

FMFP - Complete Summary

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1 Introduction & Basic Haskell Syntax

1.1 Example: GCD

The **GCD problem** is given as follows: Compute the greatest common divisor of two natural numbers. We have the following *specifications*: Let $x, y \in \mathcal{N}$ be given. The number z is the **greatest common divisor** of x and y iff. $z \mid x$ and $z \mid y$ and there is no z' , with $z' > z$, such that $z' \mid x$ and $z' \mid y$. Here, $z \mid x \equiv \exists a \in \mathcal{N}. a \cdot z = x$.

The problem specification is not **constructive**, i.e. it does not describe how the GCD should be computed.

1.1.1 Imperative GCD

```
public static int gcd(int x, int y) {
    while(x != y) {
        if(x > y) {
            x = x - y;
        } else {
            y = y - x;
        }
    }
    return x;
}
```

The **imperative GCD**, as shown above, consists of control flow statements and assignments. Assignments change the computer's *state*. To understand `gcd`, one must understand how its state changes.

Poor man's reasoning would be to simulate and track the memory content during execution. A better way would be to use *Hoare logic* in the form of $\{P\} \text{ prog } \{Q\}$. Formal reasoning is possible, but not easy!

1.1.2 Functional GCD

```
gcd x y
  | x == y    = x
  | x > y     = gcd (x - y) y
  | otherwise = gcd x      (y - x)
```

The functional way formalizes *what* should be computed, rather than *how*. This is an algorithm, provided we have also specified how functions are executed.

1.2 Basic Concepts in Functional Programming

1.2.1 Referential Transparency

Functions compute values. But functions also *are* values: we can compute and return them. It is important to note that functions in functional programming have **no side effects**: $f(x)$ always returns the same value. This in contrast to other programming languages we've known so far. Consider the following Java example:

```
class Test {
    static int y = 0;
    static int f(int x) {
        y = y + 1;
        return y;
    }
}
```

```
public static void main(String[] args) {
    System.out.println(f(0));
    System.out.println(f(0));
}
```

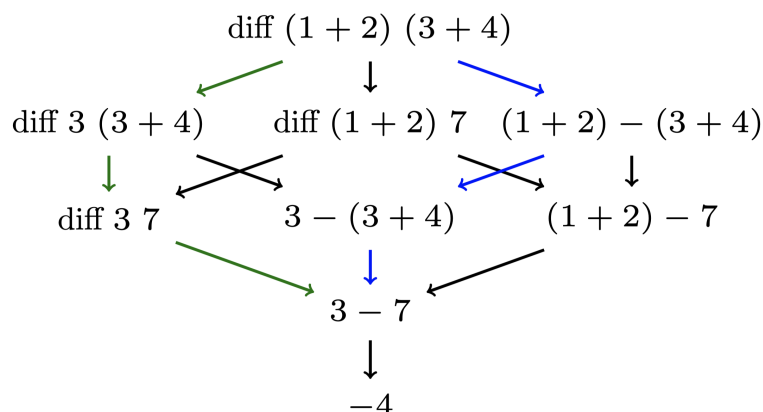
One will immediately see that this prints out 0 and then 1, which means that $f(0)$ returns different values with the same input.

Since functions have no side effects, we can reason with the more easily in mathematics. This property is also called **referential transparency**: an expression evaluates to the same value in every context.

1.2.2 Evaluation

An **evaluation strategy** defines how and when expressions are evaluated during the execution of a program. We differ between two strategies:

- *Eager evaluation*: evaluate arguments first. Also called "call-by-value", corresponds to the left (green) path in the figure below.
- *Lazy evaluation*: evaluate arguments only when needed (used by Haskell). Also called "call-by-need" (or "left-most/outermost"), corresponds to the right (blue) path in the figure below.



