Quantum field theory in curved spacetime

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Abstract

It is universally agreed that independently the theory of general relativity and quantum field theory are very successful theories. When static black holes are considered, it seems natural, that quantum field effects should have an impact on it without invoking quantum gravitational effects due to the scale of the problem. The general result is that the black hole, which was though to be static, is actually evaporating by slowly emitting particles with charge, angular momentum and energy. In this paper we consider transition of scalar field transition through a collapsing shell of matter, which latter forms a static spherically symmetric black hole. Analysis includes the use of Heun functions, which generalize hypergeometric functions.

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Chapter 1

Introduction

It has been over a century since Einstein introduced his theory of General Relativity. The premises of it are well known and are agreed upon as long as the theory of quantum particles and fields is not included. Its validity has been tested on the scales ranging from as small as one milimeter up to the size of the whole universe. It is no doubt that these are classical effects, as fully quantum effects are expected at scales many orders of magnitude tinier than those we can observe in laboratories. On the other hand, at scales 25 orders of magnitude larger than those the world is already quantum. From these two facts one can conclude that there exists a situation where gravity is still classical but matter, which is creating it or moving in a given classical geometry is fully quantum.

There are many different possibilities when these cases are real. One of them is the formation of a black hole. In 1975 Stephen Hawking [1] discovered that the black holes are not completely black, but instead are slowly evaporating. Since then many different arguments in favor of the existence of Hawking radiation have been introduced. Some of them consider the whole gravitational collapse with all the details like the origin of the aforementioned radiation, others still manage to show the same with a completely static situation [2], [3].

In this paper we trace the steps of the original paper since it appears that the differential equation the scalar field satisfies is of the form of the so-called Heun function, which generalize hypergeometric functions. Recently, many more appearances of this function have been discovered [4], mostly because of the books ([5], [6], [7]) dedicated to the properties of these functions being published and their implementation in computer algebra programs [8]. However, the quality and flexibility of any numerical work is still lagging behind. Therefore, only very rudimentary analysis can be carried out. Hence, it is the goal of this paper to find out what sort of difficulties may arise by attempting the full investigation of Hawking radiation in the simplest case.

Chapter 2

General Relativity

The modern explanation for the force of gravity is given by the theory of General Relativity ([9], [10], [11]). The main difference between it and all previous gravitational theories is the fact that space-time becomes dynamical, i.e. distances measured on it no longer follow the rules of flat space-time. Hence, particles (or fields) travel on curved manifolds, and any deviation from flat space-time predictions are assigned to the force of gravity.
The curvature of space-time is sourced by the so-called stress-energy-momentum (STM) tensor. It contains information about density and flux of energy and momentum of all fields except those of gravity. This is a result of equivalence principle, which states that at each point there exists a frame for which STM of gravity vanishes. STM tensor is conserved, meaning that its covariant divergence vanishes.

2.1 Metric formulation of gravity

Before we introduce Einstein's equations of motion, several basic results from differential geometry will be mentioned. A tensor is a multidimensional array which transforms according to a rule

$$T^{\beta_1\beta_2\ldots\beta_p}_{\alpha_1\alpha_2\ldots\alpha_q}(x^1,x^2,...,x^n) = T^{\beta'_1\beta'_2\ldots\beta'_p}_{\alpha'_1\alpha'_2\ldots\alpha'_q}(\overline{x}^1,\overline{x}^2,...,\overline{x}^n)\frac{\partial x^{\beta_1}}{\partial\overline{x}^{\beta'_1}}\frac{\partial x^{\beta_2}}{\partial\overline{x}^{\beta'_2}}\cdots\frac{\partial x^{\beta_p}}{\partial\overline{x}^{\beta'_p}}\frac{\partial\overline{x}^{\alpha'_1}}{\partial x^{\alpha_1}}\frac{\partial\overline{x}^{\alpha'_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial\overline{x}^{\alpha'_q}}{\partial x^{\alpha_q}} \tag{2.1}$$

where Einstein summation convention has been implemented.
Let us mention several frequently seen tensors and their properties.
Kronecker delta:

$$\delta_{\mu\nu} = \begin{cases} 1, & \mu=\nu \\ 0, & \mu\neq\nu \end{cases}, \quad T^{\alpha_1\ldots\mu\ldots\alpha_p}\delta_{\mu\nu} = T^{\alpha_1\ldots\nu\ldots\alpha_p}, \quad \delta_{\mu\nu}=\delta_{\nu\mu} \tag{2.2}$$

Metric tensor is an array of coefficients appearing in the expression for a length element:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.3}$$

Its properties are

$$g_{\mu\nu}=g_{\nu\mu}, \quad T^{\alpha_1\ldots\alpha_{n-1}\mu\alpha_{n+1}\ldots\alpha_p}g_{\mu\nu} = T^{\alpha_1\ldots\alpha_{n-1}}{}_{\nu}{}^{\alpha_{n+1}\ldots\alpha_p}, \quad g=\det\{g_{\mu\nu}\}, \quad g_{\mu\nu}g^{\nu\rho}=\delta^\rho_\mu \tag{2.4}$$

Levi-Civita tensor:

$$\epsilon_{\alpha_1\ldots\alpha_p} = \sqrt{|g|}\tilde{\epsilon}_{\alpha_1\ldots\alpha_p}, \quad \epsilon^{\alpha_1\ldots\alpha_p} = \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{\alpha_1\ldots\alpha_p} \tag{2.5}$$

where

$$\tilde{\epsilon} = \begin{cases} 1, \text{if } (\alpha_1,\alpha_2,...,\alpha_p) \text{ is an even permutation of } (1,2,...,p) \\ -1, \text{if } (\alpha_1,\alpha_2,...,\alpha_p) \text{ is an odd permutation of } (1,2,...,p) \\ 0, \text{ otherwise} \end{cases} \tag{2.6}$$

To preserve tensorial character, a covariant derivative has to be introduced, which maintains all covariant properties of tensors. For a general tensor its effect can be expressed as

$$\nabla_\mu T^{\beta_1\beta_2\ldots\beta_p}_{\alpha_1\alpha_2\ldots\alpha_q} = \partial_\mu T^{\beta_1\beta_2\ldots\beta_p}_{\alpha_1\alpha_2\ldots\alpha_q} + \sum_{i=1}^{p} \Gamma^{\beta_i}{}_{\lambda\mu} T^{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_p}_{\alpha_1\alpha_2\ldots\alpha_q} - \sum_{i=1}^{q} \Gamma^{\lambda}{}_{\alpha_i\mu} T^{\beta_1\beta_2\ldots\beta_p}_{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_q} \tag{2.7}$$

From this definition it is easy to notice that when tensor has no indices, i.e. if it is a scalar, then covariant derivative simplifies to an ordinary one.
Another straightforward result is noncommutativity of covariant derivatives. Not only it is in general nonvanishing, it actually represents curvature of manifold. This information is contained in the curvature tensor defined by a commutation relation

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\lambda\mu\nu}V^\lambda \tag{2.8}$$

Substitution of expression for covariant derivative leads to explicit form of curvature tensor:

$$R^\rho{}_{\lambda\mu\nu} = 2\partial_{[\mu}\Gamma^\rho{}_{\nu]\lambda} + 2\Gamma^\rho{}_{[\mu|\sigma|}\Gamma^\sigma{}_{\nu]\lambda} \tag{2.9}$$

Notice that the curvature tensor is antisymmetric with respect to last two indices. Due to this fact there exists two independent contractions:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \quad Q_{\mu\nu} = R^\rho{}_{\rho\mu\nu} \tag{2.10}$$

The former is called Ricci tensor, while the latter is known as the segmental curvature tensor. It can be shown that the segmental curvature satisfies relation

$$Q_{\mu\nu} = -N_{[\mu}{}^{\alpha\beta}{}_{,\nu]}g_{\alpha\beta}, \text{ where } N_{\mu\alpha\beta} = -\nabla_\mu g_{\alpha\beta} \tag{2.11}$$

and $N_{\mu\alpha\beta}$ is the non-metricity. Non-vanishing non-metricity neither preserves lengths nor angles, hence non-interacting fields would acquire different values at different points. This is in discordance with Einstein equivalence principle. Hence, for future discussion nonmetricity will be set to 0. This leaves only one curvature scalar, namely, Ricci scalar as a result of further contraction of Ricci tensor:

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.12}$$

Einstein-Hilbert action with matter fields and a cosmological constant reads as

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_m\right\} \tag{2.13}$$

Einstein field equations emerge as a result of variation of Einstein-Hilbert action with respect to metric provided that connection is symmetric with respect to lower indices. If this is not the case, instead as an outcome Einstein-Cartan equations pop out. Connection symbol is not fully independent from metric. Consider a sum of 3 covariant derivatives of the metric. Then it follows that

$$\nabla_\nu g_{\mu\lambda} + \nabla_\mu g_{\lambda\nu} - \nabla_\lambda g_{\mu\nu} = 0 \;\rightarrow\; \Gamma^\rho{}_{\mu\nu} = \{^\rho{}_{\mu\nu}\} + K^\rho{}_{\mu\nu} \tag{2.14}$$

where contorsion is defined by

$$K^\rho{}_{\mu\nu} = S_{\nu\mu}{}^\rho + S_{\mu\nu}{}^\rho + S^\rho{}_{\mu\nu} \tag{2.15}$$

and $S^\rho{}_{\mu\nu} = \Gamma^\rho{}_{[\mu\nu]}$ is the so-called torsion tensor and $\{^\rho{}_{\mu\nu}\}$ is the Christoffel symbol. Substitution of 2.14 into 2.13 results in

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa}(\mathring{R} + (2\mathring{\nabla}_\rho K^{\rho\nu}{}_\nu - K_\beta{}^\rho{}_\rho K^{\beta\nu}{}_\nu + K^{\rho\beta\nu}K_{\nu\beta\rho}) - 2\Lambda) + \mathcal{L}_m\right\} \tag{2.16}$$

The circular accent denotes pure christoffelian connection being used. Variation with respect to metric produces Einstein equation:

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\circ}{R} + \Lambda g_{\mu\nu} = \kappa(T_{\mu\nu} + U_{\mu\nu}) \tag{2.17}$$

where

$$U_{\mu\nu} = \frac{1}{\kappa}(K^{\rho}{}_{\mu\rho}K^{\lambda}{}_{\nu\lambda} - K^{\lambda}{}_{\mu\rho}K^{\rho}{}_{\nu\lambda} - \frac{1}{2}g_{\mu\nu}(K^{\rho}{}_{\beta\rho}K^{\lambda\beta}{}_{\lambda} - K^{\rho\beta}{}_{\lambda}K^{\lambda}{}_{\beta\rho})) \tag{2.18}$$

and

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{2.19}$$

On the other hand, varying with respect to contorsion gives rise to Cartan equation:

$$K^{\nu[\mu\rho]} + g^{\nu[\mu}K^{\rho]\gamma}{}_{\gamma} = \frac{\kappa}{2}\Pi^{\rho\mu\nu} \tag{2.20}$$

where $\Pi_{\rho}{}^{\mu\nu}$ is the spin tensor defined as

$$\Pi^{\rho\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta K_{\rho\mu\nu}} \tag{2.21}$$

Cartan equation is an algebraic one, thus an immediate conclusion is a vanishing contorsion outside of spin matter fields. Therefore, in absence of contorsion 2.17 with $U_{\mu\nu} = 0$ simplifies to field equations of General Relativity.

2.2 Riemann normal coordinates

The theory of general relativity was developed with an idea that locally an observer could not tell about the presence of gravitational forces. Therefore, it should be possible to choose a point in the spacetime such that gradients of the metric tensor vanish. These statements can be cast into two conditions, namely ([12], [13]):

$$g_{\mu\nu}|_p = \eta_{\mu\nu} \tag{2.22}$$

$$\partial_\rho g_{\mu\nu}|_p = 0. \tag{2.23}$$

The second relation sets values of Chritoffel symbols to zero. Note that this condition does not hold for their derivatives. Actually, higher derivatives of metric at a point can be related to derivatives of Christoffel symbols, and these can be related to covatiant derivatives of the Riemann tensor. Introduce Riemann normal coordinates y^α with respect to the point p, for which previous two conditions hold. Then, up to the forth power in y^α it holds that [16]

$$\begin{aligned} g_{\mu\nu}|_p &= \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma + \left[\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^{\lambda}{}_{\gamma\nu\delta}\right]y^\alpha y^\beta y^\gamma y^\delta \\ |g|_p &= 1 - \frac{1}{3}R_{\alpha\beta}y^\alpha y^\beta - \frac{1}{6}R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma + \left[\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} - \frac{1}{90}R_{\lambda\alpha\beta}{}^{\kappa}R_{\lambda\delta\gamma\kappa} - \frac{1}{20}R_{\alpha\beta;\gamma\delta}\right]y^\alpha y^\beta y^\gamma y^\delta \end{aligned} \tag{2.24}$$

where all products and derivatives of the Riemann tensor are evaluated at the point p. Semicolon stands for the covariant derivative with rightmost index being the index of last applied covariant differential operator. This expansion is one of the main tools used in the evaluation of quantum field Green functions in general spacetimes.

2.3 Frame fields

Einstein equivalence principle tells us that locally all metric tensors can be described by the Minkowski geometry. Hence it should be possible to project all tensors onto a tangent space were Lorentz transformations hold. The problem is solved with an implementation of tetrads. They should map any metric tensor to a minkowskian one:

$$e_\mu{}^A e_\nu{}^B \eta_{AB} = g_{\mu\nu} \tag{2.25}$$

where $e_\mu{}^A$ is the tetrad field ([10], [11]). here greek index represents curved manifold, while latin index stands for local tangent space. Hence, tetrads transform according to a rule

$$e_\mu{}^A = \tilde{e}_\nu{}^B \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \Lambda^A_{\ B} \tag{2.26}$$

with $\Lambda^A_{\ B}$ denoting the Lorentz transformation matrix. A tensor containing an arbitrary number of indices of both types follows analogous transformation rules. Greek indices are raised and lowered with the help of metric tensor while latin ones are managed with Minkowski metric tensor.

Tetrads contain same number of entries, like the metric tensor. However, in general they are not symmetric and thus have more degrees of freedom. This fact can be concluded from eq.(2.25), since Lorentz-transformed tetrads satisfy same equation as untransformed ones, therefore, the presence of additional freedom is attributable to rotations and boosts.

