EnpRisk - Lecture Notes Week 6

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0.0.1 Generalized Logistic Growth

Many systems exhibit succession of S-curves because the advances in technology etc. increase the carrying capacity K. One idea to *generalize the logistic growth* is to include into this the logistic equation with a population dependent carrying capacity with *delay time* τ:

$$\frac{dP}{dt} = rP(t)\big[1 - \frac{P(t)}{K(t)}\big], \text{ with } K(t) = A + BP(t-\tau)$$

In other words:

$$\frac{dx}{dt} = \alpha \cdot x(t) + \beta \cdot \frac{x^2(t)}{a + bx(t-\tau)},$$

with $x \sim P$ and parameters a, b related to r, A and B. We can distinguish four different scenarios:

- $\alpha = +1$, $\beta = -1$: gain and competition
- $\alpha = +1$, $\beta = +1$: gain and cooperation
- $\alpha = -1$, $\beta = -1$: loss and competition
- $\alpha = -1$, $\beta = +1$: loss and cooperation

Another idea is to instead of a linearly growing capacity, consider the case of exponential growth:

$$\frac{dx}{dt} = \sigma_1 x(t) - \sigma_2 \frac{x^2}{y(x)}, \text{ with } y(x) = \exp(bx(t-\tau))$$

In other words:

$$\frac{dx}{dt} = \sigma_1 x(t) - \sigma_2 x^2(t) e^{-bx(t-\tau)}$$

We can distinguish the same cases as before with $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$.

Finally, we can also consider **coupled logistic equations.** The idea is, that instead of only one species, we can also have two interacting species:

$$\frac{dx}{dt} = x - \frac{x^2}{1 + bxz} \quad \frac{dz}{dt} = z - \frac{z^2}{1 + gxz},$$

where x is the species one and y is the species two.

0.0.2 Interlude: Chaos Theory

The **logistic map** is defined as:

$$x(n+1) = \alpha x(n)[1 - x(n)]$$

It can be shown that x is *chaotic* for almost all values of $\alpha \in [3.569..., 4]$.
It can also be shown that the logistic map is nothing but a discrete version of the logistic equation

$$\frac{P(t)}{dt} = rP(t)\big[1 - \frac{P(t)}{K}\big],$$

with $\alpha = r + 1$.