University of Pavia
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TESI TRIENNALE

On the fundamental solutions for wave-like equations on curved backgrounds

$$\Box u = \delta$$

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Preface

To do

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Abstract

To do

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Mathematical Tools 1

1.1 An overview of Differential Geometry

We begin by recalling of very well known definitions in order to introduce the basic geometrical objects which are used in the text.

A **manifold** is, heuristically speaking, a space that is locally similar to $\mathbb{R}^n$. To define it we use the concepts of **topological space** and of **homeomorphism**.

1.1.1 Definition (**Topological Space**)**.** *A set* X *together with a family* $\mathcal{T}$ *(**topology**) of subsets of* X *is called a **topological space** if the following conditions are met:*

a. $\emptyset, \mathrm{X} \in \mathcal{T}$,

b. *for all* U *and* $V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$,

c. *for any index set* A*, if* $U_i \in \mathcal{T}$ *for all* $i \in A$, $\bigcup_{i \in A} U_i \in \mathcal{T}$.

An element of $\mathcal{T}$ *is called **open set**. If a point* p *is in an open set* U*, we call* U *a **neighborhood** of* p*.* ■

1.1.2 Definition (**Continuity and homomorphism**)**.** *Let* X *and* Y *be two topological spaces. A function* $f : \mathrm{X} \to \mathrm{Y}$ *is **continuous** if for any open set* U

of Y*, the preimage* $f^{-1}(U)$ *is an open set of* X*.*
A continuous and bijective map $\varphi : \mathrm{X} \to \mathrm{Y}$ *is an* ***homomorphism*** *if* $\varphi^{-1} : \mathrm{Y} \to \mathrm{X}$ *is also continuous.* ■

As for vector spaces, we can talk of a **basis** for topological space. A subset $\mathcal{B} \subset \mathcal{T}$ is a basis if any open set can be expressed as union of elements of $\mathcal{B}$. A topology is **Hausdorff** if, for any two distinct points $p, q \in \mathrm{X}$, there exist two open neighborhoods U of p and V of q such that $U \cap V = \emptyset$.

A topological space X is called **compact** if each of its open covers has a finite subcover, i.e. for any collection $\{U_i\}_{i\in A}$, (where A is a set of indexes) such that

$$\mathrm{X} \subseteq \bigcup_{i\in A} U_i,$$

there is a finite subset A' of A such that

$$\mathrm{X} \subseteq \bigcup_{i\in A'} U_i.$$

We are now ready to introduce the concept of **manifold**.

1.1.3 Definition. *An* n*-dimensional topological* ***manifold*** M *is a topological Hausdorff space (with a countable basis) that is locally homeomorphic to* $\mathbb{R}^n$*, i.e. for every* $p \in \mathrm{M}$ *there exists an open neighbourhood* U *of* p *and a homeomorphism*

$$\varphi : U \to \varphi(U),$$

such that $\varphi(U)$ *is an open subset of* $\mathbb{R}^n$*.* ■

Such homeomorphism is called a **(local) chart** of M. An **atlas** of M is a family $\{U_i, \varphi_i\}_{i\in A}$ of local charts together with an open covering of M, i.e. $\bigcup_{i\in A} U_i = \mathrm{M}$.

1.1.4 Definition. *A* ***differentiable atlas*** *of a manifold* M *is an atlas* $\{U_i, \varphi_i\}_{i\in A}$ *such that the functions*

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

are differentiable (of class C^∞*) for any* $i, j \in A$ *such that* $U_i \cap U_j \neq \emptyset$*. Each* φ_{ij} *is called* ***transition function****.* ■

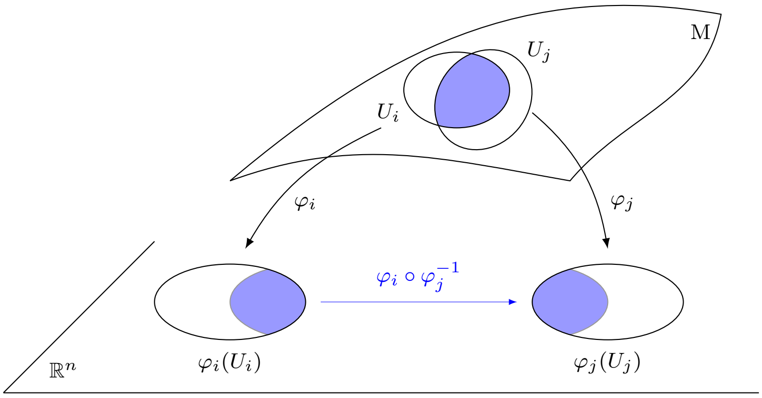


FIGURE 1.1: A differentiable atlas on a manifold M.

With this definition, each φ_{ij} is a **diffeomorphism** because one can always interchange the indexes i and j.

We are only interested in **differentiable** (or **smooth**) **manifolds**, endowed with a maximal differentiable atlas. Here maximality of the atlas means that, if φ is a chart of M and $\{U_i, \varphi_i\}_{i\in A}$ is a differentiable atlas, then φ belongs to $\{U_i, \varphi_i\}_{i\in A}$. We call a differentiable manifold with an atlas for which all chart transitions have positive Jacobian determinant an **orientable manifold**.

1.1.5 Remark. For now on, the word **manifold** will always mean **differentiable manifold** and to indicate them it will be used the letters M or N. ■

1.1.6 Definition (**Submanifold**)**.** *Let $n \leq m$. An n-dimensional **submanifold** N of M is a nonempty subset N of M such that, for every point $q \in \mathrm{N}$, there exists a local chart $\{U, \varphi\}$ of M about q such that*

$$\varphi(U \cap \mathrm{N}) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^m.$$

*If $n = m - 1$ we call N an **hypersurface** of M.* ■

1.1.7 Example. If M and N are manifolds, the Cartesian product $\mathrm{M} \times \mathrm{N}$ is endowable with canonical structure of a manifold. If $\{U_i, \varphi_i\}_{i\in A}$ is an differentiable atlas for M and $\{V_j, \psi_j\}_{j\in B}$ ia an atlas for N, then $\big\{U_i \times V_j, (\varphi_i, \psi_j)\big\}_{(i,j)\in A\times B}$ is a differentiable atlas for $\mathrm{M} \times \mathrm{N}$. ■

As in the Euclidean case, one can introduce the notion of **differentiable** map between manifolds:

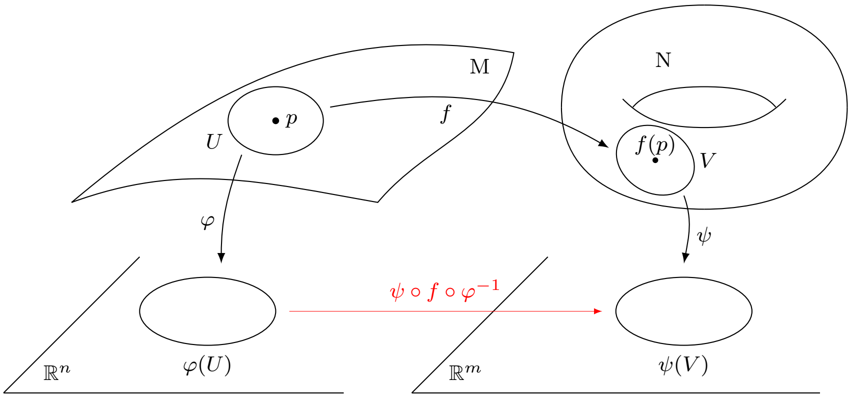


FIGURE 1.2: The concept of differentiable map f between manifolds M and N.

1.1.8 Definition. *A continuous map* $f : \mathrm{M} \to \mathrm{N}$ *between two manifolds* M *and* N *is* ***differentiable*** *at* $p \in \mathrm{M}$ *if there exist local charts* $\{U, \varphi\}$ *and* $\{V, \psi\}$ *about* p *in* M *and about* $f(p)$ *in* N *respectively, such that* $f(U) \subset V$ *and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V),$$

is differentiable (of class C^∞*) at* $\varphi(p)$*. The function* f *is said to be differentiable on* M *if it is differentiable at every point of* M. ■

The space of differentiable functions between two manifolds is denoted by $C^\infty(\mathrm{M}, \mathrm{N})$, and if $\mathrm{N} = \mathbb{C}$ simply by $C^\infty(\mathrm{M})$.

We introduce the **tangent space** of a point of a manifold. It will be constructed using the derivatives of curves which pass through the point. DA AMPLIARE

1.1.9 Definition (Tangent space). *Let* $p \in \mathrm{M}$ *and let* I *be an interval containing* 0*. We indicate* $\mathcal{C}_p = \{c \in C^\infty(I, \mathrm{M}) |\ c(0) = p\}$ *the set of differentiable curves passing through* p*.*
We consider the equivalence relation $(\sim)$*, according to which two curves* $c_1, c_2 \in \mathcal{C}_p$ *are equivalent if there exists a local chart* φ *about* p *such that* $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$*.*
The ***tangent space*** *of* M *at* p *is the set* $\mathrm{T}_p\mathrm{M} := \mathcal{C}_p / \sim$ *.* ■

One can check that the definition of the equivalence relation does not depend on the choice of local chart. In fact, if $\{U, \varphi\}$ and $\{V, \psi\}$ are local charts at p,

$$(\varphi \circ c)'(0) = (\varphi \circ \psi^{-1} \circ \psi \circ c)'(0) = D(\varphi \circ \psi^{-1})(\psi(p)) \cdot (\psi \circ c)'(0),$$

where $D(\varphi \circ \psi^{-1})(\psi(p))$ stands for the Jacobian of the transition function calculated at $\psi(p)$. It holds that $(\varphi \circ c_1)'(0)$ and $(\varphi \circ c_2)'(0)$ coincide if and only if $(\psi \circ c_1)'(0)$ and $(\psi \circ c_2)'(0)$ coincide.

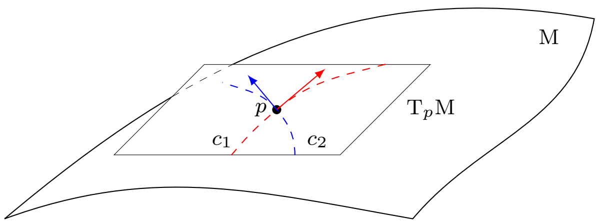


FIGURE 1.3: Tangent space $\mathrm{T}_p\mathrm{M}$ where $c_1 \nsim c_2$.

