

From networks to thermometry: precision in quantum technologies

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Key works:

- arXiv:2111.11921
Phys. Rev. Lett. **127**, 190402 (2021)
J. Phys. A: Math. Theor., 53 344001 (2020)
JPhys. Rev. A 101, 032114 (2020)

QUINFOG seminar
Instituto de Física Fundamental

14th Dec 2021

Our plan for today

- 0. Quantum estimation theory *à la* Bayes
- I. Quantum sensing networks: imaging and qubits
 - > Local and global properties
 - > Geometry and correlations
- II. Quantum thermometry – a tale in three acts:
 - > the practical,
 - > the local,
 - > and the global
- III. Quantum metrology of scale parameters
 - > Phases, locations and ... scales?
 - > How does nature do it: optimal strategies in scale estimation

Q

uantum Estimation Theory

à la Bayes

Z

Preparation and measurement

Initial state

ρ_0

Parameter-encoded state

$\rho(\Theta) = \rho(\theta_1, \dots, \theta_d)$

Measurement (POVM)

$E(m) = E(m_1, \dots, m_\mu)$

Outcomes

$m = (m_1, \dots, m_\mu)$

①



②



③



④

$\Theta \equiv$ hypothesis about the true but unknown values Θ

Estimation within the Bayesian paradigm

- Prior information: $p(\theta)$
 - > Before looking at the measurement outcomes
 - > In many cases: maximum ignorance
- Likelihood function: $p(m|\theta) = \text{Tr}[E(m)\rho(\theta)]$
 - > Links unknown parameter with measured quantity
 - > Given by the Born rule (quantum systems)
- Bayes theorem

$$p(\theta|m) \propto p(\theta) p(m|\theta) = p(\theta) \text{Tr}[E(m)\rho(\theta)]$$

An information summary

- What is the goal?

- > Estimator: $\mathbf{g}(\mathbf{m})$

- > Post-processing error

$$\epsilon(\mathbf{m}) = \int d\boldsymbol{\theta} p(\boldsymbol{\theta}|\mathbf{m}) \mathcal{D}[\mathbf{g}(\mathbf{m}), \boldsymbol{\theta}, \mathcal{W}]$$

- How do we get there?

- > Minimise the *overall* uncertainty

$$\bar{\epsilon} = \int d\boldsymbol{\theta} d\mathbf{m} p(\boldsymbol{\theta}, \mathbf{m}) \mathcal{D}[\mathbf{g}(\mathbf{m}), \boldsymbol{\theta}, \mathcal{W}]$$

That is, we need to calculate:

Relevant in experiments

$$\min_{g,E} \bar{\epsilon} = ?$$

Relevant for theorists

Note that...

- Deviation function for **phases** lying within $\theta_i \in \left[-\frac{L}{2}, \frac{L}{2}\right]$, with $L < 2$

- Single parameter **square error**:

$$\mathcal{D}[g(\mathbf{m}), \theta] \approx [g(\mathbf{m}) - \theta]^2$$

- For multiple parameters, with weighting matrix W :

$$\mathcal{D}[\mathbf{g}(\mathbf{m}), \boldsymbol{\theta}, \mathcal{W}] \approx \text{Tr} \left\{ \mathcal{W} [\mathbf{g}(\mathbf{m}) - \boldsymbol{\theta}] [\mathbf{g}(\mathbf{m}) - \boldsymbol{\theta}]^\top \right\}$$

- A useful approximation (Cramér-Rao asymptotic limit):

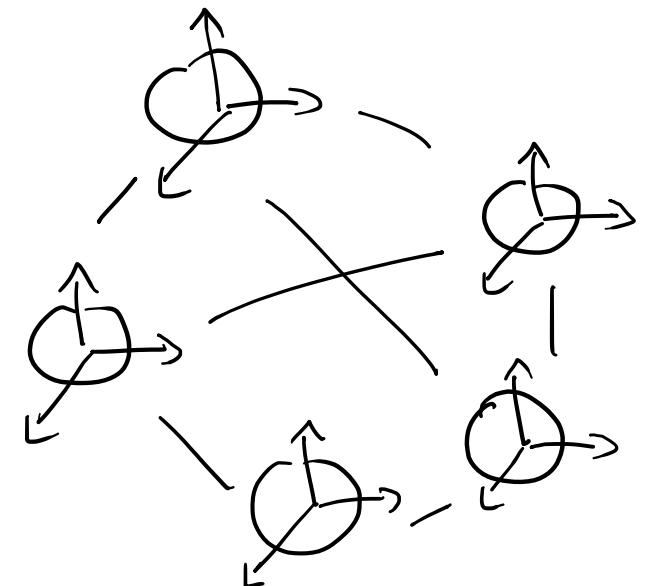
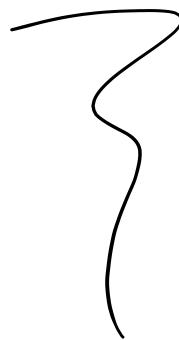
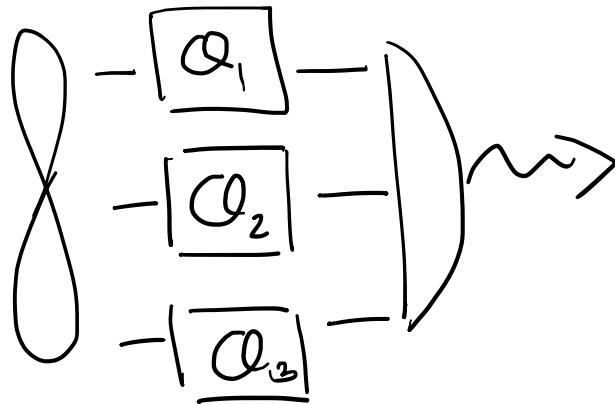
$$\bar{\epsilon}_{\text{mse}} \gtrsim \frac{1}{\mu} \text{Tr}(WF_q^{-1})$$

\uparrow
 $\mu \gg 1$

arXiv:1912.02324

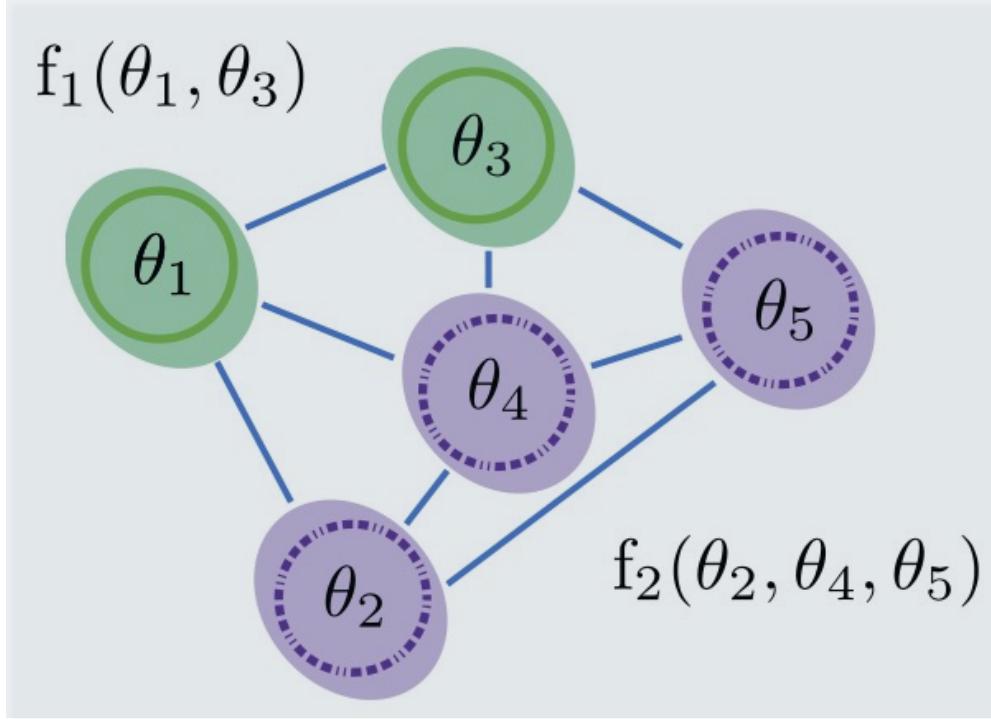
J. Phys. A: Math. Theor. 53 (2020) 344001

Quantum sensing networks: imaging and qubits



(with P. A. Knott, T. Proctor, J. Dunningham)

Quantum sensing networks: local and global properties

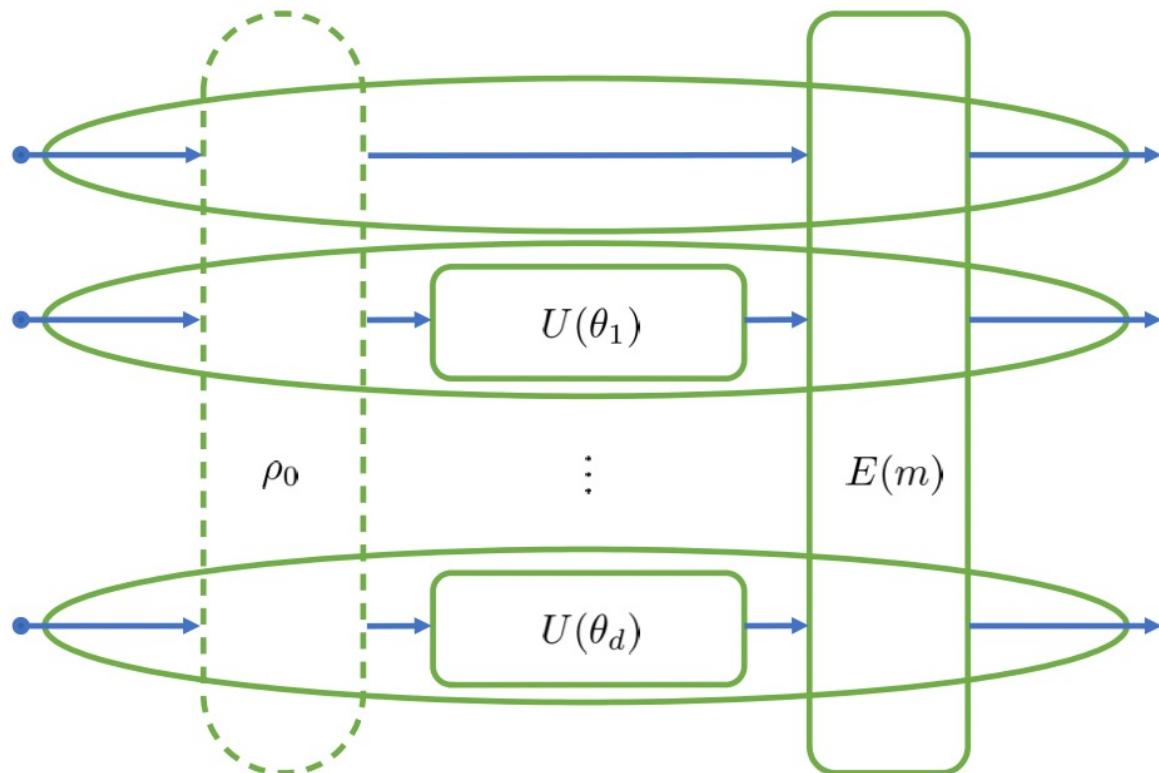


- **Local properties:**
each individual sensor
- **Global properties:**
several sensors involved

J. Phys. A: Math. Theor. **53** (2020) 344001

T. J. Proctor, P. A. Knott, and J. A. Dunningham, Networked quantum sensing, arXiv:1702.04271 (2017).

(Discrete) Quantum imaging



- Quantum enhancement **because of** using generalised NOON states

$$|\psi_0\rangle = \frac{1}{\sqrt{d + \alpha^2}} (\alpha |\bar{n} 0 \cdots 0\rangle + \cdots + |0 \cdots 0 \bar{n}\rangle)$$

P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, Quantum enhanced multiple phase estimation, *Phys. Rev. Lett.* **111**, 070403 (2013).

