Introduction to Quantum Backflow

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Answers:

- In classical physics: $P(t)$ is always decreasing with time.
- In quantum physics: not necessarily.

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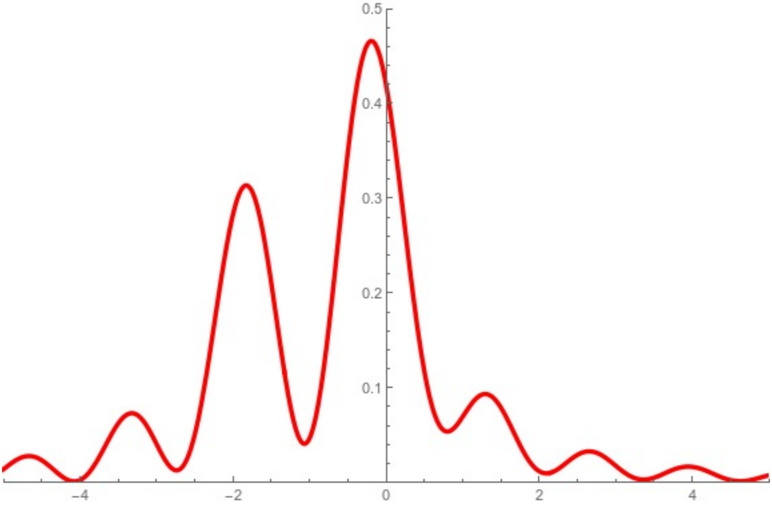

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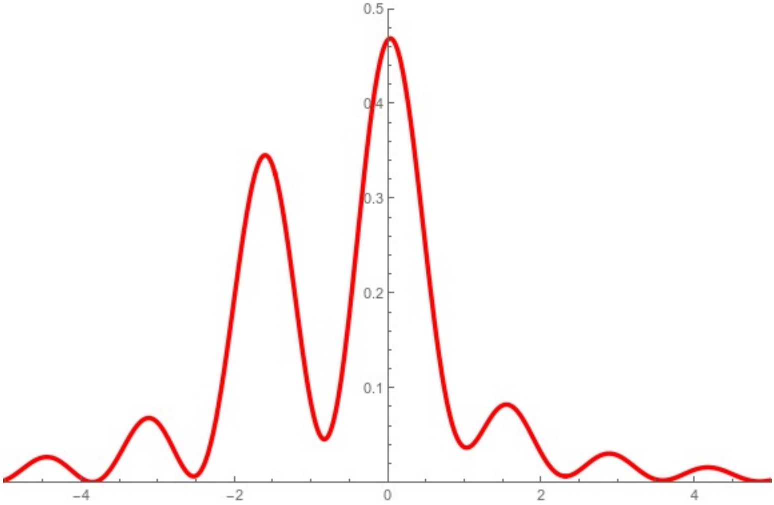

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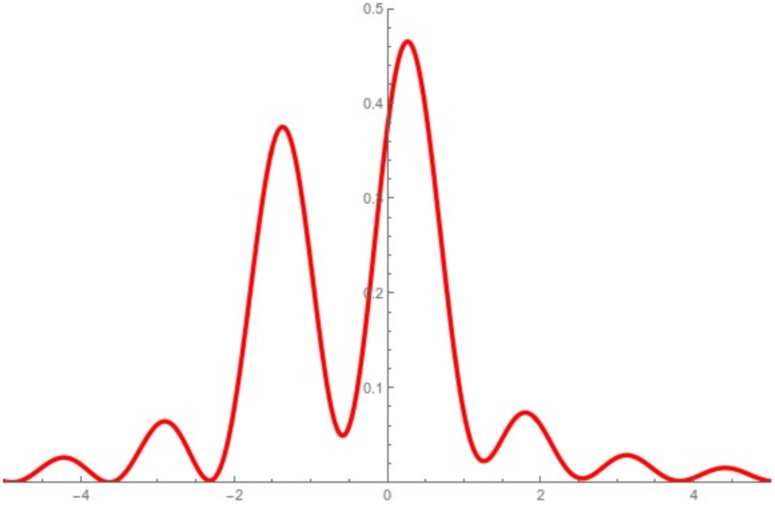

