WuS - Lecture Notes Week 2

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0.1 Examples of Probability Space

0.1.1 Example with Ω Finite

We discuss a particular type of probability spaces where the sample space Ω is an arbitrary **finite** set, and all the outcomes have the **same** probability $p_\omega = \frac{1}{|\Omega|}$.

Definition: Let Ω be a finite sample space. The **Laplace model** on Ω is the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where:

- $\mathcal{F} = \mathcal{P}(\Omega)$,
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is defined by

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

Example: We consider $n \geq 3$ points on a circle, from which we select 2 at random. What is the probability that these two points selected are neighbors? We consider the Laplace model one

$$\Omega = \{E \subset \{1, 2, ..., n\} : |E| = 2\}.$$

The event "the two points of E are neighbors" is given by

$$A = \{\{1, 2\}, \{2, 3\}, ..., \{n-1, n\}, \{n, 1\}\}$$

and we have

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{n}{\binom{n}{2}} = \frac{2}{n-1}.$$

0.1.2 Example with Ω Infinite Countable

Example: We throw a biased coin multiple times, at each throw, the coin falls on head with probability p, and it falls on tail with probability $1 - p$ (p is a fixed parameter in $[0, 1]$). We stop at the first time we see a tail. The probability that we stop exactly at time k is given by

$$p_k = p^{k-1}(1 - p).$$

For this experiment, one possible probability space is given by:

- $\Omega = \mathbb{N} \setminus \{0\} = \{1, 2, 3, ...\}$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- for $A \in \mathcal{F}$, $\mathbb{P}[A] = \sum_{k \in A} p_k$

0.2 Properties of Events

0.2.1 Operations on Events and Interpretation

The following propositions asserts that the different well-known set operations are allowed.

Proposition (Consequences of the definition): Let $\mathcal{F}$ be a sigma-algebra on Ω. We have:

- **P4.** $\emptyset \in \mathcal{F}$
- **P5.** $A_1, A_2, ... \in \mathcal{F} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$
- **P6.** $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$
- **P7.** $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$

A short summary of the common set-operations is given below:

- A^C : A does not occur.
- $A \cap B$: A and B occur.
- $A \cup B$: A or B occurs
- $A \Delta B$: one and only one of A or B occurs
- $A \subset B$: If A occurs, then B occurs
- $A \cap B = \emptyset$: A and B cannot occur at the same time
- $\Omega = A_1 \cup A_2 \cup A_3$ with A_1, A_2, A_3 pairwise disjoint: for each outcome ω, one and only one of the events A_1, A_2, A_3 is satisfied.

0.3 Properties of Probability Measures

0.3.1 Direct Consequences of the Definition

Proposition: Let $\mathbb{P}$ be an arbitrary measure on $(\Omega, \mathcal{F})$. We have:

- **P3.** $\mathbb{P}[\emptyset] = 0$.
- **P4. (additivity)** Let $k \geq 1$. let $A_1, ..., A_k$ be k pairwise disjoint events, then $\mathbb{P}[A_1 \cup \cdots \cup A_k] = \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_k]$.
- **P5.** Let A be an event, then $\mathbb{P}[A^C] = 1 - \mathbb{P}[A]$.
- **P6.** If A and B are two events (not necessarily disjoin), then $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$.

0.3.2 Useful Inequalities

Proposition (Monotonicity): Let A, $B \in \mathcal{F}$, then

$$A \subset B \implies \mathbb{P}[A] \leq \mathbb{P}[B].$$

Proposition (Union bound): Let A_1, $A_2, ...$ be a sequence of events (not necessarily disjoint), then we have

$$\mathbb{P}[\bigcup_{i=1}^{\infty} A_i] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i].$$

Remark: The union bound also applies to a *finite* collection of events.

0.3.3 Continuity Properties of Probability Measures

Proposition: Let (A_n) be an increasing sequence of events (i.e. $A_n \subset A_{n+1}$ for every n). then

$$\lim_{n \to \infty} P[A_n] = \mathbb{P}[\bigcup_{n=1}^{\infty} A_n]. \quad \text{(increasing limit)}$$

Let (B_n) be a decreasing sequence of events (i.e. $B_n \supset B_{n+1}$ for every n). Then

$$\lim_{n \to \infty} P[B_n] = \mathbb{P}[\bigcap_{n=1}^{\infty} B_n]. \quad \text{(decreasing limit)}$$

Remark: By monotonicity, we have $\mathbb{P}[A_n] \leq \mathbb{P}[A_{n+1}]$ and $\mathbb{P}[B_n] \geq \mathbb{P}[B_{n+1}]$ for every n. Hence the limits in the proposition are well defined as monotone limits.

0.4 Conditional Probabilities

Definition (Conditional probability): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let A, B be two events with $\mathbb{P}[B] > 0$. The **conditional probability of A given B** is defined by

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Remark: $\mathbb{P}[B \mid B] = 1$.

Proposition: Let Ω, $\mathcal{F}$, $\mathbb{P}$ be some probability space. Let B be an event with positive probability. Then $\mathbb{P}[\,.\mid B]$ is a probability measure on Ω.

Proposition (Formula of total probability): Let $B_1, ..., B_n$ be a partition of the sample space Ω with $\mathbb{P}[B_i] > 0$ for every $1 \leq i \leq n$. Then, one has

$$\forall A \in \mathcal{F} : \mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \mid B_i]\mathbb{P}[B_i].$$

Here, a *partition* B_i is such that $\Omega = B_1 \cup \cdots \cup B_n$ and the events are pariwise disjoint.

Proposition (Bayes formula): Let $B_1, ..., B_n \in \mathcal{F}$ be a partition of Ω with $\mathbb{P}[B_i] > 0$ for every i. For every event A with $\mathbb{P}[A] > 0$, we have

$$\forall i = 1, ..., n : \mathbb{P}[B_i \mid A] = \frac{\mathbb{P}[A \mid B_i] \cdot \mathbb{P}[B_i]}{\sum_{j=1}^{n} \mathbb{P}[A \mid B_j] \cdot \mathbb{P}[B_j]}.$$

0.5 Independence

0.5.1 Independence of Events

Definition (Independence of two events): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two events A and B are said to be **independent** If

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B].$$

Remark: If $\mathbb{P}[A] \in \{0, 1\}$, then A is independent of every event, i.e. $\forall B \in \mathcal{F} : \mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$. Furthermore we might also state, that A is independent of B if and only if A is independent of B^C.

Proposition: Let A, $B \in \mathcal{F}$ be two events with $\mathbb{P}[A], \mathbb{P}[B] > 0$. Then the following are equivalent:

- $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$ (A and B are independent)
- $\mathbb{P}[A \mid B] = \mathbb{P}[A]$ (the occurrence of B has no influence on A)
- $\mathbb{P}[B \mid A] = \mathbb{P}[B]$ (the occurrence of A has no influence on B)

Definition: Let I be an arbitrary set of indices. A collection of events $(A_i)_{i \in I}$ is said to be **independent** if

$$\forall J \subset I \text{ infinite} : \mathbb{P}[\bigcap_{j \in J} A_j] = \prod_{j \in J} \mathbb{P}[A_j].$$