It can be proven that (for a fixed atlas φ about p) the following map is a linear isomorphism between $\mathrm{T}_p\mathrm{M}$ and $\mathbb{R}^n$:

$$\begin{aligned} \Theta_\varphi : \ \mathrm{T}_p\mathrm{M} &\to \mathbb{R}^n, \\ [c] &\mapsto (\varphi \circ c)'(0). \end{aligned}$$

Hence we can think of $\mathrm{T}_p\mathrm{M}$ as being a copy of $\mathbb{R}^n$ attached to the point p on the manifold.
For reasons that we will make clear later, we denote the basis of $\mathrm{T}_p\mathrm{M}$ as

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}.$$

Collecting all tangent spaces, one builds the **tangent bundle** of a manifold M, defined as:

$$\mathrm{TM} = \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M} = \bigcup_{p \in \mathrm{M}} \{p\} \times \mathrm{T}_p\mathrm{M}.$$

1.1.10 Definition. *Let $f : \mathrm{M} \to \mathrm{N}$ be a differentiable map and let $p \in \mathrm{M}$. The **differential** of f at p is the linear map*

$$\mathrm{d}_p f : \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{N}, \quad [c] \mapsto [f \circ c] \ \cong (f \circ c)'(0).$$

*The **differential** of f is the map $\mathrm{d}f : \mathrm{TM} \to \mathrm{TN}$ such that $\mathrm{d}f|_{\mathrm{T}_p\mathrm{M}} = \mathrm{d}_p f$.* ■

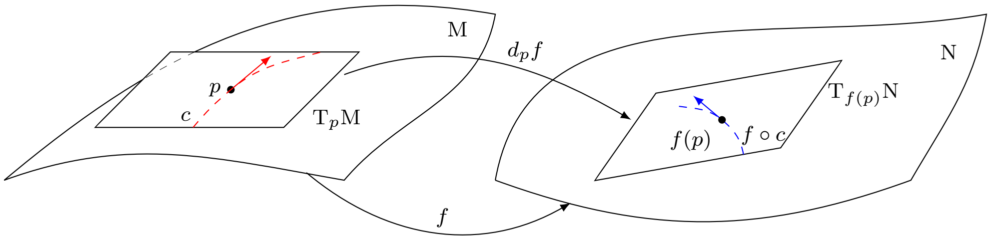


Figure 1.4: A scheme of a differential map.

Given M and a chart $\{U, \varphi\}$ near p, fix $X = [c] \in T_pM$. If we identify $T_p\mathbb{R} \cong \mathbb{R}$, we can interpret the differential $d_pf(X)$ of a function $f \in C^\infty(M)$ at a point p as the **derivative** in the direction of X:

$$\partial_X f(p) := d_pf(X).$$

A functional which is linear and follows Leibniz rule, such as $\partial_X : C^\infty(M) \to \mathbb{R}$, is called a **derivation**. The set of all derivations at p is denoted as Der_p and it is a vector space. The map $X \in T_pM \mapsto \partial_X$ is an isomorphism between T_pM and Der_p.
Let M be n-dimensional and, a **coordinate system** is a set of functions
Define:

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R}, \quad f \mapsto \left.\frac{\partial f}{\partial x^i}\right|_p = \partial_X f(p),$$

where $X = [c]$ and $c(t) = \varphi^{-1}(\varphi(p) + te_i)$ (e_i is the basis vector).
Note that, from the definition of the differential, it holds

$$\partial_X f(p) = (f \circ c)'(0) = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(p)},$$

which shows that the object we defined can be seen as a partial derivative in the usual sense.
It can be shown that the set of derivations $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ forms a basis for Der_p and, due to the isomorphism, for T_pM, X can be expressed as

$$X = X^i \frac{\partial}{\partial x^i},$$

where Einstein summation has been employed.
Observe that linearity entails that

$$\partial_X f(p) = d_pf(X) = X^i d_pf\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = X^i \left.\frac{\partial f}{\partial x^i}\right|_p.$$

$$\begin{tikzcd}
 & \mathrm{T}_p\mathrm{M} \arrow[dl, "X\mapsto\partial_X"'] \arrow[dr, "\Theta_\varphi"] & \\
\mathrm{Der}_p & & \mathbb{R}^n
\end{tikzcd}$$

FIGURE 1.5: Isomorphism relations for the tangent space.

1.1.11 Definition. *Let* M *be a manifold, we define a projection map* $\pi : \mathrm{TM} \to \mathrm{M}$ *such that* $\pi(\mathrm{T}_p\mathrm{M}) = p$, *and we call a* ***section*** *in the tangent bundle a map* $s : \mathrm{M} \to \mathrm{TM}$ *such that* $\pi \circ s = \mathrm{id}_\mathrm{M}$. ■

The dual space of the tangent space $\mathrm{T}_p\mathrm{M}$ is called the **cotangent space**, denoted with $\mathrm{T}^*_p\mathrm{M}$, which has a canonical basis denoted with $\{\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p\}$. The elements of such basis act on any element of the tangent space basis at p as follows:

$$\mathrm{d}x^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{ij}.$$

Similarly is defined the **cotangent bundle** $\mathrm{T}^*\mathrm{M}$ as the disjoint union of cotangent spaces.

1.1.12 Definition. *Sections in the tangent bundle, denoted by* $C^\infty(\mathrm{M}, \mathrm{TM})$, *are called* ***vector fields****, whereby sections in the cotangent bundle are called* 1-***forms****.* ■

Vector fields are locally expressed in terms of linear combinations of

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\} =: \{\partial_1, \ldots, \partial_n\},$$

where $\partial_i = \partial_i|_p$ at any point p, whereas 1-forms are expressed as linear combinations of

$$\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\},$$

where $\mathrm{d}x^i$ is the 1-form that acts at p as $\mathrm{d}x^i|_p$.

1.1.13 Definition. *We define the derivative in the direction of* X *as an operator* $\partial_X : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ *such that*

$$\partial_X f = \mathrm{d}f(X),$$

for any vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$. ■

It follows immediately that Leibniz's rule holds: $\partial_X(f \cdot g) = g\,\partial_X f + f\,\partial_X g$, and again holds the useful formula

$$\partial_X f = \mathrm{d}f(X) = X^i \mathrm{d}f(\partial_i) = X^i \frac{\partial f}{\partial x^i}.$$

1.1.14 Observation. Given two vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$, there is a unique vector field $[X, Y] \in C^\infty(\mathrm{M}, \mathrm{TM})$ such that

$$\partial_{[X,Y]} f = \partial_X \partial_Y - \partial_Y \partial_X f$$

for all $f \in C^\infty(\mathrm{M})$. The map $[\cdot,\cdot] : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) \to C^\infty(\mathrm{M}, \mathrm{TM})$ is called the **Lie bracket**, it is bilinear, skew-symmetric and satisfies the *Jacobi identity*: for any $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

■

1.1.15 Definition. *An **affine connection** or **covariant derivative** on a manifold* M *is a bilinear map*

$$\begin{aligned} \nabla : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) &\to C^\infty(\mathrm{M}, \mathrm{TM}) \\ (X, Y) &\mapsto \nabla_X Y\,, \end{aligned}$$

such that for all smooth functions $f \in C^\infty(\mathrm{M})$ and all vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$:

- $\nabla_{fX} Y = f \nabla_X Y$*, i.e., ∇ is $C^\infty(\mathrm{M})-$linear in the first variable;*
- $\nabla_X f = \partial_X f$*;*
- $\nabla_X (fY) = \partial_X f + f \nabla_X Y$*, i.e., ∇ satisfies the* Leibniz rule *in the second variable.*

■

The covariant derivative on the direction of the basis vector fields $\{\partial_1, \ldots, \partial_n\}$ is indicated

$$\nabla_j := \nabla_{\partial_j}.$$

We are now ready to introduce metric structures on manifolds.

1.2 Lorentzian Manifolds

We start in the simple case of **Minkowski spacetime**.

1.2.1 Definition. *Let V be an n-dimensional vector space. A* ***Lorentzian scalar product*** *is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ with signature $(- + \cdots +)$, i.e. such that one can find a basis $\{e_1, \ldots, e_n\}$ such that*

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_i, e_j \rangle = \delta_{ij} \text{ if } i, j > 1.$$

■

The **Minkowski product** $\langle x, y \rangle_0$, defined by the formula

$$\langle x, y \rangle_0 = \eta_{ik} x^i y^k = -x_1 \, y_1 + x_2 \, y_2 + \cdots + x_n \, y_n,$$

with $\eta := \operatorname{diag}(-1, 1, \ldots, 1, 1)$ is the simplest example of Lorentzian scalar product on $\mathbb{R}^n$. The n-dimensional Minkowski space, denoted by $\mathbb{M}^n$ is simply $\mathbb{R}^n$ equipped with Minkowski product.

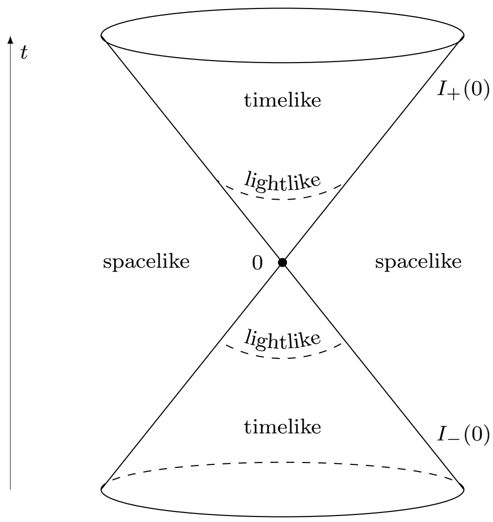


FIGURE 1.6: Minkowski time orientation.

1.2.2 Definition. *We call the* ***negative squared length*** *of a vector $X \in V$*

$$\gamma(X) = - \|X\|^2 := - \langle X, X \rangle .$$

A vector $X \in V \setminus \{0\}$ is called

- ***timelike*** *if* $\gamma(X) > 0$,
- ***lightlike*** *if* $\gamma(X) = 0$,
- ***spacelike*** *if* $\gamma(X) < 0$ *or* $X = 0$,
- ***causal*** *if it is either timelike or lightlike.*

■

This definition will mostly be used for tangent vectors, in case V is the tangent space of a Lorentzian manifold at a point.
For $n \geq 2$ the set of timelike vectors $I(0)$ consists of two connected components. A **time orientation** is the choice of one of these two components, that we call $I_+(0)$.

1.2.3 Definition. *We call*

- $J_+(0) := \overline{I_+(0)}$ *(elements are called* ***future-directed****),*
- $C_+(0) := \partial I_+(0)$ *(upper* ***light cone****),*
- $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$ *(elements are called* ***past-directed****),*
- $C_-(0) := -C_+(0)$ *(lower* ***light cone****).*

■

1.2.4 Definition. *A* ***metric*** g *on a manifold* M *is the assignment of a scalar product at each tangent space*

$$g : \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$$

which depends smoothly on the base point p*. We call it a* ***Riemannian metric*** *if the scalar product is pointwise positive definite, and a* ***Lorentzian metric*** *if it is a Lorentzian scalar product.*
A pair (M, g)*, where* M *is a manifold and* g *is a Lorentzian (Riemannian) metric is called a* ***Lorentzian (Riemannian) manifold****.* ■

The request of smooth dependence on p may be specified as follows: given any chart $\{U, \varphi = (x^1, \ldots, x^n)\}$ about p, the functions $g_{ij} : \varphi(U) \to \mathbb{R}$ defined by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, for any $i, j = 1, \ldots, n$ should be differentiable. With respect to these coordinates one writes

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j \equiv \sum_{i,j} g_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j.$$

The scalar product of two tangent vectors $v, w \in \mathrm{T}_p\mathrm{M}$, with coordinate chart $\varphi = (x^1, \dots x^n)$, such that $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$ is

$$\langle v, w \rangle = g_{ij}(\varphi(p)) v^i w^j.$$

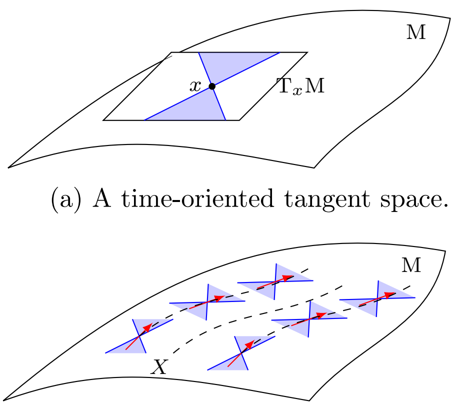
When the choice of the chart is clear we will often write, with abuse of notation $g_{ij}(p) := g_{ij}(\varphi(p))$. We will indicate $(g^{ij})_{i,j=1,\dots,n} :=)^{-1}$.