Even though tetrads allow us to project various tensors onto local Minkowski space, this is valid only for point being in consideration. Thus, parallel transport between different points is not equivalent to that in a globally flat space, and should be reflected in a similar manner as in the case of the covariant derivative. The covariant derivative of a tensor projected on a local flat space is defined as

$$\nabla_\mu T^{A_1 \dots A_n}_{B_1 \dots B_m} = \partial_\mu T^{A_1 \dots A_n}_{B_1 \dots B_m} + \sum_{i=1}^{n} \omega^{A_i}_{\ C\mu} T^{A_1 \dots C \dots A_n}_{B_1 \dots B_m} - \sum_{i=1}^{m} \omega^{C}_{\ B_i \mu} T^{A_1 \dots A_n}_{B_1 \dots C \dots B_m} \tag{2.27}$$

where $\omega^A_{\ B\mu}$ is called the spin connection. It plays a similar role like the affine connection. If a tensor is partially projected on a tangent space, it will contain both latin and greek indices. In that case the covariant derivative will contain a mixture of both connections. We will state an example in case of a tetrad:

$$\nabla_\mu e_\nu{}^A = \partial_\mu e_\nu{}^A - \Gamma^\rho_{\ \nu\mu} e_\rho{}^A + \omega^A_{\ C\mu} e_\nu{}^C \tag{2.28}$$

However, specifically for tetrads there is an exact relation, namely

$$\nabla_\mu e_\nu{}^A = 0 \tag{2.29}$$

which, although is often called "a postulate", is an exact constraint. It provides a way to calculate the spin connection if the affine connection is known and a tetrad representation has been chosen. Similarly to general metric compatibility, Minkowskian metric compatibility should hold:

$$\nabla_\mu \eta_{AB} = 0 \quad \rightarrow \quad \omega_{(AB)\mu} = 0 \tag{2.30}$$

Manipulation of eq.(2.29) results in an explicit formula for spin connection, which in the presence of torsion, is equal to

$$\omega_{\alpha\mu\nu} = [\alpha\mu\nu] + K_{\alpha\mu\nu} \tag{2.31}$$

where,

$$[\alpha\mu\nu] = \Omega_{\alpha\mu\nu} + 2\Omega_{(\mu\nu)\alpha} \tag{2.32}$$

and, under the name of Ricci rotation coefficients,

$$\Omega_{\alpha\mu\nu} = e_{\alpha A}\partial_{[\mu}e_{\nu]}^{\ A}. \tag{2.33}$$

Analogously to the metric formulation of all the curvature tensors and scalars, tetradic formulation can be written down where tetrads and spin connections act as fundamental building blocks.

2.4 Fermions in a curved spacetime

Recall that in a flat space-time Dirac equation for fermions reads as

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{2.34}$$

where gamma matrices satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.35}$$

In order to generalize Dirac theory for a curved space-time, two changes have to be applied. Firstly, covariant gamma matrices become space-time dependent due to an arbitrary metric tensor:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.36}$$

Secondly, partial derivative should be replaced by a covariant one, defined by

$$\nabla_\mu\psi = \partial_\mu\psi + \Gamma_\mu\psi, \quad \nabla_\mu\psi^\dagger = \partial_\mu\psi^\dagger + \psi^\dagger\Gamma_\mu^\dagger \tag{2.37}$$

where Γ_μ is required for covariantisation. It should not be confused with once contracted Christoffel symbol. We would like to impose several conditions on Γ_μ. From quantum field theory of fermions it is known that $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\lambda\psi$ transform as a scalar and a vector respectively. For these two transformation properties it follows that

$$\nabla_\mu(\overline{\psi}\psi) = \partial_\mu(\overline{\psi}\psi) \ \rightarrow \ \gamma^0\Gamma_\mu\gamma^0 = -\Gamma_\mu^\dagger \tag{2.38}$$

and

$$\nabla_\mu(\overline{\psi}\gamma^\nu\psi) = \partial_\mu(\overline{\psi}\gamma^\nu\psi) + \Gamma^\nu{}_{\lambda\mu}(\overline{\psi}\gamma^\lambda\psi) \ \rightarrow \ \nabla_\mu\gamma^\nu = \partial_\mu\gamma^\nu + \Gamma^\nu{}_{\lambda\mu}\gamma^\lambda + [\Gamma_\mu, \gamma^\nu] \tag{2.39}$$

Usually it is assumed that gamma matrices are covariantly conserved ($\nabla_\mu\gamma^\nu = 0$). Then it follows, that one can write a solution of the form

$$\Gamma_\mu = -\frac{1}{2}\omega_{AB\mu}[\gamma^A, \gamma^B] \tag{2.40}$$

Since eq.(2.29) allows us to compute the spin connection, the covariant derivative of fermionic fields can be readily evaluated.

Chapter 3

Black holes, coordinate charts, conformal diagrams

3.1 Schwarzschild black hole

The first nontrivial solution for Einstein field equations ever found was discovered by Karl Schwarzschild. It describes gravitational field in vacuum generated by a spherically symmetric matter configuration. Its most known representation in terms of metric tensor reads as

$$g_{\mu\nu} = diag\left(-\left[1-\frac{R_S}{r}\right], \frac{1}{1-\frac{R_S}{r}}, r^2, r^2\sin^2\theta\right), \tag{3.1}$$

where R_S is the so-called Schwarzschild radius. Comparison with Newtonian potential reveals that it is equal to double the mass of the gravitational field generating body. The physical interpretation of this chart is best explained by an example of a stationary observer far away from the black hole, using asymptotic set of coordinates $\{t, r, \theta, \phi\}$.

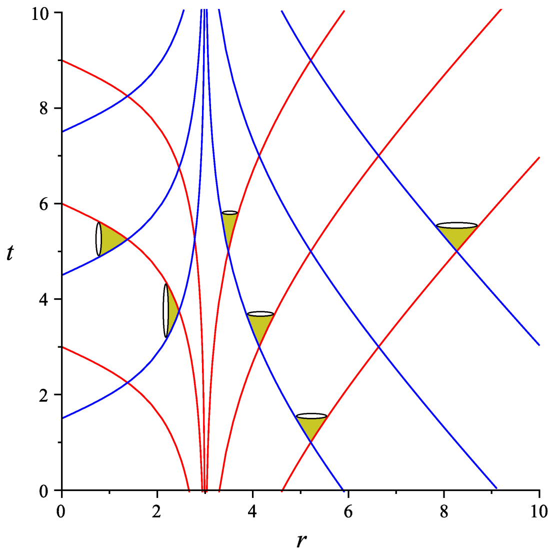


Figure 3.1: Lightray trajectories in Schwarzschild t-r coordinates

Curves in the figure (3.1) represent lightray trajectories. Blue lines stand for infalling light rays, while red lines represent outgoing lightrays. At each intersection of these two trajectories a lightcone can be drawn which locally defines future domain of spacetime accessible through a timelike or null travel. All lines are asymptotically tangent to the horizon surface denoted by $r = 1$ line. This agrees with the interpretaion of infalling object gradually slowing down and eventually halting at the horizon, since the projection of the lightcone onto r axis is arbitrary small. Furthermore, inside the blackhole all lightcones are tilted towards the singularity, disallowing the possibilty of escaping the black hole, at least classically. Notice the sudden tilt of the lightcone when passing the horizon. This is a sign of singular coordinate system used, which is in accord with the existence of unbounded components both at $r = R_S$ and $r = 0$. Since we require the solution to asymptotically resemble the flat space, the region $r < R_S$ is not described by this chart.

3.2 Eddington-Finkelstein coordinate chart

Upon solving radial eq.(2.3) in Schwarzschild coordinates in the case of light, i.e. $ds^2 = 0$, one discovers that there are 2 solutions to this equation, namely

$$v = t + r_*, \quad u = t - r_* \tag{3.2}$$

where r_* is called the tortoise coordinate and is defined as

$$r_* = r + R_S \log\left[\frac{r}{R_S} - 1\right] \tag{3.3}$$

such that it satisfies the differential relation

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{R_S}{r}}. \tag{3.4}$$

It maps a semi-infinite interval $r \in [R_S, \infty)$ onto an infinite interval $r_* \in (-\infty, \infty)$. Hence, the horizon has been pushed to an infinity. Constant v solutions penetrate the future horizon, while constant u solutions penetrate the past horizon. Eddington-Finkelstein coordinates are obtained by using v or u rather than t.

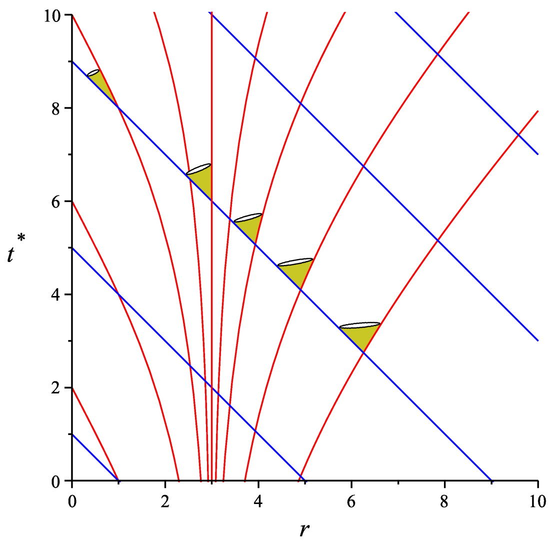


Figure 3.2: Lightray trajectories in incoming Eddington-Finkelstein v-r coordinates (t^* has the same meaning as v)

This results into ingoing or outgoint Eddington-Finkelstein coordinates:

$$ds^2 = -\left[1 - \frac{R_S}{r}\right] dv^2 + 2dvdr + dr^2 + r^2 d\Omega_{S^2}^2 \tag{3.5}$$

$$ds^2 = -\left[1 - \frac{R_S}{r}\right] du^2 - 2dudr + dr^2 + r^2 d\Omega_{S^2}^2 \tag{3.6}$$

where $d\Omega_{S^2}^2 = d\theta^2 + \sin^2\theta d\phi^2$.

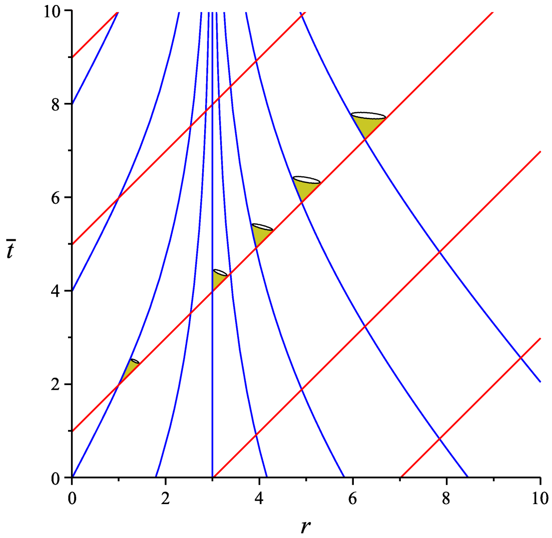


Figure 3.3: Lightray trajectories in outgoing Eddington-Finkelstein u-r coordinates ($\bar{t}$ has the same meaning as u)

Figures (3.2) and (3.3) show lightray worldlines in corresponding coordinates. Straight lines are either radially "incoming" or "outgoing" lightrays, while the curved ones travel in an opposite direction. Notice that outside of the horizon lightcone projects onto both increasing or decreasing values of r, while inside the projection is limited to one direction. Finally, outgoing coordinates represent a whitehole instead of a black hole, because lightcones always have a projection outwards. Since metric is finite everywhere except of central singularity, these coordinates can be used everywhere.

3.3 Painlevé-Gulstrand coordinates

It can be shown using invariants of general relativity (such as Kretschmann scalar, i.e. $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$), that $r = R_S$ is not a true singularity. Therefore, there should not be any physical difference for an observer who is crossing the horizon. Assuming the observer is radially infalling and requiring that his/her time matches the proper time, one gets

$$ds^2 = -\left[1 - \frac{R_S}{r}\right] dt_r^2 + 2\sqrt{\frac{R_S}{r}} dt_r dr + dr^2 + r^2 d\Omega_{S_2}^2 \tag{3.7}$$

where

$$t_r = t + 2R\left(\sqrt{\frac{r}{R_S}} + \frac{1}{2}\log\left|\frac{\frac{r}{R_S} - 1}{\frac{r}{R_S} - 1}\right|\right) \tag{3.8}$$

Indeed, $r = R_S$ has no special effect on the observer or his/her measurements. Furthermore, hypersurfaces of constant t_r represent a flat geometry. It means that the spacetime diagram would differ from Schwarzschild one by having its infalling trajectory continuous through the horizon.

3.4 Lemaître coordinates

Again, begin with Schwarzschild spacetime and apply the following transformation:

$$\begin{aligned} d\tau &= dt + \sqrt{\frac{R_S}{r}} \frac{1}{1 - \frac{R_S}{r}} dr \\ d\rho &= dt + \sqrt{\frac{r}{R_S}} \frac{1}{1 - \frac{R_S}{r}} dr. \end{aligned} \tag{3.9}$$

The resulting metric acquires the following form:

$$ds^2 = -d\tau^2 + \frac{R_S}{r} d\rho^2 + r^2 \Omega^2 \tag{3.10}$$

The metric is singular only at the center of black hole. The coordinates correspond to a freely falling observer which starts with vanishing velocity at spatial infinity. At every point this observer moves with a velocity equal to the escape velocity at that point. For constant ρ trajectories τ is the proper time for the chosen trajectory.