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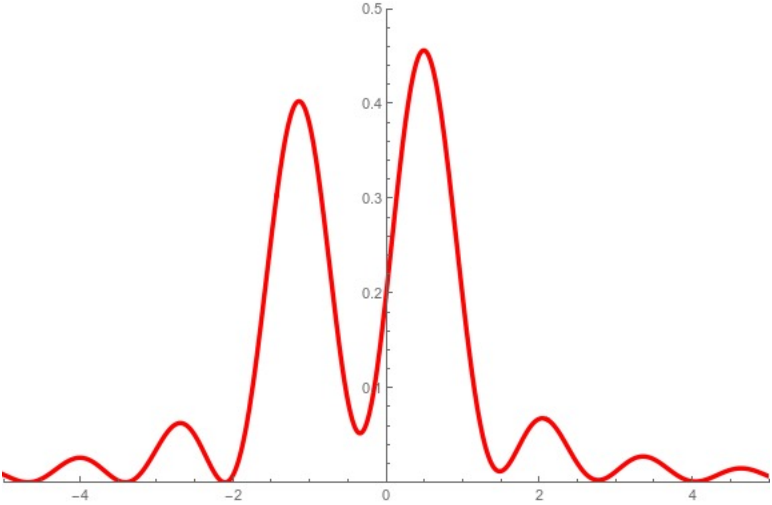

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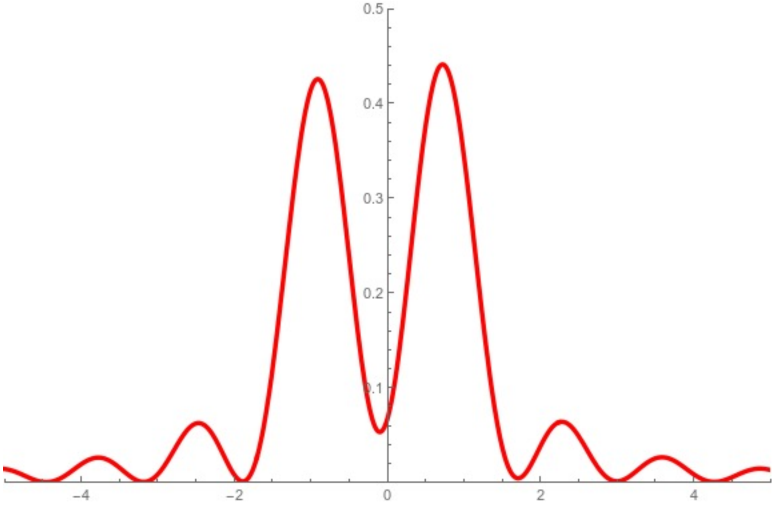

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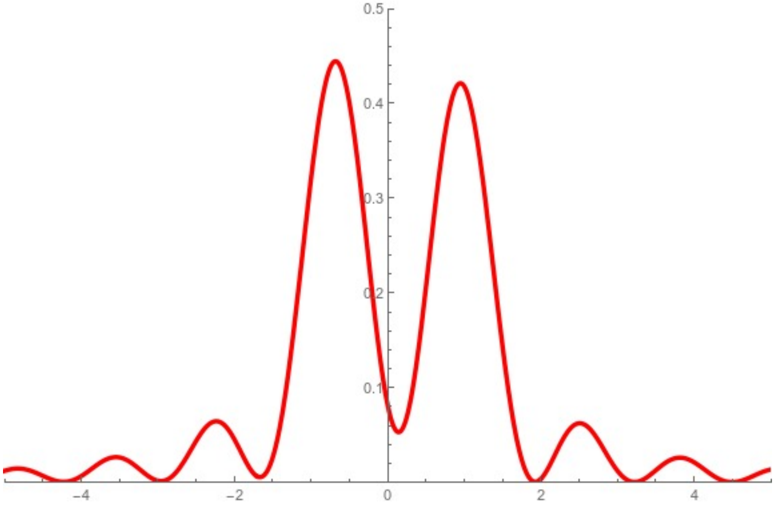

