1 Inverted Pendulum Control System

This section describes the mathematical model and control structure used to stabilize an inverted pendulum mounted on a cart. The system is modeled as a linear time-invariant (LTI) state–space system obtained by linearizing the nonlinear dynamics about the upright position.

1.1 State Variables and Measurements

The state vector is defined as

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix},$$

where

- x is the horizontal cart position (m),
- $\dot{x}$ is the cart velocity (m/s),
- ϕ is the pendulum angle measured from the upright position (rad),
- $\dot{\phi}$ is the angular velocity (rad/s).

The encoder system provides direct measurements of

$$\mathbf{y} = \begin{bmatrix} x \\ \phi \end{bmatrix}.$$

Thus, the output matrix C extracts the first and third states.

1.2 State-Space Dynamics

The continuous-time linearized dynamics of the system are

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \qquad \mathbf{y} = C\mathbf{x},$$

with matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.6369 & 0.8427 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.8198 & 30.4076 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 6.369 \\ 0 \\ 18.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The input variable u represents the horizontal force applied to the cart. In the physical implementation, this force is produced by a DC motor and mapped to PWM voltage commands.

1.3 State-Feedback Control

To stabilize the pendulum in the upright configuration, a full-state feedback law is used:

$$u = -K(\hat{\mathbf{x}} - \mathbf{x}_{\text{ref}}),$$

where

$$\mathbf{x}_{\text{ref}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and K is the feedback gain obtained via pole placement:

$$K = \begin{bmatrix} -4.7099 & -3.6774 & 13.1543 & 2.4607 \end{bmatrix}.$$

This controller stabilizes the upright equilibrium using the estimated states $\hat{\mathbf{x}}$ provided by the observer.

1.4 Luenberger State Observer

Only two states (x and ϕ) are directly measured. The velocity states $\dot{x}$ and $\dot{\phi}$ must be reconstructed. A Luenberger observer is used:

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L\left(\mathbf{y} - C\hat{\mathbf{x}}\right),$$

where L is the observer gain matrix chosen to place the observer poles faster than the controller poles.

The observer gain used in this implementation is

$$L = \begin{bmatrix} 43.7420 & 3.8482 \\ 452.3589 & 82.9844 \\ 2.3003 & 43.6211 \\ 11.1322 & 487.1874 \end{bmatrix}.$$

The estimation error dynamics follow

$$\dot{\tilde{\mathbf{x}}} = (A - LC)\tilde{\mathbf{x}}, \quad \tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}},$$

and decay exponentially since $(A - LC)$ is stable.

1.5 Closed-Loop System

Combining the state feedback controller and the observer yields the observer-based feedback loop (also known as the observer–controller or compensator):

$$\begin{aligned} u &= -K\hat{\mathbf{x}}, \\ \dot{\hat{\mathbf{x}}} &= (A - LC)\hat{\mathbf{x}} + Bu + L\mathbf{y}. \end{aligned}$$

The resulting closed-loop system ensures that:

- The state estimation error converges to zero.
- The cart balances the pendulum around $\phi = 0$.
- The system rejects small disturbances and maintains stability.

1.6 Implementation Notes

In the microcontroller implementation:

- The observer is integrated using Euler discretization at $T_s = 5\,\text{ms}$.
- Encoder counts are converted into meters (cart) and radians (pendulum).
- The control input u is scaled to a normalized PWM command.
- The controller runs entirely in discrete time, but uses the continuous-time model for computation of A, B, C, K, and L.