Introduction to Quantum Backflow

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In classical physics: $P(t)$ is always decreasing with time.

In quantum physics: not necessarily.

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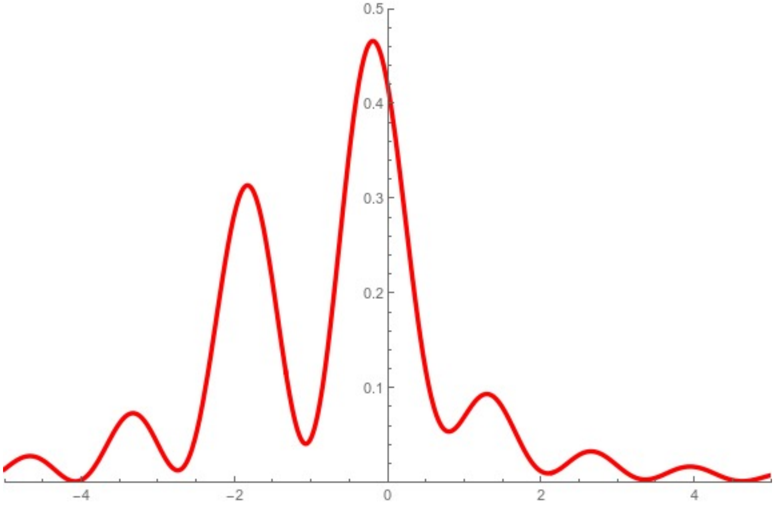

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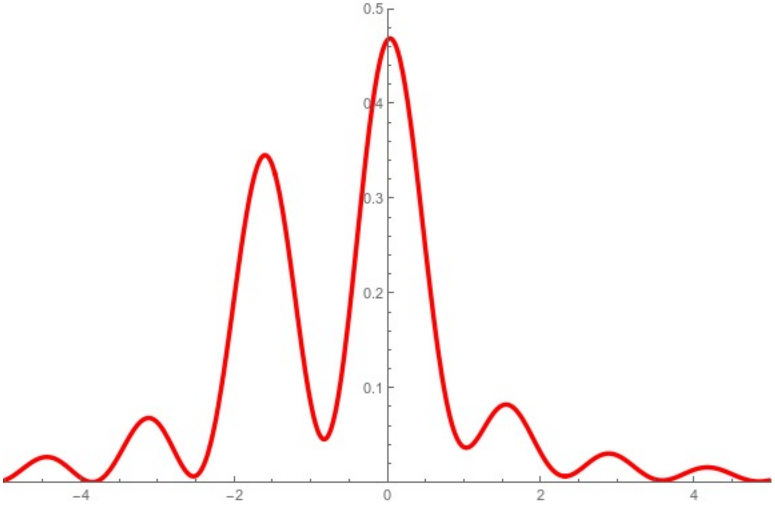

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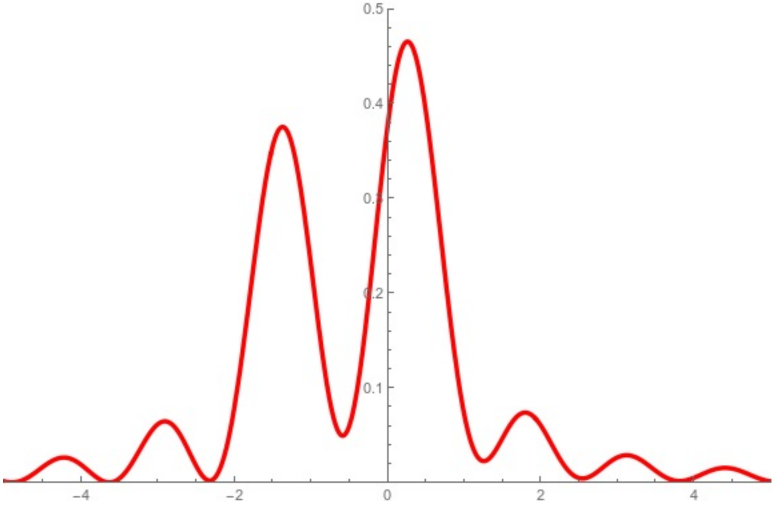