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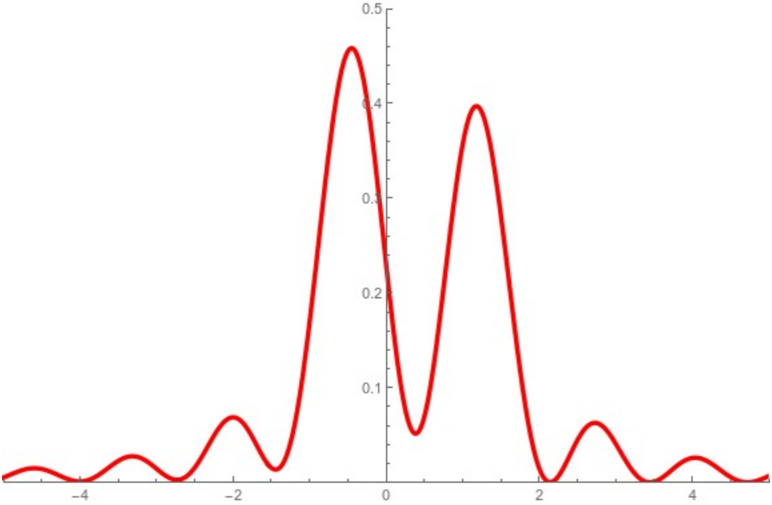

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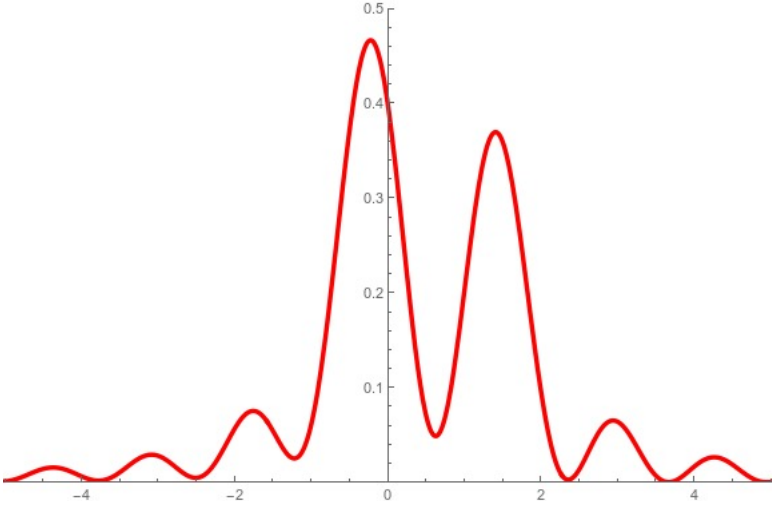

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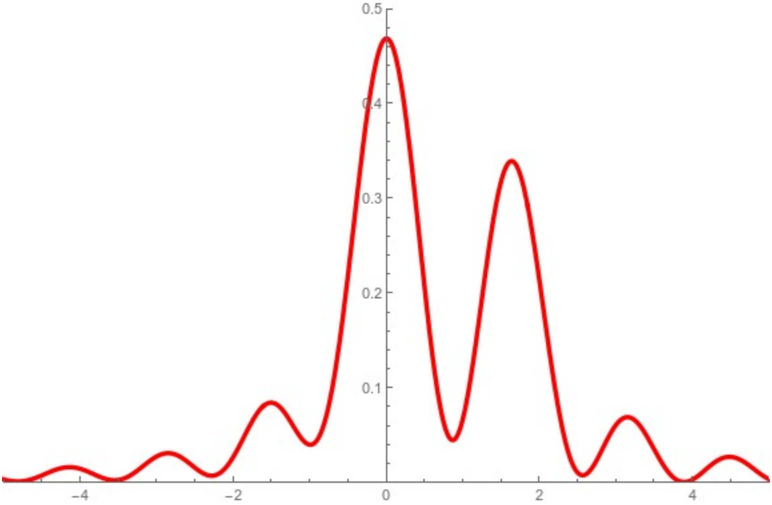

