On Maxwell Equations on Globally Hyperbolic Spacetimes with Timelike Boundary

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Abstract

We study Maxwell's equation as a theory for smooth k-forms on globally hyperbolic spacetimes with timelike boundary as defined by Aké, Flores and Sanchez [AFS18]. In particular we start by investigating on these backgrounds the D'Alembert - de Rham wave operator $\Box_k$ and we highlight the boundary conditions which yield a Green's formula for $\Box_k$. Subsequently, we characterize the space of solutions of the associated initial and boundary value problem under the assumption that advanced and retarded Green operators do exist. This hypothesis is proven to be verified by a large class of boundary conditions using the method of boundary triples and under the additional assumption that the underlying spacetime is ultrastatic. Subsequently we focus on the Maxwell operator. First we construct the boundary conditions which entail a Green's formula for such operator and then we highlight two distinguished cases, dubbed δd-tangential and δd-normal boundary conditions. Associated to these we introduce two different notions of gauge equivalence and we prove that in both cases, every equivalence class admits a representative abiding to the Lorentz gauge. We use this property and the analysis of the operator $\Box_k$ to construct and to classify the space of gauge equivalence classes of solutions of the Maxwell's equations with the prescribed boundary conditions. As a last step and in the spirit of future applications in the framework of algebraic quantum field theory, we construct the associated unital $*$-algebras of observables proving in particular that, as in the case of the Maxwell operator on globally hyperbolic spacetimes with empty boundary, they possess a non-trivial center.

Keywords: classical field theory on curved backgrounds, Maxwell equations, globally hyperbolic spacetimes with timelike boundary

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1 Introduction

Electromagnetism and the associated Maxwell's equations, written both in terms of the Faraday tensor or of the vector potential, represent one of the most studied models in mathematical physics. On the one hand they are of indisputable practical relevance, while, on the other hand, they are the prototypical example of a gauge theory, which can be still thoroughly and explicitly investigated thanks to the Abelian nature of the underlying gauge group.

On curved backgrounds the study of this model has attracted a lot of attention not only from the classical viewpoint but also in relation to its quantization. Starting from the early work of Dimock [Dim92], the investigation of Maxwell's equations, generally seen as a theory for differential forms, has been thorough especially in the framework of algebraic quantum field, *e.g.* [FP03, Pfe09]. One of the key reasons for such interest is related to the fact that electromagnetism has turned out to be one of the simplest examples where the principle of general local covariance, introduced in [BFV03], does not hold true on account of topological obstructions – see for example [BDHS14, BDS14, DL12, SDH12].

A closer look at all these references and more generally to the algebraic approach unveils that most of the analyses rest on two key data: the choice of a gauge group and of an underlying globally hyperbolic background of arbitrary dimension. While the first one is related to the interpretation of electromagnetism as a theory for the connections of a principal $U(1)$-bundle, the second one plays a key rôle in the characterization of the space of classical solutions of Maxwell's equations and in the associated construction of a unital $*$-algebra of observables. More precisely, every solution of Maxwell's equation identifies via the action of the gauge group an equivalence class of differential forms. Each of these classes admits a distinguished representative, namely a coclosed form which solves a normally hyperbolic partial differential equation, ruled by the D'Alembert - de Rham operator. Most notably, since the underlying spacetime is globally hyperbolic, one can rely on classical results, see for example [BGP07], to infer that the D'Alembert - de Rham operator admits unique advanced and retarded fundamental solutions. Not only these can be used to characterize the kernel of such operator, but they also allow both to translate the requirement of considering only coclosed form as a constraint on the admissible initial data and to give an explicit representation for the space of the gauge equivalence classes of solutions of Maxwell's equation. At a quantum level, instead, the fundamental solutions represent the building block to implement the canonical commutation relations within the algebra of observables, *cf.* [Dim92].

Completely different is the situation when we drop the assumption of the underlying background being globally hyperbolic since especially the existence and uniqueness results for the fundamental solutions are no longer valid. In this paper we will be working in this framework, assuming in particular that the underlying manifold (M, g) is globally hyperbolic and it possesses a timelike boundary, that is $(\partial M, \iota_M^* g)$, where $\iota_M : \partial M \hookrightarrow M$, is a Lorentzian smooth submanifold. From a geometric viewpoint this class of spacetimes has been formalized recently in [AFS18] and it contains several notable examples, such as anti-de Sitter (AdS) or asymptotically AdS spacetimes. These play a key rôle in several models that have recently attracted a lot of attention especially for the study of the properties of the wave or of the Klein-Gordon equation, see for example [Bac12, Hol12, Wro17, Vas12]. From a classical point of view, in order to construct the solutions for any of these equations, initial data assigned on a Cauchy surface are no longer sufficient and it is necessary to supplement them with the choice of a boundary condition. This particular feature prompts the question whether these systems still admit fundamental solutions and, if so, whether they are unique and whether they share the same structural properties of their counterparts in a globally hyperbolic spacetime with empty boundary.

For the wave operator acting on real scalar functions a complete answer to this question has been given in [DDF19] for static globally hyperbolic spacetimes with a timelike boundary combining spectral calculus with boundary triples, introduced by Grubb in [Gru68].

In this work we will be concerned instead with the study of Maxwell's equations acting on generic k-forms, with $0 \leq k < m = \dim M$, see [HLSW15] for an analysis in terms of the Faraday tensor on an anti-de Sitter spacetime. In comparison with the scalar scenario, the situation is rather different. First of all the dynamics is ruled by the operator $\delta_{k+1}\mathrm{d}_k$ where d_k is the differential acting on $\Omega^k(M)$, the space of smooth k-forms while δ_{k+1} is the codifferential acting on $(k+1)$-forms. To start with, one can observe that this operator is not formally self-adjoint and thus boundary conditions must be imposed. The admissible ones are established by a direct inspection of the Green's formula for the Maxwell-operator. In between the plethora of all possibilities we highlight two distinguished choices, dubbed δd-tangential and δd-normal boundary conditions which, in the case $k = 0$, reduce to the more common Dirichlet and Neumann boundary conditions.

As second step we recognize that a notion of gauge group has to be introduced. While a more geometric approach based on interpreting Maxwell's equation in terms of connections on a principal $U(1)$-bundle might be the most desirable approach, we decided to investigate this viewpoint in a future work. We focus instead only on Maxwell's equation as encoding the dynamics of a theory for differential k-forms. If the underlying manifold (M, g) would have no boundary the gauge group would be chosen as $\mathrm{d}\Omega^{k-1}(M)$. While at first glance one might wish to keep the same choice, it is immediate to realize that this is possible only for the δd-normal boundary condition which is insensitive to any shift of a form by an element of the gauge group. On the contrary, in the other cases, one needs to reduce the admissible gauge transformations so to ensure compatibility with the boundary conditions.

The next step in our analysis mimics the counterpart when the underlying globally hyperbolic spacetime (M, g) has empty boundary, namely we construct the space of gauge equivalence classes of solutions for Maxwell equations and we prove that each class admits a non unique representative which is a coclosed k-form ω, such that $\Box_k\omega$=0. Using a standard nomenclature, we say that we consider a Lorenz gauge fixing. Here $\Box_k = \mathrm{d}_{k-1}\delta_k + \delta_{k+1}\mathrm{d}_k$ is the D'Alembert - de Rham wave operator which is known to be normally hyperbolic, see e.g. [Pfe09]. On the one hand we observe that the mentioned non uniqueness is related to a residual gauge freedom which can be fully accounted for. On the other hand we have reduced the characterization of the equivalence classes of solutions of Maxwell's equations to studying the d'Alembert - de Rham wave operator.

This can be analyzed similarly to the wave operator acting on scalar functions as one could imagine since the two operators coincide for $k = 0$. Therefore we study $\Box_k$ independently, first identifying via a Green's formula a collection of admissible boundary conditions, Subsequently, under the assumption that advanced and retarded Green's operators exist, we characterize completely the space of solutions of the equation $\Box_k\omega = 0$, $\omega \in \Omega^k(M)$ with prescribed boundary condition. At this stage we highlight the main technical obstruction which forces us to consider only two distinguished boundary conditions for the Maxwell operator. As a matter of facts, we show that, although at an algebraic level it holds always $\delta_k \circ \Box_k = \Box_{k-1} \circ \delta_k$, the counterpart at the level of fundamental solutions is verified only for specific choices of the boundary condition. This leads to an obstruction in translating the Lorenz gauge condition of working only with coclosed k-form to a constraint in the admissible initial data. This failure does not imply that the Lorenz gauge is ill-defined, but only that, for a large class of boundary conditions, one needs to envisage a strategy different from the one used on globally hyperbolic spacetimes with empty boundary in order to study the underlying problem.

It it important to mention that it is beyond our current knowledge verifying whether our assumption on the existence of fundamental solutions is always true. We expect that a rather promising avenue consists of adapting to the case in hand the techniques and the ideas discussed in [DS17] and in [GW18], but this is certainly a challenging task, which we leave for future work. On the contrary we test our assumption in the special case of ultrastatic, globally hyperbolic spacetimes with timelike boundary. In this scenario we adopt the techniques used in [DDF19] proving that advanced and retarded fundamental solutions do exist for a large class of boundary conditions, including all those of interest for our analysis.

Finally we give an application of our result inspired by the quantization of Maxwell's equations in the algebraic approach to quantum field theory. While this framework has been extremely successful on a generic globally hyperbolic spacetime with empty boundary, only recently the case with a timelike boundary has been considered, see *e.g.* [BDS18, DF18, DW18, MSTW19, Za15]. In particular we focus on the construction of a unital $*$-algebra of observables for Maxwell's equations both with δd-tangential and δd-normal boundary condition and we prove that in both cases one can always find a non trivial Abelian ideal. This is the signature that, also in presence of a timelike boundary, one cannot expect that the principle of general local covariance holds true in its original form.

The paper is organized as follows: In Section 2 we introduce the notion of globally hyperbolic spacetime with timelike boundary as well as all the relevant space of differential k-forms. In addition we recall the basic definitions of differential and codifferential operator and we introduce two distinguished maps between bulk and boundary forms. Section 3 contains the core of this paper. For clarity purposes, we start in Subsection 3.1 from the analysis of the D'Alembert - de Rham wave operator $\Box_k$. To begin with we study a class of boundary conditions which implement the Green's formula, hence making $\Box_k$ a formally self-adjoint operator. Subsequently we assume that, for a given boundary condition, advanced and retarded Green's operators exist and we codify the information of the space of classical solutions of the underlying dynamics in terms of a short exact sequence, similar to the standard one when the underlying globally hyperbolic spacetime has no boundary, *cf.* [BGP07]. In addition we discuss the interplay between the fundamental solutions and the differential/codifferential operator. In Subsection 3.2 we focus instead on Maxwell's equations. First we investigate which boundary conditions can be imposed so that the operator ruling the dynamics is formally self-adjoint. Subsequently we introduce the δd-tangential and the δd-normal boundary conditions together with an associated gauge group. Using these data, we prove that the equivalence classes of solutions of Maxwell's equations, always admit a representative in the Lorenz gauge, which obeys an equation of motion ruled by $\Box_k$. Such solution is non unique in the sense that a residual gauge freedom exists. Nonetheless, using the fundamental solutions of the D'Alembert - de Rham wave operator, we are able to characterize the above equivalence classes in terms of suitable initial data. To conclude, in Subsection 3.3.1 and 3.3.2 we use the results from the previous parts to construct a unital $*$-algebra of observables associated to Maxwell's equation with δd-tangential and δd-normal boundary conditions. In particular we prove that in all cases there exists an Abelian $*$-ideal. In Appendix A we prove that our assumption on the existence of fundamental solutions is verified whenever the underlying spacetime is static. Finally in Appendix B it is proven an explicit decomposition for k-forms on globally hyperbolic spacetimes, which plays a key rôle is some proofs in the main body of the paper. In Appendix C we recall the basic notion of relative cohomology for manifolds with boundaries as well as the associated Poincaré-Lefschetz duality.

2 Geometric Data

In this subsection, our goal is to fix notations and conventions, as well as to summarize the main geometric data, which play a key rôle in our analysis. Following the standard definition, see for example [Lee00, Ch. 1], M indicates a smooth, second-countable, connected, oriented manifold of dimension $m > 1$, with smooth boundary ∂M, assumed for simplicity to be connected. We assume also that M admits a finite good cover. A point $p \in M$ such that there exists an open neighbourhood U containing p, diffeomorphic to an open subset of $\mathbb{R}^m$, is called an *interior point* and the collection of these points is indicated with $\mathrm{Int}(M) \equiv \mathring{M}$. As a consequence $\partial M \doteq M \setminus \mathring{M}$, if non-empty, can be read as an embedded submanifold $(\partial M, \iota_{\partial M})$ of dimension $m-1$ with $\iota_{\partial M} \in C^\infty(\partial M; M)$.

In addition we endow M with a smooth Lorentzian metric g of signature $(-,+,...,+)$ and consider only those cases in which $\iota^*_{\partial M} g$ identifies a Lorentzian metric on ∂M and (M,g) is time oriented. As a consequence $(\partial M, \iota^*_{\partial M} g)$ acquires the induced time orientation and we say that (M,g) has a *timelike boundary.*

Since we will be interested particularly in the construction of advanced and retarded fundamental solutions for normally hyperbolic operators, we focus our attention on a specific class of Lorentzian manifolds with timelike boundary, namely those which are globally hyperbolic. While, in the case of $\partial M = \emptyset$ this is a standard concept, in presence of a timelike boundary it has been properly defined and studied recently in [AFS18]. Summarizing part of their constructions and results, we say that a time-oriented, Lorentzian manifold with timelike boundary (M,g) is *causal* if it possesses no closed, causal curve, while it is *globally hyperbolic* if it is causal and, for all $p, q \in M$, $J^+(p) \cap J^-(q)$ is either empty or compact. These conditions entail the following consequences, see [AFS18, Th. 1.1 & 3.14]:

Theorem 1: *Let (M,g) be a time-oriented Lorentzian manifold with timelike boundary of dimension* $\dim M = n \geq 2$. *Then the following conditions are equivalent:*

1. *(M,g) is globally hyperbolic ;*
2. *(M,g) possesses a Cauchy surface, namely an achronal subset of M which is intersected only once by every inextensible timelike curve ;*
3. *(M,g) is isometric to $\mathbb{R} \times \Sigma$ endowed with the line-element*

$$\mathrm{d}s^2 = -\beta \mathrm{d}\tau^2 + h_\tau \,, \tag{1}$$

where $\tau : M \to \mathbb{R}$ is a Cauchy temporal function[1]*, whose gradient is tangent to ∂M, $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0,\infty))$ while $\mathbb{R} \ni \tau \to (\{\tau\} \times \Sigma, h_\tau)$ identifies a one-parameter family of $(n-1)-$dimensional spacelike, Riemannian manifolds with boundaries. Each $\{\tau\} \times \Sigma$ is a Cauchy surface for (M,g).*

Henceforth we will be tacitly assuming that, when referring to a globally hyperbolic spacetime with timelike boundary (M,g), we work directly with (1) and we shall refer to τ as the time coordinate. Furthermore each Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ acquires an orientation induced from that of M. In addition we shall say that (M,g) is *static* if it possesses a timelike Killing vector field $\chi \in \Gamma(TM)$ whose restriction to ∂M is tangent to the boundary, *i.e.* $g_p(\chi, \nu) = 0$ for all $p \in \partial M$ where ν is the outward pointing, unit vector, normal to the boundary at p. With reference to (1) and for simplicity, we identify χ with ∂_τ. Thus the condition of being static translates into the constraint that both β and h_τ are independent from τ. If in addition $\beta = 1$ we call (M,g) *ultrastatic.*

On top of a Lorentzian spacetime (M,g) with timelike boundary we consider $\Omega^k(M)$, $0 \leq k \leq \dim M$, the space of real valued smooth k-forms endowed with the standard, metric induced, pairing $(\,,\,) : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}$. A particular rôle will be played by the support of the forms that we consider. In the following definition we introduce the different possibilities that we will consider, which are a generalization of the counterpart used for scalar fields which correspond in our scenario to $k = 0$, *cf.* [Bär15].

Definition 2: *Let (M,g) be a Lorentzian spacetime with timelike boundary. We denote with*

1. *$\Omega^k_c(M)$ the space of smooth k-forms with compact support in M while with $\Omega^k_c(\mathring{M}) \subset \Omega^k_c(M)$ the collection of smooth and compactly supported k-forms ω such that $supp(\omega) \cap \partial M = \emptyset$.*

[1] Given a generic time oriented Lorentzian manifold $(N, \tilde{g})$, a Cauchy temporal function is a map $\tau : M \to \mathbb{R}$ such that its gradient is timelike and past-directed, while its level surfaces are Cauchy hypersurfaces.

2. $\Omega^k_{\mathrm{spc}}(M)$ *(resp.* $\Omega^k_{\mathrm{sfc}}(M)$*) the space of strictly past compact (resp. strictly future compact) k-forms, that is the collection of* $\omega \in \Omega^k(M)$ *such that there exists a compact set* $K \subseteq M$ *for which* $J^+(supp(\omega)) \subseteq J^+(K)$ *(resp.* $J^-(supp(\omega)) \subseteq J^-(K)$*), where* $J^\pm$ *denotes the causal future and the causal past in* M*. Notice that* $\Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M) = \Omega^k_{\mathrm{c}}(M)$*.*

3. $\Omega^k_{\mathrm{pc}}(M)$ *(resp.* $\Omega^k_{\mathrm{fc}}(M)$*) denotes the space of future compact (resp. past compact) k-forms, that is,* $\omega \in \Omega^k(M)$ *for which* $\mathrm{supp}(\omega) \cap J^-(K)$ *(resp.* $\mathrm{supp}(\omega) \cap J^+(K)$*) is compact for all compact* $K \subset M$*.*

4. $\Omega^k_{\mathrm{tc}}(M) \doteq \Omega^k_{\mathrm{fc}}(M) \cap \Omega^k_{\mathrm{pc}}(M)$*, the space of timelike compact k-forms.*

5. $\Omega^k_{\mathrm{sc}}(M) \doteq \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_{\mathrm{spc}}(M)$*, the space of spacelike compact k-forms.*

We indicate with $\mathrm{d}_k \colon \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative and, being (M,g) oriented, we can identify a unique, metric-induced, Hodge operator $\star_k : \Omega^k(M) \to \Omega^{m-k}(M)$, $m = \dim M$ such that, for all $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge \star_k \beta = g^\sharp(\alpha,\beta)\mu_g$, where $\wedge$ is the exterior product of forms and μ_g the metric induced volume form. In addition one can define a pairing between k-forms as

$$(\alpha,\beta)_k \doteq \int \alpha \wedge \star_k \beta,$$

where $\alpha, \beta \in \Omega^k(M)$ are such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact. Since M is endowed with a Lorentzian metric it holds that, when acting on smooth k-forms, $\star_k^{-1} = (-1)^{k(m-k)-1} \star_{m-k}$. Combining these data first we define the *codifferential* operator $\delta_k : \Omega^{k+1}(M) \to \Omega^k(M)$ as $\delta_k \doteq (-1)^k \star_{k-1}^{-1} \circ \mathrm{d}_{m-k} \circ \star_k$. Secondly we introduce the *D'Alembert-de Rham* wave operator $\Box_k : \Omega^k(M) \to \Omega^k(M)$ such that $\Box_k \doteq \mathrm{d}_{k-1}\delta_k + \delta_{k+1}\mathrm{d}_k$, as well as the *Maxwell* operator $\delta_{k+1}\mathrm{d}_k : \Omega^k(M) \to \Omega^k(M)$. Observe, furthermore, that $\Box_k$ differs by the more commonly used D'Alembert wave operator acting on k-forms by 0-order term built out of the metric and whose explicit form depends on the value of k, see for example [Pfe09, Sec. II].