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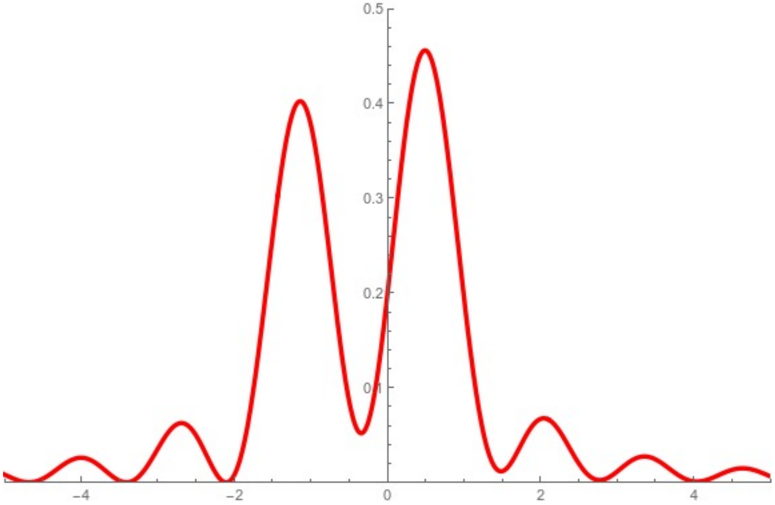

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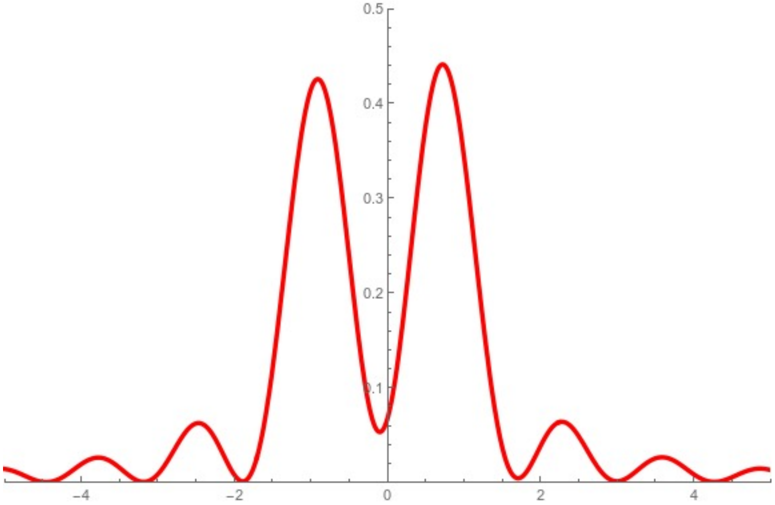

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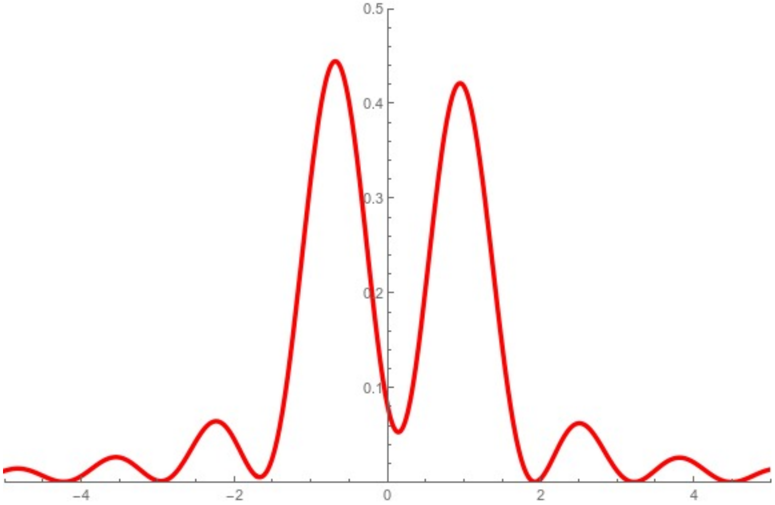