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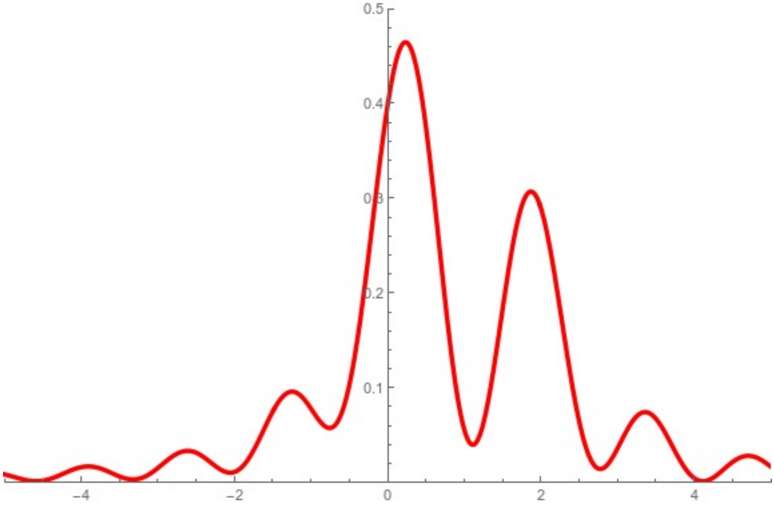

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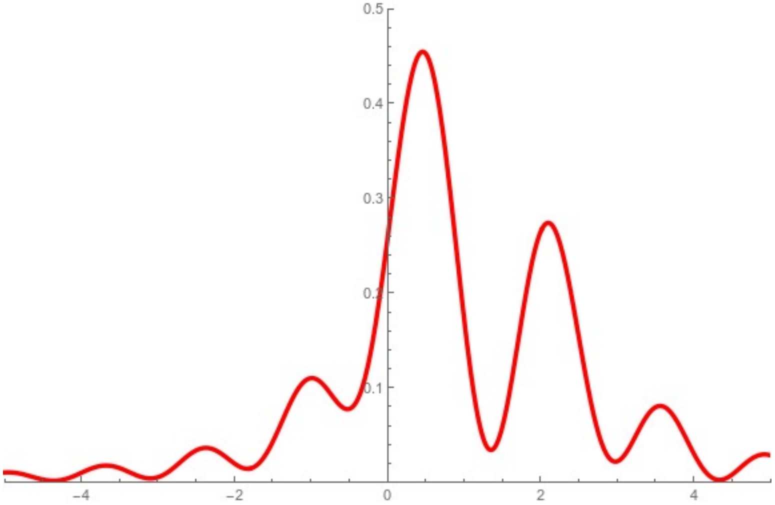

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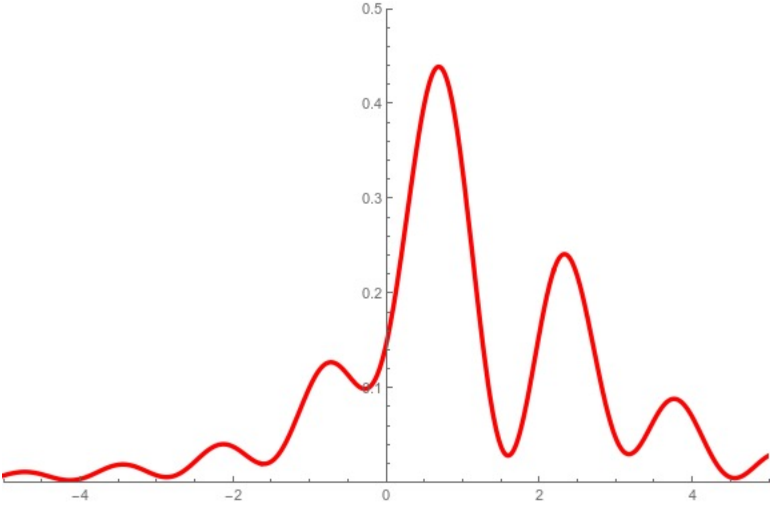