Remark 3: *For notational convenience, in the following we shall drop all sub-index* $_k$ *since the relevant value will be clear case by case from the context. Hence, unless stated otherwise, all statements of this paper apply to all* k *such that* $0 \leq k \leq \dim M$*.*

To conclude the section, we focus on the boundary ∂M and on the interplay with k-forms lying in $\Omega^k(M)$. The first step consists of defining two notable maps. These relate k-forms defined on the whole M with suitable counterparts living on ∂M and, in the special case of $k = 0$, they coincide either with the restriction to the boundary of a scalar function or with that of its derivative along the direction normal to ∂M. For later convenience we consider in the following definition a slightly more general scenario, namely a codimension 1 smoothly embedded submanifold $N \hookrightarrow M$.

Remark 4: *Since we feel that some confusion might arise, we denote the paring between forms on* ∂M *with* $(\ ,\ )_\partial$*.*

Definition 5: *Let* (M, g_M) *be a smooth Lorentzian manifold and let* $\iota_N \colon N \to M$ *be a codimension* 1 *smoothly embedded submanifold of* M *with induced metric* $g_N := \iota_N^* g_M$*. We define the* tangential *and* normal *component relative to* N *as the maps defined by*

$$\mathrm{t}_N \colon \Omega^k(M) \to \Omega^k(N)\,, \qquad \mathrm{t}_N\omega := \iota_N^*\omega\,, \tag{2a}$$

$$\mathrm{n}_N \colon \Omega^k(M) \to \Omega^{k-1}(N)\,, \qquad \mathrm{n}_N\omega := \star_N^{-1}\mathrm{t}_N \star_M \omega\,, \tag{2b}$$

where $\star_M, \star_N$ *denotes the Hodge duals over* M, N *respectively. In particular, for all* $k \in \mathbb{N} \cup \{0\}$ *we define*

$$\Omega^k_{\mathrm{t}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}_N\omega = 0\}\,, \qquad \Omega^k_{\mathrm{n}_N}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}_N\omega = 0\}\,. \tag{3}$$

Similarly we will use the symbols $\Omega^k_{c,t_N}(M)$ *and* $\Omega^k_{c,n_N}(M)$ *when we consider only smooth, compactly supported* k*-forms.*

Remark 6: *In this paper the rôle of* N *will be played often by* ∂M*. In this case, we shall drop the subscript form Equation (2), namely* $\mathrm{t} \equiv \mathrm{t}_{\partial M}$ *and* $\mathrm{n} \equiv \mathrm{n}_{\partial M}$*. Furthermore, the differential and the codifferential operators on* ∂M *will be denoted, respectively, as* d_∂*,* δ_∂*.*

Remark 7: *With reference to Definition 5, observe that the following linear map is surjective:*

$$\Omega^k(M) \ni \omega \to (\mathrm{n}\omega, \mathrm{t}\omega, \mathrm{t}\delta\omega, \mathrm{n}\mathrm{d}\omega) \in \Omega^{k-1}(\partial M) \times \Omega^k(\partial M) \times \Omega^{k-1}(\partial M) \times \Omega^k(\partial M)\,.$$

Remark 8: *The normal map* $\mathrm{n} : \Omega^k(M) \to \Omega^{k-1}(\partial M)$ *can be equivalently read as the restriction to* ∂M *of the contraction* $\nu \lrcorner \omega$ *between* $\omega \in \Omega^k(M)$ *and the vector field* $\nu \in \Gamma(TM)|_{\partial M}$ *which corresponds at each point* $p \in \partial M$ *to the outward pointing unit vector, normal to* ∂M*.*

As last step, we observe that (2) together with (3) entail the following series of identities on $\Omega^k(M)$ for all $k \in \mathbb{N} \cup \{0\}$.

$$\star\,\delta = (-1)^k \mathrm{d}\star\,, \quad \delta\star = (-1)^{k+1} \star \mathrm{d}\,, \tag{4a}$$

$$\star_\partial\, \mathrm{n} = \mathrm{t}\star\,, \quad \star_\partial \mathrm{t} = (-1)^{(m-k)}\mathrm{n}\star\,, \quad \mathrm{d}_\partial \mathrm{t} = \mathrm{t}\mathrm{d}\,, \quad \delta_\partial \mathrm{n} = -\mathrm{n}\delta\,. \tag{4b}$$

A notable consequence of (4b) is that, while on globally hyperbolic spacetimes with empty boundary, the operators d and δ are one the formal adjoint of the other, in the case in hand, the situation is different. A direct application of Stokes' theorem yields that

$$(\mathrm{d}\alpha, \beta) - (\alpha, \delta\beta) = (\mathrm{t}\alpha, \mathrm{n}\beta)_\partial\,, \tag{5}$$

where the pairing in the right-hand side is the one associated to forms living on ∂M and where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$ are arbitrary, though such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact. In connection to the operators d and δ we shall employ the notation

$$\Omega^k_{\mathrm{d}}(M) = \{\omega \in \Omega^k(M) \mid \mathrm{d}\omega = 0\}\,, \qquad \Omega^k_\delta(M) = \{\omega \in \Omega^k(M) \mid \delta\omega = 0\}\,, \tag{6}$$

where $k \in \mathbb{N}$. Similarly we shall indicate with $\Omega^k_{\sharp,\delta}(M) \doteq \Omega^k_\sharp(M) \cap \Omega^k_\delta(M)$ and $\Omega^k_{\sharp,\mathrm{d}}(M) \doteq \Omega^k_\sharp(M) \cap \Omega^k_{\mathrm{d}}(M)$ where $\sharp \in \{\mathrm{c}, \mathrm{sc}, \mathrm{pc}, \mathrm{fc}, \mathrm{tc}\}$.

3 Maxwell Equations and Boundary Conditions

In this section we analyze the space of solutions of Maxwell equations for arbitrary k-forms on a globally hyperbolic spacetime with timelike boundary (M, g). We proceed in two separate steps. First we focus our attention on the D'Alembert - de Rham wave operator $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ acting on $\Omega^k(M)$. We identify a class of boundary conditions which correspond to imposing that the underlying system is closed (*i.e.* the symplectic flux across ∂M vanishes) and we characterize the kernel of the operator in terms of its advanced and retarded fundamental solutions. These are assumed to exist and, following the same strategy employed in [DDF19] for the scalar wave equation, we prove that this is indeed the case whenever (M, g) is a ultrastatic spacetime, *cf.* Appendix A.

In the second part of the section we focus instead on the Maxwell operator $\delta\mathrm{d}\colon \Omega^k(M) \to \Omega^k(M)$. In order to characterize its kernel we will need to discuss the interplay between the choice of boundary condition and that of gauge fixing. This represents the core of this part of our work.

3.1 On the D'Alembert–de Rham wave operator

Consider the operator $\Box : \Omega^k(M) \to \Omega^k(M)$, where (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = m \geq 2$. Then, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, the following Green's formula holds true:

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = (\mathrm{t}\delta\alpha, \mathrm{n}\beta)_\partial - (\mathrm{n}\alpha, \mathrm{t}\delta\beta)_\partial - (\mathrm{n}\mathrm{d}\alpha, \mathrm{t}\beta)_\partial + (\mathrm{t}\alpha, \mathrm{n}\mathrm{d}\beta)_\partial\,, \tag{7}$$

where t, n are the maps introduced in Definition 5, while $(,)$ and $(,)_\partial$ are the standard, metric induced pairing between k-forms respectively on M and on ∂M. In view of Definition 5, it descends that the right-hand side of (7) vanishes automatically if we restrict our attention to $\alpha \in \Omega^k_{\mathrm{c}}(\mathring{M})$ or $\beta \in \Omega^k_{\mathrm{c}}(\mathring{M})$, but boundary conditions must be imposed for the same property to hold true on a larger set of k-forms. From a physical viewpoint this requirement is tantamount to imposing that the system described by k-forms obeying the D'Alembert–de Rham wave equation is closed.

Lemma 9: *Let $f, f' \in C^\infty(\partial M)$ and let*

$$\Omega^k_{f,f'}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\,,\ \mathrm{t}\delta\omega = f'\mathrm{n}\omega\}. \tag{8}$$

Then, $\forall \alpha, \beta \in \Omega^k(M)$, $0 \leq k \leq n = \dim M$ such that supp$(\alpha) \cap$ supp(β) is compact, it holds

$$(\Box\alpha, \beta) - (\alpha, \Box\beta) = 0.$$

Proof. This is a direct consequence of (7) together with the property that, for every $f \in C^\infty(\partial M)$ and for every $\alpha \in \Omega^k(\partial M)$, $\star_\partial(f\alpha) = f(\star_\partial\alpha)$. In addition observe that the assumption on the support of α and β descends also to the forms present in each of the pairing in the right hand side of (7). $\Box$

Remark 10: *In Lemma 9 two cases are quite peculiar. As a matter of fact, if $k = m = \dim M$ the first condition becomes empty since $\mathrm{d}\omega = \mathrm{t}\omega = 0$ for all $\omega \in \Omega^m(M)$. Similarly, if $k = 0$, the second condition does not bring any constraint since $\delta\omega = \mathrm{n}\omega = 0$ for all $\omega \in \Omega^0(M)$. In this case equation (8) reduces to Robin boundary conditions, which were studied in [DDF19].*

Remark 11: *It is important to stress that the boundary conditions defined in Lemma 9 are not the largest class which makes the right hand side (7) vanish. As a matter of fact one can think of additional possibilities similar to the so-called Wentzell boundary conditions, which were considered in the scalar scenario, see e.g. [DDF19, DFJ18, Za15].*

Lemma (9) individuates therefore a class of boundary conditions which makes the operator $\Box$ formally self-adjoint. In between all these possibilities we highlight those which are of particular interest to our analysis – *cf.* Theorem 16.

Definition 12: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $f \in C^\infty(\partial M)$. We call*

1. *space of k-forms with* Dirichlet *boundary condition*

$$\Omega^k_{\mathrm{D}}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{n}\omega = 0\}\,, \tag{9}$$

2. *space of k-forms with* $\Box$-tangential *boundary condition*

$$\Omega^k_{\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\omega = 0\ ,\ \mathrm{t}\delta\omega = 0\}\,, \tag{10}$$

3. *space of k-forms with* $\Box$-normal *boundary condition*

$$\Omega^k_{\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\omega = 0\ ,\ \mathrm{n}\mathrm{d}\omega = 0\}\,. \tag{11}$$

4. *space of k-forms with* Robin $\Box$-tangential *boundary condition*

$$\Omega^k_{f_\parallel}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{t}\delta\omega = f\mathrm{n}\omega\ ,\ \mathrm{t}\omega = 0\}\,, \tag{12}$$

5. *space of k-forms with* Robin $\Box$-normal *boundary condition*

$$\Omega^k_{f_\perp}(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{n}\mathrm{d}\omega = f\mathrm{t}\omega\ ,\ \mathrm{n}\omega = 0\}\,, \tag{13}$$

Whenever the domain of the operator $\Box$ is restricted to one of these space we shall indicate it with symbol $\Box_\sharp$ where $\sharp \in \{\mathrm{D}, \|, \perp, f_\parallel, f_\perp\}$.

Remark 13: *Since per definition $\delta\Omega^0(M) = \{0\} = \mathrm{n}\Omega^0(M)$, we observe that $\Omega^0_{\mathrm{D}}(M) = \Omega^0_\parallel(M)$. In particular we have*

$$\Omega^0_{\mathrm{D}}(M) \doteq \{\omega \in C^\infty(M) \mid \mathrm{t}\omega = \omega|_{\partial M} = 0\}, \qquad \Omega^0_\perp(M) \doteq \{\omega \in C^\infty(M) \mid \mathrm{n}\mathrm{d}\omega = \nu(\mathrm{d}\omega)|_{\partial M} = 0\}\,,$$

where, for all $p \in \partial M$, ν_p coincides with the outward pointing unit vector, normal to the boundary. These two options coincide with the standard Dirichlet and Neumann boundary conditions for scalar functions. Moreover for $f = 0$ we have $\Omega^k_{f_\parallel}(M) = \Omega^k_\parallel(M)$ as well as $\Omega^k_{f_\perp}(M) = \Omega^k_\perp(M)$.
Finally it is worth to mention that, for a static spacetime (M, g), the boundary conditions 1-3*, introduced in Definition 12, are themselves static, that is they do not depend explicitly on the time coordinate τ. A similar statement holds true for $f_\perp$, $f_\parallel$ boundary conditions provided that $f \in C^\infty(\partial M)$ and $\partial_\tau f = 0$. This will play a key rôle when we will verify that Assumption 16 is valid on static spacetime – cf. Proposition 48 in Appendix A.*

Remark 14: *It is interesting to observe that different boundary conditions can be related via the action of the Hodge operator. In particular, using Equation* (4) *and* (8)*, one can infer that, for any $f, f' \in C^\infty(\partial M)$ it holds that*

$$\star\Omega^k_{f,f'}(M) = \Omega^{m-k}_{-f',-f}(M).$$

At the same time, with reference, to the space of k-forms in Definition 12 it holds

$$\star\Omega^k_{\mathrm{D}}(M) = \Omega^{m-k}_{\mathrm{D}}(M)\,, \qquad \star\Omega^k_\parallel(M) = \Omega^{m-k}_\perp(M)\,, \qquad \star\Omega^k_{f_\parallel}(M) = \Omega^{m-k}_{-f_\perp}(M)\,. \tag{14}$$

For later convenience we prove the following lemma.

Lemma 15: *Let $\sharp \in \{\mathrm{D}, \|, \perp, f_\parallel, f_\perp\}$, with $f \in C^\infty(\partial M)$. The following statements hold true:*

1. *for all $\omega \in \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{spc}}(M) \cap \Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

2. *for all $\omega \in \Omega^k_\sharp(M)$ there exists $\omega^+ \in \Omega^k_{\mathrm{pc}}(M) \cap \Omega^k_\sharp(M)$ and $\omega^- \in \Omega^k_{\mathrm{fc}}(M) \cap \Omega^k_\sharp(M)$ such that $\omega = \omega^+ + \omega^-$.*

Proof. We prove the result in the first case, the second one can be proved in complete analogy. Let $\omega \in \Omega^k_{\mathrm{sc}}(M) \cap \Omega^k_\sharp(M)$. Consider Σ_1, Σ_2, two Cauchy surfaces on M – *cf.* [AFS18, Def. 3.10] – such that $J^+(\Sigma_1) \subset J^+(\Sigma_2)$. Moreover, let $\varphi_+ \in \Omega^0_{\mathrm{pc}}(M)$ be such that $\varphi_+|_{J^+(\Sigma_2)} = 1$ and $\varphi_+|_{J^-(\Sigma_1)} = 0$. We define $\varphi_- := 1 - \varphi_+ \in \Omega^0_{\mathrm{fc}}(M)$. Notice that we can always choose φ so that, for all $x \in M$, $\varphi(x)$ depends only on the value $\tau(x)$, where τ is the global time function defined in Theorem 1. We set $\omega_\pm \doteq \varphi_\pm\omega$ so that $\omega^+ \in \Omega^k_{\mathrm{spc}}(M) \cap \Omega^k_\sharp(M)$ while $\omega^- \in \Omega^k_{\mathrm{sfc}}(M) \cap \Omega^k_\sharp(M)$. This is automatic for $\sharp = \mathrm{D}$ on account of the equality

$$\mathrm{t}\omega^\pm = \varphi_\pm \mathrm{t}\omega = 0\,, \qquad \mathrm{n}\omega^\pm = \varphi_\pm \mathrm{n}\omega = 0\,.$$

We now check that $\omega^\pm \in \Omega^k_\sharp(M)$ for $\sharp=\perp$. The proof for the remaining boundary conditions $\perp, f_\parallel, f_\perp$ follows by a similar computation – or by duality *cf.* Remark 14. It holds

$$\mathrm{n}\omega_\pm = \varphi_\pm|_{\partial M}\mathrm{n}\omega = 0\,, \qquad \mathrm{nd}\omega_\pm = \mathrm{n}(\mathrm{d}\chi\wedge\omega) = \partial_\tau\chi\,\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega = 0\,.$$

In the last equality $\mathrm{t}_{\Sigma_\tau}\colon \Omega^k(M)\to\Omega^k(\Sigma_\tau)$ and $\mathrm{n}_{\partial\Sigma_\tau}\colon\Omega^k(\Sigma_\tau)\to\Omega^{k-1}(\partial\Sigma_\tau)$ are the maps from Definition 5 with $N\equiv\Sigma_\tau\doteq\{\tau\}\times\Sigma$, where $M=\mathbb{R}\times\Sigma$. The last identity follows because the condition $\mathrm{n}\omega=0$ is equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega=0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega=0$ for all $\tau\in\mathbb{R}$ – *cf.* Lemma 50 in Appendix B. □

In the following we shall make a key assumption on the existence of distinguished fundamental solutions for the operator $\Box_\sharp$ for $\sharp\in\{\mathrm{D},\|,\perp,f_\parallel,f_\perp\}$. Subsequently we shall prove that such hypothesis holds true whenever the underlying globally hyperbolic spacetime with timelike boundary is ultrastatic and $f\in C^\infty(\partial\Sigma)$ has definite sign – *cf.* Appendix A. Recalling both Definition 2 and Definition 12 we require the following:

Assumption 16: *For all $f\in C^\infty(\partial M)$ and for all $k\in\mathbb{N}\cup\{0\}$, there exist advanced $(-)$ and retarded $(+)$ fundamental solutions for the d'Alembert-de Rham wave operator $\Box_\sharp$ where $\sharp\in\{\mathrm{D},\|,\perp,f_\parallel,f_\perp\}$, $G^\pm_\sharp\colon\Omega^k_{\mathrm{c}}(M)\to\Omega^k_{\mathrm{sc},\sharp}(M)\doteq\Omega^k_{\mathrm{sc}}(M)\cap\Omega^k_\sharp(M)$ such that*

$$\Box\circ G^\pm_\sharp = \mathrm{Id}_{\Omega^k_{\mathrm{c}}(M)}\,,\qquad G^\pm_\sharp\circ\Box_{\mathrm{c},\sharp}=\mathrm{Id}_{\Omega^k_{\mathrm{c},\sharp}(M)}\,,\qquad \mathrm{supp}(G^\pm_\sharp\omega)\subseteq J^\pm(\mathrm{supp}(\omega))\,,\tag{15}$$

for all $\omega\in\Omega^k_{\mathrm{c}}(M)$ where $J^\pm$ denote the causal future and past and where $\Box_{\mathrm{c},\sharp}$ indicates that the domain of $\Box$ is restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$.