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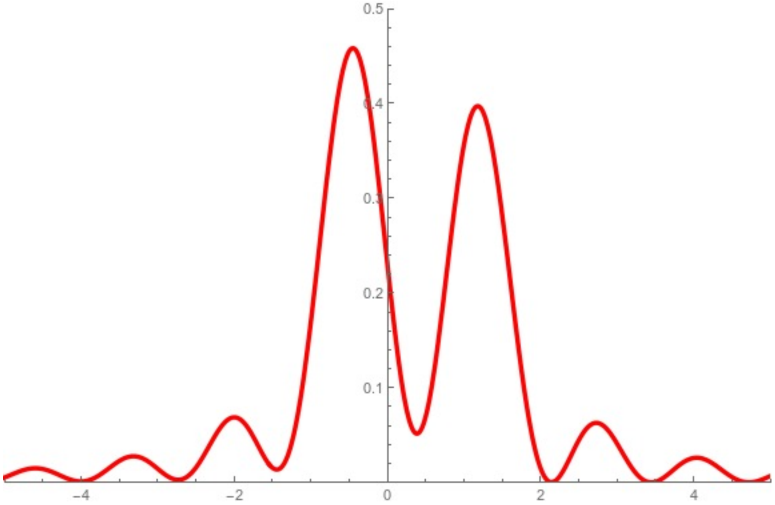

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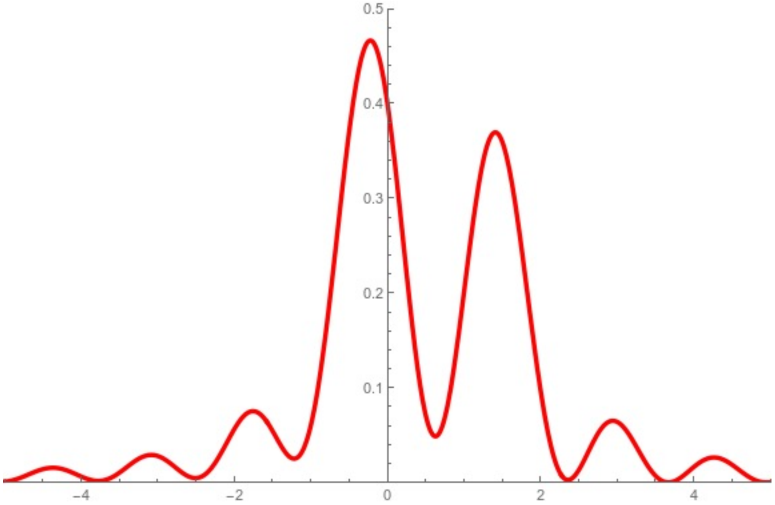

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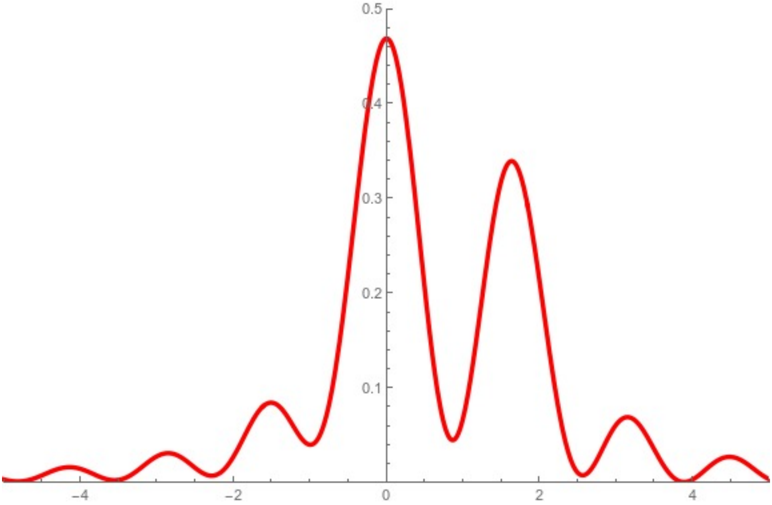

