UNIVERSITÀ DEGLI STUDI DI PAVIA

On the role of boundary conditions in the construction of fundamental solutions for Maxwell's equations on spacetimes with timelike boundary

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Tesi per la Laurea Magistrale di:
Rubens Longhi

Table of Contents

Table of Contents

Introduction (1)

This thesis deals with the characterization of the space of solutions of Maxwell's equations in terms of Green functions in spacetimes with timelike boundary. The aims are:

- Set Maxwell's equations in non-Minkowskian spacetimes with boundary
- Prove existence and uniqueness of Green functions for $\Box$ with certain classes of boundary conditions
- Construct the space of classical solutions for Maxwell's equations
- Construct the classical and quantum algebra of observables of the system

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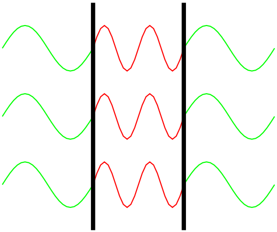
Applications to the study of several physical models in regions where the flux of physical quantities through the boundary is zero.

- Classical and quantum fields in curved spacetimes with boundary
- Electromagnetic Casimir effect
- Quantum Hall effect



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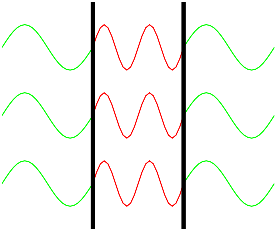

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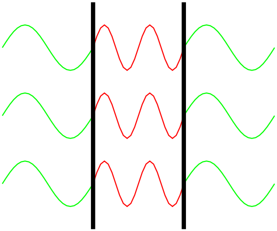

Table of Contents

Globally Hyperbolic Spacetimes with timelike boundary

We consider globally hyperbolic spacetimes M, $\dim M = m \geq 2$ with timelike boundary.

Definition.

A spacetime with boundary (M, g) is globally hyperbolic if it is time-oriented, causal, and any causal diamond is compact[1].

M has a timelike boundary if ∂M is itself a time-oriented spacetime with the induced metric $\iota^* g$, $\iota : \partial M \to M$ being the immersion map.

Theorem.

M can be split as

$$M = \mathbb{R} \times \Sigma,$$

where Σ is a Riemannian manifold with boundary $\partial\Sigma$.

[1] for all $p, q \in M$ $J^+(p) \cap J^-(q)$ is compact, $J^\pm(p)$ being the future and past of the event p

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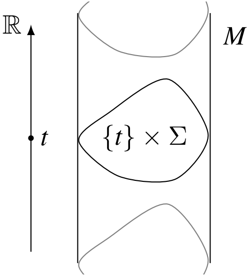

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Static spacetimes

We restrict ourselves to static spacetimes:

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A globally hyperbolic spacetime with boundary (M, g) is static if ∂_t is global timelike irrotational vector field, i.e. $\mathcal{L}_{\partial_t}(g) = 0$.

Examples:

- Half Minkowski spacetime $(\mathbb{R}^m_+ = \mathbb{R}^{m-1} \times [0, +\infty), \eta)$
- Any globally hyperbolic sub-region with timelike boundary of Minkowski spacetime
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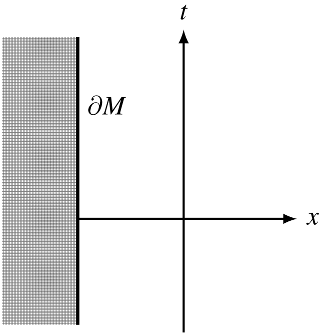


Table of Contents

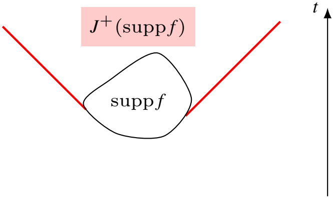
Classical fields on spacetimes

Classical fields are solutions to partial differential equations of motion.

Examples:

Schroedinger field	$(i\partial_t - H)\psi = 0$
Klein-Gordon field	$(\Box + m^2)\varphi = f$
Dirac field	$(i\gamma^\mu \partial_\mu - m)\psi = f$

They are all of the form $P\psi = f$, where P is a differential operator and $f \in C_c^\infty(M)$ is an external source. We are interested in providing a solutions that propagates the signal from the source at finite speed.



Green functions or Fundamental solutions

To solve $P\psi = f$ we look for the inverses of P, namely fundamental solutions or Green operators G. If a G exists, it provides a solution of $P\psi = f$:

$$\psi = \mathrm{Id}\,\psi = GP\psi = G(f).$$

Definition.