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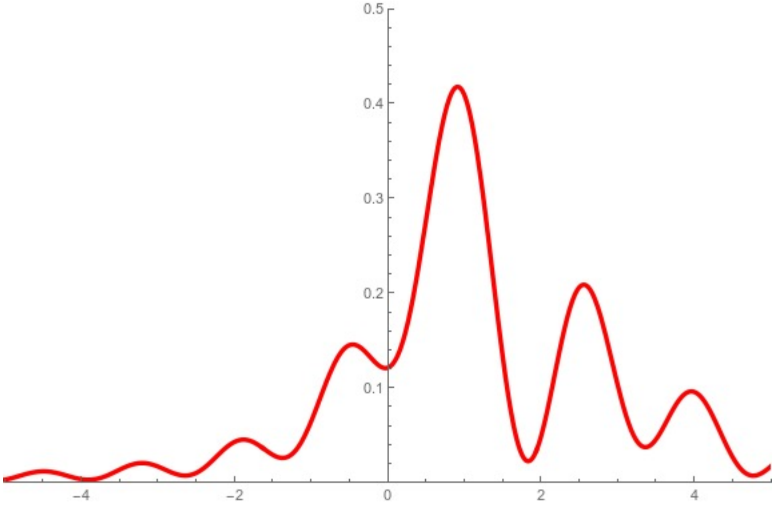

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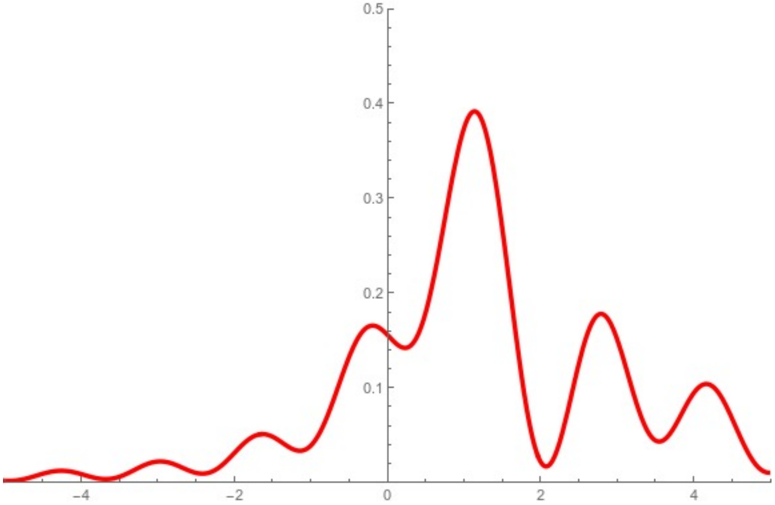

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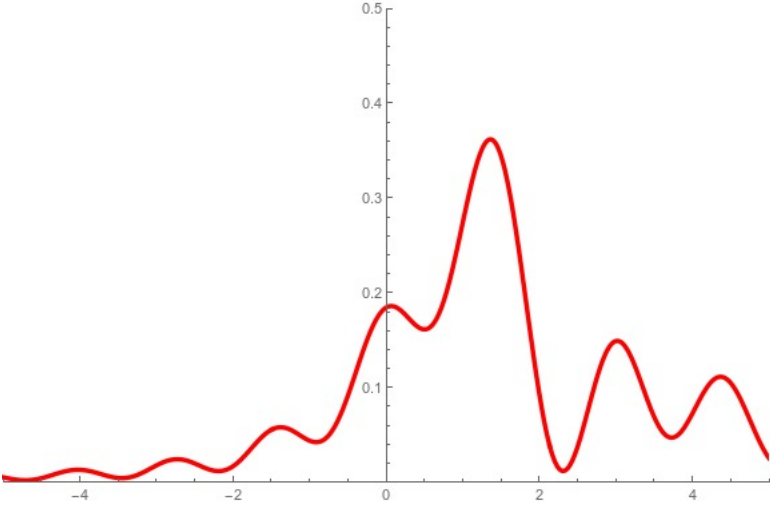

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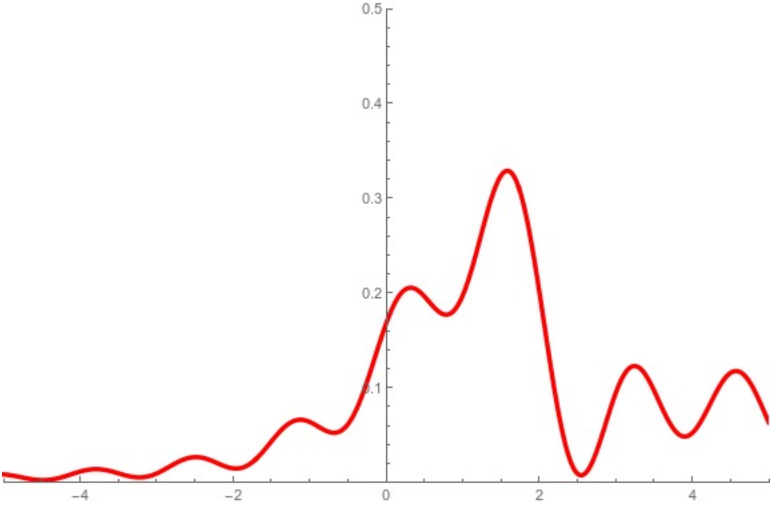