- Entanglement is **not** needed for such enhancement

$$\rho_0^{\text{ref}} \otimes \rho_0^{(1)} \otimes \cdots \otimes \rho_0^{(d)}$$

$$\rho_0^{(i)} = |\phi_0^{(i)}\rangle\langle\phi_0^{(i)}|$$

$$|\phi_0\rangle = \left[\sqrt{1 - \frac{\bar{n}}{N(d+1)}} |0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}} |N\rangle \right]$$

P. A. Knott, T. J. Proctor, A. J. Hayes, J. F. Ralph, P. Kok, and J. A. Dunningham, Local versus global strategies in multiparameter estimation, *Phys. Rev. A* **94**, 062312 (2016).

A new multi-parameter quantum bound

$$\bar{\epsilon}_{\text{mse}} \geq \sum_{i=1}^d w_i \left[\int d\theta p(\theta) \theta_i^2 - \text{Tr}(\rho S_i^2) \right]$$

❖ For generalised NOON states:

$$\bar{\epsilon}_{\text{mse}} \geq \frac{1}{\bar{n}^2} \left[\frac{\pi^2}{3} - \frac{4}{(1 + \sqrt{d})^2} \right] \xrightarrow{d \gg 1} \frac{1}{\bar{n}^2} \left(\frac{\pi^2}{3} - \frac{4}{d} \right)$$

The local strategy can be as good but not arbitrarily precise!

❖ For the local strategy:

$$\bar{\epsilon}_{\text{mse}} \geq [\pi^2/3 - f(N, \bar{n}, d)]/\bar{n}^2$$

(a) if $N = \bar{n}$, then $f(N, \bar{n}, d) = 4d/(1 + d)^2$, and

$$\bar{\epsilon}_{\text{mse}} \geq \frac{1}{\bar{n}^2} \left[\frac{\pi^2}{3} - \frac{4d}{(1 + d)^2} \right] \xrightarrow{d \gg 1} \frac{1}{\bar{n}^2} \left(\frac{\pi^2}{3} - \frac{4}{d} \right);$$

(b) if $N \rightarrow \infty$, then $f(N, \bar{n}, d) \rightarrow 0$, so

$$\bar{\epsilon}_{\text{mse}} \xrightarrow[N \rightarrow \infty]{} \frac{\pi^2}{3\bar{n}^2} = \frac{1}{d} \sum_{i=1}^d \Delta\theta_{p,i}^2.$$

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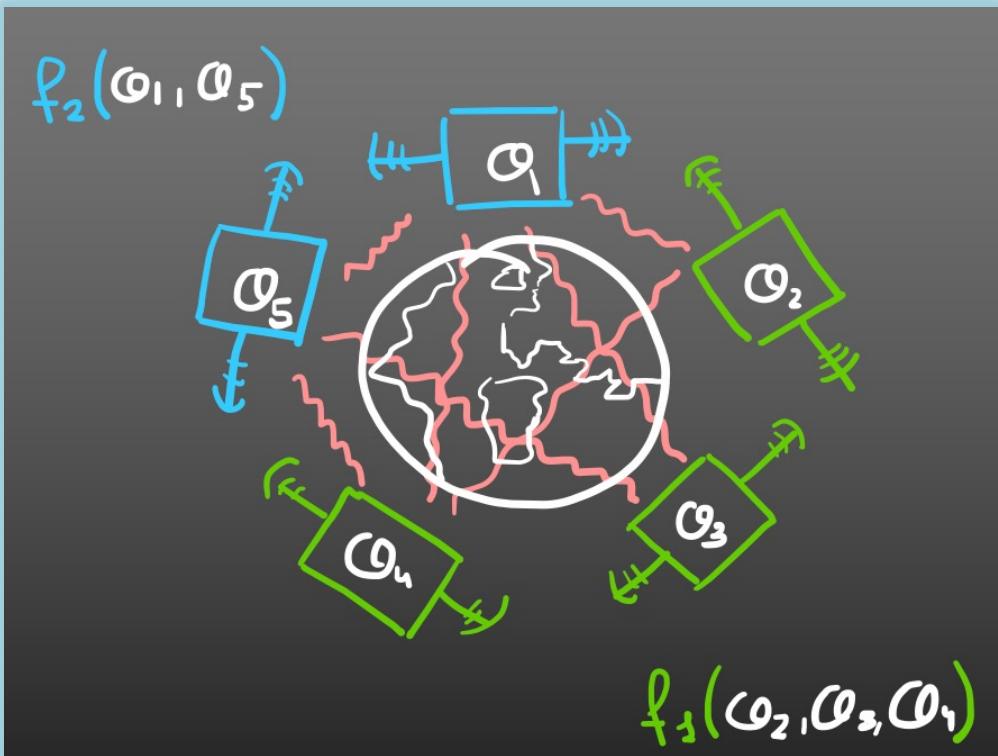
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*The metrological power of vacuum-number superpositions
is, at best, equivalent to that of NOON states.*

Qubit sensing network



$$1) \quad \rho_0 = |\Psi_0 \times \Psi_0|$$



$$2) \quad \rho(\vec{\omega}) = e^{-\vec{\vec{\omega}}^T \vec{\vec{\omega}}} \rho_0 e^{\vec{\vec{\omega}}^T \vec{\vec{\omega}}}$$

$$\left[e^{-\vec{\vec{\omega}}^T \vec{\vec{\omega}}} = e^{-\sigma_z \Omega_1/2} \otimes e^{-\sigma_z \Omega_2/2} \otimes \dots ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

* Inter-sensor correlations

$$\mathcal{J}_{ij} = \frac{\langle \kappa_i \kappa_j \rangle - \langle \kappa_i \rangle \langle \kappa_j \rangle}{\Delta \kappa_i \Delta \kappa_j} \Rightarrow$$

For local strategies,
 $\mathcal{J}_{ij} = 0, \forall i, j$

$$(\langle * \rangle = \text{Tr}[\rho_0 *] ; \Delta \kappa_i^2 = \langle \kappa_i^2 \rangle - \langle \kappa_i \rangle^2)$$

Sensor-symmetric states

$$\langle \kappa_i \kappa_j \rangle - \langle \kappa_i \rangle \langle \kappa_j \rangle = c \Rightarrow$$

$$\langle \kappa_i^2 \rangle - \langle \kappa_i \rangle^2 = \sigma$$

$$\nabla_{ij}$$

$$\mathcal{J}_{ij} = \mathcal{J} = \frac{c}{\sigma}$$

state $\leftrightarrow (\sigma, \mathcal{J})$
 $(\vec{\kappa} \text{ fixed})$

* Inter-sensor correlations

$$\frac{\langle \kappa_i \kappa_j \rangle - \langle \kappa_i \rangle \langle \kappa_j \rangle}{\Delta \kappa_i \Delta \kappa_j} \Rightarrow$$

For local strategies,
 $J_{i,j} = 0, \forall i,j$

Protocols for estimating multiple functions with quantum sensor networks: Geometry and performance

Jacob Bringewatt, Igor Boettcher, Pradeep Niroula, Przemyslaw Bienias, and Alexey V. Gorshkov
Phys. Rev. Research 3, 033011 – Published 2 July 2021

Sensor symmetry

$$\langle \kappa_i \kappa_j \rangle - \langle \kappa_i \rangle \langle \kappa_j \rangle = c \Rightarrow$$

$$\langle \kappa_i^2 \rangle - \langle \kappa_i \rangle^2 = \sigma$$

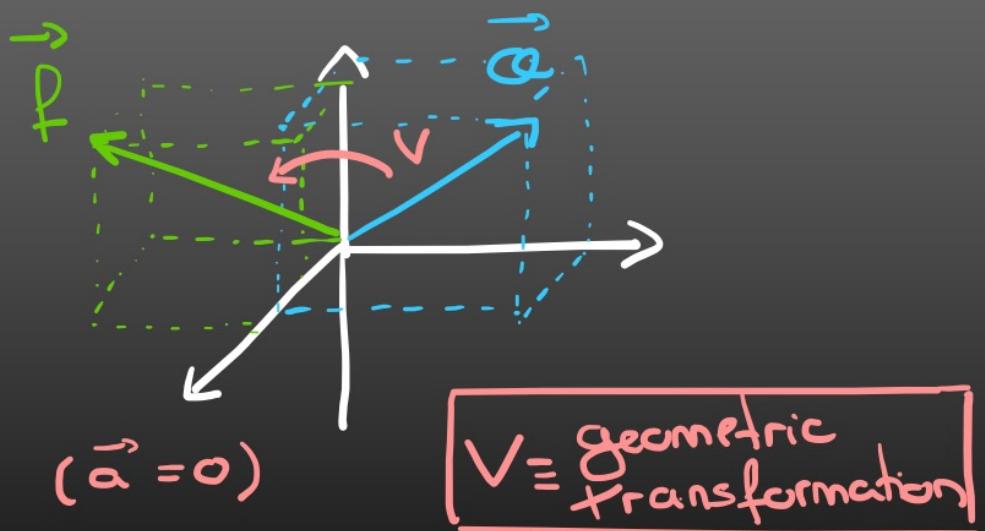
$$\Delta_{ij}$$

{ state ←
($\vec{\kappa}$ fixed)}

1) Linear functions: exact case

$$\overrightarrow{f}(\vec{\vartheta}) = V^T \vec{\vartheta} + \vec{a}$$

$$\begin{pmatrix} \quad \\ l \times 1 \end{pmatrix} = \begin{pmatrix} \quad \\ l \times d \end{pmatrix} \begin{pmatrix} \quad \\ d \times 1 \end{pmatrix} + \begin{pmatrix} \quad \\ l \times 1 \end{pmatrix}$$



2) Linear approximation

* general $\overrightarrow{f}(\vec{\vartheta})$

*



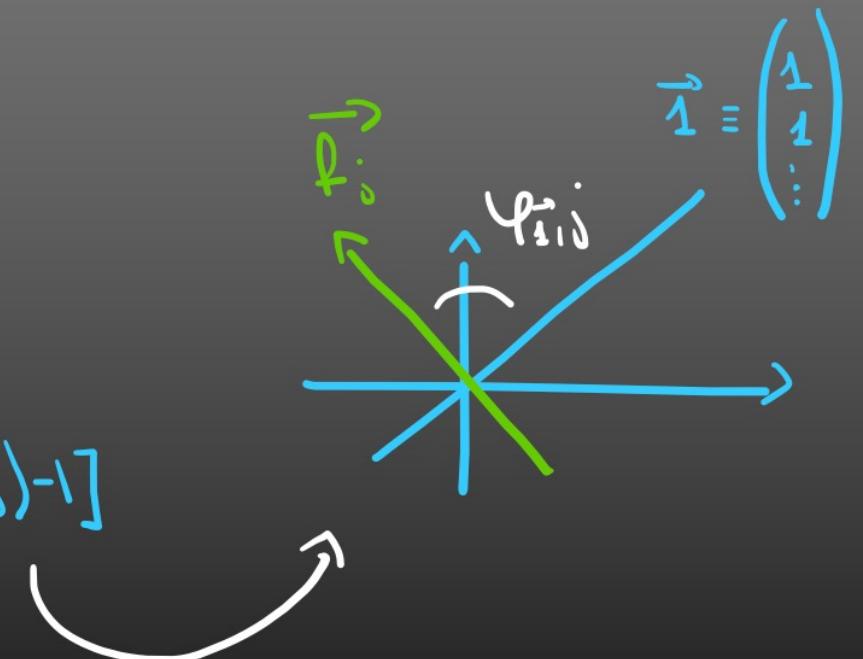
$$\begin{aligned} \overrightarrow{f}(\vec{\vartheta}) &\simeq \overrightarrow{f}(\vec{b}) + \sum_{i=1}^l \frac{\partial \overrightarrow{f}(\vec{b})}{\partial \vartheta_i} (\vartheta_i - b_i) \\ &= V^T \vec{\vartheta} + \vec{a} \end{aligned}$$