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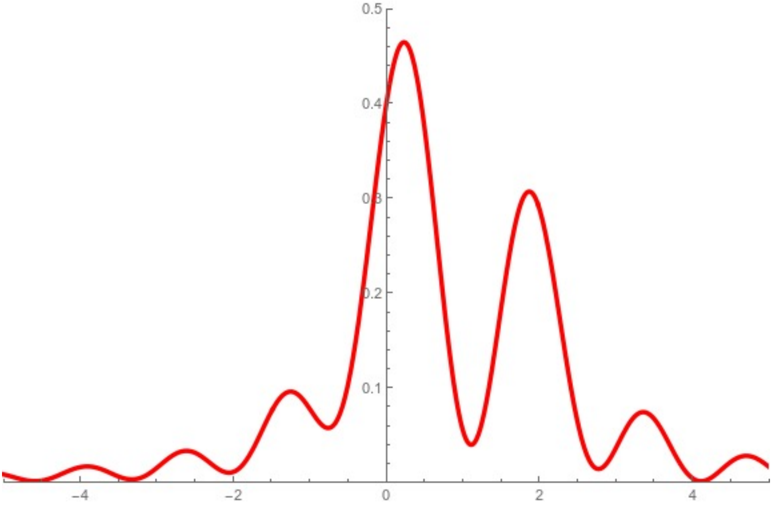

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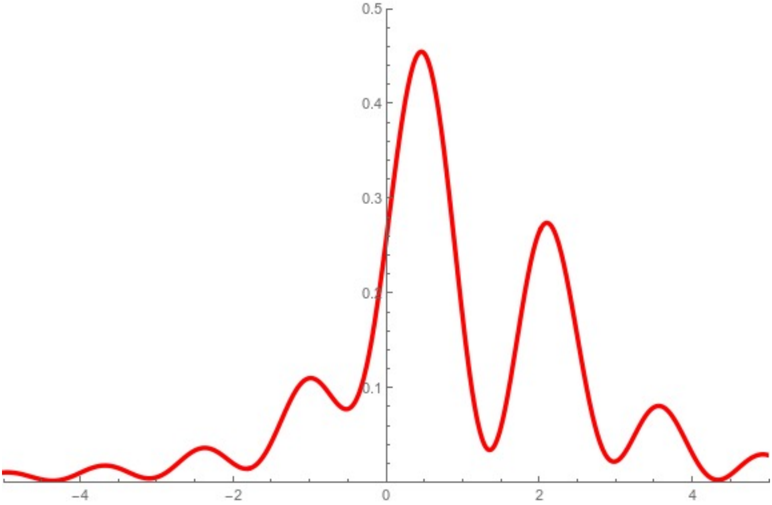

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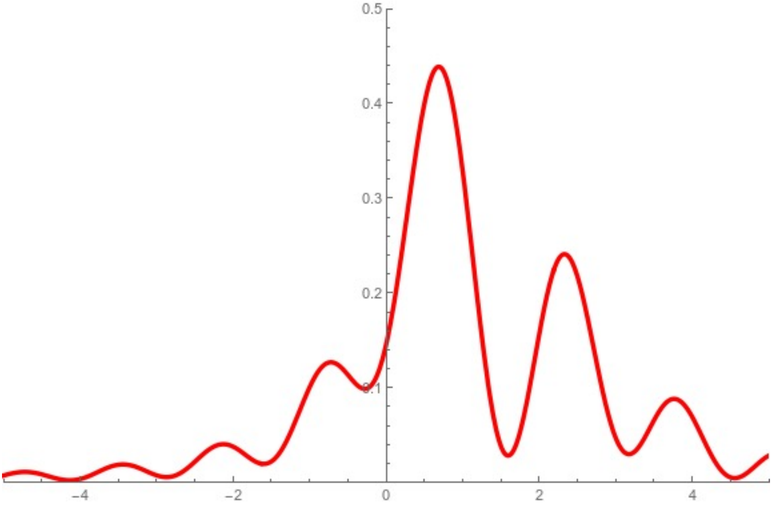