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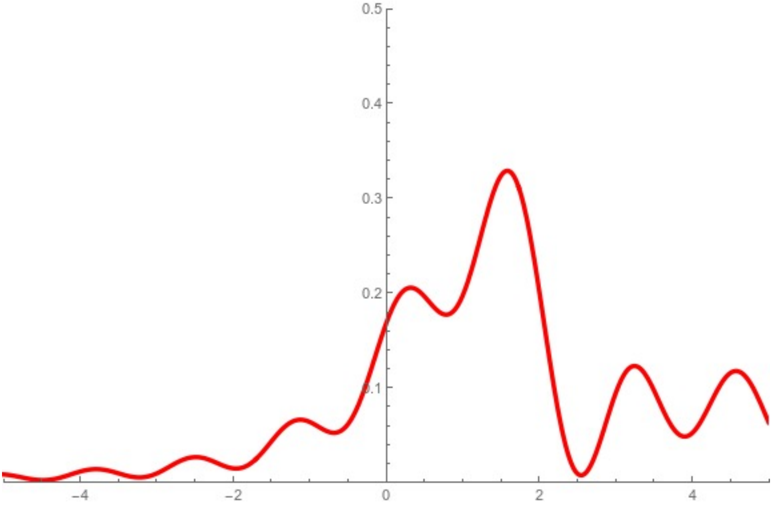

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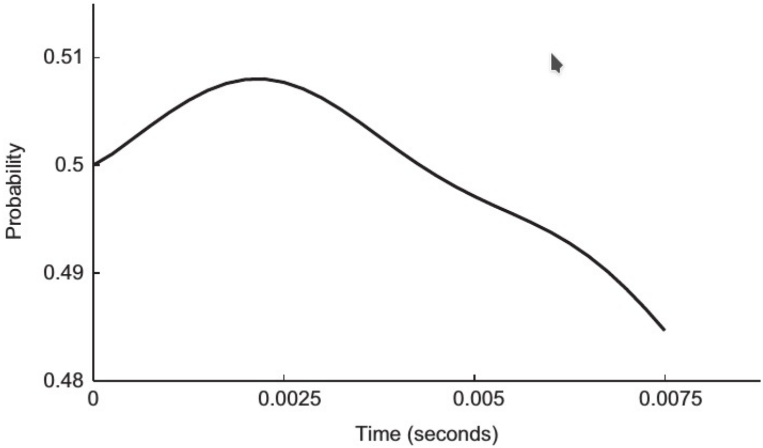

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Definition

We call $E_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the operator such that:

$$\mathcal{F}[E_\pm \psi](p) = \vartheta(\pm p)\widehat{\psi}(p)\ \forall \psi \in L^2(\mathbb{R}),$$

where ϑ is the Heaviside function.

Temporal boundedness of backflow

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Proposition

Let $K : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be the integral operator:

$$(Kf)(u) = -\frac{1}{\pi} \int_0^\infty \frac{\sin(u^2 - v^2)}{u - v} f(v) \, \mathrm{d}v \quad \forall f \in L^2(\mathbb{R}_+).$$

Then K is bounded and self-adjoint, and $\lambda = \sup \sigma(K)$.