From now on M will always indicate a **Lorentzian manifold**.

1.2.5 Definition. *A vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$ is called timelike, spacelike, lightlike or causal, if $X(p)$ is timelike, spacelike, lightlike or causal, respectively, at every point $p \in \mathrm{M}$.*

A differentiable curve $c : I \to \mathrm{M}$ is called timelike, lightlike, spacelike, causal, future-directed or past-directed if $[c](t) \in \mathrm{T}_{c(t)}\mathrm{M}$ is, for all $t \in I$, timelike, lightlike, spacelike, causal, future-directed or past-directed, respectively.

*A Lorentzian manifold M is called **time-oriented** if there exists a nowhere vanishing timelike vector field on M. If a manifold is time-oriented, we refer to it as **spacetime**.* ■



(a) A time-oriented tangent space.

(b) A time-oriented manifold together with field lines of a timelike vector field X.

FIGURE 1.7: Time orientations.

The **causality relations** on M are defined as follows. Let $p, q \in \mathrm{M}$,

- $p \ll q$ iff there exists a future-directed timelike curve from p to q,
- $p < q$ iff there is a future-directed causal curve from p to q,
- $p \leq q$ iff $p < q$ or $p = q$.

The causality relation "$<$" is a strict weak ordering and the relation "$\leq$" makes the manifold a partially ordered set.

1.2.6 Definition. *The* ***chronological future*** *of a point $x \in \mathrm{M}$ is the set $I_+^M(x)$ of points that can be reached by future-directed timelike curves, i.e.*

$$I_+^M(x) = \{y \in \mathrm{M} \mid x < y\}.$$

The ***causal future*** *$J_+^M(x)$ of a point $x \in \mathrm{M}$ is the set of points that can be reached by future-directed causal curves from x, i.e.,*

$$J_+^M(x) = \{y \in \mathrm{M} \mid x \leq y\}.$$

Given a subset $A \subset \mathrm{M}$ the ***chronological future*** *and the* ***causal future*** *of A are respectively*

$$I_+^M(A) = \bigcup_{x \in A} I_+^M(x), \qquad J_+^M(A) = \bigcup_{x \in A} J_+^M(x).$$

In a similar way, one defines the ***chronological*** *and* ***causal pasts*** *of a point x of a subset $A \subset \mathrm{M}$ by replacing future-directed curves with past directed curves. They are denoted by $I_-^M(x), I_-^M(A), J_-^M(x)$, and $J_-^M(A)$, respectively. We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$.* ■

Any subset Ω of a spacetime M is a spacetime itself, if one restricts the metric to Ω. So $J_\pm^\Omega(x)$ are well defined.

1.2.7 Definition. *A subset $\Omega \subset \mathrm{M}$ of a spacetime is called* ***causally compatible*** *if for any point $x \in \Omega$ holds*

$$J_\pm^\Omega(x) = J_\pm^{\mathrm{M}}(x) \cap \Omega,$$

where it can be noted that the inclusion "$\subset$" always holds. ■

The condition we defined means that taken two points in Ω that can be joined by a causal curve in M, there also exists a causal curve connecting them inside Ω.

We now can introduce the concept of **geodesics** and **exponential map**.

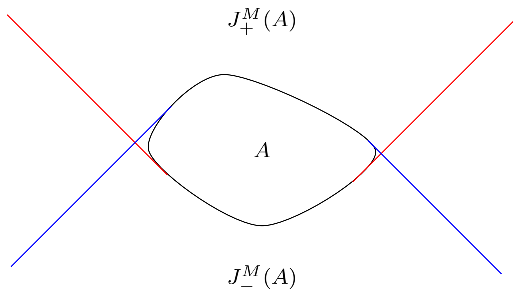


FIGURE 1.8: Causal future J_+^M and causal past J_-^M of a subset $A \subset \mathrm{M}$.

1.2.8 Definition. *Let $c : [a, b] \to \mathrm{M}$ be a curve on a Lorentzian manifold* M*. The length $L[c]$ is defined by (with Einstein summation convention)*

$$L[c] = \int_a^b \sqrt{|g([c](t), [c](t))|}\, dt = \int_a^b \sqrt{\left| g_{ik}(c(t)) \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} \right|}\, dt,$$

*where $x^i(t) := (\varphi \circ c)^i(t)$ are the coordinates of the point $c(t)$ in a chart φ. Given $p, q \in \mathrm{M}$, if $p \le q$ we define the **time-separation** between p and q as*

$$\tau(p, q) = \sup\{L[c] \mid c \textit{ is a future directed causal curve from } p \textit{ to } q\},$$

and 0 otherwise.
*A **geodesic** between two points $p, q \in \mathrm{M}$ such that $p \le q$, if it exists, is a curve c such that $L[c] = \tau(p, q)$, i.e. the curve of maximum time-separation.* ■

The request on the geodesics implies that (in variational sense) $\delta L[c] = 0$. It can be demonstrated that the stationary problem for the functional $L[c]$ is equivalent to $\delta E[c] = 0$ for the functional, called **energy**, defined by

$$E[c] = \frac{1}{2} \int_a^b |g([c](t), [c](t))|\, dt.$$

Since the Euler-Lagrange equations for a functional $I[c] = \int_a^b f(t, c(t), [c](t))\, dt$ read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}^i}\right) - \frac{\partial f}{\partial x^i} = 0,$$

being $c = (x^1, \ldots, x^n)$, then, in our case, setting $f(t, c, [c]) = g([c], [c])$:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0.$$

Here $\Gamma^i_{jk} \in C^\infty(U \subset \mathrm{M})$ are the **Christoffel symbols**, defined in the chart $\varphi = (\xi^1, \ldots, \xi^n)$ as

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lj}}{\partial \xi^k} + \frac{\partial g_{lk}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l} \right).$$

1.2.9 Definition. *A connection ∇ on a manifold* M *with a metric g is said to be a* ***metric connection*** *if for all $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds the following* Leibniz rule*:*

$$\partial_X \, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The unique metric connection which is also torsion-free, i.e.,

$$T := \nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

is called the ***Levi-Civita connection****.* ■

Another way to define **geodesics** is to say a geodesic between two points p, q on a manifold M with the Levi-Civita connection ∇ is the curve c which links p and q such that parallel transport along it preserves the tangent vector to the curve, i.e.

$$\nabla_{[c](t)}[c](t) = 0 \quad \text{for all } t \in [a, b]. \tag{1.1}$$

More precisely, in order to define the covariant derivative of $[c]$ it is necessary first to extend $[c]$ to a smooth vector field in an open set containing the image of the curve, but it can be shown that the derivative is independent of the choice of the extension.

1.2.10 Observation. We can express the Christoffel symbols in terms of the Levi-Civita connection:

$$\nabla_j \partial_k = \Gamma^i_{jk} \partial_i \tag{1.2}$$

in a local chart $\varphi = (x^1, \ldots, x^n)$. ■

1.2.11 Proposition. *Let ∇ be a connection over a manifold* M *and $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$ be vector fields. It holds*

$$\nabla_X Y = \left(X^j \partial_j Y^k + X^j Y^i \Gamma^k_{ij} \right) \partial_k,$$

in particular

$$(\nabla_j Y)^i = \partial_j Y^i + Y^i \Gamma^k_{ij}.$$

■

Proof. From Definition (1.1.15) holds:

$$\nabla_X Y = \nabla_{X^j e_j} Y^i e_i = X^j \partial_j Y^i e_i = X^j Y^i \nabla_j e_i + X^j e_i \partial_j Y^i =$$
$$= X^j Y^i \Gamma^k_{ij} e_k + (X^j \partial_j Y^k) e_k.$$

1.2.12 Proposition. *Let us consider $p \in \mathrm{M}$ and a tangent vector $\xi \in \mathrm{T}_p\mathrm{M}$. Then there exists $\varepsilon > 0$ and precisely one geodesic*

$$c_\xi : [0, \varepsilon] \to \mathrm{M},$$

such that $c_\xi(0) = p$ and $\dot{c}_\xi(0) = \xi$. ∎

1.2.13 Definition. *In the conditions of the proposition above, if we put*

$$\mathcal{D}_p = \{\xi \in \mathrm{T}_p\mathrm{M} \mid c_\xi \text{ is defined on } [0,1]\} \subset \mathrm{T}_p\mathrm{M},$$

*the **exponential map** at point p is defined as $\exp_p : \mathcal{D}_p \to \mathrm{M}$ such that $\exp_p(\xi) = c_\xi(1)$.*
*The local coordinates defined by the chart $\{U := \exp_p(\mathcal{D}_p), \exp_p^{-1}\}$ are called **normal coordinates** centered at p.* ∎

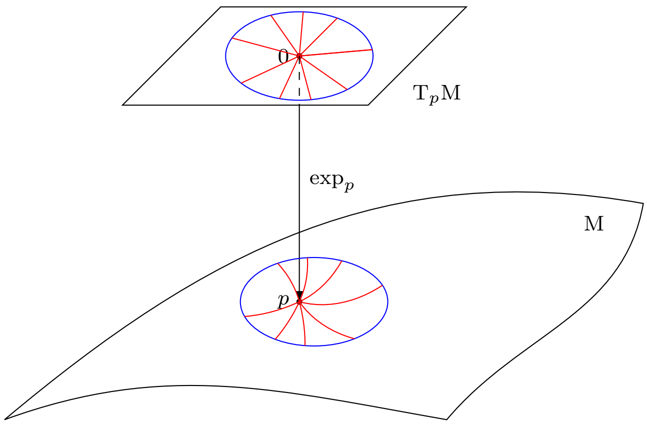


FIGURE 1.9: The exponential maps from the tangent space to the manifold.

1.2.14 Proposition. *Given normal coordinates centered at $p \in \mathrm{M}$, it holds*

$$g_{ij}(p) := g_{ij}(\exp_p(0)) = \eta_{ij},$$
$$\Gamma^i_{jk} = 0,$$

for all indexes i, j, k. ∎

We are now ready to talk about **geodesically starshaped** sets.

1.2.15 Definition. *An open subset $\Omega \subset \mathrm{M}$ is called **geodesically starshaped** with respect to a point $p \in \mathrm{M}$ if there exists an open subset $\Omega' \subset \mathrm{T}_p\mathrm{M}$, starshaped with respect to 0, such that the exponential map*

$$\exp_p|_{\Omega'} : \Omega' \to \Omega,$$

*is a diffeomorphism. If Ω is geodesically starshaped with respect to all of its points, one calls it **convex**.* ■

1.2.16 Proposition. *Under the conditions of the last definition, let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to point $p \in \mathrm{M}$. Then one has*

$$I_{\pm}^{\Omega}(p) = \exp_p(I_{\pm}(0) \cap \Omega'),$$

$$J_{\pm}^{\Omega}(p) = \exp_p(J_{\pm}(0) \cap \Omega').$$

■

1.2.1 Causality and Global Hyperbolicity

Now we introduce causal domains, because they will appear in the theory of wave equations. The local construction of fundamental solutions is always possible on causal domains, provided they are small enough.