3.5 Kruskal-Szekeres coordinates

One more example of a coordinate system valid everywhere except for central sigularity is Kruskal-Szekeres coordinate chart $\{T,\ X,\ r,\ \theta\}$:

$$ds^2 = \frac{4R_S^3}{r} e^{-\frac{r}{R_S}} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2, \quad X^2 - T^2 = \left(1 - \frac{r}{R_S}\right) e^{-\frac{r}{R_S}} \tag{3.11}$$

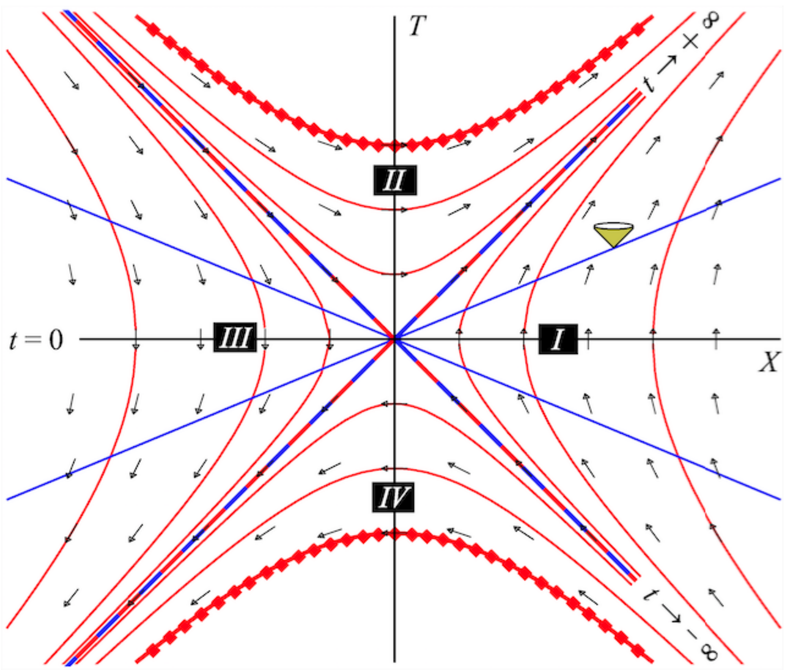


Figure 3.4: Kruskal-Szekeres spacetime diagram in T-R coordinates. Thin red hyperbolas depict points of constant radius, blue lines are slices of constant time. Blue-red lines are the horizons, while thick red hyperbolas are singularities of both white and black holes.

In addition to this, this chart also maximally extends the solutions. In other words, all geodesics can only end at true gravitational singularities, unlike in previous cases, where validity of coordinates is limited by physical reasons. The price of this extension is an addition of a white hole and another "Universe" outside of both black and white holes.

3.6 Conformal diagrams

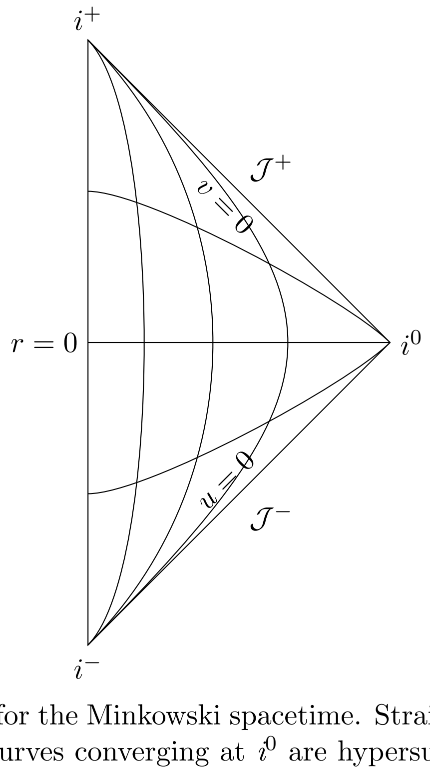
Space-times of various geometries can be comfortably analyzed using Carter-Penrose (or conformal) diagrams. Consider an infinite diagram with respect to an ingoing and outgoing null rays. Then the transformation is made using

$$\tan(u_{\text{conformal}}) = u_{\text{infinite}}, \tag{3.12}$$

where u is any of null coordinates. This map brings points infinitely far away into a diagram of a finite size. Some regions in the diagrams have standardized abreviations. The following is a list of those abreviations:

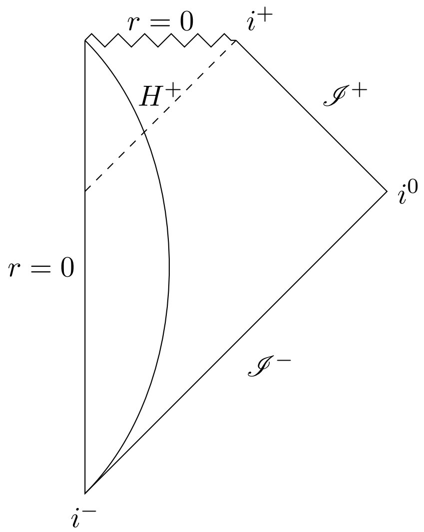
- spacelike infinity i^0; all spacelike curves terminate at this point
- future timelike infinity i^+; all timelike curves eventually terminate there, unless they cross the future horizon
- past timelike infinity i^-; all timlike curves originate there unless they come out of the past horizon
- past horizon H^-; denotes the outer limit of a white hole
- future horizon H^+; denotes the outer limit of a black hole
- past null infinity $\mathscr{I}^-$; all ingoing null lines originate on this line
- future null infinity $\mathscr{I}^+$; all outgoing null lines terminate on this line.

To illustrate the use of conformal diagrams, consider first a flat spacetime.



This is a conformal diagram for the Minkowski spacetime. Straigth vertical line stands for some central point of reference. Curves converging at i^0 are hypersurfaces of a constant time, while the ones emerging from i^- and merging at i^+ are hypersurfaces of constant radius. Since there are no singular points in the metric or corresponding invariants, the conformal diagram does not contain inaccessible or inescapable regions.

Let us dive now into the case of formation of a black hole due to a collapsing timelike shell.



Vertically aligned curve denotes the position of the timelike shell of collapsing matter. Dashed diagonal line stands for horizon, while the zigzag line on the top is the singularity. The singularity zigzag is horizontal, since it is singularity in spacetime, no just space. Namely, any object that crosses the horizon will eventually reach singularity in a finite amount of time. Future timelike infinity in this diagaram is where all timelike curves, which do not cross the horizon, terminate.

Chapter 4

Quantum field theory

4.1 Canonical quantisation of the scalar field on the flat spacetime

Consider a massive scalar field on a flat spacetime. It is described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{4.1}$$

Extremisation of the action $S = \int d^n\mathcal{L}$ for this field leads to

$$(\Box - m^2)\phi = 0 \tag{4.2}$$

which, in Cartesian coordinates, is solved by

$$\phi_k = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{e^{i\mathbf{kx}-i\omega t}}{\sqrt{2\omega}}, \tag{4.3}$$

where $\omega = \sqrt{k^2+m^2}$ and the bold product stands for scalar inner product of respective vectors. The numerical factor in front is chosed in such a way that the following normalisation holds ([14], [15]):

$$(\phi_k, \phi_{k'}) = i\int dx^3\{\phi_k\partial_t\phi^*_{k'} - \partial_t\phi_k\phi^*_{k'}\} = \delta^3(k-k') \tag{4.4}$$

Define the canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \tag{4.5}$$

Quantisation proceduce is continued by employing the equal time canonical commutation relations:

$$\begin{aligned}
[\pi(t,x), \phi(t,x')] &= 0\\
[\pi(t,x), \pi(t,x')] &= 0\\
[\phi(t,x), \phi(t,x')] &= i\delta^3(x-x')
\end{aligned} \tag{4.6}$$

Functions (4.3) and their complex conjugates form a complete basis with respect to (4.4). Thus we can expand any ϕ in this basis:

$$\phi = \sum_k (a_k\phi_k + a_k^\dagger\phi_k^*) \tag{4.7}$$

Equal time commutation relations for ϕ imply that

$$\begin{aligned}
[a_k, a_{k'}] &= 0\\
[a_k^\dagger, a_{k'}^\dagger] &= 0\\
[a_k, a_{k'}^\dagger] &= \delta^{n-1}(k-k').
\end{aligned} \tag{4.8}$$

In the Heisenberg picture time-independent quantum states span a Hilbert space. We will use Fock representation, where a multiparticle state is construced by acting with creation operators of various momenta on the so-called vacuum state, which is defined by

$$a_k |0\rangle = 0, \forall k, \tag{4.9}$$

where a_k is an anihilation operator, which reduces the number of particles with momentum k by one. Analogously, the creation operator $a_k^\dagger$ increases the number by one.
Any Fock state with the implementation of Bose statistics is expressible as

$$\left|n_{k_1}, ..., n_{k_j}\right\rangle = \prod_{i=1}^{j} \frac{(a_{k_i}^\dagger)^{n_{k_i}}}{\sqrt{n_{k_i}!}} |0\rangle \tag{4.10}$$

and is normalised according to

$$\left\langle n_{k_1}...n_{k_j} \middle| m_{k'_1}...m_{k'_l} \right\rangle = \delta_{j,l} \sum_{\mathcal{P}} \prod_{i=1}^{j} \delta_{n_{k_i}, m_{k'_{\mathcal{P}(i)}}} \delta_{k_i, k'_{\mathcal{P}(i)}}, \tag{4.11}$$

where we sum over all permutations $\mathcal{P}(i)$.

4.2 Fock states in observables

Fock states can be used to construct observables like particle number, energy and momentum of the field. Number operator is defined as

$$\begin{aligned} N_{\mathbf{k}} &= a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \text{ (\# of particles with momentum } \mathbf{k}) \\ N &= \sum_{\mathbf{k}} N_{\mathbf{k}} \text{ (\# of particles over all momenta)} \end{aligned} \tag{4.12}$$

From eqs. (4.9) and (4.10) if follows that

$$\begin{aligned} &\langle 0| N_{\mathbf{k}_i} |0\rangle = 0, \forall i \\ &\left\langle n_{\mathbf{k}_1}...n_{\mathbf{k}_j} \right| N_{\mathbf{k}_i} \left| n_{\mathbf{k}_1}...n_{\mathbf{k}_j} \right\rangle = n_{\mathbf{k}_i}. \end{aligned} \tag{4.13}$$

The symmetric energy-momentum tensor of the scalar field reads as

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\eta^{\rho\lambda} \partial_\rho \phi \partial_\lambda \phi - m^2 \phi^2) \tag{4.14}$$

The momentum density of the field is the T_{ti} component of the energy-momentum tensor. For the scalar field it is equal to $\partial_t \phi \partial_i \phi$. Hence, after the substitution of plane wave expansion, the total momentum for a constant time hypersurface is

$$P_i = \int_t dx^3 T_{ti} = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} k_i \tag{4.15}$$

and vanishes when the field is in its vacuum state. The energy density of the field is defined as T_{tt} component of the energy-momentum tensor. In the case of the scalar field it equals to $\frac{1}{2}\{(\partial_t \phi)^2 + (\vec{\nabla}\phi)^2 - m^2\phi^2\}$. The aforementioned substitution of plane wave expansion leads to the following form of the total energy on a spacelike hypersurface:

$$H = \int_t dx^3 T_{tt} = \sum_{\mathbf{k}} \left\{ a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right\} \omega_{\mathbf{k}}. \tag{4.16}$$

This quantity is infinite even in the field's vacuum state. However, for flat spacetime physics this is not important, because only excitations are physically observed. Therefore some renormalisation method should be used to define the relevant part of the total energy. Most popular method is normal ordering procedure, where all creation operators are placed to the left of all annihilation operators and vice versa. Then it is said that we observe this normal ordered total energy:

$$: H := \int_t dx^3 : T_{tt} := \sum_{\mathbf{k}} \{a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\} \omega_{\mathbf{k}}. \tag{4.17}$$

4.3 Generating functional of Green function

Consider a functional integral

$$Z[J] = \langle out, 0 | in, 0 \rangle = \int \mathcal{D}[\phi] \; e^{iS[\phi] + i\int d^4x J(x)\phi(x)}. \tag{4.18}$$

Z[J] is a generating functional, which gives the transition amplitude of propagation from $|in, 0\rangle$ to $\langle out, 0|$ with a particle production present and described by $J(x)$. In the absence of sources we have the transition between two Minkowski vacuum states. Functional differentiation of Z with respect J results in

$$\langle 0 | \mathcal{T} \prod_{i=1}^{n} \phi(x_i) |0\rangle_c = \left. \frac{\delta Z[J]}{i^n \delta J(x_1) ... \delta J(x_n)} \right|_{J=0}, \tag{4.19}$$

where on the left we have time ordered, connected Green functions.
For a scalar field the generating functional does not converge, hence a regularisation is needed, which is achieved by an additional term $-\frac{1}{2}\epsilon\phi^2$. Furthermore, action can be rewritten as

$$iS = \frac{i}{2} \int d^4x (\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 + i\epsilon\phi^2) = -\frac{i}{2} \int d^4x \; \phi(x)[\partial^\mu\partial_\mu + m^2 + i\epsilon]\phi(x) + \text{ b.t.} \tag{4.20}$$

Here "b.t." stands for boundary terms, which vanish for fields vanishing at the boundary. The objective now is to group operator in brackets with currents J instead of fields. Shift $\phi \to \phi + \phi_0$ such that $[\Box + m^2 - i\epsilon]\phi_0 = J$. Then, after integration by parts, it boils down to

$$Z[J] = e^{i\int d^4x J\phi_0} \int \mathcal{D}[\phi] e^{-\frac{i}{2}\int d^4x (\phi[\Box + m^2 - i\epsilon]\phi)} = \frac{N}{\sqrt{\det K}} e^{i\int d^4x J\phi_0}, \tag{4.21}$$

where N is a numerical factor with a value that sets $Z[0] = 1$ and K is related to Feynman propagator through

$$\frac{1}{\sqrt{\det K}} = \sqrt{\det(-G_F)} = \exp\left[\frac{1}{2} \operatorname{Tr}\log(-G_F)\right]. \tag{4.22}$$

Since Feynman propagator satisfies $[\Box + m^2 - i\epsilon]G_F(x) = -i\delta(x)$, the solution for ϕ_0 is

$$\phi_0(x) = i \int \Delta(x-y)J(y), \tag{4.23}$$

hence the final form of the generating functional is

$$Z[J] = \sqrt{\det(-G_F)} N e^{-\frac{i}{2}\int d^4x \int d^4y J(x) G_F(x-y) J(y)}. \tag{4.24}$$

Feynman propagator is recovered by

$$G_F(x-y) = \langle 0 | \mathcal{T}\phi(x)\phi(y) |0\rangle = \left. \frac{\delta Z[J]}{i^2 \delta J(x) \delta J(y)} \right|_{J=0} \tag{4.25}$$

4.4 Scalar field quantisation on an arbitrary spacetime

Assume that a spacetime is an infinitely differentiable, pseudo-Riemannian, globally hyperbolic manifold. This is required for an existence of differential equations and Cauchy hypersurfaces. For an arbitrary metric lagrangian of the scalar field becomes

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\{\nabla^\mu \phi \nabla_\mu \phi - m^2\phi^2\}. \tag{4.26}$$

By following the action extremisation procedure, one arrives at an equation, which formally looks like eq.(4.2). However, $\Box\phi$ in general stands for

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi). \tag{4.27}$$

The normalisation (4.4) is generalised to

$$(\phi_1, \phi_2) = -i\int_\Sigma \sqrt{-g_\Sigma}(\phi_1\partial_\mu\phi_2^* - \partial_\mu\phi_1\phi_2^*)d\Sigma^\mu. \tag{4.28}$$

Here integration is carried over a spacelike hypersurface. $d\Sigma^\mu$ stands for $n^\mu d\Sigma$ and n^μ is a future-directed unit vector orthogonal to Σ. Apart from covariant generalisation, $(\phi_k, \phi_{k'}) = \delta^3(\mathbf{k}-\mathbf{k}')$ still holds.