Temporal boundedness of backflow

Theorem (Temporal boundedness of backflow)

Let $\lambda = \sup \sigma(K)$*. For any right-mover* $\psi \in L^2(\mathbb{R})$ *such that* $\psi = E_+\psi$ *and for any* $T > 0$ *it holds*

$$\int_0^T j_\psi(0,t)\,\mathrm{d}t \geq -\lambda > -\infty.$$

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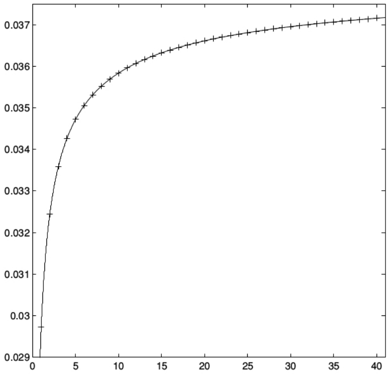
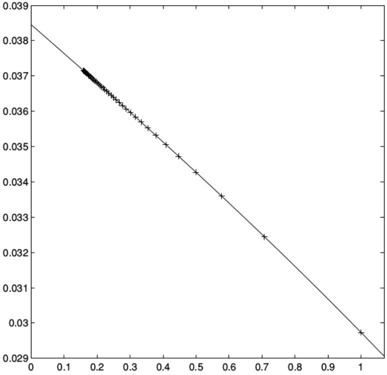
How to approximate λ? Approximating K to an Hermitian operator.





$$\lambda \approx 0.0384517$$

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Lemma

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Proposition

For any $f \in \mathcal{S}(\mathbb{R})$, $f \geq 0$, there exists a constant $\beta_0(f) \in (f) \in (-\infty, 0)$ such that $(\psi|E_+ J(f) E_+\psi) \geq \beta_0(f)$.

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- exists under regularity and short-range assumptions on V.
- links "free" solutions of Schrödinger equation with "interacting" solutions.

Asymptotic right-movers

Definition

Let $V \in L^1(\mathbb{R})$ be a potential. We referred to V as a "short-range" potential (indicated $V \in L^{1+}(\mathbb{R})$) if it satisfies $\|V\|_{1+} = \int_{\mathbb{R}} \mathrm{d}x \, (1 + |x|)|V(x)| < +\infty$.

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Theorem

Let $V \in L^{1+}(\mathbb{R})$. Then

(a) Ω_V *exists.*

(b) $[-\partial_x^2 + 2V(x) - k^2]\psi(x) = 0$ *has unique solutions*

$$\varphi_k(x) = \begin{cases} T_V(k)e^{ikx} + o(1) & \text{for } x \to +\infty \\ e^{ikx} + R_V(k)e^{-ikx} + o(1) & \text{for } x \to -\infty \end{cases}$$

(c) *For any* $\widehat{\psi} \in C_0^\infty(\mathbb{R})$, $(\Omega_V E_+ \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_k(x)\widehat{\psi}(k)\,\mathrm{d}k$.

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Expanding $E_+ \Omega_V^* J(f) \Omega_V E_+$ we have

$$(\psi | E_+ \Omega_V^* J(f) \Omega_V E_+ \psi) \geq \beta_0(f) - 2 \| J(f)(i+P)^{-1} \| [2 + \| P(\Omega_V - T_V) E_+ \|].$$

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Lemma

Let $V \in L^{1+}(\mathbb{R})$. Then, there exists $c_V \in \mathbb{R}$ such that

$$\|P(\Omega_V - T_V)E_+\| \leq 2c_V\|V\|_{1+}$$

Theorem (Boundedness of Backflow in scattering scenarios)

For any potential $V \in L^{1+}(\mathbb{R})$ and for any non-negative $f \in \mathcal{S}(\mathbb{R})$, there exists a constant $\beta_V(f) \in (-\infty, 0)$ such that

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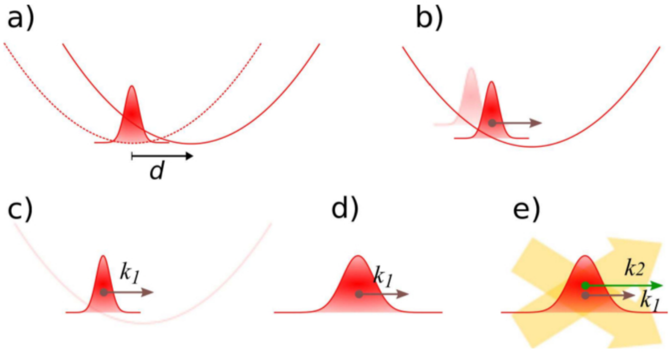
- What about experimental observations? (Bose-Einstein condensate, Bragg pulse, superposition of different momentum sates...)

Experimental set-up



Thank you for your attention!