1.3 Basic Haskell Syntax

1.3.1 Syntax and Types

We present the basic syntax principles in the following code example:

```
gcd x y      -- functions and arguments start with lower-case letters
  | x == y    = x
  | x > y      = gcd (x - y) y      -- arguments are written in sequence and
  | otherwise = gcd x      (y - x)  -- separated by whitespace
```

Furthermore, functions consist of different cases and a program consists of several definitions:

```
myConstant = 5

afunction y1 y2 ... ym
  | guard1 = expr1
  | guard2 = expr2
  ...
  | guardm = exprm

anotherFucntion z1 z2 ... zk = ...
```

Indentation determines the separation of definitions. All function definitions must start at the same indentation level. If a definition requires $n > 1$ lines, we indent lines 2 to n further. This leads to the following *recommended layout*:

```
f1 x1 x2
  | a long guard which may go over
    a number of lines
    = a long expression that can also go over
      several lines
  | g2 = expr2

f2 x1 x2 x3 = ...
```

1.3.2 Functions

Functions live in a global scope. This means that a function can be called from any other. Example:

```
f x y = ...
g x = ... h ...
h z = ... f ... g ...
```

We can define functions and variables in local scope with **let** and **where**:

```
let x1 = e1
    ...
    xn = en
in e
```

2 Natural Deduction

2.1 Introduction to Natural Deduction

2.1.1 Abstract Example (without Assumptions)

Consider the following "meaningless" language:

$$\mathcal{L} = \{\oplus, \otimes, \times, +\}$$

We furthermore state the following *rules*:

- α : If $+$, then \otimes
- β : If $+$, then \times
- γ : If \otimes and \times , then \oplus
- δ : $+$ holds

Our goal is to prove \oplus . We might proceed as follows:

1. $+$ holds by γ .
2. \otimes holds by α with 1.
3. \times holds by β with 1.
4. \oplus holds by γ with 2 and 3.

$$\frac{\frac{\frac{\delta}{+}}{\alpha}}{\otimes} \quad \frac{\frac{\frac{\delta}{+}}{\beta}}{\times} \quad \frac{\quad}{\oplus} \gamma$$

We revisit the previous example by slightly changing one of our rules:

- We can build the following proof system. In this system, Γ is the set of assumptions we make during our proof:

$$\begin{array}{c} \overline{\dots, A, \dots \vdash A} \text{ axiom} \\ \\ \frac{\Gamma \vdash +}{\Gamma \vdash \otimes} \alpha \qquad \frac{\Gamma \vdash +}{\Gamma \vdash \times} \beta \\ \\ \frac{\Gamma \vdash \otimes \quad \Gamma \vdash \times}{\Gamma \vdash \oplus} \gamma \qquad \frac{\Gamma, + \vdash \oplus}{\Gamma \vdash \oplus} \delta \end{array}$$

$$\frac{\frac{\frac{}{+ \vdash +} \textit{axiom}}{+ \vdash +} \alpha}{+ \vdash \otimes} \quad \frac{\frac{\frac{}{+ \vdash +} \textit{axiom}}{+ \vdash +} \beta}{+ \vdash \times} \gamma}{+ \vdash \oplus} \delta$$

A **proof** is a derivation whose root has no assumptions.

2.2 Propositional Logic

2.2.1 Syntax

Propositions are built from a collection of variables and closed under disjunction, conjunction, implication, etc. More formally, let a set \mathcal{V} of variables be given. \mathcal{L}_P , the **language of propositional logic**, is the smallest set where:

- $X \in \mathcal{L}_P$ if $X \in \mathcal{V}$
- $\perp \in \mathcal{L}_P$
- $A \wedge B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$
- $A \vee B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$
- $A \rightarrow B \in \mathcal{L}_P$ if $A \in \mathcal{L}_P$ and $B \in \mathcal{L}_P$

In the following: X ranges over variables, A and B over formulae.

2.2.2 Semantics

A **valuation** $\sigma : \mathcal{V} \rightarrow \{\text{True}, \text{False}\}$ is a function mapping variables to truth values. Valuations are simple kinds of models (or interpretations). We denote the set of valuations as **Valuations**.

