On account of Remark 3.1.3, the uniqueness of the Green operators as per Corollary 3.1.17 entails (3.28).

2. Equation (3.29) is a consequence of the following chain of identities valid for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$

$$(\alpha, G^{\pm}_{\sharp}\beta) = (\Box G^{\mp}_{\sharp}\alpha, G^{\pm}_{\sharp}\beta) = (G^{\mp}_{\sharp}\alpha, \Box G^{\pm}_{\sharp}\beta) = (G^{\mp}_{\sharp}\alpha, \beta)\,,$$

where we used both the support properties of the Green operators and Lemma 3.1.1.

3. The exactness of the series is proven using the properties already established for the Green operators $G^{\pm}_{\sharp}$. The left exactness of the sequence is a consequence of the second identity in Equation (3.9) which ensures that $\Box_{\sharp}\alpha = 0$, $\alpha \in \Omega^k_{\mathrm{c},\sharp}(M)$, entails $\alpha = G^{+}_{\sharp}\Box_{\sharp}\alpha = 0$. In order to prove that $\ker G_{\sharp} = \Box\Omega^k_{\mathrm{c},\sharp}$, we first observe that $G_{\sharp}\Box_{\sharp}\Omega^k_{\mathrm{c},\sharp}(M) = \{0\}$ on account of Equation (3.9). Moreover, if $\beta \in \Omega^k_{\mathrm{c}}(M)$ is such that $G_{\sharp}\beta = 0$, then $G^{+}_{\sharp}\beta = G^{-}_{\sharp}\beta$. Hence, in view of the support properties of the Green operators $G^{+}_{\sharp}\beta \in \Omega^k_{\mathrm{c},\sharp}(M)$ and $\beta = \Box_{\sharp}G^{+}_{\sharp}\beta$. Subsequently we need to verify that $\ker\Box = G_{\sharp}\Omega^k_{\mathrm{c}}(M)$. Once more $\Box_{\sharp}G_{\sharp}\Omega^k_{\mathrm{c}}(M) = \{0\}$ follows from Equation (3.9). Conversely, let $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ be such that $\Box_{\sharp}\omega = 0$. On account of Lemma B.0.2 we can split $\omega = \omega^{+} + \omega^{-}$ where $\omega^{+} \in \Omega^k_{\mathrm{spc},\sharp}(M)$. Then $\Box_{\sharp}\omega^{+} = -\Box_{\sharp}\omega^{-} \in \Omega^k_{\mathrm{c},\sharp}(M)$ and

$$G_{\sharp}\Box_{\sharp}\omega^{+} = G^{+}_{\sharp}\Box_{\sharp}\omega^{+} + G^{-}_{\sharp}\Box_{\sharp}\omega^{-} = \omega\,.$$

To conclude we need to establish the right exactness of the sequence. Consider any $\alpha \in \Omega^k_{\mathrm{sc}}(M)$ and the equation $\Box_{\sharp}\omega = \alpha$. Consider the function $\eta(\tau)$ as in Remark 3.1.18 and let $\omega \doteq G^{+}_{\sharp}(\eta\alpha) + G^{-}_{\sharp}((1-\eta)\alpha)$. In view of Remark 3.1.18 and of the support properties of the Green operators, $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ and $\Box_{\sharp}\omega = \alpha$. ∎

Remark 3.1.20. Following the same reasoning as in [Bär15] together with minor adaptation of the proofs of [DDF19], one may extend $G_{\sharp}$ to an operator $G_{\sharp}\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k_{\sharp}(M)$ for all $\sharp \in \{\mathrm{D}, \|, \perp, f_{\|}, f_{\perp}\}$. As a consequence the exact sequence of Proposition 3.1.19 generalizes as

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_{\sharp}(M) \xrightarrow{\Box_{\sharp}} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_{\sharp}} \Omega^k_{\sharp}(M) \xrightarrow{\Box_{\sharp}} \Omega^k(M) \to 0\,. \tag{3.31}$$

Remark 3.1.21. Proposition 3.1.19 and Remark 3.1.20 ensure that $\ker_{\mathrm{c}}\Box_{\sharp} \subseteq \ker_{\mathrm{tc}}\Box_{\sharp} = \{0\}$. In other words, there are no timelike compact solutions to the equation $\Box\omega = 0$ with $\sharp$-boundary conditions. More generally it can be shown that $\ker_{\mathrm{c}}\Box \subseteq \ker_{\mathrm{tc}}\Box = \{0\}$, namely there are no timelike compact solutions regardless of the boundary condition. This follows by standard arguments using a suitable energy functional defined on the solution space – *cf.* [DDF19, Thm. 30] for the proof for $k = 0$.

In studying Maxwell's equations for A, we will make extensively use of the Green operators for $\Box$ and we will need to intertwine the propagators and the differential operators d, δ. This in

general does not happen for any boundary condition, hence we will consider only the conditions $\perp$, $\|$ individuated in Definition 3.1.2. This will dictate the class of boundary conditions such that standard techniques applies in solving Maxwell's equations.
In view of the applications to the Maxwell operator, it is worth focusing specifically on the boundary conditions $\perp$, $\|$ individuated in Definition 3.1.2 since it is possible to prove a useful relation between the associated propagators and the operators d,δ.

Lemma 3.1.22. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G^{\pm}_{\|} \circ \mathrm{d} = \mathrm{d} \circ G^{\pm}_{\|} \quad \text{on } \Omega^k_{\mathrm{t}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^{\pm}_{\|} \circ \delta = \delta \circ G^{\pm}_{\|} \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,, \tag{3.32}$$

$$G^{\pm}_{\perp} \circ \delta = \delta \circ G^{\pm}_{\perp} \quad \text{on } \Omega^k_{\mathrm{n}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^{\pm}_{\perp} \circ \mathrm{d} = \mathrm{d} \circ G^{\pm}_{\perp} \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,. \tag{3.33}$$

Proof. From Equation (3.28) it follows that equations (3.32-3.33) are dual to each other via the Hodge operator. Hence we shall only focus on Equation (3.32).
For every $\alpha \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{t}}(M)$, $G^{\pm}_{\|}\mathrm{d}\alpha$ and $\mathrm{d}G^{\pm}_{\|}\alpha$ lie both in $\Omega^k_{\|}(M)$. In particular, using Equation (1.5b), $\mathrm{t}\delta\mathrm{d}G^{\pm}_{\|}\alpha = \mathrm{t}(\Box_{\|} - \mathrm{d}\delta)G^{\pm}_{\|}(\alpha) = \mathrm{t}\alpha = 0$ while the second boundary condition is automatically satisfied since $\mathrm{t}\mathrm{d}G^{\pm}_{\|} = \mathrm{d}\mathrm{t}G^{\pm}_{\|} = 0$. Hence, considering $\beta = G^{\pm}_{\|}\mathrm{d}\alpha - \mathrm{d}G^{\pm}_{\|}\alpha$, it holds that $\Box\beta = 0$ and $\beta \in \Omega^k_{\|} \cap \Omega^k_{\mathrm{pc/fc}}(M)$. In view of Remark 3.1.18, this entails $\beta = 0$. ∎

We conclude this section with a corollary to Lemma 3.1.22 which shows that, when considering the difference between the advanced and the retarded Green operators, the support restrictions present in equations (3.32-3.33) disappear.

Corollary 3.1.23. *Under the hypotheses of Assumption 3.1.4 it holds that*

$$G_{\sharp} \circ \mathrm{d} = \mathrm{d} \circ G_{\sharp} \quad \text{on } \Omega^k_{\mathrm{tc}}(M)\,, \qquad G_{\sharp} \circ \delta = \delta \circ G_{\sharp} \quad \text{on } \Omega^k_{\mathrm{tc}}(M) \qquad \sharp \in \{\|, \perp\}\,. \tag{3.34}$$

Proof. In all cases the reasoning is similar as in the proof of Equation (3.32), but it requires the following characterization of $G_{\sharp}$. Since $M \simeq \mathbb{R} \times \Sigma$ – *cf.* Theorem 1.1.2 – let $\tau_0 \in \mathbb{R}$ and consider $\alpha_0 \in \Omega^k_{\mathrm{c}}(\Sigma_0)$, where $\Sigma_0 := \{\tau_0\} \times \Sigma$. Setting $\alpha := \alpha_0 \wedge \delta_{\tau_0}\mathrm{d}\tau$ we define a distribution-valued k-form and, following [Bär15, Lem. 4.1., Thm. 4.3], we can consider $G_{\sharp}\alpha$. It turns out that $G_{\sharp}\alpha$ is the unique solution to the Cauchy problem

$$\Box\psi = 0\,, \qquad \mathrm{t}_{\Sigma_0}\psi = 0\,, \qquad \mathrm{t}_{\Sigma_0}\mathcal{L}_{\partial_\tau}(\psi) = \alpha_0\,, \qquad \sharp\text{-boundary conditions for } \psi\,, \tag{3.35}$$

where $\mathrm{t}_{\Sigma_0} \colon \Omega^k(M) \to \Omega^k(\Sigma_0)$ is defined in (1.3) with $N \equiv \Sigma_0$, while $\mathcal{L}_{\partial_\tau}$ denotes the Lie derivative along the vector field ∂_τ.
With this characterization we can prove Equation (3.34). Focusing for simplicity on the first identity of (3.34) for $\sharp =\|$, we need to show that $\mathrm{d}G_{\|}\alpha$ and $G_{\|}\mathrm{d}\alpha$ solve the same Cauchy

problem (3.35). While the analysis of the equation of motion and of the initial data do not differ from the counterpart on globally hyperbolic spacetimes with empty boundary, the only additional necessary information comes from $\mathrm{t}\delta\mathrm{d}G_{\|}^{\pm}\alpha = \mathrm{t}(\Box - \mathrm{d}\delta)G_{\|}^{\pm}\alpha = \mathrm{t}\alpha$, for all $\alpha \in \Omega^k_{\mathrm{tc}}(M)$. This entails that, being $G_{\|} = G_{\|}^{+} - G_{\|}^{-}$, $\mathrm{t}\delta\mathrm{d}G_{\|}\alpha = 0$. ∎

3.2 On the Maxwell operator

Maxwell's equations for the vector potential $A \in \Omega^1(M)$ with vanishing source read $\delta\mathrm{d}A = 0$. Hence, studying the space of solutions amounts to characterize the kernel of the Maxwell operator $\delta\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$ in connection both to the D'Alembert - de Rham wave operator $\Box$ and to the identification of suitable boundary conditions. We shall keep the assumption that (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = m \geq 2$ – *cf.* Theorem 1.1.2. Notice that, if $k = m$, then the Maxwell operator becomes trivial, while, if $k = 0$, is coincides with the D'Alembert - de Rham operator $\Box$. Hence this case falls in the one studied in the preceding section and in [DDF19]. Therefore, unless stated otherwise, henceforth we shall consider only $0 < k < m = \dim M$.

3.2.1 Spaces of solutions for selected boundary conditions

In analogy to the analysis of $\Box$, we observe that, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, from Equation (1.6), one can obtain the following Green formula:

$$(\delta\mathrm{d}\alpha, \beta) - (\alpha, \delta\mathrm{d}\beta) = (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_{\partial} - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_{\partial} \,. \tag{3.36}$$

In the same spirit of Lemma 3.1.1, the operator $\delta\mathrm{d}$ becomes formally self-adjoint if we restrict its domain to

$$\Omega^k_f(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\} \,, \tag{3.37}$$

where $f \in C^\infty(\partial M)$ is arbitrary but fixed. In what follows we will consider two particular boundary conditions which are directly related to the $\Box$-tangential and to the $\Box$-normal boundary conditions for the D'Alembert - de Rham operator, labeled $\|, \perp$, respectively – *cf.* Definition 3.1.2. We selected a particular class of boundary conditions from the entire domain (3.37) since in general it is not clear whether there exist intertwining relations between the propagators and the differential operators such as that of Lemma 3.1.22. This is an important obstruction to adapt our analysis to more general cases.