* Geometric reinterpretation:

$$V = \begin{pmatrix} \vec{f}_1 & \vec{f}_2 & \dots & \vec{f}_d \\ \downarrow & \downarrow & & \downarrow \\ f_1 & f_2 & \dots & f_d \end{pmatrix} \longleftrightarrow f_j(\vec{c}) = \sum_{i=1}^d V_{ij} c_i \equiv \vec{f}_j^T \vec{c}$$

$$\Rightarrow \text{Tr}(\Psi V^T V) = \sum_{j=1}^l w_j |\vec{f}_j|^2$$

$$\text{Tr}(\Psi V^T \chi V) = \sum_{j=1}^l w_j |\vec{f}_j|^2 [d \cos^2(\varphi_{i,j}) - 1]$$



→ Normalisation term

$$N := \text{Tr}(w v^T v)$$

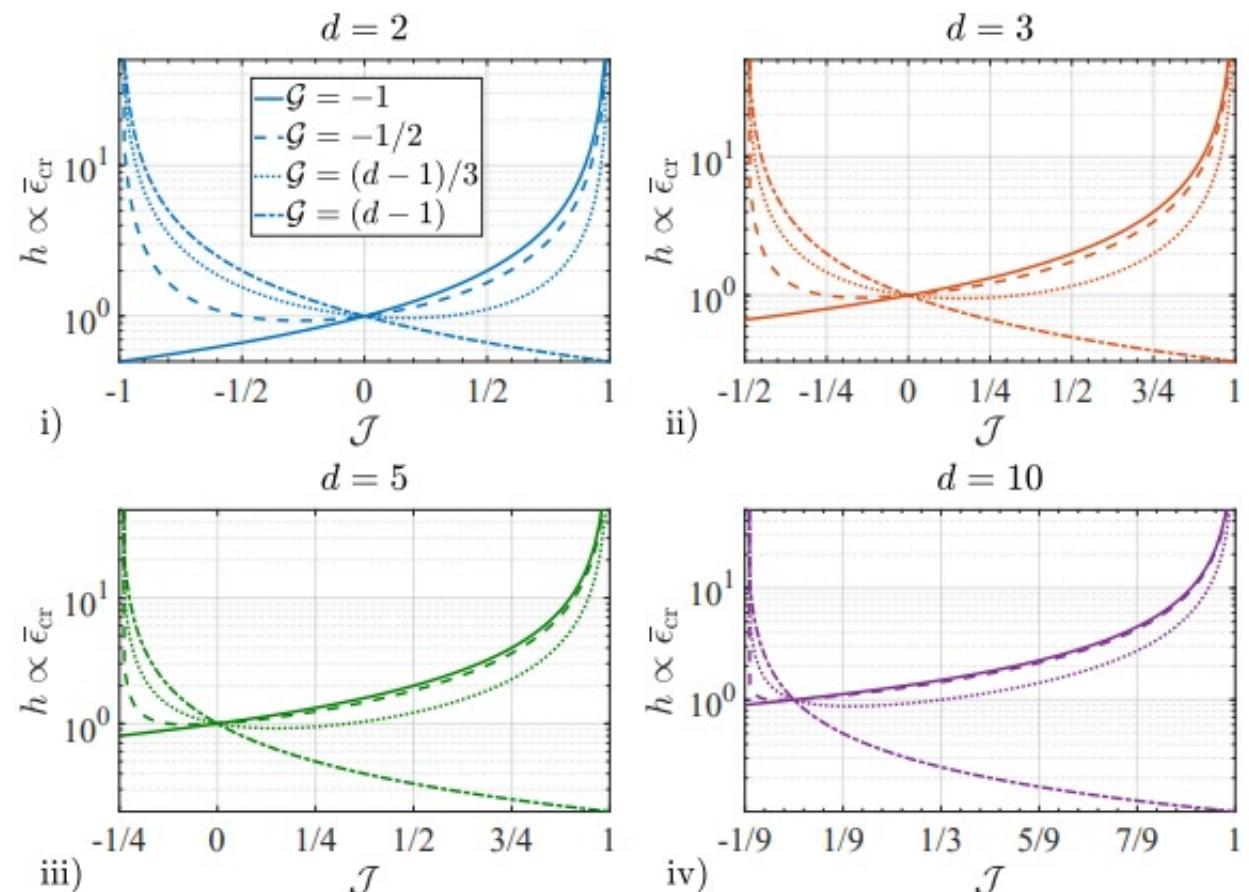
→ Geometry parameter

$$G := \frac{1}{N} \text{Tr}[w v^T x v]$$

→ Final uncertainty

$$\bar{\epsilon}_{\text{cr}} = \frac{N}{4M} \frac{[1 + (d - G)\mathcal{J}]}{(1 - \mathcal{J})[1 + (d - 1)\mathcal{J}]}$$

$\equiv h(\mathcal{J}, G, d)$



QUESTION: Given (N, G) , optimal $(\mathcal{J}, \mathcal{I})$?

\downarrow geometry of linear functions \downarrow state β_0 (\vec{v} fixed)

$$\bar{\epsilon}_{cr} \geq \frac{N}{M} h(\mathcal{J}, G, d)$$

\uparrow
 $\mathcal{I} \leq \frac{L}{4}$

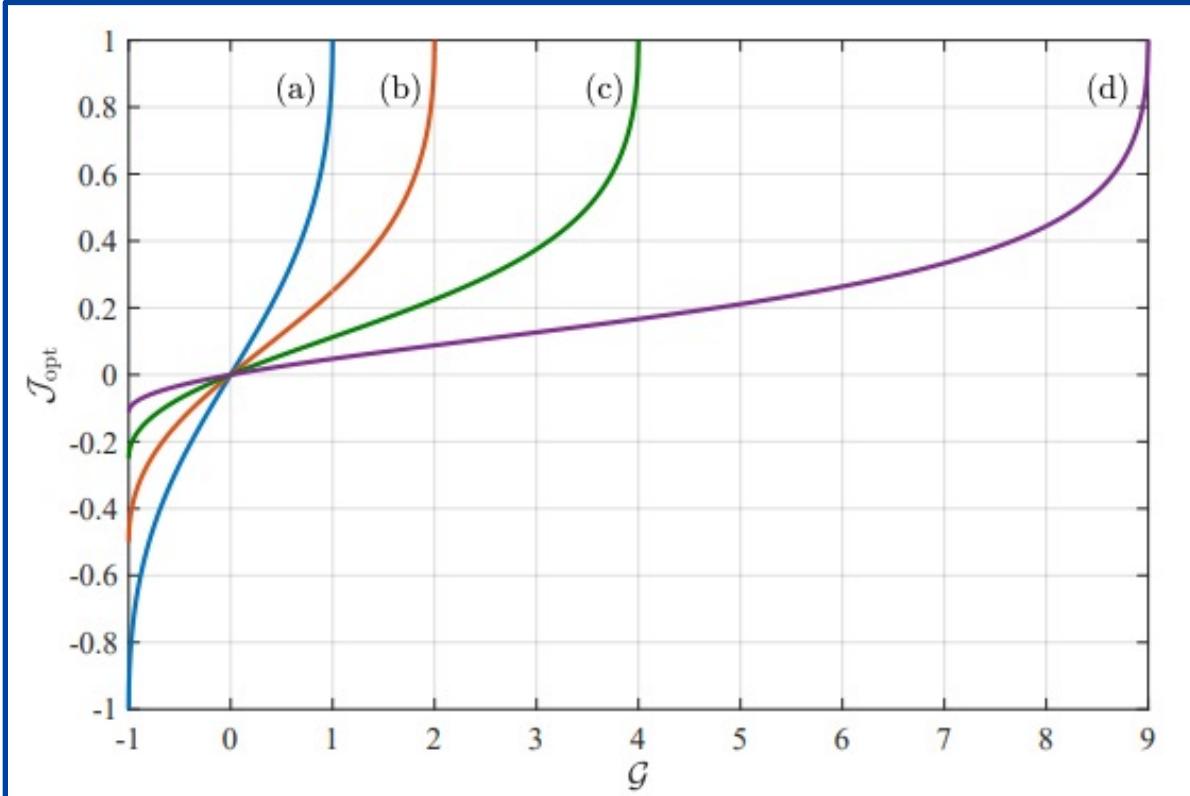
\Rightarrow
 minimisation
 of h
 w.r.t. \mathcal{J}

$$\boxed{\mathcal{J}_{opt} = \frac{1}{G+2-d} \left[1 - \sqrt{\frac{(G+1)(d-1-G)}{d-1}} \right]}$$

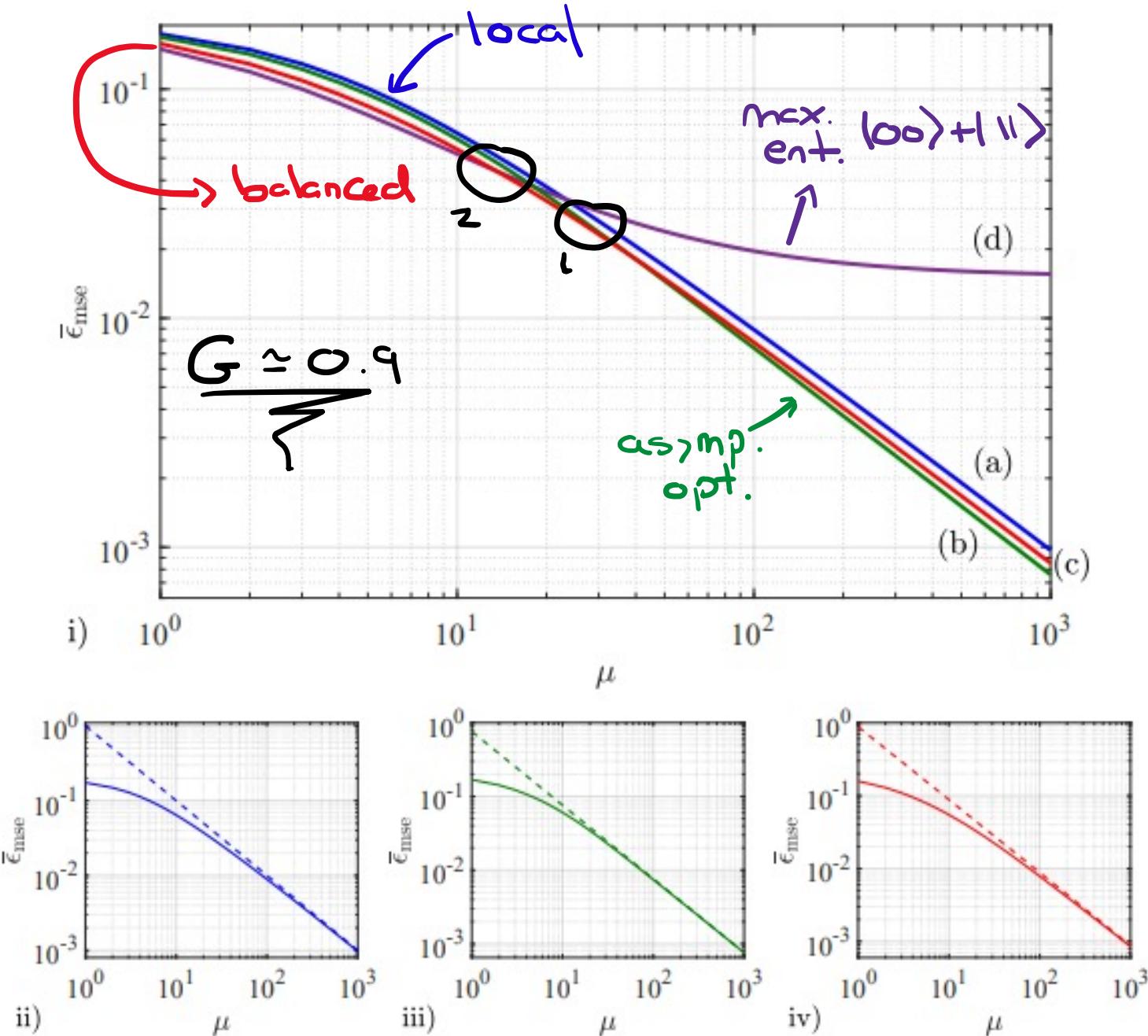
QUESTION: Given (N, G) , optimal $(\mathcal{G}, \mathcal{J})$?

