University of Pavia
Department of Physics

On the fundamental solutions for wave-like equations on curved backgrounds

$$\Box u = \delta$$

Dissertation of:
Rubens Longhi

Advisor:
Dott. Claudio Dappiaggi

Co-advisor:
Nicoló Drago

July 2017

Abstract

La tesi si propone di analizzare la risolubilitá di equazioni differenziali alle derivate parziali di tipo ondulatorio e le loro proprietá tramite la costruzione di soluzioni fondamentali distribuzionali. Dapprima si affronterá il caso nello spazio di Minkowski n-dimensionale e poi si trasporteranno, per quanto possibile, le principali proprietá delle soluzioni su particolari varietá curve di interesse per le loro applicazioni fisiche.

The aim of the thesis is to analyse the solvability of wave-like partial differential equations and their properties via the construction of distributional fundamental solutions. Initially will be explored the n-dimensional Minkowski case and then the main properties of the solutions will be translated, when possible, on particular manifolds, interesting for their physical applications.

Contents

Introduction 1

Wave-like equations are a class of differential equations which describe many physical processes, from electromagnetism to quantum field theory. Such equations contain a differential operator P, which is the generalization of d'Alembert wave operator

$$\square := \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

that describes the propagation of a wave with velocity $c \equiv 1$ in an n dimensional space. In this thesis will be presented the constructive method of fundamental solutions to solve wave-like differential equations, firstly on a flat background and then on certain manifolds.
Fundamental solutions can be thought as the distributional solutions to a differential equation in which the source is a point-like instantaneous perturbation, namely the Dirac delta. Precisely, if P is a differential operator, a fundamental solution is a solution of the equation

$$Pu = \delta_x.$$

Once a fundamental solution is found, it is easy to find the desired solution of the equation with a generic source ψ.
We will focus not only on looking for explicit fundamental solutions, which not always can be found, but rather on their support properties and what they tell about the features of the solutions to the equation with a source. In particular

we will find that any wave-like operator has two independent fundamental solutions, that will be called **retarded** and **advanced**, that manifest the causal nature of the wave propagation.
Our interest does not limit to the case where the background in which the waves move is flat, but we will generalize the results on a class of manifolds that respect precise causal properties and make the solution physically acceptable.

In the first chapter, it will be presented an overview of the main mathematical notions needed in order to set the discussion on a curved background. The main topics will be *differentiable manifolds*, *tangent space*, *Lorentzian manifolds*, *causality and global hyperbolicity*, *operators and integration*.

In the second chapter, it will be discussed the concept of fundamental solution and we will focus on the particular case of the d'Alembert wave operator in Minkowski spacetime. Two approaches will be followed: one relies on the Fourier transform and the other on a particular class of distributions, the Riesz distributions, that solve the problem for any dimension. At last, an overview of the Cauchy problem for the wave operator will be discussed in the simplest cases.

In the third chapter, Riesz distributions will be transported on manifolds and combined to solve for fundamental solutions of a generalized d'Alembert operator, first locally and then globally on suitable manifolds. The Cauchy problem will be solved in local and global setting, and it will give informations on the regularity and the support of the solutions.

Mathematical Tools 2

2.1 An overview of Differential Geometry

We begin by recalling of very well known definitions in order to introduce the basic geometrical objects which are used in the text.
A **manifold** is, heuristically speaking, a space that is locally similar to $\mathbb{R}^n$. To define it we use the concepts of **topological space** and of **homeomorphism**.

2.1.1 Definition (**Topological Space**). *A set* X *together with a family* $\mathcal{T}$ *(**topology**) of subsets of* X *is called a **topological space** if the following conditions are met:*

a. $\emptyset, \mathrm{X} \in \mathcal{T}$,

b. *for all* U *and* $V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$,

c. *for any index set* A, *if* $U_i \in \mathcal{T}$ *for all* $i \in A$, $\displaystyle\bigcup_{i \in A} U_i \in \mathcal{T}$.

An element of $\mathcal{T}$ *is called **open set**. If a point* p *is in an open set* U, *we call* U *a **neighborhood** of* p.

2.1.2 Definition (**Continuity and homomorphism**). *Let* X *and* Y *be two topological spaces. A function* $f : \mathrm{X} \to \mathrm{Y}$ *is **continuous** if for any open set* U *of* Y, *the preimage* $f^{-1}(U)$ *is an open set of* X.
A continuous and bijective map $\varphi : \mathrm{X} \to \mathrm{Y}$ *is an **homomorphism** if* $\varphi^{-1} : \mathrm{Y} \to \mathrm{X}$ *is also continuous.*

As for vector spaces, we can talk of a **basis** for topological space. A subset $\mathcal{B} \subset \mathcal{T}$ is a basis if any open set can be expressed as union of elements of $\mathcal{B}$. A topology is **Hausdorff** if, for any two distinct points $p, q \in \mathrm{X}$, there exist two open neighborhoods U of p and V of q such that $U \cap V = \emptyset$.

A topological space X is called **compact** if each of its open covers has a finite subcover, i.e. for any collection $\{U_i\}_{i\in A}$, (where A is a set of indexes) such that

$$\mathrm{X} \subseteq \bigcup_{i\in A} U_i,$$

there is a finite subset A' of A such that

$$\mathrm{X} \subseteq \bigcup_{i\in A'} U_i.$$

The closure $\overline{\Omega}$ of a subset $\Omega \subset \mathrm{X}$ is the smallest closed set that contains Ω. We say Ω is **relatively compact** if its closure $\overline{\Omega}$ is a compact subset.

We are now ready to introduce the concept of **manifold**.

2.1.3 Definition. *An n-dimensional topological* ***manifold*** M *is a topological Hausdorff space (with a countable basis) that is locally homeomorphic to $\mathbb{R}^n$, i.e. for every $p \in \mathrm{M}$ there exists an open neighbourhood U of p and a homeomorphism*

$$\varphi : U \to \varphi(U),$$

such that $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

Such homeomorphism is called a **(local) chart** of M. An **atlas** of M is a family $\{U_i, \varphi_i\}_{i\in A}$ of local charts together with an open covering of M, i.e. $\bigcup_{i\in A} U_i = \mathrm{M}$.

2.1.4 Definition. *A* ***differentiable atlas*** *of a manifold* M *is an atlas $\{U_i, \varphi_i\}_{i\in A}$ such that the functions*

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

are differentiable (of class C^∞) for any $i, j \in A$ such that $U_i \cap U_j \neq \emptyset$. Each φ_{ij} is called ***transition function****.*

With this definition, each φ_{ij} is a **diffeomorphism** because one can always interchange the indexes i and j.

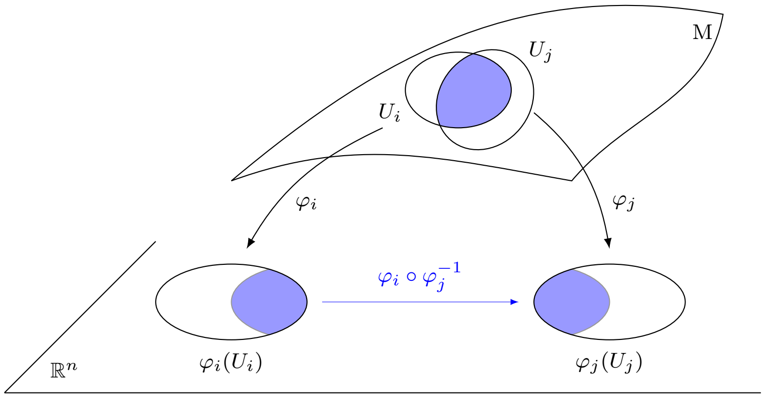


FIGURE 2.1: A differentiable atlas on a manifold M.

We are only interested in **differentiable** (or **smooth**) **manifolds**, endowed with a maximal differentiable atlas. Here maximality of the atlas means that, if φ is a chart of M and $\{U_i, \varphi_i\}_{i\in A}$ is a differentiable atlas, then φ belongs to $\{U_i, \varphi_i\}_{i\in A}$. We call a differentiable manifold with an atlas for which all chart transitions have positive Jacobian determinant an **orientable manifold**.

2.1.5 Remark. For now on, the word **manifold** will always mean **differentiable manifold** and to indicate them it will be used the letters M or N.

2.1.6 Definition (**Submanifold**)**.** *Let $n \leq m$. An n-dimensional **submanifold** N of M is a nonempty subset N of M such that, for every point $q \in$ N, there exists a local chart $\{U, \varphi\}$ of M about q such that*

$$\varphi(U \cap \mathrm{N}) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^m.$$

*If $n = m - 1$ we call N an **hypersurface** of M.*

2.1.7 Example. If M and N are manifolds, the Cartesian product M $\times$ N is endowable with canonical structure of a manifold. If $\{U_i, \varphi_i\}_{i\in A}$ is an differentiable atlas for M and $\{V_j, \psi_j\}_{j\in B}$ ia an atlas for N, then $\big\{U_i \times V_j, (\varphi_i, \psi_j)\big\}_{(i,j)\in A\times B}$ is a differentiable atlas for M $\times$ N.

As in the Euclidean case, one can introduce the notion of **differentiable** map between manifolds:

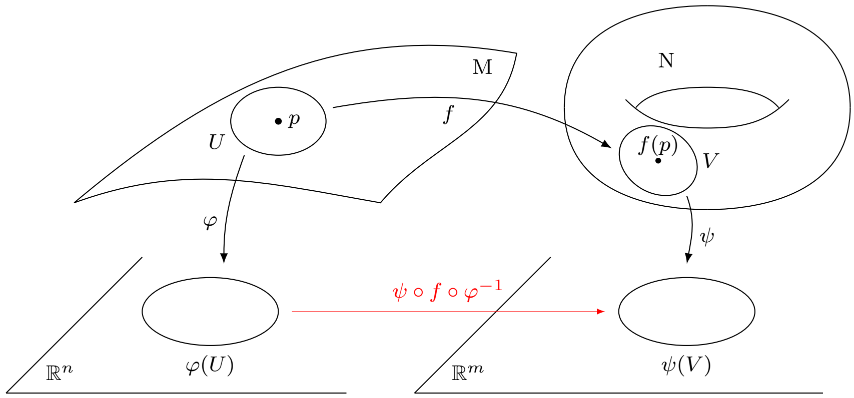


FIGURE 2.2: The notion of differentiable map f between manifolds M and N.

2.1.8 Definition. *A continuous map $f : \mathrm{M} \to \mathrm{N}$ between two manifolds M and N is **differentiable** at $p \in \mathrm{M}$ if there exist local charts $\{U, \varphi\}$ and $\{V, \psi\}$ about p in M and about $f(p)$ in N respectively, such that $f(U) \subset V$ and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V),$$

is differentiable (of class C^∞) at $\varphi(p)$. The function f is said to be differentiable on M if it is differentiable at every point of M.

The space of differentiable functions between two manifolds is denoted by $C^\infty(\mathrm{M}, \mathrm{N})$, and if $\mathrm{N} = \mathbb{C}$ simply by $C^\infty(\mathrm{M})$. If $\psi \circ f \circ \varphi^{-1}$ is of class C^k, we say $f \in C^k(\mathrm{M}, \mathrm{N})$.

We introduce the **tangent space** of a point of a manifold. It will be constructed using the derivatives of curves which pass through the point. A tangent vector at a point p is thought of as the velocity of a curve passing through the point. We can therefore define a tangent vector as an equivalence class of curves passing through p while being tangent to each other at p. .

2.1.9 Definition (Tangent space). *Let $p \in \mathrm{M}$ and let I be an interval containing 0. We indicate $\mathcal{C}_p = \{c \in C^\infty(I, \mathrm{M}) |\ c(0) = p\}$ the set of differentiable curves passing through p.*
We consider the equivalence relation $(\sim)$, according to which two curves $c_1, c_2 \in \mathcal{C}_p$ are equivalent if there exists a local chart φ about p such that $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$.
*The **tangent space** of M at p is the set $\mathrm{T}_p\mathrm{M} := \mathcal{C}_p / \sim$.*

We will refer at the equivalence class of curves with $\dot{c}$, as they are uniquely determined by the velocity $(\phi \circ c)'(0)$ at which they pass through the point p. One can check that the definition of the equivalence relation does not depend on the choice of local chart. In fact, if $\{U, \varphi\}$ and $\{V, \psi\}$ are local charts at p,

$$(\varphi \circ c)'(0) = (\varphi \circ \psi^{-1} \circ \psi \circ c)'(0) = D(\varphi \circ \psi^{-1})(\psi(p)) \cdot (\psi \circ c)'(0),$$

where $D(\varphi \circ \psi^{-1})(\psi(p))$ stands for the Jacobian of the transition function calculated at $\psi(p)$. It holds that $(\varphi \circ c_1)'(0)$ and $(\varphi \circ c_2)'(0)$ coincide if and only if $(\psi \circ c_1)'(0)$ and $(\psi \circ c_2)'(0)$ coincide.

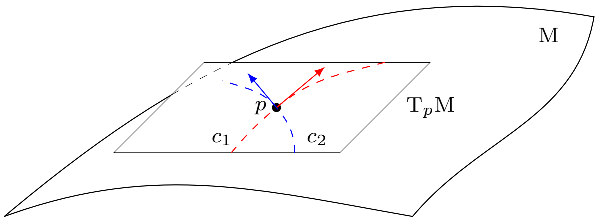


FIGURE 2.3: Tangent space $\mathrm{T}_p\mathrm{M}$ where $c_1 \nsim c_2$.

It can be proven that (for a fixed atlas φ about p) the following map is a linear isomorphism between $\mathrm{T}_p\mathrm{M}$ and $\mathbb{R}^n$:

$$\begin{aligned} \Theta_\varphi : \ \mathrm{T}_p\mathrm{M} &\to \mathbb{R}^n, \\ \dot{c} &\mapsto (\varphi \circ c)'(0). \end{aligned}$$

Hence we can think of $\mathrm{T}_p\mathrm{M}$ as being a copy of $\mathbb{R}^n$ attached to the point p on the manifold.
For reasons that we will make clear later, we denote the basis of $\mathrm{T}_p\mathrm{M}$ as

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}.$$

Collecting all tangent spaces, one builds the **tangent bundle** of a manifold M, defined as:

$$\mathrm{TM} = \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M} = \bigcup_{p \in \mathrm{M}} \{p\} \times \mathrm{T}_p\mathrm{M}.$$

2.1.10 Definition. *Let $f : \mathrm{M} \to \mathrm{N}$ be a differentiable map and let $p \in \mathrm{M}$. The **differential** of f at p is the linear map*

$$\mathrm{d}_p f : \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{N}, \quad \dot{c} \mapsto [f \circ c] \cong (f \circ c)'(0).$$

*The **differential** of f is the map* $\mathrm{d}f : \mathrm{TM} \to \mathrm{TN}$ *such that* $\mathrm{d}f|_{\mathrm{T}_p\mathrm{M}} = \mathrm{d}_p f$.

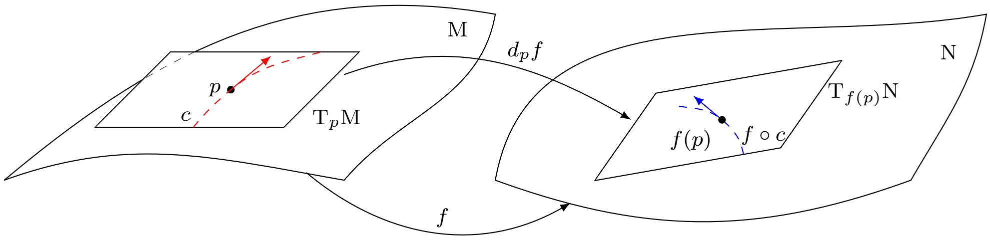


FIGURE 2.4: A scheme of a differential map.

Given M and a chart $\{U, \varphi\}$ near p, fix $X = \dot{c} \in \mathrm{T}_p\mathrm{M}$. If we identify $\mathrm{T}_p\mathbb{R} \cong \mathbb{R}$, we can interpret the differential $\mathrm{d}_p f(X)$ of a function $f \in C^\infty(\mathrm{M})$ at a point p as the **derivative** in the direction of X:

$$\partial_X f(p) := \mathrm{d}_p f(X).$$

A functional which is linear and follows Leibniz rule, such as $\partial_X : C^\infty(\mathrm{M}) \to \mathbb{R}$, is called a **derivation**. The set of all derivations at p is denoted as Der_p and it is a vector space. The map $X \in \mathrm{T}_p\mathrm{M} \mapsto \partial_X$ is an isomorphism between $\mathrm{T}_p\mathrm{M}$ and Der_p.
Define:

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(\mathrm{M}) \to \mathbb{R}, \quad f \mapsto \left.\frac{\partial f}{\partial x^i}\right|_p = \partial_X f(p),$$

where $X = \dot{c}$ and $c(t) = \varphi^{-1}(\varphi(p) + te_i)$ (e_i is the i-th canonical basis vector). Note that, from the definition of the differential, it holds

$$\partial_X f(p) = (f \circ c)'(0) = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(p)},$$

which shows that the object we defined can be seen as a partial derivative in the usual sense.
It can be shown that the set of derivations $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ forms a basis for Der_p and, due to the isomorphism, for $\mathrm{T}_p\mathrm{M}$, X can be expressed as

$$X = X^i \frac{\partial}{\partial x^i},$$

$$\begin{tikzcd} & \mathrm{T}_p\mathrm{M} \arrow[dl, "X \mapsto \partial_X"'] \arrow[dr, "\Theta_\varphi"] & \\ \mathrm{Der}_p & & \mathbb{R}^n \end{tikzcd}$$

FIGURE 2.5: Isomorphism relations for the tangent space.

where Einstein summation has been employed.
Observe that linearity entails that

$$\partial_X f(p) = \mathrm{d}_p f(X) = X^i \mathrm{d}_p f \left(\frac{\partial}{\partial x^i}\Big|_p \right) = X^i \frac{\partial f}{\partial x^i}\Big|_p .$$

2.1.11 Definition. *Let* M *be a manifold, we define a projection map* π : $\mathrm{TM} \to \mathrm{M}$ *such that* $\pi(\mathrm{T}_p\mathrm{M}) = p$*, and we call a* ***section*** *in the tangent bundle a map* $s : \mathrm{M} \to \mathrm{TM}$ *such that* $\pi \circ s = \mathrm{id}_{\mathrm{M}}$.

The dual space of the tangent space $\mathrm{T}_p\mathrm{M}$ is called the **cotangent space**, denoted with $\mathrm{T}_p^*\mathrm{M}$, which has a canonical basis denoted with $\{\mathrm{d}x^1|_p, \dots, \mathrm{d}x^n|_p\}$. The elements of such basis act on any element of the tangent space basis at p as follows:

$$\mathrm{d}x^i|_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta_{ij}.$$

Similarly is defined the **cotangent bundle** $\mathrm{T}^*\mathrm{M}$ as the disjoint union of cotangent spaces.

2.1.12 Definition. *Sections in the tangent bundle, denoted by* $C^\infty(\mathrm{M}, \mathrm{TM})$*, are called* ***vector fields****, whereby sections in the cotangent bundle are called* 1***-forms****.*

Vector fields are locally expressed in terms of linear combinations of

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} =: \{\partial_1, \dots, \partial_n\} ,$$

where $\partial_i = \partial_i|_p$ at any point p, whereas 1-forms are expressed as linear combinations of

$$\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\},$$

where $\mathrm{d}x^i$ is the 1-form that acts at p as $\mathrm{d}x^i|_p$.

2.1.13 Definition. *We define the derivative in the direction of X as an operator $\partial_X : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ such that*

$$\partial_X f = \mathrm{d}f(X),$$

for any vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$.