Definition 3.2.1. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $0 < k < \dim M$. We call*

1. *space of k-forms with $\delta\mathrm{d}$-tangential boundary condition, $\Omega^k_{\mathrm{t}}(M)$ as in Equation (1.4) with $N = \partial M$:*

$$\Omega^k_{\mathrm{t}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\} \,. \tag{3.38}$$

2. *space of k-forms with $\delta\mathrm{d}$-normal boundary condition*

$$\Omega^k_{\mathrm{nd}}(M) \doteq \{\omega \in \Omega^k(M) | \, \mathrm{nd}\omega = 0\}\,. \tag{3.39}$$

In the following our first goal is to characterize the kernel of the Maxwell operator with a prescribed boundary condition, *cf.* Equation (3.37). To this end we need to focus on the *gauge invariance* of the underlying theory.
As we recalled in Section 1.5, gauge freedom is a tool that, in globally hyperbolic spacetimes with empty boundary, ensures that Maxwell's equations can be written as an hyperbolic system $\Box A = 0$ together with a constraint on the initial data $\delta A = 0$, which is the Lorenz gauge condition.
In the following, we generalize the concept of gauge invariance when the background has a non-vanishing timelike boundary. It turns out that there exist different notions of gauge invariance that depends on the choice of the boundary condition that we require on the solutions of Maxwell's equations. In particular, if we require a solution to be $A \in \Omega^k_{\mathrm{nd}}(M)$, we notice that the boundary condition $\mathrm{nd}A = 0$ is invariant under the most general gauge transformation $A \mapsto A + \mathrm{d}\chi$, $\chi \in \Omega^{k-1}(M)$, while for $A \in \Omega^k_{\mathrm{t}}(M)$ the condition $\mathrm{t}A = 0$ is not. For this reason, this scenario is distinguished and we give to the space of solutions in $\Omega^k_{\mathrm{nd}}(M)$ a definition of gauge equivalence that is totally analogous to that with empty boundary. On the other hand, for solutions that lies in $\Omega^k_{\mathrm{t}}(M)$, one must restrict the gauge group and our choice is the space forms with vanishing tangential component. Such choice is fact not unique when working at a level of k-forms.

To avoid this quandary, one should resort to a more geometrical formulation of Maxwell's equations for $A \in \Omega^1(M)$, namely as originating from a theory for the connections of a principal $U(1)$-bundle over the underlying globally hyperbolic spacetime with timelike boundary, *cf.* [Ben+14; BDS14] for the case with empty boundary. Following the nomenclature of the cited works and with reference to Equation (1.26), one should focus on the gauge transformations of the form $A \mapsto A' = A + \eta$, where there exists $f \in C^\infty(M, U(1))$ such that $\eta = f^*(\mu_{U(1)})$, with f^* denoting the pull-back of f and $\mu_{U(1)}$ is the Maurer-Cartan form on $U(1)$. Hence, the most general gauge group is the following:

$$B_{U(1)} = \{\eta = f^*(\mu_{U(1)}) \,|\, f \in C^\infty(M, U(1))\}.$$

This space certainly includes $\mathrm{d}\Omega^0(M)$, but it is characterized explicitly in [Ben+14] as

$$B_{U(1)} = \{\eta \in \Omega^1_{\mathrm{d}}(M) \,|\, [\eta] \in H^1(M, 2\pi i\mathbb{Z})\},$$

where $H^1(M, 2\pi i\mathbb{Z})$ denotes the first de Rham cohomology group (see Appendix A) with coefficients in $2\pi i\mathbb{Z}$, so that the integral modulo $2\pi i$ is an integer. In this case the gauge group is not

a vector space but a $\mathbb{Z}$-module.
We point out that the analytic description employed in this thesis meets that of a gauge theory if one regards electromagnetism as a connection on a principal bundle with $\mathbb{R}$ as structure group. In that case the gauge group $B_{\mathbb{R}}$ reduces simply to $\mathrm{d}\Omega^0(M)$, that is the description classically used for the gauge group in globally hyperbolic manifolds with empty boundary and that we try to include in our definitions.

In the case in hand this translates in the following characterization.

Definition 3.2.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $\delta\mathrm{d}$ be the Maxwell operator acting on $\Omega^k(M)$, $0 < k < \dim M$. We say that*

1. *$A \in \Omega^k_{\mathrm{t}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{t}}(M)$ if $A - A' \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$, namely if there exists $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-tangential boundary conditions is denoted by*

$$\mathrm{Sol}_{\mathrm{t}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)}\,. \tag{3.40}$$

2. *$A \in \Omega^k_{\mathrm{nd}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{nd}}(M)$ if there exists $\chi \in \Omega^{k-1}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-normal boundary conditions is denoted by*

$$\mathrm{Sol}_{\mathrm{nd}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}(M)}\,. \tag{3.41}$$

Similarly the space of spacelike supported solutions with $\delta\mathrm{d}$-tangential (resp. $\delta\mathrm{d}$-normal) boundary conditions are

$$\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t,sc}}(M)}\,, \tag{3.42}$$

$$\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M)}\,. \tag{3.43}$$

The forms that solve the equations of motion, in the sense that they belong to the spaces defined above, are called *on-shell configurations*. On the other hand, the spaces of all possible configurations of the field, respectively $\Omega^k_{\mathrm{t}}(M)$, $\Omega^k_{\mathrm{nd}}(M)$ are called the spaces of *off-shell configurations*.

At this point we ask ourselves whether it is possible to use the Green operators of $\Box$, studied in Section 3.1, to characterize the spaces of solutions in Definition 3.2.2. The answer is positive for the selected boundary conditions since it is possible to find a representative in the gauge equivalence classes $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ (*resp.* $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$) that satisfies the Lorenz gauge $\delta A = 0$ – *cf.* [Ben16, Lem. 7.2].

In addition we provide a connection between $\delta\mathrm{d}$-tangential (*resp.* $\delta\mathrm{d}$-normal) boundary conditions with $\Box$-tangential (*resp.* $\Box$-normal) boundary conditions. These proofs rely heavily on the fact that the propagator G and the operator δ intertwine for our choice of boundary conditions, as shown in Corollary 3.1.23. Recalling Definition 3.1.2 of the $\Box$-tangential boundary condition, the following holds true.

Proposition 3.2.3. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A]\in\mathrm{Sol}_{\mathrm{t}}(M)$ there exists a representative $A'\in[A]$ such that*

$$\Box_{\parallel}A'=0\,,\qquad \delta A'=0\,. \tag{3.44}$$

Moreover, the same result holds true for $[A]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$.

Proof. We focus only on the first assertion, the proof of the second one being similar. Let $A\in[A]\in\mathrm{Sol}_{\mathrm{t}}(M)$, that is, $A\in\Omega^k(M)$, $\delta\mathrm{d}A=0$ and $\mathrm{t}A=0$. Consider any $\chi\in\Omega^{k-1}_{\mathrm{t}}(M)$ such that

$$\Box\chi=-\delta A,\qquad \delta\chi=0,\qquad \mathrm{t}\chi=0\,. \tag{3.45}$$

In view of Assumption 3.1.4 and of Remark 3.1.18, we can fix $\chi=-\sum_{\pm}G^{\pm}_{\parallel}\delta A^{\pm}$, where $A^{\pm}$ is defined as in Remark 3.1.18. Per definition of $G^{\pm}_{\parallel}$, $\mathrm{t}\chi=0$ while, on account of Corollary 3.1.23, $\delta\chi=-\sum_{\pm}\delta G^{\pm}_{\parallel}\delta A^{\pm}=0$.
Hence A' is gauge equivalent to A as per Definition 3.2.2. ■

The proof of the analogous result for $\Omega^k_{\mathrm{nd}}(M)$ is slightly different and, thus, we discuss it separately. Recalling Definition 3.1.2 of the $\Box$-normal boundary conditions, the following statement holds true.

Proposition 3.2.4. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A]\in\mathrm{Sol}_{\mathrm{nd}}(M)$ there exists a representative $A'\in[A]$ such that*

$$\Box_{\perp}A'=0\,,\qquad \delta A'=0\,. \tag{3.46}$$

Moreover, the same result holds true for $[A]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$.

Proof. As in the previous proposition, we can focus only on the first point. Let A be a representative of $[A]\in\mathrm{Sol}_{\mathrm{nd}}(M)$. Hence $A\in\Omega^k(M)$ so that $\delta\mathrm{d}A=0$ and $\mathrm{nd}A=0$. Consider first $\chi_0\in\Omega^{k-1}(M)$ such that $\mathrm{nd}\chi_0=-\mathrm{n}A$. The existence is guaranteed since the map nd is surjective – *cf.* Remark 1.2.5. As a consequence we can exploit the residual gauge freedom to select $\chi_1\in\Omega^{k-1}(M)$ such that

$$\Box\chi_1=-\delta\widetilde{A}\,,\qquad \delta\chi_1=0\,,\qquad \mathrm{nd}\chi_1=0\qquad \mathrm{n}\chi_1=0\,, \tag{3.47}$$

where $\widetilde{A}=A+\mathrm{d}\chi_0$. Let $\eta\equiv\eta(\tau)$ be a smooth function such that $\eta=0$ if $\tau<\tau_0$ while $\eta=1$

if $\tau > \tau_1$, *cf.* Remark 3.1.18. Since $\mathrm{n}\widetilde{A} = 0$ we can fine tune η in such a way that both $\widetilde{A}^+ \doteq \eta\widetilde{A}$ and $\widetilde{A}^- \doteq (1-\eta)\widetilde{A}$ satisfy $\mathrm{n}\widetilde{A}^\pm = 0$. Equation (1.5b) entails that $\mathrm{n}\delta A^\pm = -\delta\mathrm{n}A^\pm = 0$. Hence we can apply Lemma 3.1.22 and set $\chi_1 = -\sum_\pm G_\perp^\pm\delta\widetilde{A}^+$. Calling $A' = A + \mathrm{d}(\chi_0 + \chi_1)$ we obtained the desired result. ■

As it is well known, in globally hyperbolic spacetimes with empty boundary Lorenz gauge leaves a residual freedom in the choice of A'. Classically, as recalled in Section 1.5, if the boundary is empty, Lorenz gauge is imposed by requiring that, for any but fixed $A \in \Omega^1(M)$, there exists $\chi \in \Omega^0$ such that $\delta A' = \delta(A + \mathrm{d}\chi) = 0$. Since the equation $\delta\mathrm{d}\chi = \Box\chi = -\delta A$ is solvable, the existence of χ is ensured, but the uniqueness is not, since χ is determined modulo solutions of the homogeneous equations $\Box\chi = 0$.
A direct inspection of (3.45) and of (3.46) unveils that even in the case with non-empty boundary a residual freedom is left. This amount either to

$$\mathcal{G}_{\mathrm{t}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0\,,\ \mathrm{t}\chi = 0\}\,,$$

or, in the case of a $\delta\mathrm{d}$-normal boundary condition, to

$$\mathcal{G}_{\mathrm{nd}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0,\ \mathrm{n}\chi = 0\ ,\ \mathrm{nd}\chi = 0\}\,. \tag{3.48}$$

Observe that, in the definition of $\mathcal{G}_{\mathrm{nd}}(M)$, we require χ to be in the kernel of $\delta\mathrm{d}$. Nonetheless since the actual reduced gauge group is $\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)$ we can work with $\chi_0 \in \Omega^{k-1}(M)$ such that $\Box\chi_0 = 0$. As a matter of fact for all $\chi \in \mathcal{G}_{\mathrm{nd}}$ we can set $\chi_0 \doteq \chi + \mathrm{d}\lambda$ where $\lambda \in \Omega^{k-2}(M)$ is such that $\Box\lambda = -\delta\chi$ and $\mathrm{n}\lambda = \mathrm{nd}\lambda = 0$ – *cf.* Proposition 3.2.4. In addition $\mathrm{d}\chi = \mathrm{d}\chi_0$.
To better codify the results of the preceding discussion, it is also convenient to introduce the following linear spaces:

$$\mathcal{S}_{\mathrm{t}}^{\Box}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{t}A = 0\}\,, \tag{3.49}$$

$$\mathcal{S}_{\mathrm{nd}}^{\Box}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{n}A = 0,\ \ \mathrm{nd}A = 0\}\,. \tag{3.50}$$

Hence Propositions 3.2.3-3.2.4 can be summarized as stating the existence of the following isomorphisms:

$$\mathcal{S}_{\mathcal{G}_{\mathrm{t}},k}(M) \doteq \frac{\mathcal{S}_{\mathrm{t}}^{\Box}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{t}}(M)} \simeq \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad \mathcal{S}_{\mathcal{G}_{\mathrm{nd}},k}(M) \doteq \frac{\mathcal{S}_{\mathrm{nd}}^{\Box}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)} \simeq \mathrm{Sol}_{\mathrm{nd}}(M)\,. \tag{3.51}$$

3.3 Introduction to the algebraic formalism

In this section we give an overview on the algebraic approach to quantum field theory, with the aim to associate a unital $*$-algebra both to $\mathrm{Sol}_{\mathrm{t}}(M)$ and to $\mathrm{Sol}_{\mathrm{nd}}(M)$, whose elements are interpreted as the observables of the underlying quantum system. We recall that the corresponding

question, when the underlying background (M, g) is globally hyperbolic manifold with $\partial M = \emptyset$ has been thoroughly discussed in the literature – *cf.* [Ben16; DS13; HS13; SDH14].
For the principal definitions we follow mainly [KM15].

