$ *is said to be a two-sided* $*$*-ideal if it is also closed with respect to the involution. In other words* $a^* \in \mathcal{I}$ *if* $a \in \mathcal{I}$*. An algebra* $\mathcal{A}$ *is simple if it does not admit proper two-sided ideals different form* $\{0\}$ *and* $\mathcal{A}$ *itself.*

We will deal mainly with universal tensor $*$-algebras generated by a complex vector space $\mathcal{O}$. These are defined as $\mathcal{T}[\mathcal{O}] \doteq \bigoplus_{n=0}^{\infty} \mathcal{O}^{\otimes n}$, with $\mathcal{O}^{\otimes 0} \equiv \mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. To obtain the quantum algebra of observable we will quotient by a $*$-ideal generated in such a way to encode the canonical commutation relations (CCR).

3.3.1 The generators of the algebra of observables

Our goal is to find, for both $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions, a $*$-algebra $\mathcal{A}$ of functionals (generated by $\mathcal{O}$) which can be thought of as *classical observables*. To wit one can extract any information about a given field configuration by means of these functionals and, at the same time, each of them provides some information which cannot be detected by any other functional. To this end, we require the pairing between $\mathcal{A}$ and the space of solutions to Maxwell's equations with prescribed boundary conditions to be *optimal*. Optimality has to be understood in the following sense. Let Sol denote the space of solutions of a linear hyperbolic PDE.

Definition 3.3.3. *A $*$-algebra of observables $\mathcal{A}$ generated by a vector space $\mathcal{O}$ is* optimal *if*

1. *$\mathcal{O}$ is* separating. *In other words it contains enough functional to distinguish between different on-shell configurations, namely*

$$(\alpha, A) = 0 \textit{ for any } \alpha \in \mathcal{O} \textit{ implies } A = 0 \in \mathrm{Sol};$$

2. *$\mathcal{O}$ is* non redundant. *In other words*

$$(\alpha, A) = 0 \textit{ for any } A \in \mathrm{Sol} \textit{ implies } \alpha = 0 \in \mathcal{O}.$$

It will turn out that the correct optimal generators for the algebra of observables for the two boundary conditions considered (labeled respectively t, nd) are

$$\mathcal{O}_{\mathrm{t}}(M) \doteq \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \quad \mathcal{O}_{\mathrm{nd}}(M) \doteq \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}. \tag{3.52}$$

Moreover, we will prove that $\mathcal{O}_{\mathrm{t}}(M)$ is isomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and it can be endowed with a symplectic form $\widetilde{G}_{\|}$, while $\mathcal{O}_{\mathrm{nd}}(M)$ does possess only a presymplectic structure.

In the following we justify from an intuitive point of view the choice of the generators in (3.52). For an account in the case of empty boundary, see [Ben16, Sec. 7.2].
The following discussion will be focused at first on $\delta\mathrm{d}$-tangential boundary conditions. As a starting point, we consider the linear functional on the space of off-shell configurations $\Omega^k_{\mathrm{t}}(M)$. For $\alpha \in \Omega^k_{\mathrm{c}}(M)$

$$F_\alpha : \Omega^k_{\mathrm{t}}(M) \longrightarrow \mathbb{C}\,, \tag{3.53}$$

$$\beta \longmapsto (\alpha, \beta)\,, \tag{3.54}$$

where $(\ ,\ )$ is the pairing between k-forms as per equation (1.2). We require the functionals F_α to be invariant under gauge transformations, hence we impose that the space from which we choose α satisfies

$$F_\alpha\left[\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)\right] = \{0\}.$$

Since we imposed boundary conditions, the right-hand side of Equation (1.6) vanishes and hence we require

$$F_\alpha(\mathrm{d}\beta) = (\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta) = 0\,,\ \forall \beta \in \Omega^{k-1}_{\mathrm{t}}(M).$$

This implies that $\delta\alpha = 0$ and hence the space of linear functionals invariant under gauge equivalence is chosen to be $\{F_\alpha \,|\, \alpha \in \Omega^k_{\mathrm{c},\delta}(M)\} \simeq \Omega^k_{\mathrm{c},\delta}(M) \doteq \Omega^k_{\mathrm{c}}(M) \cap \ker\delta$.
The next step is to include the dynamics encoded by $\delta\mathrm{d}A = 0$, so that the functionals can be evaluated on $\mathrm{Sol}_{\mathrm{t}}(M)$. To this end we force the functionals not to be defined if $\delta\mathrm{d}A \neq 0$,

$A \in \Omega^k_{\mathrm{t}}(M)$. This is obtained by taking as space of functionals the quotient

$$\mathcal{O}_{\mathrm{t}}(M) = \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}. \tag{3.55}$$

The evaluation of $[\alpha] \in \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ and $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ will be defined as the product (α, A) for arbitrary representatives $\alpha \in [\alpha]$, $A \in [A]$. The proof that the pairing is well defined will be carried out in Section 3.4.

Adapting the previous arguments to the $\delta\mathrm{d}$-normal boundary condition requires a particular discussion. As before, we require the functionals F_α to be invariant under the action of the gauge group, which in this case is the whole $\mathrm{d}\Omega^{k-1}(M)$. Hence we impose that the space from which we choose α satisfies

$$F_\alpha\left[\mathrm{d}\Omega^{k-1}(M)\right] = \{0\}.$$

This time the right-hand side of Equation (1.6) does not vanish if we do not impose further restrictions, namely $\mathrm{n}\alpha = 0$. Hence, if $\mathrm{n}\alpha = 0$, $(\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta)$ and we have

$$F_\alpha(\mathrm{d}\beta) = (\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta) = 0\,,\ \forall\beta \in \Omega^{k-1}_{\mathrm{nd}}(M).$$

This implies that $\delta\alpha = 0$ (i.e. $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M) = \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{n}}(M) \cap \ker\delta$) and imposing the equations of motion we obtain

$$\mathcal{O}_{\mathrm{nd}}(M) = \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}. \tag{3.56}$$

This is antithetical to the case of $\delta\mathrm{d}$-tangential boundary conditions, where β is required to satisfy $\mathrm{t}\beta = 0$ – *cf.* Definition 3.2.2 – and therefore α is not forced to satisfy any boundary condition. Actually, $\delta\alpha = 0$ and $\mathrm{n}\alpha = 0$ are necessary to ensure gauge-invariance, namely $(\alpha, \mathrm{d}\beta) = 0$ for all $\beta \in \Omega^k(M)$.

Symplectic structures

In this subsection, we characterize the spaces $\mathrm{Sol}_{\mathrm{t,nd}}(M)$ as symplectic spaces and we overview technical results that connect them to the generators of the algebra of observables that we are looking for, (3.55) and (3.56).

In view of Definition 2.3.3, we recall that, given a complex vector space S and a map $\sigma\colon \mathsf{S}\times\mathsf{S} \to \mathbb{C}$, the pair (S, σ) is called complex symplectic space if σ is sesquilinear, non-degenerate[2] and $\sigma(x, y) = -\overline{\sigma(y, x)}$ for all $x, y \in \mathsf{S}$. If we do not require σ to be non-degenerate, we call (S, σ) a presymplectic space.

[2] σ is non-degenerate if $\sigma(x, y) = 0$ for all $y \in \mathsf{S}$ implies $x = 0$.

It is noteworthy that both $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ can be endowed with a presymplectic form – *cf.* [HS13, Prop. 5.1]. The presence of symplectic spaces is motivated by the analogy with classical mechanics, in particular the spaces $\mathrm{Sol}_{\mathrm{t,nd}}(M)$ are seen as classical phase spaces. The spaces $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$, $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$ are, respectively, the symplectic and pre-symplectic spaces of observables describing the classical theory of the Maxwell field on M, which is the starting point for the quantization scheme, which in the Bosonic case is based on the existence of a CCR-representation algebra of the aforementioned symplectic spaces – *cf.* [HS13, Def. 4.3].

Proposition 3.3.4. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Let $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and, for $A_1 \in [A_1]$, let $A_1 = A_1^+ + A_1^-$ be any decomposition such that $A^+ \in \Omega^k_{\mathrm{spc,t}}(M)$ while $A^- \in \Omega^k_{\mathrm{sfc,t}}(M)$ – cf. Lemma B.2. Then the following map $\sigma_{\mathrm{t}}\colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)^{\times 2} \to \mathbb{R}$ is a presymplectic form:*

$$\sigma_{\mathrm{t}}([A_1], [A_2]) = (\delta \mathrm{d} A_1^+, A_2)\,, \qquad \forall [A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,. \tag{3.57}$$

A similar result holds for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ and we denote the associated presymplectic form σ_{nd}. In particular for all $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ it holds $\sigma_{\mathrm{nd}}([A_1], [A_2]) \doteq (\delta \mathrm{d} A_1^+, A_2)$ where $A_1 \in [A_1]$ is such that $A \in \Omega^k_{\mathrm{sc},\perp}(M)$.

Proof. See Appendix C, Prop. C.1 ∎

Working either with $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{t}}(M)$ or $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{nd}}(M)$ leads to the natural question whether it is possible to give an equivalent representation of these spaces in terms of compactly supported k-forms. Using Assumption 3.1.4, the following proposition holds true:

Proposition 3.3.5. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then the following linear maps are isomorphisms of vector spaces*

$$G_\parallel \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad G_\parallel \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{c,t}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,, \tag{3.58}$$

$$G_\perp \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{tc,nd}}(M)} \to \mathrm{Sol}_{\mathrm{nd}}(M)\,, \qquad G_\perp \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)\,, \tag{3.59}$$

Proof. See Appendix C, Prop. C.2 ∎

The following proposition shows that the isomorphisms introduced in Proposition 3.3.5 for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ lift to isomorphisms of presymplectic spaces.