It follows immediately that Leibniz's rule holds: $\partial_X(f \cdot g) = g\,\partial_X f + f\,\partial_X g$, and again holds the useful formula

$$\partial_X f = \mathrm{d}f(X) = X^i \mathrm{d}f(\partial_i) = X^i \frac{\partial f}{\partial x^i}.$$

2.1.14 Observation. Given two vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$, there is a unique vector field $[X, Y] \in C^\infty(\mathrm{M}, \mathrm{TM})$ such that

$$\partial_{[X,Y]} f = \partial_X \partial_Y - \partial_Y \partial_X f$$

for all $f \in C^\infty(\mathrm{M})$. The map $[\cdot, \cdot] : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) \to C^\infty(\mathrm{M}, \mathrm{TM})$ is called the **Lie bracket**, it is bilinear, skew-symmetric and satisfies the *Jacobi identity*: for any $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

2.1.15 Definition. *An* ***affine connection*** *or* ***covariant derivative*** *on a manifold* M *is a bilinear map*

$$\begin{aligned} \nabla : C^\infty(\mathrm{M}, \mathrm{TM}) \times C^\infty(\mathrm{M}, \mathrm{TM}) &\to C^\infty(\mathrm{M}, \mathrm{TM}) \\ (X, Y) &\mapsto \nabla_X Y\,, \end{aligned}$$

such that for all smooth functions $f \in C^\infty(\mathrm{M})$ and all vector fields $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$:

- $\nabla_{fX} Y = f \nabla_X Y$, *i.e., ∇ is $C^\infty(\mathrm{M})-$linear in the first variable;*
- $\nabla_X f = \partial_X f$*;*
- $\nabla_X(fY) = Y \partial_X f + f \nabla_X Y$, *i.e., ∇ satisfies the* Leibniz rule *in the second variable.*

The covariant derivative on the direction of the basis vector fields $\{\partial_1, \ldots, \partial_n\}$ is indicated

$$\nabla_j := \nabla_{\partial_j}.$$

We are now ready to introduce metric structures on manifolds.

2.2 Lorentzian Manifolds

We start in the simple case of **Minkowski spacetime**.

2.2.1 Definition. *Let V be an n-dimensional vector space. A* ***Lorentzian scalar product*** *is a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ with signature $(-+\cdots+)$, i.e. such that one can find a basis $\{e_1,\ldots,e_n\}$ such that*

$$\langle e_1, e_1\rangle = -1, \quad \langle e_i, e_j\rangle = \delta_{ij} \;\text{ if } i,j>1.$$

The **Minkowski product** $\langle x,y\rangle_0$, defined by the formula

$$\langle x,y\rangle_0 = \eta_{ik}x^i y^k = -x_1\,y_1 + x_2\,y_2 + \cdots + x_n\,y_n,$$

with $\eta := \operatorname{diag}(-1,1,\ldots,1,1)$ is the simplest example of Lorentzian scalar product on $\mathbb{R}^n$. The n-dimensional Minkowski space, denoted by $\mathbb{M}^n$ is simply $\mathbb{R}^n$ equipped with Minkowski product.

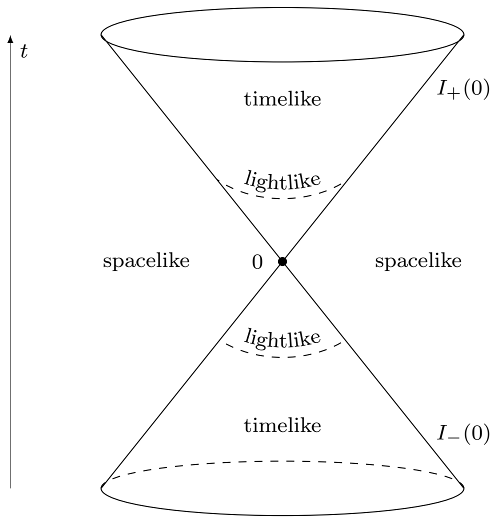


Figure 2.6: Minkowski time orientation.

2.2.2 Definition. *We call the* ***negative squared length*** *of a vector $X \in V$*

$$\gamma(X) = -\|X\|^2 := -\langle X,X\rangle.$$

A vector $X \in V\setminus\{0\}$ is called

- ***timelike*** *if $\gamma(X) > 0$,*

- ***lightlike*** *if* $\gamma(X) = 0$,
- ***spacelike*** *if* $\gamma(X) < 0$ *or* $X = 0$,
- ***causal*** *if it is either timelike or lightlike.*

This definition will mostly be used for tangent vectors, in case V is the tangent space of a Lorentzian manifold at a point.
For $n \geq 2$ the set of timelike vectors $I(0)$ consists of two connected components. A **time orientation** is the choice of one of these two components, that we call $I_+(0)$.

2.2.3 Definition. *We call*

- $J_+(0) := \overline{I_+(0)}$ *(elements are called* ***future-directed****),*
- $C_+(0) := \partial I_+(0)$ *(upper* ***light cone****),*
- $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$ *(elements are called* ***past-directed****),*
- $C_-(0) := -C_+(0)$ *(lower* ***light cone****).*

2.2.4 Definition. *A* ***metric*** g *on a manifold* M *is the assignment of a scalar product at each tangent space*

$$g : \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$$

which depends smoothly on the base point p. *We call it a* ***Riemannian metric*** *if the scalar product is pointwise positive definite, and a* ***Lorentzian metric*** *if it is a Lorentzian scalar product.*
A pair (M, g), *where* M *is a manifold and* g *is a Lorentzian (Riemannian) metric is called a* ***Lorentzian (Riemannian) manifold****.*

The request of smooth dependence on p may be specified as follows: given any chart $\{U, \varphi = (x^1, \dots, x^n)\}$ about p, the functions $g_{ij} : \varphi(U) \to \mathbb{R}$ defined by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, for any $i, j = 1, \dots, n$ should be differentiable. With respect to these coordinates one writes

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j \equiv \sum_{i,j} g_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j.$$

The scalar product of two tangent vectors $v, w \in \mathrm{T}_p\mathrm{M}$, with coordinate chart $\varphi = (x^1, \dots x^n)$, such that $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$ is

$$\langle v, w \rangle = g_{ij}(\varphi(p)) v^i w^j.$$

When the choice of the chart is clear we will often write, with abuse of notation $g_{ij}(p) := g_{ij}(\varphi(p))$. We will indicate $(g^{ij})_{i,j=1,\ldots,n} = (g_{ij})^{-1}$.

The **negative squared length** of a tangent vector X at $p \in \mathrm{M}$ generalizes naturally as follows:

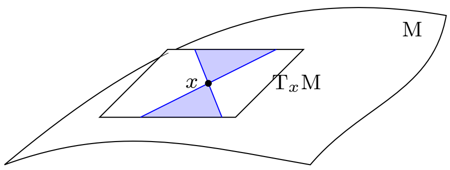
$$\gamma(X) = -\langle X, X\rangle. \tag{2.1}$$

From now on M will always indicate a **Lorentzian manifold**.

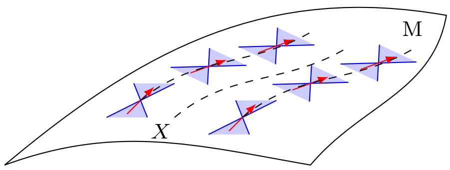
2.2.5 Definition. *A vector field $X \in C^\infty(\mathrm{M}, \mathrm{TM})$ is called timelike, spacelike, lightlike or causal, if $X(p)$ is timelike, spacelike, lightlike or causal, respectively, at every point $p \in \mathrm{M}$.*

A differentiable curve $c : I \to \mathrm{M}$ is called timelike, lightlike, spacelike, causal, future-directed or past-directed if $\dot{c}(t) \in \mathrm{T}_{c(t)}\mathrm{M}$ is, for all $t \in I$, timelike, lightlike, spacelike, causal, future-directed or past-directed, respectively.

*A Lorentzian manifold M is called **time-oriented** if there exists a nowhere vanishing timelike vector field on M. If a manifold is time-oriented, we refer to it as **spacetime**.*



(a) A time-oriented tangent space.



(b) A time-oriented manifold together with field lines of a timelike vector field X.

FIGURE 2.7: Time orientations.

The **causality relations** on M are defined as follows. Let $p, q \in \mathrm{M}$,

- $p \ll q$ iff there exists a future-directed timelike curve from p to q,

- $p < q$ iff there is a future-directed causal curve from p to q,
- $p \leq q$ iff $p < q$ or $p = q$.

The causality relation "$<$" is a strict weak ordering and the relation "$\leq$" makes the manifold a partially ordered set.

2.2.6 Definition. *The* ***chronological future*** *of a point $x \in \mathrm{M}$ is the set $I_+^M(x)$ of points that can be reached by future-directed timelike curves, i.e.*

$$I_+^M(x) = \{y \in \mathrm{M} \mid x < y\}.$$

The ***causal future*** *$J_+^M(x)$ of a point $x \in \mathrm{M}$ is the set of points that can be reached by future-directed causal curves from x, i.e.,*

$$J_+^M(x) = \{y \in \mathrm{M} \mid x \leq y\}.$$

Given a subset $A \subset \mathrm{M}$ the ***chronological future*** *and the* ***causal future*** *of A are respectively*

$$I_+^M(A) = \bigcup_{x \in A} I_+^M(x), \qquad J_+^M(A) = \bigcup_{x \in A} J_+^M(x).$$

In a similar way, one defines the ***chronological*** *and* ***causal pasts*** *of a point x of a subset $A \subset \mathrm{M}$ by replacing future-directed curves with past directed curves. They are denoted by $I_-^M(x), I_-^M(A), J_-^M(x)$, and $J_-^M(A)$, respectively. We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$.*

Any subset Ω of a spacetime M is a spacetime itself, if one restricts the metric to Ω. So $I_\pm^\Omega(x)$ and $J_\pm^\Omega(x)$ are well defined.

2.2.7 Definition. *A subset $\Omega \subset \mathrm{M}$ of a spacetime is called* ***causally compatible*** *if for any point $x \in \Omega$ holds*

$$J_\pm^\Omega(x) = J_\pm^\mathrm{M}(x) \cap \Omega,$$

where it can be noted that the inclusion "$\subset$" always holds.

The condition we defined means that taken two points in Ω that can be joined by a causal curve in M, there also exists a causal curve connecting them inside Ω.

We now can introduce the concept of **geodesics** and **exponential map**.

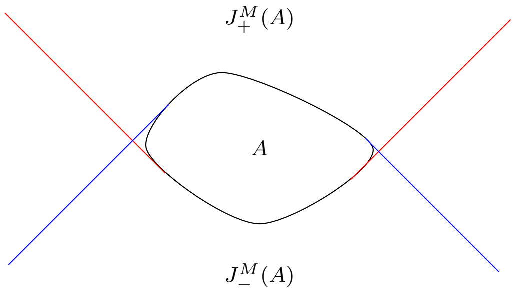


FIGURE 2.8: Causal future J_+^M and causal past J_-^M of a subset $A \subset \mathrm{M}$.

2.2.8 Definition. *Let $c : [a, b] \to \mathrm{M}$ be a curve on a Lorentzian manifold M. The length $L\dot{c}$ is defined by (with Einstein summation convention)*

$$L\dot{c} = \int_a^b \sqrt{|g(\dot{c}(t), \dot{c}(t))|}\, dt = \int_a^b \sqrt{\left| g_{ik}(c(t)) \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} \right|}\, dt,$$

*where $x^i(t) := (\varphi \circ c)^i(t)$ are the coordinates of the point $c(t)$ in a chart φ. Given $p, q \in \mathrm{M}$, if $p \leq q$ we define the **time-separation** between p and q as*

$$\tau(p, q) = \sup\{L[c] \mid c \text{ is a future directed causal curve from } p \text{ to } q\},$$

and 0 otherwise.
*A **geodesic** between two points $p, q \in \mathrm{M}$ such that $p \leq q$, if it exists, is a curve c such that $L[c] = \tau(p, q)$, i.e. the curve of maximum time-separation.*

The request on the geodesics implies that (in variational sense) $\delta L\dot{c} = 0$. It can be demonstrated that the stationary problem for the functional $L\dot{c}$ is equivalent to $\delta E\dot{c} = 0$ for the functional, called **energy**, defined by

$$E\dot{c} = \frac{1}{2} \int_a^b |g(\dot{c}(t), \dot{c}(t))|\, dt.$$

Since the Euler-Lagrange equations for a functional $I[c] = \int_a^b f(t, c(t), \dot{c}(t))\, dt$ read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}^i}\right) - \frac{\partial f}{\partial x^i} = 0,$$

being $c = (x^1, \dots, x^n)$, then, in our case, setting $f(t, c, \dot{c}) = g(\dot{c}, \dot{c})$:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0.$$

Here $\Gamma^i_{jk} \in C^\infty(U \subset \mathrm{M})$ are the **Christoffel symbols**, defined in the chart $\varphi = (\xi^1, \dots, \xi^n)$ as

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lj}}{\partial \xi^k} + \frac{\partial g_{lk}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l} \right).$$

2.2.9 Definition. *A connection ∇ on a manifold* M *with a metric g is said to be a* ***metric connection*** *if for all $X, Y, Z \in C^\infty(\mathrm{M}, \mathrm{TM})$ holds the following* Leibniz rule*:*

$$\partial_X\, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The unique metric connection which is also torsion-free, i.e.,

$$T := \nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

is called the ***Levi-Civita connection****.*

Another way to define **geodesics** is to say a geodesic between two points p, q on a manifold M with the Levi-Civita connection ∇ is the curve c which links p and q such that parallel transport along it preserves the tangent vector to the curve, i.e.

$$\nabla_{\dot c(t)} \dot c(t) = 0 \quad \text{for all } t \in [a, b]. \tag{2.2}$$

More precisely, in order to define the covariant derivative of $\dot c$ it is necessary first to extend $\dot c$ to a smooth vector field in an open set containing the image of the curve, but it can be shown that the derivative is independent of the choice of the extension.

2.2.10 Observation. We can express the Christoffel symbols in terms of the Levi-Civita connection:

$$\nabla_j \partial_k = \Gamma^i_{jk} \partial_i \tag{2.3}$$

in a local chart $\varphi = (x^1, \dots, x^n)$.

2.2.11 Proposition. *Let ∇ be a connection over a manifold* M *and $X, Y \in C^\infty(\mathrm{M}, \mathrm{TM})$ be vector fields. It holds*

$$\nabla_X Y = \left(X^j \partial_j Y^k + X^j Y^i \Gamma^k_{ij} \right) \partial_k,$$

in particular

$$(\nabla_j Y)^i = \partial_j Y^i + Y^i \Gamma^k_{ij}.$$

Proof. From Definition (2.1.15) holds:

$$\nabla_X Y = \nabla_{X^j e_j} Y^i e_i = X^j \partial_j Y^i e_i = X^j Y^i \nabla_j e_i + X^j e_i \partial_j Y^i =$$
$$= X^j Y^i \Gamma^k_{ij} e_k + (X^j \partial_j Y^k) e_k.$$

2.2.12 Proposition. *Let us consider $p \in \mathrm{M}$ and a tangent vector $\xi \in \mathrm{T}_p\mathrm{M}$. Then there exists $\varepsilon > 0$ and precisely one geodesic*

$$c_\xi : [0, \varepsilon] \to \mathrm{M},$$

such that $c_\xi(0) = p$ and $\dot{c}_\xi(0) = \xi$.

2.2.13 Definition. *In the conditions of the proposition above, if we put*

$$\mathcal{D}_p = \{\xi \in \mathrm{T}_p\mathrm{M} \mid c_\xi \text{ is defined on } [0,1]\} \subset \mathrm{T}_p\mathrm{M},$$

the ***exponential map*** *at point p is defined as $\exp_p : \mathcal{D}_p \to \mathrm{M}$ such that $\exp_p(\xi) = c_\xi(1)$.*
The local coordinates defined by the chart $\{U := \exp_p(\mathcal{D}_p), \exp_p^{-1}\}$ are called ***normal coordinates*** *centered at p.*

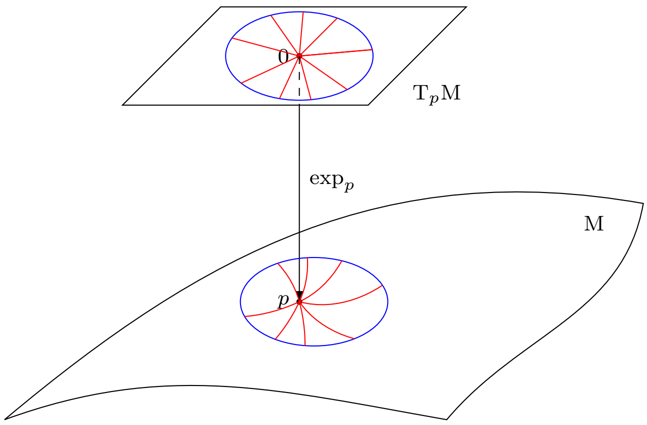


FIGURE 2.9: The exponential maps from the tangent space to the manifold.

2.2.14 Proposition. *Given normal coordinates centered at $p \in \mathrm{M}$, it holds*

$$g_{ij}(p) := g_{ij}(\exp_p(0)) = \eta_{ij},$$
$$\Gamma^i_{jk} = 0,$$

for all indexes i, j, k.

We are now ready to talk about **geodesically starshaped** sets.

2.2.15 Definition. *An open subset $\Omega \subset \mathrm{M}$ is called* ***geodesically starshaped*** *with respect to a point $p \in \mathrm{M}$ if there exists an open subset $\Omega' \subset \mathrm{T}_p\mathrm{M}$, starshaped with respect to 0, such that the exponential map*

$$\exp_p|_{\Omega'} : \Omega' \to \Omega,$$

is a diffeomorphism. If Ω is geodesically starshaped with respect to all of its points, one calls it ***convex****.*

2.2.16 Proposition. *Under the conditions of the last definition, let $\Omega \subset \mathrm{M}$ be geodesically starshaped with respect to point $p \in \mathrm{M}$. Then one has*

$$I_\pm^\Omega(p) = \exp_p(I_\pm(0) \cap \Omega'),$$

$$J_\pm^\Omega(p) = \exp_p(J_\pm(0) \cap \Omega').$$

We continue our preparations with another function that we are going to need in Chapter (3).

2.2.17 Definition. *Let $\Omega \subset \mathrm{M}$ be open and geodesically starshaped with respect to $x \in \Omega$. We define*

$$\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R},$$

where $\gamma : \mathrm{T}_x\mathrm{M} \to \mathbb{R}$ is defined in Equation (2.1).

2.2.1 Causality and Global Hyperbolicity

Now we introduce causal domains, because they will appear in the theory of wave equations. The local construction of fundamental solutions is always possible on causal domains, provided they are small enough.

2.2.18 Definition. *A domain $\Omega \subset \mathrm{M}$ is called* ***causal*** *if its closure $\overline{\Omega}$ is contained in a convex domain Ω' and for any $p, q \in \overline{\Omega}$ $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.*
A subset $A \subset \mathrm{M}$ is called ***past-compact*** *(respectively* ***future-compact****) if, for all $p \in \mathrm{M}$, $A \cap J_-^{\mathrm{M}}(p)$ (respectively $A \cap J_+^{\mathrm{M}}(p)$) is compact.*

We can notice that, if we look at compact spacetimes, something physically unsound occurs:

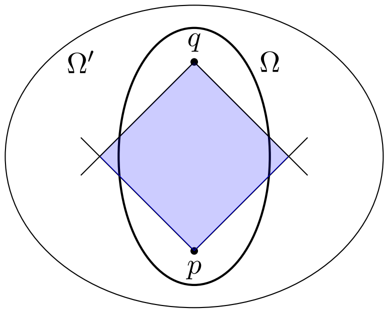


Figure 2.10: Convex, but non causal, domain

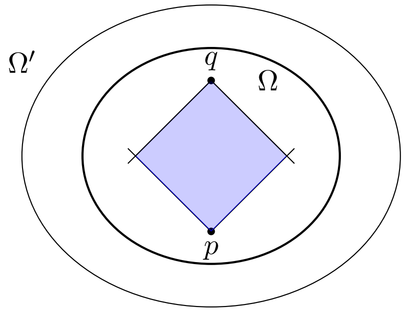


Figure 2.11: Causal domain

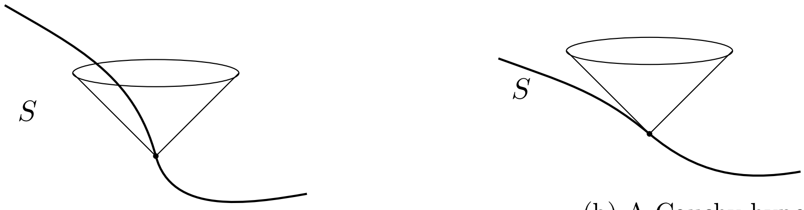
2.2.19 Proposition. *If a spacetime* M *is compact, there exists a closed timelike curve in* M.