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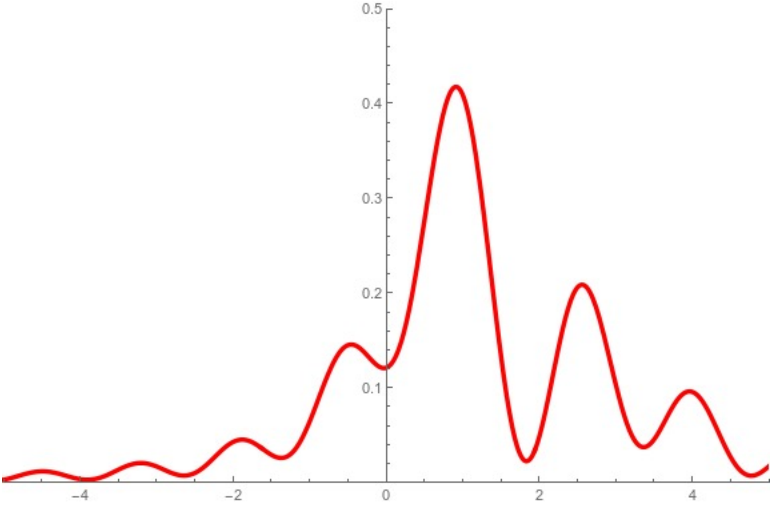

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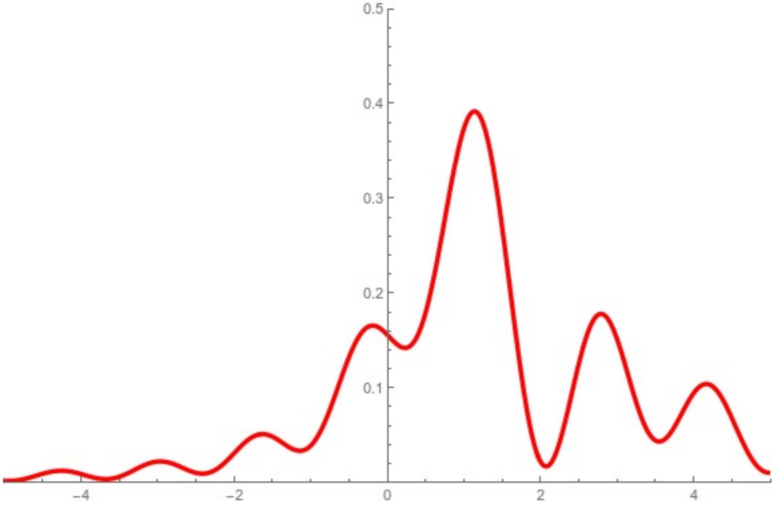