Definition 3.3.1. *An* algebra $\mathcal{A}$ *is a complex vector space endowed with an associative product* $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, *distributive with respect to the vector sum and that satisfies* $\alpha ab = (\alpha a)b = a(\alpha b)$ *if* $\alpha \in \mathbb{C}$ *and* $a, b \in \mathcal{A}$.
$\mathcal{A}$ *is a* $*$*-algebra if it admits an* involution, *i.e. an anti-linear map,* $a \to a^*$ *which is involutive -* $(a^*)^*$ *- and* $(ab)^* = b^* a^*$, *for any* $a, b \in \mathcal{A}$. *Moreover* $\mathcal{A}$ *is* unital *if contains a multiplicative unit* $\mathbb{I} \in \mathcal{A}$.
A set $G \subset \mathcal{A}$ *is said to generate the algebra* $\mathcal{A}$, *and the elements of* G *are said generators of* $\mathcal{A}$, *if each element of* $\mathcal{A}$*is a finite complex linear combination of products (with arbitrary number of factors) of elements of* G. *The centre of the algebra* $\mathcal{A}$ *is the set of elements* $z \in \mathcal{A}$ *commuting with all elements of* $\mathcal{A}$.

Traditional quantum mechanics deals with operators on Hilbert spaces. In particular, the observables of a quantum system are self-adjoint operators. Such operators, as it is well known from basic examples such as position and momentum are not bounded by some operator norm, but many of the features of the quantum theory can be understood focusing on an algebra of bounded operators.
This is a special realization of the so-called C^*-algebras. Indeed a C^*-algebra is a $*$-algebra which is a Banach space with respect to a norm $\| \ \|$ which satisfies $\|ab\| \leq \|a\| \, \|b\|$ and $\|a^* a\| = \|a\|^2$. This implies $\|a^*\| = \|a\|$ and if a C^*-algebra is unital, $\|\mathbb{I}\| = 1$. A unital $*$-algebra admits a unique norm making it C^*-algebra.
For the reasons we mentioned, we shall not use C^*-algebras, even if they still have a theoretical interest, but we will focus on the construction of a $*$-algebra.

Definition 3.3.2. *A two-sided ideal of an algebra* $\mathcal{A}$ *is a linear complex subspace* $\mathcal{I} \subset \mathcal{A}$ *such that* $ab \in \mathcal{I}$ *and* $ba \in \mathcal{I}$ *if* $a \in \mathcal{A}$ *and* $b \in \mathcal{I}$. *In a* $*$*-algebra, a two-sided ideal* $\mathcal{I}$ *is said to be a two-sided* $*$*-ideal if it is also closed with respect to the involution, in other words* $a^* \in \mathcal{I}$ *if* $a \in \mathcal{I}$. *An algebra* $\mathcal{A}$ *is simple if it does not admit two-sided ideals different form* $\{0\}$ *and* $\mathcal{A}$ *itself.*

We will mainly deal with universal tensor $*$-algebras generated by some complex vector space $\mathcal{O}$. Universal tensor algebras generated by $\mathcal{O}$ are defined as $\mathcal{T}[\mathcal{O}] \doteq \bigoplus_{n=0}^{\infty} \mathcal{O}^{\otimes n}$, with $\mathcal{O}^{\otimes 0} \equiv \mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. To obtain the quantum algebra of observable we will quotient a $*$-ideal generated in such a way that canonical commutation relations (CCR) are imposed.

3.3.1 The generators of the algebra of observables

Our goal is to find, for $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions respectively, a $*$-algebra $\mathcal{A}$ of functionals (generated by $\mathcal{O}$) which can be thought of as *classical observables* in the sense that one can extract any information about a given field configuration by means of these functionals and, at the same time, each of them provides some information which cannot be detected by any other functional. To this end, we require the pairing between $\mathcal{A}$ and the space of solutions to Maxwell's equations with prescribed boundary conditions to be *optimal*. Optimality has to be understood in the following sense:

Definition 3.3.3. *A $*$-algebra of observables $\mathcal{A}$ generated by a vector space $\mathcal{O}$ is* optimal *if*

1. *$\mathcal{O}$ is* separating. *In other words contains many functional to distinguish between different on-shell configurations (for the moment called elements of Sol), namely*

$$(\alpha, A) = 0 \text{ for any } \alpha \in \mathcal{O} \text{ implies } A = 0 \in \mathrm{Sol};$$

2. *$\mathcal{O}$ is* non redundant. *In other words*

$$(\alpha, A) = 0 \text{ for any } A \in \mathrm{Sol} \text{ implies } \alpha = 0 \in \mathcal{O}.$$

It will turn out that the correct optimal generators for the algebra of observables for the two boundary conditions considered (labeled respectively t, nd) are

$$\mathcal{O}_{\mathrm{t}}(M) \doteq \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \quad \mathcal{O}_{\mathrm{nd}}(M) \doteq \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}. \tag{3.52}$$

Moreover, we will prove that $\mathcal{O}_{\mathrm{t}}(M)$ is isomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and can be endowed by a symplectic form $\widetilde{G}_{\|}$, while $\mathcal{O}_{\mathrm{nd}}(M)$ do possess only a presymplectic structure.

In the following we justify from an intuitive point of view the choice of the generators in (3.52). For an account in the case of empty boundary, see [Ben16, Sec. 7.2].
The following discussion will be focused at first on $\delta\mathrm{d}$-tangential boundary conditions. As a starting point, we consider the linear functional on the space of off-shell configurations $\Omega^k_{\mathrm{t}}(M)$. For $\alpha \in \Omega^k_{\mathrm{c}}(M)$

$$F_\alpha : \Omega^k_{\mathrm{t}}(M) \longrightarrow \mathbb{C}\,, \tag{3.53}$$

$$\beta \longmapsto (\alpha, \beta)\,. \tag{3.54}$$

We require the functionals F_α to be invariant under gauge transformations, hence we impose that the space from which we choose α satisfies

$$F_\alpha\left(\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)\right) = \{0\}.$$

Since we imposed boundary conditions, the right-hand side of Equation (1.6) vanishes and hence we require

$$F_\alpha(\mathrm{d}\beta) = (\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta) = 0\,,\ \forall \beta \in \Omega^{k-1}_{\mathrm{t}}(M).$$

This implies that $\delta\alpha = 0$ and hence the space of linear functionals invariant under gauge equivalence is chosen to be $\{F_\alpha \,|\, \alpha \in \Omega^k_{\mathrm{c},\delta}(M)\} \simeq \Omega^k_{\mathrm{c},\delta}(M) \doteq \Omega^k_{\mathrm{c}}(M) \cap \ker \delta$.
The next step is to include the dynamic $\delta\mathrm{d}A = 0$, so that the functionals can be evaluated on $\mathrm{Sol}_{\mathrm{t}}(M)$. To do that we force the functionals not to be defined if $\delta\mathrm{d}A \neq 0$, $A \in \Omega^k_{\mathrm{t}}(M)$. This is obtained by taking as space of functionals the quotient

$$\mathcal{O}_{\mathrm{t}}(M) = \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}. \tag{3.55}$$

The evaluation of $[\alpha] \in \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ and $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ will be defined as the product (α, A) for arbitrary representatives $\alpha \in [\alpha]$, $A \in [A]$. The proof that the pairing is well defined will be carried out in Section 3.4.

Adapting the former arguments to the $\delta\mathrm{d}$-normal boundary conditions requires a particular discussion. As before, we require the functionals F_α to be invariant under the action of the gauge group, which in this case is the whole $\mathrm{d}\Omega^{k-1}(M)$, hence we impose that the space from which we choose α satisfies

$$F_\alpha\left(\mathrm{d}\Omega^{k-1}(M)\right) = \{0\}.$$

This time the right-hand side of Equation (1.6) does not vanish if we do not impose further restrictions, namely $\mathrm{n}\alpha = 0$. Hence, if $\mathrm{n}\alpha = 0$, $(\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta)$ and we have

$$F_\alpha(\mathrm{d}\beta) = (\alpha, \mathrm{d}\beta) = (\delta\alpha, \beta) = 0\,,\ \forall \beta \in \Omega^{k-1}_{\mathrm{nd}}(M).$$

This implies that $\delta\alpha = 0$ (i.e. $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M) = \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{n}}(M) \cap \ker \delta$) and imposing the equations of motions we obtain

$$\mathcal{O}_{\mathrm{nd}}(M) = \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}. \tag{3.56}$$

This goes opposite to the case of $\delta\mathrm{d}$-tangential boundary conditions, where β is required to satisfy $\mathrm{t}\beta = 0$ – *cf.* Definition 3.2.2 – and therefore α is not forced to satisfy any boundary conditions. Actually, the conditions $\delta\alpha = 0$ and $\mathrm{n}\alpha = 0$ are necessary to ensure gauge-invariance, namely $(\alpha, \mathrm{d}\beta) = 0$ for all $\beta \in \Omega^k(M)$.

Symplectic structures

In this subsection, we characterize the spaces $\mathrm{Sol}_{\mathrm{t,nd}}(M)$ as symplectic spaces and we overview technical results that connect them to the generators of the algebra of observables that we are looking for, (3.55) and (3.56).

In view of Definition 2.3.3, we recall that, given a complex vector space S and a map $\sigma \colon \mathsf{S} \times \mathsf{S} \to \mathbb{C}$, the pair (S, σ) is called complex symplectic space if σ is sesquilinear, non-degenerate[2] and $\sigma(x, y) = -\overline{\sigma(y, x)}$ for all $x, y \in \mathsf{S}$. If we do not require σ to be non-degenerate, we call (S, σ) a presymplectic space.
It is noteworthy that both $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ can be endowed with a presymplectic form – *cf.* [HS13, Prop. 5.1]. The presence of symplectic spaces is motivated in analogy with classical mechanics, in particular the spaces $\mathrm{Sol}_{\mathrm{t,nd}}(M)$ are seen as classical phase spaces. The spaces $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$, $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$ are the, respectively, symplectic and pre-symplectic spaces of observables describing the classical theory of the Maxwell field on M, which is the starting point for the quantization scheme, which in the bosonic case is based on the existence of a CCR-representation algebra of the aforementioned symplectic spaces – *cf.* [HS13, Def. 4.3].

Proposition 3.3.4. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Let $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and, for $A_1 \in [A_1]$, let $A_1 = A_1^+ + A_1^-$ be any decomposition such that $A^+ \in \Omega^k_{\mathrm{spc,t}}(M)$ while $A^- \in \Omega^k_{\mathrm{sfc,t}}(M)$ – cf. Lemma B.0.2. Then the following map $\sigma_{\mathrm{t}} \colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)^{\times 2} \to \mathbb{R}$ is a presymplectic form:*

$$\sigma_{\mathrm{t}}([A_1], [A_2]) = (\delta \mathrm{d} A_1^+, A_2)\,, \qquad \forall [A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,. \tag{3.57}$$

A similar result holds for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ and we denote the associated presymplectic form σ_{nd}. In particular for all $[A_1], [A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ we have $\sigma_{\mathrm{nd}}([A_1], [A_2]) \doteq (\delta \mathrm{d} A_1^+, A_2)$ where $A_1 \in [A_1]$ is such that $A \in \Omega^k_{\mathrm{sc},\perp}(M)$.

Proof. See Appendix C, Prop. C.0.1 ■

Working either with $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{t}}(M)$ or $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{nd}}(M)$ leads to the natural question whether it is possible to give an equivalent representation of these spaces in terms of compactly supported k-forms. Using Assumption 3.1.4, the following proposition holds true:

[2] σ is non-degenerate if $\sigma(x, y) = 0$ for all $y \in \mathsf{S}$ implies $x = 0$.