↓
geometry of
linear functions

↓
state \vec{s}_0
(\vec{v} fixed)

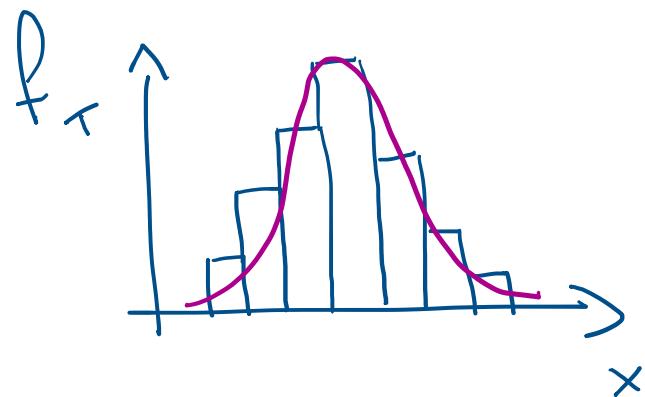


$$\boxed{\mathcal{J}_{\text{opt}} = \frac{1}{G+2-d} \left[1 - \sqrt{\frac{(G+1)(d-1-G)}{d-1}} \right]}$$



Quantum Thermometry

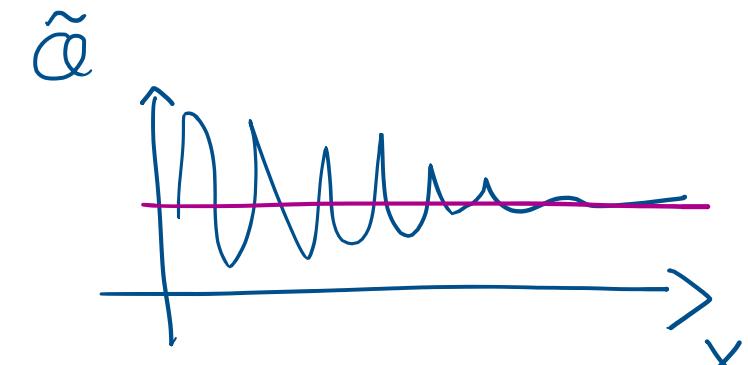
The practical



The local

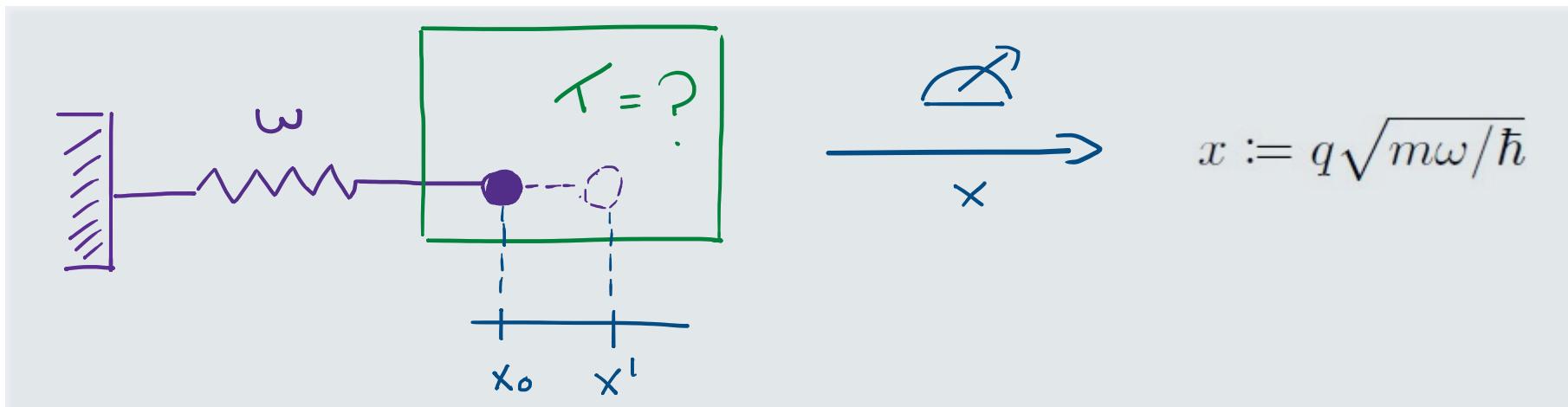


The global



(with L. A. Correa and J. Anders)

Quantum harmonic oscillator in thermal equilibrium



- $\theta =$ hypothesis about the true value of T
- Protocol statistics fully described by:

$$p(x|\theta)dx = \frac{\exp\{-x^2/[2\sigma(\theta)^2]\}}{\sqrt{2\pi\sigma(\theta)^2}} dx$$

$$\sigma(\theta) = \sqrt{\frac{1}{2} \coth \left(\frac{\hbar\omega}{2k_B\theta} \right)}$$

Usual procedure in practice:

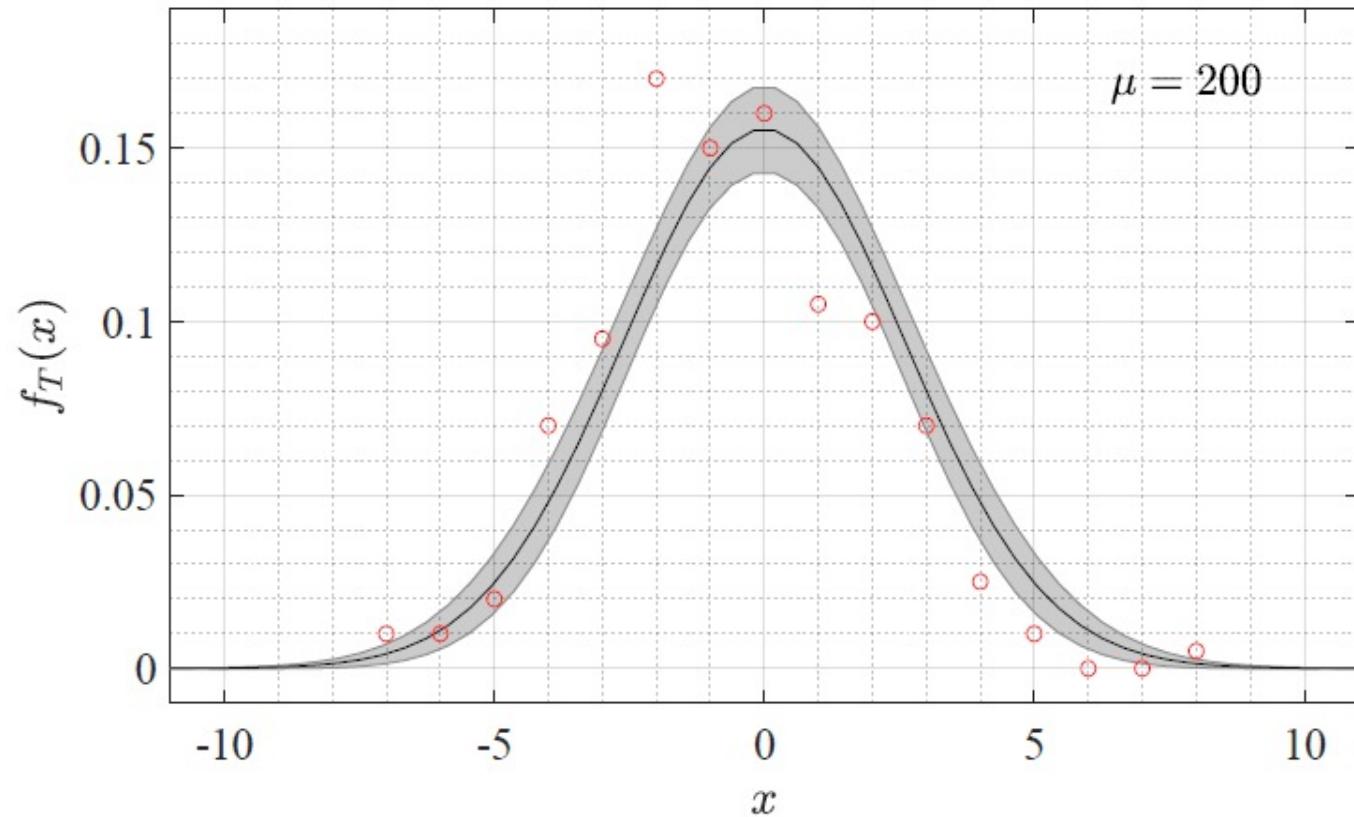
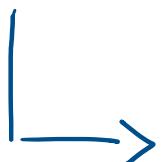
> Measure the position

$$\bar{X} = (X_1, X_2, \dots, X_n)$$

> Build a position histogram

> Fit the temperature-dependent probability to such histogram

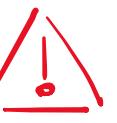
[i.e., to $p(x|T)$]



$$k_B(\tilde{\theta}_F \pm \Delta\tilde{\theta}_F)/(\hbar\omega) = 7 \pm 1$$

Least
Squares

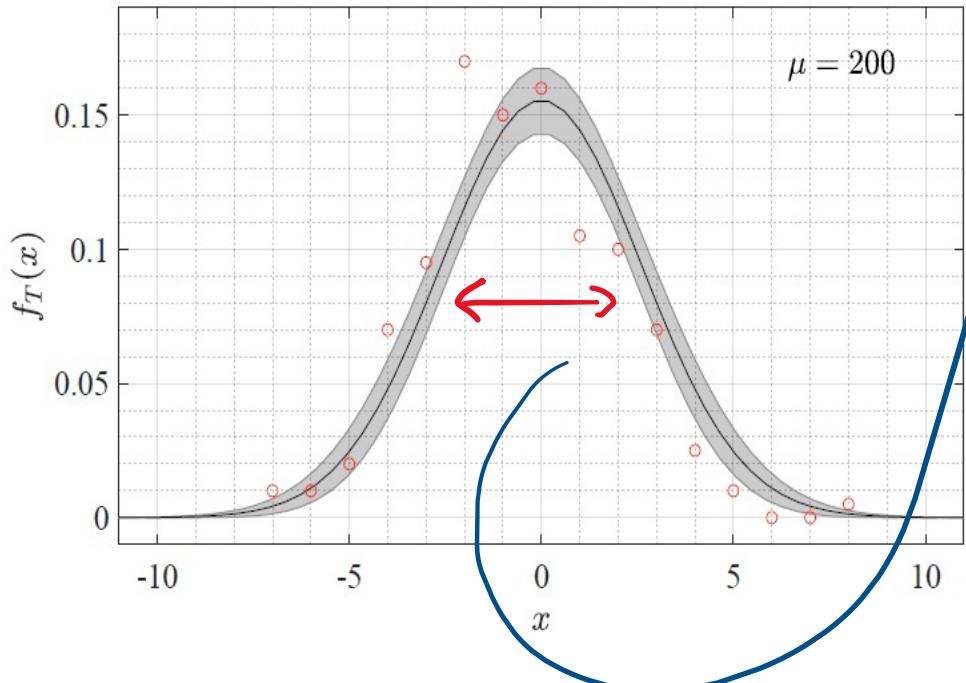
Caveats



- Histogram-based approaches:
 - > Bin selection
 - > Sufficiently large number of measurements

Local quantum thermometry: how does it help?

- Cramér-Rao bounds:



$$\Delta \tilde{\theta}^2 \geq \frac{1}{\mu F(T)} \geq \frac{1}{\mu F_q(T)}$$

measurement-dependent
Fisher info.

state-dependent
Fisher info.

Local quantum thermometry: how does it help?

- Direct experimental design:

Given the dynamics (Hamiltonian)



find:

- state $\rho(T)$
- measurement $M(x)$

s.t. $F_q(T)$ is maximum.