In a few words, there are manifolds, such as compact spacetimes, where there are timelike loops that can produce science fictional paradoxes. To avoid such unphysical and unrealistic things we require suitable causality conditions:

2.2.20 Definition. *A spacetime satisfies the* ***causality condition*** *if it does not contain any closed causal curve. A spacetime* M *satisfies the* ***strong causality condition*** *if there are no almost closed causal curves, i.e. if for any* $p \in \mathrm{M}$ *there exists a neighborhood* U *of* p *such that there exists no timelike curve that passes through* U *more than once.*

It is clear that the strong causality condition implies the causality condition.

2.2.21 Definition. *A spacetime* M *that satisfies the strong causality condition and such that for all* $p, q \in \mathrm{M}$ $J^{\mathrm{M}}_{+}(p) \cap J^{\mathrm{M}}_{-}(q)$ *is compact is called* ***globally***



(a) A non-Cauchy hypersurface

(b) A Cauchy hypersurface

Figure 2.12: Hypersurfaces.

hyperbolic.

It can be demonstrated that in globally hyperbolic manifolds for any $p \in \mathrm{M}$ and any compact set $K \subset \mathrm{M}$ the sets $J^{\mathrm{M}}_{\pm}(p)$ and $J^{\mathrm{M}}_{\pm}(K)$ are closed.

2.2.22 Definition. *A subset S of a connected time-oriented Lorentzian manifold* M *is a* ***Cauchy hypersurface*** *if each inextensible timelike curve (i.e. no reparametrisation of the curve can be continuously extended) in* M *meets S at exactly one point.*

In other words, no point of a Cauchy hypersurface is in the light cone of another point of the surface.

2.2.23 Theorem. *Let* M *be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

- M *is globally hyperbolic.*
- *There exists a Cauchy hypersurface in* M.
- M *is isometric to* $\mathbb{R} \times S$ *with metric* $g = -\beta \mathrm{d}t^2 + b_t$, *where β is a smooth positive function, b_t is a Riemannian metric on S depending smoothly on t and each $\{t\} \times S$ is a smooth Cauchy hypersurface in* M.

In such case there exists a smooth function $h : \mathrm{M} \to \mathbb{R}$ whose gradient is past-directed timelike at every point and all of whose level sets are Cauchy hypersurfaces.

2.2.24 Example. The Minkowski spacetime $\mathbb{M}^n$ is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. We have $\mathbb{M}^n = \mathbb{R} \times S$ with $S = \mathbb{R}^{n1}$, endowed with the time-independent Euclidean metric.

Let S be a Riemannian manifold with metric b and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $\mathrm{M} = \mathrm{Im} \ \times S$ with the metric $g = dt^2 + f(t)^2 b$, called **cosmological spacetime**, is globally hyperbolic if and only if S is a complete space, see [2, Lem A.5.14]. This applies in particular if S is compact.

2.3 Operators and integration on manifolds

We call $C_0^\infty(\mathrm{M})$ the set of C^∞ functions on a manifold with compact support. The integral map is defined as the unique map

$$\int_{\mathrm{M}} \cdot \, \mathrm{d}\mu : C_0^\infty(\mathrm{M}) \to \mathbb{C},$$

such that it is linear and for any local chart $\{U, \varphi\}$ and for any $f \in C_0^\infty(U)$ holds

$$\int_{\mathrm{M}} f \, \mathrm{d}\mu = \int_{\varphi(U)} (f \circ \varphi^{-1})(x) \, \mu_x \, \mathrm{d}x,$$

where we define

$$\mu_x := |\det g(x)|^{1/2}. \tag{2.4}$$

In this section we introduce the **generalized d'Alembert** operators, whose general form in local coordinates is given by

$$P = -g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + a_j(x) \frac{\partial}{\partial x^j} + b(x). \tag{2.5}$$

The d'Alembert wave operator $\Box$ is defined for smooth functions f as

$$\Box f = -\operatorname{div} \operatorname{grad} f,$$

where $\operatorname{grad} f$ is a vector field defined by the requirement that the formula

$$\langle \operatorname{grad} f, X \rangle = \partial_X f$$

holds for any vector field X. At the same time div is defined as follows

2.3.1 Definition. *The* ***divergence*** *of a vector field $Z = Z^i \partial_i$ is defined as*

$$\operatorname{div} Z = \sum_j (\nabla_j Z)_j = \partial_j Z^j + \Gamma^i_{ij} Z^j.$$

2.3.2 Proposition. *The following formula holds:*

$$\operatorname{div} Z = \mu_x^{-1} \frac{\partial}{\partial x^j} \left(\mu_x Z^j\right), \tag{2.6}$$

and the definition of divergence is consistent with that of integral.

Proof. Let $h \in C_0^\infty(\mathrm{M})$; using integration by parts

$$\int_{\mathrm{M}} h \cdot \operatorname{div}(Z) \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mathrm{d}\mu = -\int_{\mathrm{M}} Z^j \partial_j h \, \mu_x \, \mathrm{d}x.$$

Now integrating by parts again in the chart it holds:

$$-\int_{\mathrm{M}} \partial_j h \; Z^j \mu_x \, \mathrm{d}x = \int_{\mathrm{M}} h \; \partial_j(\mu_x \, Z^j) \, \mathrm{d}x = \int_{\mathrm{M}} h \; \mu_x^{-1} \partial_j(\mu_x \; Z^j) \, \mathrm{d}\mu.$$

Since this is true for any function h, the formula is proven. ∎

From the definition of gradient one can show

$$g_{ij} (\operatorname{grad} f)^i X^j = \partial_X f = X^j \frac{\partial f}{\partial x^j}, \quad \operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \partial_j.$$

Hence it holds

$$\Box f = -\mu_x^{-1} \frac{\partial}{\partial x^j} \left(\mu_x \, g^{ij} \frac{\partial f}{\partial x^i} \right).$$

In Minkowski spacetime, where $g = \eta$,

$$\Box f = -\frac{\partial}{\partial x^j} \left(\eta^{jj} \frac{\partial f}{\partial x^j} \right) = \frac{\partial^2 f}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial x_i^2} = -\partial^i \partial_i f.$$

Fundamental solutions in Minkowski spacetime 3

In this chapter, we illustrate the concept of fundamental solutions and their use in solving initial value problems. We focus on the case of the d'Alembert wave operator $\Box$ built on top of the n-dimensional Minkowski spacetime $\mathbb{M}^n$. Two different approaches will be followed. The first relies on the Fourier transform. It is useful to build explicit formulas for the fundamental solutions in the lower dimensional cases and to show the existence of such solutions in the general case. The second approach, via Riesz distributions, is useful for its generality and because it will be used in the next chapter to construct fundamental solutions on suitable Lorentzian manifolds, although it is more abstract and does not lead to explicit formulas.
We will find two independent fundamental solutions, the *retarded* and the *advanced* one, that preserve the causal structure of the spacetime, i.e. that propagate the source of the equation respectively in the causal future and in the causal past, in accordance with the causality principle.

3.1 Fundamental solutions

3.1.1 Definition. *Let P be a differential operator on a manifold* M *and* $x_0 \in \mathrm{M}$. *A* ***fundamental solution*** *for P at x_0 is a distribution $u_{x_0} \in \mathcal{D}'(\mathrm{M})$ such that*

$$Pu_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac delta distribution in x_0, i.e. $(\delta_{x_0}, f) = f(x_0)$ for all $f \in \mathcal{D}(\mathrm{M})$.

A fundamental solution $u_x \in \mathcal{D}'(\mathrm{M})$, with $x \in \mathrm{M}$ defines a distributional kernel $u \in \mathcal{D}'(\mathrm{M} \times \mathrm{M})$??????

Under the assumptions of the previous definition, the distribution $F_\psi(x) = (u_x, \psi)$, where u_x is a fundamental solution of P at x with a continuous dependence on x (i.e. the function $x \mapsto (u_x, \varphi)$ is continuous for all $\varphi \in \mathcal{D}(\mathrm{M})$) and $\psi \in \mathcal{D}'(\mathrm{M})$, is a solution for the differential equation

$$PF_\psi = \psi.$$

This comes applying the operator P on F_ψ, for which one obtains

$$PF_\psi = (u_x, P^*\psi) = (Pu_x, \psi) = (\delta_x, \psi) = \psi(x),$$

where P^* stands for the formal adjoint of P.

3.2 The d'Alembert wave operator in Minkowski

In order to be more concrete, we begin by computing the fundamental solution of $\Box$ in $\mathbb{M}^n$ with $2 \leq n \leq 4$.
As we recalled in the previous chapter, the d'Alembert wave operator is defined in $\mathbb{M}^n$, with respect to the variables $x = (t, \mathbf{x})$, as

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} = -\partial^i \partial_i.$$

We will find two fundamental solutions $G^+_{x_0}, G^-_{x_0} \in \mathcal{D}'(\mathbb{M}^n)$ for $\Box$ at $x_0 \in \mathbb{M}^n$ with the following properties

$$\operatorname{supp}(G^+_{x_0}) \subset J_+(x_0), \qquad \operatorname{supp}(G^-_{x_0}) \subset J_-(x_0). \tag{3.1}$$

Such solutions will be called respectively **retarded** (G_+) and **advanced** (G_-) fundamental solutions.
In order to find the fundamental solutions at x_0, the following proposition guarantees it suffices to solve the problem $\Box u_0 = \delta_0$.

3.2.1 Proposition. *Let $x_0 \in \mathbb{M}^n$ and T_{x_0} be the translation operator as in Definition A.0.3. Then $[\Box, T_{x_0}] = 0$ (i.e. $\Box$ and T_{x_0} commute) and a fundamental solution for $\Box$ at x_0 is*

$$u_{x_0} = T_{x_0} u_0,$$

where u_0 is a fundamental solution at 0.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{M}^n)$, then

$$(\Box T_{x_0} u, \varphi(x)) = (T_{x_0} u, \Box\varphi(x)) = (u, \Box\varphi(x + x_0)) =$$

$$= (\Box u, \varphi(x + x_0)) = (T_{x_0} \Box u, \varphi(x)) \, ,$$

where it was used the fact that $\Box$ is formally self-adjoint and invariant under translations when acting on smooth functions. Hence, it holds

$$\Box \left(T_{x_0} u_0\right) = T_{x_0} \left(\Box u_0\right) = T_{x_0} \delta_0 = \delta_{x_0},$$

because $\Box$ and T_{x_0} commute, so $T_{x_0} u_0$ is a fundamental solution at x_0. ■

3.2.2 Proposition. *Let $\psi \in \mathcal{D}(\mathbb{M}^n)$. Then*

$$F_\psi = u_0 * \psi \in C^\infty(\mathbb{M}^n) \tag{3.2}$$

is a (smooth) solution for the differential equation $PF_\psi = \psi$ (here $*$ denotes convolution).

Proof. Since $u_x = T_x u_0$ and $F_\psi = (u_x, \psi) = (T_x u_0, \psi)$ is a solution to the equation, the thesis follows immediately noting that, by definition, $(T_x u_0, \psi) = (u_0 * \psi)(x)$. The smoothness of the solution follows from Theorem A.0.7. ■

3.3 The Fourier transform approach

In order to employ the theory of Fourier transforms, it is necessary to work with distributions in $\mathcal{S}'(\mathbb{M}^n)$.
We shall begin with a lemma that helps in the computations:

3.3.1 Lemma. *For $u \in \mathcal{S}'(\mathbb{M}^n)$ if we let $x = (t, \mathbf{x}) = (t, x_1, \ldots, x_{n-1})$ and $k = (\omega, \mathbf{k}) = (\omega, k_1, \ldots, k_{n-1})$, it holds*

$$\widehat{\Box\, u}(k) = \|k\|^2 \, \widehat{u} = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k). \tag{3.3}$$

Proof. For any test function $f \in \mathcal{S}(\mathbb{M}^n)$, and for any $u \in \mathcal{S}'(\mathbb{M}^n)$ $(\Box u, f) = (u, \Box f)$, from which it descends

$$(\Box u, e^{-i\langle k,x\rangle_0}) = \left(u, \Box e^{-i\langle k,x\rangle_0}\right) = \left(u, \Box e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\right) =$$

$$= \left(u, (|\mathbf{k}|^2 - \omega^2) e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2) \left(u, e^{-i\langle k,x\rangle_0}\right) = (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k).$$

■

We start by transforming the equation:

$$\widehat{\Box u}(k) = \widehat{\delta}(k) \Rightarrow (|\mathbf{k}|^2 - \omega^2)\widehat{u}(k) = 1 \tag{3.4}$$

The difference of two solutions $\widehat{u} \in \mathcal{S}'(\mathbb{M}^n)$, is a solution $\widehat{v}$ to the equation

$$(|\mathbf{k}|^2 - \omega^2)\widehat{v}(k) = 0. \tag{3.5}$$

This equation can be seen as the Fourier transform of the correspondent homogeneous equation $\Box v = 0$. Hence $\widehat{u} + \widehat{v}$ is the transform of the sum of a particular solution for $\Box$ and a solution of the homogeneous equation, then it is a solution for Equation (3.4).

If we concentrate on the solutions for Equation (3.5), it is easy to see with a direct computation that any distribution of the form

$$\widehat{v}(k) = A(k)\delta(|\mathbf{k}|^2 - \omega^2),$$

where $A(k)$ is any suitably regular function of k, solves the equation, because the Dirac delta is supported on $\{k|\, |\mathbf{k}|^2 - \omega^2 = 0\}$. Any solution to the homogeneous wave equation can be obtained by the inverse transform of $\widehat{v}$:

$$v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0} A(k)\delta(|\mathbf{k}|^2 - \omega^2)\, \mathrm{d}k.$$

Making use of formula (A.1), the last expression becomes

$$\begin{aligned} v(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle k,x\rangle_0} \frac{A(k)}{2|\mathbf{k}|} \left[\delta(|\mathbf{k}| - \omega) + \delta(|\mathbf{k}| + \omega)\right] \mathrm{d}k = \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{e^{i|\mathbf{k}|t} A(k)|_{|\mathbf{k}|=\omega} + e^{-i|\mathbf{k}|t} A(k)|_{|\mathbf{k}|=-\omega}}{|\mathbf{k}|}. \end{aligned} \tag{3.6}$$

To solve for $\widehat{u}$ it is tempting to write

$$\widehat{u}(k) = \frac{1}{|\mathbf{k}|^2 - \omega^2} = \frac{1}{(|\mathbf{k}| - \omega)(|\mathbf{k}| + \omega)},$$

which is ill-defined as a distribution wherever $\langle k, k\rangle_0 = 0$, i.e. on the light-cone of the Fourier space. Hence, we will define $\widehat{u}$ as limit of a sequence of distributions depending on the parameter ε

$$\widehat{u}_\varepsilon = \frac{1}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} = \frac{1}{(|\mathbf{k}| \mp i\varepsilon - \omega)(|\mathbf{k}| \pm i\varepsilon + \omega)}$$

promoting ω to a complex variable and taking the limit for $\varepsilon \to 0^+$ after performing the inverse transform. The choice of the signs in such expressions leads to different fundamental solutions.

3.4 Fundamental solutions via Fourier transform

The distributions G_+ and G_- in $\mathcal{S}'(\mathbb{M}^n)$, defined respectively as the weak limit for $\varepsilon \to 0^+$ of

$$G_\varepsilon^+(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega + i\varepsilon)^2} \, \mathrm{d}k, \tag{3.7}$$

$$G_\varepsilon^-(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega - i\varepsilon)^2} \, \mathrm{d}k, \tag{3.8}$$

are respectively a **retarded** and an **advanced** fundamental solutions at $x_0 = 0$ for the d'Alembert wave operator.
The aim is to prove that $\mathrm{supp}(G_+) \subset J_+(0)$ and $\mathrm{supp}(G_-) \subset J_-(0)$, and we proceed firstly by calculating the explicit formula for $2 \leq n \leq 4$ and then discuss the general case via Riesz distributions.

We will show that, if we denote with $d := n-1$ the spatial dimensions, the retarded fundamental solutions $G^\pm_{(d)}$ turn out to be

$$G^+_{(1)}(t,x) = \frac{\Theta(t-|x|)}{2}, \quad G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi} \frac{\Theta(t^2 - |\mathbf{x}|^2)}{\sqrt{t^2 - |\mathbf{x}|^2}},$$

$$G^+_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{4\pi} \frac{\delta(t - |\mathbf{x}|)}{|\mathbf{x}|}.$$

We compute $G_\pm$ as a limit of the inverse of the Fourier transform:

$$G_\varepsilon^\pm(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\langle k,x\rangle_0}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}k$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \mathrm{d}\mathbf{k} \, e^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}\omega. \tag{3.9}$$

Computing the complex integrals

In order to calculate the inner integral in the former expression,

$$\tilde{G}_\varepsilon^\pm(t,\mathbf{k}) := \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega \pm i\varepsilon)^2} \, \mathrm{d}\omega,$$

we make use of techniques of complex analysis as follows.
Denote

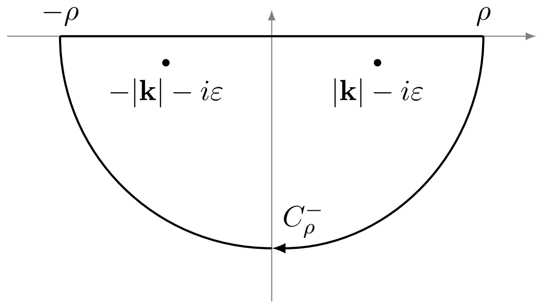


Figure 3.1: The circuit to compute $\tilde{G}^+_\varepsilon(t, \mathbf{k})$ for $t > 0$.

- C_ρ^+ the upper half-circle of radius ρ centered at $\omega = 0$ which has Im $(\omega) > 0$, oriented counter-clockwise;
- C_ρ^- the lower half-circle of radius ρ centered at $\omega = 0$ which has Im $(\omega) < 0$, oriented clockwise;
- $[-\rho, \rho]$ the interval of the real line connecting $-\rho$ and ρ, oriented from left to right.

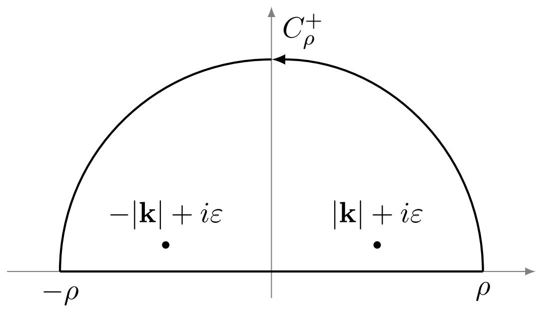


Figure 3.2: The circuit to compute $\tilde{G}^-_\varepsilon(t, \mathbf{k})$ for $t < 0$.