Like in flat case, there exists a basis in which general configuration of ϕ could be expanded. However, since in principle the spacetime can be lacking any symmetries whatsoever, the vacuum state defined by operators of some expansion will not be the same as the one of the flat case. Furthermore, if the spacetime does not contain a timelike Killing vector, then even for the same spacetime vacua defined for different instants of time will differ. Consider two different expansions in the same spacetime with their corresponding vacua:

$$\begin{aligned}\phi &= \sum_i (a_i\phi_i + a_i^\dagger\phi_i^*), \quad a_i|0_a\rangle = 0\\ \phi &= \sum_j (b_j\psi + b_j^\dagger\psi_j^*), \quad b_j|0_b\rangle = 0\end{aligned} \tag{4.29}$$

Since both basis are complete, it is possible to expand basis functions of one expansion in term of basis functions of the other expansion, namely

$$\begin{aligned}\psi_i &= \sum_j (\alpha_{ij}\phi_j + \beta_{ij}\phi_j^*)\\ \psi_i^* &= \sum_j (\alpha_{ij}^*\phi_j - \beta_{ij}\phi_j^*).\end{aligned} \tag{4.30}$$

α and β are complex numbers and are known as Bogolyubov coefficients. From orthonormality as well as eqs. (4.29) and (4.30) similar relations for creation and anihilation operators can be devised:

$$\begin{aligned}a_i &= \sum_j (\alpha_{ji}b_j + \beta_{ji}^* b_j^\dagger)\\ b_j &= \sum_i (\alpha_{ji}^* a_i - \beta_{ji}a_i^\dagger)\\ &\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij}\\ &\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0.\end{aligned} \tag{4.31}$$

Different vacua result in different notions of particles. Assume, that there no particles with respect to vacuum $|0_a\rangle$. It means that $\langle 0_a| N_i |0_a\rangle = 0$. Then in vacuum $|0_b\rangle$ it holds that

$$\langle 0_b| N_i |0_b\rangle = \sum_j |\beta_{ji}|^2, \tag{4.32}$$

thus, if β is different from 0, then there are particles even in the vacuum state.

4.5 Expectation value of the stress-energy tensor

In a semiclassical approach, stress-energy tensor is replaced with its expectation value, i.e.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa \langle \psi | T_{\mu\nu} |\psi\rangle . \tag{4.33}$$

In a classical case the right-hand side is defined as a variational derivative of matter action with respect to the metric tensor. Similar definition will be used for its expectation value. To see why this is the case, let us remember that $Z[0] = \langle out, 0|in, 0\rangle_{J=0}$, which was equal to unity in a flat spacetime. But we know, that in general vacua are not different, therefore $\langle out, 0|in, 0\rangle \neq 1$. Let us vary $Z[J]$ like we would do in the case of action:

$$\frac{2}{\sqrt{-g}} \frac{\delta Z[0]}{\delta g^{\mu\nu}} = \int \mathcal{D}[\phi] \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} e^{iS_m[\phi]} = \langle out, 0| T_{\mu\nu} |in, 0\rangle \tag{4.34}$$

Identify

$$Z[0] = e^{iW} \text{ such that } W = -i \log \langle out, 0|in, 0\rangle \tag{4.35}$$

where W is an effective action. Then

$$\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle out, 0| T_{\mu\nu} |in, 0\rangle}{\langle out, 0|in, 0\rangle} \tag{4.36}$$

$Z[0]$ above is calculated as usual except for a few changes due to covariantisation, namely

$$d^4x \to d^4x\sqrt{-g(x)}, \quad \delta^4(x-y) \to \frac{\delta^4(x-y)}{\sqrt{-g(y)}} \tag{4.37}$$

and, of course, a different Green function due to arbitrary geometry.

4.6 Adiabatic expansion of Green functions

General spacetime does not admit momentum representation due to symmetries not necessarily matching those of Minkowski spacetime. However, it is possible to compute corrections to the flat spacetime Green functions locally, which what is more useful in practise, since higher energy experiments probe more local properties. Denote $\sqrt{-g(x)} G_F(x, x') = \mathcal{G}_F(x, x')$ and transform to momentum space [16]:

$$\mathcal{G}_F(x, x') = \frac{1}{(2\pi)^4} \int d^4k e^{iky} \mathcal{G}_F(k), \quad ky = \eta^{\alpha\beta} k_\alpha y_\beta . \tag{4.38}$$

Starting from $(\Box - m^2) G_F(x, x') = -\frac{\delta^4(x-x')}{\sqrt{-g(x)}}$ and using normal coordinates, one can write down expansion up to any number of derivatives of metric. Below we limit the expansion with 4 derivatives of metric:

$$\begin{aligned} \mathcal{G}_F(k) \approx \frac{1}{k^2 - m^2} - \frac{R}{6[k^2 - m^2]^2} + \frac{i}{12} R_{;\alpha} \partial^\alpha \frac{1}{[k^2 - m^2]^2} - \frac{a_{\alpha\beta}}{3} \partial^\alpha \partial^\beta \frac{1}{[k^2 - m^2]^2} + \\ + \left(\frac{R^2}{36} + \frac{2}{3} a_\lambda{}^\lambda \right) \frac{1}{[k^2 - m^2]^3} \end{aligned} \tag{4.39}$$

with

$$a_{\alpha\beta} = -\frac{1}{40}\left[3R_{\alpha\beta} - R_{\alpha\beta;\lambda}{}^{\lambda}\right] - \frac{1}{60}\left[2R_{\alpha}{}^{\lambda}R_{\lambda\beta} - R^{\kappa}{}_{\alpha}{}^{\lambda}{}_{\beta}R_{\kappa\lambda} - R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta}\right] \tag{4.40}$$

Given expansion in momentum space have to be used to find out the form of Feynman propagator. From eqs.(4.39) and (4.40) one finds out that

$$\mathcal{G}_F(x,x') = \int \frac{d^4k}{2\pi}^4 e^{-iky}\left[a_0(x,x') + a_1(x,x')\left(-\frac{\partial}{\partial m^2}\right) + a_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\right]\frac{1}{k^2-m^2} \tag{4.41}$$

with

$$\begin{aligned} a_0(x,x') &= 1 \\ a_1(x,x') &= \frac{R}{6} - \frac{1}{12}R_{;\alpha}y^{\alpha} - \frac{1}{3}a_{\alpha\beta}y^{\alpha}y^{\beta} \\ a_2(x,x') &= \frac{1}{72}R^2 + \frac{1}{3}a^{\lambda}{}_{\lambda} \end{aligned} \tag{4.42}$$

and all geometric quantities are evaluated at x'. By using integral representation

$$\frac{1}{k^2-m^2} = -i\int_0^{\infty} ds\ e^{-is(k^2-m^2)} \tag{4.43}$$

integration with respect to momenta can be replaced with integration alond s, which ends up with a differently looking expansion

$$\mathcal{G}_F(x,x') = \frac{-1}{(4\pi)^2}\int_0^{\infty}\frac{ds}{s^2}e^{-ims^2+\frac{\sigma}{2is}}F(x,x';is) \tag{4.44}$$

where $\sigma(x,x') = \frac{1}{2}y_{\alpha}y^{\alpha}$, namely, one half of a square of proper distance between points x and x', while function F can be expanded in a series of the following form:

$$F(x,x';is) = \sum_{j=0}^{N} a_j(x,x')(is)^j \tag{4.45}$$

Reinserting G_F back results into general coordinate expansion due to DeWitt and Schwinger

$$G_F^{DS}(x,x') = \frac{\Delta(x,x')}{(4\pi)^2}\int_0^{\infty}\frac{ds}{s^2}e^{-ims^2+\frac{\sigma}{2is}}F(x,x';is). \tag{4.46}$$

$\Delta(x,x') = -\frac{det[\partial_{\mu}\partial_{\nu}\sigma]}{\sqrt{g(x)g(x')}}$ stands for van Vleck-Morete determinant, which in the case of Riemann normal coordinates reduces to $\frac{1}{\sqrt{-g(x)}}$.

4.7 One loop efective action

Since we have associated W with an effective action, in principle we could define an effective lagrangian as usual, namely

$$W = \int d^4x\mathcal{L}_{eff} = \int d^4x\sqrt{-g}L_{eff}. \tag{4.47}$$

On the other hand, we have expressions (4.24) and (4.35) telling us that

$$W = -i\log\sqrt{\det(-G_F)} + \text{const} = -\frac{i}{2}\,\text{Tr}\log(-G_F) + \text{const}. \tag{4.48}$$

Here const stands for metric-independent terms, which do not affect equations of motion, and from now on will be neglected. To establish connection between last two equations, a clarification on trace operation is needed. For an operator M we define trace as

$$\mathrm{Tr}\, M = \int d^4x\sqrt{-g(x)}M(x,x) = \int d^4x\sqrt{-g(x)}\,\langle x|\, M\, |x\rangle\,. \tag{4.49}$$

Then if Feynman propagator is defined as

$$G_F(x,x') = \langle x|\, G_F\, |x'\rangle \tag{4.50}$$

where G_F is an operator, then $G_F(x,x)$ is easily acquired by taking the limit $x' \to x$. Following DeWitt and Schwinger, the effective action reads like

$$W = \frac{i}{2}\int_{m^2}^{\infty} dm^2 \int d^4x\sqrt{-g}\; G_F^{DS}(x,x) \tag{4.51}$$

m is the mass of the field. Effective Lagrangians can be read off easily now:

$$L_{eff}(x) = (\sqrt{-g(x)})^{-1}\mathcal{L}_{eff}(x) = \frac{i}{2}\lim_{x'\to x}\int_{m^2}^{\infty} dm^2 G_F^{DS}(x,x') \tag{4.52}$$

However, it is divergent. Inspection of (4.45) & (4.46) reveals that terms up to s^2 diverge in the limit of $x' \to x$ when the σ term vanishes. To be more exact, the divergent part of the action is

$$L_{div} = -\lim_{x'\to x}\frac{\sqrt{\Delta(x,x')}}{32\pi^2}\int_0^{\infty}\frac{ds}{s^3}e^{-i(m^2 s - \frac{\sigma}{2s})}[a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2] \tag{4.53}$$

Coefficients in the equation are composed of purely geometric objects which are local. This makes sense because expansion was valid for high frequency modes, which can not capture topological features of the universe.

4.8 Renormalisation of the effective action

Since the divergence of the effective action has been established, an attempt to renormalise cosmological and gravitational constants can be pursued. This will be done by using dimensional regularistion. For that we have to begin with an effective action in arbitrary number of dimensions:

$$L_{eff} = \lim_{x'\to x}\frac{\sqrt{\Delta(x,x')}}{2(4\pi)^{n/2}}\sum_{j=0}^{\infty} a_j(x,x')\int_0^{\infty}(is)^{j-1-n/2}e^{-i(m^2 s - \sigma/2s)}ids \tag{4.54}$$

In the limit $x' \to x$ first $\frac{n}{2}+1$ terms diverge. Assume that n can be continuosly varied. Then formally the expression is integrable:

$$L_{eff} = \frac{1}{2}(4\pi)^{-n/2}\sum_{j=0}^{\infty} a_j(x)(m^2)^{n/2-j}\Gamma(j-n/2). \tag{4.55}$$

Next a new mass scale μ has to be introduced to fix dimensions of the lagrangian to $(\text{length})^{-4}$:

$$L_{eff} = \frac{1}{2}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\sum_{j=0}^{\infty} a_j(x)m^{4-2j}\Gamma(j-n/2) \tag{4.56}$$

One can notice that divergence arrises from Gamma functions. First three of them diverge as

$$\begin{aligned}
\Gamma\left(-\frac{n}{2}\right) &= \frac{4}{n(n-2)}\left(\frac{2}{4-n}-\gamma\right)+O(n-4) \\
\Gamma\left(1-\frac{n}{2}\right) &= \frac{2}{2-n}\left(\frac{2}{4-n}-\gamma\right)+O(n-4) \\
\Gamma\left(2-\frac{n}{2}\right) &= \frac{2}{4-n}-\gamma+O(n-4).
\end{aligned} \tag{4.57}$$

Similarly, $(m/\mu)^{n-4}$ can be expanded:

$$\left(\frac{m}{\mu}\right)^{n-4} = 1+\frac{1}{2}(n-4)\log\left(\frac{m^2}{\mu^2}\right)+O((n-4)^2). \tag{4.58}$$

Finally, the isolated divergent part of action is

$$L_{div} = -(4\pi)^{-n/2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\log\left(\frac{m^2}{\mu^2}\right)\right]\right\}\left(\frac{4m^4a_0(x)}{n(n-2)}-\frac{2m^2a_1(x)}{n-2}+a_2(x)\right). \tag{4.59}$$

Recall that coefficients a_0, a_1, a_2 depend on various contractions of curvature tensor up to the second power in R. Therefore, even though this addition arrises from the quantum matter part, it should be moved to the geometrical side of Einstein equations. Adding together divergent Lagrangian density with the gravitational part results in

$$\sqrt{-g}\left[\left(B+\frac{1}{16\pi G_b}\right)R-\left(A+\frac{\Lambda_b}{8\pi G_b}\right)-\frac{a_2(x)}{(4\pi)^{n/2}}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\log\left(\frac{m^2}{\mu^2}\right)\right]\right\}\right]. \tag{4.60}$$

Here Λ_b and G_b are bare values appearing in the original Lagrangian, while A and B depend only on n, m and μ. A and B diverge by themselves, so by assuming that bare values of physical constants were infinite in the very beginning, absorption of A and B into redefinition makes the total values finite.
The third term, however, can not be combined with any of preexistent parameters. In order to keep analysis consistent, the gravitational lagrangian has to be modified apriori. To be more specific, the left side of Einstein field equations should have the form

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}+\alpha A_{\mu\nu}+\beta B_{\mu\nu}+\gamma H_{\mu\nu} \tag{4.61}$$

α, β and γ are constants which are as important as Λ and G are. Now they can be renormalised in the same way as done previously. New tensors appearing are defined as

$$\begin{aligned}
A_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}R^2d^nx \\
B_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}R^{\alpha\beta}R_{\alpha\beta}d^nx \\
H_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}d^nx
\end{aligned} \tag{4.62}$$

We are interested in 4 dimensions, and it happens that Gauss-Bonnet theorem, a colloraly of which is

$$\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}+R^2-4R_{\alpha\beta}R^{\alpha\beta})=0, \tag{4.63}$$

thus not all of eq.(4.62) are independent. To conclude, gravitational plus effective lagrangian minus divergent part of it governs the physics of the semiclassical theory of gravity.

