Formal Methods and Functional Programming - Week 1 (Lectures)

- Author: Ruben Schenk
- Date: 04.03.2021
- Contact: ruben.schenk@inf.ethz.ch

Introduction

Basic concepts in functional programming

One important notion in functional programming is that functions have no side effect, i.e. $f(x)$ always returns the same value (for the same x). This allows us to reason as in mathematics, i.e. if $f(0) = 2$ then $f(0) + f(0) = 2 + 2$.

The above mentioned property is called `referential transparency`, i.e. *an expression evaluates to the same value in every context.*

Another basic concept is that we use `recursion instead of iteration`. This can be shown with an easy example of `gcd`. In Java we might use an iterative function:

```java
public static int gcd (int x, int y) {
    while(x != y) {
        if (x > y) {
            x = x - y;
        } else {
            y = y - x;
        }
    }
    return x;
}
```

whereas in functional programming we strictly use recursion:

```haskell
gcd x y
    | x == y    = x
    | x > y     = gcd (x - y) y
    | otherwise = gcd x (x - y)
```

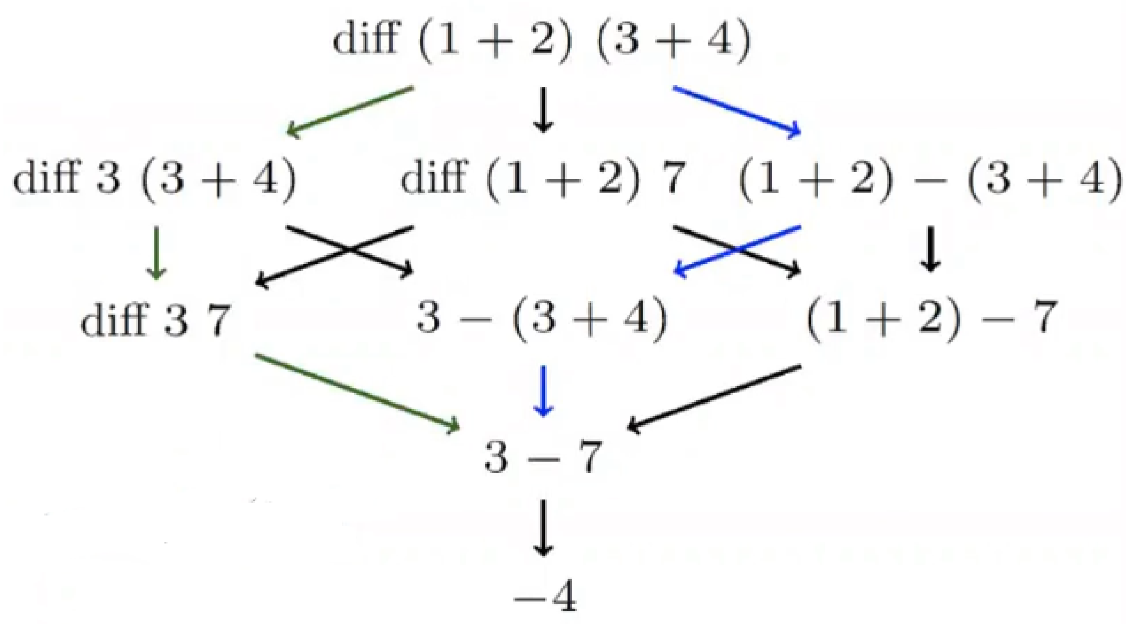
Introduction to functional programming

Expression evaluation

In general, expression evaluation in Haskell works the same way as in mathematics, e.g. for $f(x, y) = x - y$, if we want to compute $f(5, 7)$ we substitute 5 for x and 7 for y.

We differ between two types of `evaluation strategies`:

- `Eager evaluation`, where we evaluate arguments first (corresponds to the *green path* in the picture below)
- `Lazy evaluation`, which is used in Haskell, where we evaluate expression from the left and *only when needed* (corresponds to the blue path in the picture below)



Syntax and Types

The basic syntax of Haskell consists of the following two rules:

- Functions consist of different cases:

```
functionName x1 ... x2
    | guard1 = expr1
    | guard2 = expr2
    ...
    | guardm = exprm
```

- Programs consist of several definitions:

```
myConstant = 5

aFunction y1 ... ym
    | guard1 = expr1
    | guard2 = expr2

anotherFunction z1 ... zk
    ...
```

In Haskell, *indentation determines separation of definitions:

- All function definitions must start at the same indentation level.
- If a definition requires $n > 1$ lines, we indent the lines 2 to n further.