Proposition 3.3.5. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. Then the following linear maps are isomorphisms of vector spaces*

$$G_{\parallel}\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad G_{\parallel}\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,, \tag{3.58}$$

$$G_{\perp}\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,nd}}(M)} \to \mathrm{Sol}_{\mathrm{nd}}(M)\,, \qquad G_{\perp}\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)\,, \tag{3.59}$$

Proof. See Appendix C, Prop. C.0.2 ∎

The following proposition shows that the isomorphisms introduced in Proposition 3.3.5 for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ lift to isomorphisms of presymplectic spaces.

Proposition 3.3.6. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. The following statements hold true:*

1. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_{\parallel}([\alpha],[\beta]) \doteq (\alpha, G_{\parallel}\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \widetilde{G}_{\parallel}\right)$ *is symplectomorphic to* $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$.

2. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_{\perp}([\alpha],[\beta]) \doteq (\alpha, G_{\perp}\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_{\perp}\right)$ *is pre-symplectomorphic to* $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Proof. See Appendix C, Prop. C.0.3 ∎

Remark 3.3.7. Notice that, on account of Propositions 3.3.4-3.3.6, $(\mathcal{O}_{\mathrm{nd}}, \widetilde{G}_{\perp})$ is a presymplectic proper subspace of $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_{\perp}\right)$ and therefore it is not symplectomorphic to $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Remark 3.3.8. Following [HS13, Cor. 5.3], σ_{t} (*resp.* σ_{nd}) do not define in general a symplectic form on the space of spacelike compact solutions $\mathrm{Sol}_{\mathrm{t}}(M)$ (*resp.* $\mathrm{Sol}_{\mathrm{nd}}(M)$). A direct characterization of this deficiency is best understood by introducing the following quotients:

$$\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{t}} := \frac{\{A\in\Omega^k_{\mathrm{sc}}(M)\mid \delta\mathrm{d}A=0\,,\ \mathrm{t}A=0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)\cap\Omega^k_{\mathrm{sc}}(M)}\,,\qquad \widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}} := \frac{\{A\in\Omega^k_{\mathrm{sc}}(M)\mid \delta\mathrm{d}A=0\,,\ \mathrm{nd}A=0\}}{\mathrm{d}\Omega^{k-1}(M)\cap\Omega^k_{\mathrm{sc}}(M)}\,, \tag{3.60}$$

Focusing on $\delta\mathrm{d}$-normal boundary conditions, it follows that $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}} \subseteq \widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$. Moreover, $\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ is symplectic with respect to the form $\sigma_{\mathrm{nd}}([A_1],[A_2]) = (\delta\mathrm{d}A_1^+, A_2)$. This can be shown as follows: if $\sigma_{\mathrm{nd}}([A_1],[A_2]) = 0$ for all $[A_1]\in\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ then, choosing $A_1 = G_{\perp}\alpha$ with $\alpha\in\Omega^k_{\mathrm{c,n},\delta}(M)$ leads to $0=\sigma_{\perp}([G_{\perp}\alpha],[A_2]) = (\alpha, A_2)$ – *cf.* Proposition 3.3.6. This entails

$\mathrm{d}A_2 = 0$ as well as $A_2 = 0 \in H_{k,\mathrm{c,n}}(M)^* \simeq H^k(M)$ – *cf.* Appendix A. Therefore $A_2 = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}(M)$ that is $A_2 = 0$ in $\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}(M)$. A similar result holds, mutatis mutandis, for $\parallel$.
The net result is that $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$ (*resp.* $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$) is symplectic if and only if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ (*resp.* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$). This is in agreement with the analysis in [BDS14] for the case of globally hyperbolic spacetimes (M, g) with $\partial M = \emptyset$.

Example 3.3.9. We give an example where $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subsetneq \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a proper inclusion – *cf.* [HS13, Ex. 5.7]. Consider half-Minkowski spacetime $\mathbb{R}^m_+ := \mathbb{R}^{m-1} \times \overline{\mathbb{R}_+}$ with flat metric and let $p \in \mathring{\mathbb{R}}^m_+$. We introduce $M := \mathbb{R}^m_+ \setminus J(p)$ endowed with the restriction to M of the Minkowski metric. This spacetime is still globally hyperbolic with timelike boundary. Let now $p \in B_1 \subset B_2$, where B_1, B_2 are open balls in $\mathbb{R}^{m-1}_+$ centered at p.
We consider $\psi \in \Omega^0(M)$ such that $\psi|_{J(B_1 \cap M)} = 1$ and $\psi|_{J(B_2 \cap M)} = 0$. In addition we introduce $\varphi \in \Omega^0_{\mathrm{tc}}(M)$ such that: (a) for all $x \in M$, $\varphi(x)$ depends only on $\tau(x)$ – *cf.* Theorem 1.1.2; (b) $\chi := \varphi\psi \in \Omega^0_{\mathrm{tc}}(M)$ is such that $\mathrm{t}\chi = \chi|_{\partial M} = 0$; (c) there exists an interval $I \subset \mathbb{R}$ such that $\varphi|_I = 1$.
In other words φ plays the rôle of a cut-off function so that $\chi \equiv \chi(\tau)$ does not vanish only for values of τ whose associated Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ is such that $(\Sigma_\tau \cap J(B_2)) \cap \partial M = \emptyset$. It follows that $\mathrm{d}\chi \in \Omega^1_{\mathrm{c}}(M) \subseteq \Omega^1_{\mathrm{sc}}(M)$. Yet there does not exist $\zeta \in \Omega^1_{\mathrm{sc}}(M)$ such that $\mathrm{d}\zeta = \mathrm{d}\chi$. Indeed, let us consider the curve $\gamma_s \subseteq M$ parametrized by $(s, x, 0, \dots) \in M$ where $s \in I \subset \mathbb{R}$ is such that $\varphi(s) = 1$ for all $s \in I$, while $x \in (x(p), +\infty)$ – $x(p)$ denotes the x-coordinate of p. Integration along γ_s yields

$$\int_{\gamma_s} \iota^*_{\gamma_s} \mathrm{d}\chi = -1 \,, \qquad \int_{\gamma_s} \iota^*_{\gamma_s} \mathrm{d}\zeta = 0 \,.$$

■

3.4 The algebra of observables for $\mathrm{Sol}_{\mathrm{t}}(M)$ and for $\mathrm{Sol}_{\mathrm{nd}}(M)$

In this section we prove that the generators in Equation (3.52) generate indeed optimal $*$-algebras of observables, whose elements are interpreted as the observables of the underlying quantum system. We study their key structural properties and we comment on their significance. On account of the different behaviour of $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions we discuss each algebra separately.

The algebra of observables for $\mathrm{Sol}_{\mathrm{t}}(M)$

Our analysis mimic that of [Ben16; DS13; HS13; SDH14] for globally hyperbolic spacetimes with empty boundary.
Following the discussion in Subsection 3.3.1, we prove that a unital $*$-algebra $\mathcal{A}_{\mathrm{t}}(M)$ built out of distinguished linear functionals over $\mathrm{Sol}_{\mathrm{t}}(M)$, whose collection is optimal when tested on configurations in $\mathrm{Sol}_{\mathrm{t}}(M)$, is of the form (3.55).
Taking into account the discussion in the preceding sections, particularly Equation (3.52), we introduce the following structures.

Definition 3.4.1. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{t}}(M)$, *the associative, unital $*$-algebra*

$$\mathcal{A}_{\mathrm{t}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]}\,, \qquad \mathcal{O}_{\mathrm{t}}(M) = \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}\,. \tag{3.61}$$

Here $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)] \doteq \bigoplus_{n=0}^{\infty}\mathcal{O}_{\mathrm{t}}(M)^{\otimes n}$ is the universal tensor algebra with $\mathcal{O}_{\mathrm{t}}(M)^{\otimes 0}\equiv\mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. In addition $\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]$ is the $*$-ideal generated by elements of the form $[\alpha]\otimes[\beta]-[\beta]\otimes[\alpha]-i\widetilde{G}_{\|}([\alpha],[\beta])\mathbb{I}$, where $[\alpha],[\beta]\in\mathcal{O}_{\mathrm{t}}(M)$ while $\widetilde{G}_{\|}$ is defined in Proposition 3.3.6 and $\mathbb{I}$ is the identity of $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]$.

As recalled in Section 3.3, the ideal is generated so that Canonical Commutation Relations are imposed:

$$[\alpha]\otimes[\beta]-[\beta]\otimes[\alpha] = i\widetilde{G}_{\|}([\alpha],[\beta])\mathbb{I},\, [\alpha],[\beta]\in\mathcal{A}_{\mathrm{t}}(M).$$

On account of its definition, to study the properties of the algebra it suffices to focus mainly on the properties of the generators $\mathcal{O}_{\mathrm{t}}(M)$. In particular, in the next proposition we follow the rationale advocated in [Ben16] proving that $\mathcal{O}_{\mathrm{t}}(M)$ is *optimal*:

Proposition 3.4.2. *Let $\mathcal{O}_{\mathrm{t}}(M)$ be as per Definition 3.4.1. Then, calling with $(\ ,\ )$ the natural pairing between $\mathcal{O}_{\mathrm{t}}(M)$ and $\mathrm{Sol}_{\mathrm{t}}(M)$ induced from that between k-forms, $\mathcal{O}_{\mathrm{t}}(M)$ is* **optimal**, *namely, recalling Definition 3.3.3:*

1. *$\mathcal{O}_{\mathrm{t}}(M)$ is* separating, *that is*

$$([\alpha],[A]) = 0 \quad \forall[\alpha]\in\mathcal{O}_{\mathrm{t}}(M) \Longrightarrow [A]=[0]\in\mathrm{Sol}_{\mathrm{t}}(M)\,. \tag{3.62}$$

2. $\mathcal{O}_\mathrm{t}(M)$ *is* non redundant, *that is*

$$([\alpha],[A]) = 0 \quad \forall [A] \in \mathrm{Sol_t}(M) \Longrightarrow [\alpha] = [0] \in \mathcal{O}_\mathrm{t}(M)\,. \tag{3.63}$$

Proof. As starting point observe that the pairing $([\alpha],[A]) := (\alpha, A)$ is well-defined. Indeed let consider two representatives $A \in [A] \in \mathrm{Sol_t}(M)$ and $\alpha \in [\alpha] \in \mathcal{O}_\mathrm{t}(M)$. The pairing (α, A) is finite being $\mathrm{supp}(\alpha)$ compact and there is no dependence on the choice of representative. As a matter of facts, if $\mathrm{d}\chi \in \mathrm{d}\Omega^{k-1}_\mathrm{t}(M)$ and $\eta \in \Omega^k_{\mathrm{c,t}}(M)$, it holds

$$(\alpha, \mathrm{d}\chi) = (\delta\alpha, \chi) + (\mathrm{n}\alpha, \mathrm{t}\chi)_\partial = 0\,, \qquad (\delta\mathrm{d}\eta, A) = (\eta, \delta\mathrm{d}A) + (\mathrm{t}\eta, \mathrm{n}\mathrm{d}A)_\partial - (\mathrm{n}\mathrm{d}\eta, \mathrm{t}A)_\partial = 0\,,$$

where in the first equation we used the fact that $\mathrm{t}\chi = 0$ as well as $\delta\alpha = 0$, while in the second equation we used $\delta\mathrm{d}A = 0$ as well as $\mathrm{t}A = \mathrm{t}\eta = 0$.
Having established that the pairing between the equivalence classes is well-defined we prove the remaining two items separately.

1. Assume $\exists [A] \in \mathrm{Sol_t}(M)$ such that $([\alpha],[A]) = 0$, $\forall [\alpha] \in \mathcal{O}_\mathrm{t}(M)$. Working at the level of representative, since $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$ we can choose $\alpha = \delta\beta$ with $\beta \in \Omega^{k+1}_\mathrm{c}(M)$. As a consequence $0 = (\delta\beta, A) = (\beta, \mathrm{d}A)$ where we used implicitly (1.6) and $\mathrm{t}A = 0$. The arbitrariness of β and the non-degeneracy of $(\ ,\ )$ entails $\mathrm{d}A = 0$. Hence A individuates a de Rham cohomology class in $H^k_\mathrm{t}(M)$, *cf.* Appendix A. Furthermore, $([\alpha],[A]) = 0$ entails $\langle[\alpha],[A]\rangle = 0$ where $\langle\ ,\ \rangle$ denotes the pairing between $H_{k,\mathrm{c}}(M)$ and $H^k_\mathrm{t}(M)$ – *cf.* Appendix A. On account of Remark A.0.4 it holds that $\langle\ ,\ \rangle$ is non-degenerate and therefore $[A] = 0$.