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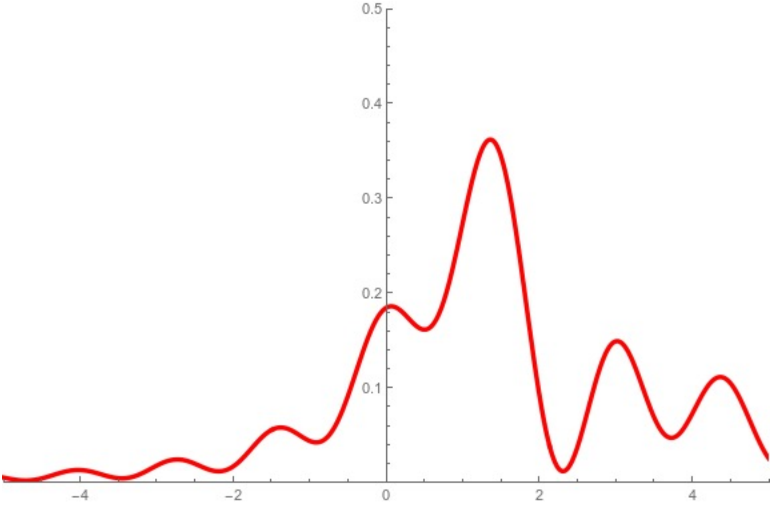

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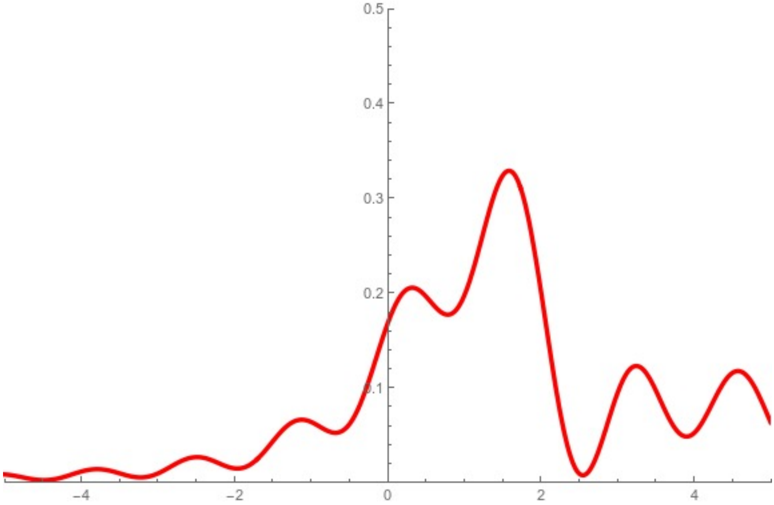

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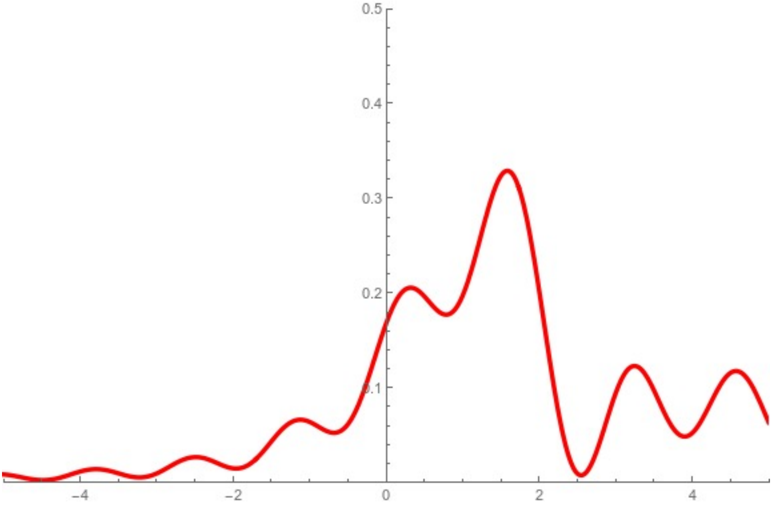