Remark 17: *Notice that domain of $G^\pm_\sharp$ is not restricted to $\Omega^k_{\mathrm{c},\sharp}(M)$. Furthermore the second identity in* (15) *cannot be extended to $G^\pm_\sharp\circ\Box=\mathrm{Id}_{\Omega^k_{\mathrm{c}}(M)}$ since it would entail $G^\pm_\sharp\Box\omega=\omega$ for all $\omega\in\Omega^k_{\mathrm{c}}(M)$. Yet the left hand side also entails that $\omega\in\Omega^k_{\mathrm{c},\sharp}$, which is manifestly a contradiction.*

Corollary 18: *Under the same hypotheses of Assumption 16, if the fundamental solutions $G^\pm_\sharp$ exist, they are unique.*

Proof. Suppose that, beside $G^-_\sharp$, there exists a second map $\widetilde{G}^-_\sharp\colon\Omega^k_{\mathrm{c}}(M)\to\Omega^k_{\mathrm{sc},\sharp}(M)$ enjoying the properties of equation (15). Then, for any but fixed $\alpha\in\Omega^k_{\mathrm{c}}(M)$ it holds

$$(\alpha,G^+_\sharp\beta)=(\Box G^-_\sharp\alpha,G^+_\sharp\beta)=(G^-_\sharp\alpha,\Box G^+_\sharp\beta)=(G^-_\sharp\alpha,\beta),\quad\forall\beta\in\Omega^k_{\mathrm{c}}(M)\,,$$

where we used both the support properties of the fundamental solutions and Lemma 9 which guarantees that $\Box$ is formally self-adjoint on $\Omega^k_\sharp(M)$. Similarly, replacing $G^-_\sharp$ with $\widetilde{G}^-_\sharp$, it holds $(\alpha,G^+_\sharp\beta)=(\widetilde{G}^-_\sharp\alpha,\beta)$. It descends that $((\widetilde{G}^-_\sharp-G^-_\sharp)\alpha,\beta)=0$, which entails $\widetilde{G}^-_\sharp\alpha=G^-_\sharp\alpha$ being the pairing between $\Omega^k(M)$ and $\Omega^k_{\mathrm{c}}(M)$ separating. A similar result holds for the retarded fundamental solution. □

This corollary can be also read as a consequence of the property that, for all $\omega\in\Omega^k_{\mathrm{c}}(M)$, $G^\pm_\sharp\omega\in\Omega^k_{\mathrm{sc},\sharp}(M)$ can be characterized as the unique solution to the Cauchy problem

$$\Box\psi=\omega\,,\qquad\mathrm{supp}(\psi)\cap M\setminus J^\pm(\mathrm{supp}(\omega))=\emptyset\,,\qquad\psi\in\Omega^k_\sharp(M)\,.\tag{16}$$

Remark 19: *The fundamental solution $G^+_\sharp$ (resp. $G^-_\sharp$) can be extended to $G^+_\sharp\colon\Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$ (resp. $G^-_\sharp\colon\Omega^k_{\mathrm{pc}}(M)\to\Omega^k_{\mathrm{pc}}(M)\cap\Omega^k_\sharp(M)$) – cf. [Bär15, Thm. 3.8]. As a consequence the problem $\Box\psi=\omega$ with $\omega\in\Omega^k(M)$ always admits a solution lying in $\Omega^k_\sharp(M)$. As a matter of facts, consider any* *smooth function $\eta \equiv \eta(\tau)$, where $\tau \in \mathbb{R}$, cf. equation (1), such that $\eta(\tau) = 1$ for all $\tau > \tau_1$ and $\eta(\tau) = 0$ for all $\tau < \tau_0$. Then calling $\omega^+ \doteq \eta\omega$ and $\omega^- = (1-\eta)\omega$, it holds $\omega^+ \in \Omega^k_{\mathrm{pc}}(M)$ while $\omega^- \in \Omega^k_{\mathrm{fc}}(M)$. Hence $\psi = G^+_\sharp \omega^+ + G^-_\sharp \omega^- \in \Omega^k_\sharp(M)$ is the sought solution.*

We prove the main result of this section, which characterizes the kernel of $\Box_\sharp$ on the space of smooth k-forms with prescribed boundary condition $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$.

Proposition 20: *Whenever Assumption 16 is fulfilled, then, for all $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$, setting $G_\sharp \doteq G^+_\sharp - G^-_\sharp : \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{sc},\sharp}(M)$, the following statements hold true:*

1. *for all $f \in C^\infty(\partial M)$ the following duality relations hold true:*

$$\star G^\pm_{\mathrm{D}} = G^\pm_{\mathrm{D}} \star , \qquad \star G^\pm_\| = G^\pm_\perp \star , \qquad \star G^\pm_{f_\|} = G^\pm_{f_\perp} \star \, . \tag{17}$$

2. *for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$ it holds*

$$(\alpha, G^\pm_\sharp \beta) = (G^\mp_\sharp \alpha, \beta) \, . \tag{18}$$

3. *the interplay between $G_\sharp$ and $\Box_\sharp$ is encoded in the short exact sequence:*

$$0 \to \Omega^k_{\mathrm{c},\sharp}(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{c}}(M) \xrightarrow{G_\sharp} \Omega^k_{\mathrm{sc},\sharp}(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{sc}}(M) \to 0 \, , \tag{19}$$

where $\Omega^k_{\mathrm{c},\sharp}(M) \doteq \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_\sharp(M)$.

Proof. We prove the different items separately. Starting from *1.*, we observe that $\star\Box = \Box\star$. Together with Remark 14, this entails that, for all $\alpha \in \Omega^k_{\mathrm{c}}(M)$,

$$\Box \star G^\pm_\sharp \alpha = \star \Box G^\pm_\sharp \star \alpha = \alpha \, .$$

On account of Remark 14, the uniqueness of the fundamental solutions as per Corollary 18 entails (17).

2. Equation (18) is a consequence of the following chain of identities valid for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(M)$

$$(\alpha, G^\pm_\sharp \beta) = (\Box G^\mp_\sharp \alpha, G^\pm_\sharp \beta) = (G^\mp_\sharp \alpha, \Box G^\pm_\sharp \beta) = (G^\mp_\sharp \alpha, \beta) \, ,$$

where we used both the support properties of the fundamental solutions and Lemma 9.

3. The exactness of the series is proven using the properties already established for the fundamental solutions $G^\pm_\sharp$. The left exactness of the sequence is a consequence of the second identity in equation (15) which ensures that $\Box_\sharp \alpha = 0$, $\alpha \in \Omega^k_{\mathrm{c},\sharp}(M)$, entails $\alpha = G^+_\sharp \Box_\sharp \alpha = 0$. In order to prove that $\ker G_\sharp = \Box \Omega^k_{\mathrm{c},\sharp}$, we first observe that $G_\sharp \Box_\sharp \Omega^k_{\mathrm{c},\sharp}(M) = \{0\}$ on account of equation (15). Moreover, if $\beta \in \Omega^k_{\mathrm{c}}(M)$ is such that $G_\sharp \beta = 0$, then $G^+_\sharp \beta = G^-_\sharp \beta$. Hence, in view of the support properties of the fundamental solutions $G^+_\sharp \beta \in \Omega^k_{\mathrm{c},\sharp}(M)$ and $\beta = \Box_\sharp G^+_\sharp \beta$. Subsequently we need to verify that $\ker \Box = G_\sharp \Omega^k_{\mathrm{c}}(M)$. Once more $\Box_\sharp G_\sharp \Omega^k_{\mathrm{c}}(M) = \{0\}$ follows from equation (15). Conversely, let $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ be such that $\Box_\sharp \omega = 0$. On account of Lemma 15 we can split $\omega = \omega^+ + \omega^-$ where $\omega^+ \in \Omega^k_{\mathrm{spc},\sharp}(M)$. Then $\Box_\sharp \omega^+ = -\Box_\sharp \omega^- \in \Omega^k_{\mathrm{c},\sharp}(M)$ and

$$G_\sharp \Box_\sharp \omega^+ = G^+_\sharp \Box_\sharp \omega^+ + G^-_\sharp \Box_\sharp \omega^- = \omega \, .$$

To conclude we need to establish the right exactness of the sequence. Consider any $\alpha \in \Omega^k_{\mathrm{sc}}(M)$ and the equation $\Box_\sharp \omega = \alpha$. Consider the function $\eta(\tau)$ as in Remark 19 and let $\omega \doteq G^+_\sharp(\eta\alpha) + G^-_\sharp((1-\eta)\alpha)$. In view of Remark 19 and of the support properties of the fundamental solutions, $\omega \in \Omega^k_{\mathrm{sc},\sharp}(M)$ and $\Box_\sharp \omega = \alpha$. $\square$

Remark 21: *Following the same reasoning as in [Bär15] together with minor adaptation of the proofs of [DDF19], one may extend* $G_\sharp$ *to an operator* $G_\sharp\colon \Omega^k_{\mathrm{tc}}(M) \to \Omega^k_\sharp(M)$ *for all* $\sharp \in \{\mathrm{D}, \|, \perp, f_\|, f_\perp\}$*. As a consequence the exact sequence of Proposition 20 generalizes as*

$$0 \to \Omega^k_{\mathrm{tc}}(M) \cap \Omega^k_\sharp(M) \xrightarrow{\Box_\sharp} \Omega^k_{\mathrm{tc}}(M) \xrightarrow{G_\sharp} \Omega^k_\sharp(M) \xrightarrow{\Box_\sharp} \Omega^k(M) \to 0\,. \tag{20}$$

Remark 22: *Proposition 20 and Remark 21 ensure that* $\ker_{\mathrm{c}} \Box_\sharp \subseteq \ker_{\mathrm{tc}} \Box_\sharp = \{0\}$*. In other words, there are no timelike compact solutions to the equation* $\Box\omega = 0$ *with* $\sharp$*-boundary conditions. More generally it can be shown that* $\ker_{\mathrm{c}} \Box \subseteq \ker_{\mathrm{tc}} \Box = \{0\}$*, namely there are no timelike compact solutions regardless of the boundary condition. This follows by standard arguments using a suitable energy functional defined on the solution space – cf. [DDF19, Thm. 30] for the proof for* $k = 0$*.*

In view of the applications to the Maxwell operator, it is worth focusing specifically on the boundary conditions $\perp$, $\|$ individuated in Definition 12 since it is possible to prove a useful relation between the associated propagators and the operators d,δ.

Lemma 23: *Under the hypotheses of Assumption 16 it holds that*

$$G^\pm_\| \circ \mathrm{d} = \mathrm{d} \circ G^\pm_\| \quad \text{on } \Omega^k_{\mathrm{t}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^\pm_\| \circ \delta = \delta \circ G^\pm_\| \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,, \tag{21}$$

$$G^\pm_\perp \circ \delta = \delta \circ G^\pm_\perp \quad \text{on } \Omega^k_{\mathrm{n}}(M) \cap \Omega^k_{\mathrm{pc/fc}}(M)\,, \qquad G^\pm_\perp \circ \mathrm{d} = \mathrm{d} \circ G^\pm_\perp \quad \text{on } \Omega^k_{\mathrm{pc/fc}}(M)\,. \tag{22}$$

Proof. From equation (17) it follows that equations (21-22) are dual to each other via the Hodge operator. Hence we shall only focus on equation (21).

For every $\alpha \in \Omega^k_{\mathrm{c}}(M) \cap \Omega^k_{\mathrm{t}}(M)$, $G^\pm_\| \mathrm{d}\alpha$ and $\mathrm{d}G^\pm_\| \alpha$ lie both in $\Omega^k_\|(M)$. In particular, using equation (4b), $\mathrm{t}\delta\mathrm{d}G^\pm_\|\alpha = \mathrm{t}(\Box_\| - \mathrm{d}\delta)G^\pm_\|(\alpha) = \mathrm{t}\alpha = 0$ while the second boundary condition is automatically satisfied since $\mathrm{t}\mathrm{d}G^\pm_\| = \mathrm{d}\mathrm{t}G^\pm_\| = 0$. Hence, considering $\beta = G^\pm_\|\mathrm{d}\alpha - \mathrm{d}G^\pm_\|\alpha$, it holds that $\Box\beta = 0$ and $\beta \in \Omega^k_\| \cap \Omega^k_{\mathrm{pc/fc}}(M)$. In view of Remark 19, this entails $\beta = 0$. □

We conclude this section with a corollary to Lemma 23 which shows that, when considering the difference between the advanced and the retarded fundamental solutions, the support restrictions present in equations (21-22) disappear.

Corollary 24: *Under the hypotheses of Assumption 16 it holds that*

$$G_\sharp \circ \mathrm{d} = \mathrm{d} \circ G_\sharp \quad \text{on } \Omega^k_{\mathrm{tc}}(M)\,, \qquad G_\sharp \circ \delta = \delta \circ G_\sharp \quad \text{on } \Omega^k_{\mathrm{tc}}(M) \qquad \sharp \in \{\|, \perp\}\,. \tag{23}$$

Proof. In all cases the reasoning is similar as in the proof of equation (21), but it requires the following characterization of $G_\sharp$. Since $M \simeq \mathbb{R} \times \Sigma$ – *cf.* Theorem 1 – let $\tau_0 \in \mathbb{R}$ and consider $\alpha_0 \in \Omega^k_{\mathrm{c}}(\Sigma_0)$, where $\Sigma_0 := \{\tau_0\} \times \Sigma$. Setting $\alpha := \alpha_0 \wedge \delta_{\tau_0}\mathrm{d}\tau$ we define a distribution-valued k-form and, following [Bär15, Lem. 4.1., Thm. 4.3], we can consider $G_\sharp\alpha$. It turns out that $G_\sharp\alpha$ is the unique solution to the Cauchy problem

$$\Box\psi = 0\,, \qquad \mathrm{t}_{\Sigma_0}\psi = 0\,, \qquad \mathrm{t}_{\Sigma_0}\mathcal{L}_{\partial_\tau}(\psi) = \alpha_0\,, \qquad \sharp\text{-boundary conditions for } \psi\,, \tag{24}$$

where $\mathrm{t}_{\Sigma_0}\colon \Omega^k(M) \to \Omega^k(\Sigma_0)$ is defined in (2) with $N \equiv \Sigma_0$, while $\mathcal{L}_{\partial_\tau}$ denotes the Lie derivative along the vector field ∂_τ.

With this characterization we can prove equation (23). Focusing for simplicity on the first identity of (23) for $\sharp = \|$, we need to show that $\mathrm{d}G_\|\alpha$ and $G_\|\mathrm{d}\alpha$ solve the same Cauchy problem (24). While the analysis of the equation of motion and of the initial data do not differ from the counterpart on globally hyperbolic spacetimes with empty boundary, the only additional necessary information comes from $\mathrm{t}\delta\mathrm{d}G^\pm_\|\alpha = \mathrm{t}(\Box - \mathrm{d}\delta)G^\pm_\|\alpha = \mathrm{t}\alpha$, for all $\alpha \in \Omega^k_{\mathrm{tc}}(M)$. This entails that, being $G_\| = G^+_\| - G^-_\|$, $\mathrm{t}\delta\mathrm{d}G_\|\alpha = 0$. □

3.2 On the Maxwell operator

In this section we focus our attention on the Maxwell operator $\delta\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$ studying its kernel in connection both to the D'Alembert - de Rham wave operator $\Box$ and to the identification of suitable boundary conditions. We shall keep the assumption that (M, g) is a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = m \geq 2$ – *cf.* Theorem 1. Notice that, if $k = m$, then the Maxwell operator becomes trivial, while, if $k = 0$, is coincides with the D'Alembert - de Rham operator $\Box$. Hence this case falls in the one studied in the preceding section and in [DDF19]. Therefore, unless stated otherwise, henceforth we shall consider only $0 < k < m = \dim M$.

In complete analogy to the analysis of $\Box$, we observe that, for any pair $\alpha, \beta \in \Omega^k(M)$ such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is compact, the following Green's formula holds true:

$$(\delta\mathrm{d}\alpha, \beta) - (\alpha, \delta\mathrm{d}\beta) = (\mathrm{t}\alpha, \mathrm{nd}\beta)_\partial - (\mathrm{nd}\alpha, \mathrm{t}\beta)_\partial \,. \tag{25}$$

In the same spirit of Lemma 9, the operator $\delta\mathrm{d}$ becomes formally self-adjoint if we restrict its domain to

$$\Omega^k_f(M) \doteq \{\omega \in \Omega^k(M) \mid \mathrm{nd}\omega = f\mathrm{t}\omega\} \,, \tag{26}$$

where $f \in C^\infty(\partial M)$ is arbitrary but fixed. In what follows we will consider two particular boundary conditions which are directly related to the $\Box$-tangential and to the $\Box$-normal boundary conditions for the D'Alembert - de Rham operator – *cf.* Definition 12. The discussion of the general case is related to the Robin $\Box$-tangential / Robin $\Box$-normal boundary conditions. However, in these case, it is not clear whether a generalization of Lemma 23 holds true. This is an important obstruction to adapt our analysis to these cases.

Definition 25: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $0 < k < \dim M$. We call*

1. *space of k-forms with $\delta\mathrm{d}$-tangential boundary condition, $\Omega^k_{\mathrm{t}}(M)$ as in equation (3) with $N = \partial M$.*
2. *space of k-forms with $\delta\mathrm{d}$-normal boundary condition*

$$\Omega^k_{\mathrm{nd}}(M) \doteq \{\omega \in \Omega^k(M) |\ \mathrm{nd}\omega = 0\} \,. \tag{27}$$

In the following our first goal is to characterize the kernel of the Maxwell operator with a prescribed boundary condition, *cf.* Equation (26). To this end we need to focus on the *gauge invariance* of the underlying theory. In the case in hand this translates in the following characterization.