Satisfiability is the smallest relation $\models \subseteq \text{Valuations} \times \mathcal{L}_P$ such that:

- $\sigma \models X$ if $\sigma(X) = \text{True}$
- $\sigma \models A \wedge B$ if $\sigma \models A$ and $\sigma \models B$
- $\sigma \models A \vee B$ if $\sigma \models A$ or $\sigma \models B$
- $\sigma \models A \rightarrow B$ if whenever $\sigma \models A$ then $\sigma \models B$

Note that $\sigma \not\models \perp$ for every $\sigma \in \text{Valuations}$.

We furthermore introduce the following characteristics about propositional logic:

- A formula $A \in \mathcal{L}_P$ is **satisfiable** if $\sigma \models A$, for some valuation σ
- A formula $A \in \mathcal{L}_P$ is **valid** (a **tautology**) if $\sigma \models A$, for all valuations σ
- **Semantic entailment:** $A_1, \dots, A_n \models A$ if for all σ , if $\sigma \models A_1, \dots, \sigma \models A_n$ then $\sigma \models A$

Examples:

- $X \wedge Y$ is satisfiable as $\sigma \models X \wedge Y$ for $\sigma(X) = \sigma(Y) = \text{True}$
- $X \rightarrow X$ is valid
- $\neg X, X \vee Y \models Y$ holds as $\sigma \models \neg X$ and $\sigma \models X \vee Y$ constraint σ to $\sigma(X) = \text{False}$ and $\sigma(Y) = \text{True}$, so $\sigma \models Y$

2.2.3 Requirements

We need some **requirements** for *deductive systems*. The main requirement is that syntactic entailment \vdash (derivation rules) and semantic entailment *vDash* (truth tables) should agree. This requirement has two parts:

- **Soundness:** If $\Gamma \vdash A$ can be derived, then $\Gamma \models A$.
- **Completeness:** If $\Gamma \models A$, then $\Gamma \vdash A$ can be derived.

Here, $\Gamma \equiv A_1, \dots, A_n$ is some collection of formulae.

2.2.4 Natural Deduction for Propositional Logic

A **sequent** is an assertion (judgement) of the form $A_1, \dots, A_n \vdash A$, where all A, A_1, \dots, A_n are propositional formulae. A **proof** of A is a derivation tree with root $\vdash A$. If the deductive system is sound, then A is a tautology.

Conjunction **Conjunction** proposes rules of two kinds: *introduce* and *eliminate* connectives. The rules are given as follows:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-I} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-EL} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-ER}$$

Example: The following figure shows an example derivation using conjunction rules.

$$\frac{\frac{\frac{\Gamma \vdash X \wedge (Y \wedge Z)}{\Gamma \vdash X} \wedge\text{-EL} \quad \frac{\frac{\frac{\Gamma \vdash X \wedge (Y \wedge Z)}{\Gamma \vdash Y \wedge Z} \wedge\text{-ER} \quad \frac{\Gamma \vdash Y \wedge Z}{\Gamma \vdash Z} \wedge\text{-ER}}{\Gamma \vdash Z} \wedge\text{-ER}}{\frac{X \wedge (Y \wedge Z) \vdash X \wedge Z}{\equiv \Gamma} \wedge\text{-I}} \text{axiom}$$

Implication The rules for **implication** are given as follows:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow\text{-I} \quad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow\text{-E}$$

Disjunction The rules for **disjunction** are given as follows:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee\text{-IL} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee\text{-IR} \quad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee\text{-E}$$

2.3 First-Order Logic

2.3.1 Syntax

In **first-order logic** we have two syntactic categories: **terms** and **formulae**.

A **signature** consists of a set of function symbols \mathcal{F} and a set of predicate symbols \mathcal{P} . We write f^k (or p^k) to indicate function symbol f (or predicate symbol p) has arity $k \in \mathcal{N}$. Constants are 0-ary function symbols.

Now, let \mathcal{V} be a set of variables. Then:

Definition: *Term*, the **terms of first-order logic**, is the smallest set where:

1. $x \in \text{Term}$ if $x \in \mathcal{V}$, and
2. $f^n(t_1, \dots, t_n) \in \text{Term}$ if $f^n \in \mathcal{F}$ and $t_i \in \text{Term}$, for all $1 \leq i \leq n$.