- Indirect experimental design:

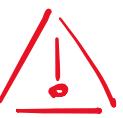
Given a practical $M(x)$ and a specific state $\rho(T)$



calculate $F(T), F_q(T);$

- if $F(T) = F_q(T)$, the scheme is optimal
- if $F(T) \neq F_q(T)$, keep searching

Caveats



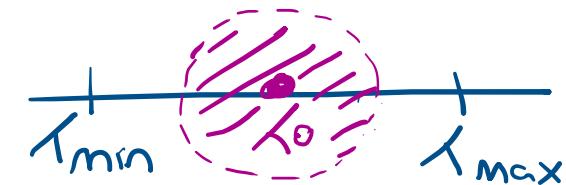
- Histogram-based approaches:

- > Bin selection
- > Sufficiently large number of measurements

- Local quantum thermometry:

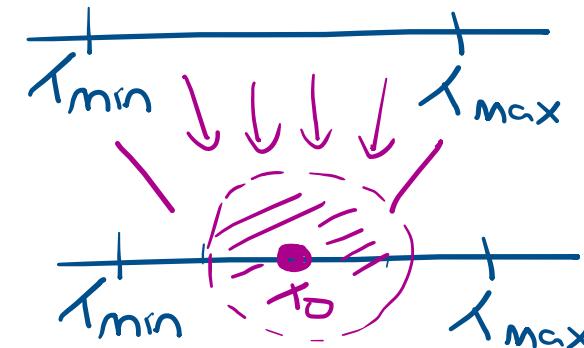
- > Exact but **very** restrictive:
exponential family + unbiasedness
- > Local prior information

(i)



- > Asymptotically large data set

(i)



(f)



- > Dependence on true temperature

A more general starting point: the Bayesian paradigm

- Prior information $\rightarrow \theta \in [\theta_{\min}, \theta_{\max}]$
- > Maximum ignorance for scale parameters $\rightarrow p(\theta) \propto \frac{1}{\theta}$
- Assessing the (overall) uncertainty of scale parameters: **logarithmic errors**

$$\bar{\epsilon}_{\text{mle}} = \int dE d\theta p(E, \theta) \left[\log^2 \left[\frac{\tilde{\theta}(E)}{\theta} \right] \right]$$

generalised relative error or noise-to-signal ratio

$$\min_{\tilde{\theta}(E)} \bar{\epsilon}_{\text{mle}} = ?$$

A two-line solution

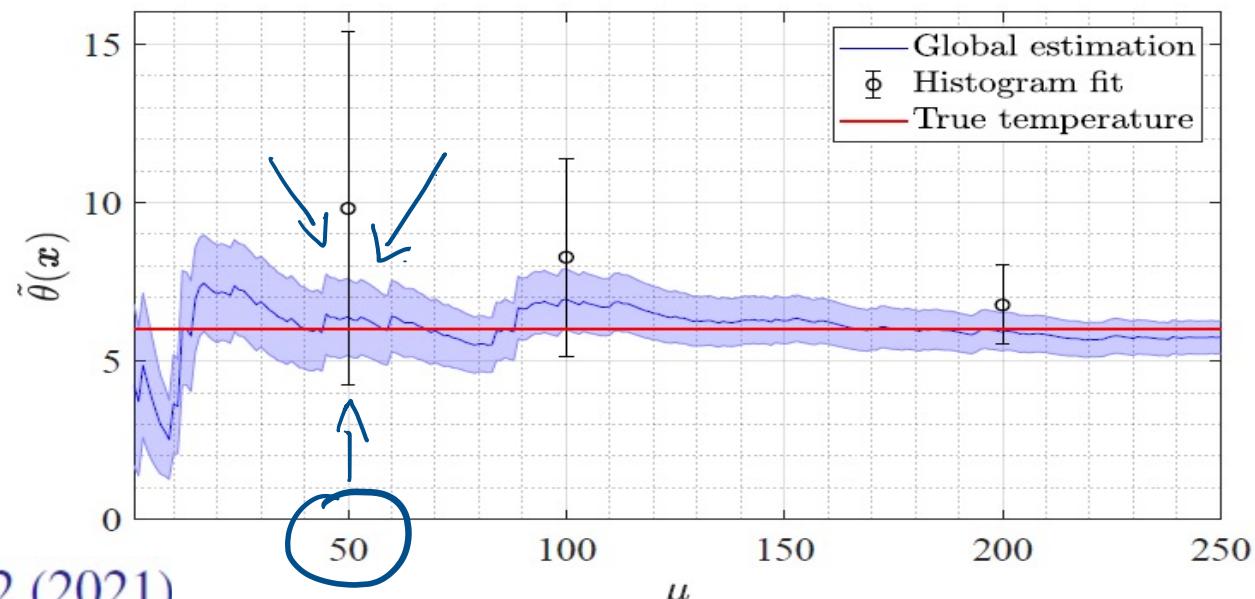
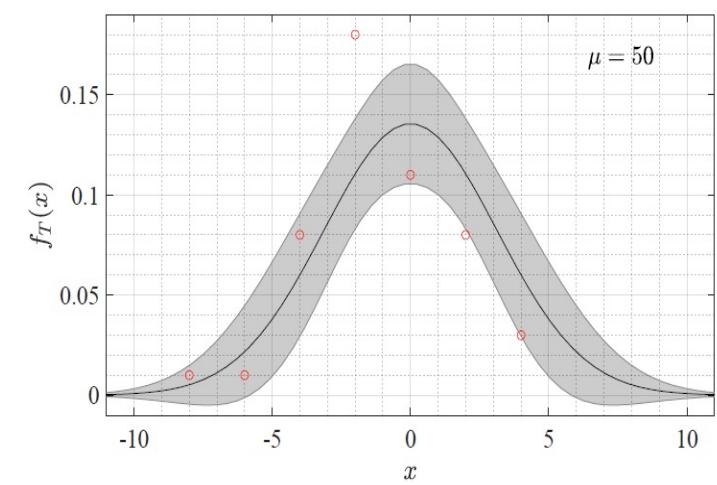
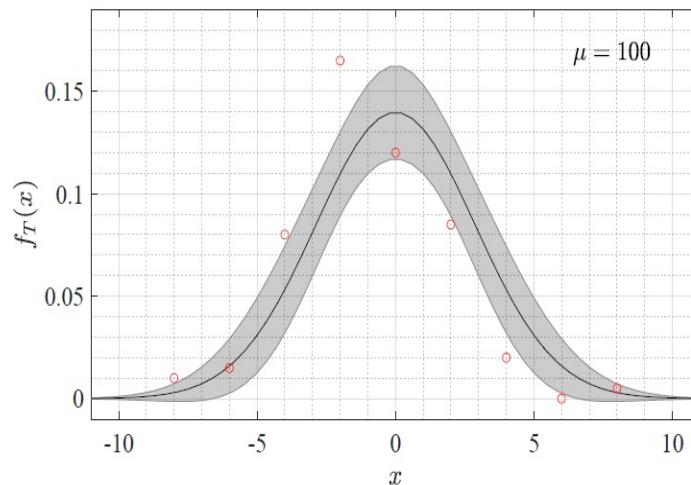
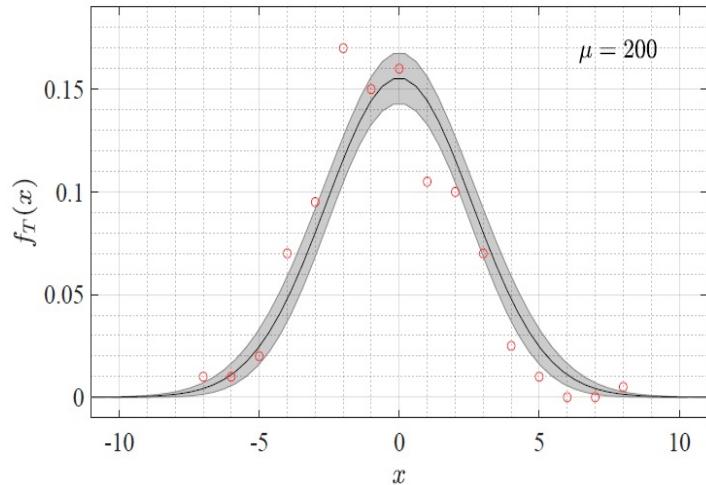
- **Optimal rule to post-process measurement outcomes into a temperature reading**
 - > Universally valid

$$\frac{k_B \tilde{\vartheta}(E)}{\varepsilon_0} = \exp \left[\int d\theta p(\theta|E) \log \left(\frac{k_B \theta}{\varepsilon_0} \right) \right]$$

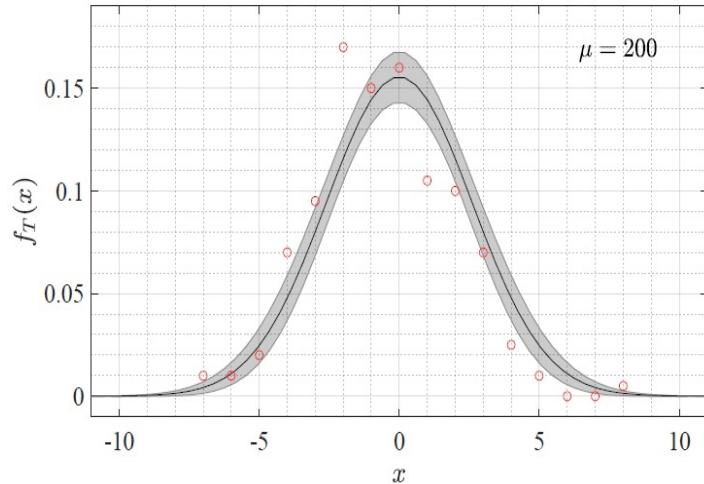
- **Minimum uncertainty *overall* (not just a bound)**
 - > Useful to study fundamental limits to the precision
 - > Universally valid *for a given measurement*
 - > Not just a bound!

$$\bar{\epsilon}_{\text{mle}} \gg \bar{\epsilon}_{\text{opt}} = \bar{\epsilon}_p - \mathcal{K}$$

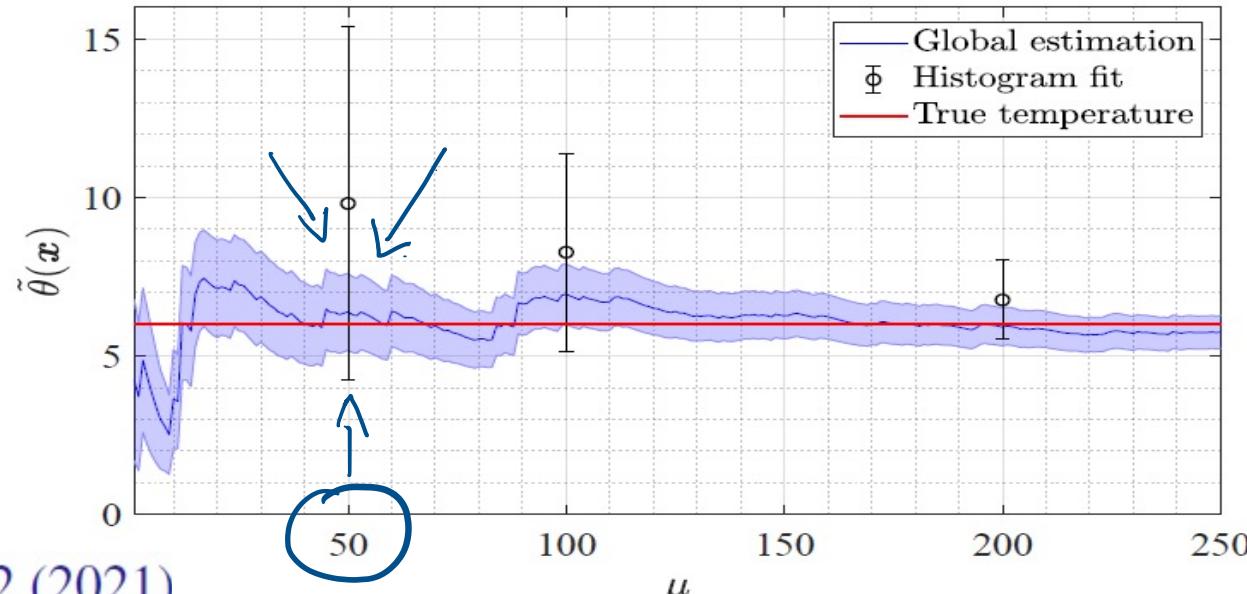
Revisiting the harmonic oscillator in thermal equilibrium