Definition 26: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary and let $\delta\mathrm{d}$ be the Maxwell operator acting on $\Omega^k(M)$, $0 < k < \dim M$. We say that*

1. *$A \in \Omega^k_{\mathrm{t}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{t}}(M)$ if $A - A' \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$, namely if there exists $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-tangential boundary conditions is denoted by*

$$\mathrm{Sol}_{\mathrm{t}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0 \,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)} \,. \tag{28}$$

2. *$A \in \Omega^k_{\mathrm{nd}}(M)$, is* gauge equivalent *to $A' \in \Omega^k_{\mathrm{nd}}(M)$ if there exists $\chi \in \Omega^{k-1}(M)$ such that $A' = A + \mathrm{d}\chi$. The space of solutions with $\delta\mathrm{d}$-normal boundary conditions is denoted by*

$$\mathrm{Sol}_{\mathrm{nd}}(M) \doteq \frac{\{A \in \Omega^k(M) |\ \delta\mathrm{d}A = 0 \,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}(M)} \,. \tag{29}$$

Similarly the space of spacelike supported solutions with δd-tangential (resp. δd-normal) boundary conditions are

$$\mathrm{Sol}_{\mathrm{t}}^{\mathrm{sc}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M)|\ \delta \mathrm{d} A = 0\,, \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t,sc}}(M)}\,, \quad \mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M) \doteq \frac{\{A \in \Omega^k_{\mathrm{sc}}(M)|\ \delta \mathrm{d} A = 0\,, \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M)}\,. \tag{30}$$

Remark 27: *Notice that in Definition 26 we have employed two different notions of gauge equivalence in the construction of $\mathrm{Sol}_{\mathrm{nd}}(M)$ and of $\mathrm{Sol}_{\mathrm{t}}(M)$, which are related to the different choices of boundary conditions. It is worth observing that the first one is the same used for Maxwell's equations written as a theory of k-forms on a globally hyperbolic spacetime without boundary. In a physical language, the underlying reason is that the boundary condition $\mathrm{nd}\omega = 0$ is a gauge-invariant identity with respect to the standard gauge transformations used when the boundary of the spacetime is empty. For this reason such scenario is certainly distinguished. As a matter of fact, when working with $\mathrm{Sol}_{\mathrm{t}}(M)$, contrary to $\mathrm{nd}\omega = 0$, the boundary condition $\mathrm{t}\omega = 0$ is not gauge invariant. Hence, in comparison to the scenario of a globally hyperbolic spacetime with empty boundary, one must introduce a reduced gauge group. When working at the level of k-forms such choice is not unique. To avoid this quandary, one should resort to a more geometrical formulation of Maxwell's equations, namely as originating from a theory for the connections of a principal $U(1)$-bundle over the underlying globally hyperbolic spacetime with timelike boundary, cf. [BDHS14, BDS14] for the case with empty boundary. Since this analysis would require a whole paper on its own we postpone it to future work.*

The following propositions discuss the existence of a representative fulfilling the Lorenz gauge condition of an equivalence classes $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ (*resp.* $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$) – *cf.* [Ben16, Lem. 7.2]. In addition we provide a connection between δd-tangential (*resp.* δd-normal) boundary conditions with $\Box$-tangential (*resp.* $\Box$-normal) boundary conditions. Recalling Definition 12 of the $\Box$-tangential boundary condition, the following holds true.

Proposition 28: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$ there exists a representative $A' \in [A]$ such that*

$$\Box_{\parallel} A' = 0\,, \qquad \delta A' = 0\,. \tag{31}$$

Moreover, the same result holds true for $[A] \in \mathrm{Sol}_{\mathrm{t}}^{\mathrm{sc}}(M)$.

Proof. We focus only on the first assertion, the proof of the second one being similar. Let $A \in [A] \in \mathrm{Sol}_{\mathrm{t}}(M)$, that is, $A \in \Omega^k(M)$, $\delta \mathrm{d}A = 0$ and $\mathrm{t}A = 0$. Consider any $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$ such that

$$\Box\chi = -\delta A, \qquad \delta\chi = 0, \qquad \mathrm{t}\chi = 0\,. \tag{32}$$

In view of Assumption 16 and of Remark 19, we can fix $\chi = -\sum_{\pm} G^{\pm}_{\parallel} \delta A^{\pm}$, where $A^{\pm}$ is defined as in Remark 19. Per definition of $G^{\pm}_{\parallel}$, $\mathrm{t}\chi = 0$ while, on account of Corollary 24, $\delta\chi = -\sum_{\pm} \delta G^{\pm}_{\parallel} \delta A^{\pm} = 0$. Hence A' is gauge equivalent to A as per Definition 26. $\Box$

The proof of the analogous result for $\Omega^k_{\mathrm{nd}}(M)$ is slightly different and, thus, we discuss it separately. Recalling Definition 12 of the $\Box$-normal boundary conditions, the following statement holds true.

Proposition 29: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then for all $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ there exists a representative $A' \in [A]$ such that*

$$\Box_{\perp} A' = 0\,, \qquad \delta A' = 0\,. \tag{33}$$

Moreover, the same result holds true for $[A] \in \mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$.

Proof. As in the previous proposition, we can focus only on the first point. Let A be a representative of $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. Hence $A \in \Omega^k(M)$ so that $\delta\mathrm{d}A = 0$ and $\mathrm{nd}A = 0$. Consider first $\chi_0 \in \Omega^{k-1}(M)$ such that $\mathrm{nd}\chi_0 = -\mathrm{n}A$. The existence is guaranteed since the map nd is surjective – *cf.* Remark 7. As a consequence we can exploit the residual gauge freedom to select $\chi_1 \in \Omega^{k-1}(M)$ such that

$$\Box\chi_1 = -\delta\widetilde{A}\,, \qquad \delta\chi_1 = 0\,, \qquad \mathrm{nd}\chi_1 = 0 \qquad \mathrm{n}\chi_1 = 0\,, \tag{34}$$

where $\widetilde{A} = A + \mathrm{d}\chi_0$. Let $\eta \equiv \eta(\tau)$ be a smooth function such that $\eta = 0$ if $\tau < \tau_0$ while $\eta = 1$ if $\tau > \tau_1$, *cf.* Remark 19. Since $\mathrm{n}\widetilde{A} = 0$ we can fine tune η in such a way that both $\widetilde{A}^+ \doteq \eta\widetilde{A}$ and $\widetilde{A}^- \doteq (1-\eta)\widetilde{A}$ satisfy $\mathrm{n}\widetilde{A}^\pm = 0$. Equation (4b) entails that $\mathrm{n}\delta A^\pm = -\delta\mathrm{n}A^\pm = 0$. Hence we can apply Lemma 23 and set $\chi_1 = -\sum_\pm G^\pm_\perp \delta\widetilde{A}^+$. Calling $A' = A + \mathrm{d}(\chi_0 + \chi_1)$ we obtained the desired result. ☐

Remark 30: *A direct inspection of* (32) *and of* (33) *unveils that choosing a solution to these equations does not fix completely the gauge and a residual freedom is left. This amount either to*

$$\mathcal{G}_{\mathrm{t}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0\,,\ \mathrm{t}\chi = 0\}\,,$$

or, in the case of a $\delta\mathrm{d}$*-normal boundary condition, to*

$$\mathcal{G}_{\mathrm{nd}}(M) \doteq \{\chi \in \Omega^{k-1}(M) \mid \delta\mathrm{d}\chi = 0,\ \mathrm{n}\chi = 0\ ,\ \mathrm{nd}\chi = 0\}\,.$$

Observe that, in the definition of $\mathcal{G}_{\mathrm{nd}}(M)$*, we require* χ *to be in the kernel of* $\delta\mathrm{d}$*. Nonetheless since the actual reduced gauge group is* $\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)$ *we can work with* $\chi_0 \in \Omega^{k-1}(M)$ *such that* $\Box\chi_0 = 0$*. As a matter of fact for all* $\chi \in \mathcal{G}_{\mathrm{nd}}$ *we can set* $\chi_0 \doteq \chi + \mathrm{d}\lambda$ *where* $\lambda \in \Omega^{k-2}(M)$ *is such that* $\Box\lambda = -\delta\chi$ *and* $\mathrm{n}\lambda = \mathrm{nd}\lambda = 0$ *– cf. Proposition 29. In addition* $\mathrm{d}\chi = \mathrm{d}\chi_0$*.*

To better codify the results of the preceding discussion, it is also convenient to introduce the following linear spaces:

$$\mathcal{S}^\Box_{\mathrm{t}}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{t}A = 0\}\,, \tag{35}$$

$$\mathcal{S}^\Box_{\mathrm{nd}}(M) \doteq \{A \in \Omega^k(M) \mid \Box A = 0\ ,\ \delta A = 0\ ,\ \mathrm{n}A = 0,\ \ \mathrm{nd}A = 0\}\,. \tag{36}$$

where $f \in C^\infty(\partial M)$*. Hence Propositions 28-29 can be summarized as stating the existence of the following isomorphisms:*

$$\mathcal{S}_{\mathcal{G}_{\mathrm{t}},k}(M) \doteq \frac{\mathcal{S}^\Box_{\mathrm{t}}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{t}}(M)} \simeq \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad \mathcal{S}_{\mathcal{G}_{\mathrm{nd}},k}(M) \doteq \frac{\mathcal{S}^\Box_{\mathrm{nd}}(M)}{\mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)} \simeq \mathrm{Sol}_{\mathrm{nd}}(M)\,. \tag{37}$$

It is noteworthy that both $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ can be endowed with a presymplectic form – *cf.* [HS13, Prop. 5.1].

Proposition 31: *Let* (M,g) *be a globally hyperbolic spacetime with timelike boundary. Let* $[A_1],[A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ *and, for* $A_1 \in [A_1]$*, let* $A_1 = A_1^+ + A_1^-$ *be any decomposition such that* $A^+ \in \Omega^k_{\mathrm{spc,t}}(M)$ *while* $A^- \in \Omega^k_{\mathrm{sfc,t}}(M)$ *– cf. Lemma 15. Then the following map* $\sigma_{\mathrm{t}}\colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)^{\times 2} \to \mathbb{R}$ *is a presymplectic form:*

$$\sigma_{\mathrm{t}}([A_1],[A_2]) = (\delta\mathrm{d}A_1^+, A_2)\,, \qquad \forall [A_1],[A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,. \tag{38}$$

A similar result holds for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ *and we denote the associated presymplectic form* σ_{nd}*. In particular for all* $[A_1],[A_2] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ *we have* $\sigma_{\mathrm{nd}}([A_1],[A_2]) \doteq (\delta\mathrm{d}A_1^+, A_2)$ *where* $A_1 \in [A_1]$ *is such that* $A \in \Omega^k_{\mathrm{sc},\perp}(M)$*.*

Proof. We shall prove the result for σ_{nd}, the proof for σ_{t} being the same mutatis mutandis.

First of all notice that for all $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ there exists $A' \in [A]$ such that $A' \in \Omega^k_{\perp}(M)$. This is realized by picking an arbitrary $A \in [A]$ and defining $A' \doteq A + \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$ is such that $\mathrm{nd}\chi = -\mathrm{n}A$ – *cf.* Remark 7. We can thus apply Lemma 15 in order to split $A' = A'_+ + A'_-$ where $A'_+ \in \Omega^k_{\mathrm{spc,nd}}(M)$ and $A'_- \in \Omega^k_{\mathrm{sfc,nd}}(M)$. Notice that this procedure is not necessary for $\delta\mathrm{d}$-tangential boundary condition since we can always split $A \in \Omega^k_{\mathrm{sc,t}}(M)$ as $A = A^+ + A^-$ with $A_+ \in \Omega^k_{\mathrm{spc,t}}(M)$ and $A_- \in \Omega^k_{\mathrm{sfc,t}}(M)$ without invoking Lemma 15.

After these preliminary observations consider the map

$$\sigma_{\mathrm{nd}} \colon (\ker\delta\mathrm{d} \cap \Omega^k_{\mathrm{sc},\perp}(M))^{\times 2} \ni (A_1, A_2) \mapsto (\delta\mathrm{d}A_1^+, A_2)\,,$$

where we used Lemma 15 and we split $A_1 = A_1^+ + A_1^-$, with $A_1^+ \in \Omega^k_{\mathrm{spc},\perp}(M)$ while $A_1^- \in \Omega^k_{\mathrm{sfc},\perp}(M)$. The pairing $(\delta\mathrm{d}A_1^+, A_2)$ is finite because A_2 is a spacelike compact k-form while $\delta\mathrm{d}A_1^+$ is compactly supported on account of A_1 being on-shell. Moreover, $(\delta\mathrm{d}A_1^+, A_2)$ is independent from the splitting $A_1 = A_1^+ + A_1^-$ and thus σ_{nd} is well-defined. Indeed, let $A_1 = \widetilde{A}_1^+ + \widetilde{A}_1^-$ be another splitting: it follows that $A_1^+ - \widetilde{A}_1^+ = -(A_1^- - \widetilde{A}_1^-) \in \Omega^k_{\mathrm{c,nd}}(M)$. Therefore

$$(\delta\mathrm{d}\widetilde{A}_1^+, A_2) = (\delta\mathrm{d}A_1^+, A_2) + (\delta\mathrm{d}(\widetilde{A}_1^+ - A_1^+), A_2) = (\delta\mathrm{d}A_1^+, A_2)\,,$$

where in the last equality we used the self-adjointness of $\delta\mathrm{d}$ on $\Omega^k_{\mathrm{nd}}(M)$.

We show that $\sigma_{\mathrm{nd}}(A_1, A_2) = -\sigma_{\mathrm{nd}}(A_2, A_1)$ for all $A_1, A_2 \in \ker\delta\mathrm{d} \cap \Omega^k_{\mathrm{sc},\perp}(M)$. For that we have

$$\begin{aligned}
\sigma_{\mathrm{nd}}(A_1, A_2) = (\delta\mathrm{d}A_1^+, A_2) &= (\delta\mathrm{d}A_1^+, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-)\\
&= -(\delta\mathrm{d}A_1^-, A_2^+) + (\delta\mathrm{d}A_1^+, A_2^-)\\
&= -(A_1^-, \delta\mathrm{d}A_2^+) + (A_1^+, \delta\mathrm{d}A_2^-)\\
&= -(A_1^-, \delta\mathrm{d}A_2^+) - (A_1^+, \delta\mathrm{d}A_2^+)\\
&= -(A_1, \delta\mathrm{d}A_2^+) = -\sigma_{\mathrm{nd}}(A_1, A_2)\,,
\end{aligned}$$

where we exploited Lemma 15 and $A_1^\pm, A_2^\pm \in \Omega^k_{\mathrm{sc,nd}}(M)$.

Finally we prove that $\sigma_{\mathrm{nd}}(A_1, \mathrm{d}\chi) = 0$ for all $\chi \in \Omega^k_{\mathrm{sc}}(M)$. Together with the antisymmetry shown before, this entails that σ_{nd} descends to a well-defined map $\sigma_{\mathrm{nd}} \colon \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)^{\times 2} \to \mathbb{R}$ which is bilinear and antisymmetric. Therefore it is a presymplectic form. To this end let $\chi \in \Omega^{k-1}_{\mathrm{sc}}(M)$: we have

$$\sigma_{\mathrm{nd}}(A, \mathrm{d}\chi) = (\delta\mathrm{d}A_1^+, \mathrm{d}\chi) = (\delta^2\mathrm{d}A_1^+, \chi) + (\mathrm{n}\delta\mathrm{d}A^+, \mathrm{t}\chi) = 0\,,$$

where we used equation (5) as well as $\mathrm{n}\delta\mathrm{d}A = -\delta\mathrm{nd}A = 0$. □

Working either with $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{t}}(M)$ or $\mathrm{Sol}^{(\mathrm{sc})}_{\mathrm{nd}}(M)$ leads to the natural question whether it is possible to give an equivalent representation of these spaces in terms of compactly supported k-forms. Using Assumption 16, the following proposition holds true:

Proposition 32: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. Then the following linear maps are isomorphisms of vector spaces*

$$G_\parallel \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)\,, \qquad G_\parallel \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)\,, \tag{39}$$

$$G_\perp \colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc,nd}}(M)} \to \mathrm{Sol}_{\mathrm{nd}}(M)\,, \qquad G_\perp \colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)} \to \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)\,, \tag{40}$$

Proof. Mutatis mutandis, the proof of the four isomorphisms is the same. Hence we focus only on $G_\parallel\colon \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{t}}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$.

A direct computation shows that $G_\parallel\left[\Omega^k_{\mathrm{tc},\delta}(M)\right] \subseteq \mathcal{S}^\Box_{\mathrm{t},k}(M)$. The condition $\delta G_\parallel\omega = 0$ follows from Corollary 24. Moreover, $G_\parallel$ descends to the quotient since for all $\eta \in \Omega^k_{\mathrm{tc},\mathrm{t}}(M)$ we have $G_\parallel\delta\mathrm{d}\eta = -G_\parallel\mathrm{d}\delta\eta = -\mathrm{d}G_\parallel\delta\eta \in \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ on account of Corollary 24.

We prove that $G_\parallel$ is surjective. Let $[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$. In view of Proposition 28 there exists $A' \in [A]$ such that $\Box_\parallel A' = 0$ as well as $\delta A' = 0$. Proposition 20 ensures that there exists $\alpha \in \Omega^k_{\mathrm{tc}}(M)$ such that $A' = G_\parallel\alpha$. Moreover, condition $\delta A' = 0$ and Corollary 24 implies that $\delta\alpha \in \ker G_\parallel$, therefore $\delta\alpha = \Box_\parallel\eta$ for some $\eta \in \Omega^k_{\mathrm{tc},\parallel}(M)$ – *cf.* Proposition 20 and Remark 21. Applying δ to the equality $\delta\alpha = \Box_\parallel\eta$ we find $\Box\delta\eta = 0$, that is, $\delta\eta = 0$ – *cf* Remark 22. It follows that $\delta\alpha = \delta\mathrm{d}\eta$. Moreover we have $[A] = [G_\parallel\alpha] = [G_\parallel\alpha - \mathrm{d}G_\parallel\eta] = [G_\parallel(\alpha - \mathrm{d}\eta)]$, where now $\alpha - \mathrm{d}\eta \in \Omega^k_{\mathrm{tc},\delta}(M)$.

Finally we prove that $G_\parallel$ is injective: let $[\alpha] \in \frac{\Omega^k_{\mathrm{tc},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{tc},\mathrm{t}}(M)}$ be such that $[G_\parallel\alpha] = [0]$. This entails that there exists $\chi \in \Omega^{k-1}_{\mathrm{tc},\mathrm{t}}(M)$ such that $G_\parallel\alpha = \mathrm{d}\chi$. Corollary 24 and $\alpha \in \Omega^k_{\mathrm{tc},\delta}(M)$ ensures that $\delta\mathrm{d}\chi = 0$, therefore $\chi \in \mathrm{Sol}_{\mathrm{t}}(M)$. Proposition 28, Remark 21 and Corollary 24 ensures that $\mathrm{d}\chi = \mathrm{d}G_\parallel\beta$ with $\beta \in \Omega^k_{\mathrm{tc},\delta}(M)$. It follows that $\alpha - \mathrm{d}\beta \in \ker G_\parallel$ and therefore $\alpha - \mathrm{d}\beta = \Box_\parallel\eta$ for $\eta \in \Omega^k_{\mathrm{tc},\parallel}(M)$ – *cf.* Remark 21. Applying δ to the last equality we find $\Box\beta = \Box\delta\eta$, hence $\beta = \delta\eta$ because of Remark 22. It follows that $\alpha = \delta\mathrm{d}\eta$ with $\eta \in \Omega^k_{\mathrm{c},\mathrm{t}}(M)$, that is, $[\alpha] = [0]$. $\Box$

The following proposition shows that the isomorphisms introduced in Proposition 32 for $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ and $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ lift to isomorphisms of presymplectic spaces.

Proposition 33: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. The following statements hold true:*

1. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\mathrm{t}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_\parallel([\alpha],[\beta]) \doteq (\alpha, G_\parallel\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\mathrm{t}}(M)}, \widetilde{G}_\parallel\right)$ *is symplectomorphic to* $(\mathrm{Sol}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$.