Definition: *Form*, the **formulae of first-order logic**, is the smallest set where:

1. $\perp \in \text{Form}$,
2. $p^n(t_1, \dots, t_n) \in \text{Form}$ if $p^n \in \mathcal{P}$ and $t_j \in \text{Term}$, for all $1 \leq j \leq n$,
3. $A \circ B \in \text{Form}$ if $A \in \text{Form}$, $B \in \text{Form}$, and $\circ \in \{\wedge, \vee, \rightarrow\}$, and
4. $Qx.A \in \text{Form}$ if $A \in \text{Form}$, $x \in \mathcal{V}$, and $Q \in \{\forall, \exists\}$.

Each occurrence of each variable in a formula is either **bound** or **free**. A variable occurrence x in a formula A is **bound** if x occurs within a subformula B of A of the form $\exists x.B$ or $\forall x.B$.

2.3.2 Binding and α -conversion

Names of bound variables are irrelevant, they just encode the binding structure. We can rename *bound* variables, this process is called **α -conversion**.

It is important to note that the renaming must *preserve the binding structure!*

Some notes on bindings and parentheses:

- \wedge binds stronger than \vee , and \vee binds stronger than \rightarrow .
- \rightarrow associates to the right, *land* and *lor* to the left.
- Negation binds stronger than binary operators.
- Quantifiers extend to the right as far as possible: to the end of the line or ')'

$$\begin{array}{c}
 \frac{\left(p \vee \left(\underline{q \wedge (\neg r)} \right) \right)}{p \rightarrow \left(\underline{(q \vee p) \rightarrow r} \right)} \rightarrow (p \vee q) \\
 \frac{p \wedge \left(\forall x. \left(\underline{q(x) \vee r} \right) \right)}{\neg \left(\forall x. \left(p(x) \wedge \left(\forall x. \left(\underline{(q(x) \wedge r(x)) \wedge s} \right) \right) \right) \right)}
 \end{array}$$

2.3.3 Semantics

A **structure** is a pair $\mathcal{S} = \langle U_{\mathcal{S}}, I_{\mathcal{S}} \rangle$ where $U_{\mathcal{S}}$ is a nonempty set, the **universe**, and $I_{\mathcal{S}}$ is a mapping where:

1. $I_{\mathcal{S}}(p^n)$ is an n -ary relation on $U_{\mathcal{S}}$, for $p^n \mathcal{P}$, and
2. $I_{\mathcal{S}}(f^n)$ is an n -ary (total) function on $U_{\mathcal{S}}$, for $f^n \in \mathcal{F}$

As a shorthand, we write $p^{\mathcal{S}}$ for $I_{\mathcal{S}}(p)$ and $f^{\mathcal{S}}$ for $I_{\mathcal{S}}(f)$.

An **interpretation** is a pair $\mathcal{I} = \langle \mathcal{S}, v \rangle$, where $\mathcal{S} = \langle U_{\mathcal{S}}, I_{\mathcal{S}} \rangle$ is a structure and $v : \mathcal{V} \rightarrow U_{\mathcal{S}}$ is a valuation.

The **value** of a term t under the interpretation $\mathcal{I} = \langle \mathcal{S}, v \rangle$ is written as $\mathcal{I}(t)$ and defined by:

1. $\mathcal{I}(x) = v(x)$, for $x \in \mathcal{V}$, and
2. $\mathcal{I}(f(t_1, \dots, t_n)) = f^{\mathcal{S}}(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n))$.

Satisfiability is the smallest relation $\models \subseteq \text{Interpretations} \times \text{Form}$ satisfying:

- $\langle \mathcal{S}, v \rangle \models p(t_1, \dots, t_n)$ if $(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n)) \in p^{\mathcal{S}}$, where $\mathcal{I} = \langle \mathcal{S}, v \rangle$.
- $\langle \mathcal{S}, v \rangle \models \forall x.A$ if $\langle \mathcal{S}, v[x \rightarrow a] \rangle \models A$, for all $a \in U_{\mathcal{S}}$.
- $\langle \mathcal{S}, v \rangle \models \exists x.A$ if $\langle \mathcal{S}, v[x \rightarrow a] \rangle \models A$, for some $a \in U_{\mathcal{S}}$.