Proposition 3.3.6. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. The following statements hold true:*

1. *$\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta \mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ is a pre-symplectic space if endowed with the bilinear map $\widetilde{G}_\parallel([\alpha], [\beta]) \doteq (\alpha, G_\parallel \beta)$.*

Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \widetilde{G}_\parallel\right)$ *is symplectomorphic to* $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$.

2. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_\perp([\alpha],[\beta]) \doteq (\alpha, G_\perp\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_\perp\right)$ *is pre-symplectomorphic to* $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Proof. See Appendix C, Prop. C.3 ∎

Remark 3.3.7. Notice that, on account of Propositions 3.3.4-3.3.6, $(\mathcal{O}_{\mathrm{nd}}, \widetilde{G}_\perp)$ is a presymplectic proper subspace of $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_\perp\right)$ and therefore it is not symplectomorphic to $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Remark 3.3.8. Following [HS13, Cor. 5.3], σ_{t} (*resp.* σ_{nd}) do not define in general a symplectic form on the space of spacelike compact solutions $\mathrm{Sol}_{\mathrm{t}}(M)$ (*resp.* $\mathrm{Sol}_{\mathrm{nd}}(M)$). A direct characterization of this deficiency is best understood by introducing the following quotients:

$$\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{t}} := \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) \mid \delta\mathrm{d}A = 0\,,\ \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M) \cap \Omega^k_{\mathrm{sc}}(M)}\,, \tag{3.60}$$

$$\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}} := \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) \mid \delta\mathrm{d}A = 0\,,\ \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}(M) \cap \Omega^k_{\mathrm{sc}}(M)}\,, \tag{3.61}$$

$\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ is symplectic with respect to the form $\sigma_{\mathrm{nd}}([A_1],[A_2]) = (\delta\mathrm{d}A_1^+, A_2)$. This can be shown as follows: If $\sigma_{\mathrm{nd}}([A_1],[A_2]) = 0$ for all $[A_1] \in \widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ then, choosing $A_1 = G_\perp\alpha$ with $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$ leads to $0 = \sigma_\perp([G_\perp\alpha],[A_2]) = (\alpha, A_2)$ – *cf.* Proposition 3.3.6. This entails $\mathrm{d}A_2 = 0$ as well as $A_2 = 0 \in H_{k,\mathrm{c,n}}(M)^* \simeq H^k(M)$ – *cf.* Appendix A. Therefore $A_2 = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}(M)$ that is $A_2 = 0$ in $\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}(M)$. A similar result holds, mutatis mutandis, for $\|$. The net result is that $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$ (*resp.* $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$) is symplectic if and only if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ (*resp.* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$). This is in agreement with the analysis in [BDS14] for the case of globally hyperbolic spacetimes (M, g) with $\partial M = \emptyset$.

Example 3.3.9. We give an example where $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subsetneq \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a proper inclusion – *cf.* [HS13, Ex. 5.7]. Consider half-Minkowski spacetime $\mathbb{R}^m_+ := \mathbb{R}^{m-1} \times \overline{\mathbb{R}_+}$ with flat metric and let $p \in \mathring{\mathbb{R}}^m_+$. We introduce $M := \mathbb{R}^m_+ \setminus J(p)$ endowed with the restriction to M of the Minkowski metric. This spacetime is still globally hyperbolic with timelike boundary. Let now $p \in B_1 \subset B_2$, where B_1, B_2 are open balls in $\mathbb{R}^{m-1}_+$ centered at p.
We consider $\psi \in \Omega^0(M)$ such that $\psi|_{J(B_1 \cap M)} = 1$ and $\psi|_{J(B_2 \cap M)} = 0$. In addition we introduce $\varphi \in \Omega^0_{\mathrm{tc}}(M)$ such that: (a) for all $x \in M$, $\varphi(x)$ depends only on $\tau(x)$ – *cf.* Theorem 1.1.2; (b) $\chi := \varphi\psi \in \Omega^0_{\mathrm{tc}}(M)$ is such that $\mathrm{t}\chi = \chi|_{\partial M} = 0$; (c) there exists an interval $I \subset \mathbb{R}$ such that $\varphi|_I = 1$.

In other words φ plays the rôle of a cut-off function so that $\chi \equiv \chi(\tau)$ does not vanish only for values of τ whose associated Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ is such that $(\Sigma_\tau \cap J(B_2)) \cap \partial M = \emptyset$. It follows that $\mathrm{d}\chi \in \Omega^1_{\mathrm{c}}(M) \subseteq \Omega^1_{\mathrm{sc}}(M)$. Yet there does not exist $\zeta \in \Omega^1_{\mathrm{sc}}(M)$ such that $\mathrm{d}\zeta = \mathrm{d}\chi$. Indeed, let us consider the curve $\gamma_s \subseteq M$ parametrized by $(s, x, 0, \dots) \in M$ where $s \in I \subset \mathbb{R}$ is such that $\varphi(s) = 1$ for all $s \in I$, while $x \in (x(p), +\infty) - x(p)$ denotes the x-coordinate of p. Integration along γ_s yields

$$\int_{\gamma_s} \iota^*_{\gamma_s} \mathrm{d}\chi = -1\,, \qquad \int_{\gamma_s} \iota^*_{\gamma_s} \mathrm{d}\zeta = 0\,.$$

■

3.4 The algebra of observables for $\mathrm{Sol}_{\mathrm{t}}(M)$ and for $\mathrm{Sol}_{\mathrm{nd}}(M)$

In this section we prove that the generators in equation (3.52) gives rise to optimal $*$-algebras of observables. We study their key structural properties and we comment on their significance. On account of the different behaviour of $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions we discuss each algebra separately.

The algebra of observables for $\mathrm{Sol}_{\mathrm{t}}(M)$

Our analysis mimic that of [Ben16; DS13; HS13; SDH14] for globally hyperbolic spacetimes with empty boundary.

Following the discussion in Subsection 3.3.1, we prove that a unital $*$-algebra $\mathcal{A}_{\mathrm{t}}(M)$ built out of distinguished linear functionals over $\mathrm{Sol}_{\mathrm{t}}(M)$, whose collection is optimal when tested on configurations in $\mathrm{Sol}_{\mathrm{t}}(M)$, is of the form (3.55).

Taking into account the discussion in the preceding sections, particularly Equation (3.52), we introduce the following structures.

Definition 3.4.1. *Let* (M, g) *be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{t}}(M)$*, the associative, unital* $*$*-algebra*

$$\mathcal{A}_{\mathrm{t}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]}\,, \qquad \mathcal{O}_{\mathrm{t}}(M) = \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}\,. \tag{3.62}$$

Here $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)] \doteq \bigoplus_{n=0}^{\infty} \mathcal{O}_{\mathrm{t}}(M)^{\otimes n}$ *is the universal tensor algebra with* $\mathcal{O}_{\mathrm{t}}(M)^{\otimes 0} \equiv \mathbb{C}$*, while the* $*$*-operation is the one induced from complex conjugation. In addition* $\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]$ *is the* $*$*-ideal generated by elements of the form* $[\alpha] \otimes [\beta] - [\beta] \otimes [\alpha] - i\widetilde{G}_{\|}([\alpha], [\beta])\mathbb{I}$*, where* $[\alpha], [\beta] \in \mathcal{O}_{\mathrm{t}}(M)$ *while* $\widetilde{G}_{\|}$ *is defined in Proposition 3.3.6 and* $\mathbb{I}$ *is the identity of* $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]$.

As recalled in Section 3.3, the ideal is generated so that canonical commutation relations are imposed:

$$[\alpha] \otimes [\beta] - [\beta] \otimes [\alpha] = i\widetilde{G}_{\|}([\alpha], [\beta])\mathbb{I},\, [\alpha], [\beta] \in \mathcal{A}_{\mathrm{t}}(M).$$

On account of its definition, to study the properties of the algebra it suffices to focus mainly on the properties of the generators $\mathcal{O}_{\mathrm{t}}(M)$. In particular, in the next proposition we follow the rationale advocated in [Ben16] proving that $\mathcal{O}_{\mathrm{t}}(M)$ is *optimal*:

Proposition 3.4.2. *Let $\mathcal{O}_{\mathrm{t}}(M)$ be as per Definition 3.4.1. Then, calling with $(\ ,\ )$ the natural pairing between $\mathcal{O}_{\mathrm{t}}(M)$ and $\mathrm{Sol}_{\mathrm{t}}(M)$ induced from that between k-forms, $\mathcal{O}_{\mathrm{t}}(M)$ is* **optimal**, *namely, recalling Definition 3.3.3:*

1. *$\mathcal{O}_{\mathrm{t}}(M)$ is* separating, *that is*

$$([\alpha],[A]) = 0 \quad \forall[\alpha] \in \mathcal{O}_{\mathrm{t}}(M) \Longrightarrow [A] = [0] \in \mathrm{Sol}_{\mathrm{t}}(M)\,. \tag{3.63}$$