The singularities are

$$\text{for } \tilde{G}^+_\varepsilon : \quad \omega = \pm|\mathbf{k}| - i\varepsilon,$$
$$\text{for } \tilde{G}^-_\varepsilon : \quad \omega = \pm|\mathbf{k}| + i\varepsilon.$$

Hence we have

$$\text{for } t < 0 \quad \tilde{G}^+_\varepsilon(t, \mathbf{k}) = \lim_{\rho \to \infty} \int_{C_\rho^+ + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2 - (\omega + i\varepsilon)^2} \, \mathrm{d}\omega = 2\pi i \sum \mathrm{Res} = 0,$$

where the sum is extended to the singularities in the upper half-plane. The expression vanishes because we choose the circuit such that the integral on C_ρ^+ vanishes in virtue of Jordan's lemma and there are no singularities in the region bounded by the circuit.
For the same reasons

$$\text{for } t>0,\ \tilde{G}_\varepsilon^-(t,\mathbf{k}) = \lim_{\rho\to\infty}\int_{C_\rho^- + [-\rho,\rho]} \frac{e^{-i\omega t}}{|\mathbf{k}|^2-(\omega-i\varepsilon)^2}\,\mathrm{d}\omega = -2\pi i\sum \mathrm{Res} = 0,$$

where the sum is extended th the singularities of the function in the lower half-plane and the minus sign arises because of the clockwise circuit.
The non-zero integrals are

$$\tilde{G}_\varepsilon^+(t,\mathbf{k}),\ \text{for } t>0, \qquad \tilde{G}_\varepsilon^-(t,\mathbf{k}),\ \text{for } t<0.$$

The first is computed via the lower circuit in Figure (3.1): $C_\rho^- + [-\rho,\rho]$, the second via the upper counterpart in Figure (3.2): $C_\rho^+ + [-\rho,\rho]$, in order to get rid of contributes from the half-circles.

The results are

$$\text{for } t>0 \quad \tilde{G}_\varepsilon^+(t,\mathbf{k}) = 2\pi i e^{-\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = 2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t},$$

$$\text{for } t<0 \quad \tilde{G}_\varepsilon^-(t,\mathbf{k}) = -2\pi i e^{\varepsilon|\mathbf{k}|t}\left(\frac{e^{-i|\mathbf{k}|t}-e^{i|\mathbf{k}|t}}{2|\mathbf{k}|}\right) = -2\pi\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\varepsilon|\mathbf{k}|t}.$$

Summing up everything in one formula:

$$\tilde{G}_\varepsilon^\pm(t,\mathbf{k}) = \pm 2\pi\Theta(\pm t)\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{\mp\varepsilon|\mathbf{k}|t}, \tag{3.10}$$

where Θ is the Heaviside step-function.
It can be noticed that

$$\tilde{G}_\varepsilon^+(t,\mathbf{k}) = \tilde{G}_\varepsilon^-(-t,\mathbf{k}),$$

because of the parity of sine function. So, we can deduce that the **advanced** solution can be calculated from the retarded one via time inversion:

$$G_-(t,\mathbf{x}) = G_+(-t,\mathbf{x}). \tag{3.11}$$

We can now show that the support $G_\pm$ is included in $J_+(0)\cup J_-(0)$.

3.4.1 Proposition. *If $x \in \mathbb{M}^n$ satisfies $\gamma(x) < 0$ (where γ is defined in Definition 2.2.2), $G_\pm(x) = 0$.*

Proof. Consider a reference frame R in which $x = (t, \mathbf{x})$ and suppose for now $t \geq 0$. It descends $G_+(x) = 0$. Since $G_\pm$ are manifestly Lorentz invariant and $\gamma(x) < 0$, one can find a reference frame R' in which $x = (t', \mathbf{x}')$ and $t' < 0$, so that $G_-(x) = 0$. The converse can be treated similarly. ■

We can focus once more on equation (3.9) to show explicit solutions for spatial dimensions d ranging from 1 to 3:

$$G^+_{(d)}(x) = \frac{1}{(2\pi)^{d+1}} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{G}^+_\varepsilon(t, \mathbf{k}) =$$
$$= \lim_{\varepsilon\to 0^+} \frac{\Theta(t)}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} e^{-\varepsilon|\mathbf{k}|t}.$$

Dimension $n = 1 + 1$ - wave on a line

The integral we have to make is:

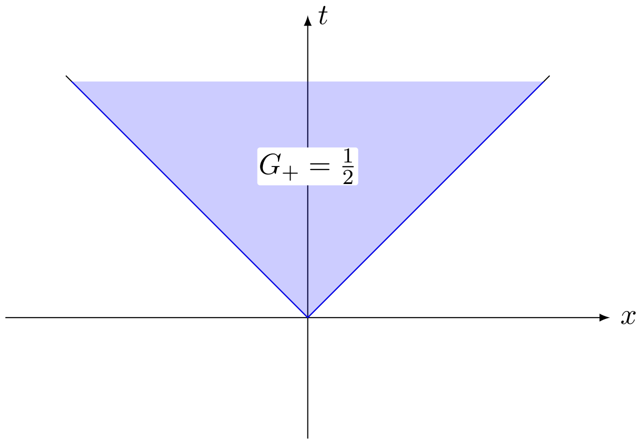


Figure 3.3: The support of G_+ in 1+1 dimensional case.

$$G^+_{(1)}(t, x) = \frac{\Theta(t)}{2\pi} \lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot x} \frac{\sin kt}{k} e^{-\varepsilon kt}.$$

It holds

$$\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x+i\varepsilon t)} \frac{\sin kt}{k} \overset{k\to k'=kt}{=} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ik\cdot(x/t+i\varepsilon)} \frac{\sin k}{k} \underset{\varepsilon\to 0^+}{\longrightarrow}$$

$$\longrightarrow \pi\chi_{[-1,1]}\left(\frac{x}{t}\right) = \pi\chi_{[-t,t]}(x),$$

where

$$\chi_{[a,b]}(z) = \begin{cases} 1, & \text{if } z \in [a,b] \\ 0, & \text{otherwise,} \end{cases}$$

Finally the integrals become

$$G^+_{(1)}(t,x) = \frac{\Theta(t)}{2}\chi_{[-t,t]}(x) = \frac{\Theta(t-|x|)}{2}.$$

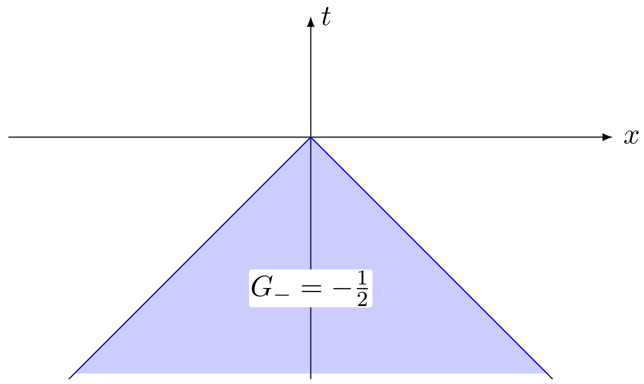


FIGURE 3.4: The support of G_- in 1+1 dimensional case.

From FIGURES (3.3) and (3.4) one can infer that the fundamental solutions are supported respectively on $J_+(0)$ and $J_-(0)$.

Dimension $n = 1+2$ - wave on a surface

The integral is two dimensional:

$$G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{(2\pi)^2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t}.$$

To evaluate it we switch to polar coordinates $\mathbf{k} = (k\cos\varphi, k\sin\varphi)$. With the integral measure

$$\mathrm{d}\mathbf{k} = k\,\mathrm{d}k\,\mathrm{d}\varphi,$$

the integral becomes ($x := |\mathbf{x}|$)

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{+\infty}\mathrm{d}k\, e^{ikx\cos\varphi}\sin kt\, e^{-\varepsilon kt}.$$

It holds that

$$\int_0^\infty \mathrm{d}k\, e^{ik(y+i\varepsilon)} \sin kt = \frac{1}{2i}\left[\int_0^{+\infty} e^{ik(y+i\varepsilon+t)}\,\mathrm{d}k + \int_0^{+\infty} e^{ik(y+i\varepsilon-t)}\,\mathrm{d}k\right] =$$

$$= \frac{1}{2}\left[I_\varepsilon(y+t) + I_\varepsilon(y-t)\right],$$

where we set $I_\varepsilon(y) := \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k$. Since

$$I_\varepsilon(y) = \frac{1}{i}\int_0^{+\infty} e^{ik(y+i\varepsilon)}\,\mathrm{d}k = \frac{1}{y+i\varepsilon},$$

the integral which needs to be evaluated is

$$\frac{1}{2}\int_0^{2\pi}\left[\frac{1}{x\cos\varphi + t + i\varepsilon} + \frac{1}{x\cos\varphi - t + i\varepsilon}\right]\mathrm{d}\varphi.$$

Such integral has a counterpart over a unit circle in the complex plane with the substitutions $\mathrm{d}\varphi = -i\mathrm{d}z/z$ and $\cos\varphi = (z+z^{-1})/2$. Hence, using Cauchy residue theorem

$$\int_0^{2\pi} \frac{1}{x\cos\varphi \pm t + i\varepsilon}\,\mathrm{d}\varphi = -2i\oint \frac{\mathrm{d}z}{xz^2 + 2(\pm t + i\varepsilon) + x} = \frac{2\pi}{\sqrt{(t\mp i\varepsilon)^2 - x^2}}.$$

Putting everything together in the weak limit $\varepsilon \to 0$ it holds

$$G^+_{(2)}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi}\frac{\Theta(t^2-|\mathbf{x}|^2)}{\sqrt{t^2-|\mathbf{x}|^2}} = \frac{\Theta(t)}{2\pi}\frac{\Theta(\gamma(x))}{\sqrt{\gamma(x)}}, \tag{3.12}$$

where $\Theta(t^2-|\mathbf{x}|^2)$ stems from Proposition 3.4.1. As a by-product, $\mathrm{supp}(G_\pm) \subseteq J_\pm(0)$.

Dimension $n = 1+3$ - spherical wave

The three-dimensional integral is

$$G^+_{(3)}(t,\mathbf{x}) = \frac{\Theta(t)}{(2\pi)^3}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\sin|\mathbf{k}|t}{|\mathbf{k}|}e^{-\varepsilon|\mathbf{k}|t}.$$

Again, we make a change of coordinate, switching to the spherical ones: $\mathbf{k} = (k\sin\vartheta\cos\varphi, k\sin\vartheta\sin\varphi, k\cos\vartheta)$. The integral measure reads

$$\mathrm{d}\mathbf{k} = k^2\sin\vartheta\,\mathrm{d}k\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

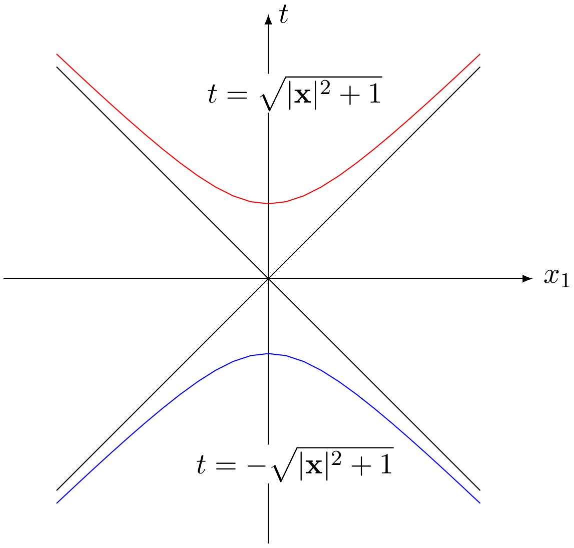


FIGURE 3.5: The level set $G_{\pm}(\mathbf{x}, t) = 1$ in the 1+2 dimensional case, plotted for one spatial axis.

and the integral to calculate is ($x := |\mathbf{x}|$)

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{+\infty} \mathrm{d}k\, k \sin kt \int_{-1}^{1} e^{ikx\cos\vartheta} \mathrm{d}(\cos\vartheta) = \frac{4\pi}{x + i\varepsilon} \int_0^{+\infty} \sin kt \sin k(x+i\varepsilon)\, \mathrm{d}k.$$

Hence we can write using the exponential function

$$\sin kt \sin kx = \frac{1}{4}\left\{\left[e^{ik(x+i\varepsilon-t)} + e^{-ik(x+i\varepsilon-t)}\right] - \left[e^{ik(x+i\varepsilon+t)} + e^{-ik(x+i\varepsilon+t)}\right]\right\},$$

and with the change of variables $k \leftrightarrow -k$ it holds

$$\frac{4\pi}{x} \int_0^{+\infty} \sin kt \sin kx\, \mathrm{d}k = \frac{2\pi^2}{x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x+i\varepsilon-t)} - e^{ik(x+i\varepsilon+t)}\, \mathrm{d}k \underset{\varepsilon\to 0^+}{\longrightarrow}$$

$$\longrightarrow \frac{2\pi^2}{x}\left[\delta(t-x) - \delta(t+x)\right].$$

To find the correct retarded and advanced fundamental solutions we notice that the second term, $\delta(t+x)$, vanishes for G_+ because $x > 0$ and $t > 0$. Conversely the first term $\delta(t-x)$ vanishes when computing G_-. In view of these considerations, the general formula for the $1+3$-case becomes

$$G^+_{(3)}(t, \mathbf{x}) = \frac{\Theta(t)}{4\pi} \frac{\delta(t - |\mathbf{x}|)}{|\mathbf{x}|}. \tag{3.13}$$

One can verify that $G^{\pm}_{(3)}$ vanish outside the support of the delta distribution. Hence they are supported respectively on the upper and on the lower light cones $C_+(0)$ and $C_-(0)$. This is a particularity of the odd spatial dimensions, as we shall prove in the next section. This feature is known as the **Huygens' principle**. It states that in general, we have for spatial dimensions $d \neq 1$

$$\operatorname{supp} G^{\pm}_{(d)} = J_{\pm}(0) \quad \text{for } d \text{ even},$$

$$\operatorname{supp} G^{\pm}_{(d)} = C_{\pm}(0) \quad \text{for } d \text{ odd}.$$

Physically, we can see δ_0 as a point source at 0 of a signal that propagates with constant speed. Inside the future light cone the solution is zero, so if d is even, the wave propagates strictly on the cone. In case d is odd, the signal of a point source propagates also inside the light cone. For an observer, the wave is noticeable not only at a single moment but still after the signal has arrived. An example of such waves are the 2-dimensional ones like water waves.

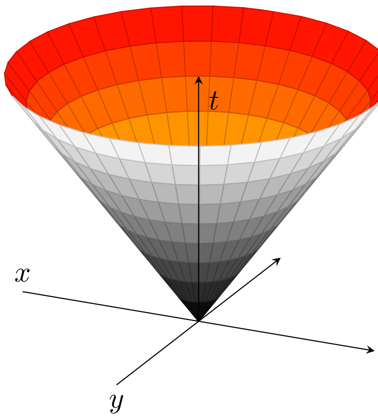


FIGURE 3.6: The support of G_+ in 1+3 dimensional case, i.e. the upper light cone $C_+(0)$, plotted for two spatial axis.

The method of *descent*

The lower dimensional advanced and retarded distributions can be directly deduced from the $d = 3$ case as we shall see. In general if we know the explicit solution to the d-dimensional case we can find the $d-1$-dimensional counterpart with the formula

$$G^{\pm}_{(d-1)}(t, x_1, \ldots, x_{d-1}) = \int_{-\infty}^{\infty} G^{\pm}_{(d)}(t, x_1, \ldots, x_d)\, \mathrm{d}x_d.$$

This technique is called *method of descent*. The last assertion can be proven taking the fundamental solution equation

$$\Box_d \, G_{(d)}(t, x_1, \ldots, x_d) = \delta(t)\delta(x_1) \cdots \delta(x_d),$$

where $\Box_d$ stands for $(\partial_t^2 - \partial_1^2 - \cdots - \partial_d^2)$. Integrating $G_{(d)}$ on the last variable against a test-function $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\int \Box_d \, G_{(d)} \, \varphi \, \mathrm{d}x_d = \Box_{(d-1)} \int G_{(d)} \, \varphi \, \mathrm{d}x_d - \int \partial_d^2 G_{(d)} \, \varphi \, \mathrm{d}x_d =$$

$$= \delta(t)\delta(x_1) \cdots \delta(x_{d-1}) \int \delta(x_d)\varphi \, \mathrm{d}x_d.$$

By letting the test-function become a sequence of cut-off functions covering the real axis, the formula

$$\Box_{(d-1)} \int G_{(d)} \, \mathrm{d}x_d = \delta(t)\delta(x_1) \cdots \delta(x_{d-1})$$

is proven.

To calculate the $d = 2$ fundamental solution from the $d = 3$ case, according to (3.13) and using Equation (A.1) we can write for the retarded fundamental solution

$$\Theta(t)\frac{\delta(t - |\mathbf{x}|)}{2|x|} = \Theta(t)\delta(t^2 - |\mathbf{x}|^2) = \Theta(t)\delta(t^2 - x_1^2 - x_2^2 - x_3^2) =$$

$$= \Theta(t)\Theta(t^2 - x_1^2 - x_2^2)\frac{\delta(x_3 - \sqrt{t^2 - x_1^2 - x_2^2}) + \delta(x_3 + \sqrt{t^2 - x_1^2 - x_2^2})}{2\sqrt{t^2 - x_1^2 - x_2^2}},$$

where we insert the Heaviside step function to take into account Proposition 3.4.1. Hence, integrating over the third variable yields

$$G^+_{(2)}(t, x_1, x_2) = \frac{\Theta(t)}{2\pi} \frac{\Theta(t^2 - x_1^2 - x_2^2)}{\sqrt{t^2 - x_1^2 - x_2^2}},$$

which is identical to Equation (3.12) as we expected. Similarly, the expression for the case $d = 1$ can be derived again.

If we now apply to the general expression for $G_{(d)}$ the Fourier transform in the last variable, similarly we obtain

$$\int_{-\infty}^{+\infty} \partial_d^2 G_{(d)}(t, x_1, \ldots, x_d) e^{-imx_d} \, \mathrm{d}x_d =$$

$$= -m^2 \int_{-\infty}^{+\infty} G_{(d)}(t, x_1, \ldots, x_d) e^{-imx_d} \, \mathrm{d}x_d =: -m^2 \widehat{G}_{d-1}(t, x_1, \ldots, x_{d-1}, m).$$

Hence, the Fourier transform on the last variable $\widehat{G}_{d-1}(t, x_1, \ldots, x_{d-1}, m)$ is, for a fixed $m \in \mathbb{R}$, a fundamental solution for the $d-1$ dimensional **Klein-Gordon** operator

$$\Box + m^2,$$

that describes the motion of spinless particles with mass m.

3.5 The Riesz distributions

To discuss explicit and useful formulas for the fundamental solution in the general n-dimensional case, the approach we adopted in the last section is not very effective. We outline a method devised by M. Riesz in the first half of the 20th century in order to find solutions to a certain class of differential equations.

3.5.1 Definition. *For $\alpha \in \mathbb{C}$ with Re $\alpha > n$ let $R_\pm(\alpha)$ be the complex-valued continuous functions defined for any $x \in \mathbb{M}^n$ by*

$$R_\pm(\alpha)(x) := \begin{cases} C(\alpha, n)\gamma(x)^{\frac{\alpha-n}{2}} & \text{if } x \in J_\pm(0) \\ 0 & \text{otherwise,} \end{cases} \tag{3.14}$$

where γ is defined in Definition 2.2.2,

$$C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1)},$$

and $z \mapsto \Gamma(z)$ is the Gamma function.

3.5.2 Remark. The functions $R_\pm(\alpha)$ are continuous because $\gamma = -\langle \cdot, \cdot \rangle_0$ vanishes on the boundary of $J_\pm(0)$ and the exponent $(\alpha - n)/2$ is assumed to have positive real part. If we increase the real part of the exponent then higher derivatives of the function vanishes at the boundary and the functions become more regular. As a matter of facts $R_\pm(\alpha) \in C^k(\mathbb{M}^n)$ whenever Re $\alpha > n + 2k$.