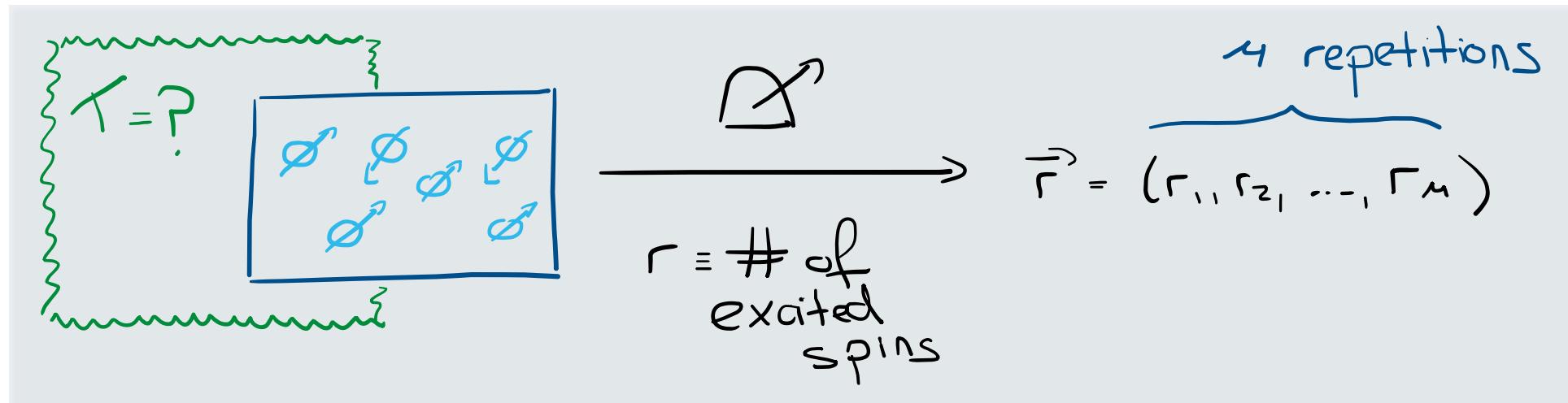
Revisiting the harmonic oscillator in thermal equilibrium



- Least square method: biased for finite statistics
- Bayesian approach: as good or better than traditional methods

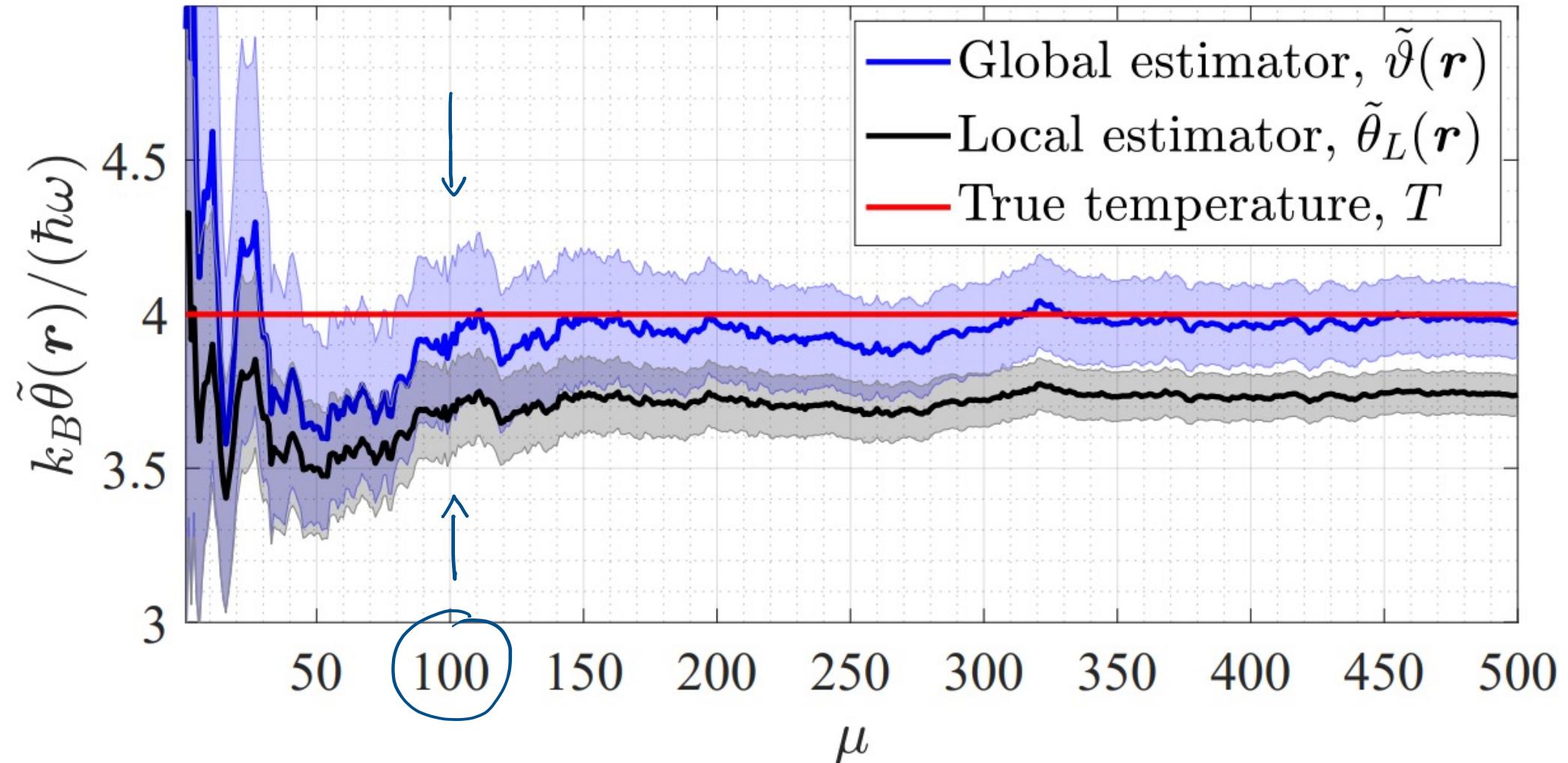


The perils of local thermometry: non-interacting spin-1/2 gas

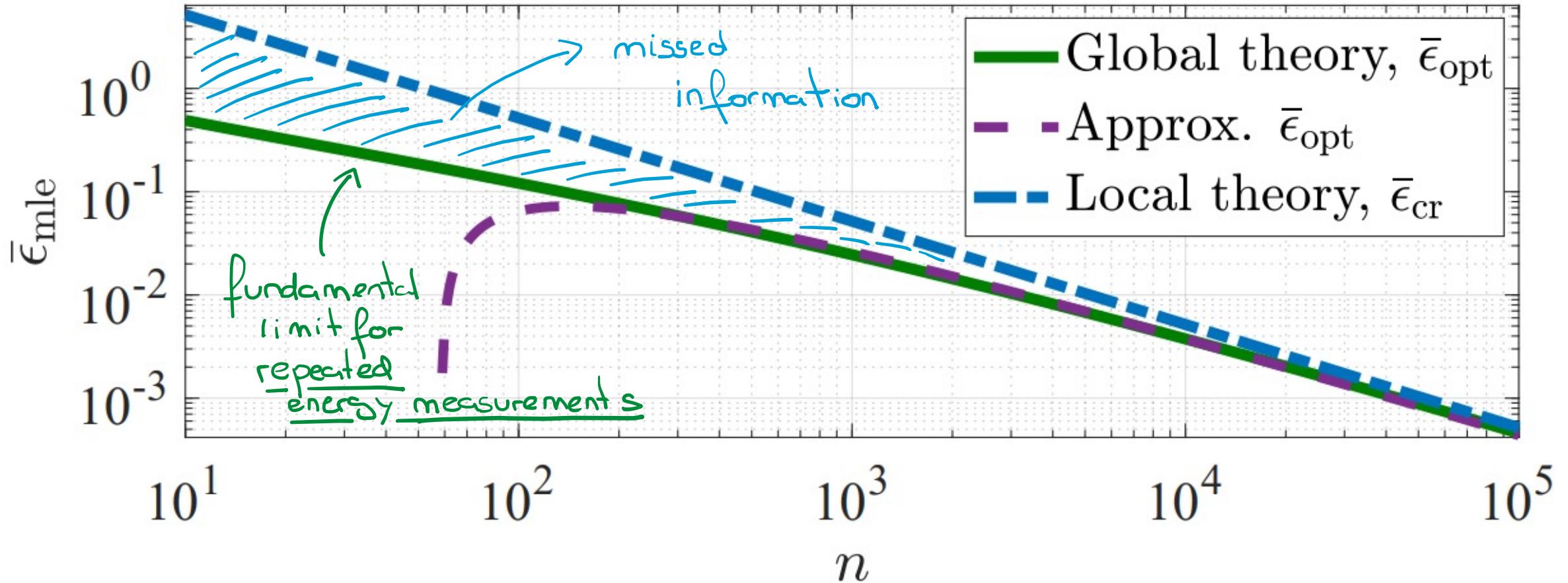


- Prior information: $p(\varphi) \propto \frac{1}{\theta}$; $\frac{k_B T}{\hbar\omega} \in [0.1, 10]$
- Measurement information: $p(r|\theta) = \binom{n}{r} \frac{\exp[-r\hbar\omega/(k_B\theta)]}{Z[\hbar\omega/(k_B\theta)]}$

The perils of local thermometry: non-interacting spin-1/2 gas



The perils of local thermometry: non-interacting spin-1/2 gas



Where to next? From thermometry to scale estimation

Bayesian
quantum
thermometry



Global Quantum Thermometry

Jesús Rubio,^{1,*} Janet Anders,^{1,2} and Luis A. Correa¹

¹Department of Physics and Astronomy, University of Exeter, Stocker Road, Exeter EX4 4QL, United Kingdom.

²Institut für Physik und Astronomie, University of Potsdam, 14476 Potsdam, Germany.

(Dated: 30th November 2020)

Where to next? From thermometry to scale estimation

Bayesian
quantum
thermometry



Global Quantum Thermometry

Jesús Rubio,^{1,*} Janet Anders,^{1,2} and Luis A. Correa¹
¹Department of Physics and Astronomy, University of Exeter, Stocker Road, Exeter EX4 4QL, United Kingdom.

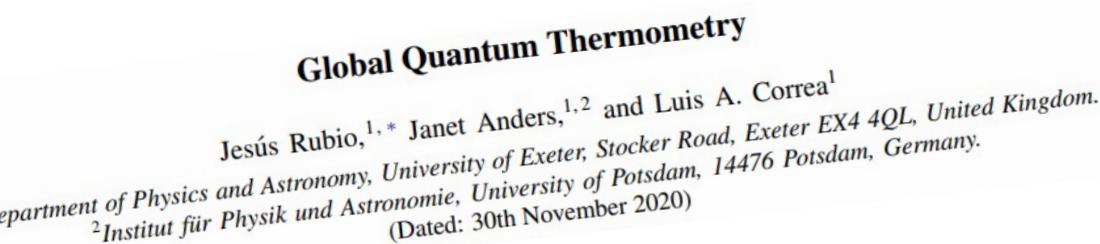
²Institut für Physik und Astronomie, University of Potsdam, 14476 Potsdam, Germany.
(Dated: 30th November 2020)

Bayesian estimation for collisional thermometry

Gabriel O. Alves^{1,*} and Gabriel T. Landi^{1,†}
¹Instituto de Física da Universidade de São Paulo, 05314-970 São Paulo, Brazil.