Following an example of a recommended layout:

```
f1 x1 x2
    | a long guard which may go over
      a number of lines
          = a long expression that also can
            go over several lines
    | g2  = e2
    | g3  = e3
```

Spaces are therefore very important, one should ***not use tabs!***

In Haskell we can use the following `types` :

- `Int` for integers with at least the range of $\{-2^{29}, \dots, 2^{29} - 1\}$ and the functions `+, *, ^, -, div, mod, abs`
- `Integer` for unbounded integers and the same functions as `Int`
- `Bool` with the values `True, False` and the binary operators `&&, ||` and the unary operator `not`
- `Char` for single characters as expected, i.e. `'a', 'b', 'c', ...`
- `String` for strings as expected, i.e. `"hello", "wordl", ...`
- `Doubles` for double precision numbers and functions like `+, -, *, /, abs, acos, ...`

We furthermore examine the type `tuple` more in detatil. Tuples are represented in `()` brackets. For example we might represente a student as a triple with his name, student id, and starting year as follows:

```
(String, Int, Int)
```

The above example is also called a `type constructor`. We can build and `element` of a student if we give it specific values, i.e.

```
("Ueli Naef", 1234, 2015)
```

The above example is also called a `term constructor`. Functions can take tuples as arguments and/or return tupled values as shown in the example below:

```
addPair :: (Int, Int) -> Int
addPair (x, y) = x + y
-----------------------------
? addPair (3, 4)
7
```

Function scope

Functions have a `global scope`, i.e. a function can be called from any other function.

We can force a `local scope` with the keywords `let` and `where` as shown in the example below:

```
f x = let sq y = y * y
      in sq x + sq x
```

In the above code, `f x` is defined as `sq x + sq x` where `sq y` is *locally* defined as `y * y`. This means that we can evaluate `f 10` but not `sq 10`.

The keyword `where` comes directly after a function definition and is used to define bindings over all guards:

```
f p1 p2 ... pm
    | g1 = e1
    | g2 = e2
    ...
    | gk = ek
    where
        v1 = r1
        v2 = r2
        ...
```

Natural Deduction

To carry out `formal reasoning` about systems we need three essential parts:

1. Language
2. Semantics
3. Deductive system for carrying out proofs

As an introduction to this topic we look at an abstract example of a formal proof:

- Language: $\mathcal{L} = \{\bigoplus, \bigotimes, +, \times\}$
- Rules:
  - α : If $+$, then $\bigotimes$.
  - β : If $+$, then $\times$.
  - γ : If $\bigotimes$ and $\times$, then $\bigoplus$.
  - δ : $+$ holds.

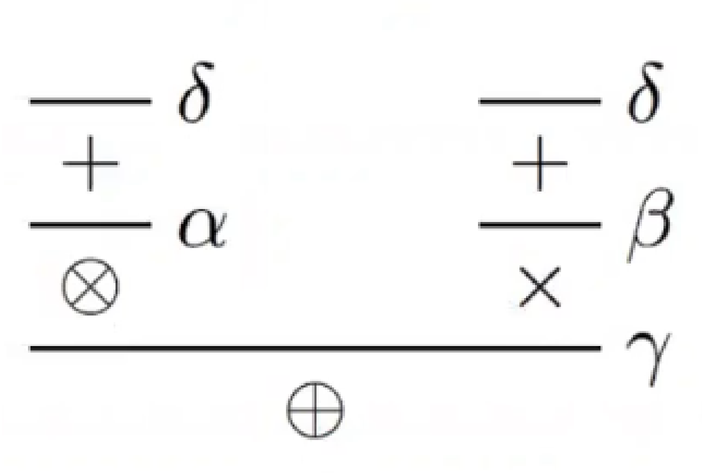
Given the task "*Prove* $+$" we can proceed as follows:

1. $+$ holds by δ
2. $\bigotimes$ holds by α with 1.
3. $\times$ holds by β with 1.
4. $\bigoplus$ holds by γ with 2 and 3.

Now, in a `deductive proof system` our rules look the following way:

$$\frac{+}{\bigotimes}\alpha \qquad \frac{+}{\times}\beta \qquad \frac{\bigotimes \quad \times}{\bigoplus}\gamma \qquad \frac{}{+}\delta$$

Our prove can then be displayed as a `derivation tree` (following in *Parwitz style*):



We now change the rules a little bit and look at another example of a formal proof:

- Language: $\mathcal{L} = \{\bigoplus, \bigotimes, +, \times\}$
- Rules:
  - α : If $+$, then $\bigotimes$.
  - β : If $+$, then $\times$.
  - γ : If $\bigotimes$ and $\times$, then $\bigoplus$.
  - δ : **We may assume $+$ when proving $\bigotimes$.**

Our linear proof changes therefore to:

1. **Assume** $+$ holds by δ
2. $\bigotimes$ holds by α with 1.
3. $\times$ holds by β with 1.