Now we discuss the first properties of $R_\pm(\alpha)$.

3.5.3 Proposition. *For all $\alpha \in \mathbb{C}$ with Re $\alpha > n$ it holds*

(1) $\gamma\, R_\pm(\alpha) = \alpha(\alpha - n + 2) R_\pm(\alpha + 2)$;

(2) $(\operatorname{grad} \gamma)\, R_\pm(\alpha) = 2\alpha \operatorname{grad} R_\pm(\alpha + 2)$;

(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.

Moreover, the map

$$\begin{aligned} \mathbb{X}_n &\to \mathcal{D}'(\mathbb{M}^n) \\ \alpha &\mapsto R_\pm(\alpha) \end{aligned}$$

(where $\mathbb{X}_n := \{\alpha \in \mathbb{C} \,|\, Re\ \alpha > n\}$) extends uniquely the whole complex plain as a holomorphic family of distributions, i.e. for each test-function $\varphi \in \mathcal{D}(\mathbb{M}^n)$, the function $\alpha \mapsto (R_\pm(\alpha), \varphi)$ is holomorphic.

Proof. To prove (1), we evaluate $\gamma\, R_\pm(\alpha)$ inside $J_\pm(0)$, because both sides of the equation vanish outside. By definition one has

$$\gamma\, R_\pm(\alpha) = C(\alpha, n)\gamma(x)^{\frac{\alpha+2-n}{2}} = \frac{C(\alpha, n)}{C(\alpha + 2, n)} R_\pm(\alpha + 2),$$

and, in virtue of the fact that $z\Gamma(z - 1) = \Gamma(z)$,

$$\frac{C(\alpha, n)}{C(\alpha + 2, n)} = \frac{2^{1-\alpha}\pi^{\frac{2-n}{n}}}{(\frac{\alpha}{2} - 1)!(\frac{\alpha-n}{2})!} \frac{(\frac{\alpha+2}{2} - 1)!(\frac{\alpha+2-n}{2})!}{2^{1-\alpha-2}\pi^{\frac{2-n}{n}}} =$$

$$= 4\frac{\alpha}{2}\frac{\alpha + 2 - n}{2} = \alpha(\alpha - n + 2).$$

For the second identity we evaluate $\partial_i\gamma \cdot R_\pm(\alpha)$. In view of Remark 3.5.2, $R_\pm(\alpha + 2) \in C^1(\mathbb{M}^n)$ For any φ integrating by parts yields:

$$\begin{aligned} \partial_i\gamma \cdot (R_\pm(\alpha), \varphi) &= C(\alpha, n) \int_{J_\pm} \gamma(x)^{\frac{\alpha-n}{2}} \partial_i\gamma(x)\varphi(x)\, \mathrm{d}x \\ &= \frac{2C(\alpha, n)}{\alpha + 2 - n} \int_{J_\pm} \partial_i(\gamma(x))^{\frac{\alpha-n+2}{2}} \varphi(x)\, \mathrm{d}x \\ &= -2C(\alpha + 2, n) \int_{J_\pm} \gamma(x)^{\frac{\alpha-n+2}{2}} \partial_i\varphi(x)\, \mathrm{d}x \\ &= -2\alpha(R_\pm(\alpha + 2), \partial_i\varphi) \\ &= 2\alpha(\partial_i R_\pm(\alpha), \varphi). \end{aligned}$$

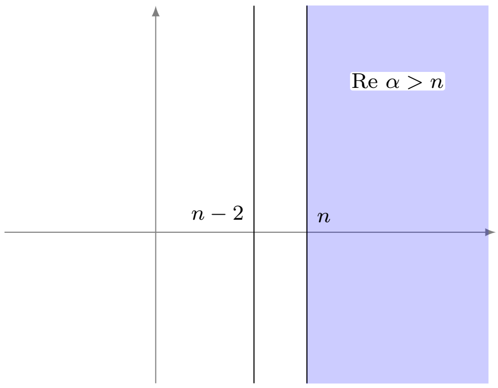


FIGURE 3.7: Iterative extension of $R_{\pm}(\alpha)$ on $\mathbb{C}$.

To prove the third formula, from (2) we have

$$\begin{aligned}\partial_i^2 R_{\pm}(\alpha+2) &= \partial_i\left(\frac{1}{2\alpha}\partial_i\gamma\cdot R_{\pm}(\alpha)\right)\\ &= \frac{1}{2\alpha}\partial_i^2\gamma\cdot R_{\pm}(\alpha)+\frac{1}{4\alpha(\alpha-2)}(\partial_i\gamma)^2\frac{(\alpha-2)(\alpha-n)}{\gamma}\cdot R_{\pm}(\alpha)\\ &= \left(\frac{1}{2\alpha}\partial_i^2\gamma+\frac{\alpha-n}{4\alpha}\cdot\frac{(\partial_i\gamma)^2}{\gamma}\right)\cdot R_{\pm}(\alpha).\end{aligned}$$

Applying $\Box$ we find

$$\Box R_{\pm}(\alpha+2)=\left(\frac{n}{\alpha}+\frac{\alpha-n}{4\alpha}\cdot\frac{4\gamma}{\gamma}\right)R_{\pm}(\alpha)=R_{\pm}(\alpha),$$

as claimed.
The last identity allows us to extend $R_{\pm}(\alpha)$ for every $\alpha\in\mathbb{C}$. For Re $\alpha>n-2$ we set

$$\widetilde{R}_{\pm}(\alpha):=\Box R_{\pm}(\alpha+2),$$

and the extension is holomorphic on $\mathbb{X}_{n-2}$. Now, proceeding by induction over n one can extend the function over the whole complex plane. ■

3.5.4 Definition. *The distributions $R_+(\alpha)$ and $R_-(\alpha)$ defined in the last proposition are called respectively the* ***retarded*** *and* ***advanced*** *Riesz distributions for the parameter $\alpha\in\mathbb{C}$.*

The Riesz distributions do not have an immediate explicit formula, but next lemma shows a more comfortable way to evaluate them when the test-function has a particular form.

3.5.5 Lemma. *Denote* $x = (t, \mathbf{x})$, *with* $\mathbf{x} \in \mathbb{R}^{n-1}$. *Let* $f \in \mathcal{D}(\mathbb{R})$ *and let* $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$ *be such that* $\varphi(x) := f(t)\psi(\mathbf{x}) \in \mathcal{D}(\mathbb{M}^n)$ *and* $\varphi(x) = f(t)$ *on* $J_+(0)$. *If Re* $\alpha > 1$ *it holds*

$$(R_\pm(\alpha), \varphi) = \frac{1}{(\alpha-1)!} \int_0^{+\infty} t^{\alpha-1} f(t)\, \mathrm{d}t.$$

To link Riesz distributions to fundamental solutions the following facts are noteworthy.

3.5.6 Proposition. *The Riesz distributions satisfy*

(1) for any $\alpha \in \mathbb{C}$, *supp* $R_\pm(\alpha) \subset J_\pm(0)$

(2) $R_\pm(0) = \delta_0$

(3) $\Box R_\pm(2) = \delta_0$, *in particular* $R_+(2)$ *and* $R_-(2)$ *are respectively a* ***retarded*** *and an* ***advanced*** *fundamental solution for* $\Box$ *at* 0.

Proof. The first assertion descends from the definition of Riesz distributions. To prove (2) fix $K \subset \mathbb{M}^n$ compact subset. Let $\sigma_K \in \mathcal{D}(\mathbb{M}^n)$ such that $\sigma_K|_K = 1$. For any $\varphi \in \mathcal{D}(\mathbb{M}^n)$ with $\operatorname{supp}\varphi \subset K$ one finds suitable smooth functions φ_j such that

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x^j \varphi_j(x),$$

then it holds

$$\begin{aligned}
(R_\pm(0), \varphi) &= (R_\pm(0), \sigma_K \varphi) \\
&= \left(R_\pm(0), \varphi(0) + \sum_{j=1}^{n} x^j \varphi_j(x) \right) \\
&= \varphi(0) \overbrace{(R_\pm(0), \sigma_K)}^{=:c_K} + \sum_{j=1}^{n} \left(\overbrace{x^j R_\pm(0)}^{=0}, \sigma_K \varphi_j \right) \\
&= c_K \varphi(0),
\end{aligned}$$

where $x^j R_\pm(0)$ vanishes because of Equation (1) in Proposition 3.5.3 one can show that c_K does not depend on the choice of K since for $K' \supset K$ and

$\operatorname{supp}\varphi \subset K \subset K'$,

$$c'_K\varphi(0) = (R_+(0), \varphi) = c_K\varphi(0).$$

It descends $c_K = c'_K =: c$. To show $c = 1$, concentrating on the case of a retarded distribution, using test-functions as in Lemma 3.5.5,

$$\begin{aligned} c \cdot \varphi(0) &= (R_+(0), \varphi) \\ &= (\Box R_+(2), \varphi) = (R_+(2), \Box\varphi) \\ &= \int_0^{+\infty} t\, f''(t)\, \mathrm{d}t = -\int_0^{+\infty} f'(t)\, \mathrm{d}t \\ &= f(0) = \varphi(0), \end{aligned}$$

which concludes the proof.
The third assertion is obtained considering (1) and making use of Equation (3) in Proposition 3.5.3. ■

3.5.7 Remark. We will prove in Chapter 4 that the retarded and the advanced fundamental solutions are **unique**. Hence, we have

$$G_\pm = R_\pm(2).$$

3.5.8 Remark. As one expects, if $\alpha \in \mathbb{R}$, then $(R_\pm(\alpha), \varphi)$ is real for any $\varphi \in \mathcal{D}(\mathbb{M}^n, \mathbb{R})$ i.e. $R_\pm(\alpha)$ is a real-valued distribution.

We are now ready to prove **Huygens' principle**, that we already mentioned before. One can restate it as follows.

3.5.9 Theorem (Huygens' principle). *If $n \geq 4$ is even, $\operatorname{supp} G_\pm = C_\pm(0)$. If $n \geq 3$ is odd, $\operatorname{supp} G_\pm = J_\pm(0)$.*

To prove it we work with Riesz distributions and we need the following lemma

3.5.10 Lemma. *The following holds:*

(1) for every $\alpha \in \mathbb{C}\backslash(\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$ we have $\operatorname{supp} R_\pm(\alpha) = J_\pm(0)$;

(2) for $n \geq 3$ and $\alpha = n-2, n-4, \dots, 1$ if n is odd or $\alpha = n-2, n-4, \dots, 2$ if n is even, we have $\operatorname{supp} R_\pm(\alpha) = C_\pm(0)$.

Proof of Theorem 3.5.9. The fundamental solutions are $G^\pm = R_\pm(2)$, so $\alpha = 2$. Since $2 = (n-2)+(4-n)$, if $n \geq 4$ and n is even, $2 \in \{n-2, n-4, \dots\}$; conversely, if n is odd 2 is not in $\{n-2, n-4, \dots, 1\}$. So the theorem follows from the last lemma. ■

3.6 General solution and Cauchy problem

We found the fundamental solution for the d'Alembert wave operator for the point $x_0 = 0$. To find the generic solution at a point $y \in \mathbb{M}^n$, as we have seen in Proposition 3.2.2, it suffices to write

$$G_y^+(x) = T_y G_+ = G_+(x - y).$$

Hence, to find the retarded solution $u_+(x)$ to the wave equation $\Box u = \psi$, where ψ is a distribution, we simply evaluate the convolution $G_+ * \psi$. The general solution is obtained by adding the solutions of the homogeneous equation as in Formula (3.6).

Now we discuss the uniqueness of the distributional solution and for its regularity we refer to Proposition 3.2.2.
To begin, we shall prove the following

3.6.1 Theorem. *Let $\psi \in \mathcal{D}'(\mathbb{M}^n)$ such that $\psi(t, \mathbf{x}) = 0$ if $t < 0$. Then ψ and G_+ can be convoluted and $u_+ = G_+ * \psi$ is the unique solution to the wave equation with source ψ such that $u_+(t, \mathbf{x}) = 0$ for $t < 0$.*

Proof. At fixed x, the distribution $G_+(x-y)\psi(y)$ has compact support in the variable y, so G_+ and ψ can be convoluted. Since $\operatorname{supp} G_+ \subset J_+(0)$, $u_+ = 0$ for $t < 0$.
The solution is unique because if there were another u solving the equation and satisfying the requested conditions, then $\phi := u_+ - u$ would be a solution to the homogeneous equation, i.e. $\Box\phi = 0$, and ϕ could be convoluted with G_+:

$$\phi = \phi * \delta = \phi * \Box\, G_+ = \Box\, \phi * G_+ = 0,$$

hence $u = u_+$. ∎

Remaining in the Minkowski case, since $\mathbb{M}^n$ is a globally hyperbolic manifold, we can find smooth Cauchy hypersurfaces, where we can assign initial values. Physically, it is clear why these can only be specified on a spacelike surface. A wave cannot travel from one point of the initial value surface to another and change the initial conditions.

3.6.2 Definition. *Let S be a smooth Cauchy hypersurface of $\mathbb{M}^n$ with a timelike unit normal vector field $\nu : S \to \mathrm{T}\mathbb{M}^n$.*
*Given a triple $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$, we call a **classical Cauchy***

problem *for* $\Box$ *on* S *the system of equations*

$$\begin{cases} \Box u = \psi \\ \\ u|_S = u_0 \\ \\ \partial_\nu u|_S = u_1. \end{cases} \tag{3.15}$$

In case the triple is taken in $\mathcal{D}'(\mathbb{M}^n) \oplus \mathcal{D}'(S) \oplus \mathcal{D}'(S)$, *i.e. the data are distributions, the problem is called* ***generalized Cauchy problem****.*

For simplicity, we concentrate on the case where S is the hyperplane

$$S_0 := \{(t, \mathbf{x}) \in \mathrm{M} \,|\, t = 0\},$$

and $\partial_\nu = \partial_t$. The general case will be addressed later in Section 4.3, when we will discuss of Cauchy problem on manifolds. To solve the initial value problem, we start with a lemma

3.6.3 Lemma. *Suppose* u *is a solution for the Cauchy problem on* S_0. *If we set*

$$\tilde{u}(t, \mathbf{x}) := \begin{cases} u(t, \mathbf{x}) & \textit{if} \quad t > 0 \\ 0 & \textit{otherwise}, \end{cases}$$

$$\tilde{\psi}(t, \mathbf{x}) := \begin{cases} \psi(t, \mathbf{x}) & \textit{if} \quad t > 0 \\ 0 & \textit{otherwise}, \end{cases}$$

it holds

$$\Box\, \tilde{u}(t, \mathbf{x}) = \tilde{\psi}(t, \mathbf{x}) + u_0(\mathbf{x})\delta'(t) + u_1(\mathbf{x})\delta(t). \tag{3.16}$$

Proof. Let $\varphi \in \mathcal{D}'(\mathbb{M}^n)$, then

$$(\Box\tilde{u}, \varphi) = (\tilde{u}, \Box\varphi) = \int_0^\infty \mathrm{d}t \int u\, \Box\varphi\, \mathrm{d}\mathbf{x} =$$

integrating by parts in the time variable

$$= \int_0^\infty \mathrm{d}t \int \Box u\, \varphi\, \mathrm{d}\mathbf{x} + \int \partial_t u(0, \mathbf{x})\, \varphi(0, \mathbf{x}) - u(0, \mathbf{x})\, \partial_t\varphi(0, \mathbf{x})\, \mathrm{d}\mathbf{x} =$$

$$= \int \tilde{\psi} + u_1\delta(t) + u_0\delta'(t)\, \mathrm{d}t\, \mathrm{d}\mathbf{x}.$$

■

3.6.4 Theorem. *If $u_\pm \in C^\infty(\mathbb{M}^n)$ solves the Cauchy problem on S_0 and $\operatorname{supp} u_\pm \subset J_\pm(K)$, where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$, then it holds*

$$u_\pm = G_\pm * (\psi + u_0 \otimes \delta' + u_1 \otimes \delta). \tag{3.17}$$

This leads to the following Corollary, that will be addressed in the general setting in Section 4.3.

3.6.5 Corollary. *If $u \in C^\infty(\mathbb{M}^n)$ solves the Cauchy problem on S_0 with $Pu = 0$ (i.e. $\psi = 0$), then*

$$\operatorname{supp} u \subset J_+(K) \cup J_-(K),$$

where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1$, and a solution is given by

$$u = G * (u_0 \otimes \delta' + u_1 \otimes \delta) \tag{3.18}$$

for all $\varphi \in \mathcal{D}(\mathbb{M}^n)$, where $G = G_+ - G_-$.

Now follows the general result on S_0.

3.6.6 Theorem. *For each $(\psi, u_0, u_1) \in \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S_0) \oplus \mathcal{D}(S_0)$, there exists a unique solution u to the Cauchy problem on S_0. Furthermore $\operatorname{supp} u \subset J_+(K) \cup J_-(K)$, where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$.*

Proof. We will only prove the existence. To build it with initial values u_0, u_1 we need the sum of a particular solution and the solution for the homogeneous equation.
We start by noting that a particular solution of $\Box\widetilde{\phi} = \psi$ induces initial values on S_0 given by $\widetilde{u}_0(\mathbf{x}) = G_+ * \psi(0, \mathbf{x})$ and $\widetilde{u}_1(\mathbf{x}) = \partial_t G_+ * \psi(0, \mathbf{x})$. Hence, for the homogeneous equation with initial values u_0, u_1, we apply Corollary 3.6.5 with initial values $u_0 - \widetilde{u}_0$ and $u_1 - \widetilde{u}_0$ and find $\widehat{\phi}$.
The general solution will be $\phi = \widehat{\phi} + \widetilde{\phi}$. ■

The solution, if we concentrate on the classical Cauchy problem, is smooth (Proposition 3.2.2) and depends continuously on the initial data, hence the map that gives the solution u can be seen as a **linear continuous operator**

$$\begin{aligned} \mathcal{D}(\mathbb{M}^n) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S) &\to C^\infty(\mathbb{M}^n) \\ (\psi, u_0, u_1) &\mapsto u. \end{aligned}$$

Fundamental solutions on manifolds 4

In this chapter, the Riesz distributions will be transported on suitable domains of Lorentzian manifolds to construct local fundamental solutions and it will be discussed the local and global solvability of the Cauchy problem.
Firstly we will present some examples of generalized d'Alembert operators and of globally hyperbolic manifolds, interesting for their physical importance, to show why we are interested in these kind of problems on manifolds and why we will follow this path. In fact, in this setting, the Fourier transform is unmanageable, but Riesz distributions are far more generalizable.
We will start by pulling back the Riesz distributions from the tangent space to an open neighborhood of a fixed point in a Lorentzian manifold using the exponential map and we will discuss the properties that still hold. We will see that none of the pulled back Riesz distributions defines a fundamental solution to any wave-like operator, but we can combine them in a certain way to obtain approximate fundamental solutions.
Precisely, we will make a formal ansatz: we will write a formal solution as a series of Riesz distributions $\mathcal{R}_{\pm}$ with functions as coefficients to be determined. Requiring $\mathcal{R}_{\pm}$ to be a fundamental solution, we will find recursive relations for the coefficients that will depend on the particular wave-like operator we are trying to solve, but that can be found in general. In this way a formal fundamental solution will be constructed as a series for which the higher terms must be cut in a precise way in order to get a true solution. We will show that it is always possible to find such a regularization $\widetilde{\mathcal{R}}_{\pm}$ and make the formal series converge. Unfortunately the formal series does not immediately converge to a

fundamental solution, hence we will have to get rid of the difference between the true and the approximate fundamental solution, denoted by $K_{\pm}$, inverting the integral kernel $\mathcal{K}_{\pm}$ induced by $K_{\pm}$. We will show that this can be done only locally.
This considerations will lead to solutions of local Cauchy problems and we will show the regularity of such a solution with smooth initial data. Then we will generalize the results on *globally hyperbolic* manifolds (see Definition 2.2.21). This result will be used to show that global fundamental solutions exist in this case. All the solution will have support properties that respect the causality principle: in fact the support of any solution is contained in the union of causal future and causal past of the support of its initial data.