Here, $v[x \rightarrow a]$ is the valuation v' identical to v , except that $v'(x) = a$.

When $\langle \mathcal{S}, v \rangle \models A$, we say that A is *satisfied with respect to* $\langle \mathcal{S}, v \rangle$ or *language* $\langle \mathcal{S}, v \rangle$ is a **model** of A . Note that if A does not have free variables, satisfaction does not depend on the valuation v . We write $\mathcal{S} \models A$. When every interpretation is a model, we write $\models A$ and say that A is **valid**.

A is **satisfiable** if there is at least one model for A (and said to be **contradictory** otherwise).

Example: Consider the following examples:

- $\forall x. \exists y. y * 2 = x$ satisfied w.r.t. rationals.
- $\forall x. \forall y. x < y \rightarrow \exists z. x < z \wedge z < y$ satisfied w.r.t. any dense order.
- $\exists x. x \neq 0$ satisfied w.r.t. structures \mathcal{S} with ≥ 2 elements in $U_{\mathcal{S}}$.
- $(\forall x. p(x, x)) \rightarrow p(a, a)$ is valid.

2.3.4 Substitution

Substitution describes the process of replacing in A all occurrences of a free variable x with some term t . We write $A[x \rightarrow t]$ to indicate the substitution.

Example:

$$\begin{aligned} A &\equiv \exists y. y * x = x * z \\ A[x \rightarrow 2 - 1] &\equiv \exists y. y * (2 - 1) = (2 - 1) * z \\ A[x \rightarrow z] &\equiv \exists y. y * z = z * z \end{aligned}$$

All free variables of t must still be free in $A[x \rightarrow t]$. Avoid *capture*! If necessary, α -convert A before substitution.

2.3.5 Universal Quantification

The rules are as follows:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x. A} \forall\text{-I}^* \quad \frac{\Gamma \vdash \forall x. A}{\Gamma \vdash A[x \mapsto t]} \forall\text{-E}$$

The side condition $*$ is: x must not be free in any assumption in Γ .

2.3.6 Existential Quantification

The rules are as follows:

$$\frac{\Gamma \vdash A[x \mapsto t]}{\Gamma \vdash \exists x. A} \exists\text{-I} \quad \frac{\Gamma \vdash \exists x. A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \exists\text{-E}^*$$

The side condition $*$ is: x is neither free in B nor free in Γ .

2.4 Equality

Equality is a logical symbol with associated proof rules. One speaks of *first-order logic with equality* rather than equality just being another predicate:

- Extended language: $t_1 = t_2 \in \text{Form}$ if $t_1, t_2 \in \text{Term}$
- extended definition of semantic entailment \models : $\mathcal{I} \models t_1 = t_2$ if $\mathcal{I}(t_1) = \mathcal{I}(t_2)$

Equality is an *equivalence* relation with the following rules:

$$\frac{}{\Gamma \vdash t = t} \text{ref} \quad \frac{\Gamma \vdash t = s}{\Gamma \vdash s = t} \text{sym} \quad \frac{\Gamma \vdash t = s \quad \Gamma \vdash s = r}{\Gamma \vdash t = r} \text{trans}$$

And equality is also a *congruence* on terms and all definable relations:

$$\frac{\Gamma \vdash t_1 = s_1 \quad \dots \quad \Gamma \vdash t_n = s_n}{\Gamma \vdash f(t_1, \dots, t_n) = f(s_1, \dots, s_n)} \text{cong}_1$$

$$\frac{\Gamma \vdash t_1 = s_1 \quad \dots \quad \Gamma \vdash t_n = s_n \quad \Gamma \vdash p(t_1, \dots, t_n)}{\Gamma \vdash p(s_1, \dots, s_n)} \text{cong}_2$$

2.5 Correctness

Correctness is important! But what does correctness mean? What properties should hold?