Chapter 5

Scalar field in Schwarzschild spacetime

We are considering a spherically collapsing shell as a generator of spacetime geometry. Massive scalar field is traveling in this spacetime without affecting it. According to Birkhoff theorem, any spherical dynamics of gravitational fields have exactly the same metric in the vacuum outside of matter, and it is described by Schwarzschild geometry. Due to shell-like configuration of matter, inside part is actually a flat spacetime.
In Eddington-Finkelstein coordinates, outside of collapsing shell, scalar field travels according to the following equation of motion:

$$\left[\left(1-\frac{R_S}{r}\right)\partial_r^2+2\partial_v\partial_r+\frac{2}{r}\partial_v+\frac{1}{r}\left(2-\frac{R_S}{r}\right)\partial_r+\frac{1}{r^2}\left(\partial_\theta^2+\cot\theta\partial_\theta+\frac{1}{\sin^2\theta}\partial_\phi^2\right)-M^2\right]\phi=0 \tag{5.1}$$

The procedure of expansion will be used here. Factorise ϕ into $e^{i\omega v}Y_L^m(\theta,\phi)f_{\omega Lm}(r)$. First two factors satisfy following eigenvalue equations

$$\begin{aligned}&\partial_v e^{i\omega v}=i\omega e^{i\omega v}\\&\left(\partial_\theta^2+\cot\theta\partial_\theta+\frac{1}{\sin^2\theta}\partial_\phi^2\right)Y_L^m=-L(L+1)Y_L^m\end{aligned} \tag{5.2}$$

Notice that eigenvalue m does not contribute to radial equation, which acquires the form

$$\left[\left(1-\frac{R_S}{r}\right)\partial_r^2+2i\omega\partial_r+\frac{2i\omega}{r}+\frac{1}{r}\left(2-\frac{R_S}{r}\right)\partial_r-\frac{L(L+1)}{r^2}-M^2\right]f_{\omega L}(r)=0 \tag{5.3}$$

The solutions to this differential equation including eigenfunctions defined above are:

$$\phi_{\omega Lm}(r)=e^{i\omega v}Y_L^m(\theta,\phi)e^{-i\omega r}e^{r\sqrt{M^2-\omega^2}}(C1_{\omega Lm}HC1_{\omega Lm}(r)+C2_{\omega Lm}HC2_{\omega Lm}(r)(R_S-r)^{-2i\omega R_S}). \tag{5.4}$$

$C1_{\omega Lm}$ and $C2_{\omega Lm}$ are integration constants. HC1 and HC2 are Heun functions, which in Maple are of following forms:

$$\begin{aligned}&HC1_{\omega Lm}=HeunC\bigg(-2R_S\sqrt{M^2-\omega^2},2i\omega R_S,0,R_S^2(M^2-2\omega^2),\\&-L(L+1)+R_S^2(2\omega^2-M^2),1-\frac{r}{R_S}\bigg)\\&HC2_{\omega Lm}=HeunC\bigg(-2R_S\sqrt{M^2-\omega^2},-2i\omega R_S,0,R_S^2(M^2-2\omega^2),\\&-L(L+1)+R_S^2(2\omega^2-M^2),1-\frac{r}{R_S}\bigg)\end{aligned} \tag{5.5}$$

These two functions correspond to an outgoing and ingoing waves. Since it is difficult to observe this fact, we will consider the limitting case of vanishing R and L, which corresponds to spherical waves in flat spacetime. Then the radial differential equations simplifies to

$$\left(\partial_r^2 + 2i\omega\partial_r + \frac{2i\omega}{r} + \frac{2}{r}\partial_r - M^2\right) f = 0, \tag{5.6}$$

a full solution to which is

$$\begin{aligned}\phi &= e^{i\omega v}\left(C_1 \frac{e^{-ir(\omega+\sqrt{\omega^2-M^2})}}{r} + C_2 \frac{e^{-ir(\omega-\sqrt{\omega^2-M^2})}}{r}\right) \\ &= C_1 \frac{e^{i(\omega t - r\sqrt{\omega^2-M^2})}}{r} + C_2 \frac{e^{i(\omega t + r\sqrt{\omega^2-M^2})}}{r}\end{aligned} \tag{5.7}$$

and indeed it is a linear combination of spherical ingoing and outgoing waves.
Ingoing and outgoing waves constitute 2 independent solutions, i.e. one can not be expressed in terms of the other. However, this is exactly what we want to achieve. The comparison of properties of outgoing waves with respect to those of ingoing waves is the main goal. The previous fact is true, but it is valid only before or only after they cross the collapsing matter, while we are aiming at situation when an ingoing wave becomes and outgoing one.
The "black box" which transforms one into the other is the geodesic equation for lightlike trajectories. The general equation for this reads

$$\frac{d^2x^\lambda}{d\lambda^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{5.8}$$

For each coordinate we can define conjugate momentum. Say, Lagrangian equals $\mathcal{L} = \frac{1}{2}g_{\mu\nu}x^\mu x^\nu$. Then the conjugate momentum is defined as $p^\mu = \frac{\delta\mathcal{L}}{\delta x^\mu} = g_{\mu\nu}x^\nu$. Since in Eddington-Finkelstein metric there is no dependence on v and ϕ, two of momenta should be assigned constant values ([15], [16], [17]):

$$\begin{aligned}-\left(1-\frac{R_S}{r}\right)\frac{dv}{d\lambda} + \frac{dr}{d\lambda} &= -E,\ E \geq 0 \text{ (energy)} \\ r^2\frac{d\phi}{d\lambda} &= L,\ L \geq 0 \quad \text{(angular momentum)}.\end{aligned} \tag{5.9}$$

If angle θ is fixed, say, to the equatorial plane, then the lightlike line element reads

$$-\left(1-\frac{R_S}{r}\right)\left(\frac{dv}{d\lambda}\right)^2 + 2\frac{dv}{d\lambda}\frac{dr}{d\lambda} + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0. \tag{5.10}$$

We are considering motion through the center of symmetry, hence the angular momentum is null. By the use of eq.(5.9) we uncover that

$$-\frac{dv}{d\lambda}\frac{1}{1-\frac{R_S}{r}}\left[\frac{dv}{d\lambda} - \frac{dr_*}{d\lambda}\right] = 0. \tag{5.11}$$

Thus the lightlike purely radial motion is satisfied by equations $\frac{dv}{d\lambda} = 0$ or $\frac{d}{d\lambda}(v - 2r_*) = 0$, and the latter brackets can be denoted as u from the outgoing Eddington-Finkelstein coordinates. To make sure that these two equations make sense, substitute either of them into eq.(5.9). What we get is

$$\begin{aligned}\frac{dr}{d\lambda} &= -E \text{ (ingoing)} \\ \frac{dr}{d\lambda} &= E \text{ (outgoing)}\end{aligned} \tag{5.12}$$

which are the properties of corresponding motions.
Now that the properties of radial motion has been introduced, dependence of u and v on λ will be analysed. The main point of this consideration is the continuity of the affine parameter λ, since motion in a straight line is smooth. This fact will allow us to express u as a function of v and vice versa.
Let us start from an equation for u, i.e. we want to write down an equation for an incoming lightlike trajectory as a function between u and v:

$$\frac{du}{d\lambda} = \frac{dv}{d\lambda} - 2\frac{dr_*}{d\lambda} = \left(1 - \frac{R_S}{r}\right)^{-1} E - \frac{dr_*}{d\lambda} = 2\left(1 - \frac{R_S}{r}\right)^{-1} E, \tag{5.13}$$

where we used eqs.(5.12) and (5.9). Since r as a function of λ is just a line, all we have to do is to define the value of the integration constant. We choose it in such a way that $r(0) = R_S$, i.e. $r = R_S - \lambda E$. Let us limit ourselves to an interval $\lambda < 0$. Then

$$u = 2E\lambda - 2R_S \log \frac{\lambda}{K_1},\ \ K_1 < 0 \tag{5.14}$$

where, again, K_1 is an integration constant. At first the significance of λ is amiguous. Apriori it is just an abstract parameter which orders values of u in some way. We assign it a value of v in the following way. It seems reasonable that the later the motion for some constant v is initiated the higher value of u it corresponds to through the crossing of the center of symmetry. By taking v to be the parameter of reference we can write down a linear relation between v and λ. Also, in our case the spacetime is dynamic, i.e. in the beginning the is no black hole, thus no penalty should arise from travelling through the center. There is, though, one moment, when the black hole is formed. This corresponds to one specific value of v, which we call v_0, after which any ingoing motion will cross the horizon and will be lost. With all this in mind we state that $v_0 - v = K_2\lambda$ with $K_2 < 0$ being a constant. Then the obvious substitution results in

$$u = \frac{2E}{K_2}[v_0 - v] - 2R_S \log \frac{v_0 - v}{K_1 K_2}, \tag{5.15}$$

which is valid for $v < v_0$. We know what happens when $v = v_0$ - the horizon is formed so any object that was travelling on a lightlike trajectory gets trapped right on the horizon as it leaves the shell. For anything incoming later no value of u can be associated, because this motion ends up at the singularity instead of future infinity.

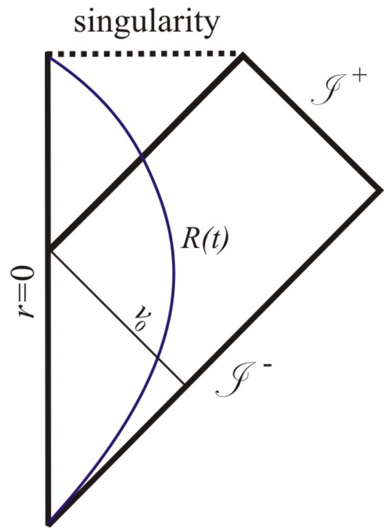


Figure 5.1: Spacetime diagram of a collapsing shell

Previously we have found solutions for a motion of a free scalar field. Also we managed to relate coordinates v and u. Now we are ready to quantify changes at past and future spatial infinities. This requires the use of Bogolyubov coefficients. We are interested in the comparison of asymptotic forms of solutions, since we expect various effects to kick in as the scalar field approaches the horizon. To fully describe the scalar field we need to choose some hypersurface which would carry information enough for the reconstruction of history of the whole spacetime. Obviously, one such hypersurface is the hypersurface of past null infinity. However, at the infinite future the future null infinity is not enough, because part of the information will enter the black hole. Therefore, the second Cauchy surface is defined as the future null infinity with the horizon included.

Along with these surfaces, several bases of functions have to be introduced. We denote the basis on the past hypersurface by $p_{\omega LM}$ and its complex conjugate together with anihilation and creation operators $\hat{a}_{\omega Lm}$ and $\hat{a}^{\dagger}_{\omega Lm}$. Similarly, the basis of the future null infinity is composed of $f_{\omega LM}$ and $q_{\omega Lm}$ and hermitian conjugate thereof, while at the surface of the black hole we have $q_{\omega Lm}$ and $c_{\omega Lm}$ and their hermitian conjugates.

Now we have to define the meaning of $p_{\omega Lm}$ and the rest. Therefore we say that $p_{\omega Lm}$ are ingoing solutions, while $f_{\omega Lm}$ and $q_{\omega Lm}$ are outgoing solutions. Then any configuration of the field ϕ can be expanded as

$$\begin{aligned} \phi &= \int \sum (a_{\omega Lm} p_{\omega Lm} + a^{\dagger}_{\omega Lm} p^{*}_{\omega Lm}) \\ &= \int \sum (b_{\omega Lm} f_{\omega Lm} + b^{\dagger}_{\omega Lm} f^{*}_{\omega Lm} + c_{\omega Lm} q_{\omega Lm} + c^{\dagger}_{\omega Lm} q^{*}_{\omega Lm}) \end{aligned} \tag{5.16}$$

Furthermore, modes at the horizon and at the null future infinity should be completely independent from each other, i.e.

$$(f_{\omega_1 L_1 m_1}, q_{\omega_2 L_2 m_2}) = 0. \tag{5.17}$$

All the functions are normalised according to

$$\begin{aligned} (p_{\omega_1 L_1 m_1}, p_{\omega_2 L_2 m_2}) &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2} \\ (f_{\omega_1 L_1 m_1}, f_{\omega_2 L_2 m_2}) &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2} \\ (q_{\omega_1 L_1 m_1}, q_{\omega_2 L_2 m_2}) &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2}. \end{aligned} \tag{5.18}$$

For pairs of complex conjugate functions an additional minus sign appears, while for a pair of a function and its complex conjugate the normalisation condition vanishes. Canonical quantisation of the field ϕ implies that

$$\begin{aligned} [a_{\omega_1 L_1 m_1}, a^{\dagger}_{\omega_2 L_2 m_2}] &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2} \\ [b_{\omega_1 L_1 m_1}, b^{\dagger}_{\omega_2 L_2 m_2}] &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2} \\ [c_{\omega_1 L_1 m_1}, c^{\dagger}_{\omega_2 L_2 m_2}] &= \delta(\omega_1 - \omega_2)\delta_{L_1,L_2}\delta_{m_1,m_2} \\ [b_{\omega_1 L_1 m_1}, c_{\omega_2 L_2 m_2}] &= 0 \end{aligned} \tag{5.19}$$

Other types of commutations are equal to 0. We already know about the ambiguity of the vacuum state. In our consideration we define a vacuum state which corresponds to the absence of particles, incoming from $\mathcal{I}^-$, i.e. $a_{\omega Lm}|0\rangle = 0$. Also, define Bogolyubov coefficients, which relate fields at $\mathcal{I}^-$ and $\mathcal{I}^+$ according to

$$f_{\omega mL} = \int d\omega' \sum_{L'=0}^{\infty} \sum_{m'=-L'}^{L'} (\alpha_{\omega\omega' LL' mm'} p_{\omega' L' m'} + \beta_{\omega\omega' LL' mm'} p^{*}_{\omega' L' m'}). \tag{5.20}$$