2. *$\mathcal{O}_{\mathrm{t}}(M)$ is* non redundant, *that is*

$$([\alpha],[A]) = 0 \quad \forall[A] \in \mathrm{Sol}_{\mathrm{t}}(M) \Longrightarrow [\alpha] = [0] \in \mathcal{O}_{\mathrm{t}}(M)\,. \tag{3.64}$$

Proof. As starting point observe that the pairing $([\alpha],[A]) := (\alpha, A)$ is well-defined. Indeed let consider two representatives $A \in [A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ and $\alpha \in [\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$. The pairing (α, A) is finite being $\mathrm{supp}(\alpha)$ compact and there is no dependence on the choice of representative. As a matter of facts, if $\mathrm{d}\chi \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ and $\eta \in \Omega^k_{\mathrm{c,t}}(M)$, it holds

$$(\alpha, \mathrm{d}\chi) = (\delta\alpha, \chi) + (\mathrm{n}\alpha, \mathrm{t}\chi)_\partial = 0\,, \qquad (\delta\mathrm{d}\eta, A) = (\eta, \delta\mathrm{d}A) + (\mathrm{t}\eta, \mathrm{n}\mathrm{d}A)_\partial - (\mathrm{n}\mathrm{d}\eta, \mathrm{t}A)_\partial = 0\,,$$

where in the first equation we used the fact that $\mathrm{t}\chi = 0$ as well as $\delta\alpha = 0$, while in the second equation we used $\delta\mathrm{d}A = 0$ as well as $\mathrm{t}A = \mathrm{t}\eta = 0$.
Having established that the pairing between the equivalence classes is well-defined we prove the remaining two items separately.

1. Assume $\exists[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ such that $([\alpha],[A]) = 0$, $\forall[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$. Working at the level of representative, since $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$ we can choose $\alpha = \delta\beta$ with $\beta \in \Omega^{k+1}_{\mathrm{c}}(M)$. As a consequence $0 = (\delta\beta, A) = (\beta, \mathrm{d}A)$ where we used implicitly (1.6) and $\mathrm{t}A = 0$. The arbitrariness of β and the non-degeneracy of $(\ ,\ )$ entails $\mathrm{d}A = 0$. Hence A individuates a de Rham cohomology class in $H^k_{\mathrm{t}}(M)$, *cf.* Appendix A. Furthermore, $([\alpha],[A]) = 0$ entails $\langle[\alpha],[A]\rangle = 0$ where $\langle\ ,\ \rangle$ denotes the pairing between $H_{k,\mathrm{c}}(M)$ and $H^k_{\mathrm{t}}(M)$ – *cf.* Appendix A. On account of Remark A.4 it holds that $\langle\ ,\ \rangle$ is non-degenerate and therefore $[A] = 0$.

2. Assume $\exists[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$ such that $([\alpha],[A]) = 0$ $\forall[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$. Working at the level of representatives, we can consider $A = G_{\|}\omega$ with $\omega \in \Omega^k_{\mathrm{c},\delta}(M)$, while $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$. Hence, in view of Proposition 3.1.20, $0 = (\alpha, A) = (\alpha, G_{\|}\omega) = -(G_{\|}\alpha, \omega)$. Choosing $\omega = \delta\beta$, $\beta \in \Omega^{k+1}_{\mathrm{c}}(M)$ and using (1.6), it descends $(\mathrm{d}G_{\|}\alpha, \beta) = 0$. Since β is arbitrary and the pairing is non degenerate $\mathrm{d}G_{\|}\alpha = 0$. Since $\mathrm{t}G_{\|}\alpha = 0$, it turns out that $G_{\|}\alpha$ individuates an equivalence class $[G_{\|}\alpha] \in H^k_{\mathrm{t}}(M)$. Using the same argument of the previous item, $(G_{\|}\alpha, \beta) = 0$ for all $\beta \in \Omega^k_{\mathrm{c},\delta}(M)$ entails that $G_{\|}\alpha = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$. Proceeding as in proof of the injectivity of $G_{\|} : \mathcal{O}_{\mathrm{t}}(M) \to \mathrm{Sol}_{\mathrm{t}}(M)$ – *cf.* Proposition 3.3.5 – it follows that $\alpha \in \delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)$ which is the sought conclusion. ■

The following corollary translates at the level of algebra of observables the degeneracy of the presymplectic spaces discussed in Proposition 3.3.6 – *cf.* Remark 3.3.8. As a matter of fact since $\widetilde{G}_{\|}$ can be degenerate, the algebra of observables $\mathcal{A}_{\mathrm{t}}(M)$ will possess a non-trivial centre. In other words

Corollary 3.4.3. *If* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ *is a strict inclusion, then the algebra* $\mathcal{A}_{\mathrm{t}}(M)$ *is not semi-simple.*

Proof. With reference to Remark 3.3.8, if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ is a strict inclusion then there exists an element $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ such that $\sigma_{\mathrm{t}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$. On account of Proposition 3.3.5 there exists $[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$ such that $[G_{\|}\alpha] = [A]$. Moreover, Proposition 3.3.6 ensures that $\widetilde{G}_{\|}([\alpha],[\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{t}}(M)$. It follows from Definition 3.4.1 that $[\alpha]$ belongs to the center of $\mathcal{A}_{\mathrm{t}}(M)$, that is, $\mathcal{A}_{\mathrm{t}}(M)$ is not semi-simple. ■

Remark 3.4.4. Corollary 3.4.3 has established that the algebra of observables possesses a non trivial center. While from a mathematical viewpoint this feature might not appear of particular significance, it has far reaching consequences from the physical viewpoint. Most notably, the existence of Abelian ideals was first observed in the study of gauge theories in [DL12] leading to an obstruction in the interpretation of these models in the language of locally covariant quantum field theories as introduced in [BFV03]. This feature has been thoroughly studied in [BHS14; BDS14; SDH14] turning out to be an intrinsic feature of Abelian gauge theories on globally hyperbolic spacetimes with empty boundary. Corollary 3.4.3 shows that the same conclusions can be drawn when the underlying manifold possesses a timelike boundary. In the next part of this section we will show that changing boundary condition does not alter the outcome.

The algebra of observable for $\mathrm{Sol}_{\mathrm{nd}}(M)$

We focus now on $\mathcal{A}_{\mathrm{nd}}(M)$, the algebra of observables associated to the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. Similarly to Definition 3.4.1, $\mathcal{A}_{\mathrm{nd}}(M)$ will be defined as a suitable quotient of the universal tensor algebra over a vector space $\mathcal{O}_{\mathrm{nd}}(M)$. However, contrary to the case of δd-tangential boundary conditions, in the case of δd-normal boundary conditions, $\mathcal{O}_{\mathrm{nd}}(M)$ will not be simplectomorphic to the configuration space $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ – *cf.* Definition 3.4.5 and Proposition 3.3.4. Nevertheless the results of Propositions 3.4.2 and 3.4.3 still hold true for $\mathcal{A}_{\mathrm{nd}}(M)$. In the last part of this section we point out another possible choice for the algebra of observables whose underlying vector space is simplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ although it requires an a priori gauge fixing.

Taking into account in particular Equation (3.52), we define

Definition 3.4.5. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{nd}}(M)$*, the associative, unital* $*$*-algebra*

$$\mathcal{A}_{\mathrm{nd}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]}\,, \qquad \mathcal{O}_{\mathrm{nd}}(M) = \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}\,. \tag{3.65}$$

where $\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)] \doteq \bigoplus_{n=0}^{\infty}\mathcal{O}_{\mathrm{nd}}(M)^{\otimes n}$ *is the universal tensor algebra with* $\mathcal{O}_{\mathrm{nd}}(M)^{\otimes 0} \equiv \mathbb{C}$*, while the* $*$*-operation is the one induced from complex conjugation. In addition* $\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]$ *is the* $*$*-ideal generated by elements of the form* $[\alpha]\otimes[\beta] - [\beta]\otimes[\alpha] - i\widetilde{G}_\perp([\alpha],[\beta])\mathbb{I}$ *, where* $[\alpha],[\beta]\in\mathcal{O}_{\mathrm{nd}}(M)$ *while* $\widetilde{G}_\perp$ *is defined in Proposition 3.3.6 and* $\mathbb{I}$ *is the identity of* $\mathcal{O}_{\mathrm{nd}}(M)$*.*

Starting from Definition 3.4.5 we can repeat, mutatis mutandis, the proof of Proposition 3.4.2.