- *Termination*: Important for many, but not all, programs.
- *Functional behavior*: Function should return "correct" value.

2.5.1 Termination

If f is defined in terms of functions g_1, \dots, g_k ($g_i \neq f$), and each g_i terminates, then so does f . The problem we encounter here is *recursion*, i.e. when some $g_i = f$.

A sufficient condition for termination is that arguments must be smaller along a well-founded order on function's domain:

- An order $>$ on a set S is **well-founded** iff. there is no infinite decreasing chain $x_1 > x_2 > x_3 > \dots$ for $x_i \in S$.

We can construct new well-founded relations from existing ones:

Let R_1 and R_2 be binary relations on a set S . The composition of R_1 and R_2 is defined as:

$$R_2 \circ R_1 \equiv \{(a, c) \in S \times S \mid \exists b \in S. a R_1 b \wedge b R_2 c\}$$

Note: For binary relation R , we write $a R b$ for $(a, b) \in R$.

Let $R \subseteq S \times S$. Define:

$$\begin{aligned} R^1 &\equiv R \\ R^{n+1} &\equiv R \circ R^n, \text{ for } n \geq 1 \\ R^+ &\equiv \bigcup_{n \geq 1} R^n \end{aligned}$$

So $a R^+ b$ iff. $a R^i b$ for some $i \geq 1$.

Lemma: Let $R \subseteq S \times S$. Let $s_0, s_i \in S$ and $i \geq 1$. Then $s_0 R^i s_i$ iff. there are $s_1, \dots, s_{i-1} \in S$ such that $s_0 R s_1 R \dots R s_{i-1} R s_i$.

Theorem: If $>$ is a well-founded order on set S , then $>^+$ is also well-founded on S .

Example: Consider the following function:

```
fac 0 = 1
fac n = n * fac (n - 1)
```

`fac n` has only `fac (n - 1)` as a recursive call, and $n > n - 1$. Here, $>$ is the standard ordering over the natural numbers. Therefore, the function terminates.

2.5.2 Proofs

Consider the following program:

```
maxi :: Int -> Int -> Int
maxi n m
  | n >= m    = n
  | otherwise = m
```

Can we prove that `maxi n m >= n`? We to a **reasoning by cases**:

We have $n \geq m \vee \neg(n \geq m)$. Now we show that `maxi n m >= n` for both cases:

- Case 1: $n \geq m$, then `max n m = n` and $n \geq n$.

- Case 2: $\neg(n \geq m)$, then $\max_i n \ m = m$. But $m > n$, so $\max_i n \ m \geq n$.

But how do we prove a formula P (with free variable n), for all $n \in \mathcal{N}$? For example, how do we prove the following equality:

$$\forall n \in \mathcal{N}. 0 + 1 + 2 + \dots + n = n \cdot (n + 1) / 2$$

We can do a **proof by induction**:

- Base case: Prove $P[n \rightarrow 0]$
- Step case: For an arbitrary m not free in P , prove $P[n \rightarrow m + 1]$ under the assumption $P[n \rightarrow m]$.

Example: We have the following conjecture: $\forall n \in \mathcal{N}. (\text{sumPowers } n) + 1 = \text{power2 } (n + 1)$ with the following code:

```
power2 :: Int -> Int
power2 0 = 1
power2 r = 2 * power2 (r - 1)

sumPowers :: Int -> Int
sumPowers 0 = 1
sumPowers r = sumPowers (r - 1) + power2 r
```

We want to proof: Let $P \equiv (\text{sumPowers } n) + 1 = \text{power2 } (n + 1)$. We show $\forall m \in \mathcal{N}. P$ by induction on n .