By simply using normalisation conditions (5.18), Bogolyubov coefficients can be extracted straightforwardly:

$$\begin{aligned} \alpha_{\omega\omega' LL' mm'} &= (p_{\omega' L' m'}, f_{\omega Lm}) \\ \beta_{\omega\omega' LL' mm'} &= -(p^{*}_{\omega' L' m'}, f_{\omega Lm}). \end{aligned} \tag{5.21}$$

To find out the values of Bogolyubov coefficients, exact forms of functions $p_{\omega Lm}$ and $f_{\omega Lm}$ have to be inserted.
As it was previously agreed, $p_{\omega Lm}$ is one specific mode, appropriately normalised, representing ingoing solutions, which has the form

$$p_{\omega Lm} = N_{\omega Lm} e^{i\omega v} Y_L^m(\theta, \phi) e^{-i\omega r} e^{r\sqrt{M^2-\omega^2}} HC1_{\omega Lm}(r) \tag{5.22}$$

with $N_{\omega Lm}$ being the normalisation constant. To determine its value, insert the given function into a generalised version of normalisation condition:

$$(p_{\omega Lm}, p_{\omega' L' m'}) = i \int_{\text{Whole space}} d^3x \sqrt{-g} \left\{ p_{\omega Lm} \partial_v p^*_{\omega' L' m'} - \partial_v p_{\omega Lm} p^*_{\omega' L' m'} \right\} \tag{5.23}$$

The first product in the curly brackets reads

$$-i\omega' N_{\omega Lm} N^*_{\omega' L' m'} e^{iv(\omega-\omega')} Y_L^m Y_{L'}^{m'*} e^{ir(\omega'-\omega)} e^{r\sqrt{M^2-\omega^2}} \left(e^{r\sqrt{M^2-\omega'^2}} \right)^* HC1_{\omega Lm} HC1^*_{\omega' L' m'} \tag{5.24}$$

while the second one looks like

$$i\omega N_{\omega Lm} N^*_{\omega' L' m'} e^{i(\omega-\omega')v} Y_L^m Y_{L'}^{m'*} e^{ir(\omega'-\omega)} e^{r\sqrt{M^2-\omega^2}} \left(e^{r\sqrt{M^2-\omega'^2}} \right)^* HC1_{\omega Lm} HC1^*_{\omega' L' m'}. \tag{5.25}$$

The product of spherical harmonics can be readily integrated to obtain Kronecker deltas:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) Y_L^m Y_{L'}^{m'*} = \delta_{m,m'} \delta_{L,L'}. \tag{5.26}$$

Hence, the inner product of scalar fields at past infinity is diagonal in m and L space. The main contribution comes from the radial part. Analytic form can not be found since orthogonality relations weighed with the square-root metric determinant are not known. There are still exact relations, though, which are weighed by singularities.

$$\begin{aligned} &(p_{\omega Lm}, p_{\omega' L' m'}) = (\omega' + \omega) \delta_{m,m'} \delta_{L,L'} |N_{\omega Lm}|^2 e^{iv(\omega-\omega')} \times \\ &\times \int_0^{\infty} r^2 dr e^{ir(\omega-\omega')} e^{r\sqrt{M^2-\omega^2}} \left(e^{r\sqrt{M^2-\omega'^2}} \right)^* HC1_{\omega Lm} HC1^*_{\omega' Lm} = \delta(\omega - \omega') \delta_{m,m'} \delta_{L,L'}. \end{aligned} \tag{5.27}$$

Similarly, functions at the infinite future can be normalised. One crucial difference, though, stems from the fact that all incoming fields for $v > v_0$ pass the horizon, therefore for these values of v Bogolyubov coefficients will bind functions $p_{\omega Lm}$ and $q_{\omega Lm}$, while for $v < v_0$ $q_{\omega Lm}$ should be replaced with $f_{\omega Lm}$. Namely, for $f_{\omega Lm}$ we have

$$f_{\omega Lm} = A_{\omega Lm} \Theta(v_0 - v) e^{i\omega v} Y_L^m e^{-i\omega r} e^{r\sqrt{M^2-\omega^2}} HC2_{\omega Lm} (R_S - r)^{-2i\omega R_S}, \tag{5.28}$$

where Θ is Heaviside step function and $A_{\omega Lm}$ is a normalisation constant. Before we dive into normalisation of $f_{\omega Lm}$, we should show that it can be simplified by introducing variable u as before:

$$e^{i\omega v} e^{-i\omega r} (R_S - r)^{-2i\omega R_S} = e^{i\omega v} e^{-2i\omega r} e^{-2i\omega R_S \log[\frac{r}{R_S} - 1]} e^{i\omega r} R_S^{-2i\omega R_S} = e^{i\omega u} e^{i\omega r} R_S^{-2i\omega R_S}. \tag{5.29}$$

Therefore, $f_{\omega Lm}$ can also be written as

$$f_{\omega Lm} = B_{\omega Lm} \Theta(v_0 - v) Y_L^m e^{i\omega u} e^{i\omega r} e^{r\sqrt{M^2-\omega^2}} HC2_{\omega Lm}. \tag{5.30}$$

Its normalisation condition, with angles already integrated out, can be written as

$$\begin{aligned} &(f_{\omega Lm}, f_{\omega' L' m'}) = \int_0^{\infty} r^2 dr e^{iu(\omega-\omega')} e^{ir(\omega'-\omega)} e^{r\sqrt{M^2-\omega'^2}} \left(e^{r\sqrt{M^2-\omega^2}} \right)^* HC2_{\omega Lm} HC2^*_{\omega' Lm} \times \\ &\times \delta_{L,L'} \delta_{m,m'} (\omega' + \omega) |B_{\omega Lm}|^2 \Theta(v_0 - v)^2 = \delta(\omega - \omega') \delta_{L,L'} \delta_{m,m'} \end{aligned} \tag{5.31}$$

Clearly, this normalisation holds only for $v < v_0$. Since $f_{\omega Lm}$ vanishes for the rest of values of v, normalisation condition can not be enforced there. Normalisation is accomplished by requiring the right-hand side to be equal to $\delta(\omega-\omega')\delta_{m,m'}\delta_{L,L'}$. Also, for simplicity we choose $\Theta(0)=0$ With both past and future functions in hand both Bogolyubov coefficients can be calculated. We will use Fourier transform due to the simplicity. We begin with the α coefficient. After the expression (5.15) is inserted, the left-hand side becomes

$$\begin{aligned}&\delta_{L,L'}\delta_{m,m'}B_{\omega Lm}N^*_{\omega' L'm'}\frac{e^{\frac{2i\omega E v_0}{K_2}}}{(K_1K_2)^{2iR_S\omega}}\int_{-\infty}^{v_0}dve^{-iv\left(\frac{2E\omega}{K_2}+\omega'\right)}(v_0-v)^{-2iR_S\omega}\times\\&\times\int_0^\infty r^2dre^{i(\omega+\omega')r}e^{r\sqrt{M^2-\omega^2}}\left(e^{r\sqrt{M^2-\omega'^2}}\right)^*HC2_{\omega Lm}HC1^*_{\omega' L'm'}\end{aligned}\tag{5.32}$$

while the right-hand one is

$$\begin{aligned}&\int_0^\infty r^2dr\int_{-\infty}^{\infty}d\omega''\sum_{L''=0}^{\infty}\sum_{m''=-L''}^{L''}\alpha_{\omega\omega''LL''mm''}\delta(\omega'-\omega'')\delta_{m'',m'}\delta_{L',L''}\times\\&\times N_{\omega''L''m''}N^*_{\omega'L'm'}e^{ir(\omega''-\omega')}\left(e^{r\sqrt{M^2-\omega'^2}}\right)^*e^{r\sqrt{M^2-\omega''^2}}HC1^*_{\omega'L'm'}HC1_{\omega''L''m''}=\\&=\frac{\alpha_{\omega\omega'LL'mm'}}{\sqrt{2\omega'}}\end{aligned}\tag{5.33}$$

and by caring out the integral with respect with v we finally get

$$\begin{aligned}&\alpha_{\omega\omega'LL'mm'}=\sqrt{2\omega'}\delta_{L,L'}\delta_{m,m'}B_{\omega Lm}N^*_{\omega'L'm'}\frac{e^{-iv_0\omega'}}{(K_1K_2)^{2iR_S\omega}}\left[-i\left(\frac{2E\omega}{K_2}+\omega'\right)\right]^{-1+2i\omega R_S}\times\\&\times\Gamma(1-2iR_S\omega)\int_0^\infty drr^2e^{ir(\omega+\omega')}e^{r\sqrt{M^2-\omega^2}}\left(e^{r\sqrt{M^2-\omega'^2}}\right)^*HC2_{\omega Lm}HC1^*_{\omega'L'm'}.\end{aligned}\tag{5.34}$$

Similarly, for the β coefficient the right-hand side is

$$\begin{aligned}&\int_0^\infty r^2dr\int_{-\infty}^{\infty}d\omega''\sum_{L''=0}^{\infty}\sum_{m''=-L''}^{L''}\beta_{\omega\omega''LL''mm''}\delta(\omega'-\omega'')\delta_{m'',m'}\delta_{L',L''}\times\\&\times N^*_{\omega''L''m''}N_{\omega'L'm'}e^{-ir(\omega''-\omega')}e^{r\sqrt{M^2-\omega'^2}}\left(e^{r\sqrt{M^2-\omega''^2}}\right)^*HC1_{\omega'L'm'}HC1^*_{\omega''L''m''}=\\&=\frac{\beta_{\omega\omega'LL'mm'}}{\sqrt{2\omega'}}\end{aligned}\tag{5.35}$$

and the left-hand side equals

$$\begin{aligned}&(-1)^{m'}\delta_{L,L'}\delta_{m,-m'}B_{\omega Lm}N_{\omega'L'm'}\frac{e^{\frac{2i\omega E v_0}{K_2}}}{(K_1K_2)^{2i\omega R_S}}\int_{-\infty}^{v_0}dve^{-iv\left(\frac{2E\omega}{K_2}-\omega'\right)}(v_0-v)^{-2iR_S\omega}\times\\&\times\int_0^\infty drr^2e^{i(\omega-\omega')r}e^{r\sqrt{M^2-\omega^2}}e^{r\sqrt{M^2-\omega'^2}}HC1_{\omega'Lm'}HC2_{\omega Lm}.\end{aligned}\tag{5.36}$$

Then coefficient β reads

$$\begin{aligned}&\beta_{\omega\omega'LL'mm'}=\sqrt{2\omega'}(-1)^{m'}\delta_{L,L'}\delta_{m,-m'}B_{\omega Lm}N_{\omega'L'm'}\frac{e^{iv_0\omega'}}{(K_1K_2)^{2i\omega R_S}}\left[-i\left(\frac{2E\omega}{K_2}-\omega'\right)\right]^{-1+2i\omega R_S}\times\\&\times\Gamma(1-2i\omega R_S)\int_0^\infty drr^2e^{i(\omega-\omega')r}e^{r\sqrt{M^2-\omega^2}}e^{r\sqrt{M^2-\omega'^2}}HC1_{\omega'L'm'}HC2_{\omega Lm}.\end{aligned}\tag{5.37}$$

After all the tedious calculations we find that both Bogolyubov coefficients are in general non-vanishing, so the number of particles is not conserved. To find the number of particles, square of absolute values is needed:

$$\begin{aligned} |\alpha_{\omega\omega' LL'mm'}|^2 = 2\omega'\delta_{L,L'}\delta_{m,m'}|B_{\omega Lm}|^2|N_{\omega' L'm'}|^2 \left|\left[-i\left(\frac{2E\omega}{K_2}+\omega'\right)\right]^{-1+2i\omega R_S}\right|^2 \times \\ \times |\Gamma(1-2iR_S\omega)|^2 \left|\mathrm{Int}_\alpha[\omega,\omega',L,L',m,m',M]\right|^2 \end{aligned} \tag{5.38}$$

$$\begin{aligned} |\beta_{\omega\omega' LL'mm'}|^2 = 2\omega'(-1)^m\delta_{L,L'}\delta_{m,-m'}|B_{\omega Lm}|^2|N_{\omega' L'm'}|^2 \left|\left[-i\left(\frac{2E\omega}{K_2}-\omega'\right)\right]^{-1+2i\omega R_S}\right|^2 \times \\ \times |\Gamma(1-2i\omega R_S)|^2 |\mathrm{Int}_\beta[\omega,\omega',L,L',m,m',M]|^2 \end{aligned} \tag{5.39}$$

Int_x stands for the radial integral belonging to the coefficient x. To help us find the number of particles in the future infinity, we can generalize eq.(4.31) with the use of eq.(5.31):

$$\begin{aligned} (f_{\omega Lm}, f_{\omega' L'm'}) = \int d\omega'' \sum_{L''=0}^{\infty}\sum_{m''=-L''}^{L''} (\alpha^*_{\omega\omega'' LL''mm''}\alpha_{\omega'\omega'' L'L''m'm''} - \beta^*_{\omega\omega'' LL''mm''}\beta_{\omega'\omega'' L'L''m'm''}) = \\ = \delta(\omega-\omega')\delta_{L,L'}\delta_{m,m'} \end{aligned} \tag{5.40}$$

In order to use this, we have to express α through β. The ratio of absolute squares is

$$\begin{aligned} \left(\frac{|\alpha_{\omega\omega' LL'mm'}|}{|\beta_{\omega\omega' LL'mm'}|}\right)^2 = (-1)^{-m}\left(\frac{|Int_\alpha[\omega,\omega',L,L',m,m',M]|}{|Int_\beta[\omega,\omega',L,L',m,m',M]|}\right)^2 \times \\ \times \left|\left[-i\left(\frac{2E\omega}{K_2}+\omega'\right)\right]^{-1+2i\omega R_S}\right|^2 \left|\left[-i\left(\frac{2E\omega}{K_2}-\omega'\right)\right]^{-1+2i\omega R_S}\right|^{-2}. \end{aligned} \tag{5.41}$$

This ratio contains complex powers. Also, here a competition between motion close to and far away from the formation of a black hole can be observed. A crude way to see it is to see what happens when the coefficient E disappears. From the ray tracking formula (5.15) one should notice that at the moment of the black hole formation the logarithmic term dominates, while for moments of time much earlier than that the E-term takes over. Thus dropping one or the other would focus the analysis on any of these limiting cases. Through the analytic continuation of the logarithm the product of complex power terms can be simplified to

$$\left(\frac{\frac{2E\omega}{K_2}+\omega'}{\frac{2E\omega}{K_2}-\omega'}\right)^2 e^{4\omega R_S[Arg\left(-i\left\{\frac{2E\omega}{K_2}-\omega'\right\}\right)-Arg\left(-i\left\{\frac{2E\omega}{K_2}+\omega'\right\}\right)]}. \tag{5.42}$$

To sum up, in principle we can say that $|\alpha|^2 = \mathcal{F}|\beta|^2$ with $\mathcal{F}$ being a nontrivial function of various parameters.