Proposition 3.4.6. *Let* $\mathcal{O}_{\mathrm{nd}}(M)$ *be as per Definition 3.4.5. Then, calling with* $(\ ,\ )$ *the natural pairing between* $\mathcal{O}_{\mathrm{nd}}(M)$ *and* $\mathrm{Sol}_{\mathrm{nd}}(M)$ *induced from those between* k*-forms,* $\mathcal{O}_{\mathrm{nd}}(M)$ *is* **optimal**, *namely*

1. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* separating, *that is*

$$([\alpha],[A]) = 0 \quad \forall[\alpha]\in\mathcal{O}_{\mathrm{nd}}(M) \Longrightarrow [A] = [0]\in\mathrm{Sol}_{\mathrm{nd}}(M)\,. \tag{3.66}$$

2. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* optimal, *that is*

$$([\alpha],[A]) = 0 \quad \forall[A]\in\mathrm{Sol}_{\mathrm{nd}}(M) \Longrightarrow [\alpha] = [0]\in\mathcal{O}_{\mathrm{nd}}(M)\,, \tag{3.67}$$

Proof. The fact that the pairing $([\alpha],[A])$ is well-defined for $[\alpha]\in\mathcal{O}_{\mathrm{nd}}(M)$ and $[A]\in\mathrm{Sol}_{\mathrm{nd}}(M)$ has already been discussed in Remark 3.3.7.
We prove the first of the two items: let $[A]\in\mathrm{Sol}_{\mathrm{nd}}(M)$ be such that $([\alpha],[A]) = 0$ for all $[\alpha]\in\mathcal{O}_{\mathrm{nd}}(M)$. This implies that $(\alpha, A) = 0$ for all $A\in[A]$ and for all $\alpha\in\Omega^k_{\mathrm{c,n},\delta}(M)$. Taking in particular $\alpha = \delta\beta$ with $\beta\in\Omega^k_{\mathrm{c,n}}(M)$ it follows $(\mathrm{d}A,\beta) = 0$. The non-degeneracy of $(\ ,\ )$ implies $\mathrm{d}A = 0$, that is A defines an element in $H^k(M)$. Moreover, the hypothesis on A implies that $\langle A,[\eta]\rangle = 0$ for all $[\eta]\in H_{k,\mathrm{c,n}}(M)$. The results in Appendix A – *cf.* Remark A.4 – ensure that $A = \mathrm{d}\chi$, therefore $[A] = [0]\in\mathrm{Sol}_{\mathrm{nd}}(M)$.
Regarding the second statement, let $[\alpha]\in\mathcal{O}_{\mathrm{nd}}(M)$ be such that $([\alpha],[A]) = 0$ for all $[A]\in\mathrm{Sol}_{\mathrm{nd}}(M)$. This implies in particular that, choosing $\alpha\in[\alpha]$ and $A = G_\perp\beta$ with $\beta\in\Omega^k_{\mathrm{c},\delta}(M)$, $0 = (\alpha, G_\perp\beta) = -(G_\perp\alpha,\beta)$. With the same argument of the first statement it follows that $G_\perp\alpha = \mathrm{d}\chi$ where $\chi\in\Omega^{k-1}(M)$ is such that $\mathrm{nd}\chi = 0$. Proceeding as in the proof of Proposition 3.3.5 it follows that $[\alpha] = [0]\in\mathcal{O}_{\mathrm{nd}}(M)$. ∎

The following corollary is analogous to Corollary 3.4.3. The proof is slightly different since in this case there does not exist a simplectomorphism between $\mathcal{O}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ – *cf.* Proposition 3.3.5 and Remark 3.3.7. The crucial part in the proof is to show that if $[A]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ is degenerate with respect to σ_{nd}, then $[A]\in G_\perp\mathcal{O}_{\mathrm{nd}}(M)$.

Corollary 3.4.7. *If* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ *is a strict inclusion, then the algebra* $\mathcal{A}_{\mathrm{nd}}(M)$ *is not semi-simple.*

Proof. On account of Remark 3.3.8, if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a strict inclusion then there exists $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ such that $\sigma_{\mathrm{nd}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. In particular we have $[A] = [\mathrm{d}\chi]$ where $\chi \in \Omega^{k-1}(M) \setminus \Omega^{k-1}_{\mathrm{sc}}(M)$ is such that $\mathrm{d}\chi \in \Omega^k_{\mathrm{sc}}(M)$.
We now prove that, up to an element in $\mathrm{d}\Omega^k_{\mathrm{sc}}(M)$, $\mathrm{d}\chi = G_\perp\alpha$ with $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$: On account of Proposition 3.3.6 it follows that $\widetilde{G}_\perp([\alpha],[\beta]) = \sigma_{\mathrm{nd}}([\mathrm{d}\chi],[G_\perp\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{nd}}(M)$. Definition 3.4.5 implies that $[\alpha] \in \mathcal{A}_{\mathrm{nd}}(M)$ lies in the center of $\mathcal{A}_{\mathrm{nd}}(M)$ which is therefore not semi-simple.
On account of Proposition 3.3.5 we have that $\mathrm{d}\chi = G_\perp\alpha + \mathrm{d}\eta$, where $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$ while $\eta \in \Omega^{k-1}_{\mathrm{sc}}(M)$. By redefining $\chi_\eta \doteq \chi - \eta$ we have $\mathrm{d}\chi_\eta = G_\perp\alpha$. Notice that this last redefinition does not spoil the property $\chi_\eta \in \Omega^{k-1}(M) \setminus \Omega^{k-1}_{\mathrm{sc}}(M)$ while $\mathrm{d}\chi_\eta \in \Omega^k_{\mathrm{sc}}(M)$ thus $\sigma_{\mathrm{nd}}([\mathrm{d}\chi_\eta],[B]) = 0$ for all $[B] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$.
The boundary conditions on $G_\perp\alpha$ implies that $\mathrm{n}\mathrm{d}\chi_\eta = \mathrm{n}G_\perp\alpha = 0$, while Corollary 3.1.24 ensures that $\delta\mathrm{d}\chi_\eta = \delta G_\perp\alpha = G_\perp\delta\alpha = 0$. It then follows that $\chi_\eta \in \mathrm{Sol}_{\mathrm{nd}}(M)$ – in degree $k-1$ – and therefore Proposition 3.3.5 entails $\chi_\eta = G_\perp\beta$ where $\beta \in \Omega^{k-1}_{\mathrm{tc},\delta}(M)$. Summing up we have $\mathrm{d}\chi_\eta = G_\perp\alpha$ as well as $\mathrm{d}\chi_\eta = G_\perp\mathrm{d}\beta$. Proposition 3.1.20 and Remark 3.1.21 imply that $\mathrm{d}\beta - \alpha = \Box_\perp\zeta$, being $\zeta \in \Omega^k_{\mathrm{tc},\perp}(M)$. Applying δ to the last equality we obtain

$$\Box\delta\zeta = \delta\Box_\perp\zeta = \delta\mathrm{d}\beta - \delta\alpha = \Box\beta\,.$$

Remark 3.1.22 ensures that $\delta\zeta = \beta$ and therefore $\alpha = -\delta\mathrm{d}\zeta$. Since $\zeta \in \Omega^k_\perp(M)$ it follows that $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$. ■

3.4.1 An alternative algebra for $\delta\mathrm{d}$-normal boundary conditions

Definition 3.4.5 identifies an algebra $\mathcal{A}_{\mathrm{nd}}(M)$ which is separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. It also satisfies most of the properties of the analogous algebra $\mathcal{A}_{\mathrm{t}}(M)$ – *cf.* Proposition 3.4.6 and Corollary 3.4.7. However, as pointed out in Remark 3.3.7, the underlying vector space $\mathcal{O}_{\mathrm{nd}}(M)$ is only a proper presymplectic subspace of $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$. This is contrary to the case of $\delta\mathrm{d}$-tangential boundary conditions where the vector space $\mathcal{O}_{\mathrm{t}}(M)$ is symplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ – *cf.* Proposition 3.3.5.
It is thus worth investigating whether there exists a different algebra $\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ still defined as a suitable quotient – *cf.* Definitions 3.4.1-3.4.5 – of the universal tensor algebra of a presymplectic vector space $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ which is presymplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$. For consistency, $\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ should be built out of a separating and non redundant collection of functionals for $\mathrm{Sol}_{\mathrm{nd}}(M)$ and the superscript gf refers to "gauge fixing" as it will become clear from the following discussion. To this end and with reference to Proposition 3.3.5, $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ can be identified as

$$\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M) \doteq \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}\,.$$

As shown in Propositions 3.3.5-3.3.6, $(\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M), \widetilde{G}_\perp)$ is a presymplectic vector space which is symplectomorphic to $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$. We can thus set

$$\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)]}{\mathcal{I}[\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)]},$$

where we refer to Definition 3.4.5 for details.
The discussion about $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ being separating and non redundant is more subtle. Indeed, the pairing between elements $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ is not well-defined – *cf.* Remark 3.3.7. However we can exploit the isomorphism identified in equation (3.51). With reference to equation (3.50), we denote with γ_{nd} the isomorphism

$$\gamma_{\mathrm{nd}} \colon \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathcal{S}_{\mathcal{G}_{\mathrm{nd}}}(M) .$$

It follows that for all $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ the following functional is well-defined:

$$F_{\gamma^*_{\mathrm{nd}}[\alpha]} \colon \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathbb{C}, \qquad F_{\gamma^*_{\mathrm{nd}}[\alpha]}([A]) := ([\alpha], [\gamma_{\mathrm{nd}} A]) .$$

Notice that the gauge-invariance of $F_{\gamma^*_{\mathrm{nd}}[\alpha]}$ is guaranteed by the combined action of γ_{nd}, which selects a "gauge-fixed" representative $\gamma_{\mathrm{nd}} A \in [A]$, and of $[\alpha]$, which remains un-effected by the residual gauge present in the choice of $\gamma_{\mathrm{nd}} A$, *i.e.* $([\alpha], \mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)) = 0$ – *cf.* Equation (3.48).
With this observation it holds that, introducing the "gauge-fixed" pairing $([\alpha], [A])_{\gamma_{\mathrm{nd}}} \doteq ([\alpha], [\gamma_{\mathrm{nd}} A])$ between $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}(M)$, the vector space $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ is indeed separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. The proof is similar to the one of Propositions 3.4.2-3.4.6 and we shall not repeat it.