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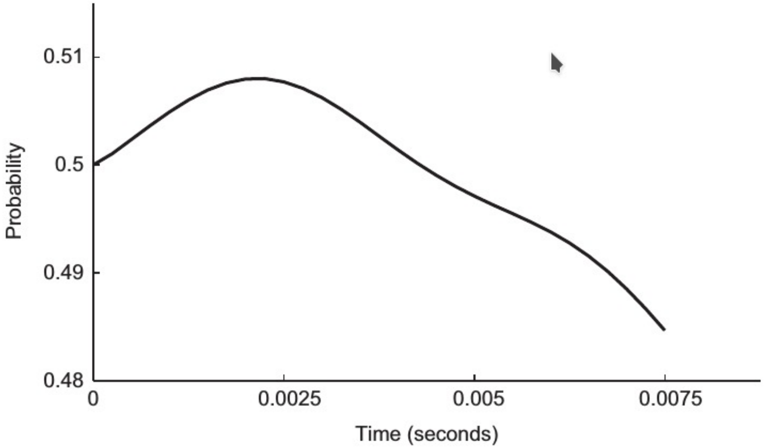

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Definition

We call $E_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the operator such that:

$$\mathcal{F}[E_\pm \psi](p) = \vartheta(\pm p)\widehat{\psi}(p)\ \forall \psi \in L^2(\mathbb{R}),$$

where ϑ is the Heaviside function.

Temporal boundedness of backflow

- Probability as a quadratic form

$$L(\psi_T) := \int_0^{+\infty} |\psi_T(x)|^2 \, \mathrm{d}x = (\widehat{\psi}| \underbrace{\widehat{U}_T^* \mathcal{F} \vartheta \mathcal{F}^{-1} \widehat{U}_T}_{:=\tilde{\vartheta}_T} \widehat{\psi}) \text{ with } \widehat{U}_T = \mathcal{F} U_T \mathcal{F}^{-1}.$$

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- **Backflow constant**: $\lambda := \sup\{(\phi|\vartheta B_T \vartheta \phi) \mid \|\phi\| = 1, \ T > 0\}$

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Theorem (Temporal boundedness of backflow)

Let $\lambda = \sup \sigma(\vartheta B \vartheta)$, where $B = B_{T=1}$ is the backflow operator. For any right-mover $\psi \in L^2(\mathbb{R})$ such that $\psi = E_+ \psi$ and for any $T > 0$ it holds

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Proof. $\sigma(\vartheta B \vartheta) \subseteq [-\|B\|, \|B\|]$ since the operator $\vartheta B \vartheta$ is bounded and self-adjoint. $\blacksquare$

Maximum backflow approximation

Proposition

Let $K : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ *be the integral operator:*

$$(Kf)(u) = -\frac{1}{\pi}\int_0^\infty \frac{\sin(u^2 - v^2)}{u - v} f(v)\,\mathrm{d}v \quad \forall f \in L^2(\mathbb{R}_+).$$

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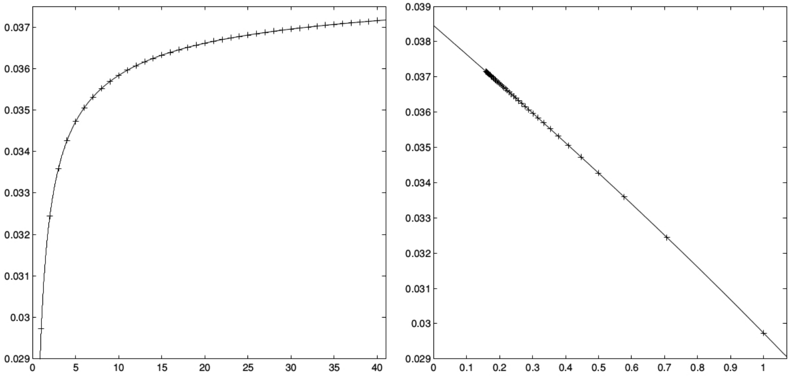
Ingredients: Hilbert transform

$$(Hf)(p) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(q)}{p - q}\,\mathrm{d}q.$$

We can use K to approximate λ.



$$\lambda \approx 0.0384517$$

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For any $f \in \mathcal{S}(\mathbb{R})$, $f \geq 0$, there exists a constant $\beta_0(f) \in (f) \in (-\infty, 0)$ such that $(\psi|E_+ J(f) E_+ \psi) \geq \beta_0(f)$.