Base case: Show $P[n \rightarrow 0]$:

$$\begin{aligned} (\text{sumPowers } 0) + 1 &= 1 + 1 = 2 \\ \text{power2 } (0 + 1) &= 2 \cdot \text{power2 } 0 = 2 \cdot 1 = 2 \end{aligned}$$

Step case: Assume $P[n \rightarrow m]$ for an arbitrary m (not in P), i.e.

$$(\text{sumPowers } m) + 1 = \text{power2 } (m + 1)$$

and prove $P[n \rightarrow m + 1]$, i.e.

$$(\text{sumPowers } (m + 1)) + 1 = \text{power2 } ((m + 1) + 1).$$

Proof:

$$\begin{aligned} (\text{sumPowers } (m + 1)) + 1 &= \text{sumPowers } ((m + 1) - 1) + \text{power2 } (m + 1) + 1 \quad (\text{def.}) \\ &= \text{sumPowers } (m) + 1 + \text{power2 } (m + 1) \quad (\text{arithmetic}) \\ &= \text{power2 } (m + 1) + \text{power2 } (m + 1) \quad (\text{ind- hypothesis}) \\ &= 2 \cdot \text{power2 } (m + 1) \quad (\text{arithmetic}) \\ &= \text{power2 } (m + 2) \quad (\text{def.}) \end{aligned}$$

We have proven $(\text{sumPowers } n) + 1 = \text{power2 } (n + 1)$.

The general schema for **well-founded induction** is given as:

- *To prove:* $\forall n \in \mathcal{N}. P$
- *Fix:* An arbitrary m not free in P
- *Assume:* $\forall l \in \mathcal{N}. l < m \rightarrow P[n \rightarrow l]$ (*induction hypothesis*)
- *Prove:* $P[n \rightarrow m]$

3 More on Haskell

3.1 Lists

3.1.1 List Type

We introduce a new type constructor: **List types**, i.e. if T is a type, then $[T]$ is a type. The elements of $[T]$ are:

- *Empty list*: $[] :: [T]$
- *Non-empty list*: $(x : xs) :: [T]$ if $x :: T$ and $xs :: [T]$

Syntactic sugar: We can write $1 : (2 : (3 : []))$ as $[1, 2, 3]$.

3.1.2 Patterns

Pattern matching has two main purposes:

- checks if an argument has the proper form
- binds values to variables

Example: `(x : xs)` matches with `[2, 3, 4]` and binds:

```
x  = 2
xs = [3, 4]
```

Patterns are *inductively* defined:

- Constants: `-2`, `'1'`, `True`, `[]`
- Variables: `x`, `foo`
- Wild card: `_`
- Tuples: `(p1, p2, ..., pk)`, where `p_i` are patterns
- Non-empty list: `(p1 : p2)`, where `p_i` are patterns

Moreover, patterns require to be **linear**, this means that each variable can occur at most once.

3.1.3 Advice on Recursion

Defining a recursion is best done by obeying the following simple steps:

- Step 1: Define the type of the function
- Step 2: Enumerate all different cases
- Step 3: Define the most simple cases
- Step 4: Define the remaining cases
- Step 5: Generalize and simplify

Example: The following code snippet shows an example of how we implement *insertion sort* recursively in Haskell:

```
isort :: [Int] -> Int
isort []      = []
isort (x : xs) = ins x (isort xs)

ins :: Int -> [Int] -> [Int]
ins a [] = [a]
ins a (x : xs)
  | a >= x    = a : (x : xs)
  | otherwise = x : ins a xs
```

Example: The following code snippet shows how we can implement *quicksort* recursively in Haskell:

```
qsort [] = []
qsort (x : xs) =
  qsort (lesseq x xs) ++ [x] ++ qsort (greater x xs)
  where
    lesseq _ [] = []
    lesseq x (y : ys)
      | (y <= x) = y : lesseq x ys
      | otherwise = lesseq x ys
    greater _ [] = []
    greater x (y : ys)
      | (y > x) = y : greater x ys
      | otherwise = greater x ys
```

3.1.4 List Comprehensions

List comprehension is a notation for sequential processing of list elements. It is analogous to set comprehension in set theory, i.e. $\{2 \cdot x \mid x \in X\}$. In Haskell, this is equivalent to $[2 * x \mid x \leftarrow xs]$.