A Green operator $G : C_c^\infty(M) \to C^\infty(M)$ for a differential operator P is such that $G \circ P = \mathrm{Id}$, $P \circ G = \mathrm{Id}$.

We want to have the properties such that the propagation speed is finite.

- advanced Green operators $\boxed{G^+}$,
  $\operatorname{supp} G^+(f) \subseteq J^+(\operatorname{supp} f)$
- retarded Green operators $\boxed{G^-}$,
  $\operatorname{supp} G^-(f) \subseteq J^-(\operatorname{supp} f)$

With $G^\pm$ we can construct causal propagator $G = G^+ - G^-$.

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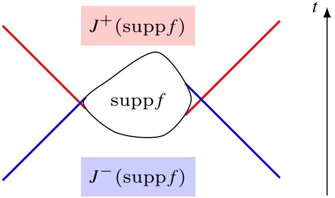

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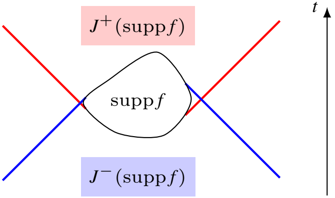

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Table of Contents

Differential forms and Maxwell's equations (1)

We denote the space of smooth differential k-forms over M as $\Omega^k(M)$, $0 \leq k \leq m$.

Electromagnetic field is regarded as the Faraday 2-form $F \in \Omega^2(M)$ (anti-symmetric covariant 2-tensor).

In terms of electric and magnetic fields it holds $\quad F = B + \mathrm{d}t \wedge E$,
with $E \in C^\infty(\mathbb{R}, \Omega^1(\Sigma))$ and $B \in C^\infty(\mathbb{R}, \Omega^2(\Sigma))$.

Maxwell's equations for F

Let $J \in \delta\Omega^2(M)$ be a 4-current. Then Maxwell's equations for the Faraday tensor $F \in \Omega^2(M)$ are

$$\mathrm{d}F = 0, \qquad \delta F = -J,$$

where $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ and $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ are the differential and codifferential operators.

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Differential forms and Maxwell's equations (2)

$$d^2\omega = 0, \quad \delta^2\omega = 0, \ \forall \omega \in \Omega^k(M).$$

In empty space $J = 0$ and F is closed ($dF = 0$) and co-closed ($\delta F = 0$).

In local components one recovers the usual covariant expressions

$$(dF)_{ijk} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad 1 \le i, j, k \le 4,$$

$$(\delta F)_k = \partial^j F_{jk} = 0, \quad 1 \le k \le 4.$$

In curved backgrounds one has to add to the usual derivatives curvature some symmetric corrections Γ:

$$\partial \longrightarrow \partial + \Gamma =: \nabla$$

Since differential forms are totally anti-symmetric, the corrections are canceled.

Maxwell's equations are invariant in form in any curved spacetime:

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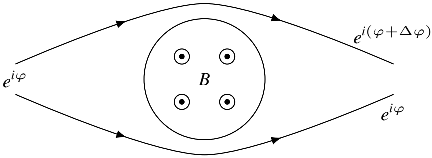
As a gauge theory, electromagnetism is formulated in terms of the potential $A \in \Omega^1(M)$. We look for a local primitive A of F, i.e.

$$F = \mathrm{d}A, \quad F_{ij} = \partial_i A_j - \partial_j A_i.$$

The Faraday tensor $F \in \Omega^2(M)$ is closed ($\mathrm{d}F = 0$), but not always exact ($F = \mathrm{d}A$), depending on the topology of M.

Aharonov-Bohm Effect

Consider M as the exterior of a solenoid run by an electric current. There is no field ($F = 0$), but $A \neq 0$ and $\mathrm{d}A = F = 0$ only locally since the space is not simply connected.



Maxwell's equations for the potential A (2)

$F = \mathrm{d}A$ implies $\mathrm{d}F = \mathrm{d}^2 A = 0$ and $\delta F = \delta \mathrm{d}A = -J$.