Conclusions

In this thesis we established, for a particular class of spacetimes with timelike boundary and for suitable boundary conditions, the existence of advanced and retarded fundamental solutions or Green operators for Maxwell's equations stated both in terms of the Faraday tensor $F \in \Omega^2(M)$ and in terms of the vector potential A. Subsequently we constructed the optimal quantum algebra of observables for the free electromagnetic field in terms of A for two selected boundary conditions.
In particular, for the equations in terms of $F \in \Omega^2(M)$, we analyzed Maxwell's equations in a framework in which the spacetime M could be split into $\mathbb{R} \times \Sigma$, with Σ being a closed Riemannian manifold with a codimension 1 interface $Z \subset \Sigma$. We separated the equations in a non-dynamical and in a dynamical part. The former has been treated using the so-called Hodge decomposition, while for the latter the fundamental solutions have been constructed using the technique of Lagrangian subspaces. These are at the hearth of a method to select boundary conditions that ensure the self-adjointness of the dynamical part of Maxwell's equations. Furthermore we gave an example of such conditions.
In the case of Maxwell's equations for a generic vector potential $A \in \Omega^k(M)$, $0 < k < \dim M$ we proceeded as follows. At first we proved of the existence of advanced and retarded fundamental solutions for $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ acting on k-forms in ultrastatic spacetimes with timelike boundary. We used the technique of boundary triples, imposing suitable classes of boundary conditions, dubbed $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal. Subsequently, we applied these results identifying the space of solutions of $\delta\mathrm{d}A = 0$ under $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions, distinguishing two different notions of gauge invariance and showing that within the two gauge equivalence classes it is always possible to find a representative that abides to Lorenz gauge. We identified, for the two boundary conditions, the optimal algebras of observables, showing that in general they do possess a non-trivial centre.

Possible extensions to this work arise for both the formulations of Maxwell's equations in terms of $F \in \Omega^2(M)$ and in terms of $A \in \Omega^k(M)$.

In particular, focusing on the results of Chapter 2, in the discussion about the non-dynamical part of Maxwell's equations for F (*cf.* Section 2.2), we assumed that the underlying Cauchy hypersurface was closed, i.e. compact and with no boundary, in order to use Hodge decomposition. Hodge decomposition was used to provide a mathematically rigorous framework for the

non-dynamical equations. It is natural to ask whether a generalization of Hodge decomposition can be used on a larger class of Cauchy hypersurfaces (*cf.* Subsection 2.2.3).
Moreover, we did not construct explicitly the algebra of observables for the Faraday tensor F, while we only gave an account on the possible strategy in Section 2.4.

On the other side, the results of Chapter 3 can be generalized in various directions. At first, we considered the wave operator $\Box$ and we indicated a class of boundary conditions, encoded by $\Omega^k_{f,f'}(M)$ which ensured the operator to be closed or, in other words, formally self-adjoint. It is important to stress that the boundary conditions $\Omega^k_{f,f'}(M)$ are not the largest class which makes the operator closed. As a matter of fact one can think of other possibilities, for example those similar to the Wentzell boundary conditions, which were studied in the scalar scenario in [DDF19; DFJA18; Zah18] and that could be interesting from a physical viewpoint. Therefore it arises the question whether there is a larger class of boundary conditions for $\Box$ such that the existence of fundamental solutions can be established.
Subsequently, in Subsection 3.1.2, we assumed the existence of distinguished fundamental solutions for the wave operator and studied their properties, which remain valid whenever the hypothesis of Assumption 3.1.4 hold. To show that the Assumption can be verified, we used a particular technique from functional analysis which is well-suited for static spacetimes. Hence, a possible follow-up is to investigate whether there are other methods that allow the construction of fundamental solutions for $\Box$ in non-ultrastatic spacetimes.
Regarding the construction of the spaces of solutions for Maxwell's equations $\delta \mathrm{d} A = 0$, we restricted the discussion to a particular pair of boundary conditions, $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal, for which it was possible to find a gauge-equivalent solution which satisfied the wave equation $\Box A = 0$. It may be interesting to investigate whether there are other boundary conditions for the Maxwell operator $\delta\mathrm{d}$ such that the construction of fundamental solutions is possible without resorting to the Lorenz gauge and thus relying on the fundamental solutions for $\Box$ – *cf.* Section 3.2.
To conclude, we think that the work should be recast regarding electromagnetism as a Yang-Mills Abelian theory for the connections on a $U(1)$-principal bundle, rather than choosing $\mathbb{R}$ as a structure group. This should affect the choice of the gauge groups and hence the formulation of the spaces of solutions and of the algebra of observables – *cf.* the discussion in Subsection 3.2.1.

Appendix A

Cohomology and Poincaré-Lefschetz duality for manifold with boundary

In this appendix we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non-empty boundary. A reader interested in more details can refer to [BT13; Sch95].

For the purpose of this appendix M refers to a smooth, oriented manifold of dimension $\dim M = m$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : \partial M \to M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work also with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_\mathrm{c}(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\ker(\mathrm{d}_k \colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d}_{k-1} \colon \Omega^{k-1}(M) \to \Omega^k(M))}, \tag{A.1}$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (1.4) and (1.5b) entail that we can define $\Omega^\bullet_\mathrm{t}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_\mathrm{t}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_\mathrm{t}(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M) \doteq \frac{\ker(\delta_k \colon \Omega^k(M) \to \Omega^{k-1}(M))}{\mathrm{Im}(\delta_{k+1} \colon \Omega^{k+1}(M) \to \Omega^k(M))}.$$

Equations (1.4) and (1.5b) entail that we can define the $\Omega^\bullet_\mathrm{n}(M)$ (*resp.* $\Omega^\bullet_\mathrm{c}(M)$, $\Omega^\bullet_{\mathrm{c,n}}(M)$), the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_\mathrm{n}(M) \subset \Omega^k(M)$ (*resp.* $\Omega^k_\mathrm{c}(M)$, $\Omega^k_{\mathrm{c,n}}(M) \subseteq \Omega^k(M)$). The associated homology groups will be denoted as $H_{k,\mathrm{n}}(M)$ (*resp.* $H_{k,\mathrm{c}}(M)$, $H_{k,\mathrm{c,n}}(M)$), $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition and on account of equation (1.5), the Hodge operator induces an isomorphism $H^k(M) \simeq H_{m-k}(M)$ which is realized

as $H^k(M) \ni [\alpha] \mapsto [\star\alpha] \in H_{m-k}(M)$. Similarly, on account of Equation (1.5b), it holds $H^k_{\mathrm{t}}(M) \simeq H_{m-k,\mathrm{n}}(M)$ and $H^k_{\mathrm{c,t}}(M) \simeq H_{m-k,\mathrm{c,n}}(M)$.
As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT13]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to

$$\Omega^k(M;\partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M;\partial M) \to \Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta) \in \Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta) = (\mathrm{d}_k\omega, \mathrm{t}\omega - \mathrm{d}_{k-1}\theta)\,. \tag{A.2}$$

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M;\partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M;\partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will always be clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT13, Prop. 6.49]:

Proposition A.1. *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M;\partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\mathrm{t}_*} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M;\partial M) \to \ldots\,, \tag{A.3}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and t_* indicate the natural counterpart of the maps π_1, π_2 and t at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition A.2. *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between $H^k_{\mathrm{t}}(M)$ and $H^k(M;\partial M)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Consider $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}$ and let $(\omega, 0) \in \Omega^k(M;\partial M)$, $k \in \mathbb{N} \cup \{0\}$. Equation (A.2) entails

$$\underline{\mathrm{d}}_k(\omega,0) = (\mathrm{d}_k\omega, \mathrm{t}\omega) = (0,0)\,.$$

At the same time, if $\omega = \mathrm{d}_{k-1}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $(\mathrm{d}_{k-1}\beta, 0) = \underline{\mathrm{d}}_{k-1}(\beta,0)$. Hence the embedding $\omega \mapsto (\omega,0)$ identifies a map $\rho : H^k_{\mathrm{t}}(M) \to H^k(M;\partial M)$ such that $\rho([\omega]) \doteq [(\omega,0)]$. To conclude, we need to prove that ρ is surjective and injective. Let thus $[(\omega',\theta)] \in H^k(M;\partial M)$. It holds that $\mathrm{d}_k\omega' = 0$ and $\mathrm{t}\omega' - \mathrm{d}_{k-1}\theta = 0$. Recalling that $\mathrm{t} : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective – *cf.* Remark 1.2.5 – for all values of $k \in \mathbb{N} \cup \{0\}$, there must exists

$\eta \in \Omega^{k-1}(M)$ such that $\mathrm{t}\eta = \theta$. Let $\omega \doteq \omega' - \mathrm{d}_{k-1}\eta$. On account of (1.5b) $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}_k$ and $(\omega, 0)$ is a representative if $[(\omega', \theta)]$ which entails that ρ is surjective.
Let $[\omega] \in H^k(M)$ be such that $\rho[\omega] = [0] \in H^k(M; \partial M)$. This implies that there exists $\beta \in \Omega^{k-1}(M)$, $\theta \in \Omega^{k-2}(\partial M)$ such that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}(\beta, \theta) = (\mathrm{d}_{k-1}\beta, \mathrm{t}\beta - \mathrm{d}_{k-2}\theta)\,.$$

Let $\eta \in \Omega^{k-2}(M)$ be such that $\mathrm{t}\eta + \theta = 0$. It follows that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}\left((\beta, \theta) + \underline{\mathrm{d}}_{k-2}(\eta, 0)\right) = \underline{\mathrm{d}}_{k-1}(\beta + \mathrm{d}_{k-2}\eta, 0)\,.$$

This entails that $\omega = \mathrm{d}_{k-1}(\beta + \mathrm{d}_{k-2}\eta)$ where $\mathrm{t}(\beta + \mathrm{d}_{k-2}\eta) = 0$. It follows that $[\omega] = 0$ that is ρ is injective.