2. Assume $\exists [\alpha] \in \mathcal{O}_\mathrm{t}(M)$ such that $([\alpha],[A]) = 0\ \forall [A] \in \mathrm{Sol_t}(M)$. Working at the level of representatives, we can consider $A = G_\|\omega$ with $\omega \in \Omega^k_{\mathrm{c},\delta}(M)$, while $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$. Hence, in view of Proposition 3.1.19, $0 = (\alpha, A) = (\alpha, G_\|\omega) = -(G_\|\alpha, \omega)$. Choosing $\omega = \delta\beta$, $\beta \in \Omega^{k+1}_\mathrm{c}(M)$ and using (1.6), it descends $(\mathrm{d}G_\|\alpha, \beta) = 0$. Since β is arbitrary and the pairing is non degenerate $\mathrm{d}G_\|\alpha = 0$. Since $\mathrm{t}G_\|\alpha = 0$, it turns out that $G_\|\alpha$ individuates an equivalence class $[G_\|\alpha] \in H^k_\mathrm{t}(M)$. Using the same argument of the previous item, $(G_\|\alpha, \beta) = 0$ for all $\beta \in \Omega^k_{\mathrm{c},\delta}(M)$ entails that $G_\|\alpha = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}_\mathrm{t}(M)$. Proceeding as in proof of the injectivity of $G_\| \colon \mathcal{O}_\mathrm{t}(M) \to \mathrm{Sol_t}(M)$ – *cf.* Proposition 3.3.5 – it follows that $\alpha \in \delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)$ which is the sought conclusion. ■

The following corollary translates at the level of algebra of observables the degeneracy of the presymplectic spaces discussed in Proposition 3.3.6 – *cf.* Remark 3.3.8. As a matter of fact since $\widetilde{G}_\|$ can be degenerate, the algebra of observables $\mathcal{A}_\mathrm{t}(M)$ will possess a non-trivial centre. In other words

Corollary 3.4.3. *If* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_\mathrm{t}(M)$ *is a strict inclusion, then the algebra* $\mathcal{A}_\mathrm{t}(M)$ *is not semi-simple.*

Proof. With reference to Remark 3.3.8, if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ is a strict inclusion then there exists an element $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ such that $\sigma_{\mathrm{t}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$. On account of Proposition 3.3.5 there exists $[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$ such that $[G_{\|}\alpha] = [A]$. Moreover, Proposition 3.3.6 ensures that $\widetilde{G}_{\|}([\alpha],[\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{t}}(M)$. It follows from Definition 3.4.1 that $[\alpha]$ belongs to the center of $\mathcal{A}_{\mathrm{t}}(M)$, that is, $\mathcal{A}_{\mathrm{t}}(M)$ is not semi-simple. ■

Remark 3.4.4. Corollary 3.4.3 has established that the algebra of observables possesses a non trivial center. While from a mathematical viewpoint this feature might not appear of particular significance, it has far reaching consequences from the physical viewpoint. Most notably, the existence of Abelian ideals was first observed in the study of gauge theories in [DL12] leading to an obstruction in the interpretation of these models in the language of locally covariant quantum field theories as introduced in [BFV03]. This feature has been thoroughly studied in [Ben+14; BDS14; SDH14] turning out to be an intrinsic feature of Abelian gauge theories on globally hyperbolic spacetimes with empty boundary. Corollary 3.4.3 shows that the same conclusions can be drawn when the underlying manifold possesses a timelike boundary. In the next part of this section we will show that changing boundary condition does not alter the outcome.

The algebra of observable for $\mathrm{Sol}_{\mathrm{nd}}(M)$

We focus now on $\mathcal{A}_{\mathrm{nd}}(M)$, the algebra of observables associated to the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. Similarly to Definition 3.4.1, $\mathcal{A}_{\mathrm{nd}}(M)$ will be defined as a suitable quotient of the universal tensor algebra over a vector space $\mathcal{O}_{\mathrm{nd}}(M)$. However, contrary to the case of δd-tangential boundary conditions, in the case of δd-normal boundary conditions, $\mathcal{O}_{\mathrm{nd}}(M)$ will not be simplectomorphic to the configuration space $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ – *cf.* Definition 3.4.5 and Proposition 3.3.4. Nevertheless the results of Propositions 3.4.2 and 3.4.3 still hold true for $\mathcal{A}_{\mathrm{nd}}(M)$. In the last part of this section we point out another possible choice for the algebra of observables whose underlying vector space is simplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ but which requires an a priori gauge fixing.
Taking into account in particular Equation (3.52), we define

Definition 3.4.5. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{nd}}(M)$*, the associative, unital $*$-algebra*

$$\mathcal{A}_{\mathrm{nd}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]}\,, \qquad \mathcal{O}_{\mathrm{nd}}(M) = \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}\,. \tag{3.64}$$

where $\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)] \doteq \bigoplus_{n=0}^{\infty} \mathcal{O}_{\mathrm{nd}}(M)^{\otimes n}$ is the universal tensor algebra with $\mathcal{O}_{\mathrm{nd}}(M)^{\otimes 0} \equiv \mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. In addition $\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]$ is the $*$-ideal generated by elements of the form $[\alpha]\otimes[\beta] - [\beta]\otimes[\alpha] - i\widetilde{G}_{\perp}([\alpha],[\beta])\mathbb{I}$, where $[\alpha],[\beta] \in \mathcal{O}_{\mathrm{nd}}(M)$ while $\widetilde{G}_{\perp}$ is defined in Proposition 3.3.6 and $\mathbb{I}$ is the identity of $\mathcal{O}_{\mathrm{nd}}(M)$.

Starting from Definition 3.4.5 we can repeat, mutatis mutandis, the proof of Proposition 3.4.2.

Proposition 3.4.6. *Let* $\mathcal{O}_{\mathrm{nd}}(M)$ *be as per Definition 3.4.5. Then, calling with* $(\ ,\ )$ *the natural pairing between* $\mathcal{O}_{\mathrm{nd}}(M)$ *and* $\mathrm{Sol}_{\mathrm{nd}}(M)$ *induced from those between* k*-forms,* $\mathcal{O}_{\mathrm{nd}}(M)$ *is* **optimal**, *namely*

1. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* separating, *that is*

$$([\alpha],[A]) = 0 \quad \forall[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M) \Longrightarrow [A] = [0] \in \mathrm{Sol}_{\mathrm{nd}}(M)\,. \tag{3.65}$$

2. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* optimal, *that is*

$$([\alpha],[A]) = 0 \quad \forall[A] \in \mathrm{Sol}_{\mathrm{nd}}(M) \Longrightarrow [\alpha] = [0] \in \mathcal{O}_{\mathrm{nd}}(M)\,, \tag{3.66}$$

Proof. The fact that the pairing $([\alpha],[A])$ is well-defined for $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$ and $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ has already been discussed in Remark 3.3.7.
We prove the first of the two items: let $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ be such that $([\alpha],[A]) = 0$ for all $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$. This implies that $(\alpha, A) = 0$ for all $A \in [A]$ and for all $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$. Taking in particular $\alpha = \delta\beta$ with $\beta \in \Omega^k_{\mathrm{c,n}}(M)$ it follows $(\mathrm{d}A, \beta) = 0$. The non-degeneracy of $(\ ,\ )$ implies $\mathrm{d}A = 0$, that is A defines an element in $H^k(M)$. Moreover, the hypothesis on A implies that $\langle A, [\eta]\rangle = 0$ for all $[\eta] \in H_{k,\mathrm{c,n}}(M)$. The results in Appendix A – *cf.* Remark A.0.4 – ensure that $A = \mathrm{d}\chi$, therefore $[A] = [0] \in \mathrm{Sol}_{\mathrm{nd}}(M)$.
Regarding the second statement, let $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$ be such that $([\alpha],[A]) = 0$ for all $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. This implies in particular that, choosing $\alpha \in [\alpha]$ and $A = G_\perp\beta$ with $\beta \in \Omega^k_{\mathrm{c},\delta}(M)$, $0 = (\alpha, G_\perp\beta) = -(G_\perp\alpha, \beta)$. With the same argument of the first statement it follows that $G_\perp\alpha = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}(M)$ is such that $\mathrm{n}\mathrm{d}\chi = 0$. Proceeding as in the proof of Proposition 3.3.5 it follows that $[\alpha] = [0] \in \mathcal{O}_{\mathrm{nd}}(M)$. ■

The following corollary is analogous to Corollary 3.4.3. The proof is slightly different since in this case there does not exist a simplectomorphism between $\mathcal{O}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ – *cf.* Proposition 3.3.5 and Remark 3.3.7. The crucial part in the proof is to show that if $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ is degenerate with respect to σ_{nd}, then $[A] \in G_\perp\mathcal{O}_{\mathrm{nd}}(M)$.

Corollary 3.4.7. *If* $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ *is a strict inclusion, then the algebra* $\mathcal{A}_{\mathrm{nd}}(M)$ *is not semi-simple.*

Proof. On account of Remark 3.3.8, if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a strict inclusion then there exists $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ such that $\sigma_{\mathrm{nd}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. In particular we have $[A] = [\mathrm{d}\chi]$ where $\chi \in \Omega^{k-1}(M) \setminus \Omega^{k-1}_{\mathrm{sc}}(M)$ is such that $\mathrm{d}\chi \in \Omega^k_{\mathrm{sc}}(M)$.
We now prove that, up to an element in $\mathrm{d}\Omega^k_{\mathrm{sc}}(M)$, $\mathrm{d}\chi = G_\perp\alpha$ with $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$: On account of Proposition 3.3.6 it follows that $\widetilde{G}_\perp([\alpha],[\beta]) = \sigma_{\mathrm{nd}}([\mathrm{d}\chi],[G_\perp\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{nd}}(M)$. Definition 3.4.5 implies that $[\alpha] \in \mathcal{A}_{\mathrm{nd}}(M)$ lies in the center of $\mathcal{A}_{\mathrm{nd}}(M)$ which is therefore not semi-simple.

On account of Proposition 3.3.5 we have that $\mathrm{d}\chi = G_\perp\alpha + \mathrm{d}\eta$, where $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$ while $\eta \in \Omega^{k-1}_{\mathrm{sc}}(M)$. By redefining $\chi_\eta \doteq \chi - \eta$ we have $\mathrm{d}\chi_\eta = G_\perp\alpha$. Notice that this last redefinition does not spoil the property $\chi_\eta \in \Omega^{k-1}(M) \setminus \Omega^{k-1}_{\mathrm{sc}}(M)$ while $\mathrm{d}\chi_\eta \in \Omega^k_{\mathrm{sc}}(M)$ thus $\sigma_{\mathrm{nd}}([\mathrm{d}\chi_\eta], [B]) = 0$ for all $[B] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$.
The boundary conditions on $G_\perp\alpha$ implies that $\mathrm{n}\mathrm{d}\chi_\eta = \mathrm{n}G_\perp\alpha = 0$, while Corollary 3.1.23 ensures that $\delta\mathrm{d}\chi_\eta = \delta G_\perp\alpha = G_\perp\delta\alpha = 0$. It then follows that $\chi_\eta \in \mathrm{Sol}_{\mathrm{nd}}(M)$ – in degree $k-1$ – and therefore Proposition 3.3.5 entails $\chi_\eta = G_\perp\beta$ where $\beta \in \Omega^{k-1}_{\mathrm{tc},\delta}(M)$. Summing up we have $\mathrm{d}\chi_\eta = G_\perp\alpha$ as well as $\mathrm{d}\chi_\eta = G_\perp\mathrm{d}\beta$. Proposition 3.1.19 and Remark 3.1.20 imply that $\mathrm{d}\beta - \alpha = \Box_\perp\zeta$, being $\zeta \in \Omega^k_{\mathrm{tc},\perp}(M)$. Applying δ to the last equality we obtain

$$\Box\delta\zeta = \delta\Box_\perp\zeta = \delta\mathrm{d}\beta - \delta\alpha = \Box\beta\,.$$

Remark 3.1.21 ensures that $\delta\zeta = \beta$ and therefore $\alpha = -\delta\mathrm{d}\zeta$. Since $\zeta \in \Omega^k_\perp(M)$ it follows that $\alpha \in \Omega^k_{\mathrm{c},\mathrm{n},\delta}(M)$. ■