1.2.17 Definition. *A domain $\Omega \subset \mathrm{M}$ is called **causal** if its closure $\overline{\Omega}$ is contained in a convex domain Ω' and for any $p, q \in \overline{\Omega}$ $J_{+}^{\Omega'}(p) \cap J_{-}^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.*
*A subset $A \subset \mathrm{M}$ is called **past-compact** (respectively **future-compact**) if, for all $p \in \mathrm{M}$, $A \cap J_{-}^{\mathrm{M}}(p)$ (respectively $A \cap J_{+}^{\mathrm{M}}(p)$) is compact.* ■

We can notice that, if we look at compact spacetimes, something physically unsound occurs:

1.2.18 Proposition. *If a spacetime M is compact, there exists a closed timelike curve in M.* ■

In a few words, in compact spacetimes there are timelike loops that can produce science fictional paradoxes. To avoid such unphysical and unrealistic things we require suitable causality conditions:

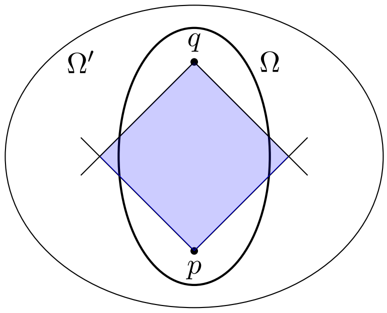


FIGURE 1.10: Convex, but non causal, domain

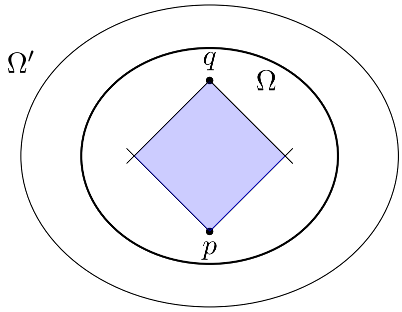


FIGURE 1.11: Causal domain

1.2.19 Definition. *A spacetime satisfies the* ***causality condition*** *if it does not contain any closed causal curve. A spacetime* M *satisfies the* ***strong causality condition*** *if there are no almost closed causal curves, i.e. if for any* $p \in \mathrm{M}$ *there exists a neighborhood* U *of* p *such that there exists no timelike curve that passes through* U *more than once.* ■

It is clear that the strong causality condition implies the causality condition.

1.2.20 Definition. *A spacetime* M *that satisfies the strong causality condition and such that for all* $p, q \in \mathrm{M}$ $J_+^{\mathrm{M}}(p) \cap J_-^{\mathrm{M}}(q)$ *is compact is called* ***globally hyperbolic****.* ■

It can be demonstrated that in globally hyperbolic manifolds for any $p \in \mathrm{M}$ and any compact set $K \subset \mathrm{M}$ the sets $J_\pm^{\mathrm{M}}(p)$ and $J_\pm^{\mathrm{M}}(K)$ are closed.

1.2.21 Definition. *A subset* S *of a connected time-oriented Lorentzian mani-*

(a) A non-Cauchy hypersurface (b) A Cauchy hypersurface

Figure 1.12

fold M *is a* ***Cauchy hypersurface*** *if each inextensible timelike curve (i.e. no reparametrisation of the curve can be continuously extended) in* M *meets* S *at exactly one point.* ■

In other words, no point of a Cauchy hypersurface is in the light cone of another point of the surface.

1.2.22 Theorem. *Let* M *be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

- M *is globally hyperbolic.*
- *There exists a Cauchy hypersurface in* M.
- M *is isometric to* $\mathbb{R} \times S$ *with metric* $g = -\beta \mathrm{d}t^2 + b_t$, *where* β *is a smooth positive function,* b_t *is a Riemannian metric on* S *depending smoothly on* t *and each* $\{t\} \times S$ *is a smooth Cauchy hypersurface in* M.

In such case there exists a smooth function $h : \mathrm{M} \to \mathbb{R}$ *whose gradient is past-directed timelike at every point and all of whose level sets are Cauchy hypersurfaces.* ■

1.3 Operators and integration on manifolds

We call $C_0^\infty(\mathrm{M})$ the set of C^∞ functions on a manifold with compact support. The integral map is defined as the unique map

$$\int_{\mathrm{M}} \cdot \, \mathrm{d}\mu : C_0^\infty(\mathrm{M}) \to \mathbb{C},$$

such that it is linear and for any local chart $\{U, \varphi\}$ and for any $f \in C_0^\infty(U)$ holds

$$\int_{\mathrm{M}} f \, \mathrm{d}\mu = \int_{\varphi(U)} (f \circ \varphi^{-1})(x) \, \mu_x \, \mathrm{d}^n x,$$

where we define

$$\mu_x := |\det g(x)|^{1/2}. \tag{1.3}$$

In this section we introduce the **generalized d'Alembert** operators, whose general form in local coordinates is given by

$$P = -g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a_j(x)\frac{\partial}{\partial x^j} + b(x).$$

The d'Alembert operator $\Box$ is defined for smooth functions f as

$$\Box f = -\operatorname{div}\operatorname{grad} f,$$

where $\operatorname{grad} f$ is a vector field defined by the requirement that the formula

$$\langle \operatorname{grad} f, X\rangle = \partial_X f$$

holds for any vector field X. At the same time div is defined as follows

1.3.1 Definition. *The* ***divergence*** *of a vector field $Z = Z^i\partial_i$ is defined as*

$$\operatorname{div} Z = \sum_j (\nabla_j Z)_j = \partial_j Z^j + \Gamma^i_{ij} Z^j.$$

■

1.3.2 Proposition. *The following formula holds:*

$$\operatorname{div} Z = \mu_x^{-1}\frac{\partial}{\partial x^j}\left(\mu_x Z^j\right)$$

and the definition of divergence is consistent with that of integral. ■

Proof. Let $h \in C_0^\infty(\mathrm{M})$; using integration by parts

$$\int_{\mathrm{M}} h \cdot \operatorname{div}(Z) \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mu_x \, \mathrm{d}x.$$

Now integrating by parts again in the chart it holds:

$$-\int_{\mathrm{M}} \partial_j h \; Z^j \mu_x \, \mathrm{d}x = \int_{\mathrm{M}} h \; \partial_j(\mu_x \, Z^j) \, \mathrm{d}x = \int_{\mathrm{M}} h \; \mu_x^{-1} \partial_j(\mu_x \; Z^j) \, \mathrm{d}\mu.$$

Since this is true for all function h, the formula is proved. ■

From the definition of gradient one can show

$$g_{ij}(\mathrm{grad}\, f)^i X^j = \partial_X f = X^j \frac{\partial f}{\partial x^j}, \quad \mathrm{grad}\, f = g^{ij} \frac{\partial f}{\partial x^i} \partial_j.$$

Hence it holds

$$\Box f = -\mu_x^{-1} \frac{\partial}{\partial x^j} \left(\mu_x\, g^{ij} \frac{\partial f}{\partial x^i} \right).$$

In Minkowski spacetime, where $g = \eta$,

$$\Box f = -\frac{\partial}{\partial x^j} \left(\eta^{jj} \frac{\partial f}{\partial x^j} \right) = \frac{\partial^2 f}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial x_i^2} = -\partial^i \partial_i f.$$

1.4 Distributions and Fourier Transform

SECTION TO EXTEND!1!!?'?

Firstly, we recall the main concepts of the theory of distributions.

1.4.1 Definition. *The space of distributions over a manifold* M *is defined as*

$$\mathcal{D}'(\mathrm{M}) = \{u : \mathcal{D}(\mathrm{M}) \to \mathbb{C} \text{ linear and continuous } \}.$$

■

The distribution $\frac{1}{x}$ is defined as

$$\frac{1}{x} = \mathrm{PV}\left(\frac{1}{x}\right) - i\pi\delta(x). \tag{1.4}$$

A particular useful formula gives

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)] \tag{1.5}$$

1.4.2 Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\rho \in \mathcal{D}(\mathbb{R}^n)$. Then $\rho * u \in C^\infty(\mathbb{R}^n)$.* ■

1.4.3 Definition. *We define the translational operator $T_{x_0} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ such that if $u \in \mathcal{D}'(\mathbb{R}^n)$,*

$$(T_{x_0} u(x), \varphi(x)) = (u(x), \varphi(x + x_0)),$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We often write $u(x - x_0) := T_{x_0} u(x)$. ■

Given a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$\widehat{f}(k) = \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle} \, \mathrm{d}x \tag{1.6}$$

We naturally extend the Fourier transform to a unitary map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and to act on tempered distributions in such a way that for $\varphi \in \mathcal{S}'(\mathcal{U})$ holds

$$\widehat{\varphi}(k) = \left(\varphi, e^{-i\langle k,x\rangle}\right).$$

If $\varphi \in \mathcal{E}'(\mathcal{U})$, $\widehat{\varphi}$ results a smooth function which extends to an entire function $\widehat{\varphi}(z)$, $z \in \mathbb{C}$.

The inverse Fourier transform is given as $\check{f}(x) := (2\pi)^{-n} \widehat{f}(-x)$ and it holds that $f = \check{\widehat{f}}$.

1.4.4 Example. The Fourier transform of the distribution $\delta \in \mathcal{E}'(\mathcal{U})$ is

$$\widehat{\delta}(k) = 1. \tag{1.7}$$

This is a straightforward computation:

$$\widehat{\delta}(k) = (\delta(x), e^{-i\langle k,x\rangle}) = e^0 = 1.$$

■

1.4.5 Proposition. *For $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ and for any multi-index α hold:*

$$\widehat{\partial^\alpha \varphi}(k) = (ik)^\alpha \widehat{\varphi}(k)$$

$$\widehat{x^\alpha \varphi}(k) = (i\partial)^\alpha \widehat{\varphi}(k).$$

■

Fundamental solutions in Minkowski spacetime 2

In this chapter, we will illustrate the concept of fundamental solutions and their use in solving initial value problems. We will focus on the case of d'Alembert operator $\Box$ when the Lorentzian manifold is flat, i.e. the Minkowski spacetime $\mathbb{M}^n$.
Two different approaches will be followed. The first, via Fourier transform, is useful to build explicit formulas for the fundamental solutions in the lower dimensional cases, but it can be also used to show the existence of such solutions in the general case and the physical importance of two of them: the *retarded* and the *advanced* one.
The second approach, via Riesz distributions, is useful for its generality and because it will be used in the next chapter to construct fundamental solutions on suitable Lorentzian manifolds, although it is more abstract and lacks of explicit formulas.

2.1 Fundamental solutions

2.1.1 Definition. *Let P be a differential operator on a manifold* M *and $x_0 \in \mathrm{M}$. A* ***fundamental solution*** *for P at x_0 is a distribution $u_{x_0} \in \mathcal{D}'(\mathrm{M})$ such that*

$$Pu_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac delta distribution in x_0, i.e. $(\delta_{x_0}, f) = f(x_0)$ for all $f \in \mathcal{D}(\mathrm{M})$. ■

Under the assumptions of the previous definition, the distribution $F_\psi(x) = (u_x, \psi)$, where u_x is a fundamental solution of P at x with a continuous dependence on x (i.e. the function $x \mapsto (u_x, \varphi)$ is continuous for all $\varphi \in \mathcal{D}(\mathbb{M})$) and $\psi \in \mathcal{D}'(\mathbb{M})$, is a solution for the differential equation

$$PF_\psi = \psi.$$

This comes applying the operator P on F_ψ, for which one obtains

$$PF_\psi = (u_x, P^*\psi) = (Pu_x, \psi) = (\delta_x, \psi) = \psi(x),$$

where P^* stands for the formal adjoint of P.