■

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau96]:

Theorem A.3. *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N} \cup \{0\}$*

$$H^{m-k}(M; \partial M) \simeq H^k_{\mathrm{c}}(M)^*\,, \qquad [\alpha] \to \left(H^k_{\mathrm{c}}(M) \ni [\eta] \mapsto \int_M \overline{\alpha} \wedge \eta \in \mathbb{C}\right). \tag{A.4}$$

where $m = \dim M$ and where on the right hand side we consider the dual of the $(m-k)$-th cohomology group built out compactly supported forms.

Remark A.4. On account of Propositions A.2-A.3 and of the isomorphisms $H^k_{(\mathrm{c})}(M) \simeq H^{m-k}_{(\mathrm{c})}(M)$ the following are isomorphisms:

$$H^k_{\mathrm{t}}(M) \simeq H^{m-k}_{\mathrm{c}}(M)^* \simeq H_{k,\mathrm{c}}(M)^*\,, \qquad H^k(M) \simeq H_{k,\mathrm{c},\mathrm{n}}(M)^*\,. \tag{A.5}$$

The proof proceeds in some steps. Let $\iota : \partial M \to M$ be the immersion map. First of all we have to check that the spaces are finite-dimensional and that the pairing $\langle\,,\,\rangle : H^{m-k}(M) \otimes H^k_{\mathrm{c}}(M, \partial M)$ defined by

$$\langle \alpha, (\omega, \theta) \rangle := \int_M \alpha \wedge \omega + \int_{\partial M} \iota^* \alpha \wedge \theta \qquad \forall \alpha \in H^{m-k}(M) \text{ and } (\omega, \theta) \in H^k_{\mathrm{c}}(M, \partial M)\,, \tag{A.6}$$

is non-degenerate, equivalently the map $\alpha \to \langle \alpha, \cdot \rangle$ should be an isomorphism.

Since a manifold M with boundary is locally homeomorphic to $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ we need Poincaré lemmas for $\mathbb{R}^m_+$.

Lemma A.5 (Poincaré lemmas for half spaces)**.** *Let* $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ *and* $k \geq 0$. *Then*

$$H^k(\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \textit{if } k = 0 \\ \{0\} & \textit{otherwise} \end{cases} \tag{A.7}$$

$$H^k_c(\mathbb{R}^m_+, \partial\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \textit{if } k = n \\ \{0\} & \textit{otherwise} \end{cases} \tag{A.8}$$

Proof. The proof for the case $n = 1$, i.e. $\mathbb{R}_+ = [0, +\infty)$ is straightforward and the n-dimensional generalisation is obtained as in ([BT13, Sec. 4]). ∎

Lemma A.6 (Mayer-Vietoris sequences)**.** *Let* M *be an orientable manifold with boundary* ∂M*, suppose* $M = U \cup V$ *with* U, V *open and denote* $\partial M_A := \partial M \cap A$*. Then the following are exact sequences:*

$$\cdots \to H^k(M, \partial M) \to H^k(U, \partial M_U) \oplus H^k(V, \partial M_V) \to H^k(U \cap V, \partial M_{U\cap V}) \to H^{k+1}(M, \partial M) \to \cdots \tag{A.9}$$

$$\cdots \leftarrow H^k_c(M, \partial M) \leftarrow H^k_c(U, \partial M_U) \oplus H^k(V, \partial M_V) \leftarrow H^k_c(U \cap V, \partial M_{U\cap V}) \leftarrow H^{k+1}_c(M, \partial M) \leftarrow \cdots \tag{A.10}$$

Proof. We will only prove the non-compact cohomology line.
We have the following Mayer-Vietoris short exact sequences for M and ∂M:

$$0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V) \longrightarrow 0$$
$$0 \to \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M_U) \oplus \Omega^{k-1}(\partial M_V) \to \Omega^{k-1}(\partial M_{U\cap V}) \to 0.$$

Hence applying the direct sum between the two sequences we obtain

$$0 \longrightarrow \Omega^k(M, \partial M) \longrightarrow \Omega^k(U, \partial M_U) \oplus \Omega^k(V, \partial M_V) \longrightarrow \Omega^k(U \cap V, \partial M_{U\cap V}) \longrightarrow 0.$$

The last row induces the desired long sequence because of the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^k(M, \partial M) \arrow[r] \arrow[d, "d"] & \Omega^k(U, \partial M_U) \oplus \Omega^k(V, \partial M_V) \arrow[r] \arrow[d, "\mathrm{d} := d \oplus d"] & \Omega^k(U \cap V, \partial M_{U\cap V}) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & \Omega^{k+1}(M, \partial M) \arrow[r] & \Omega^{k+1}(U, \partial M_U) \oplus \Omega^{k+1}(V, \partial M_V) \arrow[r] & \Omega^{k+1}(U \cap V, \partial M_{U\cap V}) \arrow[r] & 0 \end{tikzcd} \tag{A.11}$$

following the arguments in [BT13], section 2. Fix a closed form $\omega \in \Omega^k(U \cap V, \partial M_{U\cap V})$, since the first row is exact there exists a unique $\xi \in \Omega^{k+1}(M, \partial M)$ which is mapped to ω. Now, since $\mathrm{d}\omega = 0$ and the diagram is commutative $\mathrm{d}\xi$ is mapped to 0. Hence from the exactness of the second row there exists χ which is mapped to $\mathrm{d}\xi$ and it easy to see χ is closed. ∎

Lemma A.7. *If the manifold with boundary M has a* finite good cover *(see [BT13, Sec. 5]) then its (relative) cohomology and (relative) compact cohomology is finite dimensional.*

Proof. The proof is based on the existence of a Mayer-Vietoris sequence in any of the desired cases (proved in the previous proposition) and follows the outline of [BT13, Prop. 5.3.1]. ■

Lemma A.8 (Five lemma)**.** *Given the commutative diagram with exact rows*

$$\begin{tikzcd} \cdots \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "r"] & E \arrow[r] \arrow[d, "s"] & \cdots \\ \cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots \end{tikzcd} \tag{A.12}$$

if f, g, h, s are isomorphism, then so is r.

Lemma A.9. *Suppose $M = U \cup V$ with U, V open. The pairing (A.6) induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is commutative:*

$$\begin{tikzcd} \cdots \arrow[r] & H^{m-k}(M) \arrow[r] \arrow[d] & H^{m-k}(U) \oplus H^{m-k}(V) \arrow[r] \arrow[d] & H^{m-k+1}(M) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^k(M, \partial M)^* \arrow[r] & H^k(U, \partial M_U)^* \oplus H^k(V, \partial M_V)^* \arrow[r] & H^{k-1}(M)^* \arrow[r] & \cdots \end{tikzcd} \tag{A.13}$$

Proof. The proof follows that of [BT13, Lem. 5.6]. ■

Now we are ready to prove the main theorem of this section:

Proof of Poincaré-Lefschetz Duality. Follow the argument given in [BT13, Sec. 5]. By the Five lemma if Poincaré-Lefschetz duality holds for U, V and $U \cap V$, then it holds for $U \cup V$. Then it is sufficient to proceed by induction on the cardinality of a finite good cover. □

Appendix B

An explicit computation

Lemma B.1. *Let $M = \mathbb{R} \times \Sigma$ be a globally hyperbolic spacetime – cf. Theorem 1.1.2. Moreover, for all $\tau \in \mathbb{R}$, let $\mathrm{t}_{\Sigma_\tau} \colon \Omega^k(M) \to \Omega^k(\Sigma_\tau)$, $\mathrm{n}_{\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\Sigma)$ be the tangential and normal maps on $\Sigma_\tau \doteq \{\tau\} \times \Sigma$, where $M = \mathbb{R} \times \Sigma$ – cf. Definition 1.2.3. Moreover, let $\mathrm{t}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^k(\partial\Sigma_\tau)$ and let $\mathrm{n}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ be the tangential and normal maps on $\partial\Sigma_\tau \doteq \{\tau\} \times \partial\Sigma$. Let $f \in C^\infty(\partial\Sigma)$ and set $f_\tau \doteq f|_{\partial\Sigma_\tau}$. Then for $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$ it holds*

$$\omega \in \Omega^k_\sharp(M) \Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \Omega^k_\sharp(\Sigma_\tau) \quad \forall \tau \in \mathbb{R}\,. \tag{B.1}$$

More precisely this entails that

$$\begin{aligned}
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker \mathrm{n}_{\partial M} &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker \mathrm{n}_{\partial\Sigma_\tau}\,, \forall \tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{n}_{\partial M} \cap \ker \mathrm{n}_{\partial M}\mathrm{d} &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{n}_{\partial\Sigma_\tau} \cap \ker \mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\,, \forall \tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker \mathrm{t}_{\partial M}\delta &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker \mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}\,, \forall \tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{n}_{\partial M} \cap \ker(\mathrm{n}_{\partial M}\mathrm{d} - f\mathrm{t}_{\partial M}) &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{n}_{\partial\Sigma_\tau} \cap \ker(\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau} - f_t\mathrm{t}_{\partial\Sigma_\tau})\,, \forall \tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker(\mathrm{t}_{\partial M}\delta - f\mathrm{n}_{\partial M}) &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker(\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau} - f_t\mathrm{n}_{\partial\Sigma_\tau})\,, \forall t \in \mathbb{R}\,.
\end{aligned}$$