This model of a collapsing black hole is a simplified one in one essential way. Namely, the black hole, once formed, is eternal. Evaporation of the black hole is concluded aposteriori, after fluxes of particles are compared. Therefore, it seems natural, that the total number of particles of any frequency should be infinite, because the black hole is eternal and is static in size. This is why

$$\int d\omega'' \sum_{L''=0}^{\infty}\sum_{m''=-L''}^{L''} (|\alpha_{\omega\omega'' LL''mm''}|^2 - |\beta_{\omega\omega'' LL''mm''}|^2) = \lim_{\omega\to\omega'}\delta(\omega-\omega') = \infty. \tag{5.43}$$

It is meaningless to talk about an infinite number of particles. A formal division by an infinite amount of time allows us to consider fluxes over total number. Consider one of the possible definitions of the delta function:

$$\delta(\omega - \omega') = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} e^{i(\omega-\omega')t} dt = \frac{1}{\pi} \lim_{T\to\infty} \frac{\sin(T(\omega - \omega'))}{\omega - \omega'} \tag{5.44}$$

By taking the limit $\omega \to \omega'$ now, one ends up with a finite normalization, namely, T/π. Accordingly, the finiteness of number of particles per unit time can be concluded. The flux of particles can be determined by solving eq.(5.43) for $\int \sum |\beta|^2$. Of course, it is also possible just to integrate (and sum over) the expression of $|\beta|^2$ itself.

Chapter 6

Conclusions

In this paper we have analyzed Hawking radiation of a scalar field in the presence of a collapsing black hole. The radiation was quantified by comparing the number of particles (or fluxes) at the infinite past with the situation at the infinite future. It was found that creation and annihilation operators for the initial vacuum state did not match those of the future vacuum state. This fact was concluded by examining Bogolyubov coefficients, since neither of them truly vanished.

During the process of solution a chart of coordinates valid at all point in spacetime, except for the central singularity, was chosen to avoid artificial singularities in all the calculations. Eddington-Finkelstein coordinate chart was picked up as not only it satisfied the previous condition, but also still allowed a simple interpretation of the results. Apart from spherical symmetry, no other simplifications or approximations were made. This resulted in the appearance of confluent Heun functions, which are more general than most of analytic functions being used nowadays. Exact results were expressed as integrals, which cannot be evaluated further due to limited implementation of their properties in computer algebra programs.

Additionally, technical details of quantization in coordinates different than rectangular ones do not seem to be well known. Although general principles are understood, by precise statements are not easily constructed because any nonstandard conjugate pairs other than cartesian coordinates with linear momentum are usually omitted from reasonable examination. Therefore, it is advisable that any of exact analyses should include a fully developed understanding in a flat spacetime.

Chapter 7

Appendix

7.1 Frobenius solutions

Any linear second order homogeneous differential equation has a form of

$$a(z)\frac{d^2y(z)}{dz^2} + b(z)\frac{dy(z)}{dz} + c(z)y(z) = 0 \tag{7.1}$$

where a, b, c are arbitrary functions of z. For some a, b, c the differential equation may have singular points which are the roots of equation $a(z) = 0$ and, possibly, infinity. At any singularity the solutions can achieve either a finite or an infinite value. If both linearly independent solutions are finite at this point, then it is called a regular singularity. Otherwise it is an irregular singularity.

At any finite regular singularity z_j an indicial equation can be constructed:

$$\rho(\rho - 1) + p_j\rho + q_j = 0, \tag{7.2}$$

where

$$p_j = Res_{z=z_j}\frac{b(z)}{a(z)}, \quad q_j = Res_{z=z_j}(z - z_j)\frac{c(z)}{a(z)}. \tag{7.3}$$

The roots of eq.(7.2) are called characteristic exponents. At a regular singularity at infinity, the aforementioned relations are modified as

$$\rho(\rho + 1) + p_\infty\rho + q_\infty = 0, \tag{7.4}$$

where

$$p_\infty = -Res_{z=\infty}\frac{b(z)}{a(z)}, \quad q_\infty = Res_{z=\infty}z\frac{c(z)}{a(z)} \tag{7.5}$$

and roots are called as characteristic exponents at infinity.

In the neighborhood of a finite regular singularity two linearly independent Frobenius solutions can be constructed. If characteristic exponents satisfy

$$\rho_1(z_j) - \rho_2(z_j) \notin \mathbb{Z}, \tag{7.6}$$

then Frobenius solutions [6] are of the form

$$\begin{aligned} y_m(z_j, z) &= (z - z_j)^{\rho_m(z_j)}\sum_{k=0}^{\infty} c_k^m(z_j)(z - z_j), \quad |z_j| < \infty \\ y_m(\infty, z) &= z^{-\rho_m(\infty)}\sum_{k=0}^{\infty} c_k^m(\infty)z^{-k}, \quad |z_j| = \infty \end{aligned} \tag{7.7}$$

and m is an index of a linearly independent solution. However, if eq.(7.6) is not satisfied, then one of the solutions may acquire a logarithmic term:

$$\begin{aligned} y_2(z_j, z) &= (z - z_j)^{\rho_2(z_j)} \sum_{k=0}^{\infty} c_k^2 (z - z_j)^k + A_j y_1(z_j, z) \log(z - z_j), \quad |z_j| < \infty \\ y_2(\infty, z) &= z^{-\rho_2(\infty)} \sum_{k=0}^{\infty} c_k^2(\infty) z^{-k} + A_\infty y_1(\infty, z) \log(z), \quad |z_j| = \infty. \end{aligned} \tag{7.8}$$

7.2 Heun function

Since general form of these functions correspond to very general form of solutions $y(z)$, a subset of these equations, where a, b, c are polynomials has much simpler structure. Let us denote a, b, c as

$$a(z) = P_n(z),\ b(z) = P_{n-1}(z),\ c(z) = P_{n-2}(z), \tag{7.9}$$

where P_n denotes polynomial of degree n. For $n = 2$ the differential equation can be brought to an equation satisfied by hypergeometric functions, while for n=3, Heun functions are solutions. This functions contains 4 regular singularities. Manipulation of parameters allows to shift them into points $z = \{0, 1, t, \infty\}$. Then its canonical form is usually presented as

$$\frac{d^2y(z)}{dz^2} + \left\{ \frac{c}{z} + \frac{d}{z-1} + \frac{a+b+1-c-d}{z-t} \right\} \frac{dy(z)}{dz} + \frac{abz - \lambda}{z(z-1)(z-t)} y(z) = 0 \tag{7.10}$$

Coefficients a, b, c, d determines characteristic exponents of Froebenius solutions at the singularities. λ is an accessory parameter, usually serving as the spectrum of the differential operator. Taylor series can be constructed around any of singularities. For a solution of the canonical equation it holds that y(0)=1, and the series converges within radius $\min(|t|, 1)$ Other functions can be obtained by confluence of singular points. Then singly, doubly and triply confluent Heun functions come as a result. In the limit of $t \to \infty$ a singly confluent Heun function is obtained with an irregular singularity at the infinity and regular singularities at $z = 0$ and $z = 1$. This function satisfies

$$\frac{d^2y(z)}{dz^2} + \left\{ -t + \frac{c}{z} + \frac{d}{z-1} \right\} \frac{dy(z)}{dz} + \frac{\lambda - taz}{z(z-1)} y(z) = 0 \tag{7.11}$$

Here a, b, c, d, t describe the form of solutions nearby the singularities and λ is still a spectral parameter.

There exist software packages that can be used in analysis of Heun functions, but defining differential equations do not always follow canonical definitions, therefore the reader should be aware of differences between conventions in literature and computer programs.

7.3 $|\beta|^2$ dependence on the parameters of the model

Following graphs is a result of numerical calculation of $|\beta|^2$. The physical meaning of these is the density of particles in each mode denoted by ω, L and m. The total number of particles is recovered by summing and integrating over these labels.

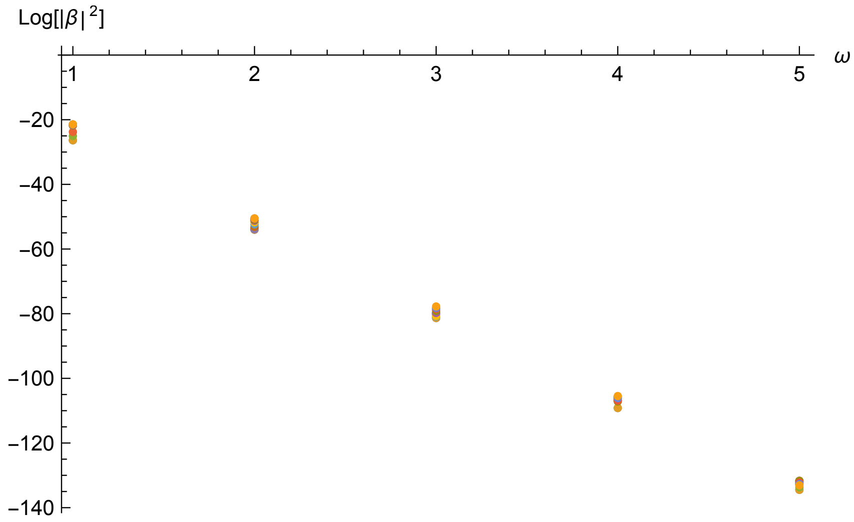


Figure 7.1: $Log[|\beta|^2]$ dependence on ω ($\omega = \omega'$ and $\frac{E}{K_2} = 0$ unless stated otherwise). For each ω a stack of points is a distribution with respect to L. Values decrease as L increases. $L \in [0, 12]$. Massless field case

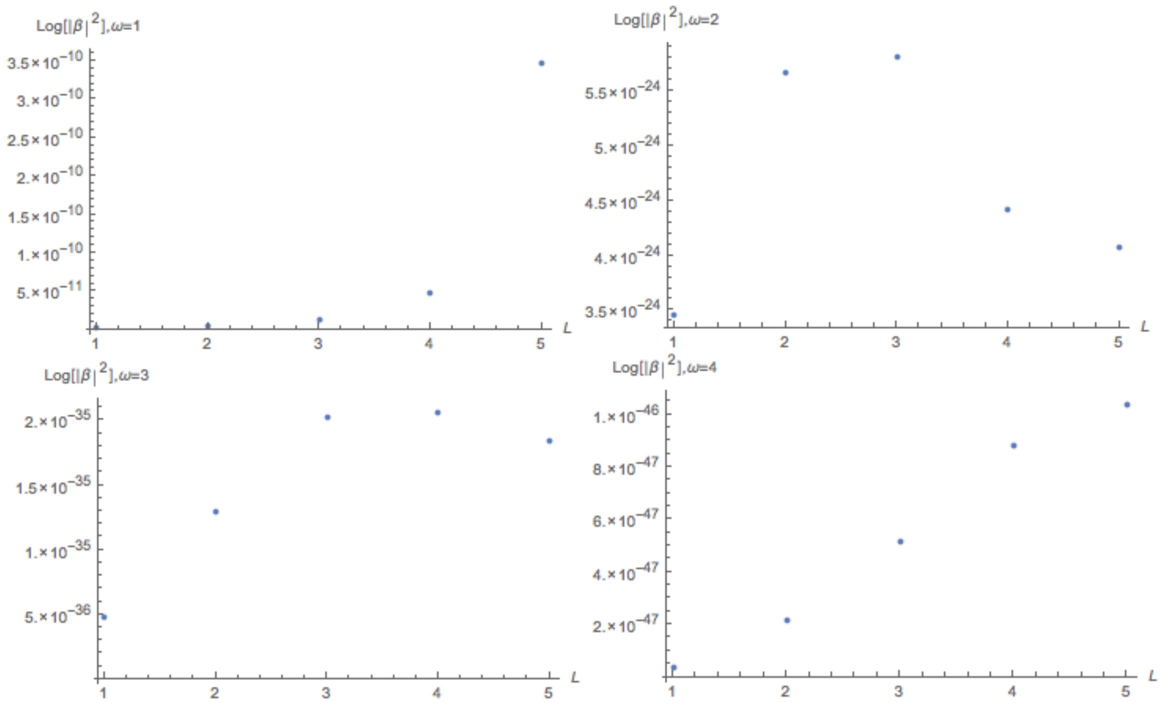


Figure 7.2: $Log[|\beta|^2]$ dependence on L. Notice that for increasing values of ω the peak is reached at higher values of L. Massless field case

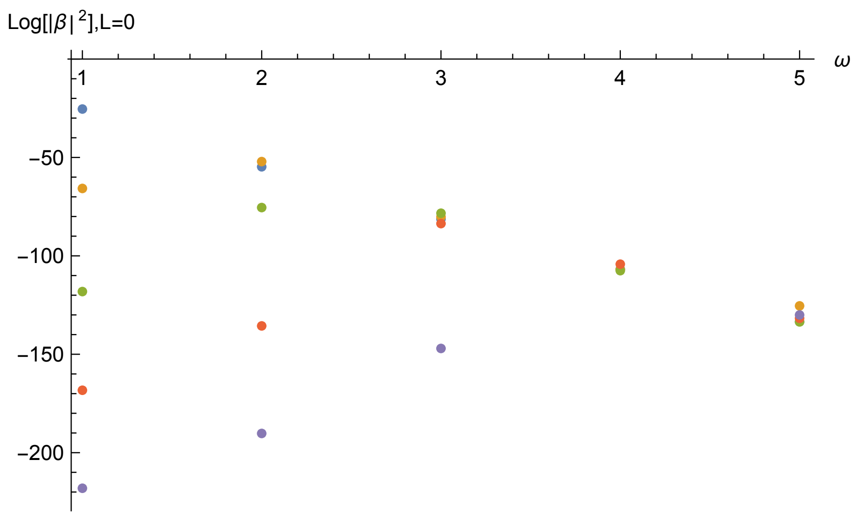


Figure 7.3: $Log[|\beta|^2]$ dependence on ω. For each ω values decrease as the mass of the field increases.