(Dated: June 24, 2021)

Where to next? From thermometry to scale estimation



Bayesian
quantum
thermometry

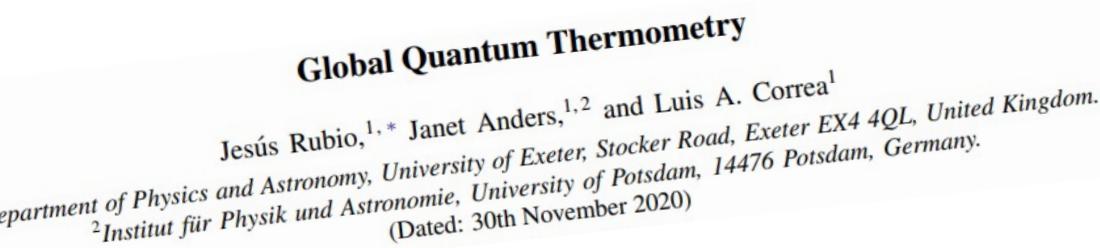


Bayesian est:
¹Instituto de Fr G₂,
Fundamental limits in Bayesian thermometry and attainability via adaptive strategies
Mohammad Mehboudi,^{1,*} Mathias R. Jørgensen,^{2,†} Stella Seah,¹
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24, 2021) **thermometry**

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24, 2021) **thermometry**

Where to next? From thermometry to scale estimation



Bayesian quantum thermometry



Fun

Bayesian estimation of attainability via adaptive strategies

Bayesian quantum thermometry based on thermodynamic length

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Gó,

Seah,¹

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Where to next? From thermometry to scale estimation

Global Quantum Thermometry

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(Dated: 30th November 2020)

Bayesian quantum thermometry

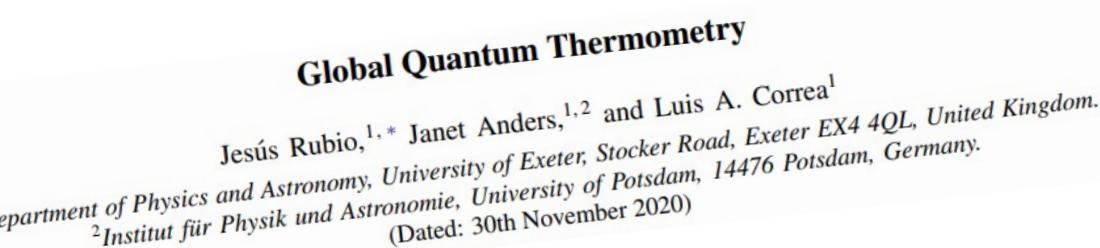
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Fun

Non-informative Bayesian Quantum Thermometry

ability
Seah,¹
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Where to next? From thermometry to scale estimation

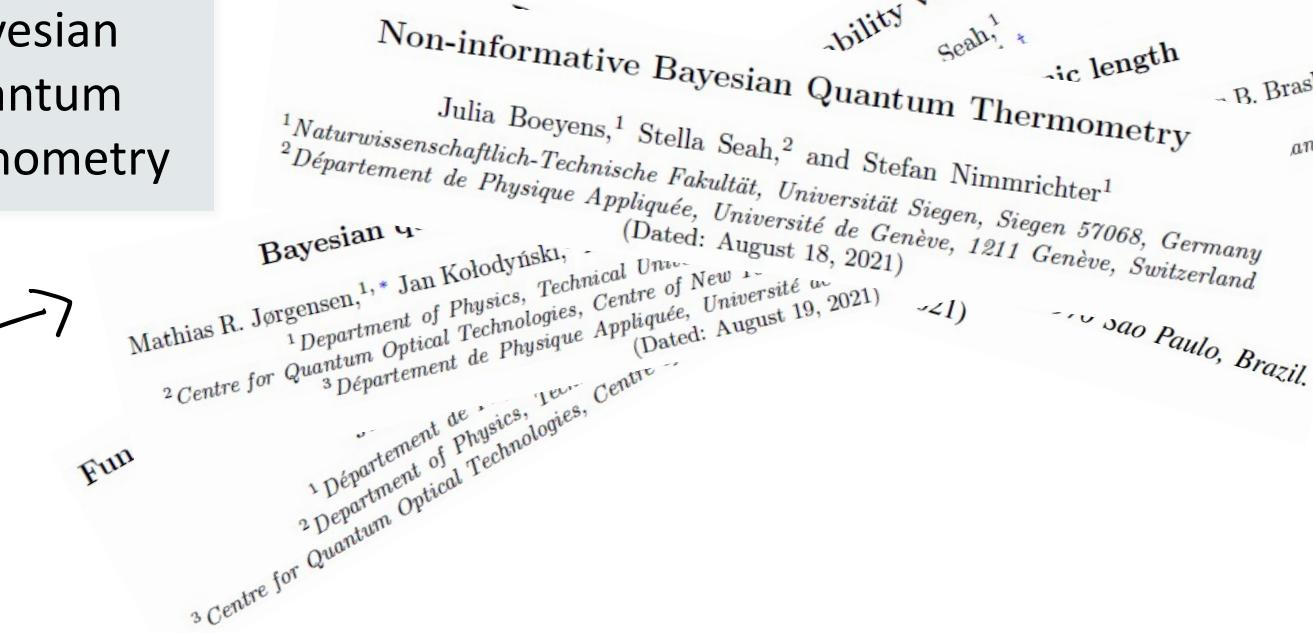


Quantum scale
estimation

Bayesian
quantum
thermometry

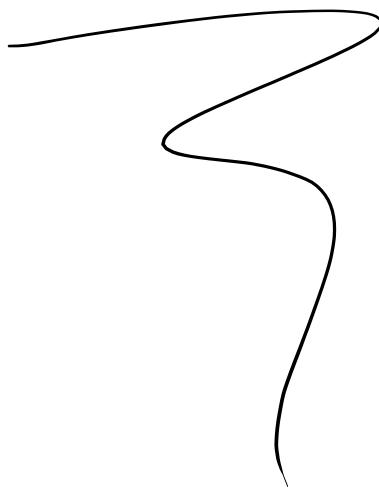


Fun



- Any scale parameter, temperature or otherwise
- Optimal measurement (not just the estimator)

Quantum metrology
of scale parameters

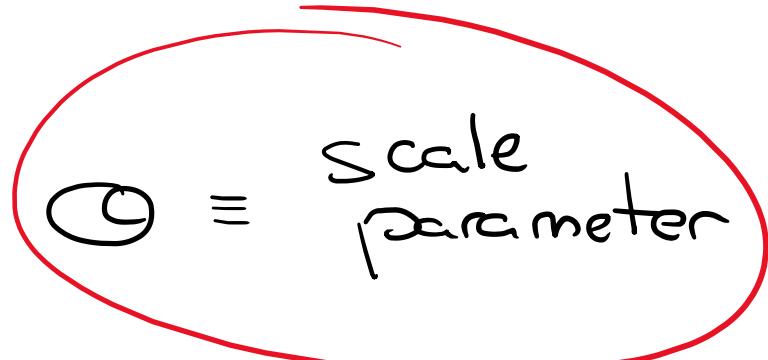


What is a scale parameter?

Examples:

- temperature: $\frac{E}{k_B T}$
- (Inverse of) Poisson rate: $kt = \frac{t}{1/k}$
- (Inverse of) decay rates: $\gamma t = \frac{t}{1/\gamma}$

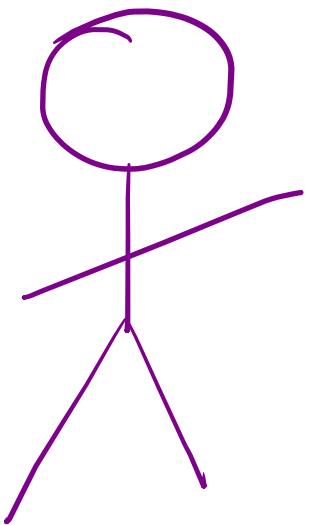
Definition:

$$\left. \begin{array}{l} Y \text{ is 'large' when } Y/\Theta \gg 1 \\ Y \text{ is 'small' when } Y/\Theta \ll 1 \end{array} \right\} \Rightarrow \Theta \equiv \text{scale parameter}$$


The **key symmetry**: scale invariance

$$\begin{aligned} Y &\mapsto Y' = \gamma Y & \rightarrow & Y'/\Theta' = Y/\Theta \\ \Theta &\mapsto \Theta' = \gamma \Theta \end{aligned}$$

Maximum ignorance about scale parameters



(A)



$$p(\alpha) d\alpha$$

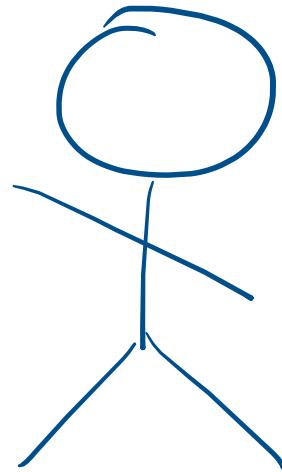
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$$\alpha' = \gamma \alpha$$



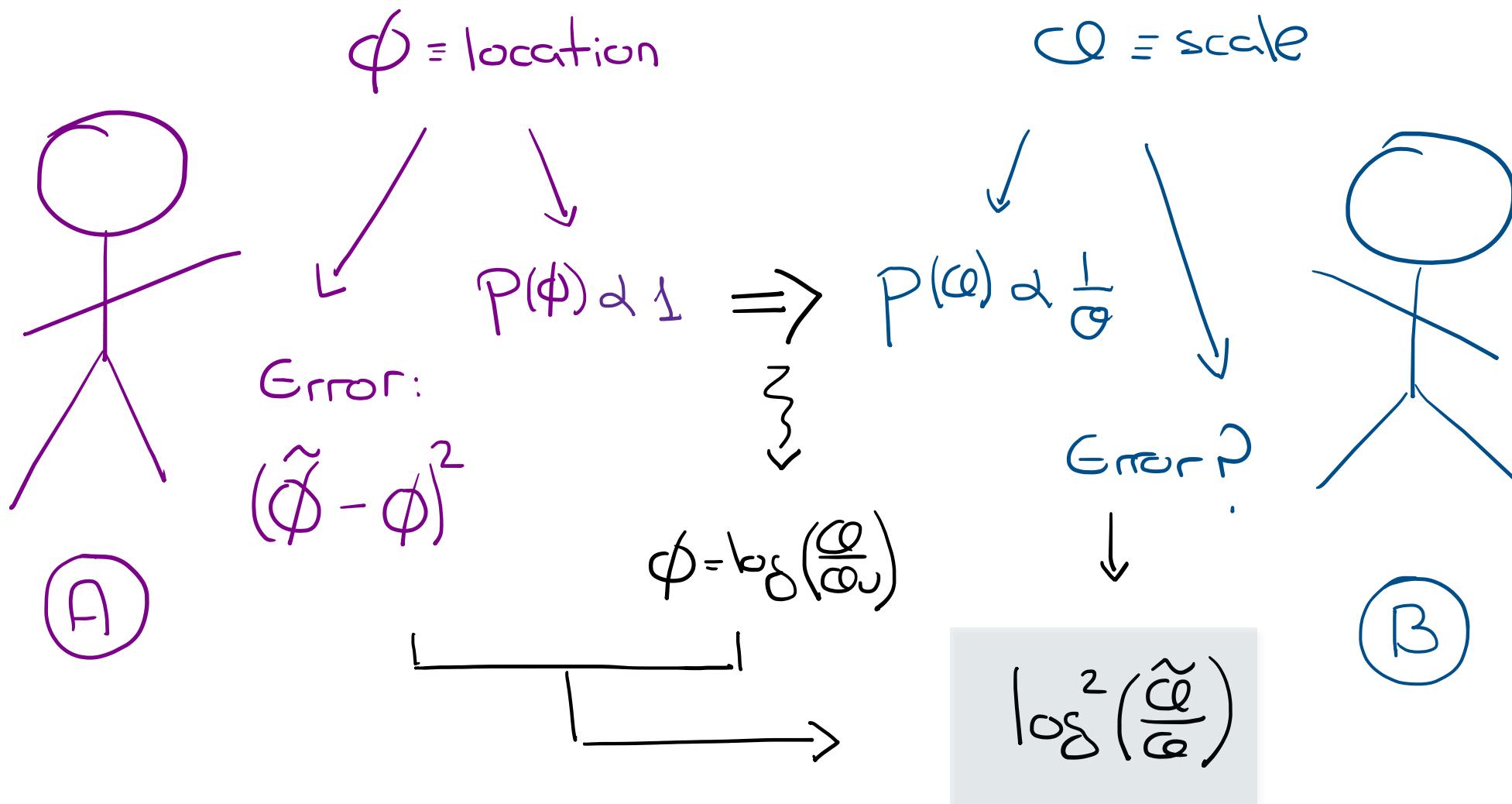
$$p(\alpha') d\alpha'$$



(B)

$$p(\alpha) = \frac{1}{\alpha}$$

Why logarithmic errors?