Proof. The equivalence (B.1) is shown for $\perp$-boundary condition. The proof for $\|$-boundary conditions follows per duality – *cf.* (3.1.3) – while the one for D-, $f_\|$-, $f_\perp$-boundary conditions can be carried out in a similar way.
On account of Theorem 1.1.2 we have that for all $\tau \in \mathbb{R}$ we can decompose any $\omega \in \Omega^k(M)$ as follows:

$$\omega|_{\Sigma_\tau} = \mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,.$$

Notice that, being the decomposition $M = \mathbb{R}\times\Sigma$ smooth we have that $\tau \to \mathrm{t}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}, \Omega^k(\Sigma))$ while $\tau \to \mathrm{n}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma))$. Here we have implicitly identified $\Sigma \simeq \Sigma_\tau$.
A similar decomposition holds near the boundary of Σ_τ. Indeed for all $(\tau, p) \in \{\tau\} \times \partial\Sigma$ we consider a neighbourhood of the form $U = [0, \epsilon_\tau) \times U_{\partial\Sigma}$. Let $U_x \doteq \{x\} \times U_{\partial\Sigma}$ for $x \in [0, \epsilon_\tau)$ and let t_{U_x}, n_{U_x} be the corresponding tangential and normal maps – *cf.* Definition 1.2.3. With

this definition we can always split $\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{\Sigma_\tau}\omega$ as follows:

$$\omega|_U = \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \wedge \mathrm{d}x + \mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}x \wedge \mathrm{d}\tau\,. \tag{B.2}$$

If p ranges on a compact set of $\partial\Sigma$ it follows that $(\tau, x) \to \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}\times[0,\epsilon), \Omega^k(\partial\Sigma))$ and similarly $\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$, $\mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$. Once again we have implicitly identified $U_{\partial\Sigma} \simeq \{x\} \times U_{\partial\Sigma}$.
According to this splitting we have

$$\begin{aligned}
\mathrm{t}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{t}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau = \mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,,\\
\mathrm{n}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau = \mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,.
\end{aligned}$$

It follows that $\mathrm{n}_{\partial M}\omega = 0$ if and only if $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and similarly $\mathrm{t}_{\partial M}\omega = 0$ if and only if $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$. This proves the thesis for Dirichlet boundary conditions. A similar computation leads to

$$\begin{aligned}
\mathrm{n}_{\partial M}\mathrm{d}\omega &= (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + (-1)^{k-1}\partial_\tau\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\\
&\quad + (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} \wedge \mathrm{d}\tau - \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\\
&= (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} \wedge \mathrm{d}\tau\,.
\end{aligned}$$

where the second equality holds true since $\mathrm{n}_{\partial M}\omega = 0$. It follows that $\mathrm{n}_{\partial M}\mathrm{d}\omega = 0$ if and only if $\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} = 0$ and $\partial_x\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} = 0$. When $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ the latter conditions are equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$. ■

Finally, we prove a very useful Lemma.

Lemma B.2. *Let $\sharp \in \{\mathrm{D}, \|, \perp, f_{\|}, f_\perp\}$, with $f \in C^\infty(\partial M)$. The following statements hold true:*

1. *for all $\omega \in \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{spc}}(M) \cap \Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*
2. *for all $\omega \in \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{pc}}(M) \cap \Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{fc}}(M) \cap \Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

Proof. We prove the result in the first case, the second one can be proved in complete analogy. Let $\omega \in \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$. Consider Σ_1, Σ_2, two Cauchy surfaces on M – *cf.* [AFS18, Def. 3.10] – such that $J^+(\Sigma_1) \subset J^+(\Sigma_2)$. Moreover, let $\varphi_+ \in \Omega^0_{\mathrm{pc}}(M)$ be such that $\varphi_+|_{J^+(\Sigma_2)} = 1$ and $\varphi_+|_{J^-(\Sigma_1)} = 0$. We define $\varphi_- := 1 - \varphi_+ \in \Omega^0_{\mathrm{fc}}(M)$. Notice that we can always choose φ so that, for all $x \in M$, $\varphi(x)$ depends only on the value $\tau(x)$, where τ is the global time function defined in Theorem 1.1.2. We set $\omega_\pm \doteq \varphi_\pm\omega$ so that $\omega^+ \in \Omega^k_{\mathrm{spc}}(M) \cap \Omega^k_\sharp(M)$ while

$\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$. This is automatic for $\sharp = \mathrm{D}$ on account of the equality

$$\mathrm{t}\omega^\pm = \varphi_\pm \mathrm{t}\omega = 0\,, \qquad \mathrm{n}\omega^\pm = \varphi_\pm \mathrm{n}\omega = 0\,.$$

We now check that $\omega^\pm \in \Omega^k_\sharp(M)$ for $\sharp = \perp$. The proof for the remaining boundary conditions $\perp, f_{\|}, f_\perp$ follows by a similar computation – or by duality *cf.* Remark 3.1.3. It holds

$$\mathrm{n}\omega_\pm = \varphi_\pm|_{\partial M}\mathrm{n}\omega = 0\,, \qquad \mathrm{n}\mathrm{d}\omega_\pm = \mathrm{n}(\mathrm{d}\chi \wedge \omega) = \partial_\tau\chi\, \mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0\,.$$

In the last equality $\mathrm{t}_{\Sigma_\tau} \colon \Omega^k(M) \to \Omega^k(\Sigma_\tau)$ and $\mathrm{n}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ are the maps from Definition 1.2.3 with $N \equiv \Sigma_\tau \doteq \{\tau\} \times \Sigma$, where $M = \mathbb{R} \times \Sigma$. The last identity follows because the condition $\mathrm{n}\omega = 0$ is equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ for all $\tau \in \mathbb{R}$ – *cf.* Lemma B.1. ■

Appendix C

Proofs of statements in Subsection 3.3.1

Proposition C.1. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Let $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and, for $A_1 \in [A_1]$, let $A_1 = A_1^+ + A_1^-$ be any decomposition such that $A^+ \in \Omega^k_{\mathrm{spc,t}}(M)$ while $A^- \in \Omega^k_{\mathrm{sfc,t}}(M)$ – cf. Lemma B.2. Then the following map $\sigma_{\mathrm{t}} \colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)^{\times 2} \to \mathbb{R}$ is a presymplectic form:*

$$\sigma_{\mathrm{t}}([A_1], [A_2]) = (\delta \mathrm{d} A_1^+, A_2)\,, \qquad \forall [A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,. \tag{C.1}$$

A similar result holds for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ and we denote the associated presymplectic form σ_{nd}. In particular for all $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ we have $\sigma_{\mathrm{nd}}([A_1], [A_2]) \doteq (\delta \mathrm{d} A_1^+, A_2)$ where $A_1 \in [A_1]$ is such that $A \in \Omega^k_{\mathrm{sc},\perp}(M)$.

Proof. We shall prove the result for σ_{nd}, the proof for σ_{t} being the same mutatis mutandis. First of all notice that for all $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ there exists $A' \in [A]$ such that $A' \in \Omega^k_{\perp}(M)$. This is realized by picking an arbitrary $A \in [A]$ and defining $A' \doteq A + \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$ is such that $\mathrm{n}\mathrm{d}\chi = -\mathrm{n}A$ – *cf.* Remark 1.2.5. We can thus apply Lemma B.2 in order to split $A' = A'_+ + A'_-$ where $A'_+ \in \Omega^k_{\mathrm{spc,nd}}(M)$ and $A'_- \in \Omega^k_{\mathrm{sfc,nd}}(M)$. Notice that this procedure is not necessary for $\delta\mathrm{d}$-tangential boundary condition since we can always split $A \in \Omega^k_{\mathrm{sc,t}}(M)$ as $A = A^+ + A^-$ with $A_+ \in \Omega^k_{\mathrm{spc,t}}(M)$ and $A_- \in \Omega^k_{\mathrm{sfc,t}}(M)$ without invoking Lemma B.2.
After these preliminary observations consider the map

$$\sigma_{\mathrm{nd}} \colon (\ker \delta\mathrm{d} \cap \Omega^k_{\mathrm{sc},\perp}(M))^{\times 2} \ni (A_1, A_2) \mapsto (\delta \mathrm{d} A_1^+, A_2)\,,$$

where we used Lemma B.2 and we split $A_1 = A_1^+ + A_1^-$, with $A_1^+ \in \Omega^k_{\mathrm{spc},\perp}(M)$ while $A_1^- \in \Omega^k_{\mathrm{sfc},\perp}(M)$. The pairing $(\delta \mathrm{d} A_1^+, A_2)$ is finite because A_2 is a spacelike compact k-form while $\delta \mathrm{d} A_1^+$ is compactly supported on account of A_1 being on-shell. Moreover, $(\delta \mathrm{d} A_1^+, A_2)$ is independent from the splitting $A_1 = A_1^+ + A_1^-$ and thus σ_{nd} is well-defined. Indeed, let $A_1 = \widetilde{A}_1^+ + \widetilde{A}_1^-$ be another splitting: it follows that $A_1^+ - \widetilde{A}_1^+ = -(A_1^- - \widetilde{A}_1^-) \in \Omega^k_{\mathrm{c,nd}}(M)$. Therefore

$$(\delta \mathrm{d} \widetilde{A}_1^+, A_2) = (\delta \mathrm{d} A_1^+, A_2) + (\delta \mathrm{d}(\widetilde{A}_1^+ - A_1^+), A_2) = (\delta \mathrm{d} A_1^+, A_2)\,,$$

where in the last equality we used the self-adjointness of $\delta \mathrm{d}$ on $\Omega^k_{\mathrm{nd}}(M)$.