4. $\bigoplus$ holds by γ with 2 and 3.

In our deductive proof system we now change our rules as follows:

$$\frac{\Gamma \vdash +}{\Gamma \vdash \bigotimes}\alpha \qquad \frac{\Gamma \vdash +}{\Gamma \vdash \times}\beta \qquad \frac{\Gamma \vdash \bigotimes \quad \Gamma \vdash \times}{\Gamma \vdash \bigoplus}\gamma \qquad \frac{\Gamma, + \vdash \bigoplus}{\Gamma \vdash \bigoplus}\delta$$

Here, Γ stands for some assumption. The first rule, read bottom-up, therefore reads as "*To prove* $\bigotimes$ *holds under some assumption* Γ*, it suffices to show that* $+$ *holds under the same assumption* Γ."

Our derivation tree, now in *Gentzen-style*, looks as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{+ \vdash +}\,\textit{axiom}}{+ \vdash \bigotimes}\,\alpha \qquad \dfrac{\dfrac{}{+ \vdash +}\,\textit{axiom}}{+ \vdash \times}\,\beta}{+ \vdash \bigoplus}\,\gamma}{\vdash \bigoplus}\,\delta$$

Propositional logic

Syntax

The formal definition is given by:

- Let a set $\mathcal{V}$ of variables be given. Then $\mathcal{L}_P$, the `language of propositional logic`, is the smallest set where:
  - $X \in \mathcal{L}_P$ if $X \in \mathcal{V}$
  - $\bot \in \mathcal{L}_P$.
  - $A \wedge B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$.
  - $A \vee B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$.
  - $A \to B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$.

Semantics

A `valuation` $\sigma : \mathcal{V} \to \{\mathrm{True}, \mathrm{False}\}$ is a function mapping variables to truth values. We furthermore let `Valuations` be the set of valuations.

`Satisfiability` describes the smalles relation $\models \subseteq Valuations \times \mathcal{L}_P$ such that:

- $\sigma \models X$ if $\sigma(X) = \mathrm{True}$
- $\sigma \models A \wedge B$ if $\sigma \models A$ and $\sigma \models B$
- $\sigma \models A \vee B$ if $\sigma \models A$ or $\sigma \models B$
- $\sigma \models A \to B$ if whenerver $\sigma \models A$ then $\sigma \models B$

We note here that $\sigma \not\models \bot$, for every $\sigma \in Valuations$.

> A formula $A \in \mathcal{L}_P$ is `satisfiable` if
> $\sigma \models A, \text{ for some valuation } \sigma$

> A formula $A \in \mathcal{L}_P$ is `valid` (a `tautology`) if
> $\sigma \models A, \text{ for all valuations } \sigma$

We furthermore respect `semantic entailment`, that is, $A_1, \ldots, A_n \models A$ if for all σ, if $\sigma \vdash A_1, \ldots, \sigma \models A_n$, then $\sigma \models A$.

Requirement for a deductive system

For a deductive system we require that *syntactic entailment* $\vdash$ (derivation rules) and *semantic entailment* $\models$ (truth tables) agree. This requirement has two parts. For $H \equiv A_1, \ldots, A_n$ some collection of formulae:

1. `Soundness`: If $H \vdash A$ can be derived, then $H \models A$
2. `Completeness`: If $H \models A$, then $H \vdash A$ can be derived

Natural deduction for propositional formulae

We define three keywords for natural deduction:

- `Sequent`: An assertion of the form $A_1, \ldots, A_n \vdash A$ where all $A, A_1, A_2, \ldots, A_n$ are propositional formulae
- `Axiom`: A starting point for building derivation trees of the form
  $\frac{}{\ldots, A, \ldots \vdash A}\ axiom$
- `Proof` (of A): A derivation tree with root $\vdash A$

Conjunction rules

We distinguish betwee ntwo kinds of `rules`:

- `introduce`, denoted with $-I$, which introduce a connective
- `eliminate`, denoted by $-EL$ or $-ER$, which eliminate connectives

Example:

$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge -I, \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge -EL, \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge -ER$

Example derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash X \wedge (Y \wedge Z)}\,\textit{axiom}}{\Gamma \vdash X}\,\wedge\text{-}\textit{EL} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash X \wedge (Y \wedge Z)}\,\textit{axiom}}{\Gamma \vdash Y \wedge Z}\,\wedge\text{-}\textit{ER}}{\Gamma \vdash Z}\,\wedge\text{-}\textit{ER}}{\underbrace{X \wedge (Y \wedge Z)}_{\equiv \Gamma} \vdash X \wedge Z}\,\wedge\text{-}\textit{I}$$

Implication rules

We have the following two `implication rules`:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to -I, \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B} \to -E$$

Disjunction rules

We have the following three `disjunction rules`:

$$\frac{\Gamma \vdash A}{\gamma \vdash A \vee B} \vee -IL, \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee -IR$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee -E$$

Falsity and negation rules

We have the following `falsity rule`:

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot - E$$

and the following `negation rules` (we define $\neg A$ as $A \to \bot$):

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash B} \neg - E$$

First-order logic

Syntax

There are two syntactic categories: `terms` and `formulae`.