2.2 The d'Alembert operator in Minkowski

In order to be more concrete, we begin by approaching the computation of the fundamental solution of $\Box$ via a Fourier transform, calculating explicit formulas for the lower dimensional Minkowski case.
As we recalled in the previous chapter, the d'Alembert operator is defined in $\mathbb{M}^n$, with the variable $x = (t, \mathbf{x})$, as

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} = -\partial^i \partial_i.$$

We look for fundamental solutions $G^+_{x_0}, G^-_{x_0} \in \mathcal{D}'(\mathbb{M}^n)$ for $\Box$ at $x_0 \in \mathbb{M}^n$ such that

$$\operatorname{supp}(G^+_{x_0}) \subset J_+^{\mathbb{M}^n}(x_0), \qquad \operatorname{supp}(G^-_{x_0}) \subset J_-^{\mathbb{M}^n}(x_0). \tag{2.1}$$

Such solutions will be called respectively **retarded** (G^+) and **advanced** (G^-) fundamental solutions.
In order to find the fundamental solution at x_0, the following proposition guarantees it suffices to solve the problem $\Box u_0 = \delta_0$, ~~and then shift the solution.~~

2.2.1 Proposition. *Let $x_0 \in \mathbb{M}^n$ and T_{x_0} be the translational operator as in Definition (1.4.3). Then $[\Box, T_{x_0}] = 0$ (i.e. $\Box$ and T_{x_0} commute) and a fundamental solution for $\Box$ at x_0 is*

$$u_{x_0} = T_{x_0} u_0,$$

where u_0 is a fundamental solution at 0. ■

Proof. Let $\varphi \in \mathcal{D}(\mathbb{M}^n)$, then

$$(\Box T_{x_0} u(x), \varphi(x)) = (T_{x_0} u(x), \Box\varphi(x)) = (u(x), \Box\varphi(x + x_0)) =$$
$$= (\Box u(x), \varphi(x + x_0)) = (T_{x_0} \Box u(x), \varphi(x)) \,,$$

where it was used the fact that $\Box$ is formally self-adjoint and invariant under translations when acting on smooth functions. Hence, it holds

$$\Box \left(T_{x_0} u_0\right) = T_{x_0} \left(\Box u_0\right) = T_{x_0} \delta_0 = \delta_{x_0},$$

because $\Box$ and T_{x_0} commute, so $T_{x_0} u_0$ is a fundamental solution at x_0. ■

2.2.2 Proposition. *Let $\psi \in \mathcal{D}(\mathbb{M}^n)$. Then*

$$F_\psi = u_0 * \psi \in C^\infty(\mathbb{M}^n) \tag{2.2}$$

is a (smooth) solution for the differential equation $PF_\psi = \psi$. ■

Proof. Since $u_x = T_x u_0$ and $F_\psi = (u_x, \psi) = (T_x u_0, \psi)$ is a solution to the equation, the thesis follows immediately noting that, by definition, $(T_x u_0, \psi) = (u_0 * \psi)(x)$. The smoothness of the solution follows from Theorem (1.4.2). ■

2.3 The Fourier transform approach

In order to perform Fourier transforms, it is necessary to work with distributions in $\mathcal{S}'(\mathbb{M}^n)$.
We shall begin with a lemma that helps in the computations:

2.3.1 Lemma. *For $u \in \mathcal{S}'(\mathbb{M}^n)$ if we let $x = (t, \mathbf{x}) = (t, x_1, \ldots, x_{n-1})$ and $k = (\omega, \mathbf{k}) = (\omega, k_1, \ldots, k_{n-1})$, it holds*

$$\widehat{\Box\, u}(k) = \|k\|^2\, \widehat{u} = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k). \tag{2.3}$$

■

Proof. For any test function $f \in \mathcal{S}(\mathbb{M}^n)$, and for any $u \in \mathcal{S}'(\mathbb{M}^n)$ $(\Box u, f) = (u, \Box f)$, so

$$(\Box u, e^{-i\langle k,x\rangle_0}) = \left(u, \Box e^{-i\langle k,x\rangle_0}\right) = \left(u, \Box e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\right) =$$

$$= \left(u, (|\mathbf{k}|^2 - \omega^2)e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\left(u, e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k).$$

■

We start transforming the equation:

$$\widehat{\Box u}(k) = \widehat{\delta}(k) \Rightarrow (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k) = 1$$

The collection of solutions $\widehat{u} \in \mathcal{S}'(\mathbb{M}^n)$, is composed by particular solutions of the inhomogeneous equation at which to add the solutions $\widehat{v}$ for the homogeneous equation $(|\mathbf{k}|^2 - \omega^2)\widehat{v}(k) = 0$.

If we concentrate on the solutions for the homogeneous equation, it is easy to see with a direct computation that any distribution of the form

$$\widehat{v}(k) = A(k)\delta(|\mathbf{k}|^2 - \omega^2),$$

where $A(k)$ is a suitable function of k, solve the equation, because the delta is supported exactly where $|\mathbf{k}|^2 - \omega^2$ vanishes. Any solution to the homogeneous wave equation can be obtained by the inverse transform of $\widehat{v}$:

$$v(x) \cancel{= \check{\widehat{v}}(x) = (2\pi)^{-n}\widehat{\widehat{v}}(-x) =} (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0} A(k)\delta(|\mathbf{k}|^2 - \omega^2)\,\mathrm{d}k.$$

Making use of formula (1.5), the last expression becomes

$$\begin{aligned} v(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0}\frac{A(k)}{2|\mathbf{k}|}\left[\delta(|\mathbf{k}| - \omega) + \delta(|\mathbf{k}| + \omega)\right]\mathrm{d}k = \\ &= (2\pi)^{-n}\int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{e^{i|\mathbf{k}|t}A(k)|_{|\mathbf{k}|=\omega} + e^{-i|\mathbf{k}|t}A(k)|_{|\mathbf{k}|=-\omega}}{|\mathbf{k}|}. \end{aligned} \tag{2.4}$$

To solve for particular solutions we are tempted to write a formula like this:

$$\widehat{u}(k) = \frac{1}{|\mathbf{k}|^2 - \omega^2} = \frac{1}{(|\mathbf{k}| - \omega)(|\mathbf{k}| + \omega)},$$

which is ill-defined where $\langle k, k\rangle_0 = 0$, i.e. on the light-cone of the Fourier space. Hence, we try to define $\widehat{u}$ as limit of a sequence of distributions depending on the parameter ε

$$\widehat{u}_\varepsilon = \frac{1}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} = \frac{1}{(|\mathbf{k}| \mp i\varepsilon - \omega)(|\mathbf{k}| \pm i\varepsilon + \omega)}$$

promoting ω to a complex variable and taking the limit for $\varepsilon \to 0^+$ after performing the inverse transform. The choice of the signs in such expressions leads to different fundamental solutions.

2.4 Fundamental solutions via Fourier transform

The distributions G^+ and G^- in $\mathcal{S}'(\mathbb{M}^n)$, defined respectively as the limit for $\varepsilon \to 0^+$ of the following sequences of distributions

$$G^+_\varepsilon(x) = \frac{1}{(2\pi)^n} \int \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}| - (\omega + i\varepsilon)^2} \,\mathrm{d}k, \text{ and} \tag{2.5}$$

$$G^-_\varepsilon(x) = \frac{1}{(2\pi)^n} \int \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}| - (\omega - i\varepsilon)^2} \,\mathrm{d}k, \tag{2.6}$$

are respectively a **retarded** and an **advanced** fundamental solutions at $x_0 = 0$ for the d'Alembert operator.
The aim is to prove that $\operatorname{supp}(G^+) \subset J^M_+(0)$ and $\operatorname{supp}(G^-) \subset J^M_-(0)$, and we proceed firstly by calculating the explicit formula for $2 \leq n \leq 4$ and then discuss the general case via Riesz distributions.
We compute $G^\pm$ as a limit of the inverse of the Fourier transform:

$$\begin{aligned} G^\pm_\varepsilon(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \,\mathrm{d}k \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \,\mathrm{d}\omega. \end{aligned} \tag{2.7}$$

Computing the complex integrals

In order to calculate the inner integral in the former expression,

$$\tilde{G}^\pm_\varepsilon(t, \mathbf{k}) := \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \,\mathrm{d}\omega,$$

we make use of techniques of complex analysis as follows.
Now we denote

- C^+_ρ the upper half-circumference of radius ρ centered at $\omega = 0$ which has Im $(\omega) > 0$, oriented counter-clockwise;
- C^-_ρ the lower half-circumference of radius ρ centered at $\omega = 0$ which has Im $(\omega) < 0$, oriented clockwise;
- $[-\rho, \rho]$ the interval of the real line connecting $-\rho$ and ρ, oriented from left to right.

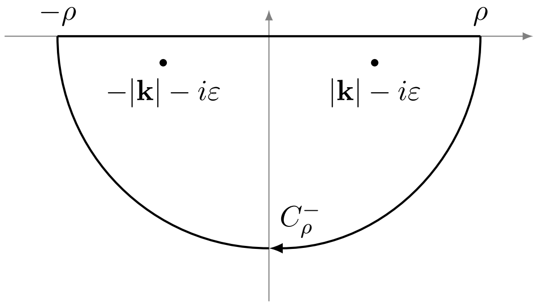


FIGURE 2.1: The circuit to compute $\tilde{G}_\varepsilon^+(t, \mathbf{k})$ for $t > 0$.

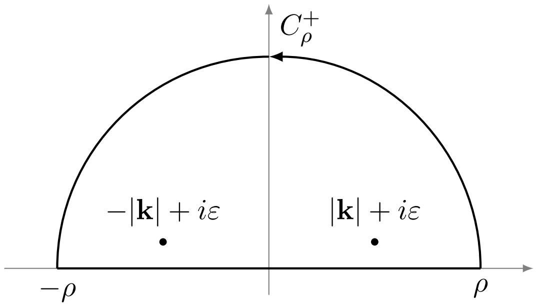


FIGURE 2.2: The circuit to compute $\tilde{G}_\varepsilon^-(t, \mathbf{k})$ for $t < 0$.

The singularities are

$$\text{for } \tilde{G}_\varepsilon^+ : \quad \omega = \pm|\mathbf{k}| - i\varepsilon,$$

$$\text{for } \tilde{G}_\varepsilon^- : \quad \omega = \pm|\mathbf{k}| + i\varepsilon.$$

Hence we have

$$\text{for } t < 0 \quad \tilde{G}_\varepsilon^+(t, \mathbf{k}) = \lim_{\rho\to\infty} \int_{C_\rho^+ + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega + i\varepsilon)^2} \mathrm{d}\omega = 2\pi i \sum \mathrm{Res} = 0,$$

where the sum is extended to the singularities in the upper half-plane. The expression vanishes because we choose the circuit such that the integral on C_ρ^+ vanishes in virtue of Jordan's lemma and there are no singularities in the region bounded by the circuit.
For the same reasons

$$\text{for } t > 0, \tilde{G}_\varepsilon^-(t, \mathbf{k}) = \lim_{\rho\to\infty} \int_{C_\rho^- + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega - i\varepsilon)^2} \mathrm{d}\omega = -2\pi i \sum \mathrm{Res} = 0,$$

where the sum is extended th the singularities of the function in the lower half-plane and the minus sign arises because of the clockwise circuit. The non-zero integrals are indeed

$$\tilde{G}^+_\varepsilon(t,\mathbf{k}),\ \text{for } t>0, \qquad \tilde{G}^-_\varepsilon(t,\mathbf{k}),\ \text{for } t<0.$$

The first is computed via the lower circuit in figure (**??**): $C^-_\rho + [-\rho,\rho]$, the second via the upper circuit in figure (**??**): $C^+_\rho + [-\rho,\rho]$, in order to get avoid contributes from the half-circumferences.