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- exists under regularity and short-range assumptions on V.
- links "free" solutions of Schrödinger equation with "interacting" solutions.

Asymptotic right-movers

Definition

Let $V \in L^1(\mathbb{R})$ be a potential. We referred to V as a "short-range" potential (indicated $V \in L^{1+}(\mathbb{R})$) if it satisfies $\|V\|_{1+} = \int_{\mathbb{R}} \mathrm{d}x\,(1+|x|)|V(x)| < +\infty$.

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Theorem

Let $V \in L^{1+}(\mathbb{R})$. Then

(a) Ω_V *exists.*

(b) $[-\partial_x^2 + 2V(x) - k^2]\psi(x) = 0$ *has unique solutions*

$$\varphi_k(x) = \begin{cases} T_V(k)e^{ikx} + o(1) & \text{for } x \to +\infty \\ e^{ikx} + R_V(k)e^{-ikx} + o(1) & \text{for } x \to -\infty \end{cases}$$

(c) *For any* $\widehat{\psi} \in C_0^\infty(\mathbb{R})$, $(\Omega_V E_+ \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_k(x)\widehat{\psi}(k)\,\mathrm{d}k$.

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Expanding $E_+ \Omega_V^* J(f) \Omega_V E_+$ we have

$$(\psi | E_+ \Omega_V^* J(f) \Omega_V E_+ \psi) \geq \beta_0(f) - 2 \| J(f)(i+P)^{-1} \| [2 + \| P(\Omega_V - T_V) E_+ \|].$$

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- we have $\|J(f)(i+P)^{-1}\| \leq \|f\|_\infty + \frac{1}{2}\|f'\|_\infty$.
- We need to evaluate $\|P(\Omega_V - T_V)E_+\|$.

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Lemma

Let $V \in L^{1+}(\mathbb{R})$. Then, there exists $c_V \in \mathbb{R}$ such that

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Sketch of proof. Rewrite the time-independent Schrödinger equation as (Lippman-Schwinger equation)

$$\varphi_k(x) = T_V(k)e^{ikx} + \int_{-\infty}^{\infty} 2V(y)G_k(y-x)\varphi_k(y)\,\mathrm{d}y,$$

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where $G_k(x) = \sin(kx)\vartheta(x)/k$, and estimate

$$(P(\Omega_V - T_V)E_+\psi)(x) = \frac{-i}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty}\mathrm{d}k\int_{\mathbb{R}}\mathrm{d}y\; V(y)G_k(x-y)\varphi_k(y)\tilde{\psi}(k)$$

with the known asymptotics for $\varphi_k(x)$

Backflow and Scattering

Theorem (Boundedness of Backflow in scattering scenarios)

For any potential $V \in L^{1+}(\mathbb{R})$ *and for any non-negative* $f \in \mathcal{S}(\mathbb{R})$*, there exists a constant* $\beta_V(f) \in (-\infty, 0)$ *such that*

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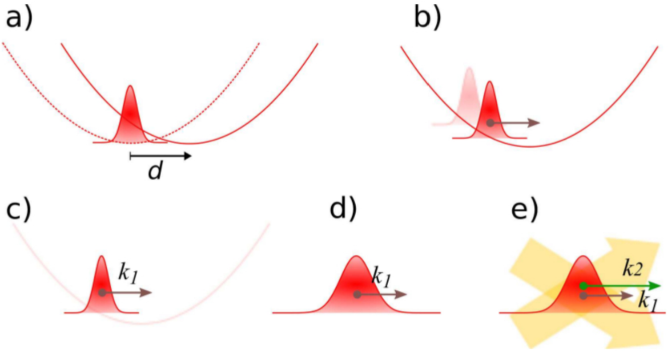
- Heuristic explanation: Backflow is a high momentum effect, but for high momentum reflection component is suppressed.
- What about experimental observations? (Bose-Einstein condensate, Bragg pulse, superposition of different momentum sates...)

Experimental set-up



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