4.1 Physical interest

We recall from Section 2.3 that a **generalized d'Alembert** operator on a manifold M is given in local coordinates by

$$P = -g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a_j(x)\frac{\partial}{\partial x^j} + b(x). \tag{4.1}$$

4.2 Local fundamental solutions

In Section 3.5 Riesz distributions have been defined on the tangent spaces of an arbitrary point of a Lorentzian manifold. We show how to extend Riesz distributions small open subsets of the Lorentzian manifold. The step from the tangent space to the manifold will be provided by the exponential map.

4.2.1 Definition. *Let $\Omega \subset \mathrm{M}$ be a geodesically starshaped open subset with respect to a point $x \in \mathrm{M}$. For $\alpha \in \mathbb{C}$ and for every $\varphi \in \mathcal{D}(\Omega\mathrm{M})$ we call Riesz distribution*

$$\left(R^{\Omega}_{\pm}(\alpha, x), \varphi\right) := \left(R_{\pm}(\alpha), (\mu_x \varphi) \circ \exp_x\right),$$

where $\exp_x : \exp_x^{-1}(\Omega) \subset \mathrm{T}_x\Omega \to \Omega$ is the exponential map at x while $\mu_x : \Omega \to \mathbb{R}$ is the volume form defined as in equation (2.4).
We call $R^{\Omega}_{+}(\alpha, x)$ and $R^{\Omega}_{-}(\alpha, x)$ respectively the ***retarded*** *and the* ***advanced*** *Riesz distributions on Ω for $\alpha \in \mathbb{C}$.*

Note that, since $\operatorname{supp}(\mu_x \varphi) \circ \exp_x \subset \Omega$, it can be regarded as a test-function on $\mathrm{T}_x\Omega$ for the Riesz distribution.

Distributions on a geodesically starshaped domain maintain some of the properties of the Minkowski counterpart, namely the ones expressed in Proposition 3.5.3 and in Proposition 3.5.6, even if with some slight differences.

4.2.2 Theorem. *Let $\Omega \subset \mathrm{M}$ be a geodesically starshaped open subset with respect to a point $x \in \mathrm{M}$. For all $\alpha \in \mathbb{C}$ it holds*

(1) If Re $\alpha > n$, then

$$R_{\pm}^{\Omega}(\alpha, x) := \begin{cases} C(\alpha, n)\Gamma_x^{\frac{\alpha-n}{2}} & \text{on } J_{\pm}^{\Omega}(x) \\ 0 & \text{elsewhere,} \end{cases} \tag{4.2}$$

where $C(\alpha, n)$ are defined in Definition 3.5.1, while Γ_x is defined in Definition 2.2.17.

(2) $\operatorname{supp} R_{\pm}^{\Omega}(\alpha, x) \subset J_{\pm}(x)$.

(3) For every $\varphi \in \mathcal{D}(\Omega)$ the map $\alpha \mapsto \big(R_{\pm}^{\Omega}(\alpha, x), \varphi\big)$ is holomorphic on $\mathbb{C}$.

(4) $\Gamma_x\, R_{\pm}^{\Omega}(\alpha, x) = \alpha(\alpha - n + 2)R_{\pm}^{\Omega}(\alpha + 2, x)$.

(5) $\operatorname{grad}\Gamma_x \cdot R_{\pm}^{\Omega}(\alpha, x) = 2\alpha \operatorname{grad} R_{\pm}^{\Omega}(\alpha + 2, x)$.

(6) $R_{\pm}^{\Omega}(0, x) = \delta_x$.

(7) $\Box R_{\pm}^{\Omega}(\alpha + 2, x) = \left(\dfrac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_{\pm}^{\Omega}(\alpha, x)$ for all $\alpha \neq 0$.

To prove the properties of Theorem 4.2.2 we need a Lemma which helps dealing with Γ_x, the function that takes the place of γ in the formulas for the Riesz distributions on a manifold.

4.2.3 Lemma. *Let $\Omega \subset \mathrm{M}$ be an open and geodesically starshaped subset with respect to $x \in \mathrm{M}$. Then it holds that*

(1) $\langle \operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\rangle = -4\Gamma_x$;

(2) $\Box\Gamma_x = 2n - \langle \operatorname{grad}\Gamma_x, \operatorname{grad}\ln(\mu_x)\rangle$.

Proof. We prove (1) in Minkowski spacetime, where we can identify Γ with the geodesic distance $\gamma = (x^1)^2 - (x^2)^2 - \cdots - (x^n)^2$. In other words

$$\operatorname{grad}\gamma = \eta^{ij}\frac{\partial\gamma}{\partial x^i}\frac{\partial}{\partial x^j} = -2x^i\frac{\partial}{\partial x^i}. \tag{4.3}$$

Hence the first formula follows from direct computation:

$$\begin{aligned}
\langle \operatorname{grad}\gamma, \operatorname{grad}\gamma\rangle &= \left\langle -2x^i \frac{\partial}{\partial x^i}, -2x^i \frac{\partial}{\partial x^i} \right\rangle \\
&= 4((x^1)^2 - (x^2)^2 - \cdots - (x^n)^2) \\
&= -4\gamma.
\end{aligned}$$

To prove (2) we notice that Leibniz rule for the divergence operator reads

$$\operatorname{div}(fZ) = f \operatorname{div} Z + \langle \operatorname{grad} f, Z\rangle, \tag{4.4}$$

for any vector field Z and for any smooth function f on M. Hence, equation (4.3) generalizes in normal coordinates as

$$\operatorname{grad}\Gamma_x = -2x^j \frac{\partial}{\partial x^j}.$$

Using (4.4) with $f = \mu_x^{-1}$ and $Z = \operatorname{grad}\Gamma_x$, we obtain

$$\operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) = \mu_x^{-1} \operatorname{div}\operatorname{grad}\Gamma_x + \langle \operatorname{grad}\mu_x^{-1}, \operatorname{grad}\Gamma_x\rangle.$$

Since

$$\begin{aligned}
\Box\Gamma_x &= -\operatorname{div}\operatorname{grad}\Gamma_x \\
&= \mu_x \langle \operatorname{grad}\mu_x^{-1}, \operatorname{grad}\Gamma_x\rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) \\
&= -\langle \operatorname{grad}\ln\mu_x, \operatorname{grad}\Gamma_x\rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x),
\end{aligned}$$

it remains to be shown that $\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) = -2n$. Using the definition of divergence and Equation (2.6),

$$\mu_x \operatorname{div}(\mu_x \operatorname{grad}\Gamma_x) = \frac{\partial}{\partial x^j}(\operatorname{grad}\Gamma_x)^j = -2\frac{\partial}{\partial x^j} x^j = -2n.$$

■

Proof of Theorem (4.2.2). Let Re $\alpha > n$, $\varphi \in \mathcal{D}(\Omega)$ and let us denote with X the tangent vectors in $\mathrm{T}_x\Omega$. Then from Definition 4.2.1,

$$\begin{aligned}
\left(R_\pm^\Omega(\alpha, x), \varphi\right) &= (R_\pm(\alpha), (\mu_x\varphi) \circ \exp_x) \\
&= C(\alpha, n) \int_{J_\pm(0)} \gamma^{\frac{\alpha-n}{2}}(X) \mu_x\, \varphi(\exp_x(X))\, \mathrm{d}X \\
&= C(\alpha, n) \int_{J_\pm^\Omega(x)} \Gamma_x^{\frac{\alpha-n}{2}} \varphi(x)\, \mathrm{d}\mu.
\end{aligned}$$

Assertions (2) and (3) follow directly from the corresponding properties of the Riesz distributions on Minkowski spacetime.

By (1), equation (4) holds when Re $\alpha > n$ on account of the corresponding counterpart in the Minkowski case. By analytic extension of the distribution, it must hold in the whole complex plain.
To prove (5), we compute for Re $\alpha > n$

$$\begin{aligned} 2\alpha \operatorname{grad} R_\pm^\Omega(\alpha+2,x) &= 2\alpha C(\alpha+2,n) \operatorname{grad} \Gamma_x^{\frac{\alpha+2-n}{2}} \\ &= 2\alpha C(\alpha+2,n) \frac{\alpha+2-n}{2} \Gamma_x^{\frac{\alpha-n}{2}} \operatorname{grad} \Gamma_x \\ &= R_\pm^\Omega(\alpha,x) \operatorname{grad} \Gamma_x. \end{aligned}$$

and again we extend it for any complex α through analyticity.
To prove (6) let $\varphi \in \mathcal{D}(\Omega)$. Then,

$$\begin{aligned} \left(R_\pm^\Omega(0,x),\varphi\right) &= (R_\pm(0),(\mu_x\varphi)\circ\exp_x) \\ &= (\delta_0,(\mu_x\varphi)\circ\exp_x) \\ &= ((\mu_x\varphi)\circ\exp_x)(0) = \varphi(x) \\ &= (\delta_x,\varphi), \end{aligned}$$

where we used Equation (2) in Proposition 3.5.6. To prove (7) we take $\alpha \in \mathbb{C}$ such that Re $\alpha > n+2$ so that $R_\pm^\Omega(\alpha+2,x)$ is C^2. by the means of analytic extension of the distribution, the relation must hold for any $\alpha \in \mathbb{C}$.

$$\begin{aligned} \Box R_\pm^\Omega(\alpha+2,x) &= -\operatorname{div}\operatorname{grad} R_\pm^\Omega(\alpha+2,x) \\ &= -\frac{1}{2\alpha}\operatorname{div}\left(R_\pm^\Omega(\alpha,x)\cdot\operatorname{grad}\Gamma_x\right) \\ &= \frac{1}{2\alpha}\Box\Gamma_x\, R_\pm^\Omega(\alpha,x) - \frac{1}{2\alpha}\langle \operatorname{grad}\Gamma_x, \operatorname{grad} R_\pm^\Omega(\alpha,x)\rangle \\ &= \frac{1}{2\alpha}\Box\Gamma_x\, R_\pm^\Omega(\alpha,x) - \frac{R_\pm^\Omega(\alpha-2,x)}{2\alpha\cdot 2(\alpha-2)}\langle \operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\rangle, \end{aligned}$$

and making use of equation (1) in Lemma (4.2.3) we have

$$\begin{aligned} &= \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x) + \frac{1}{\alpha(\alpha-2)}\Gamma_x R_\pm^\Omega(\alpha-2,x) \\ &= \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x) + \frac{(\alpha-2)(\alpha-n)}{\alpha(\alpha-2)} R_\pm^\Omega(\alpha,x) \\ &= \left(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_\pm^\Omega(\alpha,x). \end{aligned}$$

■

4.2.1 Formal ansatz

One may think, in analogy with the Minkowski scenario, that properties (2) and (6) expressed in the Theorem 4.2.2 make the Riesz distribution $R^\Omega_\pm(2,x)$ a good candidate to be a fundamental solution for $\Box$ at x. The situation is far more complicated: in fact, one can not compute $\Box R^\Omega_\pm(\alpha+2,x)$ for $\alpha=0$ unless $\Box\Gamma_x - 2n$ vanishes identically, which in general is not the case.
It turns out that $R^\Omega_\pm(2,x)$ does not suffice to construct fundamental solutions and to this end also $R^\Omega_\pm(2k+2,x)$ for $k\geq 1$ are needed.

Let Ω be a geodesically starshaped open subset with respect to a given $x\in \mathrm{M}$ so that the Riesz distributions are well defined. We look for fundamental solutions $\mathcal{R}_\pm$ for a generalized d'Alembert operator P, defined in equation (4.1). We want to combine them in order to have a formal series, with unknown coefficients, and make it converge under suitable conditions. We make the following formal ansatz: the fundamental solutions for P are of the form

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} V^k_x \cdot R^\Omega_\pm(2k+2,x), \tag{4.5}$$

where for each k, V^k_x is a smooth function yet to be found.

The series above is only formal, but applying P term to term and requiring $\mathcal{R}_\pm(x)$ to be a fundamental solution one has

$$P\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} P\left(V^k_x \cdot R^\Omega_\pm(2k+2,x)\right) = \delta_x.$$

Hence, one can translate the condition of $\mathcal{R}_\pm(x)$ being a fundamental solution at x into conditions on the V^k_x.
To do this we need a lemma.

4.2.4 Lemma. *Let P be a generalized d'Alembert operator as in equation* (4.1) *and let $f,g\in C^\infty(\Omega)$. Then it holds*

$$P(fg) = \Box f\, g - 2\partial_{\operatorname{grad} f}\, g + P(g)\, f.$$

Using the last lemma with $f = R^\Omega_\pm(2k+2,x)$, $g = V^k_x$ and properties (4), (5) and (7) in Theorem 4.2.2 we compute

$$R^\Omega_\pm(0,x) = \sum_{k=0}^{\infty} P\left(V^k_x \cdot R^\Omega_\pm(2k+2,x)\right) =$$

$$= \sum_{k=0}^{\infty} \left\{ V_x^k \cdot \Box R_\pm^\Omega(2+2k,x) - 2\partial_{\operatorname{grad} R_\pm^\Omega(2+2k,x)} V_x^k + P V_x^k \cdot R_\pm^\Omega(2k+2,x) \right\}$$
$$= V_x^0 \cdot \Box R_\pm^\Omega(2,x) - 2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 +$$
$$+ \sum_{k=1}^{\infty} \tfrac{1}{2k} \left\{ V_x^k \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right) - \partial_{\operatorname{grad}\Gamma_x} V_x^k + 2kPV_x^{k-1} \right\} R_\pm^\Omega(2k,x).$$

Identifying the coefficients in front of $R_\pm^\Omega(2k,x)$ we get the conditions

$$\text{for } k=0 \quad 2\partial_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 - \Box R_\pm^\Omega(2,x)\, V_x^0 + R_\pm^\Omega(0,x) = 0, \tag{4.6a}$$
$$\text{for } k\geq 1 \quad \partial_{\operatorname{grad}\Gamma_x} V_x^k - \left(\frac{1}{2}\Box\Gamma_x - n + 2k\right) V_x^k = 2k\, PV_x^{k-1}. \tag{4.6b}$$

For consistency with the Minkowski case, between the functions that solve (4.6a), one chooses $V_x^0(x) \equiv 1$. To see why, one has to recall that if $P = \Box$ in Minkowski, $R_\pm(2)$ are fundamental solutions (see Proposition 3.5.6), hence the correspondent series (4.5) has to stop at $k=0$ and the coefficient $V_x^0(x)$ has to be unitary.

4.2.5 Definition. *Let $\Omega \subset \mathrm{M}$ be a geodesically starshaped open subset with respect to $x \in \mathrm{M}$ and P a generalized d'Alembert operator as in equation* (4.1)*. A sequence of* ***Hadamard coefficients*** *for P at $x \in \Omega$ is a sequence $\{V_x^k\}_{k\in\mathbb{N}}$ of $C^\infty(\Omega)$ which fulfills Equation* (4.6b) *with $V_x^0(x) = 1$ for all $x \in \Omega$.*

Hence, in order to construct, at least formally, fundamental solutions, the coefficients must satisfy the differential Equation (4.6b). These turn out to be identifiable recursively without further assumption.
A simple case in which an explicit formula for the coefficients can be obtained when P has no first order derivatives.

Let $\Phi : \Omega \times [0,1] \to \Omega$ be such that $\Phi(y,s) = \exp_x(s \cdot \exp_x^{-1}(y))$, which is well defined and smooth since Ω is a geodesically starshaped set. In other words, Φ is a parametrization of the geodesic connecting x to another point $y \in \Omega$, as one can see in Figure 4.1.

4.2.6 Theorem. *Let $\Omega \subset \mathrm{M}$ be a geodesically starshaped open subset with respect to $x \in \mathrm{M}$. Let P be a generalized d'Alembert operator of the form $P = \Box + \mathrm{V}$, where $\mathrm{V} \in C^\infty(\mathrm{M})$. Then there exist unique Hadamard coefficients V_x^k for P at x given by*

$$V_x^0(y) = \mu_x^{-\frac{1}{2}}(y) \tag{4.7}$$

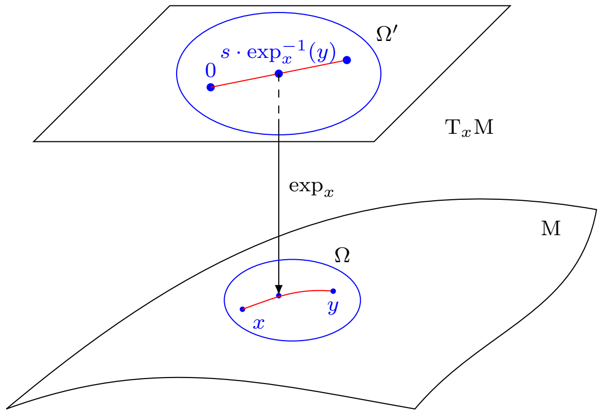


FIGURE 4.1: The action of the map Φ.

and for $k \geq 1$

$$V_x^k(y) = -k\mu_x^{-\frac{1}{2}}(y) \int_0^1 \mu_x^{\frac{1}{2}}(\Phi(y,s))\, s^{k-1} \left((PV_x^{k-1})\Phi(y,s)\right) \mathrm{d}s. \tag{4.8}$$

If the point x vary and if Ω is convex, the Riesz distributions are defined for all $x \in \Omega$. Write $V^k(x,y) := V_x^k(y)$ for the Hadamard coefficients at x, equations (4.7) and (4.8) imply that $V^k \in C^\infty(\Omega \times \Omega)$.

4.2.7 Definition. *Let* $\Omega \subset \mathrm{M}$ *be convex and* P *a generalized d'Alembert operator as in equation* (4.1). *We call*

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} V^k(x,\cdot)\, R_\pm^\Omega(2k+2,x), \tag{4.9}$$

the ***retarded*** *or* ***advanced*** *formal fundamental solutions for* P *at* $x \in \Omega$.

4.2.8 Remark. The coefficients $V^k \in C^\infty(\Omega \times \Omega)$ can be calculated by the analogous of equations (4.7) and (4.8):

$$V^0(x,y) = \mu_x^{-\frac{1}{2}}(y),$$
$$V^k(x,y) = -k\mu_x^{-\frac{1}{2}}(y) \int_0^1 \mu_x^{\frac{1}{2}}(\Phi(y,s))\, s^{k-1} \left((P_{(2)}V^{k-1})\Phi(y,s)\right) \mathrm{d}s,$$

where the subscript "(2)" in $P_{(2)}V^{k-1}$ stands for P acting on the second variable, i.e. on $s \mapsto V^{k-1}(\cdot, s)$.

4.2.2 Approximate fundamental solutions

The series defining $\mathcal{R}_\pm(x)$ may diverge, hence it does not provide any local fundamental solution. The idea is to make the series convergent by keeping the first terms of the formal series and multiplying the higher ones by suitable cut-off functions.
Let Ω' be a convex subset. Fix an integer $N \geq \frac{n}{2}$. Then, for all $k \geq N$, the distribution $R_\pm^{\Omega'}(2k+2, x)$ is continuous on Ω'. We split the formal fundamental solutions

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{N-1} V^k(x,\cdot)\, R_\pm^{\Omega'}(2k+2,x) + \sum_{k=N}^{\infty} V^k(x,\cdot)\, R_\pm^{\Omega'}(2k+2,x).$$

4.2.9 Proposition. *Let $\Omega \subset \Omega'$ be a relatively compact open subset. Let $\sigma : \mathbb{R} \to [0,1]$ be a smooth cut-off function with $\operatorname{supp}\sigma \subset [-1,1]$ and $\sigma = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Then there exists a sequence $\{\varepsilon_k\}_{k\geq N}$ in $(0,1]$, such that, for each $j \geq 0$, the series*

$$(x,y) \mapsto \sum_{k=N+j}^{\infty} \sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y) R_\pm^{\Omega'}(2+2k,x)(y) =$$

$$= \begin{cases} \sum_{k=N+j}^{\infty} C(2+2k,n)\sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y)\Gamma_x(y)^{k+1-\frac{n}{2}} & \text{if} \quad y \in J_\pm^{\Omega'}(x) \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

converges in $C^j(\overline{\Omega}\times\overline{\Omega})$. In particular the series

$$(x,y) \mapsto \sum_{k=N}^{\infty} \sigma\left(\frac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,y) R_\pm^{\Omega'}(2+2k,x)(y)$$

defines a continuous function.