Maxwell's equations for $A \in \Omega^1(M)$

$$\delta \mathrm{d}A = -J. \tag{1}$$

If $A' = A + \mathrm{d}\chi$, $\chi \in C^\infty(M)$: $\quad F_{A'} = \mathrm{d}A' = \mathrm{d}A + \mathrm{d}^2\chi = \mathrm{d}A = F_A$

Definition. (Gauge invariance with empty boundary)

If $\partial M = \emptyset,\ A, A' \in \Omega^1(M)$ solutions of (1) are gauge-equivalent if there exists $\chi \in C^\infty(M)$ such that

$$A' = A + \mathrm{d}\chi.$$

Then $A \to A + \mathrm{d}\chi$ is called gauge transformation.

Maxwell's equations for the potential A (3)

Is there a gauge transformation $A \to A'$ such that $\Box A' = -J$, where $\Box = \delta \mathrm{d} + \mathrm{d}\delta$ is the wave operator?
In Lorenz gauge $\delta A' = 0 \Rightarrow \mathrm{d}\delta A' = 0 \Rightarrow (\delta \mathrm{d} + \mathrm{d}\delta)A' = \Box A' = -J$.

The transformation $A \to A' = A + \mathrm{d}\chi$ must be such that

$$\delta A' = \delta A + \delta \mathrm{d}\chi = \delta A + \Box \chi = 0$$

If $\partial M = \emptyset$, for any fixed $A \in \Omega^1(M)$ the equation $\Box \chi = -\delta A$ is always solvable.

Theorem.

If $\partial M = \emptyset$, for any solution $A \in \Omega^1(M)$ there exists $A' \in \Omega^1(M)$ gauge equivalent to A such that A' satisfies the Lorenz gauge $\delta A' = 0$ ($\partial^k A_k = 0$) and hence the system becomes

$$\Box A' = -J, \quad \delta A' = 0.$$

Green operators for $\Box$ with empty boundary

Theorem.

If M is globally hyperbolic with $\partial M = \emptyset$, there exist unique advanced and retarded Green operators $G^{\pm}$ for $\Box$.

Solutions to Maxwell's equations are completely determined in terms of Green operators for $\Box$.

$$A = \mathrm{Id}\, A = (G^{\pm} \circ \Box)A = -G^{\pm}(J).$$

Lorenz gauge is preserved since $\delta \circ G^{\pm} = G^{\pm} \circ \delta$:

$$\delta A = -\delta G(J) = -G(\delta J) = 0,$$

with $\delta J = 0$ being the conservation of current ($\partial_j J^j = 0$).

In spacetimes with non-empty boundary $\delta \circ G^{\pm} = G^{\pm} \circ \delta$ can no longer hold depending on the boundary conditions.

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Table of Contents

Generalization to spacetimes with boundary

In spacetimes with non-empty boundary it is not always possible to solve $\Box\chi = -\delta A$. We look for boundary conditions for $\Box$ such that $G^{\pm}$ exist.

Physically sound boundary conditions are those such that the flux of physical quantities vanish:

We require the symplectic flux through the boundary to vanish (symmetric operator):

$$\sigma(\alpha,\beta) = (\Box\alpha,\beta) - (\alpha,\Box\beta) = 0,$$

$\forall\alpha,\beta \in \Omega^1(M)$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta$ compact. $(\alpha,\beta) = \int_M \overline{\alpha}\wedge\star\beta = \int_M \overline{\alpha_j}\beta^j \,\mathrm{d}\mu_g$,

$$\sigma(\alpha,\beta) = (\mathrm{t}\delta\alpha,\mathrm{n}\beta)_\partial - (\mathrm{n}\alpha,\mathrm{t}\delta\beta)_\partial - (\mathrm{nd}\alpha,\mathrm{t}\beta)_\partial + (\mathrm{t}\alpha,\mathrm{nd}\beta)_\partial,$$

Where for $\omega \in \Omega^1(M)$, $\mathrm{t}\omega$ is the projection on ∂M and $\mathrm{n}\omega$ is the projection on the normal to ∂M.

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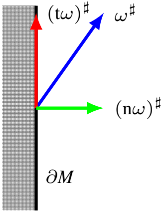

Boundary conditions for $\Box$

Symplectic Flux

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The symplectic flux vanishes if α, β are in the following spaces.

We studied, among the others:

- space of k-forms with *Dirichlet* boundary condition

$$\Omega^k_{\mathrm{D}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{n}\omega = 0\}\ ,$$

- space of k-forms with $\Box$-*tangential* boundary condition

$$\Omega^k_{\|}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{t}\delta\omega = 0\}\ ,$$

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- space of k-forms with $\Box$-*normal* boundary condition

$$\Omega^k_{\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\ ,\ \mathrm{n}\mathrm{d}\omega = 0\}\ .$$

Splitting of $\Box$ in static spacetimes

In static globally hyperbolic spacetimes it holds $M = \mathbb{R} \times \Sigma$ and $\Box$ splits as

$$\Box = \partial_t^2 - \Delta,$$

where Δ is the spatial Laplacian on Σ.