2. $\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\mathrm{nd}}(M)}$ *is a pre-symplectic space if endowed with the bilinear map* $\widetilde{G}_\perp([\alpha],[\beta]) \doteq (\alpha, G_\perp\beta)$.

   Moreover $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\mathrm{nd}}(M)}, \widetilde{G}_\perp\right)$ *is pre-symplectomorphic to* $(\mathrm{Sol}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Proof. The proof of the two statements is the same. Hence we focus only on the first one. We observe that $\widetilde{G}_\parallel$ is well-defined. As a matter of fact, let $\alpha, \beta \in \Omega^k_{\mathrm{c},\delta}(M)$, then $G_\parallel\beta \in \Omega^k_{\mathrm{sc}}(M)$ and therefore the pairing $(\alpha, G_\parallel\beta)$ is finite. Moreover if $\eta \in \Omega^k_{\mathrm{c},\parallel}(M)$ we have

$$
\begin{aligned}
(\delta\mathrm{d}\eta, G_\parallel\beta) &= (\eta, \delta\mathrm{d}G_\parallel\beta) = -(\eta, \mathrm{d}\delta G_\parallel\beta) = -(\eta, \mathrm{d}G_\parallel\delta\beta) = 0\,,\\
(\alpha, G_\parallel\delta\mathrm{d}\eta) &= -(\alpha, G_\parallel\mathrm{d}\delta\eta) = -(\alpha, \mathrm{d}G_\parallel\delta\eta) = 0\,,
\end{aligned}
$$

where we used that $G_\parallel\beta, \eta \in \Omega^k_{\mathrm{c},\mathrm{t}}(M)$ – *cf.* equation (25) – as well as $\delta G_\parallel\beta = G_\parallel\delta\beta = 0$ – *cf.* Corollary 24. Therefore $\widetilde{G}_\parallel$ is well-defined: Moreover, it is per construction bilinear and antisymmetric, therefore it induces a pre-symplectic structure.

We now show that the isomorphism $G_\parallel\colon \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\parallel}(M)} \to \mathrm{Sol}_{\mathrm{t}}(M)$ is a pre-symplectomorphism. Let $[\alpha], [\beta] \in \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c},\parallel}(M)}$. As a direct consequence of the properties of $G_\parallel = G^+_\parallel - G^-_\parallel$, calling $A_1 = G_\parallel\alpha$ and

$A_2 = G_\parallel \beta$, we can consider $A_1^\pm = G_\parallel^\pm \alpha$ in equation (38). This leads us to

$$\sigma_{\mathrm{t}}([G_\parallel\alpha],[G_\parallel\beta]) = (\delta \mathrm{d} G_\parallel^+\alpha, G_\parallel\beta) = (\Box G_\parallel^+\alpha - \mathrm{d}\delta G_\parallel^+\alpha, G_\parallel\beta) = (\alpha, G_\parallel\beta) = \widetilde{G}_\parallel([\alpha],[\beta])\,,$$

where we used Corollary 24 so that $\mathrm{d}\delta G_\parallel^+\alpha = \mathrm{d}G_\parallel^+\delta\alpha = 0$. □

Remark 34: *Following [HS13, Cor. 5.3], σ_{t} (resp. σ_{nd}) do not define in general a symplectic form on the space of spacelike compact solutions $\mathrm{Sol}_{\mathrm{t}}(M)$ (resp. $\mathrm{Sol}_{\mathrm{nd}}(M)$). A direct characterization of this deficiency is best understood by introducing the following quotients:*

$$\widehat{\mathrm{Sol}}_{\mathrm{t}}^{\mathrm{sc}} := \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) \mid \delta\mathrm{d}A = 0\,,\ \mathrm{t}A = 0\}}{\mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M) \cap \Omega^k_{\mathrm{sc}}(M)}\,,\quad \widehat{\mathrm{Sol}}_{\mathrm{nd}}^{\mathrm{sc}} := \frac{\{A \in \Omega^k_{\mathrm{sc}}(M) \mid \delta\mathrm{d}A = 0\,,\ \mathrm{nd}A = 0\}}{\mathrm{d}\Omega^{k-1}(M) \cap \Omega^k_{\mathrm{sc}}(M)}\,, \tag{41}$$

Focusing on $\delta\mathrm{d}$-normal boundary conditions, it follows that $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}} \subseteq \widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$. Moreover, $\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ is symplectic with respect to the form $\sigma_{\mathrm{nd}}([A_1],[A_2]) = (\delta\mathrm{d}A_1^+, A_2)$. This can be shown as follows: if $\sigma_{\mathrm{nd}}([A_1],[A_2]) = 0$ for all $[A_1] \in \widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}$ then, choosing $A_1 = G_\perp\alpha$ with $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$ leads to $0 = \sigma_\perp([G_\perp\alpha],[A_2]) = (\alpha, A_2)$ – cf. Proposition 33. This entails $\mathrm{d}A_2 = 0$ as well as $A_2 = 0 \in H_{k,\mathrm{c,n}}(M)^* \simeq H^k(M)$ – cf. Appendix C. Therefore $A_2 = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}(M)$ that is $A_2 = 0$ in $\widehat{\mathrm{Sol}}^{\mathrm{sc}}_{\mathrm{nd}}(M)$. A similar result holds, mutatis mutandis, for $\parallel$.

The net result is that $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$ (resp. $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M), \sigma_{\mathrm{t}})$) is symplectic if and only if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ (resp. $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) = \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$). This is in agreement with the analysis in [BDS14] for the case of globally hyperbolic spacetimes (M,g) with $\partial M = \emptyset$.

Example 35: *We give an example where $\mathrm{d}\Omega^{k-1}_{\mathrm{sc}}(M) \subseteq \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a proper inclusion – cf. [HS13, Ex. 5.7]. Consider half-Minkowski spacetime $\mathbb{R}^m_+ := \mathbb{R}^{m-1} \times \overline{\mathbb{R}_+}$ with flat metric and let $p \in \mathring{\mathbb{R}}^m_+$. We introduce $M := \mathbb{R}^m_+ \setminus J(p)$ endowed with the restriction to M of the Minkowski metric. This spacetime is still globally hyperbolic with timelike boundary. Let now $p \in B_1 \subset B_2$, where B_1, B_2 are open balls in $\mathbb{R}^{m-1}_+$ centered at p.*

We consider $\psi \in \Omega^0(M)$ such that $\psi|_{J(B_1\cap M)} = 1$ and $\psi|_{J(B_2\cap M)} = 0$. In addition we introduce $\varphi \in \Omega^0_{\mathrm{tc}}(M)$ such that: (a) for all $x \in M$, $\varphi(x)$ depends only on $\tau(x)$ – cf. Theorem 1; (b) $\chi := \varphi\psi \in \Omega^0_{\mathrm{tc}}(M)$ is such that $\mathrm{t}\chi = \chi|_{\partial M} = 0$; (c) there exists an interval $I \subset \mathbb{R}$ such that $\varphi|_I = 1$.

In other words φ plays the rôle of a cut-off function so that $\chi \equiv \chi(\tau)$ does not vanish only for values of τ whose associated Cauchy surface $\Sigma_\tau \doteq \{\tau\} \times \Sigma$ is such that $(\Sigma_\tau \cap J(B_2)) \cap \partial M = \emptyset$.

It follows that $\mathrm{d}\chi \in \Omega^1_{\mathrm{c}}(M) \subseteq \Omega^1_{\mathrm{sc}}(M)$. Yet there does not exist $\zeta \in \Omega^1_{\mathrm{sc}}(M)$ such that $\mathrm{d}\zeta = \mathrm{d}\chi$. Indeed, let us consider the curve $\gamma_s \subseteq M$ parametrized by $(s, x, 0, \ldots) \in M$ where $s \in I \subset \mathbb{R}$ is such that $\varphi(s) = 1$ for all $s \in I$, while $x \in (x(p), +\infty)$ – $x(p)$ denotes the x-coordinate of p. Integration along γ_s yields

$$\int_{\gamma_s} \iota^*_{\gamma_s}\mathrm{d}\chi = -1\,, \qquad \int_{\gamma_s} \iota^*_{\gamma_s}\mathrm{d}\zeta = 0\,.$$

3.3 The algebra of observables for $\mathrm{Sol}_{\mathrm{t}}(M)$ and for $\mathrm{Sol}_{\mathrm{nd}}(M)$

In this section we discuss an application of the results of the previous section. Motivated by the algebraic approach to quantum field theory, we associate a unital $*$-algebra both to $\mathrm{Sol}_{\mathrm{t}}(M)$ and to $\mathrm{Sol}_{\mathrm{nd}}(M)$, whose elements are interpreted as the observables of the underlying quantum system. Furthermore we study its key structural properties and we comment on their significance. We recall that the corresponding question, when the underlying background (M,g) is globally hyperbolic manifold with $\partial M = \emptyset$ has been thoroughly discussed in the literature – *cf.* [Ben16, DS11, HS13, SDH12]. On account of the different behaviour of $\delta\mathrm{d}$-tangential and $\delta\mathrm{d}$-normal boundary conditions we discuss each algebra separately.

3.3.1 The algebra of observable for $\mathrm{Sol}_{\mathrm{t}}(M)$

In this section we introduce the algebra of observables associated to the solution space $\mathrm{Sol}_{\mathrm{t}}(M)$ and we discuss its main properties. Our analysis follows closely that of [Ben16, DS11, HS13, SDH12] for globally hyperbolic spacetimes with empty boundary.
Following [Ben16] we will identify a unital $*$-algebra $\mathcal{A}_{\mathrm{t}}(M)$ built out of distinguished linear functionals over $\mathrm{Sol}_{\mathrm{t}}(M)$, whose collection is fixed so to contain enough elements to distinguish all configurations in $\mathrm{Sol}_{\mathrm{t}}(M)$ – *cf.* Proposition 37.

Taking into account the discussion in the preceding sections, particularly Equation (4b) and Definition 5 we introduce the following structures.

Definition 36: *Let (M,g) be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{t}}(M)$*, the associative, unital $*$-algebra*

$$\mathcal{A}_{\mathrm{t}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]}\,, \qquad \mathcal{O}_{\mathrm{t}}(M) \doteq \frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)}\,. \tag{42}$$

Here $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)] \doteq \bigoplus_{n=0}^{\infty}\mathcal{O}_{\mathrm{t}}(M)^{\otimes n}$ is the universal tensor algebra with $\mathcal{O}_{\mathrm{t}}(M)^{\otimes 0} \equiv \mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. In addition $\mathcal{I}[\mathcal{O}_{\mathrm{t}}(M)]$ is the $*$-ideal generated by elements of the form $[\alpha]\otimes[\beta]-[\beta]\otimes[\alpha]-i\widetilde{G}_{\|}([\alpha],[\beta])\mathbb{I}$, where $[\alpha],[\beta]\in\mathcal{O}_{\mathrm{t}}(M)$ while $\widetilde{G}_{\|}$ is defined in Proposition 33 and $\mathbb{I}$ is the identity of $\mathcal{T}[\mathcal{O}_{\mathrm{t}}(M)]$.

We study the structural properties of the algebra of observables. On account of its definition, it suffices to focus mainly on the properties of the generators $\mathcal{O}_{\mathrm{t}}(M)$. In particular, in the next proposition we follow the rationale advocated in [Ben16] proving that $\mathcal{O}_{\mathrm{t}}(M)$ is *optimal*:

Proposition 37: *Let $\mathcal{O}_{\mathrm{t}}(M)$ be as per Definition 36. Then, calling with $(\ ,\ )$ the natural pairing between $\mathcal{O}_{\mathrm{t}}(M)$ and $\mathrm{Sol}_{\mathrm{t}}(M)$ induced from that between k-forms, $\mathcal{O}_{\mathrm{t}}(M)$ is* **optimal***, namely:*

1. *$\mathcal{O}_{\mathrm{t}}(M)$ is* separating*, that is*

$$([\alpha],[A]) = 0 \quad \forall[\alpha]\in\mathcal{O}_{\mathrm{t}}(M) \Longrightarrow [A]=[0]\in\mathrm{Sol}_{\mathrm{t}}(M)\,. \tag{43}$$

2. *$\mathcal{O}_{\mathrm{t}}(M)$ is* non redundant*, that is*

$$([\alpha],[A]) = 0 \quad \forall[A]\in\mathrm{Sol}_{\mathrm{t}}(M) \Longrightarrow [\alpha]=[0]\in\mathcal{O}_{\mathrm{t}}(M)\,. \tag{44}$$

Proof. As starting point observe that the pairing $([\alpha],[A]) := (\alpha, A)$ is well-defined. Indeed let consider two representatives $A\in[A]\in\mathrm{Sol}_{\mathrm{t}}(M)$ and $\alpha\in[\alpha]\in\mathcal{O}_{\mathrm{t}}(M)$. The pairing (α,A) is finite being $\mathrm{supp}(\alpha)$ compact and there is no dependence on the choice of representative. As a matter of facts, if $\mathrm{d}\chi\in\mathring{d}\Omega^{k-1}_{\mathrm{t}}(M)$ and $\eta\in\Omega^k_{\mathrm{c,t}}(M)$, it holds

$$(\alpha,\mathrm{d}\chi) = (\delta\alpha,\chi) + (\mathrm{n}\alpha,\mathrm{t}\chi)_\partial = 0\,, \qquad (\delta\mathrm{d}\eta, A) = (\eta,\delta\mathrm{d}A) + (\mathrm{t}\eta,\mathrm{n}\mathrm{d}A)_\partial - (\mathrm{n}\mathrm{d}\eta,\mathrm{t}A)_\partial = 0\,,$$

where in the first equation we used the fact that $\mathrm{t}\chi = 0$ as well as $\delta\alpha = 0$, while in the second equation we used $\delta\mathrm{d}A = 0$ as well as $\mathrm{t}A = \mathrm{t}\eta = 0$.

Having established that the pairing between the equivalence classes is well-defined we prove the remaining two items separately.

1. Assume $\exists[A]\in\mathrm{Sol}_{\mathrm{t}}(M)$ such that $([\alpha],[A])=0$, $\forall[\alpha]\in\mathcal{O}_{\mathrm{t}}(M)$. Working at the level of representative, since $\alpha\in\Omega^k_{\mathrm{c},\delta}(M)$ we can choose $\alpha=\delta\beta$ with $\beta\in\Omega^{k+1}_{\mathrm{c}}(M)$. As a consequence $0=(\delta\beta,A)=(\beta,\mathrm{d}A)$

where we used implicitly (5) and $\mathrm{t}A = 0$. The arbitrariness of β and the non-degeneracy of $(\ ,\ )$ entails $\mathrm{d}A = 0$. Hence A individuates a de Rham cohomology class in $H^k_{\mathrm{t}}(M)$, *cf.* Appendix C. Furthermore, $([\alpha],[A]) = 0$ entails $\langle[\alpha],[A]\rangle = 0$ where $\langle\ ,\ \rangle$ denotes the pairing between $H_{k,\mathrm{c}}(M)$ and $H^k_{\mathrm{t}}(M)$ – *cf.* Appendix C. On account of Remark 54 it holds that $\langle\ ,\ \rangle$ is non-degenerate and therefore $[A] = 0$.

2. Assume $\exists[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$ such that $([\alpha],[A]) = 0\ \forall[A] \in \mathrm{Sol}_{\mathrm{t}}(M)$. Working at the level of representatives, we can consider $A = G_\|\omega$ with $\omega \in \Omega^k_{\mathrm{c},\delta}(M)$, while $\alpha \in \Omega^k_{\mathrm{c},\delta}(M)$. Hence, in view of Proposition 20, $0 = (\alpha, A) = (\alpha, G_\|\omega) = -(G_\|\alpha, \omega)$. Choosing $\omega = \delta\beta$, $\beta \in \Omega^{k+1}_{\mathrm{c}}(M)$ and using (5), it descends $(\mathrm{d}G_\|\alpha, \beta) = 0$. Since β is arbitrary and the pairing is non degenerate $\mathrm{d}G_\|\alpha = 0$. Since $\mathrm{t}G_\|\alpha = 0$, it turns out that $G_\|\alpha$ individuates an equivalence class $[G_\|\alpha] \in H^k_{\mathrm{t}}(M)$. Using the same argument of the previous item, $(G_\|\alpha, \beta) = 0$ for all $\beta \in \Omega^k_{\mathrm{c},\delta}(M)$ entails that $G_\|\alpha = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}_{\mathrm{t}}(M)$. Proceeding as in proof of the injectivity of $G_\| : \mathcal{O}_{\mathrm{t}}(M) \to \mathrm{Sol}_{\mathrm{t}}(M)$ – *cf.* Proposition 32 – it follows that $\alpha \in \delta\mathrm{d}\Omega^k_{\mathrm{c,t}}(M)$ which is the sought conclusion. □

The following corollary translates at the level of algebra of observables the degeneracy of the pre-symplectic spaces discussed in Proposition 33 – *cf.* Remark 34. As a matter of fact since $\widetilde{G}_\|$ can be degenerate, the algebra of observables $\mathcal{A}_{\mathrm{t}}(M)$ will possess a non-trivial centre. In other words

Corollary 38: *If $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ is a strict inclusion, then the algebra $\mathcal{A}_{\mathrm{t}}(M)$ is not semi-simple.*

Proof. With reference to Remark 34, if $\mathrm{d}\Omega^{k-1}_{\mathrm{sc,t}}(M) \subset \Omega^k_{\mathrm{sc}}(M) \cap \mathrm{d}\Omega^{k-1}_{\mathrm{t}}(M)$ is a strict inclusion then there exists an element $[A] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$ such that $\sigma_{\mathrm{t}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}^{\mathrm{sc}}_{\mathrm{t}}(M)$. On account of Proposition 32 there exists $[\alpha] \in \mathcal{O}_{\mathrm{t}}(M)$ such that $[G_\|\alpha] = [A]$. Moreover, Proposition 33 ensures that $\widetilde{G}_\|([\alpha],[\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{t}}(M)$. It follows from Definition 36 that $[\alpha]$ belongs to the center of $\mathcal{A}_{\mathrm{t}}(M)$, that is, $\mathcal{A}_{\mathrm{t}}(M)$ is not semi-simple. □

Remark 39: *Corollary 38 has established that the algebra of observables possesses a non trivial center. While from a mathematical viewpoint this feature might not appear of particular significance, it has far reaching consequences from the physical viewpoint. Most notably, the existence of Abelian ideals was first observed in the study of gauge theories in [DL12] leading to an obstruction in the interpretation of these models in the language of locally covariant quantum field theories as introduced in [BFV03]. This feature has been thoroughly studied in [BDHS14, BDS14, SDH12] turning out to be an intrinsic feature of Abelian gauge theories on globally hyperbolic spacetimes with empty boundary. Corollary 38 shows that the same conclusions can be drawn when the underlying manifold possesses a timelike boundary. In the next part of this section we will show that changing boundary condition does not alter the outcome.*

3.3.2 The algebra of observable for $\mathrm{Sol}_{\mathrm{nd}}(M)$

We focus now on $\mathcal{A}_{\mathrm{nd}}(M)$, the algebra of observables associated to the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. Similarly to Definition 36, $\mathcal{A}_{\mathrm{nd}}(M)$ will be defined as a suitable quotient of the universal tensor algebra over a vector space $\mathcal{O}_{\mathrm{nd}}(M)$. However, contrary to the case of $\delta\mathrm{d}$-tangential boundary conditions, in the case of $\delta\mathrm{d}$-normal boundary conditions, $\mathcal{O}_{\mathrm{nd}}(M)$ will not be simplectomorphic to the configuration space $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ – *cf.* Definition 40 and Proposition 31. Nevertheless the results of Propositions 37 and 38 still hold true for $\mathcal{A}_{\mathrm{nd}}(M)$. In the last part of this section we point out another possible choice for the algebra of observables whose underlying vector space is simplectomorphic to $\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M)$ but which requires an a priori gauge fixing.