We the main tools that we need to approximate the fundamental solutions.

4.2.10 Theorem. *Let Ω' be a convex open subset of M, let P be a generalized d'Alembert operator over M, as in equation (4.1), and σ a cut-off function per Proposition 4.2.9. Then, for every relatively compact open subset $\Omega \subset \Omega'$, there exists a positive sequence $\{\varepsilon_k\}_{j\geq N}$, such that for every $x \in \overline{\Omega}$*

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{k=0}^{N-1} V^k(x,\cdot)\, R_\pm^{\Omega'}(2k+2,x) + \sum_{k=N}^{\infty} \sigma\left(\tfrac{\Gamma_x(y)}{\varepsilon_k}\right) V^k(x,\cdot)\, R_\pm^{\Omega'}(2k+2,x) \tag{4.11}$$

defines a distribution on Ω satisfying

(1) $\operatorname{supp} \widetilde{\mathcal{R}}_{\pm}(x) \subset J^{\Omega}_{\pm}(x)$,

(2) $P_{(2)}\widetilde{\mathcal{R}}_{\pm}(x) = \delta_x + K_{\pm}(x, \cdot)$ *where* $K_{\pm} \in C^{\infty}(\overline{\Omega} \times \overline{\Omega})$,

(3) for every $\varphi \in \mathcal{D}(\Omega)$, $x \mapsto \left(\widetilde{\mathcal{R}}_{\pm}(x), \varphi\right)$ *is smooth on* Ω.

With a suitable sequence $\{\varepsilon_k\}$ the distributions defined in equation (4.11), called approximate **retarded** or **advanced** fundamental solutions, approximate the true fundamental solutions, namely the difference $P_{(2)}\widetilde{\mathcal{R}}_{\pm}(x) - \delta_x$ is a smooth function.

4.2.3 True fundamental solutions

We can turn the approximate fundamental solution into a true one getting rid of the error terms.
To start with, notice that, if a sequence $\{\varepsilon_k\}$ gives an approximate fundamental solution for Ω, the same sequence still provides an approximate fundamental solution for any $\Omega_1 \subset \Omega$.

We can read $K_{\pm} \in C^{\infty}(\overline{\Omega} \times \overline{\Omega})$ as an integral kernel defining the smooth integral operator for any $x \in \Omega$ and for any $u \in C^0(\Omega)$

$$(\mathcal{K}_{\pm}u)(x) := \int_{\Omega} K_{\pm}(x, y)u(y)\, \mathrm{d}\mu(y). \tag{4.12}$$

Since the map $\varphi \mapsto (\delta_x, \varphi) : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is the identity operator $\mathbb{1}|_{\mathcal{D}(\Omega)}$, one can rewrite Equation (2) in Theorem 4.2.10 as

$$\left(P_{(2)}\widetilde{\mathcal{R}}_{\pm}(x), \varphi\right) = (\mathbb{1} + \mathcal{K}_{\pm})\, \varphi,$$

for all $\varphi \in \mathcal{D}(\Omega)$.
We look for an inverse of $(\mathbb{1} + \mathcal{K}_{\pm})$. Its importance can be appreciated by observing that for all $\varphi \in \mathcal{D}(\Omega)$

$$(G^{\Omega}_{\pm}, \varphi) := (\mathbb{1} + \mathcal{K}_{\pm})^{-1} \left(y \mapsto \left(\widetilde{\mathcal{R}}_{\pm}(y), \varphi\right)\right). \tag{4.13}$$

Hence

$$
\begin{aligned}
\big(PG^{\Omega}_{\pm}(x),\varphi\big) &= \big(G^{\Omega}_{\pm}(x),P^{*}\varphi\big)\\
&= \Big[(\mathbb{1}+\mathcal{K}_{\pm})^{-1}\Big(y\mapsto\Big(\widetilde{\mathcal{R}}_{\pm}(y),P^{*}\varphi\Big)\Big)\Big](x)\\
&= \left[(\mathbb{1}+\mathcal{K}_{\pm})^{-1}\left(\underbrace{y\mapsto\Big(P_{(2)}\widetilde{\mathcal{R}}_{\pm}(y),\varphi\Big)}_{(\mathbb{1}+\mathcal{K}_{\pm})\,\varphi}\right)\right](x)\\
&= \varphi(x),
\end{aligned}
$$

that is $PG^{\Omega}_{\pm}(x)=\delta_x$. If we can identify an inverse for $\mathbb{1}+\mathcal{K}_{\pm}$ and if we can prove that $G^{\Omega}_{\pm}$ is a well-defined distribution, we obtain a local **exact** fundamental solution.
The idea is to use the fact that, given a bounded operator A on a Banach space, an operator of the form $(\mathbb{1}+A)$ is invertible if $\|A\|<1$. In our case this condition can be satisfied on domains with *small* volume. To be more precise we have the following proposition:

4.2.11 Proposition. *Let $\Omega\subset\Omega'$ be a relatively compact open and causal set (see Definition 2.2.18) and assume*

$$
|\overline{\Omega}|\cdot\sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}K_{\pm}(x,y)<1, \tag{4.14}
$$

where $|\overline{\Omega}|=\int_{\overline{\Omega}}\mathrm{d}\mu$. Then $(\mathbb{1}+\mathcal{K}_{\pm}):C^k(\overline{\Omega})\to C^k(\overline{\Omega})$ is an isomorphism for all $k\in\mathbb{N}$.

The main results are:

4.2.12 Theorem. *Let P be a generalized d'Alembert operator on* M *as in equation* (4.1)*. Then every point of* M *possesses a relatively compact causal open neighborhood Ω such that*

(1) $G^{\Omega}_{\pm}(x)$, defined in equation (4.13)*, are fundamental solutions for P at x over Ω,*

(2) $\operatorname{supp}G^{\Omega}_{\pm}(x)\subset J^{\Omega}_{\pm}(x)$, i.e. $G^{\Omega}_{\pm}(x)$ are a ***retarded*** *and an* ***advanced*** *fundamental solution,*

(3) $\varphi\mapsto(G^{\Omega}_{\pm}(x),\varphi)$ is smooth for all $\varphi\in\mathcal{D}(\Omega)$.

4.2.4 Asymptotic behaviour

The formal fundamental solution is asymptotic in the sense that the true fundamental solution coincides with the truncated one

$$\mathcal{R}_\pm^{N+j}(x) = \sum_{k=0}^{N-1+j} V^k(x,\cdot)\, R_\pm^{\Omega'}(2+2k,x),$$

up to an error term which is regular on the light cone. More precisely, for all $j \in \mathbb{N}$ the map

$$(x,y) \mapsto (G_\pm^\Omega(x) - \mathcal{R}_\pm^{N+j}(x))(y) \tag{4.15}$$

is in $C^k(\Omega \times \Omega)$. Moreover it holds:

4.2.13 Proposition. *For every $j \in \mathbb{N}$ there exists a constant C_j such that*

$$\sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} (G_\pm^\Omega(x) - \mathcal{R}_\pm^{N+j}(x))(y) \le C_j |\Gamma_x(y)|^j,$$

for all $(x,y) \in \Omega \times \Omega$.

To prove this proposition we need a lemma.

4.2.14 Lemma. *Let $f \in C^{3j+1}(\mathbb{M}^n)$ such that $f = 0$ if $\|x\| < 0$. Then there exists a contiunuous function $h : \mathbb{M}^n \to \mathbb{R}$ such that*

$$f(x) = h(x)\gamma(x)^j,$$

where $\gamma(x) = -\langle x, x\rangle$.

Proof of Proposition 4.2.13. Using the properties expressed in Theorem 4.2.2, we find constants C'_k such that

$$\begin{aligned}
&(G_\pm^\Omega(x) - \mathcal{R}_\pm^{N+j}(x))(y) = \\
&= (G_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3j+1}(x))(y) + \sum_{k=N+j}^{N+3j} V^k(x,y)\, R_\pm^{\Omega'}(2+2k,x)(y) \\
&= (G_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3j+1}(x))(y) + \\
&+ \sum_{k=N+j}^{N+3j} V^k(x,y)\, C'_k\, \Gamma_x(y)^j\, R_\pm^{\Omega'}(2+2(k-j),x)(y).
\end{aligned}$$

The function $h_k(x,y) := C'_k\, V^k(x,y)\, R_\pm^{\Omega'}(2+2(k-j),x)(y)$ is continuous since $2+2(k-j) \ge 2+2N \ge 2+n > n$. In view of equation (4.15), the function $(x,y) \mapsto (G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y)$ is C^{3j+1} and $\operatorname{supp}(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x)) \subset J^{\Omega}_{\pm}(x)$. We apply Lemma 4.2.14 in normal coordinates (note that $\Gamma_x(y) = \gamma(\exp_x^{-1}(y)))$ and find a continuous function h such that

$$(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+3j+1}_{\pm}(x))(y) = h(x,y)\,\Gamma_x(y)^k.$$

Hence,

$$(G^{\Omega}_{\pm}(x) - \mathcal{R}^{N+j}_{\pm}(x))(y) = \left(h(x,y) + \sum_{k=N+j}^{N+3j} h_k(x,y) \right) \Gamma_x(y)^j.$$

If we now set $C_j := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(h(x,y) + \sum_{k=N+j}^{N+3j} h_k(x,y) \right)$ the proposition is proven. ∎

4.2.5 Uniqueness and regularity

We want to solve the inhomogeneous equation $Pu = \psi$ for a given ψ with small support, *small* in the sense of Proposition 4.2.11. The support Ω is relatively compact and Equation (4.14) holds.

4.2.15 Proposition. *Under the assumption of Proposition 4.2.11, for every $\psi \in \mathcal{D}(\Omega)$ there exists a function $u_{\pm} \in C^{\infty}(\Omega)$ such that*

$$\begin{aligned} &Pu_{\pm} = \psi \\ &\mathit{supp}\, u_{\pm} \subset J^{\Omega}_{\pm}(\mathit{supp}\,\psi). \end{aligned}$$

Proof. Let

$$(u_{\pm}, \varphi) := \int_{\Omega} (G^{\Omega}_{\pm}(x), \varphi)\, \psi(x)\, \mathrm{d}\mu. \tag{4.16}$$

We will not prove that equation (4.16) defines a smooth function, rather only $Pu_{\pm} = \psi$. Let $\varphi \in \mathcal{D}(\Omega)$. It holds

$$\begin{aligned} (Pu_{\pm}, \varphi) &= (u_{\pm}, P^*\varphi) \\ &= \int_{\Omega} (G^{\Omega}_{\pm}(x), P^*\varphi)\, \psi(x)\, \mathrm{d}\mu \\ &= \int_{\Omega} (\underbrace{P_{(2)} G^{\Omega}_{\pm}(x)}_{=\delta_x}, \varphi)\, \psi(x)\, \mathrm{d}\mu \\ &= \int_{\Omega} \varphi(x)\, \psi(x)\, \mathrm{d}\mu = (\psi, \varphi). \end{aligned}$$

To prove the support condition, let $\varphi \in \mathcal{D}(\Omega)$ such that $(u_\pm, \varphi) \neq 0$, then there exists $x \in \Omega$ such that $(G^\Omega_\pm(x), \varphi)\, \psi(x) \neq 0$, which implies $\operatorname{supp} \varphi \cap \operatorname{supp} G^\Omega_\pm(x) \neq \emptyset$ and $x \in \operatorname{supp} \psi$. Hence $\operatorname{supp} \varphi \cap J^\Omega_\pm(x) \neq \emptyset$, i.e. $x \in J^\Omega_\mp(\operatorname{supp} \varphi)$, so that $J^\Omega_\mp(\operatorname{supp} \varphi) \cap \operatorname{supp} \psi$, which is equivalent to the thesis. ∎

As we mentioned in Proposition 2.2.19, the topological and geometrical properties of the Lorentzian manifold may be problematic when we are looking for solutions of a differential equation. For example, even if the manifold is not compact, the existence of closed timelike loops can make the problem ill-posed. To avoid these situations and to implement the causality conditions we restrict the discussion to the **globally hyperbolic** setting (see Definition 2.2.21), although some results may be extended to other cases.

In such case the main results are

4.2.16 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M *as in equation* (4.1). *Then every solution of the equation $PF = 0$ in $\mathcal{D}'(\mathrm{M})$ with past- or future-compact support (see Definition 2.2.18) vanishes.*

Sketch of proof. Take the case $F \in \mathcal{D}'(\Omega)$ with past-compact support. The thesis is $(F, \varphi) = 0$ for any $\varphi \in \mathcal{D}(\mathrm{M})$. The idea is to solve the inhomogeneous equation

$$\begin{aligned} &P^* u = \varphi \\ &\operatorname{supp} u \subset J^\Omega_-(\operatorname{supp} \varphi) \end{aligned}$$

(using Proposition 4.2.15) for any φ such that $\operatorname{supp} \varphi$ is small in the sense expressed in Proposition 4.2.11. If one proves that $\operatorname{supp} F \cap J^\Omega_-(\operatorname{supp} \varphi)$ is compact, it holds

$$(F, \varphi) = (F, P^* u) = (PF, u) = 0.$$

The proof of such properties involves the global hyperbolicity of the manifold. ∎

Now uniqueness is straightforward:

4.2.17 Corollary. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M *as in equation* (4.1) *and let $x \in \mathrm{M}$. Then there exist at most one retarded and at most one advanced fundamental solution for P at x.*

Proof. Let G_1 and G_2 be two retarded fundamental solutions at x. Then $G = G_1 - G_2$ is a solution for $PG = 0$. Since G_1 and G_2 are retarded solutions we know that $\operatorname{supp} G \subset \operatorname{supp} G_1 \cup \operatorname{supp} G_2 \subset J_+^{\mathrm{M}}(x)$. On a globally hyperbolic manifold $J_+^{\mathrm{M}}(x)$ is past compact. Then for Theorem 4.2.16 $G = 0$ and hence $G_1 = G_2$. ■

4.3 Local and global Cauchy problem

We now explore the solvability of the Cauchy problem, in analogy with Section 3.6, in order to obtain results needed to prove the existence and uniqueness of the fundamental solutions.

4.3.1 Local solvability

Next we prove existence and uniqueness of solutions to the Cauchy problem on small domains. To start with, it is useful to show a formula that helps controlling a solution of the problem in terms of its initial data. This theorem and its corollary are the analogous of Theorem 3.6.4 and Corollary 3.6.5, which can be regarded as particular cases.

4.3.1 Theorem. *Let P be a generalized d'Alembert operator on* M, *as in equation* (4.1), *and let S be a smooth spacelike hypersurface of* M *with a timelike oriented unit normal vector field $n : S \to \mathrm{TM}$.*
Let $\Omega \subset \mathrm{M}$ be a small subset in the sense of Proposition 4.2.11 such that $S \cap \Omega$ is a Cauchy hypersurface of Ω. If $u_\pm \in C^\infty(\Omega)$ such that $\operatorname{supp} u_\pm \subset J_\pm^\Omega(K)$ (where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$) solve the Cauchy problem

$$\begin{cases} Pu_\pm = \psi \\ \\ u_\pm|_{S\cap\Omega} = u_0 \\ \\ \partial_n u_\pm|_{S\cap\Omega} = u_1, \end{cases} \tag{4.17}$$

with $(\psi, u_0, u_1) \in \mathcal{D}(\Omega) \oplus \mathcal{D}(S\cap\Omega) \oplus \mathcal{D}(S\cap\Omega)$, then it holds

$$\begin{aligned} \int_\Omega u_\pm(x)\varphi(x)\,\mathrm{d}\mu = &\int_\Omega \left(G_\pm^\Omega(x), \varphi\right)\psi(x)\,\mathrm{d}\mu + \\ &+ \int_{S\cap\Omega} \left(\partial_n (G_\pm^\Omega(x), \varphi) u_0 - (G_\pm^\Omega(x), \varphi) u_1\right)\,\mathrm{d}\mu_S, \end{aligned} \tag{4.18}$$

for all $\varphi \in \mathcal{D}(\Omega)$*, where* $\mathrm{d}\mu_S$ *is the* $n-1$ *dimensional pull-back measure on* S.[1]

4.3.2 Corollary. *In the conditions of last lemma, if* $u \in C^\infty(\Omega)$ *solves* $Pu = 0$ *then*

$$\operatorname{supp} u \subset J_+^\Omega(K) \cup J_-^\Omega(K),$$

where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1$*, and*

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}\mu = \int_{S\cap\Omega} \big(\partial_n(G^\Omega(x),\varphi)u_0 - (G^\Omega(x),\varphi)u_1\big)\,\mathrm{d}\mu_S, \tag{4.19}$$

for all $\varphi \in \mathcal{D}(\Omega)$*, where* $G^\Omega(x) = G_+^\Omega(x) - G_-^\Omega(x)$ *and where* $\mathrm{d}\mu_S$ *is the* $n-1$ *dimensional pull-back measure on* S.

Proof. We apply Lemma **??** with $\psi = 0$, noting that one can write u in the form $u = u_+ - u_-$ such that $\operatorname{supp} u_\pm \subset J_\pm^\Omega(K)$. Using equations (4.18) for both u_+ and u_- and subtracting the outcomes, we obtain the thesis. Since $\operatorname{supp} G_\pm^\Omega(x) \subset J_\pm^\Omega(x)$, the right hand side of Equation (4.18) vanishes outside $J_+(K) \cup J_-(K)$, hence $\operatorname{supp} u \subset J_+(K) \cup J_-(K)$. ■

4.3.3 Theorem. *Under the hypotheses of Lemma 4.3.1, for each* small *open subset* $\Omega \subset \mathrm{M}$*, such that the hypotheses of Proposition 4.2.11 are satisfied, such that* $S \cap \Omega$ *is a Cauchy hypersurface of* Ω*, it holds that for all triples* $(\psi, u_0, u_1) \in \mathcal{D}(\Omega) \oplus \mathcal{D}(S\cap\Omega) \oplus \mathcal{D}(S\cap\Omega)$ *there exists a unique* $u \in C^\infty(\Omega)$ *with*

$$\begin{cases} Pu = \psi \\[2em] u|_{S\cap\Omega} = u_0 \\[2em] \partial_n u|_{S\cap\Omega} = u_1. \end{cases} \tag{4.20}$$

Moreover $\operatorname{supp} u \subset J_+^\Omega(K) \cup J_-^\Omega(K)$*, where* $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$.