Quantum scale estimation: statement of the problem

Using the Born rule,

measurement

$$\bar{\epsilon}_{\text{mle}} = \text{Tr} \left\{ \int dx M(x) W[\tilde{\theta}(x)] \right\}; \quad W[\tilde{\theta}(x)] = \underbrace{\int d\theta p(\theta)}_{\text{prior}} \underbrace{\rho(\theta)}_{\text{state}} \log^2 \left[\frac{\tilde{\theta}(x)}{\theta} \right]$$

estimator

Our goal is to find the minimum:

$$\min_{\tilde{\theta}(x), M(x)} \text{Tr} \left\{ \int dx M(x) W[\tilde{\theta}(x)] \right\} = \bar{\epsilon}_{\text{min}}$$

The optimal quantum strategy

$$\mathcal{S} = \int ds |s\rangle\langle s| s$$

Calculate as
 $\overbrace{\hspace{1cm}}^>$

$$\mathcal{S}\varrho_0 + \varrho_0\mathcal{S} = 2\varrho_1$$

$$\varrho_k = \int d\theta p(\theta) \rho(\theta) \log^k \left(\frac{\theta}{\theta_u} \right)$$

Optimal measurement

$$\mathcal{M}(s) = |s\rangle\langle s|$$

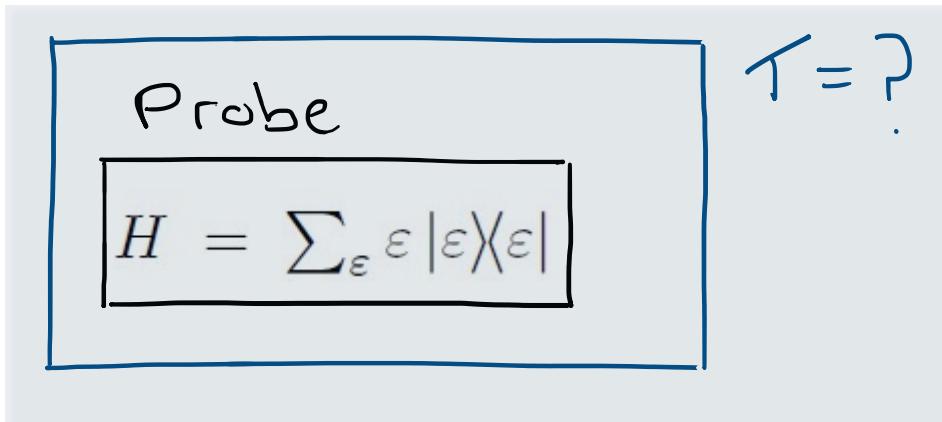
Optimal estimator

$$\tilde{\vartheta}(s) = \theta_u \exp \left[\int d\theta p(\theta|s) \log \left(\frac{\theta}{\theta_u} \right) \right]$$

Experimental error

$$\epsilon(s) = \int d\theta p(\theta|s) \log^2 \left[\frac{\tilde{\vartheta}(s)}{\theta_u} \right]$$

Revisiting equilibrium quantum thermometry



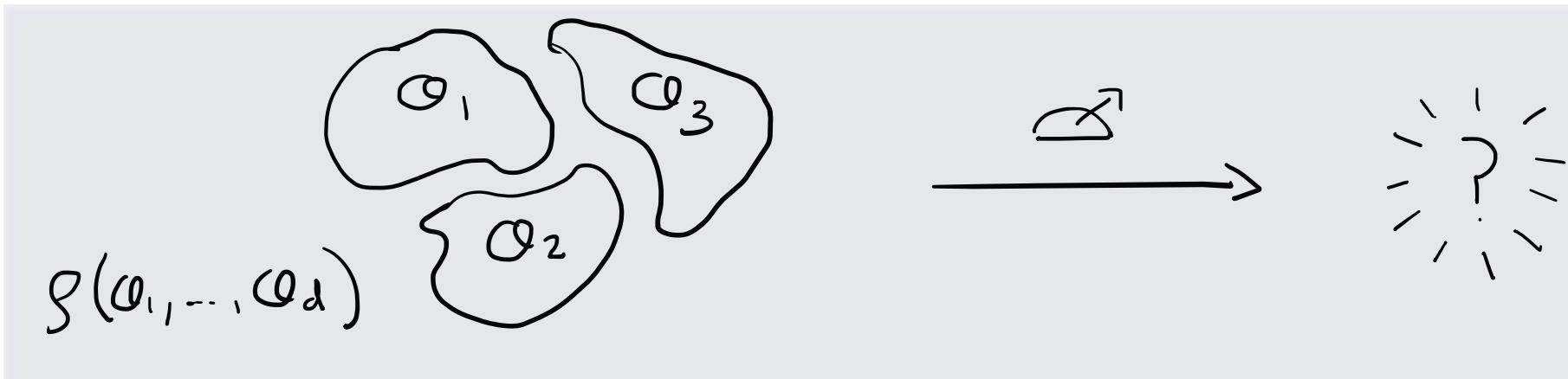
$$\tau = ?$$

$$\equiv \rho(\theta) = \frac{\exp[-H/(k_B\theta)]}{\text{Tr}\{\exp[-H/(k_B\theta)]\}}$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$
$$\mathcal{S} = \sum_{\varepsilon} |\varepsilon\rangle\langle\varepsilon| \frac{R_{\varepsilon,1}}{R_{\varepsilon,0}}$$

- For thermal states, **energy measurements are universally optimal**
- The optimal measurement may sometimes be implemented in the laboratory

Towards scale-invariant multi-parameter schemes



$$\bar{\epsilon}_{\text{mle}} \geq \frac{1}{d} \sum_{i=1}^d \left[\int d\theta p(\theta) \log^2 \left(\frac{\theta_i}{\theta_{u,i}} \right) - \text{Tr}(\rho_{0,i} \mathcal{S}_i^2) \right]$$

- Not saturable when $[S_i, S_j] \neq 0$
- Quantum compatibility: prior- and uncertainty-dependent

arXiv:2111.11921

A. Luis, Complementarity for Generalized Observables, Phys.
Rev. Lett. **88**, 230401 (2002).

Phases, locations and scales

Type of parameter	phase	location	scale
General support	$0 \leq \theta < 2\pi$	$-\infty < \theta < \infty$	$0 < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \theta + 2\gamma\pi, \gamma \in \mathbb{Z}$	$\theta \mapsto \theta' = \theta + \gamma, \gamma \in \mathbb{R}$	$\theta \mapsto \theta' = \gamma\theta, \gamma \in \mathbb{R}_*^+$
Maximum ignorance	$p(\theta) = 1/2\pi$	$p(\theta) \propto 1$	$p(\theta) \propto 1/\theta$
Deviation function $\mathcal{D}[\tilde{\theta}(x), \theta]$	$4 \sin^2 \{ [\tilde{\theta}(x) - \theta]/2 \}$	$[\tilde{\theta}(x) - \theta]^2$	$\log^2 [\tilde{\theta}(x)/\theta]$

An attractive perspective:

- > **Elementary quantities** (each its own quantum estimation theory)
- > Multi-parameter estimation with mixed models?

What have we learnt?

- There is, in networked quantum sensing, ...
 - > ... a **fundamental link between correlations and geometry**
 - > ... a trade-off between the asymptotic and non-asymptotic precisions
 - > ... a rich and unexplored area within limited-data metrology, which requires Bayesian techniques by construction
- Quantum thermometry *à la Bayes...*
 - > ... is very general (minimal assumptions)
 - > ... is **experimentally friendly**, as it provides
 - > a universal map from data sets to optimal estimates
 - > a clear and direct assessment of experimental errors
 - > ... is reliable (simulations) and **works in experiments**
 - > ... provides the key mathematics for the metrology of scale parameters
- Quantum scale estimation ...
 - > ... establishes a framework for the most precise estimation that the laws of quantum mechanics allow for scale parameters
 - > ... addresses an important gap in quantum metrology
 - > ... provides a fundamental picture : **phases, locations and scales**

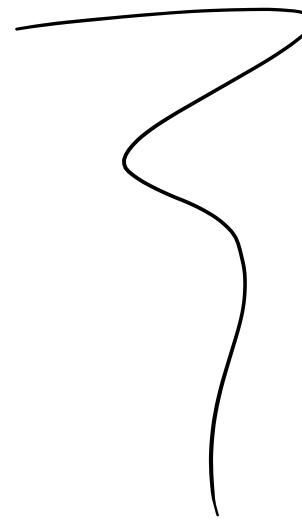
thank you for
your attention

arXiv:2111.11921

Phys. Rev. Lett. **127**, 190402 (2021)
J. Phys. A: Math. Theor., 53 344001 (2020)
JPhys. Rev. A 101, 032114 (2020)

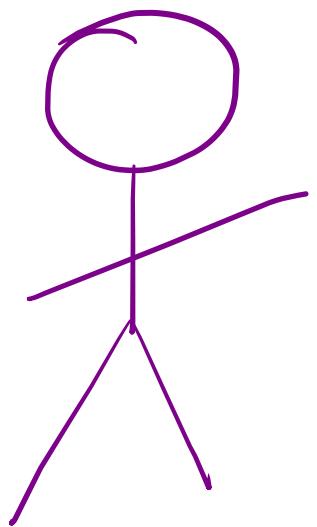
If you have any question or comment:
J.Rubio-Jimenez@exeter.ac.uk

Supplementary material



Why logarithmic errors?

Prior range: $\omega \in [0.01, 100]$; scale



(A)

$$\begin{aligned} p(\omega) d\omega \\ \downarrow \\ \int d\omega p(\omega) (\tilde{\omega} - \omega)^2 \\ \downarrow \\ \tilde{\omega} = \int d\omega p(\omega) \omega \end{aligned}$$

≈ 50

$$\begin{aligned} p(\omega) d\omega \\ \downarrow \\ \int d\omega p(\omega) \log^2\left(\frac{\tilde{\omega}}{\omega}\right) \\ \downarrow \\ \tilde{\omega} = \exp\left[\int d\omega p(\omega) \log(\omega)\right] \end{aligned}$$

(B)

= 1 ✓

Some theoretical consequences: quantum observables

$$\hat{\Theta} = \theta_u \exp(\mathcal{S}) = \dots = \int ds \mathcal{M}(s) \tilde{\vartheta}(s)$$

Diagram illustrating the components of the quantum observable $\hat{\Theta}$:

- Initial state and parameter encoding:** Represented by $\rho(\theta)$, indicated by arrows pointing to θ_u and $\exp(\mathcal{S})$.
- Prior information:** Represented by $p(\theta)$, indicated by an arrow pointing to θ_u .
- Quantum measurement + scale values:** Indicated by arrows pointing to the integral expression and the function $\tilde{\vartheta}(s)$.

- D1. The initial state and the associated parameter encoding, both captured by $\rho(\theta)$.
- D2. The prior information, represented by $p(\theta)$.
- D3. The fact that scale uncertainties are quantified using the mean logarithmic error $\bar{\epsilon}_{\text{mle}}$.

Some theoretical consequences: quantum observables



S. Personick, Application of quantum estimation theory to analog communication over quantum channels, *IEEE Transactions on Information Theory* **17**, 240 (1971)

→ phase/location observable



C. W. Helstrom, *Quantum Detection and Estimation Theory*
A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*

→ phase and time observables

→ position and momentum observables
(estimation-theoretic)



arXiv:2111.11921

Phys. Rev. Lett. **127**, 190402 (2021)

→ scale observable (e.g., temperature)