3.4.1 An alternative algebra for $\delta\mathrm{d}$-normal boundary conditions

Definition 3.4.5 identifies an algebra $\mathcal{A}_{\mathrm{nd}}(M)$ which is separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. It also satisfies most of the properties of the analogous algebra $\mathcal{A}_{\mathrm{t}}(M)$ – *cf.* Proposition 3.4.6 and Corollary 3.4.7. However, as pointed out in Remark 3.3.7, the underlying vector space $\mathcal{O}_{\mathrm{nd}}(M)$ is only a proper presymplectic subspace of $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$. This is contrary to the case of $\delta\mathrm{d}$-tangential boundary conditions where the vector space $\mathcal{O}_{\mathrm{t}}(M)$ is symplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ – *cf.* Proposition 3.3.5.
It is thus worth investigating whether there exists a different algebra $\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ still defined as a suitable quotient – *cf.* Definitions 3.4.1-3.4.5 – of the universal tensor algebra of a presymplectic vector space $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ which is presymplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$. For consistency, $\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ should be built out of a separating and non redundant collection of functionals for $\mathrm{Sol}_{\mathrm{nd}}(M)$ and the superscript gf refers to "gauge fixing" as it will become clear from the following discussion. To this end and with reference to Proposition 3.3.5, $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ can be identified as

$$\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M) \doteq \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\mathrm{nd}}(M)}\,.$$

As shown in Propositions 3.3.5-3.3.6, $(\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M), \widetilde{G}_\perp)$ is a presymplectic vector space which is symplectomorphic to $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$. We can thus set

$$\mathcal{A}^{\mathrm{gf}}_{\mathrm{nd}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)]}{\mathcal{I}[\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)]}\,,$$

where we refer to Definition 3.4.5 for details.
The discussion about $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ being separating and non redundant is more subtle. Indeed, the pairing between elements $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ is not well-defined – *cf.* Remark

3.3.7. However we can exploit the isomorphism identified in equation (3.51). With reference to equation (3.50), we denote with γ_{nd} the isomorphism

$$\gamma_{\mathrm{nd}}\colon\ \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathcal{S}_{\mathcal{G}_{\mathrm{nd}}}(M)\,.$$

It follows that for all $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ the following functional is well-defined:

$$F_{\gamma^*_{\mathrm{nd}}[\alpha]}\colon\ \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathbb{C}\,, \qquad F_{\gamma^*_{\mathrm{nd}}[\alpha]}([A]) := ([\alpha],[\gamma_{\mathrm{nd}}A])\,.$$

Notice that the gauge-invariance of $F_{\gamma^*_{\mathrm{nd}}[\alpha]}$ is guaranteed by the combined action of γ_{nd}, which selects a "gauge-fixed" representative $\gamma_{\mathrm{nd}}A \in [A]$, and of $[\alpha]$, which remains un-effected by the residual gauge present in the choice of $\gamma_{\mathrm{nd}}A$, *i.e.* $([\alpha], \mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)) = 0$ – *cf.* Equation (3.48).
With this observation it holds that, introducing the "gauge-fixed" pairing $([\alpha],[A])_{\gamma_{\mathrm{nd}}} \doteq ([\alpha],[\gamma_{\mathrm{nd}}A])$ between $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}(M)$, the vector space $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ is indeed separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. The proof is similar to the one of Propositions 3.4.2-3.4.6 and we shall not repeat it.

Appendix A

Poincaré-Lefschetz duality for manifold with boundary

In this appendix we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non-empty boundary. A reader interested in more details can refer to [BT13; Sch95].

For the purpose of this appendix M refers to a smooth, oriented manifold of dimension $\dim M = m$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : \partial M \to M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work also with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_{\mathrm{c}}(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\ker(\mathrm{d}_k \colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d}_{k-1} \colon \Omega^{k-1}(M) \to \Omega^k(M))}, \tag{A.1}$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (1.4) and (1.5b) entail that we can define $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_{\mathrm{t}}(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M) \doteq \frac{\ker(\delta_k \colon \Omega^k(M) \to \Omega^{k-1}(M))}{\mathrm{Im}(\delta_{k+1} \colon \Omega^{k+1}(M) \to \Omega^k(M))}.$$

Equations (1.4) and (1.5b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M)$ (*resp.* $\Omega^\bullet_{\mathrm{c}}(M)$, $\Omega^\bullet_{\mathrm{c,n}}(M)$), the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$ (*resp.* $\Omega^k_{\mathrm{c}}(M)$, $\Omega^k_{\mathrm{c,n}}(M) \subseteq \Omega^k(M)$). The associated homology groups will be denoted as $H_{k,\mathrm{n}}(M)$ (*resp.* $H_{k,\mathrm{c}}(M)$,

$H_{k,\mathrm{c,n}}(M)$), $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition and on account of equation (1.5), the Hodge operator induces an isomorphism $H^k(M) \simeq H_{m-k}(M)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [\star\alpha] \in H_{m-k}(M)$. Similarly, on account of Equation (1.5b), it holds $H^k_{\mathrm{t}}(M) \simeq H_{m-k,\mathrm{n}}(M)$ and $H^k_{\mathrm{c,t}}(M) \simeq H_{m-k,\mathrm{c,n}}(M)$.
As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT13]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M;\partial M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to

$$\Omega^k(M;\partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M;\partial M) \to \Omega^{k+1}(M;\partial M)$ such that for any $(\omega,\theta) \in \Omega^k(M;\partial M)$

$$\underline{\mathrm{d}}_k(\omega,\theta) = (\mathrm{d}_k\omega, \mathrm{t}\omega - \mathrm{d}_{k-1}\theta)\,. \tag{A.2}$$

Per construction, each $\Omega^k(M;\partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M;\partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M;\partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will always be clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M;\partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT13, Prop. 6.49]:

Proposition A.0.1. *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M;\partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\mathrm{t}_*} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M;\partial M) \to \ldots, \tag{A.3}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and t_* indicate the natural counterpart of the maps π_1, π_2 and t at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition A.0.2. *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between $H^k_{\mathrm{t}}(M)$ and $H^k(M;\partial M)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Consider $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}$ and let $(\omega, 0) \in \Omega^k(M;\partial M)$, $k \in \mathbb{N} \cup \{0\}$. Equation (A.2) entails

$$\underline{\mathrm{d}}_k(\omega,0) = (\mathrm{d}_k\omega, \mathrm{t}\omega) = (0,0)\,.$$

At the same time, if $\omega = \mathrm{d}_{k-1}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $(\mathrm{d}_{k-1}\beta, 0) = \underline{\mathrm{d}}_{k-1}(\beta, 0)$. Hence the embedding $\omega \mapsto (\omega, 0)$ identifies a map $\rho : H^k_{\mathrm{t}}(M) \to H^k(M;\partial M)$ such that $\rho([\omega]) \doteq$

$[(\omega, 0)]$. To conclude, we need to prove that ρ is surjective and injective. Let thus $[(\omega', \theta)] \in H^k(M; \partial M)$. It holds that $\mathrm{d}_k\omega' = 0$ and $\mathrm{t}\omega' - \mathrm{d}_{k-1}\theta = 0$. Recalling that $\mathrm{t} : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective – *cf.* Remark 1.2.5 – for all values of $k \in \mathbb{N} \cup \{0\}$, there must exists $\eta \in \Omega^{k-1}(M)$ such that $\mathrm{t}\eta = \theta$. Let $\omega \doteq \omega' - \mathrm{d}_{k-1}\eta$. On account of (1.5b) $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}_k$ and $(\omega, 0)$ is a representative if $[(\omega', \theta)]$ which entails that ρ is surjective.
Let $[\omega] \in H^k(M)$ be such that $\rho[\omega] = [0] \in H^k(M; \partial M)$. This implies that there exists $\beta \in \Omega^{k-1}(M)$, $\theta \in \Omega^{k-2}(\partial M)$ such that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}(\beta, \theta) = (\mathrm{d}_{k-1}\beta, \mathrm{t}\beta - \mathrm{d}_{k-2}\theta) .$$

Let $\eta \in \Omega^{k-2}(M)$ be such that $\mathrm{t}\eta + \theta = 0$. It follows that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}\left((\beta, \theta) + \underline{\mathrm{d}}_{k-2}(\eta, 0)\right) = \underline{\mathrm{d}}_{k-1}(\beta + \mathrm{d}_{k-2}\eta, 0) .$$

This entails that $\omega = \mathrm{d}_{k-1}(\beta + \mathrm{d}_{k-2}\eta)$ where $\mathrm{t}(\beta + \mathrm{d}_{k-2}\eta) = 0$. It follows that $[\omega] = 0$ that is ρ is injective.

■

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau96]:

Theorem A.0.3. *Under the geometric assumptions specified at the beginning of the section and assuming in addition that M admits a finite good cover, it holds that, for all $k \in \mathbb{N} \cup \{0\}$*

$$H^{m-k}(M; \partial M) \simeq H^k_{\mathrm{c}}(M)^* , \qquad [\alpha] \to \left(H^k_{\mathrm{c}}(M) \ni [\eta] \mapsto \int_M \overline{\alpha} \wedge \eta \in \mathbb{C} \right) . \tag{A.4}$$

where $m = \dim M$ and where on the right hand side we consider the dual of the $(m-k)$-th cohomology group built out compactly supported forms.

Remark A.0.4. On account of Propositions A.0.2-A.0.3 and of the isomorphisms $H^k_{(\mathrm{c})}(M) \simeq H^{m-k}_{(\mathrm{c})}(M)$ the following are isomorphisms:

$$H^k_{\mathrm{t}}(M) \simeq H^{m-k}_{\mathrm{c}}(M)^* \simeq H_{k,\mathrm{c}}(M)^* , \qquad H^k(M) \simeq H_{k,\mathrm{c,n}}(M)^* . \tag{A.5}$$

The proof proceeds in some steps. Let $\iota : \partial M \to M$ be the immersion map. First of all we have to check that the spaces are finite-dimensional and that the pairing $\langle \, , \rangle : H^{m-k}(M) \otimes H^k_{\mathrm{c}}(M, \partial M)$ defined by

$$\langle \alpha, (\omega, \theta) \rangle := \int_M \alpha \wedge \omega + \int_{\partial M} \iota^* \alpha \wedge \theta \qquad \forall \alpha \in H^{m-k}(M) \text{ and } (\omega, \theta) \in H^k_{\mathrm{c}}(M, \partial M) , \tag{A.6}$$

is non-degenerate, equivalently the map $\alpha \to \langle \alpha, \cdot \rangle$ should be an isomorphism.

Since a manifold M with boundary is locally homeomorphic to $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ we need Poincaré lemmas for $\mathbb{R}^m_+$.

Lemma A.0.5 (Poincaré lemmas for half spaces)**.** *Let* $\mathbb{R}^m_+ := \{(x_1, \ldots, x_n) \,|\, x_1 \geq 0\}$ *and* $k \geq 0$*. Then*

$$H^k(\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \{0\} & \text{otherwise} \end{cases} \tag{A.7}$$

$$H^k_c(\mathbb{R}^m_+, \partial\mathbb{R}^m_+) \simeq \begin{cases} \mathbb{R} & \text{if } k = n \\ \{0\} & \text{otherwise} \end{cases} \tag{A.8}$$

Proof. The proof for the case $n = 1$, i.e. $\mathbb{R}_+ = [0, +\infty)$ is straightforward and the n-dimensional generalisation is obtained as in ([BT13, Sec. 4]). ■

Lemma A.0.6 (Mayer-Vietoris sequences)**.** *Let M be an orientable manifold with boundary ∂M, suppose $M = U \cup V$ with U, V open and denote $\partial M_A := \partial M \cap A$. Then the following are exact sequences:*

$$\cdots \to H^k(M, \partial M) \to H^k(U, \partial M_U) \oplus H^k(V, \partial M_V) \to H^k(U \cap V, \partial M_{U\cap V}) \to H^{k+1}(M, \partial M) \to \cdots \tag{A.9}$$

$$\cdots \leftarrow H^k_c(M, \partial M) \leftarrow H^k_c(U, \partial M_U) \oplus H^k(V, \partial M_V) \leftarrow H^k_c(U \cap V, \partial M_{U\cap V}) \leftarrow H^{k+1}_c(M, \partial M) \leftarrow \cdots \tag{A.10}$$

Proof. We will only prove the non-compact cohomology line.
We have the following Mayer-Vietoris short exact sequences for M and ∂M:

$$0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V) \longrightarrow 0$$
$$0 \to \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M_U) \oplus \Omega^{k-1}(\partial M_V) \to \Omega^{k-1}(\partial M_{U\cap V}) \to 0.$$