Sketch of proof. Since causal domains are globally hyperbolic we may apply Theorem 2.2.23 to find an isometry $\Omega = \mathbb{R} \times (S \cap \Omega)$, setting the metric in the form $g = -\beta\,\mathrm{d}t^2 + b_t$. Now we look for a formal solution

$$u(t,x) = \sum_{j=0}^\infty t^j\,\widetilde{u}_j(x), \tag{4.21}$$

[1] Pull-back measure are defined in [5, Chapter 1]

where $(t,x) \in \mathbb{R} \times (S \cap \Omega)$. On $S \cap \Omega$ we set $\widetilde{u}_0 = u_0$ and $\widetilde{u}_1 = -(\beta)^{\frac{1}{2}} u_1$. The generalized d'Alembert operator can be written in the form $P = \frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y$, where Y contains at most derivatives of order 1 in t.
From the equation

$$\psi = Pu = \left(\frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y\right) u = \frac{1}{\beta}\sum_{j=2}^{\infty} j(j-1)t^{j-2}\widetilde{u}_j + Yu, \tag{4.22}$$

for $t = 0$ we have

$$2\beta(0,x)\,\widetilde{u}_2(x) = -Y(\widetilde{u}_0 + t\widetilde{u}_1)(0,x) + \psi(0,x).$$

This relation determines uniquely $\widetilde{u}_2$ from $\widetilde{u}_0$,$\widetilde{u}_1$ and $\psi|_S$. One can hence find recursive relations for $\widetilde{u}_j$ differentiating Equation (4.22) with respect to t and repeating the procedure.
In general the series in Equation (4.21) is non-convergent, but one can find a suitable positive sequence $\{\varepsilon_j\}$ such that the series

$$\widehat{u} := \sum_{j=0}^{\infty} \sigma\left(\frac{t}{\varepsilon_j}\right) t^j \widetilde{u}_j,$$

defines a smooth function on Ω and such that $P\widehat{u} - \psi$ vanishes at least on $S \cap \Omega$. Here σ is a cut-off function as in Proposition 4.2.9. Proposition 4.2.15 provides smooth solutions $\widetilde{u}_\pm$ of the inhomogeneous problems

$$\begin{aligned} &P\widetilde{u}_\pm = h_\pm \\ &\operatorname{supp}\widetilde{u}_\pm \subset J^\Omega_\pm(\operatorname{supp} h_\pm), \end{aligned}$$

where $h_\pm|_{J^\Omega_\pm(S\cap\Omega)} := P\widehat{u} - \psi$ vanishes everywhere else on Ω. One needs to show that $u_\pm := \widehat{u} - \widetilde{u}_\pm$ solves the equation $Pu_\pm = \psi$ on $J^\Omega_\pm(S \cap \Omega)$ and that it vanishes on $J^\Omega_\mp(S \cap \Omega)$. Then the function

$$u := \begin{cases} u_+ & \text{on} \quad J^\Omega_+(S \cap \Omega) \\ u_- & \text{on} \quad J^\Omega_-(S \cap \Omega) \end{cases}$$

is smooth and it solves the Cauchy problem.
Uniqueness follows from Corollary 4.3.2. In fact, if u_1 and u_2 solve the Cauchy problem, for linearity $u_1 - u_2$ solves the problem with vanishing initial data $(\psi, u_0, u_1) \equiv (0,0,0)$. Hence $u := u_1 - u_2 = 0$ because of equation (4.19). The assertion of the support follows from the corresponding one from the homogeneous problem and from Corollary 4.3.2. ■

4.3.2 Global solvability

The main result of this section generalizes what we obtained in Theorem 4.3.3 to the globally hyperbolic case.

4.3.4 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M, *as in equation* (4.1), *and let S be a smooth Cauchy hypersurface of* M *with a timelike unit normal vector field $n : S \to \mathrm{TM}$.*
Then it holds that for all triples $(\psi, u_0, u_1) \in \mathcal{D}(\mathrm{M}) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S)$ there exists a unique $u \in C^\infty(\mathrm{M})$ such that

$$\begin{cases} Pu = \psi \\ \\ u|_S = u_0 \\ \\ \partial_n u|_S = u_1. \end{cases} \tag{4.23}$$

Furthermore $\operatorname{supp} u \subset J_+^{\mathrm{M}}(K) \cup J_-^{\mathrm{M}}(K)$, where $K := \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} \psi$ and the map

$$\begin{aligned} \mathcal{D}(\mathrm{M}) \oplus \mathcal{D}(S) \oplus \mathcal{D}(S) &\to C^\infty(\mathrm{M}) \\ (\psi, u_0, u_1) &\mapsto u. \end{aligned}$$

*is **linear continuous**.*

Sketch of proof. The existence of u is proved in two steps. One constructs a solution u in the strip $(-\varepsilon, \varepsilon) \times S$ for some $\varepsilon > 0$ by gluing together local solutions obtained in Theorem 4.3.3, noting that there is only a finite number of them since the supports are compact. Then one extends u in the whole future and past of the strip. For uniqueness, an argument similar to that of Theorem 4.3.3, which made use of Corollary 4.3.2, can be used.
The continuous dependence on the initial data can be proven with methods of functional analysis. ■

4.3.5 Remark. The former result can be extended further. In fact the same thesis holds even if the triple of initial data (ψ, u_0, u_1) is in $C^\infty(\mathrm{M}) \oplus C^\infty(S) \oplus C^\infty(S)$.

4.4 Global fundamental solutions

Since uniqueness of the retarded and advanced fundamental solutions has already been proven in Corollary 4.2.17, it remains to show the global existence. We summarize the results in the following theorem.

4.4.1 Theorem. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime* M, *as in equation* (4.1)*. Then for each $x \in \mathrm{M}$ there exists a unique fundamental solution $G_+(x)$ with past-compact support and a unique one $G_-(x)$ with future-compact support.*
Furthermore, they satisfy

- *supp* $G_\pm \subset J_\pm^{\mathrm{M}}(x)$,
- *the maps $x \mapsto (G_\pm(x), \varphi)$ identify smooth functions on* M *satisfying* $P^*(G_\pm(\cdot), \varphi) = \varphi$, *for every* $\varphi \in \mathcal{D}(\mathrm{M})$.

4.4.2 Remark. From the last theorem one can prove that on a globally hyperbolic spacetime, the wave-like equation $Pu = \psi$, with $\psi \in \mathcal{D}(\mathrm{M})$ possesses a unique solution $u_\pm$ with $\operatorname{supp} u_+$ (respectively $\operatorname{supp} u_-$) being past (respectively future) compact.

4.5 Green's operators

We want to show the strict correspondence between fundamental solutions on manifolds and the so-called **Green's operators**. They are operators which can be seen as inverses of P when restricted to suitable spaces of functions.

4.5.1 Definition. *Let P be a generalized d'Alembert operator on a spacetime* M. *A retarded* ***Green's operator*** *for P on* M *is a linear map*

$$\mathbb{G}_+ : \mathcal{D}(\mathrm{M}) \to C^\infty(\mathrm{M})$$

such that satisfies

(1) $P \circ \mathbb{G}_+ = \mathbb{1}_{\mathcal{D}(\mathrm{M})}$,

(2) $\mathbb{G}_+ \circ P|_{\mathcal{D}(\mathrm{M})} = \mathbb{1}_{\mathcal{D}(\mathrm{M})}$,

1. *supp* $\mathbb{G}_+\varphi \subset J_+^{\mathrm{M}}(\text{supp}\,\varphi)$ *for all* $\varphi \in \mathcal{D}(\mathrm{M})$.

An advanced **Green's operator** $\mathbb{G}_- : \mathcal{D}(\mathrm{M}) \to C^\infty(\mathrm{M})$ *satisfies* (1) *and* (2) *and* $\mathrm{supp}\, \mathbb{G}_-\varphi \subset J_-^{\mathrm{M}}(\mathrm{supp}\, \varphi)$ *for all* $\varphi \in \mathcal{D}(\mathrm{M})$.

Next proposition shows that in fact Green's operators and fundamental solutions are two different versions of mainly the same concept.

4.5.2 Proposition. *In the frame of the above definition, retarded (resp. advanced) Green's operators for P stand in one-to-one correspondence with advanced (resp.retarded) fundamental solutions for P^*. More precisely, if $G_\pm(x)$ is a family of retarded or advanced fundamental solutions for the adjoint operator P^* and if*

- $x \mapsto (G_\pm(x), \varphi)$ *is smooth for each test-function* φ,
- $G_\pm(x)$ *satisfies the differential equation* $P(G_\pm(\cdot), \varphi) = \varphi$

, then

$$(\mathbb{G}_\pm\varphi)(x) := (G_\mp(x), \varphi), \tag{4.24}$$

defines retarded or advanced Green's operators for P respectively.
Conversely, for every Green's operators $\mathbb{G}_\pm$ for P, Equation (4.24) *defines fundamental solutions $G_\mp(x)$ such that $x \mapsto (G_\pm(x), \varphi)$ is smooth for each test-function φ and satisfies the differential equation $P(G_\pm(\cdot), \varphi) = \varphi$.*

5 Conclusions

In this thesis we ~~achieved~~ ~~important results on~~ local and global ~~solvability~~ of wave-like equations ~~in an n-dimensional flat background and in the general setting of a Lorentzian manifold as well~~, using the method of fundamental solutions.

In particular we ~~found~~ two ~~independent~~ fundamental solutions G_+ and G_-, the *retarded* and the *advanced* one, that propagate the source of the equation respectively in the causal future and in the causal past, in accordance with the causality principle.
We showed explicit formulas for the wave operator on Minkowsi spacetime with dimensions $2 \leq n \leq 4$ approaching the problem via Fourier transform. Then to deduce general properties of solutions, we introduced the Riesz distributions, a family of distributions $R_\pm(\alpha)$ depending on a complex parameter α. Setting $\alpha = 2$ we found the two fundamental solutions, retarded and advanced, that we were looking for. Via Riesz distribution we proved the properties of the support of retarded and advanced fundamental solutions, as well as Huygens' principle (see Section 3.4 and Theorem 3.5.9). We addressed the Cauchy problem on flat space, realizing that it is well posed only if the initial data are set on a particular class of hypersurfaces: Cauchy hypersurfaces (see Definition 2.2.22).

Shifting to the case on a manifold, we found out that the Fourier transform approach is unmanageable when dealing with wave-like differential equations on curved backgrounds, and to reach our target we followed another path based

on local extensions of Riesz distributions from tangent space to the manifold. In fact, initially we pulled back Riesz distribution from the tangent space of a fixed point $x \in \mathrm{M}$ on a geodesically starshaped open subset centered on x. Then we made the following formal ansatz: to find local retarded and advanced fundamental solutions for a wave-like operator one must look for infinite formal combinations of selected Riesz distributions, with coefficients that are functions V_x^k to be determined for any $k \in \mathbb{N}$. Implementing this condition formally, one finds recursive relations for the coefficients. It turns out that such relations are differential equations that can be always solved. The formal series had to be transformed into a true fundamental solution. We got rid of the difference between the asymptotic series and the fundamental solution and we found a true local fundamental solution with methods of functional analysis.
At this point we were able to solve the local Cauchy problem: we deduced that if we fix smooth initial data on a Cauchy surface of a manifold, the local solution is unique, smooth and its support is included in the causal past and future of the union of the supports of the initial data. In other words, we proved that a wave-like signal cannot locally travel faster than light.

Gluing ~~up~~ local solutions, we found that if the manifold respects the condition of *global hyperbolicity* (Definition 2.2.21), which is equivalent to the request of the existence of a Cauchy hypersurface, we can construct a global and smooth solution to the Cauchy problem, maintaining the local support properties that we mentioned earlier. Globally hyperbolic Lorentzian manifolds turned out to form a good class for the solution theory of wave-like operators. On them we have unique advanced and retarded fundamental solutions and Green's operators, i.e. operators that are the inverse of a differential operator P (see Definition 4.5.1). The Cauchy problem is well-posed.
In conclusion, we made clear how retarded and advanced fundamental solutions are strictly related to Green's operator. In fact they can be seen as two aspects of mainly the same concept.

The possible extensions and follow-ups to this work are many. Firstly, the results can be extended to differential operators that acts on sections of vector bundles. This opens the door of quantum fields theory on curved backgrounds, whose aim is to provide a partial unification of General Relativity with Quantum Physics where the gravitational field is left classical while the other fields are quantized. In particular one can deal with other operators such as Dirac operator D, whose square is a wave-like operator, or with generalized Klein-Gordon operator as well as the operators of electrodynamics equations. Other extensions may go in the direction of addressing the global problem in non-globally hyperbolic manifolds, since most basic models in General Relativity turn out to be globally hyperbolic, but there are exceptions such as anti-deSitter spacetime. ~~In such manifolds the fundamental solutions may not be unique or even exist.~~

Distributions and Fourier Transform A

Firstly, we recall the main concepts of the theory of distributions on manifolds.

For a manifold M we define $\mathcal{D}(\mathrm{M}) := C_0^\infty(\mathrm{M})$ as the space of test-functions on M. We say a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of $C^j(\mathrm{M})$ (with $j \in \mathbb{N} \cup +\infty$) converges to $\varphi \in C^j(\mathrm{M})$ if there exists a compact subset $K \subset \mathrm{M}$ such that $\operatorname{supp}\varphi_k \subset K$ and all the derivatives of φ_k up to the j-th order converge uniformly in K.
A linear map $u : \mathcal{D}(\mathrm{M}) \to \mathbb{C}$ is continuous if for all sequences $\{\varphi_k\}_{k\in\mathbb{N}}$ of $\mathcal{D}(\mathrm{M})$ that converge to $\varphi \in \mathcal{D}(\mathrm{M})$, $(u, \varphi_k) \to (u, \varphi)$, where with (u, φ) we denote the map u tested against φ.

A.0.1 Definition. *The space of distributions over* M *is defined as*

$$\mathcal{D}'(\mathrm{M}) = \{u : \mathcal{D}(\mathrm{M}) \to \mathbb{C} \text{ linear and continuous } \}.$$

The support of a distribution is the set $\mathrm{M} \setminus X$*, where* X *is the set of points* $x \in \mathrm{M}$ *such that there exists a neighborhood* U *of* x *such that* $u|_{\mathcal{D}(U)} \neq 0$*. We say* $u \in \mathcal{E}'(\mathrm{M})$ *if* $\operatorname{supp} u$ *is a compact subset of* M.

We call $u \in \mathcal{D}'(\mathrm{M})$ the **weak limit** of a sequence of distributions $\{u_i\}_{i\in\mathbb{N}}$ if for all $\varphi \in \mathcal{D}(\mathrm{M})$ holds $\lim_{i\to\infty}(u_i, \varphi) = (u, \varphi)$.

A.0.2 Remark. For any fixed $f \in C^\infty(\mathrm{M})$ the map $\varphi \mapsto \int_\mathrm{M} f(x)\,\varphi(x)\,\mathrm{d}\mu$ defines a distribution on M. We denote this distribution again by f, hence we identify $C^\infty(\mathrm{M})$ as a subset of $\mathcal{D}'(\mathrm{M})$.

A.0.3 Definition. *We define the translation operator $T_{x_0} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ such that if $u \in \mathcal{D}'(\mathbb{R}^n)$,*

$$(T_{x_0} u, \varphi(x)) = (u, \varphi(x + x_0)),$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We write $u(x - x_0) := T_{x_0} u$.

A.0.4 Example. Given $x \in \mathrm{M}$ the **Dirac delta** δ_x is a distribution defined for $\varphi \in \mathcal{D}'(\mathrm{M})$ by

$$(\delta_x, \varphi) = \varphi(x).$$

If M is isomorphic to $\mathbb{R}^n$, a particularly useful formula gives

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x - a) + \delta(x + a)]. \tag{A.1}$$

A.0.5 Definition. *We define the tensor product of two distributions $u \in \mathcal{D}'(\mathrm{M})$ and $v \in \mathcal{D}'(\mathrm{N})$ as the unique distribution $u \otimes v \in \mathcal{D}'(\mathrm{M} \times \mathrm{N})$ such that for any $g \in \mathcal{D}(\mathrm{M} \times \mathrm{N})$*

$$(u \otimes v, g(x, y)) = \big(u, (v, g(x, y))\big) = \big(v, (u, g(x, y))\big).$$

Given a differential operator $P : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$ there is a unique $P^* : C^\infty(\mathrm{M}) \to C^\infty(\mathrm{M})$, called the **formal adjoint** of P such that for any $\varphi, \psi \in \mathcal{D}(\mathrm{M})$ holds

$$\int_{\mathrm{M}} \psi(P\varphi) \,\mathrm{d}\mu = \int_{\mathrm{M}} (P^*\psi)\varphi \,\mathrm{d}\mu.$$

Any linear differential operator P extends canonically to $P : \mathcal{D}'(\mathrm{M}) \to \mathcal{D}'(\mathrm{M})$ by

$$(Pu, \varphi) = (u, P^*\varphi).$$

In particular, we define the product of a distribution $u \in \mathcal{D}'(\mathrm{M})$ with a function $f \in C^\infty(\mathrm{M})$ as the distribution $f \cdot u$ such that $(f \cdot u, \varphi) = (u, f\varphi)$ for any $\varphi \in \mathcal{D}(\mathrm{M})$.

A.0.6 Definition. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. We define the* ***convolution*** *of u and v as the unique distribution $u * v \in \mathcal{D}'(\mathbb{R}^n)$ such that*

$$(u * v, \varphi) = \big(u \otimes v, \varphi(x + y)\big) = \big(v, (u, \varphi(x + y))\big) = \big(u, (v, \varphi(x + y))\big).$$

A.0.7 Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\rho \in \mathcal{D}(\mathbb{R}^n)$. Then $\rho * u \in C^\infty(\mathbb{R}^n)$.*

Now we recall the main concepts of Fourier theory on $\mathbb{R}^n$.

We call Schwartz space $\mathcal{S}(\mathbb{R}^n)$ the set of rapidly decreasing functions, i.e. the functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\lim_{|x|\to\infty} x^\alpha \partial^\beta f(x) = 0,$$

for any multi-index $\alpha, \beta \in \mathbb{N}^n$. A sequence $\{f_j\}_{j\in\mathbb{N}}$ of rapidly decreasing functions converge to f in $\mathcal{S}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta (f_j - f)(x)| \to 0,$$

as $j \to \infty$.

A.0.8 Definition. *A distribution $u : \mathcal{D}(\mathrm{M}) \to \mathbb{C}$ is called a* ***tempered distribution*** *if for all sequences $\{f_k\}_{k\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}^n)$ that converge to $f \in \mathcal{S}(\mathbb{R}^n)$, $(u, f_k) \to (u, f)$. The set of tempered distribution is denoted with $\mathcal{S}'(\mathbb{R}^n)$.*

Given a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$\widehat{f}(k) = \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle}\, \mathrm{d}x \tag{A.2}$$

We naturally extend the Fourier transform to a unitary map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and to a map on tempered distributions in such a way that for $u \in \mathcal{S}'(\mathbb{R}^n)$ holds

$$(\widehat{u}, \varphi) = (u, \widehat{\varphi}),$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $u \in \mathcal{E}'(\mathbb{R}^n)$ holds

$$\widehat{u}(k) = \left(u, e^{-i\langle k,x\rangle}\right),$$

and $\widehat{u}$ results a smooth function which extends to an entire function $\widehat{u}(z)$, $z \in \mathbb{C}$.

The inverse Fourier transform of $f \in L^1(\mathbb{R}^n)$ is given as $\check{f}(x) := (2\pi)^{-n} \widehat{f}(-x)$ and it holds that $f = \check{\widehat{f}}$.

The Fourier transform of the distribution $\delta \in \mathcal{E}'(\mathcal{U})$ is $\widehat{\delta}(k) = 1$. This is a straightforward computation:

$$\widehat{\delta}(k) = (\delta(x), e^{-i\langle k,x\rangle}) = e^0 = 1.$$

Other useful formulas that holds for $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ and for any multi-index α are

- $\widehat{\partial^\alpha \varphi}(k) = (ik)^\alpha \widehat{\varphi}(k)$
- $\widehat{x^\alpha \varphi}(k) = (i\partial)^\alpha \widehat{\varphi}(k)$.

List of Figures

Bibliography

[1] Jost,J. *Riemannian Geometry and Geometric Analysis*, Springer, 2011.

[2] Bär, C., Ginoux, N., Pfäffle, F. *Wave equations on Lorentzian Manifolds and quantization*, EMS Publishing House, Zurich, 2007.

[3] Pfäffle,F. *Lorentzian Manifolds*, Lect. Notes Phys. **786**, 2009.

[4] Ginoux, N. *Linear Wave equations*, Lect. Notes Phys. **786**, 2009.

[5] Semyon Alesker, Joseph H.G. Fu *Integral Geometry and Valuations* Springer, 09 ott 2014