Since our boundary conditions are themselves static, the flux can be expressed with respect to the spatial derivatives only:

$$\sigma(\alpha, \beta) = (\Delta(\alpha|_\Sigma), \beta|_\Sigma)_\Sigma - (\alpha|_\Sigma, \Delta(\beta|_\Sigma))_\Sigma\,,$$

where $(\ ,\ )_\Sigma$ is the Hilbert scalar product in $\mathrm{L}^2\Omega^k(\Sigma)$.

Vanishing symplectic flux is equivalent to Δ being symmetric as a densely defined operator on $\mathrm{L}^2\Omega^k(\Sigma)$. For a unitary time evolution we look for a self-adjoint extension of Δ.

To select self-adjoint extensions we use the method of boundary triples.

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Table of Contents

Boundary triples (1)

Definition.

A boundary triple for a symmetric differential operator S on a Riemannian manifold with boundary Σ is a triple $(\mathrm{L}^2\Omega^k(\partial\Sigma), \gamma_0, \gamma_1)$ such that

$$\sigma(\alpha, \beta) = (\gamma_1\alpha, \gamma_0\beta)_\partial - (\gamma_0\alpha, \gamma_1\beta)_\partial$$

for any $\alpha, \beta \in \mathrm{dom}(S^*)$.

Self-adjoint extensions of S are in one-to-one correspondence with all physically sound boundary conditions.

The space $\ker(\mathcal{A}\gamma_1 - \mathcal{B}\gamma_0)$ parametrizes the boundary conditions that vanish the symplectic flux, where $\mathcal{A}, \mathcal{B}$ is a self-adjoint pair of operators on $\mathrm{L}^2\Omega^k(\partial\Sigma)$.

In our case, $S = \Delta$ and the following identity

$$(\gamma_1\alpha, \gamma_0\beta)_\partial - (\gamma_0\alpha, \gamma_1\beta)_\partial = (\mathrm{t}\delta\alpha, \mathrm{n}\beta)_\partial - (\mathrm{n}\alpha, \mathrm{t}\delta\beta)_\partial - (\mathrm{nd}\alpha, \mathrm{t}\beta)_\partial + (\mathrm{t}\alpha, \mathrm{nd}\beta)_\partial$$

entails that the boundary maps are $\gamma_0(\alpha) = \begin{bmatrix} \mathrm{n}\alpha \\ \mathrm{t}\alpha \end{bmatrix}$ and $\gamma_1(\alpha) = \begin{bmatrix} \mathrm{t}\delta\alpha \\ \mathrm{nd}\alpha \end{bmatrix}$.

Boundary triples (1)

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Boundary triples (2)

Boundary maps.

$$\gamma_1(\alpha) = \begin{bmatrix} \mathrm{t}\delta\alpha \\ \mathrm{n}\mathrm{d}\alpha \end{bmatrix}, \qquad \gamma_0(\alpha) = \begin{bmatrix} \mathrm{n}\alpha \\ \mathrm{t}\alpha \end{bmatrix}$$

With the following choices, if $\alpha, \beta \in \ker(\mathcal{A}\gamma_1 - \mathcal{B}\gamma_0)$

- $\mathcal{A} = 0$ and $\mathcal{B} = \mathrm{Id} \Rightarrow$ Dirichlet $\mathrm{t}\alpha = 0$, $\mathrm{n}\alpha = 0$,
- $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \Box$-tangential $\mathrm{t}\alpha = 0$, $\mathrm{t}\delta\alpha = 0$,
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The corresponding self-adjoint extensions of Δ are denoted by $\Delta_{\mathrm{D}}, \Delta_{\|}, \Delta_{\perp}$.

The boundary conditions are chosen such that for D, $\|$, $\perp$, the Green operators commute with the differential operators.