The results are

$$\text{for } t>0 \quad \tilde{G}^+_\varepsilon(t,\mathbf{k}) = 2\pi i e^{-\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = 2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t},$$

$$\text{for } t<0 \quad \tilde{G}^-_\varepsilon(t,\mathbf{k}) = -2\pi i e^{\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = -2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\varepsilon|\mathbf{k}|t}.$$

Summing up everything in one formula:

$$\tilde{G}^\pm_\varepsilon(t,\mathbf{k}) = \pm 2\pi\Theta(\pm t)\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\mp\varepsilon|\mathbf{k}|t}, \tag{2.8}$$

where Θ is the Heaviside step-function.
It can be noticed that

$$\tilde{G}^+_\varepsilon(t,\mathbf{k}) = \tilde{G}^-_\varepsilon(-t,\mathbf{k}),$$

because of the parity of sine function. So, we can deduce that the **advanced** solution can be easily calculated from the retarded one with a time inversion:

$$G^-(t,\mathbf{x}) = G^+(-t,\mathbf{x}). \tag{2.9}$$

We can now show that the support $G^\pm$ is, at least, included in $J_+(0)\cap J_-(0)$.

2.4.1 Proposition. *If $x\in\mathbb{M}^n$ is spacelike, $G^\pm(x)=0$.* ■

Proof. Consider a frame of reference R in which $x=(t,\mathbf{x})$ and suppose for now $t\geq 0$; hence $G^+(x)=0$. Since $G^\pm$ are manifestly Lorentz invariant (the calculation involves a scalar product) and x is spacelike, one can find a frame of reference R' in which $x = (t', \mathbf{x}')$ and $t' < 0$, so that $G^-(x) = 0$. The opposite case can be treated similarly. ■

Now we come back to equation (2.7) and show explicit solutions for spacial dimensions d from 1 to 3:

$$G^+_{(d)}(x) = (2\pi)^{-n} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{G}^+_\varepsilon(t,\mathbf{k}) =$$

$$= \lim_{\varepsilon\to 0^+} \frac{\Theta(t)}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} e^{-\varepsilon|\mathbf{k}|t}.$$

Dimension $n = 1 + 1$ - wave on a line

The integral we have to perform in the $n = 1 + 1$ dimensional case is:

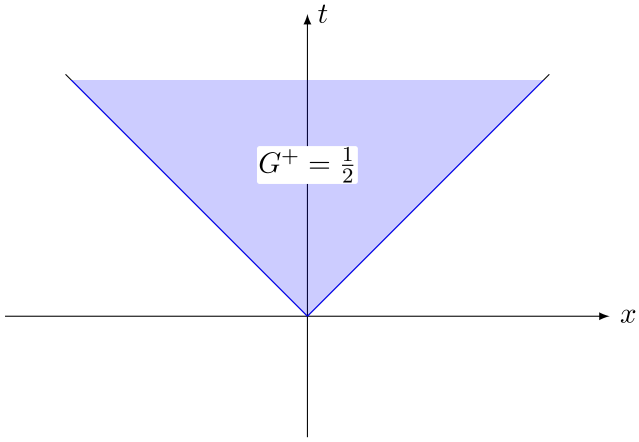


FIGURE 2.3: The support of G^+ in 1+1 dimensional case.

$$G^+_{(1)}(t,x) = \frac{\Theta(t)}{2\pi} \lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x} \frac{\sin kt}{k} e^{-\varepsilon kt}.$$

It holds

$$\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x+i\varepsilon t)} \frac{\sin kt}{k} = \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x/t+i\varepsilon)} \frac{\sin k}{k} \xrightarrow[\varepsilon\to 0^+]{}$$

$$\longrightarrow \pi\chi_{[-1,1]}\left(\frac{x}{t}\right) = \pi\chi_{[-t,t]}(x),$$

where

$$\chi_{[a,b]}(z) = \begin{cases} 1, & \text{if } z \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

Finally the integrals become

$$G^+_{(1)}(t,x) = \frac{\Theta(t)}{2}\chi_{[-t,t]}(x) = \frac{\Theta(t-|x|)}{2}$$

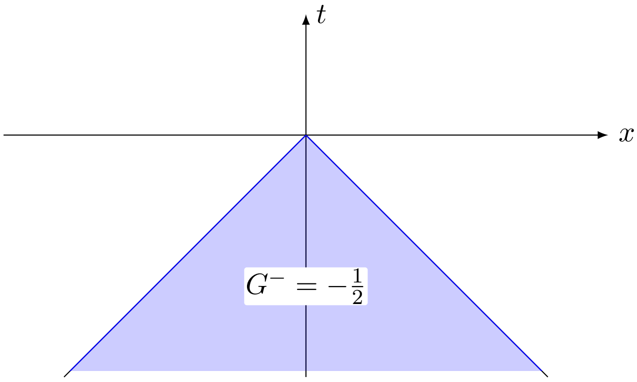


FIGURE 2.4: The support of G^- in 1+1 dimensional case.

From FIGURE (**??**) and (**??**) one can infer that the fundamental solutions are supported respectively on $J_+(0)$ and $J_-(0)$.

Dimension $n = 1+2$ - wave on a surface

The integral is two dimensional:

$$G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{(2\pi)^2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t}.$$

To evaluate it we switch to polar coordinates $\mathbf{k} = (k\cos\varphi, k\sin\varphi)$. With the substitution

$$\mathrm{d}\mathbf{k} = k\,\mathrm{d}k\,\mathrm{d}\varphi,$$

the integral becomes ($x := |\mathbf{x}|$)

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{+\infty}\mathrm{d}k\, e^{ikx\cos\varphi}\sin kt\, e^{-\varepsilon kt}.$$

It holds that

$$\int_0^\infty \mathrm{d}k\, e^{ik(y+i\varepsilon)} \sin kt = \frac{1}{2i}\left[\int_0^{+\infty} e^{ik(y+i\varepsilon+t)}\,\mathrm{d}k + \int_0^{+\infty} e^{ik(y+i\varepsilon-t)}\,\mathrm{d}k\right] =$$

$$= \frac{1}{2}\left[I_\varepsilon(y+t) + I_\varepsilon(y-t)\right],$$

where we called $I_\varepsilon(y) = \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k$. With a straightforward calculation we find

$$I_\varepsilon(y) = \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k = \frac{1}{y+i\varepsilon},$$

and the integral to calculate is now

$$\frac{1}{2}\int_0^{2\pi}\left[\frac{1}{x\cos\varphi + t + i\varepsilon} + \frac{1}{x\cos\varphi - t + i\varepsilon}\right]\mathrm{d}\varphi.$$

Such integral has a counterpart over a unit circle in the complex plane with the substitutions $\mathrm{d}\varphi = -i\mathrm{d}z/z$ and $\cos\varphi = (z+z^{-1})/2$. Hence, using the residue theorem

$$\int_0^{2\pi} \frac{1}{x\cos\varphi \pm t + i\varepsilon}\,\mathrm{d}\varphi = -2i\oint \frac{\mathrm{d}z}{xz^2 + 2(\pm t + i\varepsilon) + x} = \frac{2\pi}{\sqrt{(t\mp i\varepsilon)^2 - x^2}}.$$

Putting everything together in the limit $\varepsilon \to 0$

$$G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi}\frac{\Theta(t^2-|\mathbf{x}|^2)}{\sqrt{t^2-|\mathbf{x}|^2}} = \frac{\Theta(t)}{2\pi}\frac{\Theta(\gamma(x))}{\sqrt{\gamma(x)}}, \tag{2.10}$$

where we put the $\Theta(t^2-|\mathbf{x}|^2)$ to take care of what we demonstrated in Proposition (2.4.1). As a by-product, $\mathrm{supp}(G^\pm) \subseteq J_\pm(0)$.

Dimension $n = 1+3$ - spherical wave

The three-dimensional integral is

$$G^+_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{(2\pi)^3}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3}\mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t}.$$

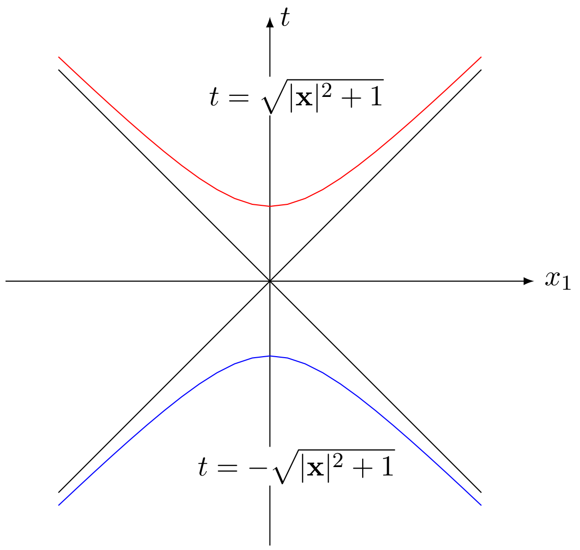


FIGURE 2.5: The level set $G^{\pm}(\mathbf{x}, t) = 1$ in the 1+2 dimensional case, plotted for one spacial axis.

Again, we make a change of coordinate, switching to the spherical ones: $\mathbf{k} = (k \sin\vartheta \cos\varphi, k \sin\vartheta \sin\varphi, k \cos\vartheta)$. The substitution is

$$\mathrm{d}\mathbf{k} = k^2 \sin\vartheta \, \mathrm{d}k \, \mathrm{d}\vartheta \, \mathrm{d}\varphi$$

and the integral to calculate is ($x := |\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}k \, k \sin kt \int_{-1}^{1} e^{ikx\cos\vartheta} \mathrm{d}(\cos\vartheta) = \frac{4\pi}{x + i\varepsilon} \int_0^{+\infty} \sin kt \sin k(x+i\varepsilon) \, \mathrm{d}k.$$

Hence we can write using the exponential function

$$\sin kt \sin kx = \frac{1}{4} \left\{ \left[e^{ik(x+i\varepsilon-t)} + e^{-ik(x+i\varepsilon-t)} \right] - \left[e^{ik(x+i\varepsilon+t)} + e^{-ik(x+i\varepsilon+t)} \right] \right\}$$

and with the change of variables $k \leftrightarrow -k$ we have

$$\frac{4\pi}{x} \int_0^{+\infty} \sin kt \sin kx \, \mathrm{d}k = \frac{2\pi^2}{x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x+i\varepsilon-t)} - e^{ik(x+i\varepsilon+t)} \, \mathrm{d}k \xrightarrow[\varepsilon \to 0^+]{}$$

$$\longrightarrow \frac{2\pi^2}{x} \left[\delta(t-x) - \delta(t+x) \right].$$

To find the correct retarded and advanced fundamental solutions we notice that the second term, $\delta(t+x)$, vanishes for G^+ because $x>0$ and $t>0$; conversely the first term $\delta(t-x)$ vanishes when computing G^-. In view of these considerations, the general formula for the $n=1+3$ case becomes

$$G^{+}_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{4\pi}\frac{\delta(t-|\mathbf{x}|)}{|\mathbf{x}|}. \tag{2.11}$$

One can verify that $G^{\pm}_{(3)}$ vanish outside the support of the delta distribution; hence they are supported respectively on the upper and on the lower light cones $C_+(0)$ and $C_-(0)$. Such supports are included in $J_\pm(0)$ as required, but the unique feature of being supported only on the light cone is a particularity of the odd spacial dimensions case, as we shall prove in the next section and it is known as the **Huygens' principle**. It states that in general, we have for spacial dimensions $d \neq 1$

$$\operatorname{supp} G^{\pm}_{(d)} = J_\pm(0) \quad \text{for } d \text{ even},$$

$$\operatorname{supp} G^{\pm}_{(d)} = C_\pm(0) \quad \text{for } d \text{ odd}.$$

Physically, we can see δ_0 as point source at 0 of a signal that propagates with constant speed. Inside the future light cone the solution is zero, so if d is even, the wave propagates strictly on the cone. In case d is odd, the signal of a point source propagates also inside the light cone. For an observer, the wave is noticeable not only at a single moment but still after the signal has arrived. An example of such waves are waves on 2-dimensional surfaces like water waves.