Hence applying the direct sum between the two sequences we obtain

$$0 \longrightarrow \Omega^k(M, \partial M) \longrightarrow \Omega^k(U, \partial M_U) \oplus \Omega^k(V, \partial M_V) \longrightarrow \Omega^k(U \cap V, \partial M_{U\cap V}) \longrightarrow 0.$$

The last row induces the desired long sequence because of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(M,\partial M) \arrow[r] \arrow[d, "d"] & \Omega^k(U,\partial M_U)\oplus\Omega^k(V,\partial M_V) \arrow[r] \arrow[d, "\mathrm{d}:=d\oplus d"] & \Omega^k(U\cap V,\partial M_{U\cap V}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(M,\partial M) \arrow[r] & \Omega^{k+1}(U,\partial M_U)\oplus\Omega^{k+1}(V,\partial M_V) \arrow[r] & \Omega^{k+1}(U\cap V,\partial M_{U\cap V}) \arrow[r] & 0
\end{tikzcd} \tag{A.11}$$

following the arguments in [BT13], section 2. Fix a closed form $\omega\in\Omega^k(U\cap V,\partial M_{U\cap V})$, since the first row is exact there exists a unique $\xi\in\Omega^{k+1}(M,\partial M)$ which is mapped to ω. Now, since $\mathrm{d}\omega=0$ and the diagram is commutative $\mathrm{d}\xi$ is mapped to 0. Hence from the exactness of the second row there exists χ which is mapped to $\mathrm{d}\xi$ and it easy to see χ is closed. ∎

Lemma A.0.7. *If the manifold with boundary M has a* finite good cover *(see [BT13, Sec. 5]) then its (relative) cohomology and (relative) compact cohomology is finite dimensional.*

Proof. The proof is based on the existence of a Mayer-Vietoris sequence in any of the desired cases (proved in the previous proposition) and follows the outline of [BT13, Prop. 5.3.1]. ∎

Lemma A.0.8 (Five lemma)**.** *Given the commutative diagram with exact rows*

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "r"] & E \arrow[r] \arrow[d, "s"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & \cdots
\end{tikzcd} \tag{A.12}$$

if f,g,h,s are isomorphism, then so is r.

Lemma A.0.9. *Suppose $M=U\cup V$ with U,V open. The pairing (A.6) induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^{m-k}(M) \arrow[r] \arrow[d] & H^{m-k}(U)\oplus H^{m-k}(V) \arrow[r] \arrow[d] & H^{m-k+1}(M) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^k(M,\partial M)^* \arrow[r] & H^k(U,\partial M_U)^*\oplus H^k(V,\partial M_V)^* \arrow[r] & H^{k-1}(M)^* \arrow[r] & \cdots
\end{tikzcd} \tag{A.13}$$

Proof. The proof follows that of [BT13, Lem. 5.6]. ∎

Now we are ready to prove the main theorem of this section:

Proof of Poincaré-Lefschetz Duality. Follow the argument given in [BT13, Sec. 5]. By the Five lemma if Poincaré-Lefschetz duality holds for U,V and $U\cap V$, then it holds for $U\cup V$. Then it is sufficient to proceed by induction on the cardinality of a finite good cover. □

Appendix B

An explicit computation

Lemma B.0.1. *Let $M = \mathbb{R}\times\Sigma$ be a globally hyperbolic spacetime – cf. Theorem ??. Moreover, for all $\tau \in \mathbb{R}$, let $\mathrm{t}_{\Sigma_\tau} : \Omega^k(M) \to \Omega^k(\Sigma_\tau)$, $\mathrm{n}_{\Sigma_\tau} : \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\Sigma)$ be the tangential and normal maps on $\Sigma_\tau \doteq \{\tau\} \times \Sigma$, where $M = \mathbb{R} \times \Sigma$ – cf. Definition 1.2.3. Moreover, let $\mathrm{t}_{\partial\Sigma_\tau} : \Omega^k(\Sigma_\tau) \to \Omega^k(\partial\Sigma_\tau)$ and let $\mathrm{n}_{\partial\Sigma_\tau} : \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ be the tangential and normal maps on $\partial\Sigma_\tau \doteq \{\tau\} \times \partial\Sigma$. Let $f \in C^\infty(\partial\Sigma)$ and set $f_\tau \doteq f|_{\partial\Sigma_\tau}$. Then for $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$ it holds*

$$\omega \in \Omega^k_\sharp(M) \Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \Omega^k_\sharp(\Sigma_\tau) \quad \forall\tau \in \mathbb{R}\,. \tag{B.1}$$

More precisely this entails that

$$\begin{aligned}
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker \mathrm{n}_{\partial M} &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker \mathrm{n}_{\partial\Sigma_\tau}\,, \forall\tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{n}_{\partial M} \cap \ker \mathrm{n}_{\partial M}\mathrm{d} &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{n}_{\partial\Sigma_\tau} \cap \ker \mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\,, \forall\tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker \mathrm{t}_{\partial M}\delta &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker \mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}\,, \forall\tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{n}_{\partial M} \cap \ker(\mathrm{n}_{\partial M}\mathrm{d} - f\mathrm{t}_{\partial M}) &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{n}_{\partial\Sigma_\tau} \cap \ker(\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau} - f_t\mathrm{t}_{\partial\Sigma_\tau})\,, \forall\tau \in \mathbb{R}\,;\\
\omega \in \ker \mathrm{t}_{\partial M} \cap \ker(\mathrm{t}_{\partial M}\delta - f\mathrm{n}_{\partial M}) &\Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \ker \mathrm{t}_{\partial\Sigma_\tau} \cap \ker(\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau} - f_t\mathrm{n}_{\partial\Sigma_\tau})\,, \forall t \in \mathbb{R}\,.
\end{aligned}$$

Proof. The equivalence (B.1) is shown for $\perp$-boundary condition. The proof for $\|$-boundary conditions follows per duality – *cf.* (3.1.3) – while the one for D-, $f_\|$-, $f_\perp$-boundary conditions can be carried out in a similar way.
On account of Theorem 1.1.2 we have that for all $\tau \in \mathbb{R}$ we can decompose any $\omega \in \Omega^k(M)$ as follows:

$$\omega|_{\Sigma_\tau} = \mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau\,.$$

Notice that, being the decomposition $M = \mathbb{R}\times\Sigma$ smooth we have that $\tau \to \mathrm{t}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}, \Omega^k(\Sigma))$ while $\tau \to \mathrm{n}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma))$. Here we have implicitly identified $\Sigma \simeq \Sigma_\tau$.
A similar decomposition holds near the boundary of Σ_τ. Indeed for all $(\tau, p) \in \{\tau\} \times \partial\Sigma$ we consider a neighbourhood of the form $U = [0, \epsilon_\tau) \times U_{\partial\Sigma}$. Let $U_x \doteq \{x\} \times U_{\partial\Sigma}$ for $x \in [0, \epsilon_\tau)$ and let t_{U_x}, n_{U_x} be the corresponding tangential and normal maps – *cf.* Definition 1.2.3. With

this definition we can always split $\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{\Sigma_\tau}\omega$ as follows:

$$\omega|_U = \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \wedge \mathrm{d}x + \mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}x \wedge \mathrm{d}\tau\,. \tag{B.2}$$

If p ranges on a compact set of $\partial\Sigma$ it follows that $(\tau, x) \to \mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega \in C^\infty(\mathbb{R}\times[0,\epsilon), \Omega^k(\partial\Sigma))$ and similarly $\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$, $\mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$. Once again we have implicitly identified $U_{\partial\Sigma} \simeq \{x\}\times U_{\partial\Sigma}$.
According to this splitting we have

$$\begin{aligned}
\mathrm{t}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{t}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau = \mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,,\\
\mathrm{n}_{\partial M}\omega|_{(\tau,p)} &= \mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau = \mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega + \mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,.
\end{aligned}$$

It follows that $\mathrm{n}_{\partial M}\omega = 0$ if and only if $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and similarly $\mathrm{t}_{\partial M}\omega = 0$ if and only if $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$. This proves the thesis for Dirichlet boundary conditions. A similar computation leads to

$$\begin{aligned}
\mathrm{n}_{\partial M}\mathrm{d}\omega = (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega + (-1)^{k-1}\partial_\tau\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\\
+ (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0}\wedge\mathrm{d}\tau - \mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\\
= (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} + (-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0}\wedge\mathrm{d}\tau\,.
\end{aligned}$$

where the second equality holds true since $\mathrm{n}_{\partial M}\omega = 0$. It follows that $\mathrm{n}_{\partial M}\mathrm{d}\omega = 0$ if and only if $\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0} = 0$ and $\partial_x\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0} = 0$. When $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ the latter conditions are equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$. ∎

Finally, we prove a very useful Lemma.

Lemma B.0.2. *Let $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$, with $f \in C^\infty(\partial M)$. The following statements hold true:*

1. *for all $\omega \in \Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{spc}}(M)\cap\Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{sfc}}(M)\cap\Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

2. *for all $\omega \in \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{fc}}(M)\cap\Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

Proof. We prove the result in the first case, the second one can be proved in complete analogy. Let $\omega \in \Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_\sharp(M)$. Consider Σ_1, Σ_2, two Cauchy surfaces on M – *cf.* [AFS18, Def. 3.10] – such that $J^+(\Sigma_1) \subset J^+(\Sigma_2)$. Moreover, let $\varphi_+ \in \Omega^0_{\mathrm{pc}}(M)$ be such that $\varphi_+|_{J^+(\Sigma_2)} = 1$ and $\varphi_+|_{J^-(\Sigma_1)} = 0$. We define $\varphi_- := 1 - \varphi_+ \in \Omega^0_{\mathrm{fc}}(M)$. Notice that we can always choose φ so that, for all $x \in M$, $\varphi(x)$ depends only on the value $\tau(x)$, where τ is the global time function defined in Theorem 1.1.2. We set $\omega_\pm \doteq \varphi_\pm\omega$ so that $\omega^+ \in \Omega^k_{\mathrm{spc}}(M)\cap\Omega^k_\sharp(M)$ while

$\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$. This is automatic for $\sharp = \mathrm{D}$ on account of the equality

$$\mathrm{t}\omega^\pm = \varphi_\pm \mathrm{t}\omega = 0\,, \qquad \mathrm{n}\omega^\pm = \varphi_\pm \mathrm{n}\omega = 0\,.$$

We now check that $\omega^\pm \in \Omega^k_\sharp(M)$ for $\sharp =\perp$. The proof for the remaining boundary conditions $\perp, f_\|, f_\perp$ follows by a similar computation – or by duality *cf.* Remark 3.1.3. It holds

$$\mathrm{n}\omega_\pm = \varphi_\pm|_{\partial M}\mathrm{n}\omega = 0\,, \qquad \mathrm{n}\mathrm{d}\omega_\pm = \mathrm{n}(\mathrm{d}\chi \wedge \omega) = \partial_\tau\chi\, \mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0\,.$$

In the last equality $\mathrm{t}_{\Sigma_\tau} \colon \Omega^k(M) \to \Omega^k(\Sigma_\tau)$ and $\mathrm{n}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ are the maps from Definition 1.2.3 with $N \equiv \Sigma_\tau \doteq \{\tau\} \times \Sigma$, where $M = \mathbb{R} \times \Sigma$. The last identity follows because the condition $\mathrm{n}\omega = 0$ is equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega = 0$ for all $\tau \in \mathbb{R}$ – *cf.* Lemma B.0.1. ■

Appendix C

Proofs of statements in Subsection 3.3.1

Proposition C.0.1. *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. Let $[A_1],[A_2]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and, for $A_1\in[A_1]$, let $A_1=A_1^+ + A_1^-$ be any decomposition such that $A^+\in\Omega^k_{\mathrm{spc,t}}(M)$ while $A^-\in\Omega^k_{\mathrm{sfc,t}}(M)$ – cf. Lemma B.0.2. Then the following map $\sigma_{\mathrm{t}}\colon\ \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)^{\times 2}\to\mathbb{R}$ is a presymplectic form:*

$$\sigma_{\mathrm{t}}([A_1],[A_2])=(\delta\mathrm{d}A_1^+,A_2)\,,\qquad\forall[A_1],[A_2]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,. \tag{C.1}$$

A similar result holds for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ and we denote the associated presymplectic form σ_{nd}. In particular for all $[A_1],[A_2]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ we have $\sigma_{\mathrm{nd}}([A_1],[A_2])\doteq(\delta\mathrm{d}A_1^+,A_2)$ where $A_1\in[A_1]$ is such that $A\in\Omega^k_{\mathrm{sc},\perp}(M)$.