Green operators for $\Box$

Recalling $\Box = \partial_t^2 - \Delta$, we exploit spectral calculus to obtain Green operators from the following bidistributions:

We set $\mathcal{G}_\sharp^+ = \vartheta(t-t')\mathcal{G}_\sharp$ and $\mathcal{G}_\sharp^- = -\vartheta(t'-t)\mathcal{G}_\sharp$, where:

$$\mathcal{G}_\sharp(\alpha,\beta) = \int_{\mathbb{R}^2} \left(\alpha|_\Sigma, \Delta_\sharp^{-1/2} \sin\left(\Delta_\sharp^{1/2}(t-t')\right)\beta|_\Sigma\right)_\Sigma \mathrm{d}t\mathrm{d}t' ,$$

for $\sharp \in \{\mathrm{D}, \|, \perp\}$.

The bidistributions define uniquely $G_\sharp^\pm$ such that

$$(G_\sharp^\pm(\alpha), \beta) = \mathcal{G}_\sharp^\pm(\alpha, \beta).$$

For $\sharp \in \{\mathrm{D}, \|, \perp\}$, the Green operators commute with the differential operators: $\delta \circ G_\sharp^\pm = G_\sharp^\pm \circ \delta$.

Table of Contents

Maxwell's equations with boundary

We apply the results to Maxwell's equations for A in empty space: $\delta \mathrm{d}A = 0$.

The symplectic flux for the $\delta\mathrm{d}$ operator is

$$(\delta \mathrm{d}\alpha, \beta) - (\alpha, \delta \mathrm{d}\beta) = (\mathrm{t}\alpha, \mathrm{nd}\beta)_\partial - (\mathrm{nd}\alpha, \mathrm{t}\beta)_\partial .$$

Boundary conditions considered:

- $\delta\mathrm{d}$-tangential ($\mathrm{t}\alpha = 0$),
- $\delta\mathrm{d}$-normal ($\mathrm{nd}\alpha = 0$)

Two different notions of gauge-invariance must be introduced.

Gauge invariance with boundary conditions

Definition.

We say two solutions A, A' of $\delta \mathrm{d} A = 0$ with $\delta\mathrm{d}$-tangential are gauge-equivalent if there exists $\chi \in \Omega^0_{\mathrm{t}}(M) = \{\omega \in C^\infty(M) \,|\, \mathrm{t}\omega = 0\}$ such that $A' = A + \mathrm{d}\chi$.

$$\mathrm{t}A' = \mathrm{t}(A + \mathrm{d}\chi) = \mathrm{t}\mathrm{d}\chi = \mathrm{d}\mathrm{t}\chi = 0$$

Definition.

We say two solutions A, A' of $\delta \mathrm{d} A = 0$ with $\delta\mathrm{d}$-normal are gauge equivalent if there exists $\chi \in C^\infty(M)$ such that $A' = A + \mathrm{d}\chi$.

$$\mathrm{n}\mathrm{d}A' = \mathrm{n}\mathrm{d}(A + \mathrm{d}\chi) = \mathrm{n}\mathrm{d}^2\chi = 0.$$

$$\mathrm{Sol}_{\mathrm{t}}(M) = \frac{\{A \in \Omega^1(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^0_{\mathrm{t}}(M)}, \quad \mathrm{Sol}_{\mathrm{nd}}(M) = \frac{\{A \in \Omega^1(M) |\ \delta\mathrm{d}A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^0(M)}$$

Gauge invariance with boundary conditions

Theorem.

Let M globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ there exists a representative $A' \in [A]$ such that

$$\Box_{\parallel} A' = 0\,, \qquad \delta A' = 0\,.$$

Theorem.

Let M be a globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ there exists a representative $A' \in [A]$ such that

$$\Box_{\perp} A' = 0\,, \qquad \delta A' = 0\,.$$

The proof of the Theorems is based on the fact that Green operators for $\Box$ commute with differential operators: $\delta \circ G^{\pm}_{\parallel} = G^{\pm}_{\parallel} \circ \delta$ and $\delta \circ G^{\pm}_{\perp} = G^{\pm}_{\perp} \circ \delta$.

Table of Contents

Conclusions

We proved that the spaces of solutions of $\delta \mathrm{d} A = 0$ with $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions are completely described by $G^{\pm}_{\parallel}$ and $G^{\pm}_{\perp}$ for the wave operator $\Box$.

- More general boundary conditions for $\Box$.
- Different methods other than boundary triples to obtain self-adjoint extensions and hence Green operators.
- Do we have to rely on wave operator to solve Maxwell's equations?
- Regarding electromagnetism as a gauge theory with $U(1)$ structure group should reduce the group of gauge transformations $A \to A + \mathrm{d}\chi$,

$$A \to A - ig^{-1}\mathrm{d}g, \; g \in C^{\infty}(M, U(1)).$$

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