The method of *descent*

The lower dimensional advanced and retarded distributions can be directly deduced from the $d=3$ case as we shall see. In general if we know the explicit solution to the d case we can find the $d-1$ solution with the formula

$$G^{\pm}_{(d-1)}(t,x_1,\dots,x_{d-1}) = \int_{-\infty}^{\infty} G^{\pm}_{(d)}(t,x_1,\dots,x_d)\,\mathrm{d}x_d.$$

This technique is called *method of descent*. The last assertion can be proven taking the fundamental solution equation

$$\Box_d\, G_{(d)}(t,x_1,\dots,x_d) = \delta(t)\delta(x_1)\cdots\delta(x_d),$$

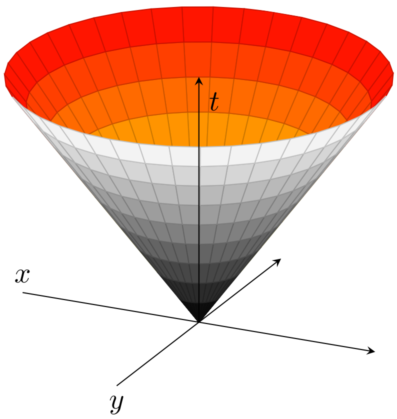


FIGURE 2.6: The support of G^+ in 1+3 dimensional case, i.e. the upper light cone $C_+(0)$, plotted for two spacial axis.

where $\Box_d$ stands for $(\partial_t^2 - \partial_1^2 - \cdots - \partial_d^2)$, and integrating it on the last variable with a test-function $\varphi \in \mathcal{D}$:

$$\int \Box_d \, G_{(d)} \, \varphi \, \mathrm{d}x_d = \Box_{(d-1)} \int G_{(d)} \, \varphi \, \mathrm{d}x_d - \int \partial_d^2 G_{(d)} \, \varphi \, \mathrm{d}x_d =$$

$$= \delta(t)\delta(x_1) \cdots \delta(x_{d-1}) \int \delta(x_d)\varphi \, \mathrm{d}x_d.$$

By letting the test-function become a sequence of cut-off functions for the domain, the formula

$$\Box_{(d-1)} \int G_{(d)} \, \mathrm{d}x_d = \delta(t)\delta(x_1) \cdots \delta(x_{d-1})$$

is proven.

To calculate the $d = 2$ fundamental solution from the $d = 3$ case, according to (2.11) and using the powerful formula (1.5) we can write for the retarded solution

$$\Theta(t)\frac{\delta(t - |\mathbf{x}|)}{2|x|} = \Theta(t)\delta(t^2 - |\mathbf{x}|^2) = \Theta(t)\delta(t^2 - x_1^2 - x_2^2 - x_3^2) =$$

$$= \Theta(t)\Theta(t^2 - x_1^2 - x_2^2)\frac{\delta(x_3 - \sqrt{t^2 - x_1^2 - x_2^2}) + \delta(x_3 + \sqrt{t^2 - x_1^2 - x_2^2})}{2\sqrt{t^2 - x_1^2 - x_2^2}},$$

where we put the Heaviside step function to take into account Proposition (2.4.1). Hence, integrating over the third variable makes the deltas disappear with a gain of a factor of 2 and we have

$$G^{+}_{(2)}(t,x_1,x_2) = \frac{\Theta(t)}{2\pi}\frac{\Theta(t^2-x_1^2-x_2^2)}{\sqrt{t^2-x_1^2-x_2^2}},$$

which is identical to Equation (2.10) as we expected. Similarly, the expression for the case $d=1$ can be derived again.

If we now apply to the general expression for $G_{(d)}$ the Fourier transform in the last variable, with similar calculation we obtain

$$\int_{-\infty}^{+\infty} \partial_d^2 G_{(d)}(t,x_1,\ldots,x_d)e^{-imx_d}\,\mathrm{d}x_d =$$

$$= -m^2\int_{-\infty}^{+\infty} G_{(d)}(t,x_1,\ldots,x_d)e^{-imx_d}\,\mathrm{d}x_d =: -m^2\widehat{G}_{d-1}(t,x_1,\ldots,x_{d-1},m),$$

hence, the Fourier transform on the last variable $\widehat{G}_{d-1}(t,x_1,\ldots,x_{d-1},m)$ is, for a fixed $m\in\mathbb{R}$, a fundamental solution for the $d-1$ dimensional **Klein-Gordon** operator

$$\Box + m^2,$$

that describes the motion of spinless particles with mass m.

2.5 The Riesz distributions

To discuss an explicit and useful formula of the fundamental solution for the wave operator in the general n-dimensional case, the approach we adopted in the last section, that made use of Fourier transform, is not very effective. The distributions $R_\pm(\alpha)$ that we here define, were introduced by M. Riesz in the first half of the 20 th century in order to find solutions to certain differential equations.

2.5.1 Definition. *For $\alpha\in\mathbb{C}$ with $\mathrm{Re}\,\alpha > n$ let $R_\pm(\alpha)$ be the ~~the~~ complex-valued continuous functions defined for any $x\in\mathbb{M}^n$ by*

$$R_\pm(\alpha)(x) := \begin{cases} C(\alpha,n)\gamma(x)^{\frac{\alpha-n}{2}} & \text{if } x\in J_\pm(0) \\ 0 & \text{otherwise,}\end{cases} \tag{2.12}$$

where

$$C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!},$$

and $z \mapsto (z-1)!$ *is the Gamma function.* ■

2.5.2 Remark. The functions $R_\pm(\alpha)$ are continuous because $\gamma = \langle\cdot,\cdot\rangle_0$ vanishes on the boundary of $J_\pm(0)$ and the exponent $(\alpha-n)/2$ is assumed to have positive real part. Indeed, if we increase the real part of the exponent then higher derivatives of the function vanishes on the boundary and the functions become more regular. Indeed it is C^k as soon as $\mathrm{Re}\,\alpha > n+2k$. ■

Now we show the first properties of $R_\pm(\alpha)$.

2.5.3 Proposition. *For all* $\alpha \in \mathbb{C}$ *with Re* $\alpha > n$ *it holds*

(1). $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha-n+2)R_\pm(\alpha+2)$;

(2). $(\mathrm{grad}\,\gamma)\cdot R_\pm(\alpha) = 2\alpha\,\mathrm{grad}\,R_\pm(\alpha+2)$;

(3). $\Box R_\pm(\alpha+2) = R_\pm(\alpha)$.

Moreover, the map

$$\begin{aligned}\{\mathit{Re}\ \alpha > n\} &\to \mathcal{D}'(\mathbb{M}^n)\\ \alpha &\mapsto R_\pm(\alpha)\end{aligned}$$

extends uniquely to all of $\mathbb{C}$ *as a holomorphic family of distributions, i.e. for each test-function* $\varphi \in \mathcal{D}(\mathbb{M}^n)$, *the function* $\alpha \mapsto (R_\pm(\alpha), \varphi)$ *is holomorphic.* ■

Proof. To prove (1), we just have to evaluate it inside $J_\pm(0)$, because both sides of the equation vanish outside. By definition one has

$$\gamma \cdot R_\pm(\alpha) = C(\alpha, n)\gamma(x)^{\frac{\alpha+2-n}{2}} = \frac{C(\alpha, n)}{C(\alpha+2, n)}R_\pm(\alpha+2),$$

and

$$\begin{aligned}\frac{C(\alpha, n)}{C(\alpha+2, n)} &= \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}\frac{(\frac{\alpha+2}{2}-1)!(\frac{\alpha+2-n}{2})!}{2^{1-\alpha-2}\pi^{\frac{2-n}{n}}} = \\ &= 4\frac{\alpha}{2}\frac{\alpha+2-n}{2} = \alpha(\alpha-n+2).\end{aligned}$$

For the second identity we evaluate $\partial_i\gamma \cdot R_\pm(\alpha)$. In view of what said in Remark (2.5.2), $R_\pm(\alpha+2)$ is C^1 on all Minkowski spacetime. We fix a test function φ and integrate by parts:

$$\begin{aligned}
\partial_i\gamma \cdot (R_\pm(\alpha),\varphi) &= C(\alpha,n)\int_{J_\pm} \gamma(x)^{\frac{\alpha-n}{2}}\partial_i\gamma(x)\varphi(x)\,\mathrm{d}x \\
&= \frac{2C(\alpha,n)}{\alpha+2-n}\int_{J_\pm} \partial_i(\gamma(x))^{\frac{\alpha-n+2}{2}}\varphi(x)\,\mathrm{d}x \\
&= -2C(\alpha+2,n)\int_{J_\pm} \gamma(x)^{\frac{\alpha-n+2}{2}}\partial_i\varphi(x)\,\mathrm{d}x \\
&= -2\alpha(R_\pm(\alpha+2),\partial_i\varphi) \\
&= 2\alpha(\partial_i R_\pm(\alpha),\varphi).
\end{aligned}$$

To prove the third formula, from (2) we have

$$\begin{aligned}
\partial_i^2 R_\pm(\alpha+2) &= \partial_i\left(\frac{1}{2\alpha}\partial_i\gamma\cdot R_\pm(\alpha)\right) \\
&= \frac{1}{2\alpha}\partial_i^2\gamma\cdot R_\pm(\alpha) + \frac{1}{4\alpha(\alpha-2)}(\partial_i\gamma)^2\frac{(\alpha-2)(\alpha-n)}{\gamma}\cdot R_\pm(\alpha) \\
&= \left(\frac{1}{2\alpha}\partial_i^2\gamma + \frac{\alpha-n}{4\alpha}\cdot\frac{(\partial_i\gamma)^2}{\gamma}\right)\cdot R_\pm(\alpha).
\end{aligned}$$

Now we evaluate the d'Alembert operator $\partial_t^2 - \sum_i \partial_i^2$ and we find

$$\Box R_\pm(\alpha+2) = \left(\frac{n}{\alpha} + \frac{\alpha-n}{4\alpha}\cdot\frac{4\gamma}{\gamma}\right)R_\pm(\alpha) = R_\pm(\alpha),$$

as claimed.
The last identity allows us to extend $R_\pm(\alpha)$ for every $\alpha\in\mathbb{C}$. For $\operatorname{Re}\alpha > n-2$ we set

$$\widetilde{R}_\pm(\alpha) := \Box R_\pm(\alpha+2),$$

and the extension is holomorphic on $\{\operatorname{Re}\alpha > n-2\}$. Now, proceeding by induction over n one can extend the function over the whole complex plane. ■

2.5.4 Definition. *The distributions $R_+(\alpha)$ and $R_-(\alpha)$ defined in the last Proposition are called respectively the* ***retarded*** *and* ***advanced*** *Riesz distributions to the parameter $\alpha\in\mathbb{C}$.* ■

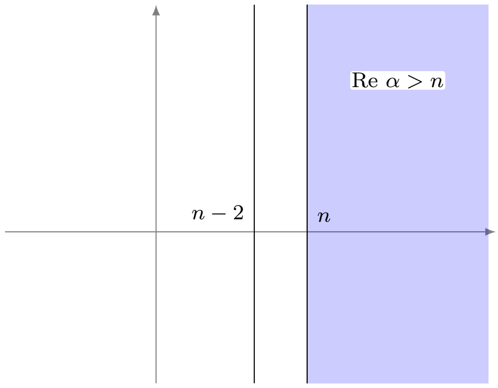


FIGURE 2.7: Iterative extension of $R_{\pm}(\alpha)$ on $\mathbb{C}$.