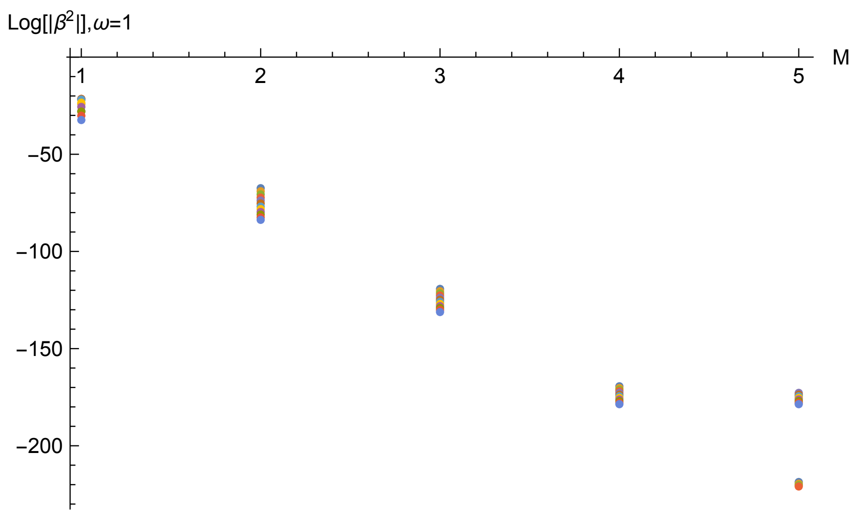


Figure 7.4: $Log[|\beta|^2]$ dependence on the mass of the field for $\omega = 1$. For each value of M a stack of points represents different values of $L \in [0, 12]$. Dependence on L is not straightforward, i.e. one can not conclude that values increase of decrease monotonically.

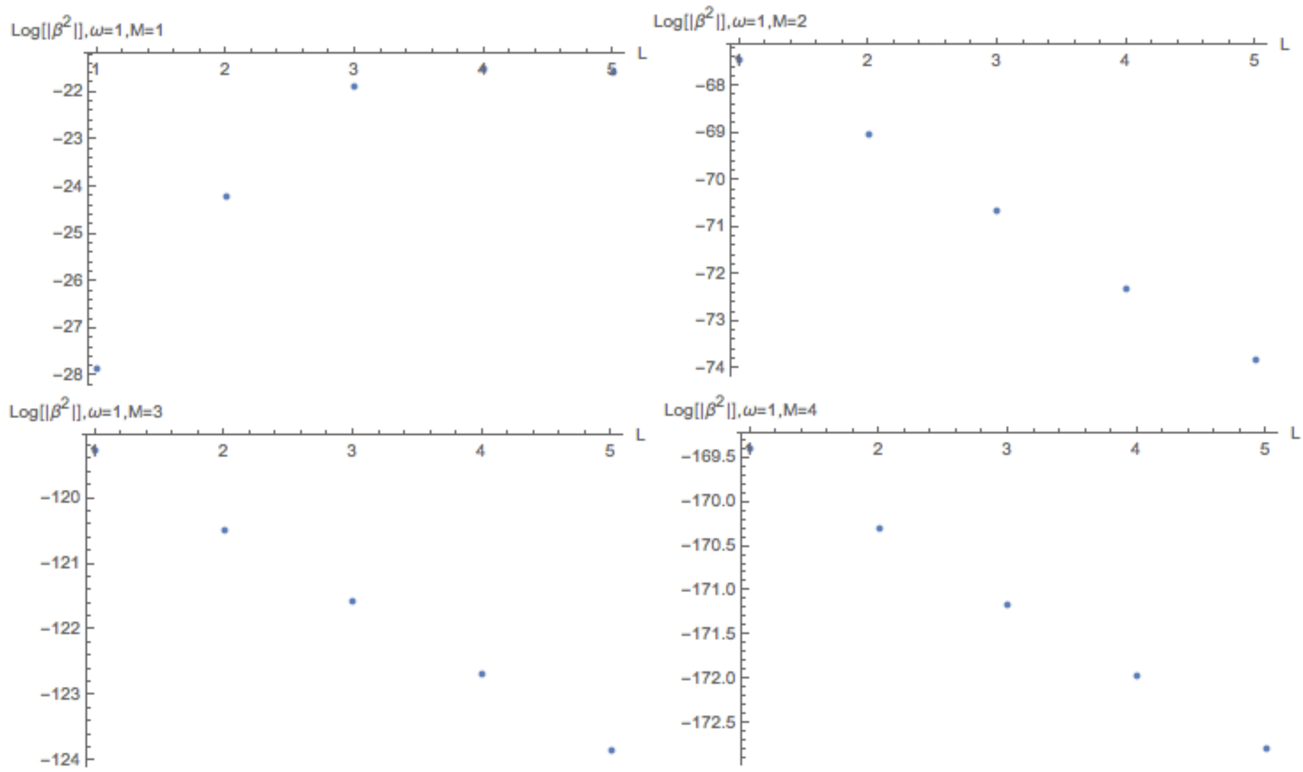


Figure 7.5: $Log[|\beta|^2]$ dependence on L. For all cases except for the first one values decrease with an increase in L.

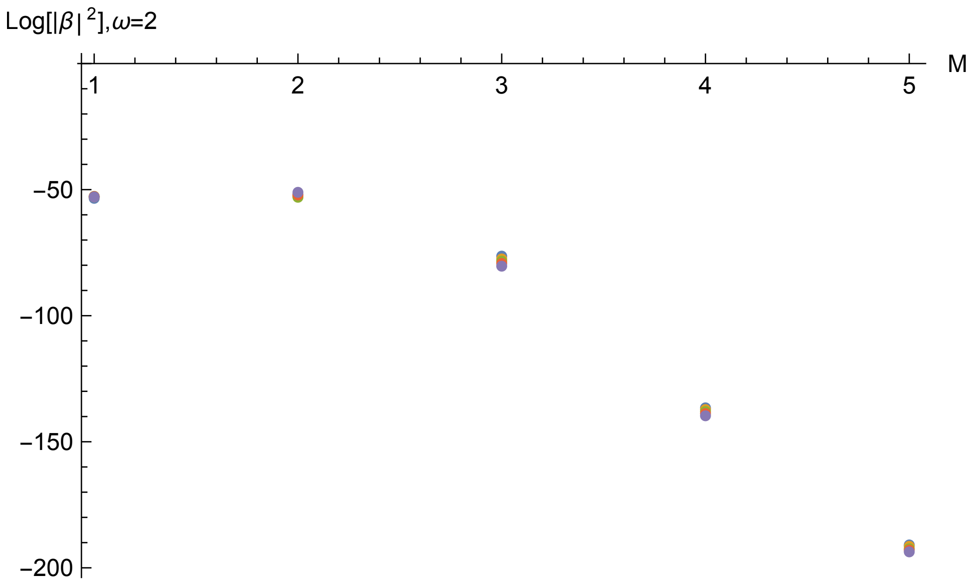


Figure 7.6: $Log[|\beta|^2]$ dependence on M. All values decrease universally. For each value of M a stack of points is represented by various values of L.

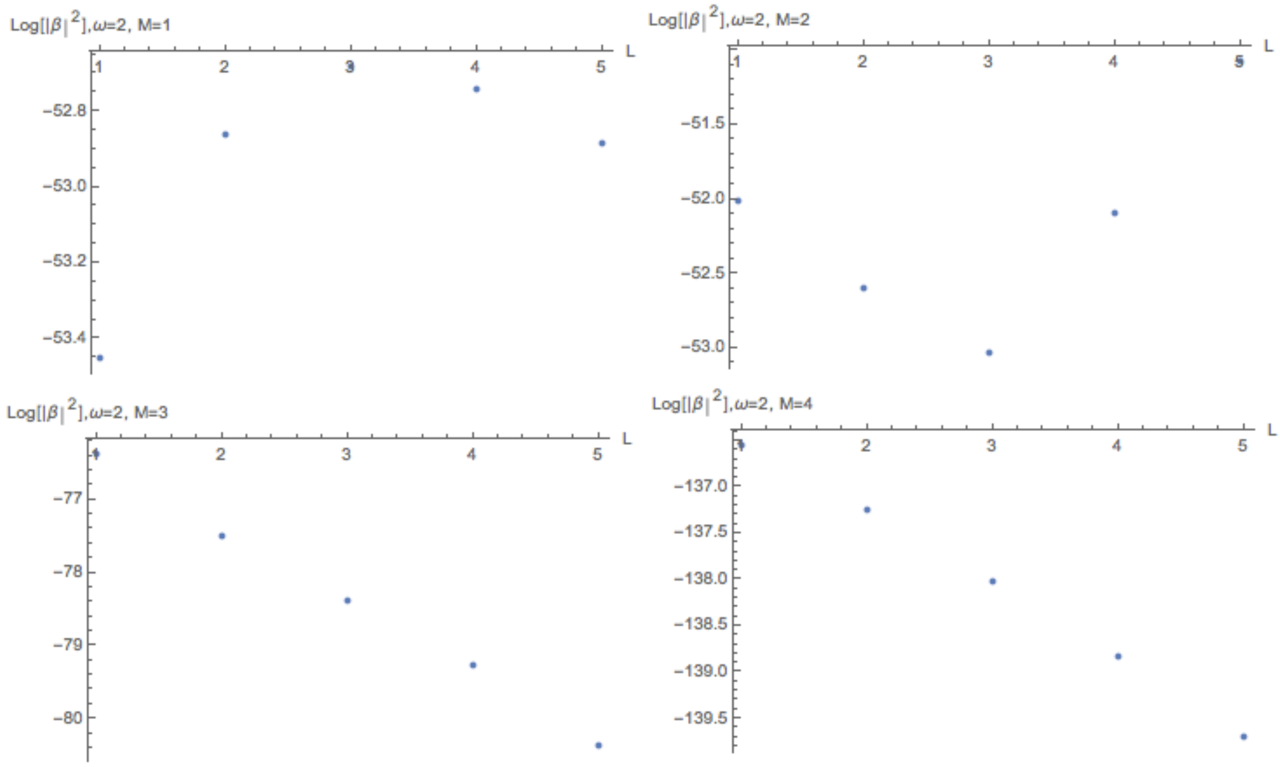


Figure 7.7: $Log[|\beta|^2]$ dependence on L. Similarly to $\omega = 1$ case, for smaller values of M there is some nontrivial behavior, but for larger values a universal drop is observed.

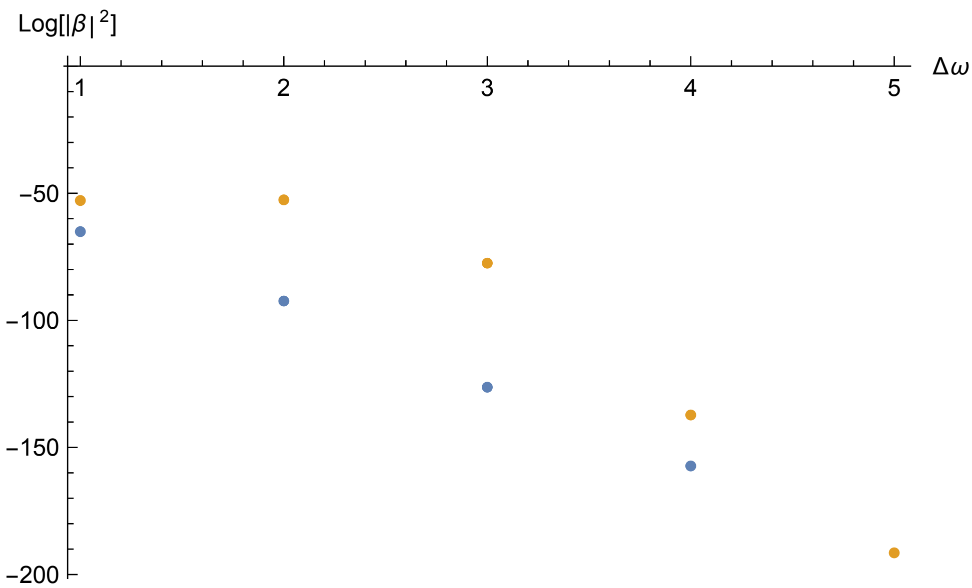


Figure 7.8: $Log[|\beta|^2]$ dependence on $\Delta\omega$. The difference in color represents different ω of reference, i.e. for top values $\omega = 1$, while for bottom values $\omega = 2$, and $\omega' = \omega + \Delta\omega$. The density of particles grows as the difference between frequencies diminishes, but in principle there is some mode mixing in ω space.

Bibliography

[1] Stephen Hawking. *Particle Creation by Black Holes.* Communications in Mathematical Physics, 43, 199-220, 1975.

[2] D. Philipp, V. Perlick. *On analytic solutions of wave equations in regular coordinate systems on Schwarzschild background.* arXiv:1503.08101v1, 2015.

[3] T. Damour, R. Ruffini. *Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism.* Phys. Rev. D, 14, 332, 1976.

[4] M. Hortacsu. *Heun Functions and their uses in Physics.* arXiv:1101.0471v8, 2011.

[5] G. Kristensson. *Second Order Differential Equations. Special functions and Their Classification.* Springer, 2010.

[6] S. Y. Slavyanov, W. Lay. *Special Functions. A Unified Theory Based on Singularities.* Oxford University Press, Oxford, 2000.

[7] Digital Library of Mathematical Functions.
`http://dlmf.nist.gov/`

[8] The five Second Order Linear Heun equations and the corresponding Heun function solutions.
`http://www.maplesoft.com/support/help/Maple/view.aspx?path=Heun`

[9] S. Carrol. *Spacetime and Geometry. An Introduction to General Relativity.*Addison-Wesley, San Francisco, 2003.

[10] M. Gasperini. *Theory of Gravitational Interactions.* Springer, 2013.

[11] M. Gasperini, V. Sabbata. *Introduction to Gravitation.* World Scientific Publishing, Singapore, 1985.

[12] L. Brewin. *Riemann normal coordinates.* 1996

[13] A. Hatzinikitas.*A note on Riemann normal coordinates.* arXiv:hep-th/0001078v1, 2000.

[14] N. D. Birrel, P. C. W. Davies. *Quantum fields in curved spacetime.* Cambridge University Press, Cambridge, 1994.

[15] L. Parker, D. Toms. *Quantum field theory in curved spacetime.* Cambridge University Press, Cambridge, 2009.

[16] T. S. Bunch, L. Parker.*Feynman propagator in curved spacetime: A momentum-space representation.* Phys. Rev. D 20, 2499, 1979.

[17] V. P. Frolov, I. D. Novikov. *Black Hole Physics. Basic Concepts and New Developments.* Kluwer Academic Publishers, Dordrecht, 1998.