Proof. We shall prove the result for σ_{nd}, the proof for σ_{t} being the same mutatis mutandis. First of all notice that for all $[A]\in\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ there exists $A'\in[A]$ such that $A'\in\Omega^k_\perp(M)$. This is realized by picking an arbitrary $A\in[A]$ and defining $A'\doteq A+\mathrm{d}\chi$ where $\chi\in\Omega^{k-1}_{\mathrm{sc}}(M)$ is such that $\mathrm{n}\mathrm{d}\chi=-\mathrm{n}A$ – *cf.* Remark 1.2.5. We can thus apply Lemma B.0.2 in order to split $A'=A'_+ + A'_-$ where $A'_+\in\Omega^k_{\mathrm{spc,nd}}(M)$ and $A'_-\in\Omega^k_{\mathrm{sfc,nd}}(M)$. Notice that this procedure is not necessary for $\delta\mathrm{d}$-tangential boundary condition since we can always split $A\in\Omega^k_{\mathrm{sc,t}}(M)$ as $A=A^+ + A^-$ with $A_+\in\Omega^k_{\mathrm{spc,t}}(M)$ and $A_-\in\Omega^k_{\mathrm{sfc,t}}(M)$ without invoking Lemma B.0.2. After these preliminary observations consider the map

$$\sigma_{\mathrm{nd}}\colon(\ker\delta\mathrm{d}\cap\Omega^k_{\mathrm{sc},\perp}(M))^{\times 2}\ni(A_1,A_2)\mapsto(\delta\mathrm{d}A_1^+,A_2)\,,$$

where we used Lemma B.0.2 and we split $A_1=A_1^+ + A_1^-$, with $A_1^+\in\Omega^k_{\mathrm{spc},\perp}(M)$ while $A_1^-\in\Omega^k_{\mathrm{sfc},\perp}(M)$. The pairing $(\delta\mathrm{d}A_1^+,A_2)$ is finite because A_2 is a spacelike compact k-form while $\delta\mathrm{d}A_1^+$ is compactly supported on account of A_1 being on-shell. Moreover, $(\delta\mathrm{d}A_1^+,A_2)$ is independent from the splitting $A_1=A_1^+ + A_1^-$ and thus σ_{nd} is well-defined. Indeed, let $A_1=\widetilde{A}_1^+ + \widetilde{A}_1^-$ be another splitting: it follows that $A_1^+-\widetilde{A}_1^+=-(A_1^- - \widetilde{A}_1^-)\in\Omega^k_{\mathrm{c,nd}}(M)$. Therefore

$$(\delta\mathrm{d}\widetilde{A}_1^+,A_2)=(\delta\mathrm{d}A_1^+,A_2)+(\delta\mathrm{d}(\widetilde{A}_1^+-A_1^+),A_2)=(\delta\mathrm{d}A_1^+,A_2)\,,$$

where in the last equality we used the self-adjointness of $\delta\mathrm{d}$ on $\Omega^k_{\mathrm{nd}}(M)$.
We show that $\sigma_{\mathrm{nd}}(A_1, A_2) = -\sigma_{\mathrm{nd}}(A_2, A_1)$ for all $A_1, A_2 \in \ker\delta \cap \Omega^k_{\mathrm{sc},\perp}(M)$. For that we have

$$\begin{aligned}
\sigma_{\mathrm{nd}}(A_1, A_2) = (\delta\mathrm{d}A_1^+, A_2) &= (\delta\mathrm{d}A_1^+, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-) \\
&= -(\delta\mathrm{d}A_1^-, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-) \\
&= -(A_1^-, \delta\mathrm{d}A_2^+) + (A_1^+, \delta\mathrm{d}A_2^-) \\
&= -(A_1^-, \delta\mathrm{d}A_2^+) - (A_1^+, \delta\mathrm{d}A_2^+) \\
&= -(A_1, \delta\mathrm{d}A_2^+) = -\sigma_{\mathrm{nd}}(A_1, A_2)\,,
\end{aligned}$$

where we exploited Lemma B.0.2 and $A_1^\pm, A_2^\pm \in \Omega^k_{\mathrm{sc,nd}}(M)$.
Finally we prove that $\sigma_{\mathrm{nd}}(A_1, \mathrm{d}\chi) = 0$ for all $\chi \in \Omega^k_{\mathrm{sc}}(M)$. Together with the antisymmetry shown before, this entails that σ_{nd} descends to a well-defined map $\sigma_{\mathrm{nd}} \colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)^{\times 2} \to \mathbb{R}$ which is bilinear and antisymmetric. Therefore it is a presymplectic form. To this end let $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$: we have

$$\sigma_{\mathrm{nd}}(A, \mathrm{d}\chi) = (\delta\mathrm{d}A_1^+, \mathrm{d}\chi) = (\delta^2\mathrm{d}A_1^+, \chi) + (\mathrm{n}\delta\mathrm{d}A^+, \mathrm{t}\chi) = 0\,,$$

where we used Equation (1.6) as well as $\mathrm{n}\delta\mathrm{d}A = -\delta\mathrm{n}\mathrm{d}A = 0$. ■

Proposition C.0.2. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then the following linear maps are isomorphisms of vector spaces*

$$G_\parallel \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad G_\parallel \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,, \tag{C.2}$$

$$G_\perp \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,nd}}(M)} \to \mathrm{Sol}_{\mathrm{nd}}(M)\,, \qquad G_\perp \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)\,, \tag{C.3}$$

Proof. Mutatis mutandis, the proof of the four isomorphisms is the same. Hence we focus only on $G_\parallel \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$.
A direct computation shows that $G_\parallel\left[\Omega^k_{\mathrm{tc},\delta}(M)\right] \subseteq \mathcal{S}^{\Box}_{\mathrm{t},k}(M)$. The condition $\delta G_\parallel\omega = 0$ follows from Corollary 3.1.23. Moreover, $G_\parallel$ descends to the quotient since for all $\eta \in \Omega^k_{\mathrm{tc,t}}(M)$ we have $G_\parallel\delta\mathrm{d}\eta = -G_\parallel\mathrm{d}\delta\eta = -\mathrm{d}G_\parallel\delta\eta \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ on account of Corollary 3.1.23.
We prove that $G_\parallel$ is surjective. Let $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$. In view of Proposition 3.2.3 there exists $A' \in [A]$ such that $\Box_\parallel A' = 0$ as well as $\delta A' = 0$. Proposition 3.1.19 ensures that there exists $\alpha \in \Omega^k_{\mathrm{tc}}(M)$ such that $A' = G_\parallel\alpha$. Moreover, condition $\delta A' = 0$ and Corollary 3.1.23 implies that $\delta\alpha \in \ker G_\parallel$, therefore $\delta\alpha = \Box_\parallel\eta$ for some $\eta \in \Omega^k_{\mathrm{tc},\parallel}(M)$ – *cf.* Proposition 3.1.19 and Remark 3.1.20. Applying δ to the equality $\delta\alpha = \Box_\parallel\eta$ we find $\Box\delta\eta = 0$, that is, $\delta\eta = 0$ – *cf* Remark 3.1.21. It follows that $\delta\alpha = \delta\mathrm{d}\eta$. Moreover we have $[A] = [G_\parallel\alpha] = [G_\parallel\alpha - \mathrm{d}G_\parallel\eta] = [G_\parallel(\alpha - \mathrm{d}\eta)]$, where now $\alpha - \mathrm{d}\eta \in \Omega^k_{\mathrm{tc},\delta}(M)$.

Finally we prove that $G_\parallel$ is injective: let $[\alpha] \in \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)}$ be such that $[G_\parallel\alpha] = [0]$. This entails that there exists $\chi \in \Omega^{k-1}_{\mathrm{tc,t}}(M)$ such that $G_\parallel\alpha = \mathrm{d}\chi$. Corollary 3.1.23 and $\alpha \in \Omega^k_{\mathrm{tc},\delta}(M)$ ensures that $\delta\mathrm{d}\chi = 0$, therefore $\chi \in \mathrm{Sol}_{\mathrm{t}}(M)$. Proposition 3.2.3, Remark 3.1.20 and Corollary 3.1.23 ensures that $\mathrm{d}\chi = \mathrm{d}G_\parallel\beta$ with $\beta \in \Omega^k_{\mathrm{tc},\delta}(M)$. It follows that $\alpha - \mathrm{d}\beta \in \ker G_\parallel$ and therefore $\alpha - \mathrm{d}\beta = \Box_\parallel\eta$ for $\eta \in \Omega^k_{\mathrm{tc},\parallel}(M)$ – *cf.* Remark 3.1.20. Applying δ to the last equality we find $\Box\beta = \Box\delta\eta$, hence $\beta = \delta\eta$ because of Remark 3.1.21. It follows that $\alpha = \delta\mathrm{d}\eta$ with $\eta \in \Omega^k_{\mathrm{c,t}}(M)$, that is, $[\alpha] = [0]$. ■

Proposition C.0.3. *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. The following statements hold true:*

1. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_\parallel([\alpha],[\beta]) \doteq (\alpha, G_\parallel\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}, \widetilde{G}_\parallel\right)$ *is symplectomorphic to* $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$.

2. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_\perp([\alpha],[\beta]) \doteq (\alpha, G_\perp\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_\perp\right)$ *is pre-symplectomorphic to* $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Proof. The proof of the two statements is the same. Hence we focus only on the first one. We observe that $\widetilde{G}_\parallel$ is well-defined. As a matter of fact, let $\alpha, \beta \in \Omega^k_{\mathrm{c},\delta}(M)$, then $G_\parallel\beta \in \Omega^k_{\mathrm{sc}}(M)$ and therefore the pairing $(\alpha, G_\parallel\beta)$ is finite. Moreover if $\eta \in \Omega^k_{\mathrm{c},\parallel}(M)$ we have

$$(\delta\mathrm{d}\eta, G_\parallel\beta) = (\eta, \delta\mathrm{d}G_\parallel\beta) = -(\eta, \mathrm{d}\delta G_\parallel\beta) = -(\eta, \mathrm{d}G_\parallel\delta\beta) = 0\,,$$
$$(\alpha, G_\parallel\delta\mathrm{d}\eta) = -(\alpha, G_\parallel\mathrm{d}\delta\eta) = -(\alpha, \mathrm{d}G_\parallel\delta\eta) = 0\,,$$

where we used that $G_\parallel\beta, \eta \in \Omega^k_{\mathrm{c,t}}(M)$ – *cf.* Equation (3.36) – as well as $\delta G_\parallel\beta = G_\parallel\delta\beta = 0$ – *cf.* Corollary 3.1.23. Therefore $\widetilde{G}_\parallel$ is well-defined: Moreover, it is per construction bilinear and antisymmetric, therefore it induces a pre-symplectic structure.
We now show that the isomorphism $G_\parallel : \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\parallel}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$ is a pre-symplectomorphism. Let $[\alpha],[\beta] \in \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\parallel}(M)}$. As a direct consequence of the properties of $G_\parallel = G^+_\parallel - G^-_\parallel$, calling $A_1 = G_\parallel\alpha$ and $A_2 = G_\parallel\beta$, we can consider $A^\pm_1 = G^\pm_\parallel\alpha$ in Equation (C.1). This leads us to

$$\sigma_{\mathrm{t}}([G_\parallel\alpha],[G_\parallel\beta]) = (\delta\mathrm{d}G^+_\parallel\alpha, G_\parallel\beta) = (\Box G^+_\parallel\alpha - \mathrm{d}\delta G^+_\parallel\alpha, G_\parallel\beta) = (\alpha, G_\parallel\beta) = \widetilde{G}_\parallel([\alpha],[\beta])\,,$$

where we used Corollary 3.1.23 so that $\mathrm{d}\delta G^+_\parallel\alpha = \mathrm{d}G^+_\parallel\delta\alpha = 0$. ■

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List of Symbols

(M, g)	globally hyperbolic spacetime of dimension m (with timelike boundary)
(Σ, h)	Cauchy hypersurface as a Riemannian manifold (with boundary)
$J^+(p)$	causal future of $p \in M$
$J^-(p)$	causal past of $p \in M$
$\mathbb{M}^m$	m-dimensional Minkowski spacetime
$\mathbb{M}^m_+$	m-dimensional half-Minkowski spacetime

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