Definition 40: *Let (M, g) be a globally hyperbolic spacetime with timelike boundary. We call* algebra of observables *associated to* $\mathrm{Sol}_{\mathrm{nd}}(M)$, *the associative, unital $*$-algebra*

$$\mathcal{A}_{\mathrm{nd}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]}, \qquad \mathcal{O}_{\mathrm{nd}}(M) \doteq \frac{\Omega^k_{\mathrm{c,n},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}. \tag{45}$$

where $\mathcal{T}[\mathcal{O}_{\mathrm{nd}}(M)] \doteq \bigoplus_{n=0}^{\infty} \mathcal{O}_{\mathrm{nd}}(M)^{\otimes n}$ is the universal tensor algebra with $\mathcal{O}_{\mathrm{nd}}(M)^{\otimes 0} \equiv \mathbb{C}$, while the $*$-operation is the one induced from complex conjugation. In addition $\mathcal{I}[\mathcal{O}_{\mathrm{nd}}(M)]$ is the $*$-ideal generated by elements of the form $[\alpha] \otimes [\beta] - [\beta] \otimes [\alpha] - i\widetilde{G}_{\perp}([\alpha], [\beta])\mathbb{I}$, where $[\alpha], [\beta] \in \mathcal{O}_{\mathrm{nd}}(M)$ while $\widetilde{G}_{\perp}$ is defined in Proposition 33 and $\mathbb{I}$ is the identity of $\mathcal{O}_{\mathrm{nd}}(M)$.

Remark 41: *Notice that, with respect to the definition of $\mathcal{O}_{\mathrm{t}}(M)$ – cf. Definition 36 – the vector space $\mathcal{O}_{\mathrm{nd}}(M)$ introduced in Definition 40 contains equivalence classes built out of forms $\alpha \in \Omega^k_{\mathrm{c}}(M)$ such that $\delta\alpha = 0$ as well as $\mathrm{n}\alpha = 0$. The last condition is sufficient and necessary to have a well-defined pairing among $\mathcal{O}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}(M)$. Indeed for all $A \in [A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ and for all $\alpha \in [\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$ we have that (α, A) is well-defined being α compactly supported. Moreover, for all $\chi \in \Omega^{k-1}(M)$ and $\eta \in \Omega^k_{\mathrm{nd}}(M)$ we have*

$$(\alpha, \mathrm{d}\chi) = (\delta\alpha, \chi) + (\mathrm{n}\alpha, \mathrm{t}\chi)_\partial = 0, \qquad (\delta\mathrm{d}\eta, A) = (\eta, \delta\mathrm{d}A) + (\mathrm{nd}\eta, \mathrm{t}A)_\partial - (\mathrm{t}\eta, \mathrm{nd}A)_\partial = 0.$$

Notice that in the first equation we used the condition $\mathrm{n}\alpha = 0$ since χ has no assigned boundary condition. This goes opposite to the case of δd-tangential boundary conditions, where χ is required to satisfy $\mathrm{t}\chi = 0$ – cf. Definition 26 – and therefore α is not forced to satisfy any boundary conditions. Actually, the conditions $\delta\alpha = 0$ and $\mathrm{n}\alpha = 0$ are necessary to ensure gauge-invariance, namely $(\alpha, \mathrm{d}\chi) = 0$ for all $\chi \in \Omega^k(M)$.

Finally notice that, on account of Propositions 31-33, $(\mathcal{O}_{\mathrm{nd}}, \widetilde{G}_{\perp})$ is a presymplectic proper subspace of $\left(\frac{\Omega^k_{\mathrm{c},\delta}(M)}{\delta\mathrm{d}\Omega^k_{\mathrm{c,nd}}(M)}, \widetilde{G}_{\perp}\right)$ and therefore it is not symplectomorphic to $(\mathrm{Sol}^{\mathrm{sc}}_{\mathrm{nd}}(M), \sigma_{\mathrm{nd}})$.

Starting from Definition 40 we can repeat, mutatis mutandis, the proof of Proposition 37.

Proposition 42: *Let $\mathcal{O}_{\mathrm{nd}}(M)$ be as per Definition 40. Then, calling with $(\ ,\ )$ the natural pairing between $\mathcal{O}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}(M)$ induced from those between k-forms, $\mathcal{O}_{\mathrm{nd}}(M)$ is* **optimal**, *namely*

1. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* separating, *that is*

$$([\alpha], [A]) = 0 \quad \forall[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M) \Longrightarrow [A] = [0] \in \mathrm{Sol}_{\mathrm{nd}}(M). \tag{46}$$

2. $\mathcal{O}_{\mathrm{nd}}(M)$ *is* optimal, *that is*

$$([\alpha], [A]) = 0 \quad \forall[A] \in \mathrm{Sol}_{\mathrm{nd}}(M) \Longrightarrow [\alpha] = [0] \in \mathcal{O}_{\mathrm{nd}}(M), \tag{47}$$

Proof. The fact that the pairing $([\alpha], [A])$ is well-defined for $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$ and $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ has already been discussed in Remark 41.

We prove the first of the two items: let $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ be such that $([\alpha], [A]) = 0$ for all $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$. This implies that $(\alpha, A) = 0$ for all $A \in [A]$ and for all $\alpha \in \Omega^k_{\mathrm{c,n},\delta}(M)$. Taking in particular $\alpha = \delta\beta$ with $\beta \in \Omega^k_{\mathrm{c,n}}(M)$ it follows $(\mathrm{d}A, \beta) = 0$. The non-degeneracy of $(\ ,\ )$ implies $\mathrm{d}A = 0$, that is A defines an element in $H^k(M)$. Moreover, the hypothesis on A implies that $\langle A, [\eta]\rangle = 0$ for all $[\eta] \in H_{k,\mathrm{c,n}}(M)$. The results in Appendix C – *cf.* Remark 54 – ensure that $A = \mathrm{d}\chi$, therefore $[A] = [0] \in \mathrm{Sol}_{\mathrm{nd}}(M)$.

Regarding the second statement, let $[\alpha] \in \mathcal{O}_{\mathrm{nd}}(M)$ be such that $([\alpha], [A]) = 0$ for all $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. This implies in particular that, choosing $\alpha \in [\alpha]$ and $A = G_{\perp}\beta$ with $\beta \in \Omega^k_{\mathrm{c},\delta}(M)$, $0 = (\alpha, G_{\perp}\beta) = -(G_{\perp}\alpha, \beta)$. With the same argument of the first statement it follows that $G_{\perp}\alpha = \mathrm{d}\chi$ where $\chi \in \Omega^{k-1}(M)$ is such that $\mathrm{nd}\chi = 0$. Proceeding as in the proof of Proposition 32 it follows that $[\alpha] = [0] \in \mathcal{O}_{\mathrm{nd}}(M)$. □

The following corollary is analogous to Corollary 38. The proof is slightly different since in this case there does not exist a simplectomorphism between $\mathcal{O}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$ – *cf.* Proposition 32 and Remark 41. The crucial part in the proof is to show that if $[A] \in \mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$ is degenerate with respect to σ_{nd}, then $[A] \in G_\perp \mathcal{O}_{\mathrm{nd}}(M)$.

Corollary 43: *If $\mathrm{d}\Omega_{\mathrm{sc}}^{k-1}(M) \subset \Omega_{\mathrm{sc}}^k(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a strict inclusion, then the algebra $\mathcal{A}_{\mathrm{nd}}(M)$ is not semi-simple.*

Proof. On account of Remark 34, if $\mathrm{d}\Omega_{\mathrm{sc}}^{k-1}(M) \subset \Omega_{\mathrm{sc}}^k(M) \cap \mathrm{d}\Omega^{k-1}(M)$ is a strict inclusion then there exists $[A] \in \mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$ such that $\sigma_{\mathrm{nd}}([A],[B]) = 0$ for all $[B] \in \mathrm{Sol}_{\mathrm{nd}}(M)$. In particular we have $[A] = [\mathrm{d}\chi]$ where $\chi \in \Omega^{k-1}(M) \setminus \Omega_{\mathrm{sc}}^{k-1}(M)$ is such that $\mathrm{d}\chi \in \Omega_{\mathrm{sc}}^k(M)$.

We now prove that, up to an element in $\mathrm{d}\Omega_{\mathrm{sc}}^k(M)$, $\mathrm{d}\chi = G_\perp \alpha$ with $\alpha \in \Omega_{\mathrm{c,n},\delta}^k(M)$: On account of Proposition 33 it follows that $\widetilde{G}_\perp([\alpha],[\beta]) = \sigma_{\mathrm{nd}}([\mathrm{d}\chi],[G_\perp\beta]) = 0$ for all $[\beta] \in \mathcal{O}_{\mathrm{nd}}(M)$. Definition 40 implies that $[\alpha] \in \mathcal{A}_{\mathrm{nd}}(M)$ lies in the center of $\mathcal{A}_{\mathrm{nd}}(M)$ which is therefore not semi-simple.

On account of Proposition 32 we have that $\mathrm{d}\chi = G_\perp\alpha + \mathrm{d}\eta$, where $\alpha \in \Omega_{\mathrm{c},\delta}^k(M)$ while $\eta \in \Omega_{\mathrm{sc}}^{k-1}(M)$. By redefining $\chi_\eta \doteq \chi - \eta$ we have $\mathrm{d}\chi_\eta = G_\perp\alpha$. Notice that this last redefinition does not spoil the property $\chi_\eta \in \Omega^{k-1}(M) \setminus \Omega_{\mathrm{sc}}^{k-1}(M)$ while $\mathrm{d}\chi_\eta \in \Omega_{\mathrm{sc}}^k(M)$ thus $\sigma_{\mathrm{nd}}([\mathrm{d}\chi_\eta],[B]) = 0$ for all $[B] \in \mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$.

The boundary conditions on $G_\perp\alpha$ implies that $\mathrm{nd}\chi_\eta = \mathrm{n}G_\perp\alpha = 0$, while Corollary 24 ensures that $\delta\mathrm{d}\chi_\eta = \delta G_\perp\alpha = G_\perp\delta\alpha = 0$. It then follows that $\chi_\eta \in \mathrm{Sol}_{\mathrm{nd}}(M)$ – in degree $k-1$ – and therefore Proposition 32 entails $\chi_\eta = G_\perp\beta$ where $\beta \in \Omega_{\mathrm{tc},\delta}^{k-1}(M)$. Summing up we have $\mathrm{d}\chi_\eta = G_\perp\alpha$ as well as $\mathrm{d}\chi_\eta = G_\perp\mathrm{d}\beta$. Proposition 20 and Remark 21 imply that $\mathrm{d}\beta - \alpha = \Box_\perp\zeta$, being $\zeta \in \Omega_{\mathrm{tc},\perp}^k(M)$. Applying δ to the last equality we obtain

$$\Box\delta\zeta = \delta\Box_\perp\zeta = \delta\mathrm{d}\beta - \delta\alpha = \Box\beta\,.$$

Remark 22 ensures that $\delta\zeta = \beta$ and therefore $\alpha = -\delta\mathrm{d}\zeta$. Since $\zeta \in \Omega_\perp^k(M)$ it follows that $\alpha \in \Omega_{\mathrm{c,n},\delta}^k(M)$. □

Definition 40 identifies an algebra $\mathcal{A}_{\mathrm{nd}}(M)$ which is separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. It also satisfies most of the properties of the analogous algebra $\mathcal{A}_{\mathrm{t}}(M)$ – *cf.* Proposition 42 and Corollary 43. However, as pointed out in Remark 41, the underlying vector space $\mathcal{O}_{\mathrm{nd}}(M)$ is only a proper presymplectic subspace of $(\mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M), \sigma_{\mathrm{nd}})$. This is contrary to the case of δd-tangential boundary conditions where the vector space $\mathcal{O}_{\mathrm{t}}(M)$ is symplectomorphic to $\mathrm{Sol}_{\mathrm{t}}^{\mathrm{sc}}(M)$ – *cf.* Proposition 32.

It is thus worth investigating whether there exists a different algebra $\mathcal{A}_{\mathrm{nd}}^{\mathrm{gf}}(M)$ still defined as a suitable quotient – *cf.* Definitions 36-40 – of the universal tensor algebra of a presymplectic vector space $\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M)$ which is presymplectomorphic to $\mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M)$. For consistency, $\mathcal{A}_{\mathrm{nd}}^{\mathrm{gf}}(M)$ should be built out of a separating and non redundant collection of functionals for $\mathrm{Sol}_{\mathrm{nd}}(M)$ and the superscript *gf* refers to "gauge fixing" as it will become clear from the following discussion. To this end and with reference to Proposition 32, $\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M)$ can be identified as

$$\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M) \doteq \frac{\Omega_{\mathrm{c},\delta}^k(M)}{\delta\mathrm{d}\Omega_{\mathrm{c,nd}}^k(M)}\,.$$

As shown in Propositions 32-33, $(\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M), \widetilde{G}_\perp)$ is a presymplectic vector space which is symplectomorphic to $(\mathrm{Sol}_{\mathrm{nd}}^{\mathrm{sc}}(M), \sigma_{\mathrm{nd}})$. We can thus set

$$\mathcal{A}_{\mathrm{nd}}^{\mathrm{gf}}(M) \doteq \frac{\mathcal{T}[\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M)]}{\mathcal{I}[\mathcal{O}_{\mathrm{nd}}^{\mathrm{gf}}(M)]}\,,$$

where we refer to Definition 40 for details.
The discussion about $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ being separating and non redundant is more subtle. Indeed, the pairing between elements $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $[A] \in \mathrm{Sol}_{\mathrm{nd}}(M)$ is not well-defined – *cf.* Remark 41. However we can exploit the isomorphism identified in equation (37) – *cf.* Remark 30. With reference to equation (36), we denote with γ_{nd} the isomorphism

$$\gamma_{\mathrm{nd}} \colon \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathcal{S}_{\mathcal{G}_{\mathrm{nd}}}(M)\,.$$

It follows that for all $[\alpha] \in \mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ the following functional is well-defined:

$$F_{\gamma^*_{\mathrm{nd}}[\alpha]} \colon \mathrm{Sol}_{\mathrm{nd}}(M) \to \mathbb{C}\,, \qquad F_{\gamma^*_{\mathrm{nd}}[\alpha]}([A]) := ([\alpha], [\gamma_{\mathrm{nd}} A])\,.$$

Notice that the gauge-invariance of $F_{\gamma^*_{\mathrm{nd}}[\alpha]}$ is guaranteed by the combined action of γ_{nd}, which selects a "gauge-fixed" representative $\gamma_{\mathrm{nd}} A \in [A]$, and of $[\alpha]$, which remains un-effected by the residual gauge present in the choice of $\gamma_{\mathrm{nd}} A$, *i.e.* $([\alpha], \mathrm{d}\mathcal{G}_{\mathrm{nd}}(M)) = 0$ – *cf.* Remark 30.
With this observation it holds that, introducing the "gauge-fixed" pairing $([\alpha], [A])_{\gamma_{\mathrm{nd}}} \doteq ([\alpha], [\gamma_{\mathrm{nd}} A])$ between $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ and $\mathrm{Sol}_{\mathrm{nd}}(M)$, the vector space $\mathcal{O}^{\mathrm{gf}}_{\mathrm{nd}}(M)$ is indeed separating and optimal for the configuration space $\mathrm{Sol}_{\mathrm{nd}}(M)$. The proof is similar to the one of Propositions 37-42 and we shall not repeat it.

Remark 44: *To conclude this section we observe that all algebras of observables that we have constructed obey to the so-called principle of* F-locality. *This concept was introduced for the first time in [Kay92] and it asserts that, given any globally hyperbolic region $\mathcal{O} \subset \mathring{M}$ the restriction to $\mathcal{O}$ of the algebra of observables built on M is $*$-isomorphic to the one which one would construct intrinsically on $(\mathcal{O}, g|_{\mathcal{O}})$. In our approach this property is implemented per construction and its proof is a direct generalization of the same argument given in [DF18]. For this reason we omit the details.*

A Existence of fundamental solutions on ultrastatic spacetimes

In this section we prove that Assumption 16 is verified in a large class of globally hyperbolic spacetimes (M, g) with timelike boundary. These can be characterized by the following two additional hypotheses:

1. (M, g) is ultrastatic, that is, with reference to Equation (1), we impose $\beta = 1$ and $h_\tau = h_0$ for all $\tau \in \mathbb{R}$. Hence ∂_τ is a timelike Killing vector field.

2. The Cauchy surface (Σ, h_0) with $\partial\Sigma \neq \emptyset$ is of *bounded geometry*, that is there exists an $(n-1)$-dimensional Riemmanian manifold $(\widehat{\Sigma}, \widehat{h})$ of bounded geometry[2] such that $\Sigma \subset \widehat{\Sigma}$ and $\widehat{h}|_\Sigma = h_0$. In addition, $\partial\Sigma$ is a smooth submanifold of bounded geometry in $\widehat{\Sigma}$.

It is worth recalling that, whenever one considers a complex vector bundle E over (Σ, h_0) endowed with both a fiberwise Hermitian product $\langle\ ,\ \rangle_E$ and a product preserving connection ∇^E, one can define a suitable notion of Sobolev spaces. Most notably, let $\Gamma_{\mathrm{me}}(E)$ denote the equivalence classes of measurable sections of E. Then, for all $\ell \in \mathbb{N} \cup \{0\}$, we define

$$H^\ell(\Sigma; E) \equiv H^\ell(E) \doteq \{u \in \Gamma_{\mathrm{me}}(E) \mid \nabla^j u \in L^2(\Sigma; E \otimes T^*\Sigma^{\otimes j}),\ j \leq \ell\}\,, \tag{48}$$

[2] Recall that a Riemannian manifold (N, h) with $\partial N = \emptyset$ is called of *bounded geometry* if the injectivity radius $r_{\mathrm{inj}}(N) > 0$ and $\|\nabla^k R\|_{L^\infty(N)} < \infty$ for all $k \in \mathbb{N} \cup \{0\}$ where R is the scalar curvature while ∇ is the Levi-Civita connection associated to h.

where we omitted the subscript E on ∇ for notational simplicity. The theory of these space has been thoroughly studied in the literature and for the case in hand we refer mainly to [GS13].