List comprehensions are very powerful! The following code snippet, again, implements *quicksort* as shown previously:

```
q [] = []
q (p : xs) = q [x | x <- xs, x <= p] ++ [p] ++ q [x | x <- xs, x > p]
```

3.1.5 Induction over Lists

How are elements in $[T]$ constructed? $[] :: [T]$ and $(y : ys) :: [T]$ if $y :: T$ and $ys :: [T]$. This corresponds to the following rule:

- Proof by induction: to prove P for all xs in $[T]$
- Base case: prove $P[xs \rightarrow []]$
- Step case: prove $\forall y :: T, ys :: [T]. P[xs \rightarrow ys] \rightarrow P[xs \rightarrow y : ys]$, i.e.
 - Fix arbitrary: $y :: T$ and $ys :: [T]$ (both not free in P)
 - Induction hypothesis: $P[xs \rightarrow ys]$
 - To prove: $P[xs \rightarrow y : ys]$

3.2 Abstractions

3.2.1 Polymorphic Types

If we consider the `length` function, it should output the length of a list of *any* type. We say that the type of the function is **polymorphic**, i.e. $[t] \rightarrow \text{Int}$ for all types t .

This is often called **parametric polymorphism**, which is different from *subtyping polymorphism*, where methods can be applied to objects of sub-classes only.

Definition: A type w for f is a **most general** (also called **principal**) type iff. for all types s for f , s is an instance of w .

It is important to note that type variables in Haskell start with a *lower-case letter*!

Example: Consider the following polymorphic types:

```
:type (++)
(++) :: [a] -> [a] -> [a]

:type zip
zip :: [a] -> [b] -> [(a, b)]

:type []
[] :: [a]
```

3.2.2 Higher-order Functions

We can distinguish the order of functions in the following way:

- First order: Arguments are base types or constructor types

```
Int -> [Int]
```

- Second order: Arguments are themselves functions

```
(Int -> Int) -> [Int]
```

- Third order: Arguments are functions, whose arguments are functions

```
((Int -> Int) -> Int) -> [Int]
```

- Higher-order functions: Functions of arbitrary order

Example: Consider the map function:

```
map :: (a -> b) -> [a] -> [b]
map f []      = []
map f (x : xs) = f x : map f xs

times2 x = 2 * x

double xs = map times2 xs
```

Example: Consider the foldr function:

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f z []      = z
foldr f z (x : xs) = f x (foldr f z xs)

sumList xs = foldr (+) 0 xs
```

3.2.3 λ -Expressions

Consider the following two functions:


```

times2 x = 2 * x
double xs = map times2 xs

atEnd x xs = xs ++ [x]
rev xs = foldr atEnd [] xs

```

Haskell provides a notation to write functions like `times2` and `atEnd` in-line via so-called **λ -expressions**:

```

? map (\x -> 2 * x) [2, 3, 4]
[4, 6, 8]

? foldr (\x xs -> xs ++ [x]) [] [1, 2, 3, 4]
[4, 3, 2, 1]

```

This is also called *Church's λ -notation*, i.e. replacing λ by the character '`\`'.

3.2.4 Functions as Values

In Haskell, functions can be returned as values! Consider the following simple example where we return the two-times-application of some function f :

```

(.) :: (b -> c) -> (a -> b) -> (a -> c)
(f . g) x = f (g x)

twice :: (t -> t) -> (t -> t)
twice f = f . f

? twice times2 3
12 :: Int

```

3.2.5 Difference Lists

Difference lists are functions `[a] -> [a]` that prepend a list to its argument.

```

type DList a = [a] -> [a]

empty :: DList a
empty = \xs -> xs -- empty list

sngl :: a -> DList a
sngl x = \xs -> x : xs -- singleton list

app :: DList a -> DList a -> DList a
ys 'app' zs = \xs -> ys (zs xs) -- concatenation

fromList :: [a] -> DList a
fromList ys = \xs -> ys ++ xs -- conversion from lists

toList :: DList a -> [a]
toList ys = ys [] -- conversion to lists

```