The Riesz distributions do not have an immediate explicit formula, but next lemma shows a more comfortable way to evaluate them when the test-function has a particular form.

2.5.5 Lemma. *Denote $x = (t, \mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^{n-1}$. Let $f \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$ such that $\varphi(x) := f(t)\psi(\mathbf{x}) \in \mathcal{D}(\mathbb{M}^n)$ and $\varphi(x) = f(t)$ on $J_+(0)$. If Re $\alpha > 1$ it holds*

$$(R_{\pm}(\alpha), \varphi) = \frac{1}{(\alpha-1)!} \int_0^{+\infty} t^{\alpha-1} f(t)\, \mathrm{d}t.$$

■

The facts of our interest on Riesz distributions, which links them to the study of fundamental solution are the following.

2.5.6 Proposition. *The Riesz distributions satisfy*

(1) for any $\alpha \in \mathbb{C}$, supp $R_{\pm}(\alpha) \subset J_{\pm}(0)$

(2) $R_{\pm}(0) = \delta_0$

(3) $\Box R_{\pm}(2) = \delta_0$, in particular $R_+(2)$ and $R_-(2)$ are respectively a ***retarded*** *and an* ***advanced*** *fundamental solution for $\Box$ at 0.*

■

Proof. The first assertion is straightforward from the definition of Riesz distributions.
To prove (2) fix $K \subset \mathbb{M}^n$ compact subset. Let $\sigma_K \in \mathcal{D}(\mathbb{M}^n)$ such that $\sigma_k|_K = 1$. For any $\varphi \in \mathcal{D}(\mathbb{M}^n)$ with $\operatorname{supp}\varphi \subset K$ one finds suitable smooth functions φ_j such that

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x^j \varphi_j(x),$$

then it holds

$$\begin{aligned}
(R_\pm(0), \varphi) &= (R_\pm(0), \sigma_K \varphi) \\
&= \left(R_\pm(0), \varphi(0) + \sum_{j=1}^{n} x^j \varphi_j(x) \right) \\
&= \varphi(0) \overbrace{(R_\pm(0), \sigma_K)}^{=: c_K} + \sum_{j=1}^{n} \left(\overbrace{x^j R_\pm(0)}^{=0}, \sigma_K \varphi_j \right) \\
&= c_K \varphi(0),
\end{aligned}$$

where $x^j R_\pm(0)$ vanishes because of Equation (1) in Proposition (2.5.3) and it is easy to show that c_K doesn't really depend on K since for $K' \supset K$ and $\operatorname{supp}\varphi \subset K \subset K'$,

$$c'_K \varphi(0) = (R_+(0), \varphi) = c_K \varphi(0),$$

so that $c_K = c'_K =: c$. To show $c = 1$, concentrating on the case of retarded distribution, using test-function as in Lemma (2.5.5),

$$\begin{aligned}
c \cdot \varphi(0) &= (R_+(0), \varphi) \\
&= (\Box R_+(2), \varphi) \\
&= (R_+(2), \Box \varphi) \\
&= \int_0^{+\infty} t\, f''(t)\, \mathrm{d}t \\
&= -\int_0^{+\infty} f'(t)\, \mathrm{d}t \\
&= f(0) \\
&= \varphi(0),
\end{aligned}$$

which concludes the proof.
The third assertion is obtained considering (1) and making use of (3) in Proposition (2.5.3). ■

2.5.7 Remark. We have already proven with the Fourier transform approach that the retarded and the advanced fundamental solutions are **unique**. Hence, we have

$$G^{\pm} = R_{\pm}(2).$$

■

2.5.8 Remark. As one expects, if $\alpha \in \mathbb{R}$, then $(R_{\pm}(\alpha), \varphi)$ is real for any $\varphi \in \mathcal{D}(\mathbb{M}^n, \mathbb{R})$ i.e. $R_{\pm}(\alpha)$ is a real-valued distribution. ■

We are now ready to prove **Huygens' principle**, that we already mentioned before. One can restate it as follows.

2.5.9 Theorem (Huygens' principle). *If $n \geq 4$ is even, $\operatorname{supp} G^{\pm} = C_{\pm}(0)$. If $n \geq 3$ is odd, $\operatorname{supp} G^{\pm} = J_{\pm}(0)$.* ■

To prove it we work with Riesz distributions and we need a Lemma

2.5.10 Lemma. *The following holds:*

(1) for every $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$ we have $\operatorname{supp} R_{\pm}(\alpha) = J_{\pm}(0)$;

(2) for $n \geq 3$ and $\alpha = n-2, n-4, \dots, 1$ if n is odd or $\alpha = n-2, n-4, \dots, 2$ if n is even, we have $\operatorname{supp} R_{\pm}(\alpha) = C_{\pm}(0)$.

■

Proof of Theorem (2.5.9). The fundamental solutions are $G^{\pm} = R_{\pm}(2)$, so $\alpha = 2$. Since $2 = (n-2) + (4-n)$, if $n \geq 4$ and n is even, $2 \in \{n-2, n-4, \dots\}$; conversely, if n is odd 2 is not in $\{n-2, n-4, \dots, 1\}$. So the theorem follows from the last Lemma. ■

2.6 General solution and Cauchy problem

We found the fundamental solution for the d'Alembert operator for the point $x_0 = 0$. To find the generic solution at a point $y \in \mathbb{M}^n$, as we have seen in Proposition (2.2.2), it suffices to write

$$G_y^+(x) = T_y G^+ = G^+(x-y).$$

Hence, to find the retarded solution $F_\psi^+(x)$ to the wave equation $\Box u = \psi$, where ψ is a distribution, we simply evaluate the convolution $G^+ * \psi$. The general solution is obtained by adding the solutions of the homogeneous equation as in Formula (2.4).

Now we discuss the uniqueness of the distributional solution and for its regularity we remind to Proposition (2.2.2).
To begin, we shall prove the following

2.6.1 Theorem. *Let $\psi \in \mathcal{D}'(\mathbb{M}^n)$ such that $\psi(t,\mathbf{x}) = 0$ if $t < 0$. Then ψ and G^+ can be convoluted and $F_\psi^+ = G^+ * \psi$ is the unique solution to the wave equation such that $F_\psi^+(t,\mathbf{x}) = 0$ for $t < 0$.*

■

Proof. At fixed x, the distribution $G^+(x-y)\psi(y)$ has compact support in the variable y, so G^+ and ψ can be convoluted and since $\operatorname{supp} G^+ \subset J_+(0)$ it is clear that $F_\psi^+ = 0$ for $t < 0$.
The solution is unique because if there was another F_ψ solving the equation and satisfying the requested conditions, then $\phi := F_\psi^+ - F_\psi$ would be a solution to the homogeneous equation, i.e. $\Box\phi = 0$, and ϕ can be convoluted with G^+. So

$$\phi = \phi * \delta = \phi * \Box\, G^+ = \Box\, \phi * G^+ = 0,$$

hence $F_\psi = F_\psi^+$. ■

Remaining in the Minkowski case, since $\mathbb{M}^n$ is a globally hyperbolic manifold, we can find smooth Cauchy hypersurfaces, where we can put some initial values.
Physically, it is clear why the initial values can only be specified on a spacelike surface: otherwise a wave could travel from one point of the initial value surface to another and change the initial conditions.

2.6.2 Definition. *Let S be a smooth Cauchy hypersurface of $\mathbb{M}^n$ with a timelike unit normal vector field $\nu : S \to \mathrm{T}\mathbb{M}^n$.*

Given a triple $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$, we call a ***classical Cauchy problem*** *for $\Box$ on S the system of equations*

$$\begin{cases} \Box u = \psi \\ \\ u|_S = u_0 \\ \\ \partial_\nu u|_S = u_1. \end{cases} \tag{2.13}$$

In case the triple is taken in $\mathcal{D}'(\mathbb{M}^n) \oplus \mathcal{D}'(S) \oplus \mathcal{D}'(S)$, i.e. the data are distributions, the problem is called ***generalized Cauchy problem****.* ■

For simplicity, we concentrate on the case where S is the hyperplane

$$\{(t, \mathbf{x}) \in \mathrm{M} \,|\, t = 0\},$$

and $\partial_\nu = \partial_t$. The general case will be addressed later, when e will discuss the general case on manifolds. To solve the Cauchy problem, we start with a lemma

2.6.3 Lemma. *Suppose u is a solution for the Cauchy problem on $\{t = 0\}$. If we set $\tilde{u} = u$ if $t > 0$ and $\tilde{u} = 0$ otherwise; $\tilde{\psi} = \psi$ if $t > 0$ and $\tilde{\psi} = 0$ otherwise, it holds*

$$\Box\, \tilde{u}(t, \mathbf{x}) = \tilde{\psi}(t, \mathbf{x}) + u_0(\mathbf{x})\delta'(t) + u_1(\mathbf{x})\delta(t). \tag{2.14}$$

■

Proof. Let $\varphi \in \mathcal{D}'(\mathbb{M}^n)$, then

$$(\Box\tilde{u}, \varphi) = (\tilde{u}, \Box\varphi) = \int_0^\infty \mathrm{d}t \int u\, \Box\varphi\, \mathrm{d}\mathbf{x} =$$

using integration by parts on the time variable

$$= \int_0^\infty \mathrm{d}t \int \Box u\, \varphi\, \mathrm{d}\mathbf{x} + \int \partial_t u(0, \mathbf{x})\, \varphi(0, \mathbf{x}) - u(0, \mathbf{x})\, \partial_t\varphi(0, \mathbf{x})\, \mathrm{d}\mathbf{x} =$$

$$= \int \tilde{\psi} + u_1\delta(t) + u_0\delta'(t)\, \mathrm{d}t\, \mathrm{d}\mathbf{x}.$$

■

Now it follows directly, applying the last Lemma our case, the solution to the Cauchy problem.

2.6.4 Theorem. *For each* $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$, *the retarded solution to the Cauchy problem on* $\{t = 0\}$ *is unique and it is given by the formula*

$$F^+_\psi = G^+ * (\psi + u_0 \otimes \delta' + u_1 \otimes \delta). \tag{2.15}$$

■

The solution, if we concentrate on the classical Cauchy problem, is smooth (Proposition **??**) and depends continuously on the initial data, hence the map defined in Equation (2.15) can be seen as a **linear continuous operator**

$$\begin{aligned} \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S) &\to C^\infty(\mathbb{M}^n) \\ (\psi, u_0, u_1) &\mapsto F^+_\psi. \end{aligned}$$

Fundamental solutions on manifolds 3

In this chapter, the Riesz distributions will be transported on suitable domains of Lorentzian manifolds to construct local fundamental solutions and discuss the local and global solvability of the Cauchy problem.

3.1 Local fundamental solutions

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