Furthermore, a `signature` consists of a set of function symbols $\mathcal{F}$ and a aset of predicate symbols $\mathcal{P}$ and we also denote the set of variables as $\mathcal{V}$.

Then *Term*, the `terms of first-order logic`, is the smalles set where

1. $x \in Term$ if $x \in \mathcal{V}$, and
2. $f^n(t_1, \ldots, t_n) \in Term$ if $f^n \in \mathcal{F}$ and $t_j \in Term$ for all $1 \leq j \leq n$

Form, the `formulae of first-order logic`, is the smallest set where

1. $\bot \in Form$
2. $p^n(t_1, \ldots, t_n) \in Form$ if $p^n \in \mathcal{P}$ and $t_j \in Term$, for all $1 \leq j \leq n$
3. $A \circ B \in Form$ if $A \in Form$, $B \in Form$, and $\circ \in \{\wedge, \vee, \to\}$
4. $Qx.\, A \in Form$ if $A \in Form$, $x \in \mathcal{V}$, and $Q \in \{\forall, \exists\}$

Each occurrence of each variable in a formula is either `bound` or `free`.

A variable occurrence x in a formula A is `bound` if x occurs within a subformula B of A of the form $\exists x.\, B$ or $\forall x.\, B$ and is said to be `free` otherwise.

α - conversion

We can rename *bound* variables at any time (called α `-conversion`). Example:
$\forall x.\, \exists y.\, p(x,\, y) \equiv \forall y.\, \exists x.\, p(y,\, x)$

Omitting parantheses

For binary operators we have the following binding strengths:

- $\wedge$ binds stronger than $\vee$ binds stronger than $\rightarrow$
- $\rightarrow$ associates to the right, $\wedge$ abd $\vee$ bind to the left
- $\neg$ binds stronger than any binary operator
- Quantifiers extend to the right as far as possible, that is, the end of the line or ")"

Semantics

A `structure` is a pair $\mathcal{S} = \langle U_\mathcal{S},\, I_\mathcal{S} \rangle$ where $U_\mathcal{S}$ is a non-empty set, the `universe`, and $I_\mathcal{S}$ is a mapping where:

1. $I_\mathcal{S}(p^n)$ is an n-ary relation on $U_\mathcal{S}$, for $p^n \in \mathcal{P}$, and
2. $I_\mathcal{S}(f^n)$ is an n-ary (total) function on $U_\mathcal{S}$, for $f^n \in \mathcal{F}$

As shorthand, we may also write $p^\mathcal{S}$ for $I_\mathcal{S}(p)$ and $f^\mathcal{S}$ for $I_\mathcal{S}(f)$.

An `interpretation` is a pair $\mathcal{I} = \langle \mathcal{S},\, v \rangle$, where $\mathcal{S} = \langle U_\mathcal{S},\, I_\mathcal{S} \rangle$ is a structure and $v : \mathcal{V} \rightarrow U_\mathcal{S}$ a valuation.

The `value` of a term t under the interpretation $\mathcal{I} = \langle \mathcal{S},\, v \rangle$ is written as $\mathcal{I}(t)$ and defined by

1. $\mathcal{I}(x) = v(x)$, for $x \in \mathcal{V}$, and
2. $\mathcal{I}(f(t_1, \ldots, t_n)) = f^\mathcal{S}(\mathcal{I}(t_1), \ldots, \mathcal{I}(t_n))$

When $\langle \mathcal{S},\, v \rangle \models A$ we say A `is satisfied with respect to` $\langle \mathcal{S},\, v \rangle$ or $\langle \mathcal{S},\, v \rangle$ `is a model of` A.

When every suitable interpretation is a model, we write $\models A$ and say A is `valid`.

A is `satisifable` if there is at leat one model for A.

Following an example of a suitable model for

$\forall x.\, p(x,\, s(x))$

- $U_\mathcal{S} = \mathcal{N}$
- $p^\mathcal{S} = \{(m,\, n) \,|\, m, n \in U_\mathcal{S} \text{ and } m < n\}$
- $s^\mathcal{S}(x) = x + 1$