We show that $\sigma_{\mathrm{nd}}(A_1, A_2) = -\sigma_{\mathrm{nd}}(A_2, A_1)$ for all $A_1, A_2 \in \ker \delta\mathrm{d} \cap \Omega^k_{\mathrm{sc},\perp}(M)$. For that we have

$$\begin{aligned}\sigma_{\mathrm{nd}}(A_1, A_2) = (\delta\mathrm{d}A_1^+, A_2) &= (\delta\mathrm{d}A_1^+, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-)\\ &= -(\delta\mathrm{d}A_1^-, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-)\\ &= -(A_1^-, \delta\mathrm{d}A_2^+) + (A_1^+, \delta\mathrm{d}A_2^-)\\ &= -(A_1^-, \delta\mathrm{d}A_2^+) - (A_1^+, \delta\mathrm{d}A_2^+)\\ &= -(A_1, \delta\mathrm{d}A_2^+) = -\sigma_{\mathrm{nd}}(A_1, A_2)\,,\end{aligned}$$

where we exploited Lemma B.2 and $A_1^\pm, A_2^\pm \in \Omega^k_{\mathrm{sc,nd}}(M)$.
Finally we prove that $\sigma_{\mathrm{nd}}(A_1, \mathrm{d}\chi) = 0$ for all $\chi \in \Omega^k_{\mathrm{sc}}(M)$. Together with the antisymmetry shown before, this entails that σ_{nd} descends to a well-defined map $\sigma_{\mathrm{nd}}\colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)^{\times 2} \to \mathbb{R}$ which is bilinear and antisymmetric. Therefore it is a presymplectic form. To this end let $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$: we have

$$\sigma_{\mathrm{nd}}(A, \mathrm{d}\chi) = (\delta\mathrm{d}A_1^+, \mathrm{d}\chi) = (\delta^2\mathrm{d}A_1^+, \chi) + (\mathrm{n}\delta\mathrm{d}A^+, \mathrm{t}\chi) = 0\,,$$

where we used Equation (1.6) as well as $\mathrm{n}\delta\mathrm{d}A = -\delta\mathrm{n}\mathrm{d}A = 0$. ■

Proposition C.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then the following linear maps are isomorphisms of vector spaces*

$$G_\parallel\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad G_\parallel\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,, \tag{C.2}$$

$$G_\perp\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,nd}}(M)} \to \mathrm{Sol}_{\mathrm{nd}}(M)\,, \qquad G_\perp\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)\,, \tag{C.3}$$

Proof. Mutatis mutandis, the proof of the four isomorphisms is the same. Hence we focus only on $G_\parallel\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$.
A direct computation shows that $G_\parallel\left[\Omega^k_{\mathrm{tc},\delta}(M)\right] \subseteq \mathcal{S}^{\Box}_{\mathrm{t},k}(M)$. The condition $\delta G_\parallel\omega = 0$ follows from Corollary 3.1.24. Moreover, $G_\parallel$ descends to the quotient since for all $\eta \in \Omega^k_{\mathrm{tc,t}}(M)$ we have $G_\parallel\delta\mathrm{d}\eta = -G_\parallel\mathrm{d}\delta\eta = -\mathrm{d}G_\parallel\delta\eta \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ on account of Corollary 3.1.24.
We prove that $G_\parallel$ is surjective. Let $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$. In view of Proposition 3.2.3 there exists $A' \in [A]$ such that $\Box_\parallel A' = 0$ as well as $\delta A' = 0$. Proposition 3.1.20 ensures that there exists $\alpha \in \Omega^k_{\mathrm{tc}}(M)$ such that $A' = G_\parallel\alpha$. Moreover, condition $\delta A' = 0$ and Corollary 3.1.24 implies that $\delta\alpha \in \ker G_\parallel$, therefore $\delta\alpha = \Box_\parallel\eta$ for some $\eta \in \Omega^k_{\mathrm{tc},\parallel}(M)$ – *cf.* Proposition 3.1.20 and Remark 3.1.21. Applying δ to the equality $\delta\alpha = \Box_\parallel\eta$ we find $\Box\delta\eta = 0$, that is, $\delta\eta = 0$ – *cf* Remark 3.1.22. It follows that $\delta\alpha = \delta\mathrm{d}\eta$. Moreover we have $[A] = [G_\parallel\alpha] = [G_\parallel\alpha - \mathrm{d}G_\parallel\eta] = [G_\parallel(\alpha - \mathrm{d}\eta)]$, where now $\alpha - \mathrm{d}\eta \in \Omega^k_{\mathrm{tc},\delta}(M)$.
Finally we prove that $G_\parallel$ is injective: let $[\alpha] \in \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)}$ be such that $[G_\parallel\alpha] = [0]$. This entails

that there exists $\chi \in \Omega^{k-1}_{\mathrm{tc,t}}(M)$ such that $G_{\|}\alpha = \mathrm{d}\chi$. Corollary 3.1.24 and $\alpha \in \Omega^k_{\mathrm{tc},\delta}(M)$ ensures that $\delta\mathrm{d}\chi = 0$, therefore $\chi \in \mathrm{Sol}_{\mathrm{t}}(M)$. Proposition 3.2.3, Remark 3.1.21 and Corollary 3.1.24 ensures that $\mathrm{d}\chi = \mathrm{d}G_{\|}\beta$ with $\beta \in \Omega^k_{\mathrm{tc},\delta}(M)$. It follows that $\alpha - \mathrm{d}\beta \in \ker G_{\|}$ and therefore $\alpha - \mathrm{d}\beta = \Box_{\|}\eta$ for $\eta \in \Omega^k_{\mathrm{tc},\|}(M)$ – *cf.* Remark 3.1.21. Applying δ to the last equality we find $\Box\beta = \Box\delta\eta$, hence $\beta = \delta\eta$ because of Remark 3.1.22. It follows that $\alpha = \delta\mathrm{d}\eta$ with $\eta \in \Omega^k_{\mathrm{c,t}}(M)$, that is, $[\alpha] = [0]$. ∎

Proposition C.3. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. The following statements hold true:*

1. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_{\|}([\alpha],[\beta]) \doteq (\alpha, G_{\|}\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \widetilde{G}_{\|}\right)$ *is symplectomorphic to* $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$.

2. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_{\perp}([\alpha],[\beta]) \doteq (\alpha, G_{\perp}\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_{\perp}\right)$ *is pre-symplectomorphic to* $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Proof. The proof of the two statements is the same. Hence we focus only on the first one. We observe that $\widetilde{G}_{\|}$ is well-defined. As a matter of fact, let $\alpha, \beta \in \Omega^k_{\mathrm{c},\delta}(M)$, then $G_{\|}\beta \in \Omega^k_{\mathrm{sc}}(M)$ and therefore the pairing $(\alpha, G_{\|}\beta)$ is finite. Moreover if $\eta \in \Omega^k_{\mathrm{c},\|}(M)$ we have

$$(\delta\mathrm{d}\eta, G_{\|}\beta) = (\eta, \delta\mathrm{d}G_{\|}\beta) = -(\eta, \mathrm{d}\delta G_{\|}\beta) = -(\eta, \mathrm{d}G_{\|}\delta\beta) = 0\,,$$
$$(\alpha, G_{\|}\delta\mathrm{d}\eta) = -(\alpha, G_{\|}\mathrm{d}\delta\eta) = -(\alpha, \mathrm{d}G_{\|}\delta\eta) = 0\,,$$

where we used that $G_{\|}\beta, \eta \in \Omega^k_{\mathrm{c,t}}(M)$ – *cf.* Equation (3.36) – as well as $\delta G_{\|}\beta = G_{\|}\delta\beta = 0$ – *cf.* Corollary 3.1.24. Therefore $\widetilde{G}_{\|}$ is well-defined: Moreover, it is per construction bilinear and antisymmetric, therefore it induces a pre-symplectic structure.
We now show that the isomorphism $G_{\|}\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\|}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$ is a pre-symplectomorphism. Let $[\alpha], [\beta] \in \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\|}(M)}$. As a direct consequence of the properties of $G_{\|} = G^+_{\|} - G^-_{\|}$, calling $A_1 = G_{\|}\alpha$ and $A_2 = G_{\|}\beta$, we can consider $A^{\pm}_1 = G^{\pm}_{\|}\alpha$ in Equation (C.1). This leads us to

$$\sigma_{\mathrm{t}}([G_{\|}\alpha], [G_{\|}\beta]) = (\delta\mathrm{d}G^+_{\|}\alpha, G_{\|}\beta) = (\Box G^+_{\|}\alpha - \mathrm{d}\delta G^+_{\|}\alpha, G_{\|}\beta) = (\alpha, G_{\|}\beta) = \widetilde{G}_{\|}([\alpha],[\beta])\,,$$

where we used Corollary 3.1.24 so that $\mathrm{d}\delta G^+_{\|}\alpha = \mathrm{d}G^+_{\|}\delta\alpha = 0$. ∎

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