In the following we study the existence of advanced and retarded fundamental solutions for the D'Alembert - de Rham wave operator $\Box = \mathrm{d}\delta + \delta\mathrm{d}$ acting on k-forms. We use a method, first employed in [DDF19] for the special case $k = 0$, based on a functional analytic tool known as boundary triple, see for example [BL12]. In order to be self-consistent, we will recall the necessary definitions and results from this paper, to which we refer for further details. The main ingredient is the following:

Definition 45: *Let H be a separable Hilbert space over $\mathbb{C}$ and let $S : D(S) \subset \mathsf{H} \to \mathsf{H}$ be a closed, symmetric, linear operator. A* **boundary triple** *for the adjoint operator S^* is a triple $(\mathsf{h}, \gamma_0, \gamma_1)$ consisting of a separable Hilbert space h over $\mathbb{C}$ and of two linear maps $\gamma_i : D(S^*) \to \mathsf{h}$, $i = 0, 1$ such that*

$$(S^* f|f')_{\mathsf{H}} - (f|S^* f')_{\mathsf{H}} = (\gamma_1 f|\gamma_0 f')_{\mathsf{h}} - (\gamma_0 f|\gamma_1 f')_{\mathsf{h}}\,, \quad \forall f, f' \in D(S^*)\,,$$

In addition the map $\gamma : D(S^*) \to \mathsf{h} \times \mathsf{h}$ such that $f \mapsto (\gamma_0(f), \gamma_1(f))$ is surjective.

Boundary triples are a convenient tool to characterize the self-adjoint extensions of a large class of linear operators. The proof of the following proposition can be found in [Mal92].

Proposition 46: *Let $S : D(S) \subseteq \mathsf{H} \to \mathsf{H}$ be a closed, symmetric operator. Then S admits a boundary triple $(\mathsf{h}, \gamma_0, \gamma_1)$ if and only if it admits self-adjoint extensions. If $\Theta : D(\Theta) \subseteq \mathsf{h} \to \mathsf{h}$ is a closed, densely defined linear relation, then $S_\Theta \doteq S^*|_{\ker(\gamma_1 - \Theta\gamma_0)}$ is a closed extension of S. In addition the map $\Theta \to S_\Theta$ is one-to-one and $S_\Theta^* = S_{\Theta^*}$. Hence there is a one-to-one correspondence between self-adjoint relations Θ and self-adjoint extensions of S.*

In order to apply these tools to the case in hand, first of all we need to recall that our goal is that of constructing advanced and retarded fundamental solutions for the D'Alembert de-Rham wave operator $\Box$ acting on k-forms. In other words, calling as $\Lambda^k T^*\mathring{M}$ the k-th exterior power of the cotangent bundle over $\mathring{M}$, $k \geq 1$, and with $\boxtimes$ the external tensor product, we look for $G^\pm \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M} \boxtimes \Lambda^k T^*\mathring{M})'$ such that

$$\Box \circ G^\pm = G^\pm \circ \Box = \mathrm{Id}\,|_{\Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M} \boxtimes \Lambda^k T^*\mathring{M})}\,,$$

while $\mathrm{supp}(G^\pm(\omega)) \subseteq J^\pm(\mathrm{supp}(\omega))$ for all $\omega \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M})$ – *cf.* Assumption 16. Working at the level of integral kernels and setting $G^\pm(\tau - \tau', x, x') = \theta[\pm(\tau - \tau')]G(\tau - \tau', x, x')$, this amounts to solving the following distributional, initial value problem

$$(\Box \otimes \mathbb{I})\, G = (\mathbb{I} \otimes \Box)\, G = 0\,, \qquad G|_{\tau=\tau'} = 0\,, \qquad \partial_\tau G|_{\tau=\tau'} = \delta_{\mathrm{diag}(\mathring{M})}\,. \tag{49}$$

where $\delta_{\mathrm{diag}(\mathring{M})}$ stands for the bi-distribution yielding $\delta_{\mathrm{diag}(\mathring{M})}(\omega_1 \boxtimes \omega_2) = (\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M})$. Since we have assumed that the underlying spacetime (M, g) is ultrastatic, Equation (1) entails that [Pfe09]

$$\Box = -\partial_\tau^2 + S,$$

where S is a uniformly elliptic operator whose local form can be found in [Pfe09]. This entails that, in order to construct solution of (49), we can follow the rationale outlined in [DDF19].

To this end we start by focusing our attention on S analysing it within the framework of boundary triples. Our first observation consists of noticing, that being (M, g) globally hyperbolic, Theorem 1 ensures that M is diffeomorphic to $\mathbb{R} \times \Sigma$. Leaving implicit the identification $M \simeq \mathbb{R} \times \Sigma$ and recalling Theorem 1, let us indicate with $\iota_\tau \colon \Sigma \to M$ the (smooth one-parameter group of) embedding maps which realizes Σ at time τ as $\iota_\tau\Sigma = \{\tau\} \times \Sigma \doteq \Sigma_\tau$. It holds $\Sigma_\tau \simeq \Sigma_{\tau'}$ for all $\tau, \tau' \in \mathbb{R}$. If follows that, on account of Theorem 1, for all $\omega \in \Omega^k(M)$ and $\tau \in \mathbb{R}$, $\omega|_{\Sigma_\tau} \in \Gamma(\iota_\tau^* \Lambda^k T^* M)$. Here $\iota_\tau^*(\Lambda^k T^* M)$ denotes the pull-back bundle over $\Sigma_\tau \simeq \Sigma$ built out of $\Lambda^k T^*M$ via ι_τ – *cf.* [Hu94]. Moreover, recalling Definition 5, it holds that $\omega|_{\Sigma_\tau}$ can be further decomposed as

$$\omega|_{\Sigma_\tau} := (\star_{\Sigma_\tau}^{-1}\iota_\tau^*\star_M)\omega \wedge \mathrm{d}\tau + \iota_\tau^*\omega = \mathrm{n}_{\Sigma_\tau}\omega \wedge \mathrm{d}\tau + \mathrm{t}_{\Sigma_\tau}\omega\,.$$

where $\mathrm{t}_{\Sigma_\tau}\omega \in \Omega^k(\Sigma_\tau)$ while $\mathrm{n}_{\Sigma_\tau}\omega \in \Omega^{k-1}(\Sigma_\tau)$ – *cf.* Definition 5. Barring the identification between Σ_τ and $\Sigma_{\tau'}$ the latter decomposition induces the isomorphisms

$$\Gamma(\iota_\tau^*\Lambda^k T^*M) \simeq \Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma)\,, \qquad \omega \to (\omega_0 \oplus \omega_1) \tag{50}$$

$$\Omega^k(M) \to C^\infty(\mathbb{R}, \Omega^{k-1}(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k(\Sigma))\,, \qquad \omega \to (\tau \mapsto \mathrm{t}_{\Sigma_\tau}\omega) \oplus (\tau \mapsto \mathrm{n}_{\Sigma_\tau}\omega)\,. \tag{51}$$

Furthermore a direct computation shows that, for all $\omega \in \Omega^k(M)$, it holds that

$$S\omega|_{\Sigma_\tau} = (\Delta_{k-1}\mathrm{t}_{\Sigma_\tau}\omega) \wedge \mathrm{d}\tau + \Delta_k \mathrm{n}_{\Sigma_\tau}\omega\,,$$

where Δ_k is the Laplace-Beltrami operator acting on k-forms, built out of h_0.

Putting together all these data and working in the language of Definition 45, we can consider the following building blocks:

1. As Hilbert space we set

$$\mathsf{H} \equiv L^2(\Omega^{k-1}(\Sigma)) \oplus L^2(\Omega^k(\Sigma))\,,$$

   where $L^2(\Omega^k(\Sigma))$ is the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ with respect to the pairing $(\ ,\ )_\Sigma$ between k-forms, *i.e.*, $(\alpha, \beta)_\Sigma = \int\limits_\Sigma \alpha \wedge *_\Sigma\beta$ for all $\alpha, \beta \in \Omega^k_{\mathrm{c}}(\Sigma)$.

2. We identify with a slight abuse of notation S with $\Delta_{k-1} \oplus \Delta_k$ where Δ_k is the Laplace-Beltrami operator built out of h_0 acting on k-forms.

Observe that S can be regarded as an Hermitian and densely defined operator on $H^2_0(\Lambda^{k-1}T^*\Sigma) \oplus H^2_0(\Lambda^k T^*\Sigma)$ where $H^2_0(\Lambda^k T^*\Sigma)$ is the closure of $\Omega^k_{\mathrm{c}}(\Sigma)$ with respect to the $H^2(\Lambda^k T^*\Sigma)$-norm – *cf.* Equation (48) with $E \equiv \Lambda^k T^*\Sigma$. In this case both the inner product and the connection are those induced from the underlying metric h_0. Hence standard arguments entail that S is a closed symmetric operator on H whose adjoint S^* is defined on the maximal domain $D(S^*) \doteq \{(\omega_0 \oplus \omega_1) \in \mathsf{H} \mid S(\omega_0 \oplus \omega_1) \in \mathsf{H}\}$. In addition $S^*(\omega_0 \oplus \omega_1) = S(\omega_0 \oplus \omega_1)$ for all $\omega_0 \oplus \omega_1 \in D(S^*)$. Hence all the requirements of Definition 45 are met and, in view of Proposition 46 S^* admits a boundary triple.

In order to realize it explicitly, we start by observing that, since $\partial\Sigma \neq \emptyset$, we can introduce the standard trace map between Sobolev spaces, *i.e.*, for every $\ell \geq \frac{1}{2}$ there exists a continuous surjective map $\mathrm{res}_\ell : H^\ell(\Lambda^k T^*\Sigma) \to H^{\ell-\frac{1}{2}}(\Lambda^k T^*\partial\Sigma)$ whose action on $\Omega^k_{\mathrm{c}}(\Sigma)$ coincides with the restriction to $\partial\Sigma$ for every ℓ. This last property allows us to better characterize the action of the restriction map, since, for every $\alpha \in \Omega^k_{\mathrm{c}}(\Sigma)$ a straightforward computations shows that, for all $\ell \geq \frac{1}{2}$

$$\alpha|_{\partial\Sigma} = \mathrm{res}_\ell\alpha = \alpha_0 + \alpha_1 \wedge \mathrm{d}x\,,$$

where, up to an irrelevant isomorphism, we can identify $\alpha_0 \equiv \mathrm{t}_{\partial\Sigma}\alpha$ and $\alpha_1 \equiv \mathrm{n}_{\partial\Sigma}\alpha$ – *cf.* Definition 5. Here, for every $p \in \partial\Sigma$, $\mathrm{d}x$ is the basis element of T^*_pM such that $\mathrm{d}x(\nu_p) = 1$ where ν_p is the outward pointing, unit vector normal to $\partial\Sigma$ at p. With this observation in mind and following mutatis mutandis the same analysis of [DDF19] for the scalar case, we can construct the following boundary triple for S^*

- $\mathsf{h} = \mathsf{h}_0 \oplus \mathsf{h}_1$ where $\mathsf{h}_0 \doteq L^2(\Omega^{k-1}(\partial\Sigma)) \oplus L^2(\Omega^{k-2}(\partial\Sigma))$ while $\mathsf{h}_1 = L^2(\Omega^{k-1}(\partial\Sigma)) \oplus L^2(\Omega^k(\partial\Sigma))$;

- the map $\gamma_0 : D(S^*) \to \mathsf{h}$ such that, for all $\omega_0 \oplus \omega_1 \in D(S^*)$,

$$\gamma_0(\omega_0 \oplus \omega_1) = (\mathrm{n}_{\partial\Sigma}\omega_0 \oplus \mathrm{t}_{\partial\Sigma}\omega_0) \oplus (\mathrm{n}_{\partial\Sigma}\omega_1 \oplus \mathrm{t}_{\partial\Sigma}\omega_1)\,. \tag{52}$$

- the map $\gamma_1 : D(S^*) \to \mathsf{h}$ such that, for all $\omega_0 \oplus \omega_1 \in D(S^*)$,

$$\gamma_1(\omega_0 \oplus \omega_1) = (\mathrm{t}_{\partial\Sigma}\delta_\Sigma\omega_0 \oplus \mathrm{n}_{\partial\Sigma}\mathrm{d}_\Sigma\omega_0) \oplus (\mathrm{t}_{\partial\Sigma}\delta_\Sigma\omega_1 \oplus \mathrm{n}_{\partial\Sigma}\mathrm{d}_\Sigma\omega_1)\,, \tag{53}$$

  where with a slight abuse of notation we denote still with d_Σ and δ_Σ the extension to the space of square-integrable k-forms of the action of the differential and of the codifferential on $\Omega^k_{\mathrm{c}}(\Sigma)$.

In view of Proposition 46 we can follow slavishly the proof of [DDF19, Th. 30] to infer the following statement:

Theorem 47: *Let (M, g) be an ultrastatic and globally hyperbolic spacetime with timelike boundary. Let $(\mathsf{h}, \gamma_0, \gamma_1))$ be the boundary triple built as per Equation (52) and (53) associated to the operator S^*. Let Θ be a self-adjoint relation on h and let $S_\Theta \doteq S^*|_{D(S_\Theta)}$ where $D(S_\Theta) = \ker(\gamma_1 - \Theta\gamma_0)$. If the spectrum of S_Θ is bounded from below, then there exists unique advanced and retarded Green's operator $G^\pm_\Theta$ associated to $-\partial^2_\tau + S_\Theta$. They are completely determined in terms of the bidistributions $G^\pm_\Theta = \theta[\pm(\tau-\tau')]G_\Theta$ where $G_\Theta \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M} \boxtimes \Lambda^k T^*\mathring{M})'$ is such that for $\omega_1, \omega_2 \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M})$,*

$$G_\Theta(\omega_1, \omega_2) = \int_{\mathbb{R}^2} \left(\omega_1|_\Sigma, S^{-\frac{1}{2}}_{k,\Theta} \sin(S^{\frac{1}{2}}_{k,\Theta}(\tau-\tau'))\omega_2|_\Sigma\right)_\Sigma \mathrm{d}\tau\mathrm{d}\tau'\,,$$

where $(\ ,\ )_\Sigma$ stands for the pairing between k-forms and where ω_2 identifies an element in $D(S_\Theta)$ via the identifications (51). Moreover it holds that

$$\gamma_1\left(G^\pm_\Theta\omega\right) = \Theta\gamma_0\left(G^\pm_\Theta\omega\right)\,, \quad \forall\omega \in \Gamma_{\mathrm{c}}(\Lambda^k T^*\mathring{M})\,. \tag{54}$$

The last step consists of proving that the boundary conditions introduced in Definition 12 fall in the class considered in Theorem 47. In the following proposition we adopt for simplicity the notation $\mathrm{t} = \mathrm{t}_{\partial\Sigma}$, $\mathrm{n} = \mathrm{n}_{\partial\Sigma}$, $\mathrm{nd} = \mathrm{n}_{\partial\Sigma}\mathrm{d}_{\partial\Sigma}$, $\mathrm{t}\delta = \mathrm{t}_{\partial\Sigma}\delta_\Sigma$.

Proposition 48: *The following relations on h are selfadjoint:*

$$\Theta_\| \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0 \oplus\ ;\ 0 \oplus \mathrm{nd}\omega_0 \oplus 0 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in D(S^*)\} \tag{55}$$

$$\Theta_\perp \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1 \oplus\ ;\ \mathrm{t}\delta\omega_0 \oplus 0 \oplus \mathrm{t}\delta\omega_1 \oplus 0) \mid \omega_0 \oplus \omega_1 \in D(S^*)\} \tag{56}$$

$$\Theta_{f_\|} \doteq \{(\mathrm{n}\omega_0 \oplus 0 \oplus \mathrm{n}\omega_1 \oplus 0 \oplus\ ;\ f\mathrm{n}\omega_0 \oplus \mathrm{nd}\omega_0 \oplus f\mathrm{n}\omega_1 \oplus \mathrm{nd}\omega_1) \mid \omega_0 \oplus \omega_1 \in D(S^*)\}, \quad f \in C^\infty(\partial\Sigma)\, f \geq 0\,. \tag{57}$$

$$\Theta_{f_\perp} \doteq \{(0 \oplus \mathrm{t}\omega_0 \oplus 0 \oplus \mathrm{t}\omega_1 \oplus\ ;\ \mathrm{t}\delta\omega_0 \oplus f\mathrm{t}\omega_0 \oplus \mathrm{t}\delta\omega_1 \oplus f\mathrm{t}\omega_1) \mid \omega_0 \oplus \omega_1 \in D(S^*)\}, \quad f \in C^\infty(\partial\Sigma)\, f \leq 0\,. \tag{58}$$

Moreover the self-adjoint extension $S_{\Theta_\sharp}$ for $\sharp \in \{\|, \perp, f_\|, f_\perp\}$ abides to the hypotheses of Theorem 47. The associated propagators $G_\sharp$, $\sharp \in \{\|, \perp, (f, 0)\}$, obey the boundary conditions as per Definition 12.

Proof. We recall that, given a relation $\Theta \subseteq \mathsf{h} \times \mathsf{h}$, the adjont relation Θ^* is defined by

$$\Theta^* \doteq \{(y_1, y_2) \in \mathsf{h} \times \mathsf{h} \mid (x_1, y_2)_{\mathsf{h}} = (x_2, y_1)_{\mathsf{h}}\,,\ \forall(x_1, x_2) \in \Theta\}\,. \tag{59}$$

The relation Θ is self-adjoint if $\Theta = \Theta^*$.

We show that $\Theta_\parallel, \Theta_\perp, \Theta_{f_\parallel}, \Theta_{f_\perp}$ are self-adjoint relations. Since the proof is very similar we shall consider only the case of $\Theta_\parallel$. A short computation shows that $\Theta_\parallel \subseteq \Theta_\parallel^*$. We prove the converse inclusion. Let $\underline{\alpha} := (\alpha_1 \oplus \dots \alpha_4\, ;\, \alpha_5 \oplus \dots \alpha_8) \in \Theta_\parallel^*$. Considering equation (59) we find

$$(\mathrm{n}\omega_0, \alpha_5) + (\mathrm{n}\omega_1, \alpha_7) = (\mathrm{nd}\omega_0, \alpha_2) + (\alpha_4, \mathrm{nd}\omega_1, \alpha_4)\,, \qquad \forall \omega_0 \oplus \omega_1 \in D(S^*)\,. \tag{60}$$

Choosing ω_1 and $\mathrm{n}\omega_0 = 0$ – this does not affect the value $\mathrm{nd}\omega_0$ on account of Remark 7 – it follows that $(\alpha_2, \mathrm{nd}\omega_0) = 0$ for all $\omega_0 \in \Omega^{k-1}_{\mathrm{c,n}}(\Sigma)$. Since nd is surjective it follows that $\alpha_2 = 0$. With a similar argument $\alpha_5 = 0$ as well as $\alpha_2 = 0$, $\alpha_4 = 0$. Finally, on account of Remark 7 there exists $\omega_0 \oplus \omega_1 \in D(S^*)$ such that

$$\mathrm{n}\omega_0 = \alpha_1\,, \qquad \mathrm{n}\omega_1 = \alpha_3\,, \qquad \mathrm{nd}\omega_0 = \alpha_6\,, \qquad \mathrm{nd}\omega_1 = \alpha_8\,.$$

It follows that $\alpha \in \Theta_\parallel$, that is, $\Theta_\parallel = \Theta_\parallel^*$.

In addition $S_{\Theta_\sharp}$ is positive definite for $\sharp \in \{\parallel, \perp, f_\parallel, f_\perp\}$. It follows from the following equality, which holds for all $\omega_0 \otimes \omega_1 \in D(S^*)$:

$$(\omega_0 \oplus \omega_1, S_{\Theta_\sharp}(\omega_0 \oplus \omega_1))_{\mathsf{H}} = \sum_{j=1}^{2} \left[\|\mathrm{d}\omega_i\|^2 + \|\delta\omega_i\|^2 + (\mathrm{n}\omega_i, \mathrm{t}\delta\omega_i) - (\mathrm{t}\omega_i, \mathrm{nd}\omega_i)\right]\,,$$

where the last two terms are non-negative because of the boundary conditions and of the hypothesis on the sign of f. Therefore we can apply Theorem 47.

Finally we should prove that the propagators $G^\pm_{\Theta_\sharp}$ associated with the relations $\Theta_\sharp$ coincide with the propagators $G^\pm_\sharp$ introduced in Theorem 16. The fulfilment of the appropriate boundary conditions is a consequence of Lemma 50. □

Remark 49: *It is worth mentioning that, although we have only considered test sections of compact support in* $\mathring{M}$, *such assumption can be relaxed allowing the support to intersect* ∂M. *In order to prove that this operation is legitimate, a rather natural strategy consists of realizing that the boundary conditions here considered fall in the (generalization of those of) Robin type. These were considered in [GW18] for the case of a real scalar field on an asymptotically anti de Sitter spacetime where, in between many results, it was proven the explicit form of the wavefront set of the advanced and retarded fundamental solutions. In particular it was shown that two point lie in the wave front set either if they are connected directly by a light geodesic or by one which is reflected at the boundary. A direct inspection of their approach suggests that the same result holds true if one considers also static globally hyperbolic spacetimes with timelike boundary and vector valued fields. A detailed proof of this statement would require a lengthy paper on its own and thus this question will be addressed explicitly in a future work.*

B An explicit decomposition

Lemma 50: *Let* $M = \mathbb{R} \times \Sigma$ *be a globally hyperbolic spacetime – cf. Theorem 1. Moreover, for all* $\tau \in \mathbb{R}$, *let* $\mathrm{t}_{\Sigma_\tau} \colon \Omega^k(M) \to \Omega^k(\Sigma_\tau)$, $\mathrm{n}_{\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\Sigma)$ *be the tangential and normal maps on* $\Sigma_\tau \doteq \{\tau\} \times \Sigma$, *where* $M = \mathbb{R} \times \Sigma$ *– cf. Definition 5. Moreover, let* $\mathrm{t}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^k(\partial\Sigma_\tau)$ *and let* $\mathrm{n}_{\partial\Sigma_\tau} \colon \Omega^k(\Sigma_\tau) \to \Omega^{k-1}(\partial\Sigma_\tau)$ *be the tangential and normal maps on* $\partial\Sigma_\tau \doteq \{\tau\} \times \partial\Sigma$. *Let* $f \in C^\infty(\partial\Sigma)$ *and set* $f_\tau \doteq f|_{\partial\Sigma_\tau}$. *Then for* $\sharp \in \{\mathrm{D}, \parallel, \perp, f_\parallel, f_\perp\}$ *it holds*

$$\omega \in \Omega^k_\sharp(M) \Longleftrightarrow \mathrm{t}_{\Sigma_\tau}\omega, \mathrm{n}_{\Sigma_\tau}\omega \in \Omega^k_\sharp(\Sigma_\tau) \quad \forall \tau \in \mathbb{R}\,. \tag{61}$$

More precisely this entails that

$$\begin{aligned}
\omega\in\ker \mathrm{t}_{\partial M}\cap\ker \mathrm{n}_{\partial M} &\iff \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker\mathrm{n}_{\partial\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker \mathrm{n}_{\partial M}\cap\ker \mathrm{n}_{\partial M}\mathrm{d} &\iff \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{n}_{\partial\Sigma_\tau}\cap\ker\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker \mathrm{t}_{\partial M}\cap\ker \mathrm{t}_{\partial M}\delta &\iff \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker \mathrm{n}_{\partial M}\cap\ker(\mathrm{n}_{\partial M}\mathrm{d}-f\mathrm{t}_{\partial M}) &\iff \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{n}_{\partial\Sigma_\tau}\cap\ker(\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}-f_t\mathrm{t}_{\partial\Sigma_\tau})\,,\forall\tau\in\mathbb{R}\,;\\
\omega\in\ker \mathrm{t}_{\partial M}\cap\ker(\mathrm{t}_{\partial M}\delta-f\mathrm{n}_{\partial M}) &\iff \mathrm{t}_{\Sigma_\tau}\omega,\mathrm{n}_{\Sigma_\tau}\omega\in\ker\mathrm{t}_{\partial\Sigma_\tau}\cap\ker(\mathrm{t}_{\partial\Sigma_\tau}\delta_{\Sigma_\tau}-f_t\mathrm{n}_{\partial\Sigma_\tau})\,,\forall t\in\mathbb{R}\,.
\end{aligned}$$

Proof. The equivalence (61) is shown for $\perp$-boundary condition. The proof for $\|$-boundary conditions follows per duality – *cf.* (13) – while the one for D-, $f_\|$-, $f_\perp$-boundary conditions can be carried out in a similar way.

On account of Theorem 1 we have that for all $\tau\in\mathbb{R}$ we can decompose any $\omega\in\Omega^k(M)$ as follows:

$$\omega|_{\Sigma_\tau}=\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,.$$

Notice that, being the decomposition $M=\mathbb{R}\times\Sigma$ smooth we have that $\tau\to\mathrm{t}_{\Sigma_\tau}\omega\in C^\infty(\mathbb{R},\Omega^k(\Sigma))$ while $\tau\to\mathrm{n}_{\Sigma_\tau}\omega\in C^\infty(\mathbb{R},\Omega^{k-1}(\Sigma))$. Here we have implicitly identified $\Sigma\simeq\Sigma_\tau$.

A similar decomposition holds near the boundary of Σ_τ. Indeed for all $(\tau,p)\in\{\tau\}\times\partial\Sigma$ we consider a neighbourhood of the form $U=[0,\epsilon_\tau)\times U_{\partial\Sigma}$. Let $U_x\doteq\{x\}\times U_{\partial\Sigma}$ for $x\in[0,\epsilon_\tau)$ and let t_{U_x}, n_{U_x} be the corresponding tangential and normal maps – *cf.* Definition 5. With this definition we can always split $\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{\Sigma_\tau}\omega$ as follows:

$$\omega|_U=\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega\wedge\mathrm{d}x+\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau+\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}x\wedge\mathrm{d}\tau\,. \tag{62}$$

If p ranges on a compact set of $\partial\Sigma$ it follows that $(\tau,x)\to\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega\in C^\infty(\mathbb{R}\times[0,\epsilon),\Omega^k(\partial\Sigma))$ and similarly $\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$, $\mathrm{n}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega$ and $\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega$. Once again we have implicitly identified $U_{\partial\Sigma}\simeq\{x\}\times U_{\partial\Sigma}$.

According to this splitting we have

$$\begin{aligned}
\mathrm{t}_{\partial M}\omega|_{(\tau,p)}&=\mathrm{t}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{t}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau=\mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,,\\
\mathrm{n}_{\partial M}\omega|_{(\tau,p)}&=\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau=\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega+\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\,.
\end{aligned}$$

It follows that $\mathrm{n}_{\partial M}\omega=0$ if and only if $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega=0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega=0$ and similarly $\mathrm{t}_{\partial M}\omega=0$ if and only if $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega=0$ and $\mathrm{t}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega=0$. This proves the thesis for Dirichlet boundary conditions. A similar computation leads to

$$\begin{aligned}
\mathrm{n}_{\partial M}\mathrm{d}\omega&=(-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0}+\mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega+(-1)^{k-1}\partial_\tau\mathrm{n}_{U_0}\mathrm{t}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\\
&\quad+(-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0}\wedge\mathrm{d}\tau-\mathrm{d}_{\partial\Sigma_\tau}\mathrm{n}_{U_0}\mathrm{n}_{\Sigma_\tau}\omega\wedge\mathrm{d}\tau\\
&=(-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0}+(-1)^k\partial_x\mathrm{t}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0}\wedge\mathrm{d}\tau\,.
\end{aligned}$$

where the second equality holds true since $\mathrm{n}_{\partial M}\omega=0$. It follows that $\mathrm{n}_{\partial M}\mathrm{d}\omega=0$ if and only if $\partial_x\mathrm{t}_{U_x}\mathrm{t}_{\Sigma_\tau}\omega|_{x=0}=0$ and $\partial_x\mathrm{n}_{U_x}\mathrm{n}_{\Sigma_\tau}\omega|_{x=0}=0$. When $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega=0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega=0$ the latter conditions are equivalent to $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{n}_{\Sigma_\tau}\omega=0$ and $\mathrm{n}_{\partial\Sigma_\tau}\mathrm{d}_{\Sigma_\tau}\mathrm{t}_{\Sigma_\tau}\omega=0$. □

C Relative de Rham cohomology

In this appendix we summarize a few definitions and results concerning de Rham cohomology and Poincaré duality, especially when the underlying manifold has a non-empty boundary. A reader interested in more details can refer to [BT82, Sch95].

For the purpose of this section M refers to a smooth, oriented manifold of dimension $\dim M = m$ with a smooth boundary ∂M, together with an embedding map $\iota_{\partial M} : M \to \partial M$. In addition ∂M comes endowed with orientation induced from M via $\iota_{\partial M}$. We recall that $\Omega^\bullet(M)$ stands for the de Rham cochain complex which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. Observe that we shall need to work also with compactly supported forms and all definitions can be adapted accordingly. To indicate this specific choice, we shall use a subscript c, *e.g.* $\Omega^\bullet_{\mathrm{c}}(M)$. We denote instead the k-th de Rham cohomology group of M as

$$H^k(M) \doteq \frac{\ker(\mathrm{d}_k \colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d}_{k-1} \colon \Omega^{k-1}(M) \to \Omega^k(M))},$$

where we introduce the subscript k to highlight that the differential operator d acts on k-forms. Equations (3) and (4b) entail that we can define $\Omega^\bullet_{\mathrm{t}}(M)$, the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{t}}(M) \subset \Omega^k(M)$. The associated de Rham cohomology groups will be denoted as $H^k_{\mathrm{t}}(M)$, $k \in \mathbb{N} \cup \{0\}$.

Similarly we can work with the codifferential δ in place of d, hence identifying a chain complex $\Omega^\bullet(M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to $\Omega^k(M)$, the space of smooth k-forms. The associated k-th homology groups will be denoted with

$$H_k(M) \doteq \frac{\ker(\delta_k \colon \Omega^k(M) \to \Omega^{k-1}(M))}{\mathrm{Im}(\delta_{k+1} \colon \Omega^{k+1}(M) \to \Omega^k(M))}.$$

Equations (3) and (4b) entail that we can define the $\Omega^\bullet_{\mathrm{n}}(M)$ (*resp.* $\Omega^\bullet_{\mathrm{c}}(M)$, $\Omega^\bullet_{\mathrm{c,n}}(M)$), the subcomplex of $\Omega^\bullet(M)$, whose degree k corresponds to $\Omega^k_{\mathrm{n}}(M) \subset \Omega^k(M)$ (*resp.* $\Omega^k_{\mathrm{c}}(M), \Omega^k_{\mathrm{c,n}}(M) \subseteq \Omega^k(M)$). The associated homology groups will be denoted as $H_{k,\mathrm{n}}(M)$ (*resp.* $H_{k,\mathrm{c}}(M)$, $H_{k,\mathrm{c,n}}(M)$), $k \in \mathbb{N} \cup \{0\}$. Observe that, in view of its definition and on account of equation (4), the Hodge operator induces an isomorphism $H^k(M) \simeq H_{m-k}(M)$ which is realized as $H^k(M) \ni [\alpha] \mapsto [\star\alpha] \in H_{m-k}(M)$. Similarly, on account of Equation (4b), it holds $H^k_{\mathrm{t}}(M) \simeq H_{m-k,\mathrm{n}}(M)$ and $H^k_{\mathrm{c,t}}(M) \simeq H_{m-k,\mathrm{c,n}}(M)$.

As last ingredient, we introduce the notion of relative cohomology, *cf.* [BT82]. We start by defining the relative de Rham cochain complex $\Omega^\bullet(M; \partial M)$ which in degree $k \in \mathbb{N} \cup \{0\}$ corresponds to

$$\Omega^k(M; \partial M) \doteq \Omega^k(M) \oplus \Omega^{k-1}(\partial M),$$

endowed with the differential operator $\underline{\mathrm{d}}_k : \Omega^k(M; \partial M) \to \Omega^{k+1}(M; \partial M)$ such that for any $(\omega, \theta) \in \Omega^k(M; \partial M)$

$$\underline{\mathrm{d}}_k(\omega, \theta) = (\mathrm{d}_k\omega, \mathrm{t}\omega - \mathrm{d}_{k-1}\theta)\,. \tag{63}$$

Per construction, each $\Omega^k(M; \partial M)$ comes endowed naturally with the projections on each of the defining components, namely $\pi_1 : \Omega^k(M; \partial M) \to \Omega^k(M)$ and $\pi_2 : \Omega^k(M; \partial M) \to \Omega^k(\partial M)$. With a slight abuse of notation we make no explicit reference to k in the symbol of these maps, since the domain of definition will always be clear from the context. The relative cohomology groups associated to $\underline{\mathrm{d}}_k$ will be denoted instead as $H^k(M; \partial M)$ and the following proposition characterizes the relation with the standard de Rham cohomology groups built on M and on ∂M, *cf.* [BT82, Prop. 6.49]:

Proposition 51: *Under the geometric assumptions specified at the beginning of the section, there exists an exact sequence*

$$\ldots \to H^k(M; \partial M) \xrightarrow{\pi_{1,*}} H^k(M) \xrightarrow{\mathrm{t}_*} H^k(\partial M) \xrightarrow{\pi_{2,*}} H^{k+1}(M; \partial M) \to \ldots, \tag{64}$$

where $\pi_{1,*}$, $\pi_{2,*}$ and t_* indicate the natural counterpart of the maps π_1, π_2 and t at the level of cohomology groups.

The relevance of the relative cohomology groups in our analysis is highlighted by the following statement, of which we give a concise proof:

Proposition 52: *Under the geometric assumptions specified at the beginning of the section, there exists an isomorphism between* $H^k_{\mathrm{t}}(M)$ *and* $H^k(M;\partial M)$ *for all* $k \in \mathbb{N} \cup \{0\}$*.*

Proof. Consider $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}$ and let $(\omega, 0) \in \Omega^k(M;\partial M)$, $k \in \mathbb{N} \cup \{0\}$. Equation (63) entails

$$\underline{\mathrm{d}}_k(\omega, 0) = (\mathrm{d}_k\omega, \mathrm{t}\omega) = (0, 0)\,.$$

At the same time, if $\omega = \mathrm{d}_{k-1}\beta$ with $\beta \in \Omega^{k-1}_{\mathrm{t}}(M)$, then $(\mathrm{d}_{k-1}\beta, 0) = \underline{\mathrm{d}}_{k-1}(\beta, 0)$. Hence the embedding $\omega \mapsto (\omega, 0)$ identifies a map $\rho : H^k_{\mathrm{t}}(M) \to H^k(M;\partial M)$ such that $\rho([\omega]) \doteq [(\omega, 0)]$.

To conclude, we need to prove that ρ is surjective and injective. Let thus $[(\omega', \theta)] \in H^k(M;\partial M)$. It holds that $\mathrm{d}_k\omega' = 0$ and $\mathrm{t}\omega' - \mathrm{d}_{k-1}\theta = 0$. Recalling that $\mathrm{t} : \Omega^k(M) \to \Omega^k(\partial M)$ is surjective – *cf.* Remark 7 – for all values of $k \in \mathbb{N} \cup \{0\}$, there must exists $\eta \in \Omega^{k-1}(M)$ such that $\mathrm{t}\eta = \theta$. Let $\omega \doteq \omega' - \mathrm{d}_{k-1}\eta$. On account of (4b) $\omega \in \Omega^k_{\mathrm{t}}(M) \cap \ker \mathrm{d}_k$ and $(\omega, 0)$ is a representative if $[(\omega', \theta)]$ which entails that ρ is surjective.

Let $[\omega] \in H^k(M)$ be such that $\rho[\omega] = [0] \in H^k(M;\partial M)$. This implies that there exists $\beta \in \Omega^{k-1}(M)$, $\theta \in \Omega^{k-2}(\partial M)$ such that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}(\beta, \theta) = (\mathrm{d}_{k-1}\beta, \mathrm{t}\beta - \mathrm{d}_{k-2}\theta)\,.$$

Let $\eta \in \Omega^{k-2}(M)$ be such that $\mathrm{t}\eta + \theta = 0$. It follows that

$$(\omega, 0) = \underline{\mathrm{d}}_{k-1}\left((\beta, \theta) + \underline{\mathrm{d}}_{k-2}(\eta, 0)\right) = \underline{\mathrm{d}}_{k-1}(\beta + \mathrm{d}_{k-2}\eta, 0)\,.$$

This entails that $\omega = \mathrm{d}_{k-1}(\beta + \mathrm{d}_{k-2}\eta)$ where $\mathrm{t}(\beta + \mathrm{d}_{k-2}\eta) = 0$. It follows that $[\omega] = 0$ that is ρ is injective.

□

To conclude, we recall a notable result concerning the relative cohomology, which is a specialization to the case in hand of the Poincaré-Lefschetz duality, an account of which can be found in [Mau80]:

Proposition 53: *Under the geometric assumptions specified at the beginning of the section and assuming in addition that* M *admits a finite good cover, it holds that, for all* $k \in \mathbb{N} \cup \{0\}$

$$H^{m-k}(M;\partial M) \simeq H^k_{\mathrm{c}}(M)^*\,, \qquad [\alpha] \to \left(H^k_{\mathrm{c}}(M) \ni [\eta] \mapsto \int_M \overline{\alpha} \wedge \eta \in \mathbb{C}\right). \tag{65}$$

where $m = \dim M$ *and where on the right hand side we consider the dual of the* $(m-k)$*-th cohomology group built out compactly supported forms.*

Remark 54: *On account of Propositions 52-53 and of the isomorphisms* $H^k_{(\mathrm{c})}(M) \simeq H^{m-k}_{(\mathrm{c})}(M)$ *we have the following isomorphism*

$$H^k_{\mathrm{t}}(M) \simeq H^{m-k}_{\mathrm{c}}(M)^* \simeq H_{k,\mathrm{c}}(M)^*\,, \qquad H^k(M) \simeq H_{k,\mathrm{c},\mathrm{n}}(M)^*\,. \tag{66}$$

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